Data Fusion Based on Fuzzy Quantifiers

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Fuzzy quantifiers (like *many*, *few*, ...) are an important research topic not only due to their abundance in natural language (NL), but also because an adequate account of these quantifiers would provide a class of powerful yet human-understandable operators for information aggregation and data fusion. We introduce the DFS theory of fuzzy quantification, present a model of the theory, and describe an algorithm for the evaluation of the resulting fuzzy quantifiers. We discuss their use for data fusion and outline some areas of application.

1 Introduction

Fuzzy quantifiers (like *many*, *few*...) are an important research topic not only due to their abundance in natural language, but also because an adequate account of these quantifiers would provide a class of powerful yet human-understandable operators for information aggregation and data fusion. Recognising their value as genuine operators for the aggregation over *sets* of gradual evaluations (i.e. irreducible to pairwise combination of results), the use of fuzzy quantifiers has been suggested in the literature for various purposes of aggregation and data fusion and in a variety of applications including multi-criteria decision making, fuzzy databases and information retrieval, fuzzy expert systems, and others. These attempts are limited in use due to their lack of theoretical foundation, which can result in counter-intuitive behaviour as reported e.g. by Ralescu [9].

In the paper, we present an axiomatic theory of fuzzy quantification, based on the novel concept of a *determiner fuzzification scheme (DFS)*. Unlike existing approaches to fuzzy quantification, DFS is

- a compatible extension of the *theory of generalized quantifiers* (TGQ [1]);
- a genuine theory of fuzzy multi-place quantification;
- not limited to absolute and proportional quantifiers;
- able to handle both quantitative and non-quantitative (i.e. qualitative) quantifiers;
- not limited to finite universes of discourse;
- based on a rigid axiomatic foundation;
- fully compatible to negation, antonyms, duals, and other important constructions on quantifiers.

We then present a model $\mathcal{M}$ of the theory and show how the resulting aggregation operators can be efficiently evaluated based on histogram computations. Finally, we discuss their application to data fusion and sketch some areas of application.

2 DFS theory

The basic idea behind DFS can be sketched as follows. By an n-ary *fuzzy quantifier* on a universe $E \neq \varnothing$ we denote a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I} = [0,1]$ which to each n-tuple of fuzzy subsets $X_i \in \widetilde{\mathcal{P}}(E)$ of E assigns a gradual result $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$. This definition is the obvious extension of generalised quantifiers (determiners) in the sense of TGQ to the fuzzy case. But which fuzzy quantifier can be arguably said to correspond to a given natural language quantifier, say "all"? In order to alleviate this problem, and to provide a representation of fuzzy quantifiers which lends itself better to understanding, DFS theory introduces the notion of a *semi-fuzzy quantifier*, i.e. a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$

which to each n-tuple of *crisp* subsets of E assigns a gradual result $Q(X_1, \ldots, X_n) \in \mathbf{I}$. Examples of semi-fuzzy quantifiers are

$$\forall_E(X) = 1 \Leftrightarrow X = E$$
$$\exists_E(X) = 1 \Leftrightarrow X \neq \varnothing$$
$$\mathbf{all}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \subseteq X_2$$
$$\mathbf{some}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \cap X_2 \neq \varnothing$$
$$\mathbf{atleast\ n}_E(X_1, X_2) = 1 \Leftrightarrow |(X_1 \cap X_2)| \geq n$$
$$\mathbf{prop}_E(X_1, X_2) = \begin{cases} \dfrac{|X_1 \cap X_2|}{|X_1|} & : \ |X_1| \neq \varnothing \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

where $E \neq \varnothing$ is a base set, $X, X_1, X_2 \in \mathcal{P}(E)$, and $|.|$ denotes cardinality. **prop**, the proportion of $X_1's$ that are X_2, is defined for finite E only; $\frac{1}{2}$ represents indeterminacy. We will usually drop the subscript E.

Semi-fuzzy quantifiers are systematically generalised to corresponding fuzzy quantifiers $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by applying a determiner fuzzification scheme.

Definition 1 (Determiner fuzzification schemes) *Suppose $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$. $\mathcal{F}$ is called a* determiner fuzzification scheme (DFS) *iff the following axioms are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

$$\mathcal{F}(Q) = Q \quad \text{if } n = 0 \qquad \text{(DFS 1)}$$
$$\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}}) = \mathrm{id}_{\mathbf{I}} \qquad \text{(DFS 2)}$$
$$\mathcal{F}(\widetilde{\neg}\, Q) = \widetilde{\neg}\, \mathcal{F}(Q) \qquad \text{(DFS 3)}$$
$$\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i \quad \text{if } i \in \{1, \ldots, n\} \qquad \text{(DFS 4)}$$
$$\mathcal{F}(Q\neg) = \mathcal{F}(Q)\widetilde{\neg} \quad \text{if } n > 0 \qquad \text{(DFS 5)}$$
$$\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\cap} \quad \text{if } n > 0 \qquad \text{(DFS 6)}$$
$$\mathcal{F}(Q \triangleleft A) = \mathcal{F}(Q) \triangleleft A \quad \text{if } n > 0,\ A \text{ crisp} \qquad \text{(DFS 7)}$$

