Fuzzy Quantifiers, Multiple Variable Binding and Branching Quantification

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Abstract. Lindström [1] introduced a very powerful notion of quantifiers, which permits multi-place quantification and the simultaneous binding of several variables. 'Branching' quantification was found to be useful by linguists e.g. for modelling reciprocal constructions like "Most men and most women admire each other". Westerståhl [2] showed how to compute the three-place Lindström quantifier for "Q_1 A's and Q_2 B's R each other" from the binary quantifiers Q_1 and Q_2, assuming crisp quantifiers and arguments. In the paper, I generalize his method to approximate quantifiers like "many" and fuzzy arguments like "young". A consistent interpretation is achieved by extending the DFS theory of fuzzy quantification [3,4], which rests on a system of formal adequacy criteria. The new analysis is important to linguistic data summarization because the full meaning of reciprocal summarizers (e.g. describing factors which are "correlated" or "associated" with each other), can only be captured by branching quantification.[1]

1 Introduction

The quantifiers found in natural language (NL) are by no means restricted to the absolute and proportional types usually considered in fuzzy set theory. The linguistic theory of quantification, i.e. the Theory of Generalized Quantifiers (TGQ) [5,6], recognizes more than thirty different types of quantifiers, including quantifiers of exception like "all except about ten", cardinal comparatives like "many more than" and many others [6]. These quantifiers can be unary (like proper names in "Ronald is X") or multi-place; quantitative (like "about ten") or non-quantitative (like "all except Lotfi"); and they can be simplex or constructed, like "most married X's are Y's or Z's". However, it is not only the diversity of possible quantifiers in NL which poses difficulties to a systematic and comprehensive modelling of NL quantification. Even in simple cases like "most", the ways in which these quantifiers interact when combined in meaningful propositions can be complex and sometimes even puzzling. Consider "Most men and most women admire each other", for example, in which we find a reciprocal predicate, "admire each other". Barwise [7] argues that so-called branching quantification is needed to capture the meaning of propositions involving reciprocal predicates. Without branching quantifiers, the above example must be linearly phrased as either

a. $[\mathrm{most}\, x : \mathrm{men}(x)][\mathrm{most}\, y : \mathrm{women}(y)]\, \mathrm{adm}(x, y)$
b. $[\mathrm{most}\, y : \mathrm{women}(y)][\mathrm{most}\, x : \mathrm{men}(x)]\, \mathrm{adm}(x, y)\,.$

[1] The proofs of all theorems cited here are listed in [4].

Neither interpretation captures the expected symmetry with respect to the men and women involved. In fact, we need a construction like

$$\left.\begin{array}{l}[Q_1\, x: \quad \mathrm{men}(x)] \\ [Q_2\, y: \mathrm{women}(y)]\end{array}\right\rangle\ \mathrm{adm}(x, y)$$

where $Q_1 = Q_2 = \mathrm{most}$ operate in parallel and independently of each other. This branching use of quantifiers can be analysed in terms of Lindström quantifiers [1], i.e. multi-place quantifiers capable of binding several variables. We then have three arguments, and Q should bind x in $\mathrm{men}(x)$, y in $\mathrm{women}(y)$ and both x, y in $\mathrm{adm}(x, y)$. Thus, the above expression can be modelled by a Lindström quantifier of type $\langle 1, 1, 2\rangle$:

$$Q_{x,y,xy}(\mathrm{men}(x), \mathrm{women}(y), \mathrm{adm}(x, y))\,.$$

Obviously, the interpretation of Q depends on the meaning of "most" (majority of), i.e. $\mathbf{most}(Y_1, Y_2) = 1$ if $|Y_1 \cap Y_2| > \frac{1}{2}|Y_1|$ and 0 otherwise, where Y_1, Y_2 are crisp subsets of the given universe $E \neq \varnothing$. The quantifier Q, on the other hand, accepts the sets $A, B \in \mathcal{P}(E)$ (e.g. men and women), and the binary relation $R \in \mathcal{P}(E^2)$ (people admiring each other in the example). Barwise [7, p. 63] showed how to define Q in a special case; see also Westerståhl [2, p. 274, (D1)]. Hence suppose that Q_1 and Q_2, like "most", are nondecreasing in their second argument, i.e. $Q(Y_1, Y_2) \leq Q(Y_1, Y_2')$ whenever $Y_2 \subseteq Y_2'$. The complex quantifier Q can then be expressed as:

