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Fundamentals of Fuzzy Quantification: Plausible Models, Constructive Principles, and Efficient Implementation

Ingo Glöckner

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Technische Fakultät der Universität Bielefeld,
Abteilung Informationstechnik, Postfach 10 01 31, 33501 Bielefeld

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Plausible Models, Constructive Principles, and Efficient Implementation

Ingo Glöckner

Universität Bielefeld, Technische Fakultät, AG Technische Informatik,
P.O.-Box 10 01 31, 33501 Bielefeld,
EMail: ingo@techfak.uni-bielefeld.de

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Contents

Preface and report overview

Natural language heavily depends on quantifying constructions. These often involve fuzzy concepts like “tall”, and they frequently refer to fuzzy quantities in agreement like “about ten”, “almost all”, “many” etc. In order to exploit this expressive power and make fuzzy quantification available to technical applications, fuzzy set theory has been enriched with various techniques which reduce fuzzy quantification to a comparison of scalar or fuzzy cardinalities [188, 124, 170, 176]. However, it soon became clear that the results of these methods fail to be plausible in some cases [2, 3, 30, 37, 47, 124, 125, 175]. This certainly hindered the spread of these ‘traditional’ approaches into commercial applications. Only recently a new solution has been pursued which showed itself immune against the pitfalls of existing approaches to fuzzy quantification. The goal of this report is to compile the material on this new theory known as the DFS theory of fuzzy quantification, which until now was scattered across several publications [46, 47, 48, 49, 50, 51]. It will stipulate a canonical terminology and notation, thus introducing the theory as a consistent whole, including latest developments. The present report covers the fundamental elements of fuzzy quantification which comprise the formal framework for describing fuzzy quantifiers; the axioms imposed on plausible models; my results on derived properties of these models; a catalogue of additional adequacy requirements; a discussion of several constructive principles along with an analysis of the generated models; an extensive list of algorithms for implementing fuzzy quantifiers in the models; and finally some illustrative examples. In particular, the missing elements have now been added, which were still needed to complete the proposed solution and make it really useful in practice. Until recently, there was a clear focus on theoretical topics and beyond some example algorithms sketched in [57, 59], no systematical discussion of implementation issues. In order to close this gap, the report now presents an in-depth treatise of the computational aspects of DFS theory. The report discloses a general strategy for implementing quantifiers in a model of interest, and it further develops a number of supplementary techniques which will optimize processing times. In particular, I describe an analysis of fuzzy quantifiers in terms of cardinality coefficients, which can be computed from histogram information. In order to illustrate the implementation strategy, the basic procedure will then be detailed for three prototypical models, and the complete algorithms for implementing the relevant quantifiers in these models will be presented. The quantifiers covered by this method not only include the familiar absolute and proportional quantifiers known from existing work on fuzzy quantification. They also include some further important types like quantifiers of exception (“all except k”) and cardinal comparatives (“many more than”) which are well-known to linguists, but innovative in the fuzzy sets framework.

The report is organized as follows. The *first chapter* presents a general introduction into the history and main issues of fuzzy natural language quantification. To this end, the basic characteristics of linguistic quantifiers are reviewed, in order to distill some requirements on a principled theory of linguistic quantification. Following this, I discuss the problem of linguistic vagueness and its modelling in terms of continuous membership grades, which is fundamental to fuzzy set theory. Zadeh’s traditional framework for fuzzy quantification [188, 190] is then explained and it is shown that both the framework and the particular approaches that evolved from it are inconsistent with the linguistic data.

The *second chapter* introduces a novel framework for fuzzy quantification in which the linguistic phenomenon can be studied with the desired comprehensiveness and formal rigor. The framework comprises: fuzzy quantifiers, which serve as the operational model or target representation of the quantifiers of interest; semi-fuzzy quantifiers, which avail us with a uniform and practical specification format; and finally QFMs (quantifier fuzzification mechanisms), which establish the link between specifications and operational quantifiers. The basic representations underlying the proposed framework are directly modelled after the generalized quantifiers familiar in linguistics [6, 8]. Compared to the two-valued linguistic concept, semi-fuzzy quantifiers add approximate quantifiers like “almost all”, while fuzzy quantifiers further admit fuzzy arguments (“tall”, “young” etc.). Thus, semi-fuzzy quantifiers and fuzzy quantifiers are the apparent extension of the original generalized quantifiers to type III and type IV quantifications in the sense of Liu and Kerre [99]. The organization of my approach into specification and operational layers greatly simplifies its application in practice. The commitments intrinsic to this analysis of fuzzy quantification are discussed at the end of the chapter, as well as the actual coverage of the proposed approach compared to the phenomenon of linguistic quantification in its full breadth.

The *third chapter* is concerned with the investigation of formal criteria which characterize a class of plausible models of fuzzy quantification. The basic strategy can be likened to the axiomatic description of t-norms [139], which constitute the plausible conjunctions in fuzzy set theory. Thus my approach is essentially algebraic. The basic idea is that of making explicit the intuitive expectations on plausible interpretations in order to eliminate the notorious difficulties of existing approaches. To this end, I will introduce a system of six basic requirements which distill a larger catalogue of linguistic desiderata. The criteria are chosen such that they capture independent aspects of plausibility and, taken together, identify a class of plausible models which answers the relevant linguistic expectations.

It remains to be shown that the proposed axioms successfully capture a class of plausible models. In the *fourth chapter*, I produced evidence that this is indeed the case. To this end, a plethora of criteria will be considered which are significant to the linguistic plausibility of the computed interpretations and to their expected coherence. All of these criteria are validated by the proposed models. This supports my choice of axioms even if some of these might appear rather ‘abstract’ at first sight. Apart from this purpose of justifying the proposed class of models, the formalization of plausibility criteria is also a topic of independent interest. By investigating such criteria, we can further our knowledge about quantifier interpretations in natural languages.

In the *fifth* chapter I will structure the space of possible models. Specifically I will place attention on certain subclasses of models, i.e. classes of models with some common structure or joint properties. The relative homogeneity of the models within these classes lets me develop some important concepts on these models, e.g. regarding the specificity of results. In addition I will identify the class of standard models, which is compliant with the standard choice of connectives in fuzzy set theory. The role of these standard models to fuzzy quantification can be likened to that of Abelian groups vs. general groups in mathematical group theory.

The *sixth chapter*, then, is devoted to the study of some additional properties, like continuity (smoothness), which are 'nice to have' from a practical perspective, but not always useful for theoretical investigations, or sometimes even awkward in this context. Consequently these properties should not be assumed in general, and this is why I did not include them into my core requirements on plausible models. The chapter further considers some 'critical' properties which cannot be satisfied by the models for reasons of incompatibility (usually these properties even fail in much weaker systems). The existence of such cases is not surprising, of course, because fuzzy logic, as a rule, can never satisfy all axioms of Boolean algebra. These difficulties will usually be resolved by showing that the critical property conflicts with very basic requirements, and by pruning the original postulate to a compatible 'core' requirement.

Having laid these theoretical foundations, I proceed to the issue of identifying prototypical models, which are potentially useful to applications. It is hence necessary to investigate certain constructive principles, which give us a grip on such concrete examples. Existing research has typically tried to explain fuzzy quantification in terms of cardinality comparisons, based on some notion of cardinality for fuzzy sets. However, such reduction is not possible for arbitrary quantifiers. Hence a comprehensive interpretation of linguistic quantifiers must rest on a more general conception.

