A Framework for Evaluating Fusion Operators Based on the Theory of Generalized Quantifiers

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Abstract

Fuzzy linguistic quantifiers – operators intended to model vague quantifying expressions in natural language like "almost all" or "few" – have gained importance as operators for information combination and the fusion of gradual evaluations. They are particularly appealing because of their ease-of-use: people are familiar with these operators, which can be applied for technical fusion purposes in the same way as in everyday language.
Because of the irregular and rather intangible phenomena it tries to model – viz, those of imprecision and uncertainty – fuzzy logic should be particularly specific about its foundations. However, work on mathematical foundations and linguistic justification of fuzzy linguistic quantifiers is scarce. In the paper, we propose a framework for evaluating approaches to fuzzy quantification which relates these to the logico-linguistic theory of generalized quantifiers (TGQ). By reformulating these approaches as fuzzification mechanisms, we can investigate properties of the fuzzification mappings which express important aspects of the meaning of natural language quantifiers.

1 Introduction

Natural language (NL) is pervaded by quantifiers. It is virtually impossible to express a natural language sentence which does not involve quantification because every nominal phrase ("most people", "almost all men" ...) has a quantificational aspect (typically expressed by a "determiner" or "generalized quantifier" such as "most", "the", "a", etc.). In addition, aggregational modes of temporal or local description such as "almost always", "everywhere" are naturally modelled through quantification.

In order to handle such cases, Zadeh [17, 18] has initiated research which tries to model natural language quantifiers by operators called "fuzzy linguistic quantifiers". Several classes of operators have been proposed as properly representing the phenomenon of "vague" or fuzzy NL quantification (a survey is provided in [11]), but there is no consensus about the proper choice, and notes on implausible behavior of these approaches are scattered over the literature [12, 13, 16, 5].

These foundational problems notwithstanding, the areas of application have been so obvious and auspicious that a broad span of systems in a variety of fields and for a variety of purposes have been implemented. Fuzzy linguistic quantifiers have been utilized for fusion tasks in

- multi-criteria decision making [15];
- data summarisation [14];
- information retrieval [3];
- fuzzy databases [9].

In our view, fuzzy quantifying operators will unfold their full potential for information aggregation and data fusion only if these operators are linguistically adequate, i.e. able to capture the meaning of corresponding NL quantifiers. For example, if an operator is labelled "most", it is essential that this operator behave like the NL quantifier "most".

2 (Two-Valued) Generalized Quantifiers

We will start our presentation from the viewpoint of the Theory of Generalized Quantifiers (TGQ [1, 2]). By an n-ary *generalized quantifier* (sometimes dubbed "determiner") on a base set $E \neq \varnothing$ we denote a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2} = \{0, 1\}$ which to each n-tuple of crisp subsets $X_1, \ldots, X_n \in \mathcal{P}(E)$ of E assigns a two-valued quantification result $Q(X_1, \ldots, X_n) \in \mathbf{2}$. Examples are

$$\begin{aligned}
&\forall_E(X) = 1 \Leftrightarrow X = E \\
&\exists_E(X) = 1 \Leftrightarrow X \neq \varnothing \\
&\textbf{all}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \subseteq X_2 \\
&\textbf{some}_E(X_1, X_2) = 1 \Leftrightarrow X_1 \cap X_2 \neq \varnothing \\
&\textbf{atleast m}_E(X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq m \\
&\textbf{all except m}_E(X_1, X_2) = 1 \Leftrightarrow |X_1 \setminus X_2| \leq m\,.
\end{aligned}$$

Whenever the base set (domain) is clear from the context, we will drop the subscript E; $|.|$ denotes cardinality. Let us remark that E might be infinite in the general case. For finite E, we can define proportional quantifiers

$$[\textbf{rate} \geq r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq r\,|X_1|$$
$$[\textbf{rate} > r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| > r\,|X_1|\,.$$

for $r \in \mathbf{I}$, $X_1, X_2 \in \mathcal{P}(E)$.