Q nonincreasing in n-th arg

$$\Rightarrow \mathcal{F}(Q) \text{ nonincr. in } n\text{-th arg, } n > 0 \qquad \text{(DFS 8)}$$
$$\mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} \hat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i) \qquad \text{(DFS 9)}$$

where $f_1, \ldots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.

In the following, we will denote by $\mathbb{F}_{E,n}$ the set of all fuzzy quantifiers $Q : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ and by $\mathbb{S}_{E,n}$ the set of all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

We will now briefly explain the DFS axioms and the operators used in the above definition.[1] A more thorough treatment, including all proofs of properties claimed, can be found in [5].

Preservation of constants The set of constant semi-fuzzy quantifiers coincides with the set of constant fuzzy quantifiers. It is natural to require that the fuzzification scheme maps every constant semi-fuzzy quantifier to itself (viewed as a fuzzy quantifier). An important consequence of (DFS 1) is that for every $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\mathcal{F}(Q)|_{\mathcal{P}(E)^n} = Q \qquad (1)$$

$\mathcal{F}(Q)$ hence corresponds to the original quantifier when all argument sets are crisp.

Identity axiom By a canonical construction which we describe now, $\mathcal{F}$ induces a unique fuzzy operator for each of the propositional connectives. Let $\{*\}$ be a given singleton set, and define $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{2}$ by $\pi_*(Y) = \chi_Y(*)$, where χ_Y is the characteristic function of $Y \in \mathcal{P}(\{*\})$. Furthermore, let $\widetilde{\pi}_* : \widetilde{\mathcal{P}}(\{*\}) \longrightarrow \mathbf{I}$ the bijection $\widetilde{\pi}_*(X) = \mu_X(*)$, where μ_X is the membership function of $X \in \widetilde{\mathcal{P}}(\{*\})$.

Definition 2 (Induced truth functions) *Suppose $f : \mathbf{2}^n \longrightarrow \mathbf{2}$ is a propositional function (e.g., $f = \wedge$). We can view f as a semi-fuzzy quantifier $f^* : \mathcal{P}(\{*\})^n \longrightarrow \mathbf{2} \subseteq \mathbf{I}$ by defining*

$$f^*(X_1, \ldots, X_n) = f(\pi_*(X_1), \ldots, \pi_*(X_n)).$$

By applying $\mathcal{F}$, f^* is generalized to a fuzzy quantifier $\mathcal{F}(f^*) : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \mathbf{I}$, from which we obtain a fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$,

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots x_n) = \mathcal{F}(f^*)(\widetilde{\pi}_*^{-1}(x_1), \ldots, \widetilde{\pi}_*^{-1}(x_n))$$

for all $x_1, \ldots, x_n \in \mathbf{I}$.

By pointwise application of the induced negation $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$, conjunction $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$, and disjunction $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$, $\mathcal{F}$ also induces a unique choice of fuzzy complement $\widetilde{\neg}$, fuzzy intersection $\widetilde{\cap}$, and fuzzy union $\widetilde{\cup}$.

(DFS 2) requires that $\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}}) = \mathrm{id}_{\mathbf{I}}$, i.e. the identity truth function is mapped to its intended fuzzy analogon. This is sufficient to ensure that all truth-functions are mapped to their intended fuzzy counterparts: in every DFS, $\widetilde{\neg}$ is a strong negation operator, $\widetilde{\wedge}$ is a t-norm, and $\widetilde{\vee}$ the dual s-norm of $\widetilde{\wedge}$.

[1]We will define these operators for $Q \in \mathbb{S}_{E,n}$ only. The definitions for $\widetilde{Q} \in \mathbb{F}_{E,n}$ are entirely analogous.

External negation Let $Q \in \mathbb{S}_{E,n}$ a semi-fuzzy quantifier. By $\widetilde{\neg}\, Q \in \mathbb{S}_{E,n}$ we denote the semi-fuzzy quantifier defined by

$$(\widetilde{\neg}\, Q)(X_1, \ldots, X_n) = \widetilde{\neg}\,(Q(X_1, \ldots, X_n))$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

(DFS 3) states that every DFS $\mathcal{F}$ commutes with the formation of (external) negations of arbitrary semi-fuzzy quantifiers. For example, from **no** $= \widetilde{\neg}$ **some** we obtain $\mathcal{F}(\textbf{no}) = \widetilde{\neg}\, \mathcal{F}(\textbf{some})$.