In chapters seven to ten, I will describe some suitable choices for such constructions which result in the increasingly broader classes of $\mathcal{M}_\mathcal{B}$-models (*chapter seven*), $\mathcal{F}_\xi$-models (*chapter eight*), and finally $\mathcal{F}_\Omega$ or $\mathcal{F}_\omega$-models (*chapter nine*). All of these models rest on a generalized supervaluationist approach based on a three-valued cutting mechanism. In the *tenth chapter*, I consider a different mechanism, based on the extension principle, which results in the constructive class of $\mathcal{F}_\psi$-models. This class of models which are definable in terms of the standard extension principle, is then shown to provide a different perspective on the class of $\mathcal{F}_\Omega$-models, to which it is coextensive. Apart from introducing these classes of prototypical models, I also demonstrate how important properties of the models, like continuity, can be expressed in terms of conditions imposed on the underlying constructions. This facilitates to check whether a model of interest is sufficiently robust, how it compares with another model by specificity, etc. In particular, this analysis reveals that all practical $\mathcal{F}_\psi$- or $\mathcal{F}_\Omega$-models belong to the $\mathcal{F}_\xi$-type.

The *eleventh chapter* presents the algorithmic part of the theory, and is thus concerned with the issue of efficient implementation. Obviously, it only makes sense to consider practical (i.e. sufficiently robust) models. Thus, I can confine myself to analyzing models of the $\mathcal{F}_\xi$-type. The general strategies for efficient implementation that I expound in this chapter, will be instantiated for three prototypical models. The considered quantifiers include the familiar absolute and proportional types, as well as quantifiers of exception and cardinal comparatives. Some application examples are also discussed at the end of the chapter.

In the *twelvth chapter*, I propose an extension of the basic framework for fuzzy quantification. My goal is that of supporting the most powerful notion of quantifiers developed by mathematicians, i.e. so-called Lindström quantifiers [97]. These quantifiers are also of potential linguistic relevance, and I explain how they can be used to model certain reciprocal constructions in natural language.

The *thirteenth chapter* will resume the main contributions of the report, and propose some directions for future research.

There are four appendices.

Appendix A presents the technical details of my results on existing approaches that were cited in the introduction. It develops an evaluation framework for those approaches to fuzzy quantification, which are based on Zadeh's traditional framework. This analysis lets me apply the plausibility criteria developed in the main part of the report to the existing approaches to fuzzy quantification described in the literature.

Appendix B discusses the basic concept of a fuzzification mechanism, which underlies my proposed framework, and it sheds some light on its ramifications.

This report presents a total of 275 theorems and it would have been impossible to list, or even sketch, the proofs of all of them. In order to keep the size of this work within limits, and also to improve its readability, the proofs of most theorems have therefore been detached from this major contribution and published in a series of more specialized technical reports [46, 48, 49, 50, 51].

Appendix C contains a theorem reference chart which connects the theorems cited here with their original proofs published in these reports.

Finally *Appendix D* lists the complete proofs of all 'new' theorems, which are mostly concerned with issues related to implementation and with branching quantification.

1 An introduction to fuzzy quantification: Origins and basic concepts

1.1 Motivation and chapter overview

From a linguistic perspective, it is *quantification* which makes all the difference between "having no dollars" and "having a lot of dollars". And it is the meaning of the quantifier "most" which eventually decides if "Most Americans voted Al Gore" or "Most Americans voted Bush" (as it stands). Natural language (NL) quantifiers like "all", "almost all", "many" etc. serve an important purpose because they permit us to talk about properties of collections, as opposed to describing specific individuals only; in technical terms, quantifiers are a 'second-order' construct. Hence the quantifying statement "Most Americans voted Bush" asserts that the set of voters of George W. Bush comprises the majority of Americans, while "Bush sneezes" only tells us something about a specific individual. By describing collections rather than individuals, quantifiers extend the expressive power of natural languages far beyond that of propositional logic, thus making them a universal communication medium. Not surprisingly, then, quantifiers are ubiquitous in everyday language, and examples of quantification come up in all areas of daily life (see Table 1). But quantifiers are not only frequently found in spoken and written language; they also have a considerable share in what is being said. Whether there are "many clouds over Italy" or "very few clouds", say, can make quite a difference for prospective tourists planning their summer vacation. Due to the crucial importance of NL quantifiers to the meaning and expressiveness of language, but also to the admissible inferences (i.e., the valid logical conclusions), it is not surprising that quantifying constructions aroused interest since the very beginning of scientific research. In fact, the analysis of quantification, which was originally considered the subject of logic and thus philosophy, can be traced back to the Ancient Greek, notably to the 'first formal logician', Aristotle [16, p. 40]. In his *Topics*, Aristotle describes the subject of logic as follows.

> *"Reasoning is an argument in which, certain things being laid down, something other than these necessarily comes about through them."*
> (English translation in Bocheński [16, (10.05), p. 44])

The early interest in universal quantification is explained by this focus on valid rules and inferential patterns, because universal propositions are necessary to describe general laws which are valid for all possible instantiations. For example, the infamous "All men are mortals" makes an assertion about arbitrary men. Similar considerations apply to existential quantification. Existential propositions let us talk about an individual without explicitly naming it, and even without knowing its identity, cf. "Peter screamed" vs. "A man screamed". In addition, existential quantification results from universal quantification and negation. These dependencies become visible in the well-known 'square of opposition' or 'Aristotelian square', see Bocheński [16, p. 59, (12.09)] and Chap. 3 below. To sum up, the universal and existential modes of quantification are of special importance to reasoning. Moreover, they represent the simplest and least peculiar examples of NL quantification. It is therefore not surprising that

Finance and Economics:
Many firms have stopped making markets (p. 75)
Business:
Most bosses assume they can change prices often and with little effort (p. 63) *Few* business schools teach pricing as a discipline (p. 63)
Politics:
Many Indians admit that they have misgoverned their only Muslim-majority state . . . (p. 25) *Several* separatist leaders . . . seem even more willing to co-operate with India (p. 26)
Science and Technology:
Some philosophers see free will as an illusion that helps people to interact with one another (p. 85) *Most* courts, for example, accept a claim of insanity as a defence in certain criminal cases (p. 85)
Literature and Arts:
Few living novelists write better than Mr Winton about the sea (p. 89)
Studies of Feminism:
Some radical women preached free love while *most* emphasised sexual purity (p. 89)

Table 1: Examples of NL quantification in various areas of everyday life. Source: *The Economist 25-31/05/2002*

logicians tentatively confined their work to these regular cases only. However, the understanding of quantification advanced at a slow pace, and it took more than 2,000 years until the puzzles of universal and existential quantification were finally solved. In part, this slow progress exemplifies the intellectual difficulties associated with a second-order abstraction like quantification. Its chief reason, however, was the focus on syllogistic reasoning. According to Bocheński, a syllogism is *"a λόγος"* [form of speech] *"in which if something is posited, something else necessarily follows. Moreover such λόγοι are there treated as formulas which exhibit variables in place of words with constant meaning"* [16, p. 2]. In practice, the syllogisms used for reasoning are composed of three parts: a major premiss, a minor premiss, and a conclusion. An example is [135, p. 206]:

> All men are mortal (Major premiss)
> Socrates is a man (Minor premiss)
> Therefore: Socrates is mortal (Conclusion).