TGQ has classified the wealth of quantificational phenomena in natural languages in order to unveil universal properties shared by quantifiers in all natural languages, or single out classes of quantifiers with specific properties (we shall describe some of these properties below). An extension to the continuous-valued case, in order to better capture the meaning of vague quantifying expressions like *almost all*, has not been an issue to TGQ.

3 Fuzzy Generalized Quantifiers

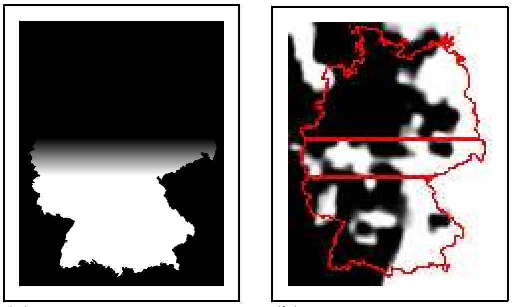
In [5], we have proposed a straightforward generalisation of generalized quantifiers to the fuzzy case. A fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ of a set E assigns to each $e \in E$ a membership degree $\mu_X(e) \in \mathbf{I} = [0,1]$; we denote by $\widetilde{\mathcal{P}}(E)$ the set of all fuzzy subsets (fuzzy powerset) of E. An *n-ary fuzzy quantifier* $\widetilde{Q}$ on a base set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of fuzzy subsets $X_1, \ldots, X_n$ of E assigns a gradual result $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$.[1] An example is

$$\widetilde{\forall}(X) = \inf_{e \in E} \mu_X(e)\,, \qquad X \in \widetilde{\mathcal{P}}(E)\,.$$

Fuzzy quantifiers catch a broad class of fusion operators. For example, if $E \neq \varnothing$ is a set of criteria (e.g. multiple sensors, experts ...), $\mu_{X_1}(e) \in \mathbf{I}$ expresses the "weight" or "relevance" of the criterion $e \in E$, and $\mu_{X_2}(e) \in \mathbf{I}$ expresses the degree to which $e \in E$ is satisfied, then every fusion operator $\widetilde{Q}$ which combines the criteria as a function $\widetilde{Q}(X_1, X_2)$ of X_1 and X_2, is a fuzzy quantifier by definition.

Let us give an example. In developing a system for the content-based retrieval of meteorological (weather information) documents [7], we have faced the problem of ranking satellite images according to accumulative criteria such as *"almost all of Southern Germany is cloudy"*. In this case, E is the set of pixel coordinates, $\mu_{X_1}(e)$ expresses the degree to which pixel $e \in E$ belongs to Southern Germany, and $\mu_{X_2}(e)$ expresses the degree to which the pixel is classified as cloudy (see Fig. 1). The desired fusion operator to combine the criteria can then be modeled by a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ suited for interpreting "almost all".

The example indicates some problems inherent to fuzzy quantifiers: Which fuzzy quantifier corresponds to a given (possibly vague) NL quantifier, e.g. "almost all"? How can we describe characteristics of fuzzy quantifiers and how can we locate a fuzzy quantifier based on a description of desired properties? Fuzzy quantifiers are possibly too rich a set of operators to investigate these questions directly, and all approaches to fuzzy quantification have therefore introduced some kind of *simplified representation.*



(a) (b)

Figure 1: (a) X_1 = southern_germany (Pixels with $\mu_{X_1}(e) = 1$ depicted white). (b) X_2 = cloudy (Pixels classified as cloudy depicted white. The contours of Germany, split in southern, intermediate and northern part, have been added to facilitate interpretation.)