Argument transposition Let $Q \in \mathbb{S}_{E,n}, n > 0$ and $i \in \{0, \ldots, n\}$. By $Q\tau_i \in \mathbb{S}_{E,n}$ we denote the semi-fuzzy quantifier defined by

$$Q\tau_i(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{i-1}, X_n, X_{i+1}, \ldots, X_{n-1}, X_i)\,,$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

Because every permutation can be expressed as a sequence of transpositions, (DFS 4) ensures that $\mathcal{F}$ commutes with arbitrary permutations of the arguments of a quantifier.

Antonyms Let $Q \in \mathbb{S}_{E,n}, n > 0$. The antonym $Q\neg \in \mathbb{S}_{E,n}$ is defined by

$$Q\neg(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{n-1}, \neg X_n)$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

(DFS 5) requires that $\mathcal{F}$ be compatible to the formation of *antonyms*. For example, from **no** $=$ **all**$\neg$ we obtain $\mathcal{F}(\textbf{no}) = \mathcal{F}(\textbf{all})\widetilde{\neg}$.

By the *dual* of a semi-fuzzy quantifier Q we mean $\widetilde{\neg} Q\neg$ (analogously for fuzzy quantifiers). Combining (DFS 3) and (DFS 5), we obtain that every DFS is also compatible to dualisation. Hence from **some** $= \widetilde{\neg}$ **all**$\neg$, we also have $\mathcal{F}(\textbf{some}) = \widetilde{\neg}\, \mathcal{F}(\textbf{all})\widetilde{\neg}$.

Internal meets Suppose $Q \in \mathbb{S}_{E,n}$, $n > 0$. The semi-fuzzy quantifier $Q\cap \in \mathbb{S}_{E,n+1}$ is defined by

$$Q\cap(X_1, \ldots, X_{n+1}) = Q(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})\,,$$

for all $(X_1, \ldots, X_{n+1}) \in \mathcal{P}(E)^{n+1}$.

(DFS 6) states that $\mathcal{F}$ is compatible to intersections in the last argument of a semi-fuzzy quantifier. For example, $\mathcal{F}(\textbf{some}) = \mathcal{F}(\exists)\widetilde{\cap}$, because the two-place quantifier **some** can be expressed as **some** $= \exists\cap$. By (DFS 4), (DFS 6) generalises to intersections in arbitrary argument positions.

Argument insertion Suppose $Q \in \mathbb{S}_{E,n}$, $n > 0$, and $A \in \mathcal{P}(E)$. By $Q \triangleleft A \in \mathbb{S}_{E,n-1}$ we denote the semi-fuzzy quantifier defined by

$$Q \triangleleft A(X_1, \ldots, X_{n-1}) = Q(X_1, \ldots, X_{n-1}, A)$$

for all $(X_1, \ldots, X_{n-1}) \in \mathcal{P}(E)^{n-1}$.

(DFS 7) expresses that every DFS commutes with the insertion of arguments. The axiom is of particular importance because it generally ensures that *boundary conditions* (with respect to the crisp case) are valid. For example, the axiom ensures that $x_1 \mathbin{\widetilde{\wedge}} 1 = x_1$ for all $x_1 \in \mathbf{I}$, which is one of the defining conditions of t-norms.

Preservation of monotonicity A semi-fuzzy quantifier $Q \in \mathbb{S}_{E,n}$ is said to be nonincreasing in its i-th argument ($i \in \{1, \ldots, n\}$, $n > 0$) iff for all $X_1, \ldots, X_n, X_i' \in \mathcal{P}(E)$ such that $X_i \subseteq X_i'$, $Q(X_1, \ldots, X_n) \geq Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$. On fuzzy quantifiers $\widetilde{Q} \in \mathbb{F}_{E,n}$, we use an analog definition, where $X_1, \ldots, X_n, X_i' \in \widetilde{\mathcal{P}}(E)$, and "$\subseteq$" is the fuzzy inclusion relation.

(DFS 8) expresses that $\mathcal{F}$ preserves decreasing monotonicity of a quantifier in its last argument. By (DFS 4), (DFS 8) generalises to the preservation of decreasing monotonicity in arbitrary argument positions. It also follows that $\mathcal{F}$ preserves increasing monotonicity, and even monotonicity properties of quantifiers which hold only locally.

Induced extension principle Every mapping $f : E' \longrightarrow E$ uniquely determines a powerset function $\widehat{f} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$, which is defined by $\widehat{f}(X) = \{f(e) : e \in X\}$, for all $X \in \mathcal{P}(E')$. The underlying mechanism which transports f to $\widehat{f}$ can be generalized to the case of fuzzy sets.