$$Q(A, B, R) = \begin{cases} 1 & : \exists U \times V \subseteq R : Q_1(A, U) = 1 \wedge Q_2(B, V) = 1 \\ 0 & : \text{else} \end{cases} \qquad (1)$$

In the following, I will extend this analysis to approximate quantifiers and fuzzy arguments ("Many young and most old people respect each other"). To this end I describe the assumed formal framework, then incorporating Lindström quantifiers which bind several variables. Finally I apply the above analysis of branching quantifiers and its generalization by Westerståhl to the modelling of fuzzy branching quantification.

2 The Linguistic Theory of Fuzzy Quantification

TGQ rests on a simple but expressive model of two-valued quantifiers, which offers a uniform representation for the diversity of NL examples mentioned above. However, TGQ was not developed with fuzzy sets in mind, and its semantic analysis is essentially two-valued. To cover a broad range of fuzzy NL quantifiers, I hence developed the 'DFS theory' of fuzzy quantification [4,3] which introduces the following basic notions.

Definition 1 *An n-ary* fuzzy quantifier $\widetilde{Q}$ *on a base set $E \neq \varnothing$ assigns a gradual interpretation $\widetilde{Q}(X_1, \ldots, X_n) \in [0, 1]$ to all fuzzy subsets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.*

($\widetilde{\mathcal{P}}(E)$ is the fuzzy powerset). Fuzzy quantifiers are expressive operators, but hard to define because the usual cardinality is not applicable to the fuzzy sets they process. We hence need simplified specifications, powerful enough to embed all quantifiers of TGQ.

Definition 2 *An n-ary* semi-fuzzy quantifier *on a base set $E \neq \varnothing$ assigns a gradual result $Q(Y_1, \dots, Y_n) \in [0,1]$ to all* crisp *subsets $Y_1, \dots, Y_n \in \mathcal{P}(E)$.*

Semi-fuzzy quantifiers are much easier to define because the usual crisp cardinality is applicable to their arguments. An interpretation mechanism is used to associate these specifications with their matching fuzzy quantifiers:

Definition 3 *A* quantifier fuzzification mechanism *(QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier Q a fuzzy quantifier $\mathcal{F}(Q)$ of the same arity and on the same base set.*

The resulting fuzzy quantifiers $\mathcal{F}(Q)$ can then be applied to fuzzy arguments. In order to ensure plausible results, the QFM should conform to all requirements of linguistic relevance. My research into various such properties converged into the following system of six basic postulates.

(Z-1) **Correct generalisation.** For all crisp arguments $Y_1, \dots, Y_n \in \mathcal{P}(E)$, we require that $\mathcal{F}(Q)(Y_1, \dots, Y_n) = Q(Y_1, \dots, Y_n)$. (Combined with the other axioms, this condition can be restricted to $n \leq 1$). Rationale: $\mathcal{F}(Q)$ should properly generalize Q for crisp arguments.

(Z-2) **Membership assessment.** The two-valued quantifier defined by $\pi_e(Y)$ =1 if $e \in Y$ and $\pi_e(Y) = 0$ otherwise for crisp Y, has the obvious fuzzy counterpart $\widetilde{\pi}_e(X) = \mu_X(e)$ for fuzzy subsets. We require that $\mathcal{F}(\pi_e) = \widetilde{\pi}_e$. Rationale: Membership assessment (crisp or fuzzy) can be modelled through quantifiers. While π_e checks if e is present in its argument, $\widetilde{\pi}_e$ returns the degree to which e is contained in its argument. It is natural to require that π_e be mapped to $\widetilde{\pi}_e$, which serves the same purpose in the fuzzy case.