4 Fuzzy Linguistic Quantifiers

Following Zadeh [17, 18, 19], most existing approaches to fuzzy quantification have chosen to define *fuzzy linguistic quantifiers* as fuzzy subsets of the non-negative reals (absolute quantifiers like **some**, with membership functions $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$), or of the unit interval (proportional quantifiers like **most**, with membership functions $\mu_Q \in \mathbf{I}^{\mathbf{I}}$).[2] These "fuzzy numbers"[3] provide the desired simplified representation. For example, we could define a proportional fuzzy linguistic quantifier **almost all** by $\mu_{\textbf{almost all}}(x) = S(x, 0.7, 0.9)$ forall $x \in \mathbf{I}$, using Zadeh's S-function (see Fig. 2).

The μ_Q are not directly applicable to fuzzy sets for the purpose of quantification. What is needed is a mechanism $\mathcal{Z}$ which maps each μ_Q to a fuzzy quantifier $\mathcal{Z}_E^{(1)}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ (monadic or unrestricted use, relative to E), or $\mathcal{Z}_E^{(2)}(\mu_Q) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ (two-place or restricted use, relative to first argument):[4]

unrestricted: $\mathcal{Z}^{(1)}(\mu_Q)(X)$, "$Q$ elements (of E) are X"
restricted: $\mathcal{Z}^{(2)}(\mu_Q)(X_1, X_2)$, "$Q$ X_1's are X_2".

[1] This definition closely resembles Zadeh's [19, pp.756] alternative view of fuzzy quantifiers as fuzzy second-order predicates, but models these as mappings in order to simplify notation. The set of all fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, given E and n, will be denoted $\mathbb{F}_{E,n}$.

[2] A^B denotes the set of mappings $f : B \longrightarrow A$.

[3] The convexity implications of fuzzy numbers are not always satisfied, as witnessed e.g. by the quantifier **an even number of**.

[4] We will use the superscripts only when necessary to discern the unrestricted and restricted use, and usually drop the subscript E.

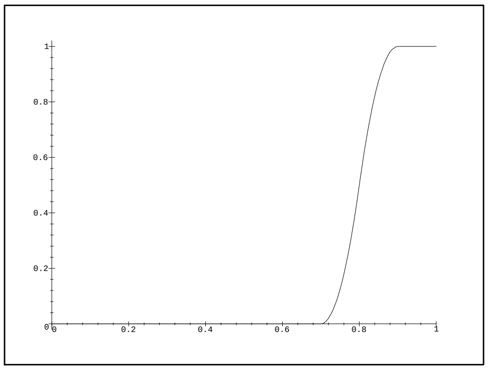


Figure 2: A possible definition of **almost all**

Zadeh has also formulated the idea that in order to evaluate a statement "$Q\ X's$ are A" (in our notation: to compute $\mathcal{Z}^{(1)}(\mu_Q)(A)$), one should instead evaluate the statement "$\mathrm{card}(X)$ is Q", where card is a scalar or fuzzy measure of the cardinality of the fuzzy set $A \in \widetilde{\mathcal{P}}(E)$ associated with the linguistic variable X.[5] The approaches described in the literature mainly differ in the measure of fuzzy cardinality used and in the way that the required comparison of fuzzy cardinalities is accomplished: Zadeh proposes the use of Σ-counts or FG-counts, and Ralescu's [12] possibilistic approach is based on FE-counts.[6] Yager [15] proposes the use of Ordered Weighted Averaging (OWA) operators.

Before discussing particular approaches, let us consider some general implications of using fuzzy linguistic quantifiers. Due to these approaches' direct relying on the computation of (some "fuzzified" notion of) cardinality and proportion, they are unable to provide a *uniform* account of absolute and proportional quantifiers, which must be represented (and hence evaluated) differently. An extension to other types of quantifiers (e.g., quantifiers of exception like **all except m**, and ternary quantifiers like **more** X **than** Y **are** Z, would require the introduction of further descriptions for each considered quantifier type, and corresponding evaluation formulas. The notions of cardinality and proportion (ratio) are not sufficient to evaluate all quantifiers of interest even in the crisp case. Apart from quantitative type, there is a variety of non-quantitative (qualitative) quantifiers.[7,8] From an information fusion perspective, non-quantitative quantifiers are of interest when the criteria cannot be viewed as being indistinguishable modulo relevance, or when there are interactions among the criteria, e.g. among gradual judgements of experts. Due to these limitations, it is not possible to view the n-ary, perhaps non-quantitative, two-valued quantifiers of TGQ as a special case of fuzzy linguistic quantifiers, which are generally monadic or two-place, absolute or proportional, and quantitative.