Definition 3 (Induced extension principle) *$\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E' \longrightarrow E$ (where $E, E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ defined by*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e) = \mathcal{F}(\chi_{\widehat{f}(\_)}(e))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E')$, $e \in E$.

(DFS 9) establishes a relation between powerset functions and $\widehat{\mathcal{F}}$ by requiring that $\mathcal{F}$ be compatible with its induced extension principle. Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $f_1, \ldots, f_n : E' \longrightarrow E$ are given ($E' \neq \varnothing$). Then

$$Q' = Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$$

is defined by

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n)),$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E')$. (DFS 9) requires that $\mathcal{F}(Q') = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$, i.e.

$$\begin{aligned}&\mathcal{F}(Q')(X_1, \ldots, X_n) = \\ &\qquad \mathcal{F}(Q)(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n))\end{aligned}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$.

(DFS 9) is of particular importance because it is the only axiom which relates the behaviour of $\mathcal{F}$ on different domains E, E'.

Definition 4 (Standard DFS) *A DFS $\mathcal{F}$ is called a* standard DFS *if it induces the standard negation $1 - x$ and the standard extension principle,*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e) = \sup\{\mu_X(v) : v \in f^{-1}(e)\},$$

for all $f : E' \longrightarrow E$, $e \in E$, $X \in \widetilde{\mathcal{P}}(E')$.

Every standard DFS induces the standard connectives, i.e. $\widetilde{\mathcal{F}}(\wedge) = \min$, $\widetilde{\mathcal{F}}(\vee) = \max$ etc.

3 DFS models

By stating the DFS axioms, we have made explicit our intuitions about "reasonable" mechanisms of fuzzy quantification. In order to show that these axioms are consistent (but also to make the theory useful for purposes of data fusion), we now present an actual model. The model uses the fuzzy median as an aggregation operator over sets of gradual evaluations.

Definition 5 (Fuzzy median)
The fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is defined by

$$\mathrm{m}_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & : \ \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & : \ \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

[2]i.e. of those $e \in E$ such that $e \in (X_i)_\gamma^{\max} \setminus (X_i)_\gamma^{\min}$

The fuzzy median can be extended to an operator (again denoted $\mathrm{m}_{\frac{1}{2}}$) which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.

Definition 6 (Extended fuzzy median)
The (extended) fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ is defined by

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X, \ \sup X)\,,$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Definition 7
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $\gamma \in \mathbf{I}$. $Q_\gamma \in \mathbb{F}_{E,n}$ is defined by

$$Q_\gamma(X_1, \ \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma\}\,,$$

where (for $i = 1, \ldots, n$),

$$\begin{aligned} \mathcal{Y}_i^\gamma &= \{Y \subseteq E : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ (X_i)_\gamma^{\min} &= \begin{cases} (X_i)_{\geq \frac{1}{2} + \frac{1}{2}\gamma} & : \ \gamma \in (0, 1] \\ (X_i)_{> \frac{1}{2}} & : \ \gamma = 0 \end{cases} \\ (X_i)_\gamma^{\max} &= \begin{cases} (X_i)_{> \frac{1}{2} - \frac{1}{2}\gamma} & : \ \gamma \in (0, 1] \\ (X_i)_{\geq \frac{1}{2}} & : \ \gamma = 0 \end{cases} \end{aligned}$$

$(X_i)_{\geq \alpha}$ α-cut, $(X_i)_{>\alpha}$ strict α-cut.

γ can be thought of as a parameter of "cautiousness". If $\gamma = 0$, the set of indeterminates[2] contains only those $e \in E$ with $\mu_{X_i}(e) = \frac{1}{2}$; all other elements of E are mapped to the closest truth value in $\{0, 1\}$. As γ increases, the set of indeterminates is increasing. For $\gamma = 1$, then, the level of maximal cautiousness is reached where all elements of E except those with $\mu_{X_i}(e) \in \{0, 1\}$ are interpreted as indeterminates.

The assignment $Q \mapsto Q_\gamma$ is not a DFS yet; the fuzzy median suppresses too much structure. We must hence take into account the results obtained at each level of cautiousness.

Definition 8
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{M}(Q) \in \mathbb{F}_{E,n}$ is defined by

$$\mathcal{M}(Q)(X_1, \ \ldots, X_n) = \int_0^1 Q_\gamma(X_1, \ \ldots, X_n)\, d\gamma\,.$$

It can be shown that the integral exists, regardless of Q and the choice of argument sets.

Theorem 1 *$\mathcal{M}$ is a standard DFS.*

Note. In [5], it was shown that $\mathcal{M}$ is a consistent generalisation of the fuzzification scheme proposed by Gaines [4].