The fuzzy connectives which best match a QFM are given by a canonical construction.

Definition 4 *The* induced truth function $\widetilde{\mathcal{F}}(f) : [0,1]^n \longrightarrow [0,1]$ *of* $f : \{0,1\}^n \longrightarrow [0,1]$ *is defined by* $\widetilde{\mathcal{F}}(f) = \mathcal{F}(f \circ \eta^{-1}) \circ \widetilde{\eta}$*, where* $\eta(y_1, \dots, y_n) = \{i : y_i = 1\}$ *for all* $y_1, \dots, y_n \in \{0,1\}$ *and* $\mu_{\widetilde{\eta}(x_1,\dots,x_n)}(i) = x_i$ *for all* $x_i \in [0,1]$*,* $i \in \{1, \dots, n\}$.

Whenever $\mathcal{F}$ is understood, I abbreviate $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$, $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ etc. These connectives are extended to fuzzy set operations in the usual ways. The desired criteria involving fuzzy complement and union can now be expressed as follows.

(Z-3) **Dualisation.** $\mathcal{F}$ preserves dualisation of quantifiers, i.e. $\mathcal{F}(Q')(X_1, \dots, X_n) = \widetilde{\neg}\, \mathcal{F}(Q)(X_1, \dots, X_{n-1}, \widetilde{\neg}\, X_n)$ for $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ given $Q'(Y_1, \dots, Y_n) = \widetilde{\neg}\, Q(Y_1, \dots, Y_{n-1}, \neg Y_n)$ for all crisp arguments. Rationale: “All A's are B” and “It is not the case that some A's are not B's” should be equivalent.

(Z-4) **Union.** $\mathcal{F}$ must be compatible with unions of arguments, i.e. we should expect that $\mathcal{F}(Q')(X_1, \dots, X_{n+1}) = \mathcal{F}(Q)(X_1, \dots, X_{n-1}, X_n \,\widetilde{\cup}\, X_{n+1})$ provided that $Q'(Y_1, \dots, Y_{n+1}) = Q(Y_1, \dots, Y_{n-1}, Y_n \cup Y_{n+1})$. Rationale: This postulate permits a compositional treatment of patterns like “Many A's are B's or C's”

(Z-5) **Monotonicity in arguments.** $\mathcal{F}$ must preserve monotonicity in arguments, i.e. if Q is nondecreasing/nonincreasing in the i-th argument, then $\mathcal{F}(Q)$ has the same property. (Combined with the other axioms, the condition can be restricted to non-increasing Q). Rationale: The interpretation of "All men are tall" and "All young men are tall" must be systematically different and the former statement expresses the stricter condition.

The last criterion which I will state requires the extension of mappings $f : E \longrightarrow E'$ to fuzzy powerset mappings $f' : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$. The usual way of doing this is by applying the standard extension principle. In this case, the extension $f' = \hat{\hat{f}}$ becomes $\mu_{\hat{\hat{f}}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}$ for all $e' \in E'$. In order to define the sixth criterion, I must admit other choices which match the existential quantifiers $\mathcal{F}(\exists)$ of $\mathcal{F}$.

Definition 5 *The* induced extension principle *of $\mathcal{F}$, denoted $\widehat{\mathcal{F}}$, maps f to the extension $\widehat{\mathcal{F}}(f)$ defined $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\pi_{e'} \circ \widehat{f})$, where $\widehat{f}(Y) = \{f(e) : e \in Y\}$ for all $Y \in \mathcal{P}(E)$.*

(Z-6) **Functional application.** $\mathcal{F}$ must be compatible with 'functional application', i.e. $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n))$, where the semi-fuzzy quantifier Q' is defined by $Q'(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n))$. Rationale: $\mathcal{F}$ must behave consistently over different domains E.

Definition 6 *A QFM $\mathcal{F}$ which satisfies (Z-1) to (Z-6) is called a* determiner fuzzification scheme (DFS).