[5] On proportional quantifiers, the cardinality measure is replaced by a measure of fuzzy proportion, usually referred to as "relative cardinality".

[6] Σ-count, FG-count and FE-count are measures of fuzzy cardinality, cf. Zadeh [18].

[7] Q is said to be quantitative if it is automorphism-invariant, i.e. $Q(\pi(X_1), \ldots, \pi(X_n)) = Q(X_1, \ldots, X_n)$ for all automorphisms (permutations) $\pi : E \longrightarrow E$.

[8] A comprehensive survey of NL quantifier types is presented in [10].

5 Semi-Fuzzy Generalized Quantifiers

We therefore have to solve the problem of providing a sufficiently simple representation of fuzzy quantifying operators, in which all two-valued quantifiers of TGQ can be embedded. We accomplish this by the following definition. An *n-ary semi-fuzzy quantifier* on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of *crisp* subsets of E assigns a gradual result $Q(X_1, \ldots, X_n) \in \mathbf{I}$.[9] Semi-fuzzy quantifiers are half-way between two-valued quantifiers and fuzzy quantifiers because they have crisp input and fuzzy (gradual) output. In particular, every two-valued quantifier of TGQ is a semi-fuzzy quantifier by definition. Our above definition of the fuzzy linguistic quantifier **almost all** (see Fig. 2) can easily be turned into an example of a (genuinely continuous-valued) semi-fuzzy quantifier **almost all** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$:

$$\textbf{almost all}(X_1, X_2) = \begin{cases} S\left(\frac{|X_1 \cap X_2|}{|X_1|}, 0.7, 0.9\right) & : \quad X_1 \neq \varnothing \\ 1 & : \quad \text{else} \end{cases}$$

Semi-fuzzy quantifiers are not subject to the restrictions of fuzzy linguistic quantifiers: they can express genuine multiplace quantification (arbitrary n); they are not restricted to the absolute and proportional types; they are not necessarily quantitative (again in the sense of automorphism-invariance); and there is no a priori restriction to finite domains. In addition, it is relatively easy to understand the input-output behavior of a semi-fuzzy quantifier because it is stated in terms of crisp argument sets.

However, being half-way between two-valued generalized quantifiers and fuzzy quantifiers, semi-fuzzy quantifiers do not accept fuzzy input, and we have to make use of a fuzzification mechanism which transports these to fuzzy quantifiers.

6 Quantifier Fuzzification Mechanisms

A *quantifier fuzzification mechanism* $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ of the same arity n and on the same base set E.

[9] Given $E \neq \varnothing$ and $n \in \mathbb{N}$, we denote the set of all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by $\mathbb{S}_{E,n}$.

By viewing approaches to fuzzy quantification as instances of quantifier fuzzification mechanisms (QFM), we are able to explore the mathematical well-behavedness of these approaches by investigating preservation and homomorphism properties of the corresponding fuzzification mappings. A comprehensive account of such adequacy conditions is given in [5, 6]. For lack of space, we shall restrict attention to some special cases required for the proofs to follow.

The most basic requirement on a QFM $\mathcal{F}$ is that of *correct generalisation*, i.e. for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\mathcal{F}(Q)|_{\mathcal{P}(E)^n} = Q\,, \tag{1}$$

i.e. $\mathcal{F}(Q)$ coincides with Q on all crisp arguments.