The belief of Aristotle and his followers that *"all deductive inference, when strictly stated, is syllogistic"* [135, p. 207] obstructed progress in at least two ways. First of all, Aristotelian logic is concerned with universal and existential quantification, but it lacks the notions of universal and existential quantifiers. In other words, the quantifiers "all" and "some" did not appear as logical abstractions in their own right, but only as structural elements of the syllogism. This hindered the development of a clear semantics for universal and existential quantification. Another factor which proved itself obstructive was Aristotle's analysis of quantifying expressions into subject and predicate. "Sokrates" and "all Greek", say, are both considered subjects, which obscures that their logical analysis must be pretty different.[1] Thus the syllogistic logic has intrinsic weaknesses and Russell even remarks that the logical work of Aristotle was *"a dead end, followed by over two thousand years of stagnation"* [135, p. 206]. G. Frege was the first to recognize the unfortunate effect of analysing quantified propositions into subject and predicate. In his *Begriffsschrift*, he remarks:

> *"In the first draft of my formula language I allowed myself to be misled by the example of ordinary language into constructing judgments out of subject and predicate. But I soon became convinced that this was an obstacle to my specific goal and led only to useless prolixity."*
> (English translation in van Heijenoort [63, p. 13])

Frege therefore replaced subjects and predicates with a novel analysis into function and argument(s), which is necessary for a convincing account of many-place functions in logic. This analysis, which has now become standard, also permitted Frege to develop the modern doctrine of quantifiers which, *"in contrast to the Aristotelian tradition, . . . are conceived as separate from the quantified function and its copula, and are so symbolized"* [16, p. 347]. Thus the existential and universal quantifiers – $\forall$ ("for all") and $\exists$ ("exists") in my notation – became first-class citizens of logic. The subsequent adoption and refinement of Freges doctrine marked a revolution in the historical development of logic which resulted in today's systems of first-order predicate logic, higher-order logics, and type theory. The new analysis in terms of function and argument(s) and the flexible use of quantifiers marked the necessary end of syllogistic reasoning. It was replaced with modern calculi which unlike syllogisms, are applicable to arbitrary formulas of predicate logic. Hence formal logic was finally equipped with a model of quantification which suits its purposes. Apart from the apparent application to mathematics, there have also been attempts to utilize the logical quantifiers $\forall$ and $\exists$ for the semantic description of natural language. One of the pioneering works is Russell's analysis of definite descriptions like "the author of Waverley". As Russell argues, these descriptions are reducible to combinations of the logical quantifiers. The proposition "Scott was the author of Waverley", for example, which involves the definite singular quantifier "the", is rephrased thus [135, p. 785]:

> There is an entity c such that the statement "x wrote Waverley" is true if x is c and false otherwise; moreover c is Scott.

In terms of the logical quantifiers, this analysis becomes

$$\exists c(\mathrm{aw}(c) \wedge \forall x(\mathrm{aw}(x) \rightarrow x = c) \wedge c = \mathrm{Scott})\,,$$

where 'aw' abbreviates "author of Waverley". The analysis of a special kind of NL quantifier in terms of the logical quantifiers may look promising. However, there is no

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[1] as witnessed by examples like $\mathrm{sleep}(\mathrm{Sokrates})$ vs. $\forall x(\mathrm{greek}(x) \rightarrow \mathrm{sleep}(x))$.

apparent generalization of Russell's approach to a broader class of NL quantifiers. In addition, Russell's theory of descriptions has been faced with substantial criticism due to its poor account of the involved presuppositions.[2]

In this chapter, I first show that comparable difficulties must be expected if one tries to express linguistic quantifiers in terms of $\forall$ and $\exists$. Specifically, the logical quantifiers fail to account for the wealth of NL quantifiers because they are too weak, i.e. unable to capture the meaning of important examples. Moreover, *"the familiar $\forall$ and $\exists$ are extremely atypical quantifiers. They have special properties which are entirely misleading when one is concerned with quantifiers in general"* [6, p. 160]. Thus we have structural differences, like the dependency of many NL quantifiers on a pattern of several arguments. And certain assumptions which $\forall$ and $\exists$ legitimate must be given up in the general case. In order to develop a practical notion of quantifiers, we must then take a look into that property of NL known as 'vagueness' or 'fuzziness'. Basically, the vagueness of NL makes a limited repository of NL concepts applicable to a potentially unbounded variety of phenomena, by allowing a partial mismatch between the descriptive means and the observed phenomenon. Thus the vagueness of language introduces imprecision or 'imperfection' but makes it a practical communication medium. In the chapter, I provide some background information on linguistic vagueness and the attempts at its modelling by presenting the fundamentals of vagueness theory. Only one of these methods has achieved practical significance, viz Zadeh's fuzzy set theory which interprets vague (or 'fuzzy') predicates in terms of mathematical models known as 'fuzzy sets'. The method is not only applicable to NL predicates, but also potentially useful for modelling NL quantifiers. Following a discussion of the origins of fuzzy quantification theory and its main directions of research, I then introduce the central problematic of this report, i.e. the *modelling problem* of establishing a system of plausible interpretations for linguistic quantifiers in a suitable framework for fuzzy quantification. I present Zadeh's traditional solution to the modelling problem and the specific interpretation methods that evolved from it. The subsequent evaluation of these approaches based on my own findings and those reported in the literature, will raise serious doubts concerning the coverage and plausibility of these methods. In concluding the chapter, I relate these difficulties to Zadeh's superordinate framework and outline the necessary improvements for a successful replacement.

1.2 Logical quantifiers vs. linguistic quantifiers

In the comparison of logical quantifiers and NL quantifiers, I will roughly follow the classic arguments of Barwise and Cooper [6] that the logical quantifiers are insufficient for modelling NL semantics, thus substantiating the need for a generalization. Before we can discuss this matter, however, we must first clarify some terminological subtleties. In the linguistic theory of quantification, quantifying elements like "all", "most" etc. are usually referred to as 'determiners'. The term 'quantifier', then, is reserved for the (interpretation of) nominal phrases only, i.e. for the class 'NP' of expressions such as proper names, descriptions, and quantified terms [45, p. 226]. From

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[2] for example, "the author of Waverley" presupposes that such an author exists. See van der Sandt [137] and Sag/Prince [136] for a survey of work on presuppositions and pointers into the literature.

a linguist's point of view, making a sharp distinction between determiners and quantifiers can be useful, especially at the crossroads of quantifiers and syntactic analysis. In other cases, however, the distinction is not that productive, and only adds complexity. For the purposes of this report, it is favourable to adopt a 'flat' model of quantifiers, which offers a uniform representation for both determiners and quantifiers corresponding to NPs, see Chap. 2 for details. Such a flat model is often used in the literature when emphasis is placed on the quantifying elements 'per se' and their semantic description [9, p. 444]. The model allows me to drop the awkward distinction and view determiners as a special case of quantifiers.

The chosen terminology is also in good conformance with existing research on fuzzy quantifiers (to be discussed below), in which expressions like "all" or "most" are generally called 'quantifiers' rather than 'determiners'. These approaches depart from the linguistic notion of quantifiers in yet another point, which is concerned with the possible ranges of quantification. When interpreting a quantifying proposition, one always needs a *base set* associated with the quantifier, i.e. a non-empty set E (for 'entities') which supplies the individuals over which the quantification ranges. For the semantic interpretation of a formula like $\forall x\, \mathrm{expensive}(x)$, say, one uses a 'model' or 'structure' which among other things, specifies the collection of 'all things' against which $\mathrm{expensive}(x)$ will be tested. Similar considerations apply to the interpretation of quantifying propositions in natural language, like the corresponding "Everything is expensive". In the usual case of quantification based on determiners and NPs, the base set of the quantifier coincides with the given universe of discourse, i.e. the collection of individuals we can talk about. In the above example of the logical quantifiers in predicate logic, the base set used for interpreting $\forall$ and $\exists$ is always the set of entities provided by the model. In fact, this is the only type of quantification considered by most linguists, which usually focus on determiners and NPs. This narrow notion of quantifiers rests on the hypothesis that determiners and NPs are the only sources of NL quantifiers over the domain of discourse, see Barwise and Cooper [6, p. 177]. However, there are other constructions in NL which arguably involve quantification, although they do not fit into the NP picture. As opposed to the 'explicit' quantifications considered so far, which are always connected to determiners like "most", these cases make 'implicit' use of quantification, which is needed for their semantical interpretation. Typical examples are *temporal adverbs* like "always", "never", "often", "a few times"; and *spatial adverbs* like "everywhere", "somewhere" etc.