(In linguistics, "most", "almost all" etc. are called 'determiners'). If $\mathcal{F}$ induces $\neg x = 1 - x$ and the standard extension principle, then it is called a *standard DFS*. Let us now consider some properties of these models. If $\mathcal{F}$ is a DFS, then

- $\mathcal{F}$ induces a reasonable set of fuzzy propositional connectives, i.e. $\widetilde{\neg}$ is a strong negation $\widetilde{\wedge}$ is a t-norm, $\widetilde{\vee}$ is an s-norm etc.
- $\mathcal{F}(\forall)$ is a T-quantifier and $\mathcal{F}(\exists)$ is an S-quantifier in the sense of Thiele [8].
- $\mathcal{F}$ is compatible with negations, e.g. "It is not the case that most A's are B's".
- $\mathcal{F}$ is compatible with the formation of antonyms, e.g. "Most A's are not B's".
- $\mathcal{F}$ is compatible with intersections of arguments, e.g. "Most A's are B's and C's".
- $\mathcal{F}$ is compatible with argument permutations. In particular, symmetry properties of a quantifier are preserved by applying $\mathcal{F}$.
- $\mathcal{F}$ is compatible with crisp adjectival restriction, e.g. "Many married A's are B's".

The models also account for some additional considerations of specifically linguistic interest. For a comprehensive discussion of semantical properties and a description of prototypical models, see [4]. These models include $\mathcal{M}_{\mathrm{CX}}$, a standard DFS which consistently generalises the Sugeno integral and hence the 'basic' FG-count approach to arbitrary n-place quantifiers. Due to its unique properties, this model is the preferred choice for all applications that need to capture NL semantics. Another interesting example $\mathcal{F}_{\mathrm{owa}}$, consistently generalises the Choquet integral and hence the 'basic' OWA approach. An efficient histogram-based method for implementing quantifiers in these models is described in [4].

3 Extension towards multiple variable binding

A *Lindström quantifier* is a class $\mathcal{Q}$ of (relational) structures of type $t = \langle t_1, \dots, t_n \rangle$, such that $\mathcal{Q}$ is closed under isomorphism [1, p. 186]. The cardinal $n \in \mathbb{N}$ specifies the number of arguments; the components $t_i \in \mathbb{N}$ specify the number of variables that the quantifier binds in its i-th argument position. For example, the existential quantifier, which accepts one argument and binds one variable, has type $t = \langle 1 \rangle$. The corresponding class $\mathcal{E}$ comprises all structures $\langle E, A \rangle$ where $E \neq \varnothing$ is a base set and $A \subseteq E$ is nonempty. In the introduction, we already met with a more complex quantifier Q of type $\langle 1, 1, 2 \rangle$. In this case, $\mathcal{Q}$ is the class of all structures $\langle E, A, B, R \rangle$ with $Q(A, B, R) = 1$, where $A, B \in \mathcal{P}(E)$, $R \in \mathcal{P}(E^2)$. To model quantifiers like "all except Lotfi", which depend on specific individuals, we must drop the assumption of isomorphism closure. Hence, in principle, a *generalized Lindström quantifier* is a class $\mathcal{Q}$ of relational structures of type $t = \langle t_1, \dots, t_n \rangle$. However, it is convenient to stipulate the following alternative notions.

Definition 7 *A* two-valued L-quantifier *of type $t = \langle t_1, \dots, t_n \rangle$ on a base set $E \neq \varnothing$ assigns a crisp quantification result $Q(Y_1, \dots, Y_n) \in \{0, 1\}$ to each choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \dots, n\}$. A* full two-valued L-quantifier *Q of type t assigns a two-valued L-quantifier Q_E of type t on E to each base set $E \neq \varnothing$.*

Hence 'full' L-quantifiers are in one-to-one correspondence with generalized Lindström quantifiers. The extension of L-quantifiers to gradual outputs should be obvious.