A semi-fuzzy quantifier $Q \in \mathbb{S}_{E,n}$ is said to be *nonincreasing in its i-th argument* ($i \in \{1, \ldots, n\}$, $n > 0$) iff for all $X_1, \ldots, X_n, X_i' \in \mathcal{P}(E)$ such that $X_i \subseteq X_i'$, $Q(X_1, \ldots, X_n) \geq Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)$. The definitions for nondecreasing monotonicity, and the adaptation of these concepts to the fuzzy case are straightforward. We say that $\mathcal{F}$ preserves *monotonicity in arguments* if nonincreasing or nondecreasing monotonicity of a semi-fuzzy quantifier in its arguments are preserved when applying $\mathcal{F}$.

Q is said to *have extension* if for each choice of base sets $E \subseteq E'$ and all $X_1 \ldots X_n \in \mathcal{P}(E)$,

$$Q_E(X_1, \ldots, X_n) = Q_{E'}(X_1, \ldots, X_n)\,. \tag{2}$$

This is a very important property possessed by virtually all NL quantifiers, which expresses some kind of context insensitivity: we can add an arbitrary number of irrelevant objects to our original domain E without altering the quantification result. A QFM $\mathcal{F}$ is said to *preserve extension* if each pair $Q \in \mathbb{S}_{E,n}$, $Q' \in \mathbb{S}_{E',n}$, $E \subseteq E'$ of semi-fuzzy quantifiers satisfying (2) is mapped to a pair of fuzzy quantifiers with the same property (adapted to fuzzy arguments).

7 The Evaluation Framework

Existing approaches to fuzzification cannot be directly viewed as quantifier fuzzification mechanisms because they are not applicable to semi-fuzzy quantifiers. We have to bridge the gap between semi-fuzzy quantifiers and fuzzy linguistic quantifiers in a systematical way.

Suppose $\mathcal{Z}$ is one of the approaches based on fuzzy linguistic quantifiers. Because the quantificational phenomena $\mathcal{Z}$ addresses are too limited (see above), it does not give rise to a "full" (totally defined) quantifier fuzzification mechanism. However, we can reconstruct a partially defined quantifier fuzzification mechanism $\mathcal{F}$ in a post-hoc fashion as follows.

Let us define the *underlying semi-fuzzy quantifier* $\mathcal{U}(\widetilde{Q}) \in \mathbb{S}_{E,n}$ of a given fuzzy quantifier $\widetilde{Q} \in \mathbb{F}_{E,n}$ by $\mathcal{U}(\widetilde{Q}) = \widetilde{Q}|_{\mathcal{P}(E)^n}$. Given the membership function μ_Q of a fuzzy linguistic quantifier, we then first obtain the corresponding semi-fuzzy quantifier (relative to $\mathcal{Z}$) as $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$, and use this to define $\mathcal{F}(Q) = \mathcal{Z}(\mu_Q)$. Obviously, the construction of $\mathcal{F}$ succeeds only if $\mathcal{U}(\mathcal{Z}(\mu_Q)) \mapsto \mathcal{Z}(\mu_Q)$ is functional, but this is a reasonable adequacy condition anyway.[10] We shall call it the *quantifier framework assumption* (QFA). In case the QFA holds unconditionally for $\mathcal{Z}$, we can use the constructed partial fuzzification mechanism $\mathcal{F}$ to establish or reject the preservation and homomorphism properties of interest. In case the QFA is violated by $\mathcal{Z}$, it can always be enforced by restricting attention to smaller subsets of considered fuzzy linguistic quantifiers μ_Q, which comply with the QFA. It therefore makes sense to say that $\mathcal{Z}$ can *represent* a semi-fuzzy quantifier $Q \in \mathbb{S}_{E,n}$ if there exists some μ_Q such that $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$. We can then refute a property of interest by proving that $\mathcal{Z}$ cannot represent Q without violating the property.