In this case, the base set of the corresponding quantifiers differs from the universe of discourse, and quantification now ranges over such things as points in time or space (the precise nature of the base sets obviously depends on the chosen modelling of time and space). Another source of examples are *dispositions*, i.e. propositions like "Slimness is attractive" which are preponderantly, but not necessarily always, true [191, p. 713]. In this case no quantifier is visible at the NL surface at all. However, the above disposition can be given the pragmatical interpretation "Usually slimness is attractive", which depends on some kind of quantification over situations or circumstances. This approach was pioneered in Zadeh's work on dispositional reasoning based on fuzzy set theory [190, 191]. Similar ideas emerged in non-monotonic logic where the modelling of default reasoning in terms of generalized quantifiers was proposed [138]. In fuzzy

NL proposition	**Logical paraphrase**
Everything is expensive	$\forall x\, \mathrm{expensive}(x)$
All bachelors are men	$\forall x(\mathrm{bachelor}(x) \rightarrow \mathrm{man}(x))$
Some men are married	$\exists x(\mathrm{man}(x) \wedge \mathrm{married}(x))$
No man is his own father	$\neg\exists x(\mathrm{man}(x) \wedge \mathrm{father}(x, x))$

Table 2: Quantifying propositions and their translations into predicate logic: Some elementary examples

set theory, it is customary to adopt a broad notion of quantifiers which also covers such ‘implicit’ cases. I will follow this practice here, thus admitting both the explicit and implicit types of quantification.

Now that the terminological commitments have been explained, we are ready to contrast the logical account of quantifiers with some facts about ‘natural’ quantifiers actually found in human languages. Let us begin with the issue of expressive power, i.e. is it possible to define the meaning of NL quantifiers in terms of the logical quantifiers $\forall$ and $\exists$ of first-order predicate logic? It should not come as a surprise that this cannot be done in general; consequently, we are interested in identifying the precise class of NL quantifiers which are first-order definable. The linguistic equivalents of the logical quantifiers, i.e. “all”, “some” as well as the derivations “no” and “not all” have trivial renderings in first-order logic, see Table 2 for some apparent examples. Noticing that “some” means “at least one”, we can try and extend the analysis to quantifying propositions of the general form “There are at least k X’s”, for some cardinal $k \in \mathbb{N}$. The following definitions of corresponding quantifiers $[\geq k]$ should be straightforward:

$$[\geq 1]x\varphi(x) \leftrightarrow \exists x\varphi(x)$$
$$[\geq 2]x\varphi(x) \leftrightarrow \exists x_1\exists x_2(\varphi(x_1) \wedge \varphi(x_2) \wedge x_1 \neq x_2)$$
$$[\geq 3]x\varphi(x) \leftrightarrow \exists x_1\exists x_2\exists x_3(\varphi(x_1) \wedge \varphi(x_2) \wedge \varphi(x_3) \wedge x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_2 \neq x_3)$$
$$\vdots$$
$$[\geq k]x\varphi(x) \leftrightarrow \exists x_1 \cdots \exists x_k(\bigwedge_{i=1}^{k} \varphi(x_i) \wedge \bigwedge_{i \neq j} x_i \neq x_j)\,.$$

Of course, it is also possible to express these quantifiers in terms of cardinalities. Hence “There are at least k X’s” asserts that the cardinality of the collection of X’s is larger than or equal to k. The example is representative of the general class of quantifiers defined in terms of absolute counts. These will be called *absolute quantifiers* in this report, c.f. [46, p. 82], [99, p. 2], [88, p. 208]; however, the terms ‘cardinal determiners’ [81, p. 98] and ‘quantifiers of the first kind’ [188, p. 149] are also common. Further examples of this class comprise apparent derivations like “more than k” and “at most k”, as well as the ‘bounding’ type of quantifiers like “exactly ten” or “between five and ten”, see Keenan and Moss [81, p. 123]. The corresponding quantifiers, denoted $[>k]$,

$[\leq k]$, $[=k]$ and $[\geq k; \leq u]$, respectively, are all first-order definable:

$$\begin{aligned}
&[>k]x\varphi(x) \leftrightarrow [\geq k+1]x\varphi(x) \\
&[\leq k]x\varphi(x) \leftrightarrow \neg[>k]x\varphi(x) \\
&[=k]x\varphi(x) \leftrightarrow [\geq k]x\varphi(x) \wedge [\leq k]x\varphi(x) \\
&[\geq k; \leq u]\varphi(x) \leftrightarrow [\geq k]x\varphi(x) \wedge [\leq u]x\varphi(x)\,.
\end{aligned}$$

Thus typical absolute quantifiers appear to be first-order definable. The issue was clarified by van Benthem [9, p. 462, Th-6.2], who has a general result on logical definability:

> "*All first-order definable quantifiers are logically equivalent to Boolean compounds of the types* **at most k (non-) A are (not) B** *and* **there are at most k (non-) A**."

Apart from its apparent assertion about absolute quantifiers, what this means is that as a rule, all other classes of NL quantifiers are not first-order definable. Consider the quantifier "most", for example. The truth of a proposition like "Most men are married" obviously depends on the relative share of those men which are married. The example is representative of the general class of quantifiers which depend on a ratio of cardinalities, so-called *proportional quantifiers*. These quantifiers are also known as 'proportional determiners' [81, p. 123], 'relative quantifiers' [99, p. 2], [88, p. 209] or 'quantifiers of the second kind' [188, p. 149]. Further examples of the proportional type comprise "half of the", "every third", "five percent", "more than ten percent", "less than twenty percent", "between ten and twenty percent" etc. Proportional quantifiers are frequently used in natural language (in particular the approximate examples "many", "few", "almost all" etc. which will be discussed later) and also of obvious relevance to applications. However, these quantifiers are not first-order definable, except for a few cases like "100 percent" and "more than 0 percent", which degenerate into the logical quantifiers "all" and "some", respectively. An exemplary proof that "more than half" is not first-order definable was presented by Barwise and Cooper [6, Th-C12, p. 213]. Hence predicate logic lacks the expressive power to model common types of quantifiers, including the proportional examples: *"It is not just that we do not see how to express them in terms of $\forall$ and $\exists$; it simply cannot be done"* [6, p. 160]. The significance of these findings mainly stems from the fact that first-order predicate logic is the strongest system of logic for which the usual proof theory can be developed [98]. Consequently, there are inherent limits to the development of logical calculi for NL quantifiers. This somewhat pessimistic prospect for *reasoning* with linguistic quantifiers should not discourage the *modelling* of such quantifiers, though. For example, it is perfectly possible that many NL quantifiers be definable in a stronger system of higher-order logic.