4 Computation of DFS quantifiers

Our theory has been designed to establish a principled account of fuzzy quantifiers in data fusion, and we have ensured this by stating it in the form of axioms. We have also provided a model $\mathcal{M}$ of the theory. In the following, we will show how to implement the resulting quantifiers $\mathcal{M}(Q)$.

4.1 Evaluation of "simple" quantifiers

Let us firstly consider the standard quantifiers. Because $\mathcal{M}$ is a standard DFS, we obtain that

$$\begin{aligned}
\mathcal{M}(\exists)(X) &= \sup\{\mu_X(e) : e \in E\} \\
\mathcal{M}(\forall)(X) &= \inf\{\mu_X(e) : e \in E\} \\
\mathcal{M}(\textbf{all})(X_1, X_2) &= \inf\{\max(1 - \mu_{X_1}(e), \\
&\qquad \mu_{X_2}(e)) : e \in E\} \\
\mathcal{M}(\textbf{some})(X_1, X_2) &= \sup\{\min(\mu_{X_1}(e), \\
&\qquad \mu_{X_2}(e)) : e \in E\}
\end{aligned}$$

for all base sets $E \neq \varnothing$ and X, X_1, $X_2 \in \widetilde{\mathcal{P}}(E)$.

In addition, if $E \neq \varnothing$ is finite, we have

$$\mathcal{M}(\textbf{atleast n})(X_1, X_2) = \mu_{(n)}$$

for all X_1, $X_2 \in \widetilde{\mathcal{P}}(E)$, where $\mu_{(i)}$ is the i-th largest element in the ordered sequence of membership values of $X_1 \mathbin{\widetilde{\cap}} X_2$, including duplicates.

4.2 Histogram-based evaluation

In the general case, $\mathcal{M}(Q)$ cannot be described by closed-form expressions (as in 4.1). However, the resulting fuzzy quantifiers can often be efficiently implemented on the basis of histogram computations if Q is quantitative.[3]

In the following, we shall assume that the base set E be finite. For simplicity of presentation, we will describe a procedure for computing DFS-quantifiers suited to integer-arithmetics. We hence assume that, for a fixed $m' \in \mathbb{N} \setminus \{0\}$, all membership values of fuzzy argument sets $X_1, \ldots, X_n$ satisfy

$$\mu_{X_i}(e) \in \left\{0, \frac{1}{m'}, \ldots, \frac{m'-1}{m'}, 1\right\} \qquad (2)$$

for all $e \in E$.

If $X \in \widetilde{\mathcal{P}}(E)$ satisfies (2), we can conveniently represent the required histogram of X as an $(m' + 1)$-dimensional array $\mathrm{Hist}_X : \{0, \ldots, m'\} \longrightarrow \mathbb{N}$, defined by

$$\mathrm{Hist}_X[j] = \left|\{e \in E : \mu_X(e) = \frac{j}{m'}\}\right|$$

for all $j = 0, \ldots, m'$. We further assume that m' is even, (i.e. $m' = 2m$ for a given $m \in \mathbb{N} \setminus \{0\}$).

4.3 Evaluation of one-place quantifiers

Suppose $Q \in \mathbb{S}_{E,1}$ is quantitative. There exists $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that $Q(X) = q(|X|)$ for all $X \in \mathcal{P}(E)$. Abbreviating $L = |X|_\gamma^{\min} = |(X)_\gamma^{\min}|$, $U = |X|_\gamma^{\max} = |(X)_\gamma^{\max}|$, we define

$$\begin{aligned}
q^{\max}(L, U) &= \max\{q(K) : L \leq K \leq U\} \\
q^{\min}(L, U) &= \min\{q(K) : L \leq K \leq U\}
\end{aligned}$$

Then $Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(q^{\min}(L, U), q^{\max}(L, U))$, and $\mathcal{M}(Q)(X)$ can be computed as follows.

```
ALGORITHM DFS-UNARY
INPUT: X
// initialise H, L, U
H = Hist_X;
L = Σ_{j=1}^{m} H[m+j];
U = L + H[m];
cq= m_½(q^min(L,U), q^max(L,U));
if( cq== ½ ) return ½;
sum = cq;
if( cq > ½ )
  for( j=1; j<m; j++ ) {
    nc= true; // "no change"
    // update clauses for L and U
    if( H[m+j] ≠ 0 )
      { L -= H[m+j]; nc= false; }
    if( H[m-j] ≠ 0 )
      { U += H[m-j]; nc= false; }
    if( nc) { sum += cq; continue; }
    // one of L or U has changed
    cq= q^min(L,U);
    if( cq ≤ ½ ) break;
    sum += cq;
    }
else
  for( j=1; j<m; j++ ) {
```