We will now present examples of the framework in action. We will focus on one of the most prominent approaches to fuzzy quantification, namely the Σ-count approach (Zadeh [17, 18]). However, the evaluation framework can also be applied to other approaches such as Zadeh's FG-count approach, Ralescu's FE-count approach and Yager's OWA approach, as shown in [6].

8 Evaluation of the Sigma-Count Approach

The Σ*-count* of a fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ (E finite), is defined as the sum of its membership values, i.e.

$$\Sigma\text{-Count}(X) = \sum_{e \in E} \mu_X(e)\,.$$

It is claimed to provide a (coarse) summary of the cardinality of the fuzzy set X, expressed as a non-negative real number.
A corresponding scalar definition of fuzzy proportion, the *relative* Σ*-count*, is defined by

$$\Sigma\text{-Count}(X_2/X_1) = \Sigma\text{-Count}(X_1 \cap X_2) / \,\Sigma\text{-Count}(X_1)\,.$$

In the Σ-count approach [17, 18], fuzzy linguistic quantifiers of the absolute and proportional kinds are treated differently. Hence, in order to obtain both the unrestricted and restricted versions of absolute and proportional quantifiers,

[10] It is rather unlikely that a natural language, say English, would provide a pair of quantifiers with the same meaning on crisp arguments, but different behaviour if fuzziness is involved.

a total of four different evaluation formulae are required.[11]

$$\begin{aligned}
\mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) &= \mu_Q\,(\Sigma\text{-Count}(X))\\
\mathrm{SC}^{(2)}_{\mathrm{abs}}(\mu_Q)(X_1, X_2) &= \mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1 \cap X_2)\\
\mathrm{SC}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) &= \mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(E, X)\\
\mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) &= \mu_Q\,(\Sigma\text{-Count}(X_2/X_1))
\end{aligned}$$

Let us now recast Yager's example [16, p.257] on counter-intuitive behaviour of the Σ-count approach in our setting. Suppose $E = \{\text{hans}, \text{maria}, \text{tom}\}$ is a set of persons, and using Zadeh's notation, blond $= \frac{1}{3}/\text{hans} + \frac{1}{3}/\text{maria} + \frac{1}{3}/\text{tom}$, i.e. all are blond to a degree of $\frac{1}{3}$. Let us now evaluate $[!\textbf{one}](\text{blond})$, where $Q = [!\textbf{one}]$ denotes the two-valued quantifier **exactly one** in its monadic use (i.e. the statement expresses "there is exactly one blond (person)"). Then, assuming that *correct generalisation* be respected, we are forced to have $\mu_Q(1) = 1$. It follows that the above statement evaluates to 1 (fully true), although there is clearly *not* exactly one blond person in the base set (which one should that be?) but rather *a total amount of blondness* of one, as Yager puts it. Now let us address some novel aspects.

Question 1: *Does the Σ-count approach comply with the QFA?*

Answer: *No.*[12] This is most apparent with absolute fuzzy linguistic quantifiers $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$: Any pair $\mu_Q \neq \mu_{Q'}$ such that $\mu_Q|_{\mathbb{N}} = \mu_{Q'}|_{\mathbb{N}}$ violates the assumption. □

The trouble is that with absolute quantifiers, the Σ-count approach requires a decision on which quantification results to assign in the case that the computed Σ-count is not a cardinal number. But intuitions are scarce in this unfamiliar case.

"Trivial" or "degenerate" cases often require particular attention. One such case is that of a quantifier supplied with an argument tuple of empty sets.