In any case, a successful modelling must account for the peculiarities of NL quantifiers, and we shall now consider some of those characteristics which set them apart from the logical quantifiers. The most conspicious difference is concerned with the quanticational structure of natural language, i.e. linguistic quantifiers typically show more complex patterns of arguments compared to the logical ones. Consider the quantifying proposition "Some men are married", for example. The proposition can be viewed as an instance of the general pattern "Some Y_1's are Y_2's", which involves two arguments Y_1, Y_2. The first argument, Y_1, serves to restrict the quantification to the individuals of interest (the set of men in our case). It is therefore called the *restriction* of the quantifier. The second argument, Y_2, asserts something about the individuals specified by Y_1 (in this case that one of these individuals is married). This second argument will be called the *scope* of the quantifier. The example thus exemplifies the *restricted use* of an NL quantifier, where the range of quantification is controlled by a restriction argument. Occasionally, one also finds the *unrestricted use* of an NL quantifier. A proposition like "Something is wrong", for example, which instantiates the general pattern "Something is Y", does not constrain the objects of interest in any way. There is only one argument, Y, which functions as the scope of the quantifier. Because no restriction is specified, the quantification ranges over the full base set, E. It is this usage of "some" after which existential quantification in predicate logic has been modelled. In formulas of the type $\exists x \varphi(x)$, there is only a scope argument $\varphi(x)$. We thus have unrestricted quantification, because x ranges over all individuals of the given domain. As it happens, the restricted use of "some", as in "Some men are married", can be reduced to the unrestricted case, i.e. to the equivalent "Something is a man and married". It is this principle which underlies the usual translation of existential propositions into predicate logic, where "Some men are married" would typically map to the unrestricted statement $\exists x(\mathrm{men}(x) \wedge \mathrm{married}(x))$. Universally quantified propositions permit a similar analysis. Consider the assertion "All bachelors are men", for example, which is an instance of the general pattern "All Y_1's are Y_2's". We hence have restricted quantification, based on the restriction "bachelor" and the scope "men". Again, the restricted case can be reduced to simple unrestricted quantification. For example, "All bachelors are men" admits the unrestricted paraphrase "All things are either not bachelors or men". This analysis corresponds to the usual translation into predicate logic, $\forall x(\mathrm{bachelor}(x) \rightarrow \mathrm{men}(x))$.

Such reductions notwithstanding, my examples demonstrate that it is the *restricted* type of quantification which prevails in natural language. Hence NL quantifiers are usually two-place and controlled by a restriction argument, rather than totally unconstrained, ranging over the full domain. The principle of restricted quantification and the general preference for this type of quantification in NL accounts for the somewhat arbitrary nature of the base set E as a whole. The base set must be large enough to supply all individuals we can talk about, and consequently there will be a lot of individuals in every given situation which are totally unrelated to the task at hand. It is therefore necessary for successful communication that one's assertions be constrained to the objects of actual interest, and restricted quantification is the linguistic device to achieve this. In fact, there are only very limited ways of expressing unrestricted quantification in NL, because the restricted form is so closely tied to NL syntax and the general pattern "Q Y_1's are Y_2's". Apart from a few special cases like "everything", "something" and "nothing", which can be considered genuinely unrestricted, and the construction "There are Q X's", which is possible for absolute quantifiers, it appears that we can usually only paraphrase the unrestricted form of quantification, by using formulations like "Q things are X", or "Q of all things are X". These facts about natural language are in apparent contrast with the situation in predicate logic, where quantifiers are always unrestricted. Of course restricted existential and universal quantification can be simulated in predicate logic, and the limitation of the logical language to the simple unrestricted case is certainly motivated by the availability of this reduction. Turning to linguistic quantifiers in general, we should therefore examine whether this reduction generalizes to arbitrary NL quantifiers. In fact, the issue has already been decided by linguistic research, and the answer came out negative: NL quantifiers which cannot be expressed in terms of their unrestricted counterparts are rather common. The prime case are again proportional quantifiers like “most”, “more than 60 percent” etc. An exemplary proof that “more than half” is irreducible to the unrestricted case was given by Barwise and Cooper [6, p. 214]:

> *There is no way to define “More than half of the V’s” in terms of “More than half of all things” and the operations of first-order logic, even if one restricts attention to finite models.*

From a different perspective, the irreducibility of many NL quantifiers to unrestricted quantification demonstrates that these quantifiers do not permit a translation into unary (one-place) quantification. Hence natural language is typically concerned with multi-place quantifiers of arities $n > 1$. Up to now, we were only concerned with multi-place quantifiers of arity $n = 2$, i.e. restricted quantification of the absolute or proportional type. However, multi-place quantifiers of arity $n > 2$ are also quite common. A typical example is “More men than women are snorers”, which is drawn from the general pattern “More Y_1’s than Y_2’s are Y_3’s”. Hence “more than” is a three-place quantifier. It is representative of a class of NL quantifiers called *cardinal comparatives* [82, p. 305] or *comparative determiners* [81, p. 123], which effect a comparison of cardinalities. For another example, consider “Most dogs and cats are either coddled or uncared-for”. In linguistics, it is customary to model such cases by complex, constructed quantifiers. The above proposition, for example, can be analysed in terms of the pattern “Most Y_1’s and Y_2’s are either Y_3’s or Y_4’s”, which corresponds to a four-place quantifier. This treatment of Boolean constructions is necessary in order to preserve the validity of Frege’s compositionality principle, see Keenan and Moss [81, p. 78-93] for linguistic motivation and an extensive discussion of the subject. To sum up, the logical quantifiers are generally unrestricted and confined to a single argument, while in natural language, we usually have restricted quantification and two or more arguments. Due to this structural difference, a more general model is needed for linguistic quantifiers, which must be treated as some kind of many-place function.

In the following, we shall consider some further properties of the logical quantifiers which do not generalize to linguistic quantifiers. The first is concerned with quantitativity: i.e. the logical quantifiers do not refer to specific elements of the domain, and can thus be defined in terms of the cardinality of their arguments or Boolean combinations. Hence $\exists x \varphi(x)$ asserts that the set of all things which satisfy $\varphi(x)$ has positive cardinality, while $\forall x \varphi(x)$ asserts that the complement of this set has zero cardinality. In general, such quantifiers which depend on cardinalities will be called *quantitative*. Obviously, absolute quantifiers like “more than ten” conform to this scheme. The same can be said about proportional quantifiers like “most”, which are defined in terms of a ratio of cardinalities (assuming that the base set E be finite). Hence many NL quantifiers depend on cardinalities, and are thus quantitative. However, the preponderance of quantitative examples does not mean that linguistic quantifiers are restricted to the quantitative type. Some counter-examples comprise "Hans" and other proper names, "all married" and other composites restricted by an adjective, and finally quantifiers on infinite base sets – as shown by van Benthem [8, p.474], most quantifiers of interest become non-quantitative in this case.[3] A comprehensive model of NL quantification should cover all of these cases regardless of quantitativity.