Definition 8 *A* semi-fuzzy L-quantifier *of type $t = \langle t_1, \dots, t_n \rangle$ on $E \neq \varnothing$ assigns a gradual result $Q(Y_1, \dots, Y_n) \in [0, 1]$ to all crisp $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \dots, n\}$.*

Thus, Q accepts crisp arguments of the indicated types, but it can express approximate quantification. Semi-fuzzy L-quantifiers establish a uniform specification medium for quantifiers with multiple variable binding. We further need operational quantifiers and fuzzification mechanisms which associate specifications and target quantifiers.

Definition 9 *A* fuzzy L-quantifier of type t on $E \neq \varnothing$ *assigns a gradual interpretation $\widetilde{Q}(X_1, \dots, X_n) \in [0, 1]$ to all fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \dots, n\}$.*

Definition 10 *An* L-QFM *$\mathcal{F}$ assigns to each semi-fuzzy quantifier Q of some type t on $E \neq \varnothing$ a fuzzy L-quantifier $\mathcal{F}(Q)$ of the same type t and on the same base set E.*

Let me now associate with every L-QFM $\mathcal{F}$ a corresponding 'ordinary' QFM $\mathcal{F}_R$. For every n-ary semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow [0, 1]$, on E, let Q' denote the n-ary quantifier on E^1 defined by $Q'(Y_1, \dots, Y_n) = Q'(\widehat{\vartheta}(Y_1), \dots, \widehat{\vartheta}(Y_n))$ for $Y_1, \dots, Y_n \in \mathcal{P}(E^1)$, where $\vartheta : E^1 \longrightarrow E$ is the mapping $\vartheta((e)) = e$ for $(e) \in E^1$. Then let

$$\mathcal{F}_R(Q)(X_1, \dots, X_n) = \mathcal{F}(Q')(\hat{\hat{\beta}}(X_1), \dots, \hat{\hat{\beta}}(X_n))$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, where $\hat{\hat{\beta}}$ is obtained from $\beta : E \longrightarrow E^1$ with $\beta(e) = (e)$ by applying the standard extension principle. The induced fuzzy connectives and extension principle of $\mathcal{F}$ are identified with the connectives and extension principle of the ordinary QFM $\mathcal{F}_R$. Based on these preparations, I can now develop criteria for plausible L-models of fuzzy quantification which parallel my requirements on QFMs. (The 'rationale' for these conditions is the same as above in each case).

(L-1) **Correct generalisation.** It is required that $\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$ for all crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ldots, n\}$; combined with the other axioms, this condition can be restricted to quantifiers of types $t = \langle\rangle$ or $t = \langle 1 \rangle$.

(L-2) **Membership assessment.** Quantifiers for membership assessment of the special form $\pi_{(e)} : \mathcal{P}(E^1) \longrightarrow \{0, 1\}$ for some $e \in E$ also qualify as two-valued L-quantifiers of type $\langle 1 \rangle$ on E. These quantifiers should be mapped to their fuzzy counterparts $\widetilde{\pi}_{(e)}$ of type $\langle 1 \rangle$ on E, i.e. we must have $\mathcal{F}(\pi_{(e)}) = \widetilde{\pi}_{(e)}$.

(L-3) **Dualisation.** $\mathcal{F}$ preserves dualisation of quantifiers, i.e. $\mathcal{F}(Q')(X_1, \ldots, X_n) = \widetilde{\neg}\, \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, \widetilde{\neg}\, X_n)$ for all fuzzy $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$ if $Q'(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)$ for all crisp $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ldots, n\}$.

(L-4) **Union.** $\mathcal{F}$ must be compatible with unions of arguments, i.e. we should expect that $\mathcal{F}(Q')(X_1, \ldots, X_{n+1}) = \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, X_n \mathbin{\widetilde{\cup}} X_{n+1})$ provided that $Q'(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$.

(L-5) **Monotonicity in arguments.** We require that $\mathcal{F}$ preserve monotonicity in arguments, i.e. if Q is nondecreasing/nonincreasing in the i-th argument, then $\mathcal{F}(Q)$ has the same property. (The condition can again be restricted to the case that Q is nonincreasing in its n-th argument).