Question 2: *Does the Σ-count approach treat consistently the case of empty argument sets?*

Answer: *No.* As an example, let us choose $0 < r_2 < r_1 < 1$ and consider the two-valued proportional quantifiers $Q_1 = [\textbf{rate} \geq r_1]$, $Q_2 = [\textbf{rate} > r_2] : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ (see p.1), which have $Q_1(\varnothing, \varnothing) = 1$, but $Q_2(\varnothing, \varnothing) = 0$. The problem is that Zadeh does not specify the denotation of Σ-Count$(\varnothing/\varnothing)$. So let us assume that Σ-Count$(\varnothing/\varnothing) = c \in \mathbf{I}$. *Correct generalization* demands that $\mu_{Q_1}(c) = 1$, i.e. $c \geq r_1$, but also that $\mu_{Q_2}(c) = 0$, i.e. $c \leq r_2$, which contradicts our assumption $r_2 < r_1$. □

In addition, the Σ-count approach yields potentially satisfying results only if μ_Q is genuinely fuzzy, because a two-valued quantifier (with corresponding two-valued membership functions μ_Q) is mapped to a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{2}$, the results of which are *always crisp*.[13] Proponents of the Σ-count approach might now object that, although a two-valued quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is to be modelled, an adequate choice of μ_Q should be continuous-valued.

Question 3: *Can we avoid this pitfall of the Σ-count approach by using a continuous-valued μ_Q?*

Answer: *No.* Let us assume that the two-valued quantifier Q to be "fuzzified" is of the proportional type; we will utilize the fact that such quantifiers have extension. Now suppose $\mu_Q \in \mathbf{I}^{\mathbf{I}}$ is a proper choice for interpreting Q, and $q \in \mathbb{Q} \cap \mathbf{I}$ is some rational number in $\mathbf{I}$. Firstly, we can choose $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ such that Σ-Count$(X_2/X_1) = q$. Because q is rational and nonnegative, there exist $z, n \in \mathbb{N}$ such that $q = z/n$; we may also require that $n \geq |E|$. Extend E by arbitrary elements to some superset $E' \supseteq E$ with $|E'| = n$, and choose an arbitrary *crisp* subset $Z \in \mathcal{P}(E')$ with $|Z| = z$. *Correct generalisation* yields $\mu_{Q_{E'}}(q) = \mu_{Q_{E'}}(\Sigma\text{-Count}(Z/E)) = Q_{E'}(E, Z) \in \mathbf{2}$. *Preservation of extensions* yields $\mu_{Q_E}(\Sigma\text{-Count}(X_2/X_1)) = \mu_{Q_{E'}}(\Sigma\text{-Count}(X_2/X_1)) = \mu_{Q_{E'}}(q) \in \mathbf{2}$. The membership functions suited for modelling two-valued proportional quantifiers are therefore restricted to $\{0, 1\}$ on $\mathbf{I} \cap \mathbb{Q}$. The option of selecting values in the open interval $(0, 1)$ for the remaining "definition gaps" on $\mathbf{I} \setminus \mathbb{Q}$ has few practical relevance. □

Question 4: *Is it possible to model quantifiers of exception using the Σ-count approach?*

Answer: *No.* Suppose $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$, $|E| \geq 2$, is a two-valued quantifier of exception, say $Q = $ **all except m**, $0 < m < |E|$, and define $Q' : \mathcal{P}(E) \longrightarrow \mathbf{2}$ by $Q'(Z) = Q(E, Z)$. Because Q is a quantifier of exception, we have $Q(X_1, X_2) = Q'({X_1}^c \cup X_2)$, for all $X_1, X_2 \in \mathcal{P}(E)$. But every absolute quantifier would have $Q(X_1, X_2) = Q'(X_1 \cap X_2)$, if Q' be defined in this way. By *correct generalisation*, it follows that there is no $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$ such that $Q = \mathcal{U}(\mathrm{SC}^{(2)}_{\mathrm{abs}}(\mu_Q))$. Let us now show that there is also no $\mu_Q \in \mathbf{I}^{\mathbf{I}}$ such that $Q = \mathcal{U}(\mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q))$, i.e. the Sigma-Count approach cannot represent Q subject to our conditions. We will need the lemma stated below. Noting that $Q = $ **all except m** is nonincreasing in its first argument, a reasonable choice of $\widetilde{Q}$ should preserve this property. We can apply the lemma to obtain $\mu_Q(x) = \text{const}$ for all $x \in (0, 1)$. Combining this with our results on Question 3, there are 8 choices left for μ_Q, corresponding to choices of $c_0, c_1, c_* \in \mathbf{2}$ with $\mu_Q(0) = c_0$, $\mu_Q(1) = c_1$, and $\mu_Q(x) = c_*$ for all $x \in (0, 1)$. It is easily checked that $\mathcal{U}(\mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)) \neq Q$ in these cases. □