Finally the usual treatment of universal and existential quantification in predicate logic suggests that the meaning of quantifiers be 'cast in stone' and totally independent of factors like the model chosen for interpretation. This conception is certainly appropriate for quantifiers like $\forall$ and $\exists$, which can indeed be defined without reference to the model. In the following, we shall denote all quantifiers with this invariance property as *logical quantifiers*, i.e. quantifiers which can be treated as logical symbols. It is important to notice that many NL quantifiers do not fit into this category. Examples comprise the quantifier "every man" in the pattern "every man is Y" and other noun phrases; the quantifier "all married" in the pattern "All married Y_1's are Y_2's" and other quantifiers restricted by adjectives; and finally "Peter" in the pattern "Peter is Y", as well as other proper names. In all of these cases, it is only the chosen model (i.e. 'structure', 'interpretation') which supplies an interpretation for the non-logical symbols "men" and "married" (predicates) and "Peter" (treated as a constant), thus fixing the meaning of the above quantifiers. For example, we cannot interpret propositions of the form "Every man is X" unless we know the denotation of "man", i.e. the collection of individuals it refers to. Consequently, "every" is a logical symbol, but the derived quantifier "every man" is a non-logical symbol. Similar considerations apply to the remaining examples. In the examples discussed so far, knowing the model was sufficient to decide upon the meaning of the non-logical quantifiers. There are quantifiers, however, the meaning of which can vary even when the model is fixed, and which thus show some kind of *context dependence*. In fact, such quantifiers can even be found in mathematics. Consider the quantifier $Qx\varphi(x)$ of Sgro [140], for example, which asserts that the set of all things which satisfy $\varphi(x)$ contains a non-empty open set. As pointed out by Barwise and Cooper, the meaning of Q is *"determined not by logic, but by some underlying notion of distance, or, more precisely, by an underlying 'topology'."* [6, p. 162]. Hence a richer notion of model is necessary to fix the meaning of the quantifier. In linguistics, similar methods must be employed for assigning an interpretation to proportional quantifiers like "more than 50 percent" when the base sets are infinite, see e.g. Barwise/Cooper [6, p. 163] and van Benthem [8, p. 474-477]. Furthermore, the meaning of such quantifiers as "many" or "few" is context-dependent even when the base set is finite. To be specific these quantifiers involve some kind of comparison. It is this 'implied' comparison which brings about their context-dependence, because *"in simple uses at least, the standard of comparison is usually not given"* [82, p. 258]. Keenan and Stavi conclude from this observation that quantifying propositions involving "many" or "most" cannot be interpreted at all and thus assume no determi-

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[3] we shall see below on p. 15 how the criterion for quantitativity can be formalized such that it also works for infinite base sets.

nate truth value. However, these quantifiers are very common in ordinary language and people normally understand them without difficulty. This indicates that we need not resign and join the pessimism of Keenan and Stavi. By contrast, the ease with which people deal with such expressions rather suggests the use of the same strategy as above, thus postulating a richer source of information which fixes the meaning of these quantifiers as well. In order to remain agnostic about the precise nature of these 'enriched' models and the particular kinds of information needed for disambiguation, it is useful to delegate this type of problem to some external means, thus leaving the basic notion of a logical model intact. Following Barwise and Cooper [6, p. 163], I will therefore assume that there is a rich *context* which fixes the meaning of all quantifiers of interest, and leave it to later specialized research to formalize a suitable notion of contexts.

The examples presented above demonstrate that a generalized notion of quantifiers is needed to capture the meaning of linguistic quantifiers, because the familiar predicate calculus covers only a small fraction of NL quantification. In addition, some informal requirements on such a generalized notion have already been identified (e.g. concerning argument structure). In fact, there are proposals for a generalization of quantifiers in mathematical logic; the pioneering work in this area was done in the 1950's by Mostowski [108]. The Mostowskian notion of a quantifier is apparent from the following consideration. Let us denote by $\{x : \varphi(x)\}$ the set of all things which satisfy $\varphi(x)$. It is then apparent that the logical quantifiers can be expressed in terms of $\{x : \varphi(x)\}$, i.e. $\forall x\varphi(x)$ asserts that $\{x : \varphi(x)\} = E$, and $\exists x\varphi(x)$ asserts that $\{x : \varphi(x)\} \neq \varnothing$. Hence both $\forall$ and $\exists$ are definable in terms of a suitable mapping $Q : \mathcal{P}(E) \longrightarrow \{0, 1\}$, where $\mathcal{P}(E)$ denotes the powerset (set of subsets) of E, i.e. $Qx\varphi(x)$ is satisfied in the given model if and only if $Q(Y) = 1$, where $Y = \{x : \varphi(x)\}$. This suggests an apparent abstraction to the class of generalized quantifiers defined in terms of all such mappings $Q : \mathcal{P}(E) \longrightarrow \{0, 1\}$. Basically, we have now arrived at Mostowski's notion of a *quantifier restricted to* E, where E is the given base set (to be precise, Mostowski uses a somewhat different notation). However, Mostowski further constrains the admissible choices of Q by admitting quantitative examples only. Mostowski was the first to formalize the intuitive criterion of quantitativity in terms of *automorphism invariance*. Hence every quantifier restricted to E in Mostowski's sense has $Q(\beta(Y)) = Q(Y)$, for all bijections $\beta : E \longrightarrow E$ and all $Y \in \mathcal{P}(E)$, i.e. Q cannot refer to any specific individuals in E. Mostowski's proposal has become the standard formalization of quantitativity, because it offers a straightforward definition for both finite and infinite base sets. The 'quantifiers restricted to E' so defined, i.e. automorphism-invariant mappings $Q : \mathcal{P}(E) \longrightarrow \{0, 1\}$, still depend on the base set $E \neq \varnothing$. A (full) *quantifier* Q in the sense of Mostowski, then, assigns to each base set $E \neq \varnothing$ a corresponding quantifier Q_E restricted to E. This definition eliminates the dependence of Mostowskian quantifiers on any specific base set, a feature which is probably more important for mathematical applications than for linguistic description. Resuming, Mostowski has proposed a straightforward generalization of the logical quantifiers, which increases expressiveness and flexibility. His notion of generalized quantifers is taylored to mathematics, though. Mostowski's own examples are limited to 'numerical' or 'cardinality' quantifiers like "at most finitely many" and "at most denumerably many" [108, p. 14/15]. Subsequent work on mathematical generalized quantifiers comprises Keisler's discussion of cardinality quantifiers [84], as well as the topological quantifiers studied by Sgro [140]. The Mostowskian picture is still to narrow for a thorough treatment of quantification in NL, however. In particular, it neither accounts for restricted or multi-place quantification, nor for the non-quantitativity of certain NL quantifiers. Moreover, Mostowski's assumption that every quantifier be defined on every possible base set E, is likely not appropriate for NL quantifiers, which might be tied to certain choices of base sets. Lindström 1966 [97] introduces an even more powerful notion of quantifier, which takes the generalization begun by Mostowski to an extreme. The quantifiers of Lindström are able to express multiplace quantification, i.e. they accept several arguments. In addition, Lindström quantifiers are capable of binding more than one variable at a time (the simultaneous binding of several variables by a quantifier has first been described by Rosser and Turquette [132]). A Lindström quantifier of type $\langle 1, 0, 2\rangle$, say, lets us construct logical formulas like

$$Q_{x,,yz}(\varphi(x), \psi, \chi(y, z)) ,$$

where the three-place quantifier Q binds one variable x in the first argument $\varphi(x)$, no variables in ψ, and two variables y and z in the third argument $\chi(y, z)$. Thus, Lindström quantifiers are indeed very expressive. However, they too are not suited as a general model of linguistic quantifiers because Lindström quantifiers are invariant under isomorphisms and thus quantitative. Hence both Mostowski's and Lindström's generalizations of quantifiers were not developed with linguistic applications in mind. Nevertheless, it should be obvious that similar modelling devices are necessary for the thorough description of quantification in NL. A generalization of quantifiers suited for linguistics should support all kinds of multi-place quantification, quantitative as well as non-quantitative examples, and it should not force NL quantifiers to be defined for arbitrary models. In any case, what we need is a practical model of quantifiers the application of which is not confined to the ethereal realm of mathematics. In order to cope with actual language use, this model must give up some idealizations which are legitimate in mathematical logic. The logical quantifiers are always supplied with precisely defined inputs and always determine a clear-cut result in response to such inputs; in this sense, they are idealized quantifiers. But natural language is different.