(L-6) **Functional application.** Given a semi-fuzzy L-quantifier Q of type $t = \langle t_1, \ldots, t_n \rangle$ on E, another type $t' = \langle t'_1, \ldots, t'_n \rangle$ (same n), a set $E' \neq \varnothing$, and mappings $f_i : {E'}^{t'_i} \longrightarrow E^{t_i}$ for $i \in \{1, \ldots, n\}$, we can define a quantifier Q' of type t' on E' by $Q'(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n))$ for all $Y_i \in \mathcal{P}({E'}^{t'_i})$, $i \in \{1, \ldots, n\}$. It is required that $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f)_1(X_1), \ldots, \widehat{\mathcal{F}}(f)_n(X_n))$ for all fuzzy arguments $X_i \in \widetilde{\mathcal{P}}({E'}^{t'_i})$, $i \in \{1, \ldots, n\}$.

Definition 11 *An L-QFM which satisfies (L-1) to (L-6) is called an* L-DFS.

Theorem 1 *For every L-DFS $\mathcal{F}$, the corresponding QFM $\mathcal{F}_R$ is a DFS.*

Hence the generalized models are also suitable for carrying out 'ordinary' quantification. Now let $\mathcal{F}$ be an ordinary QFM and let Q be a semi-fuzzy L-quantifier of type $t = \langle t_1, \ldots, t_n \rangle$ on $E \neq \varnothing$. Let $m = \max\{t_1, \ldots, t_n\}$ and define $\zeta_i : E^{t_i} \longrightarrow E^m$ and $\kappa_i : E^m \longrightarrow E^{t_i}$ by

$$\zeta_i(e_1, \ldots, e_{t_i}) = (e_1, \ldots, e_{t_i - 1}, e_{t_i}, e_{t_i}, \ldots, e_{t_i})$$
$$\kappa_i(e_1, \ldots, e_m) = (e_1, \ldots, e_{t_i})$$

for $i \in \{1, \ldots, n\}$. I introduce an n-ary semi-fuzzy quantifier Q' on E^m defined by

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{\kappa}_1(Y_1 \cap \zeta_1(E^{t_1})), \ldots, \widehat{\kappa}_n(Y_n \cap \zeta_n(E^{t_n})))\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E^m)$.

Definition 12 *For every QFM $\mathcal{F}$, the L-QFM $\mathcal{F}_L$ is defined by*

$$\mathcal{F}_L(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_n(X_n))$$

for all $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \ldots, n\}$.

Theorem 2 *If $\mathcal{F}$ is a DFS, then $\mathcal{F}_{LR} = \mathcal{F}$, i.e. $\mathcal{F}_L$ properly generalizes $\mathcal{F}$.*

Theorem 3 *If $\mathcal{F}$ is a DFS, then $\mathcal{F}_L$ is an L-DFS, i.e. we obtain plausible models.*

Theorem 4 *If $\mathcal{F}$ is an L-DFS, then $\mathcal{F}_{RL} = \mathcal{F}$.*

Hence every L-DFS $\mathcal{F}'$ can now be expressed as $\mathcal{F}' = \mathcal{F}_L$. The canonical construction of $\mathcal{F}_L$ thus permits the re-use of $\mathcal{M}_{\mathrm{CX}}$ and $\mathcal{F}_{\mathrm{owa}}$ to handle fuzzy L-quantification.