[3] i.e. $Q(\beta(X_1), \ldots, \beta(X_n)) = Q(X_1, \ldots, X_n)$ for all automorphisms $\beta : E \longrightarrow E$.

```
        nc= true;
        ⋮ // update clauses etc. as above
        cq= q^max(L,U);
        if( cq ≥ 1/2 ) break;
        sum += cq;
        }
return (sum + 1/2*(m-j)) / m;
END
```

If Q fulfills some additional requirements, computation of $q^{\min}$ and $q^{\max}$ can be simplified. Firstly, if Q is unimodal (i.e. there is an $j_{\mathrm{pk}} \in \{0, \ldots, |E|\}$ such that q is nondecreasing for all $i \leq j_{\mathrm{pk}}$ and nonincreasing for all $i \geq j_{\mathrm{pk}}$), then

$$q^{\min}(L, U) = \min(q(L), q(U))$$

$$q^{\max}(L, U) = \begin{cases} q(j_{\mathrm{pk}}) & : \ L \leq j_{\mathrm{pk}} \leq U \\ q(U) & : \ U < j_{\mathrm{pk}} \\ q(L) & : \ L > j_{\mathrm{pk}} \end{cases}$$

Examples for unimodal quantifiers are **exactly n**, **about n**, **between n and m** (one-place use).

A further simplification is possible if Q is nondecreasing. Then $q^{\min}(L, U) = q(L)$ and $q^{\max}(L, U) = q(U)$, i.e. we can omit the updating of U in the first for-loop and likewise omit L in the second for-loop. Nonincreasing quantifiers permit similar simplifications.

4.4 Evaluation of two-place quantifiers

Absolute quantifiers Suppose $Q \in \mathbb{S}_{E,2}$ and there is a quantitative $Q' \in \mathbb{S}_{E,1}$ such that $Q(X_1, X_2) = Q'(X_1 \cap X_2)$ for all $X_1, X_2 \in \mathcal{P}(E)$. The DFS axioms ensure that $\mathcal{M}(Q)(X_1, X_2) = \mathcal{M}(Q')(X_1 \tilde{\cap} X_2)$ for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, i.e. in order to compute $\mathcal{M}(Q)(X_1, X_2)$, we can use the above algorithm DFS-UNARY for computing $\mathcal{M}(Q')$.

Quantifiers of exception In addition to absolute quantifiers, one often encounters semi-fuzzy quantifiers $Q \in \mathbb{S}_{E,2}$ such that the antonym $Q\neg$ of Q is an absolute quantifier.

Quantifier	Antonym (absolute)
all	**no**
all except exactly n	**exactly n**
all except about n	**about n**
all except at most n	**at most n**

By the DFS axioms, $\mathcal{M}(Q)(X_1, X_2) = \mathcal{M}(Q\neg)(X_1, \tilde{\neg} X_2)$ for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. We can use the above algorithm for evaluating the absolute quantifier $Q\neg$ in order to compute $\mathcal{M}(Q)$.

Proportional quantifiers Let us now turn to genuine two-place quantifiers, i.e. semi-fuzzy quantifiers which are irreducible to one-place quantifiers. The most important examples are the so-called *proportional quantifiers*. These are defined by $Q(X_1, X_2) = q(|X_1|, |X_1 \cap X_2|)$, where $q : \{0, \ldots, |E|\}^{\mathbf{2}} \longrightarrow \mathbf{I}$ has the form

$$q(a, b) = \begin{cases} v_0 & : \ a = 0 \\ f(b/a) & : \ a \neq 0 \end{cases} \tag{3}$$

An example is **prop** where $v_0 = \frac{1}{2}$ and $f = \mathrm{id}_{\mathbf{I}}$. We shall restrict attention here to those proportional quantifiers where $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing.[4] Suppose Q is such a quantifier and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. We are using abbreviations $Z_1 = X_1$, $Z_2 = X_1 \tilde{\cap} X_2$ and $Z_3 = X_1 \tilde{\cap} \tilde{\neg} X_2$; let $L_k = |Z_k|_\gamma^{\min}$ and $U_k = |Z_k|_\gamma^{\max}$, $k \in \{1, 2, 3\}$. Then

$$Q_\gamma(X_1, X_2) = \mathrm{m}_{\frac{1}{2}}(q^{\min}(L_1, L_2, U_1, U_3), \\ q^{\max}(L_1, L_3, U_1, U_2))\,.$$

Abbreviating $f^{\min} = f(L_2/(L_2 + U_3))$, $q^{\min}\{0, \ldots, |E|\}^4 \longrightarrow \mathbf{I}$ is defined as follows.