1.3 The vagueness of language

The familiar systems of logic rest on simplifying assumptions which can no longer be upheld in the case of natural language. Specifically, classical logic is intended for sharply defined, unambiguous concepts, which are totally independent of context. This assumption of predicate logic is well-suited for mathematics; and Abelian groups, vector fields and the like are the prime cases of such artificial, idealized concepts. It is remarkable that this precision of artificial, logical languages is totally absent in natural languages. By contrast, the meanings of NL terms are typically non-idealized or 'imperfect'. There are several ways in which this imprecision shows up in NL:

1. vagueness or 'fuzziness'
2. underspecificity

3. ambiguity
4. context-dependence

Let us start by discussing the notion of vagueness, the remaining factors will be considered later on. Vagueness is a typical characteristic of NL concepts. It shows up in adjectives like “young”, “bald”, “tall”, “expensive”, “smart”, “important”, but also in nouns like “heap”, “child” or “beauty”. In all of these cases, we experience vagueness. For example, it is hard to tell the admissible ages of persons who qualify as “young”, and it is equally difficult to relate the baldness of a person to the number of hair on the person’s head or similar factors. It is notoriously difficult, but ethically relevant, to decide upon the precise age an embryo must reach to become a “child”. According to Keefe and Smith [80, Chap. 1], whose reasoning I will now sketch, vague predicates like the above can be recognized from the following characteristics. As opposed to so-called precise predicates, vague predicates have *borderline cases*, they have *fuzzy boundaries*, and they are *susceptible to Sorites paradoxes*. Let me now explain these criteria in turn. Borderline cases, to begin with, are cases in which we find ourselves unable or ‘unwilling’ to decide whether or not the property of interest applies. Hence a borderline bald person is neither clearly bald nor clearly not bald. It is important to understand that the unclarity associated with borderline cases is not due to a lack of information, because a borderline bald person will remain a borderline case even if we gather further information, e.g. by inspecting his scalp with a magnifying-glass. Similarly, deciding if a borderline tall person is tall will not be simplified if we measure the person’s height with the utmost precision. For example, 1.84 metres vs. 1.84000001 metres is not likely to make a difference. Consequently, assertions like “Bill is bald” or “Tim is tall” (assuming these are borderline cases), must be considered neither true nor false, but rather ‘indeterminate’ or ‘in-between’. Hence vague predicates violate the law of ‘tertium non datur’, thus suggesting an extension of the two-valued system of truth values. Vague predicates further lack sharp boundaries. For example, there is no exact point on a scale of ages which separates “young” persons from those no longer young. Rather we have a *fuzzy boundary*, i.e. a blurred area of gradual transition. In general, then, this means that a vague predicate F not only has boundary cases; moreover, there is not even a clear separation between the clear cases and the borderline cases. It rather appears that in the boundary area, there is a gradual shift of F-ness which runs the gamut from clear positives to clear negatives. Finally vague predicates are susceptible to Sorites paradoxes. According to Keefe and Smith [80, p. 9/10],

> A paradigm Sorites set-up for the predicate F is a sequence of objects x_i, such that the two premises
>
> (1) Fx_1
> (2) For all i, if Fx_i, then Fx_{i+1}
>
> both appear true, but, for some suitable large n, the putative conclusion
>
> (3) Fx_n
>
> seems false.

For example, (1) a person of age ten is young; (2) a young person remains young when aging by 1 millisecond; (3) a ninety-year old is young. By instantiating the general scheme, similar examples can be constructed for arbitrary vague predicates.

In the above discussion of vagueness, I preferred ‘simple’ predicates which mainly depend on a single dimension – like “tall” (height), “young” (age) or “bald” (which was supposed to depend on the number of hair for simplicity). Obviously, multidimensional predicates like “big”, or predicates like “beautiful” or “important”, where the relevant dimensions are not even clear, also exhibit vagueness, and the three criteria for vagueness also cover these cases. In addition, the vagueness of natural language is not restricted to common nouns like “heap” or “importance” and adjectives like “tall” or “young”. By contrast, vagueness is ubiquituous in NL and can be found right across all syntactic categories. For examples, verbs like “hurry”, adverbs like “slowly”, modifiers like “very” and quantifiers like “many” or “almost all” are all vague. At this point, I would like to explain the classificatory difference between vagueness and the other sources of imperfection in NL.

- *Underspecificity,* to begin with, is *“a matter of being less than adequately informative for the purpose in hand”*, examples being *“Someone said something”* and *“an integer greater than thirty”*, see [80, p. 5]. These examples demonstrate that underspecificity per se has nothing to do with fuzzy boundaries, borderline cases, and the Sorites paradox.

- *Ambiguity* refers to one-to-many relationships between words and word senses. “Liver”, for example, can denote someone who lives, as in “loose liver”, but it can also refert to the part of the body, as in “liver complaint”.[4] The individual word senses of an ambiguous term can be either vague or not, just like ordinary word senses. Hence, ambiguity as such has nothing to do with vagueness.

- *Context dependence* describes the variability in the meaning of NL terms which can be attributed to a change in external factors. (We have already met this phenomenon when discussing quantifiers like “many”). For example, a small basket-ball player may well exceed the height of a tall equestrian. Obviously, many vague predicates are context-dependent. Relational adjectives like “tall”, “young”, “expensive” in particular, depend on a standard of comparison which is usually not stated explicitly, and must hence be resolved from context. However, as pointed out by Keefe and Smith [80, p. 6]:

  > *“Fix on a context which can be made as definite as you like (in particular, choose a specific comparison class): “tall” will remain vague, with borderline cases and fuzzy boundaries, and the sorites paradox will retain its force. This indicates that we are unlikely to understand vagueness or solve the paradox by concentrating on context-dependence.*”

In communication, the imperfection of NL results in some uncertainty regarding the information conveyed. It is therefore instructive to relate the above notions of un-

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[4] the possible association of “loose livers” and “liver complaints” is not a semantical issue but rather a matter of spirits.

derspecificity, ambiguity, context-dependence and vagueness to the 'classical' model of uncertainty, i.e. probability theory. In my view, underspecificity, ambiguity and context-dependence all introduce a number of *alternatives* regarding the intended interpretation. In the case of underspecificity, these alternatives result from the use of overly general terms; in the case of ambiguity, the alternatives express different word senses; and in the case of context-dependence, we have the option of choosing different comparison classes. In order to decide between these alternatives, we cannot do better but rely on experience and hence, expectations. This indicates that probability theory, which is concerned with the formalization of expectations, might provide a suitable framework for discussing these phenomena. Vagueness, however, is not primarily an epistemic question and caused by a lack of information. For example, if someone is borderline tall, then *"no amount of further information about his exact height (and the heights of others) could help us decide whether he* is *tall"* [80, p. 2]. Consequently, a probabilistic model which fills in expectations of the missing pieces of information will not help here. Moreover, the probabilistic model assumes a clear distinction between positive and negative outcomes of an event. Vagueness, however, means a lack of such a definite or sharp distinction, thus undermining the very preconditions of probabilistic modelling. Finally, vagueness does not seem to involve any kind of "events". Consequently, it is generally agreed that vagueness falls outside the realm of probability theory, and needs a different methodology for its modelling.