[11] Stated in our notation. The "abs"-versions apply to the absolute kind where $\mu_Q \in \mathbf{I}^{\mathbb{R}^+}$, the "prp"-versions to the proportional kind $\mu_Q \in \mathbf{I}^{\mathbf{I}}$.

[12] It should be noted that all other approaches to fuzzy quantification do comply with the QFA [6].

[13] This problem has been obscured by Zadeh's use of the quantifier *most*, which he views as being genuinely fuzzy.

Lemma. If $\widetilde{Q} = \mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)$ is nondecreasing (nonincreasing) in its first argument and $|E| \geq 2$, then $\mu_Q|_{(0,1)} = \mathrm{const}$.

Proof. Suppose that $\widetilde{Q}$ is nondecreasing in its first argument.

a. Suppose $0 < b < a < 1$, choose some $* \in E$ and let $X_1 = 1/*$, $X'_1 = \frac{a}{b}/*$ and $X_2 = a/*$. Then Σ-Count$(X_2/X_1) = a > \Sigma$-Count$(X_2/X'_1) = b$.

Because $\widetilde{Q} = \mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)$ is nondecreasing in its first argument and $X'_1 \supseteq X_1$, we have

$$\mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(X'_1, X_2) \geq \mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2)\,,$$

i.e. $\mu_Q(b) \geq \mu_Q(a)$ for all $0 < b < a < 1$.

b. To show that μ_Q is also nondecreasing in $(0,1)$, suppose $0 < p < q < 1$, choose $e_1 \neq e_2 \in E$ and let $X_1 = (1-q)/e_1$, $X'_1 = (1-q)/e_1 + 1/e_2$ and $X_2 = p(1-q)/e_1 + (q + (1-q)(q-p))/e_2$. Then Σ-Count$(X_2/X_1) = p$ and Σ-Count$(X_2/X'_1) = q$.

From $X_1 \subseteq X_2$ we obtain $\mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) \leq \mathrm{SC}^{(2)}_{\mathrm{prp}}(\mu_Q)(X'_1, X_2)$ because $\widetilde{Q}$ is nondecreasing. Therefore $\mu_Q(\Sigma\text{-Count}(X_2/X_1)) \leq \mu_Q(\Sigma\text{-Count}(X_2/X'_1))$, i.e. $\mu_Q(p) \leq \mu_Q(q)$.

The proof for nonincreasing $\widetilde{Q}$ is analogous. □

9 Conclusion

Fuzzy quantifiers form an interesting class of operators because of their promise to provide a numerical interpretation of the fusion operators of natural language. By presenting a framework for evaluating formal properties of approaches to fuzzy quantification, we have rendered possible the investigation of these approaches based on formal and linguistic considerations.

In particular, we have argued that fuzzy linguistic quantifiers are too limited to express important classes of quantifiers. An exemplary investigation of the Σ-count approach within our proposed framework, as well as our findings on the other approaches reported in [6], have substantiated our doubts on the adequacy of approaches based on fuzzy linguistic quantifiers. We believe that our framework can direct future research to approaches conforming to formal and linguistic requirements. In particular, it might be advantageous to abandon fuzzy linguistic quantifiers completely, in favor of a theory of fuzzy quantification built directly on semi-fuzzy quantifiers and quantifier fuzzification mechanisms, due to their better compliance with TGQ. A promising attempt to do so has been presented in [5, 8]. Its application to information combination tasks in a content-based retrieval system is described in [7].

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