1. $L_1 > 0$. Then $q^{\min} = f^{\min}$.
2. $L_1 = 0$.
   a. $L_2 + U_3 > 0$. Then $q^{\min} = \min(v_0, f^{\min})$.
   b. $L_2 + U_3 = 0$.
      i. $U_1 > 0$. Then $q^{\min} = \min(v_0, f(1))$.
      ii. $U_1 = 0$. Then $q^{\min} = v_0$.

Note. If $v_0 \leq f(1)$, then $\min(v_0, f(1)) = v_0$, i.e. we need not distinguish 2.b.i and 2.b.ii.

For $q^{\max}(L_1, L_3, U_1, U_2)$, we have

1. $L_1 > 0$. Then $q^{\max} = f^{\max}$.
2. $L_1 = 0$.
   a. $U_2 + L_3 > 0$. Then $q^{\max} = \max(v_0, f^{\max})$.

[4] If f is nonincreasing, we can compute $\mathcal{M}(Q) = \tilde{\neg}\mathcal{M}(\tilde{\neg} Q)$, noting that $\tilde{\neg} Q$ of Q is proportional and nondecreasing.

b. $U_2 + L_3 = 0$.

i. $U_1 > 0$. Then $q^{\max} = \max(v_0, f(0))$.

ii. $U_1 = 0$. Then $q^{\max} = v_0$.

where $f^{\max} = f(U_2/(U_2 + L_3))$.

Note. If $f(0) \leq v_0$, then 2.b.i and 2.b.ii need not be distinguished.

A slight modification of DFS-UNARY will suffice to evaluate proportional quantifiers:

1. In the initialisation part, compute H_k, L_k and U_k for $k \in \{1, 2, 3\}$.
2. Replace both occurrences of $q^{\min}(\dots)$ by $q^{\min}(L_1, L_2, U_1, U_3)$ and both occurrences of $q^{\max}(\dots)$ by $q^{\max}(L_1, L_3, U_1, U_2)$.
3. In the first for-loop, use update clauses for L_1, L_2, U_1, U_3.
4. In the second for-loop, use update statements for L_1, L_3, U_1, U_2.

5 Application of DFS to data fusion

In the following, we will establish the relationship between fuzzy quantifiers and data fusion. Suppose E denotes a set of experts (sensors, algorithms...) and p a proposition to be evaluated, e.g. $p =$ "Feature x_l corresponds to feature y_r". Each expert $e \in E$ has an associated degree of "competence" $\mu_C(e) \in \mathbf{I}$ with respect to evaluating p, and each expert provides a gradual evaluation $\mu_T(e) \in \mathbf{I}$ of the truth of p. We can view $\mu_C, \mu_T : E \longrightarrow \mathbf{I}$ as membership functions of corresponding fuzzy subsets $C, T \in \widetilde{\mathcal{P}}(E)$. The problem is to determine an improved global evaluation $G = \widetilde{Q}(C, T) \in \mathbf{I}$ of the degree of truth of p which provides increased robustness against noise, failure, and other sources of erroneous information. Obviously, the problem of data fusion becomes that of finding a suitable fuzzy quantifier $\widetilde{Q} \in \widetilde{\mathbb{F}}_{E,2}$.

DFS simplifies this task in that it only requires a description of the desired behaviour of $\widetilde{Q}$ on crisp arguments, i.e. in terms of a semi-fuzzy quantifier $Q \in \mathbb{S}_{E,2}$. For example, if all competent experts must confirm p in order to regard p as true, the proper choice of Q is $Q = \textbf{all}$. If we only require that m competent experts confirm the truth of p, the proper choice is $Q = \textbf{atleast m}$. If the evaluation of p corresponds to the proportion of competent experts which assert p, we can choose $Q = \textbf{prop}$. From Q we obtain the desired $\widetilde{Q} = \mathcal{M}(Q)$, which generalises the behaviour of the fusion operator from crisp sets to the case of degrees of competence and truth.

In the general case where we have several dimensions of relevance, data fusion can be accomplished by an n-ary fuzzy quantifier $\widetilde{Q} \in \widetilde{\mathbb{F}}_{E,n}$. DFS can also handle the required multiplace quantification. Furthermore, DFS is able to model non-quantitative quantifiers. The need for such quantifiers arises when the experts cannot be viewed as indistinguishable (modulo competence) or when there are interactions among the expert's judgements.[5] If $Q \in \mathbb{F}_{E,n}$ is non-quantitative, we can still apply $\mathcal{M}$ to obtain $\mathcal{M}(Q)$. The histogram-based algorithm is not applicable in this case. However, if $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ satisfy (2), we can compute

$$\mathcal{M}(Q)(X_1, \dots, X_n) = \frac{1}{m} \sum_{j=0}^{m-1} Q_{\overline{\gamma}_j}(X_1, \dots, X_n)$$

where $\overline{\gamma}_j = \dfrac{2j+1}{2m}$.