4 Application to fuzzy branching quantification

Let me now reconsider the motivating example, "Many young and most old people respect each other". In this case, we have semi-fuzzy quantifiers $Q_1 = \textbf{many}$, defined by $\textbf{many}(Y_1, Y_2) = |Y_1 \cap Y_2|/|Y_1|$, say, and $Q_2 = \textbf{most}$. Both quantifiers are nondecreasing in their second argument, i.e. we can adopt eq. (1). The modification to gradual truth values will be accomplished in the usual way, i.e. by replacing existential quantifiers with $\sup$ and conjunctions with $\min$. The semi-fuzzy L-quantifier Q of type $\langle 1, 1, 2 \rangle$ constructed from Q_1, Q_2 then becomes

$$Q(A, B, R) = \sup\{\min(Q_1(A, U), Q_2(B, V)) : U \times V \subseteq R\}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. By applying $\mathcal{F}$, we then obtain the fuzzy L-quantifier $\mathcal{F}(Q)$ suitable for computing interpretations. In the example, we have fuzzy subsets **young**, **old** $\in \widetilde{\mathcal{P}}(E)$ of young and old people, and a fuzzy relation **rsp** $\in \widetilde{\mathcal{P}}(E^2)$ of people who respect each other. The interpretation of "Many young and most old people respect each other" is then given by $\mathcal{F}(Q)(\textbf{young}, \textbf{old}, \textbf{rsp})$.

Finally let me describe how Westerståhl's generic method for interpreting branching quantifiers can be applied in the fuzzy case. Hence let Q_1, Q_2 be arbitrary semi-fuzzy quantifiers of arity $n = 2$. I introduce nondecreasing and nonincreasing approximations of the Q_i's, defined by $Q_i^+(Y_1, Y_2) = \sup\{Q_i(Y_1, L) : L \subseteq Y_2\}$ and $Q_i^-(Y_1, Y_2) = \sup\{Q_i(Y_1, U) : U \supseteq Y_2\}$, respectively. With the usual replacement of existential quantification with $\sup$ and conjunction with $\min$, Westerståhls formula [2, p. 281, Def. 3.1] becomes:

$$\begin{aligned} Q(A, B, R) = \sup\{&\min\{Q_1^+(A, U_1), Q_2^+(B, V_1), Q_1^-(A, U_2), Q_2^-(B, V_2)\} : \\ &(U_1 \cap A) \times (V_1 \cap B) \subseteq R \cap (A \times B) \subseteq (U_2 \cap A) \times (V_2 \cap B)\} \end{aligned}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. We can then apply $\mathcal{F}$ to fetch $\mathcal{F}(Q)$. As shown by Westerståhl [2, p.284], his method results in meaningful interpretations provided that (a) Q_1 and Q_2 are 'logical', i.e. $Q_i(Y_1, Y_2)$ can be expressed as a function of $|Y_1|$

and $|Y_1 \cap Y_2|$; and (b) the Q_i's satisfy $Q_i(Y_1, Y_2) \geq \min(Q_i(Y_1, L), Q_i(Y_1, U)$ for all $L \subseteq Y_2 \subseteq U$. The latter condition ensures that Q_1 and Q_2 can be recovered from their nondecreasing approximations Q_i^+ and their nonincreasing approximations Q_i^-, i.e. $Q_i = \min(Q_i^+, Q_i^-)$. This is the case when Q_1 and Q_2 are nondecreasing in their second argument ("many"), nonincreasing ("few"), or of unimodal shape ("about ten", "about one third"). An example with unimodal quantifiers, which demand the generic method, is "About fifty young and about sixty old persons respect each other".

5 Conclusion

In the paper, I proposed an extension of the DFS theory of fuzzy quantification with Lindström-like quantifiers. Westerståhl's method based on Lindström quantifiers which assigns a meaningful interpretation to branching NL quantification was then extended to approximate quantifiers and fuzzy arguments. The proposed analysis of reciprocal constructions in terms of fuzzy branching quantifiers is important to linguistic data summarization [9,10]. Many summarizers of interest express mutual (or symmetric) relationships and can therefore be verbalized by a reciprocal construction. An ordinary summary like "Q_1 X_1's are strongly correlated with Q_2 X_2's" neglects the resulting groups of mutually correlated objects. The proposed analysis in terms of branching quantifiers, by contrast, permits me to support a novel type of summary specialized on groups of interrelated objects. Branching quantification, in this view, is a natural language technique for detecting such groups in the data. A possible summary involving a reciprocal predicate is "The intake of most vegetables and many health-related indicators are strongly associated with each other".

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