6 Comparison to existing approaches

We shall now compare DFS to other approaches to fuzzy quantification which have been applied in the area of data fusion. These approaches build on Zadeh's idea [11] of representing fuzzy quantifiers by fuzzy subsets $\mu_Q \in \widetilde{\mathcal{P}}(\mathbb{R}^+)$ of the non-negative reals (absolute quantifiers) or of the unit interval ($\mu_Q \in \widetilde{\mathcal{P}}(\mathbf{I})$, proportional quantifiers). In order to make these fuzzy numbers applicable to fuzzy sets for the purpose of quantification, a mechanism (which we denote by $\mathcal{Z}$) is needed which maps μ_Q to a fuzzy quantifier $\mathcal{Z}(\mu_Q) \in \widetilde{\mathbb{F}}_{E,1}$ (unrestricted use, relative to E), or $\mathcal{Z}(\mu_Q) \in \widetilde{\mathbb{F}}_{E,2}$ (restricted use, relative to first argument). Zadeh defines $\mathcal{Z}$ in terms of Σ-Counts or FG-Counts [11], while Yager [10] uses ordered weighted averaging (OWA) operators.

Some benefits of DFS compared to these approaches have been mentioned in the introduction; notably, only DFS is based on an axiomatic

[5] See Grabisch [7] for a discussion of interaction in the framework of fuzzy integrals.

foundation. Another benefit of DFS in data fusion applications is its broad coverage of quantificational operators. All approaches to fuzzy quantification try to facilitate the definition of fuzzy quantifiers by introducing a more accessible level of description (semi-fuzzy quantifiers in DFS vs. fuzzy sets μ_Q in other approaches). This raises the question of whether these approaches can represent *enough* quantifiers, in the sense that for every $\widetilde{Q} \in \mathbb{F}_{E,n}$, there exists a $Q \in \mathbb{S}_{E,n}$ such that $\mathcal{M}(Q)$ is sufficiently close to $\widetilde{Q}$ (likewise for $\mathcal{Z}(\mu_Q)$).

A rough account of "closeness" is given as follows. Let us define the underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q}) \in \mathbb{S}_{E,n}$ of $\widetilde{Q} \in \mathbb{F}_{E,n}$ by $\mathcal{U}(\widetilde{Q}) = \widetilde{Q}|_{\mathcal{P}(E)^n}$. We may define an equivalence relation $\sim$ on $\mathbb{F}_{E,n}$ by

$$\widetilde{Q} \sim \widetilde{Q}' \Leftrightarrow \mathcal{U}(\widetilde{Q}) = \mathcal{U}(\widetilde{Q}')\,.$$

$\widetilde{Q}$, $\widetilde{Q}'$ are thus considered similar if they meet the same boundary conditions (to the crisp case).

In DFS, (1) ensures that for every $\widetilde{Q} \in \mathbb{F}_{E,n}$, there exists a semi-fuzzy quantifier $Q \in \mathbb{S}_{E,n}$ such that $\widetilde{Q} \sim \mathcal{M}(Q)$, viz. $Q = \mathcal{U}(\widetilde{Q})$. The other approaches, however, are limited to absolute and proportional quantifiers, and have not yet been generalised to quantifiers of arbitrary arities n. In addition, $\mathcal{Z}(\mu_Q)$ is always *quantitative*, regardless of μ_Q. It follows that if $\widetilde{Q}$ is a fuzzy quantifier such that $\mathcal{U}(\widetilde{Q})$ is not quantitative, no μ_Q exists such that $\widetilde{Q} \sim \mathcal{Z}(\mu_Q)$. We consider this a serious limitation because (as stated above), the use of non-quantitative fuzzy quantifiers can be perfectly reasonable in data fusion.

7 Perspective

We have presented a principled account of fuzzy quantification, which results in a novel class of operators for data fusion. There is a wide area of applications of these fuzzy quantifiers. In an experimental multimedia retrieval system (Glöckner & Knoll [6]) for meteorological documents, DFS quantifiers are utilized for various purposes of information aggregation and data fusion. For example, a user of this system might ask for satellite images in which it is "cloudy in southern Bavaria". In this case, E is the set of pixel coordinates, C is the fuzzy set of pixels which belong to southern Bavaria, T is the fuzzy set of pixels classified as "cloudy". **prop** is used as the default fusion operator. Furthermore, we are currently working on the application of DFS quantifiers in area-based stereo image matching. Other fields of potential application have been described in the literature. Kacprzyk et al. [8] present a group decision-support system in which the Σ-Count approach is used for the fusion of "opinions" of a group's individuals. In Bordogna & Pasi [2], an information retrieval system is described which uses OWA-operators to determine an improved measure of search term-document relevance by fusing gradual relevance judgements obtained for the document's sections.

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