Compared to the precision achieved in logic, the phenomenon of vagueness appears as a deficiency or 'imperfection' at first sight. Nevertheless, vagueness appears to be a universal principle inherent to all natural languages. In contrast to artificial languages designed on the logician's desk, natural languages had to stand the test of actual language use; these languages were never meant to be 'ideal', they had to be practical, robust and flexible in the first place. This hints at the potential relevance of vagueness to the functioning of NL and its utility as a communication medium, i.e. there seems to be a purpose served by vague predicates, which makes vagueness more than an eliminable feature of NL, but rather one of its essential components. And indeed, from a different perspective, the seeming weakness of NL turns into one of its greatest strengths: while artificial systems of logic essentially depend on everything to be made 100% precise, this requirement is alien to NL, which easily accomodates vagueness and imprecision. The demand for accuracy intrinsic to classical logic might thus create a burden of 'precisification' and formalization which makes its application impractical or even not feasible. In addition, the vagueness of NL accounts for the imprecision of our senses. Not surprisingly, then, perceptual predicates like "red" which directly refer to perceptual categories are typically vague. Wright [160] considers such predicates 'tolerant', because there is "a notion of degree of change too small to make any difference" to their applicability.

Resuming, natural languages profit from incorporating this kind of tolerance, which lets them absorb a certain degree of the variability and imprecision that we face. It hence appears that it is two-valued logic and 'classical' digital computers modelled after it, which have some deficiency concerning real-world application due to their intrinsic brittleness. In sum, then, there is much to be gained from a mathematical modelling of vagueness and its application in computer programs with increased robustness which are capable of solving complex tasks in real-world environments.

The models of vagueness described in the literature can be classified into epistemic approaches, supervaluationist approaches, and finally models based on many-valued logics, usually either three-valued or continuous-valued. The latter proposals are also known as 'degree theories of vagueness'. From the *epistemic* point of view [23, 24, 142, 159], borderline propositions are either true or false, but they are unknowingly so. Thus, vague predicates are not inherently different from exact predicates; we are only ignorant of their precise boundaries. Obviously, this epistemic view, which denies the very phenomenon of vagueness, does not contribute much to the goal of utilizing vagueness for robust computer programs. When trying to model vague predicates in a two-valued logic, we will be forced to introduce an artificial precise boundary, which gives a wrong impression of total accuracy, and also results in a substantial loss of information. The *supervaluationist approach* attempts to avoid the bias made by committing to such an artificial boundary:

> *"Replacing vagueness by precision would involve fixing a sharp boundary between the positive and negative extensions and thereby deciding which way to classify each of the borderline cases. However, adopting any* one *of these ways of making a vague predicate precise – any one of "precisification" or "sharpening" – would be arbitrary. For there are* many*, equally good, sharpenings. According to supervaluationism, our treatment of vague predicates should take account of* all *of them"* Keefe and Smith [80, p. 24].

Based on the resulting alternatives, the truth of a proposition is then determined by the following *principle of supervaluation*:

> *"A sentence is true iff it is true on all precisifications, false iff false on all precisifications, and neither true nor false otherwise"* Keefe and Smith [80, p. 24].

This 'third case' admitted in supervaluationism is usually not considered a genuine truth value on a par with "true" and "false", but rather a truth value gap [42], i.e. no truth value at all, or a truth value 'glut' [13], i.e. both true and false. However, if the third case is viewed as a genuine third truth value, say $\frac{1}{2}$, this naturally takes us to the modelling of vagueness in terms of a three-valued logic, as proposed by Tye [155]. In a theoretical set-up which only supports three cases ("true"; "false"; and "gap", "glut", or third truth value), we are forced to introduce a sharp boundary between the clear positives and the borderline cases (similarly between borderline cases and clear negatives). This appears unnatural and contradicts the principle of 'tolerance' mentioned above. Considering the vague predicate "tall", for example, there is a strong intuition that changing the height of a person by 0.1mm cannot decide if someone is clearly tall or not clearly tall; 1 millisecond of elapsed time will not decide if someone is clearly young or not clearly young, etc. This observation suggests the use of a continuous-valued model, in which the various shades of F-ness of a vague predicate F can be represented by real numbers raging from 0 (complete falsity) to 1 (fully true). The resulting *degree theories* of vagueness obviously account for borderline cases (which result in intermediate truth values) and fuzzy boundaries (which can be modelled as a smooth transition between the extreme cases of 0 and 1). The Sorites paradox can now be resolved as follows: the inductive premise (2), "If Fx_i then Fx_{i+1}", is not completely true, but only very nearly true. Although there is never a substantial drop in degrees of truth between consecutive x_i's, repeated application of (2) will accumulate these effects. Consequently, the truth of Fx_i will decrease as i becomes larger, until it approaches complete falsity, as witnessed by the falsity of the putative conclusion Fx_n. Thus, the Sorites paradox resolves nicely, and the continuous-valued model gives a satisfactory account of all three characteristics of vagueness. In the literature, there are different proposals for degree theories, and defences of using a continuous-valued model [13, 41, 60, 96, 101, 179]. However, only one of these approaches has acquired practical significance outside philosophical debates, and is highly visible both in scientific research and commercial applications.

1.4 Fuzzy set theory: A model of linguistic vagueness

In the 1930's, the philosopher M. Black made a first suggestion that vague predicates be modelled by continuous degrees of F-ness [13]. However, Black's speculations went almost unnoticed for some decades and he did not found a degree-based school of vagueness modelling. Thus the origins of the degree-based model of vagueness are commonly seen in L.A. Zadeh's independent work on the subject. In 1965, Zadeh published a seminal paper [179], in which he introduces the basic notions of a fuzzy set (i.e., mathematical representation of a vague predicate). He further proposes generalizations of the classical set operations to such fuzzy sets, thus unfolding the rudiments of a set theory which incorporates 'fuzziness'.

The fundamental concept of fuzzy set theory is easily explained: a *fuzzy subset* X of some base set E assigns to each individual $e \in E$ a *membership degree* $\mu_X(e)$ in the continuous range $\mathbf{I} = [0, 1]$. The mapping $\mu_X : E \longrightarrow [0, 1]$ so defined is called the *membership function* of X. For simplicity, we shall call X a *fuzzy set* (rather than fuzzy subset of E) when the choice of E is clear from the context or inessential. Two-valued sets can be viewed as a special case of fuzzy sets, which only assume membership degrees in the set $\mathbf{2} = \{0, 1\}$. In this case, μ_X coincides with the characteristic function, or indicator function, of X. Two-valued sets are often called *crisp* in fuzzy set theory. The term 'fuzzy' is usually taken to include both the crisp and the genuinely fuzzy cases; and only when directly contrasting crisp and fuzzy sets, I will assume a dichotomy of (genuinely) fuzzy vs. crisp/precise. Obviously, a fuzzy set X is uniquely determined by the function μ_X; and membership functions, which are nothing but ordinary mappings $f : E \longrightarrow [0, 1]$, are a possible representation of fuzzy sets. Many authors therefore identify fuzzy sets and their membership functions. For reasons to be explained in Chap. 2 p. 69, I will not enforce this identification, and usually prefer the μ_X-based notation. The collection of all fuzzy subsets of a given set E, called the *fuzzy powerset* of E, will be symbolized $\widetilde{\mathcal{P}}(E)$. The fuzzy counterparts of the classical set-theoretical operations intersection, union and complementation are mappings $\cap, \cup : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ and $\neg : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$. They are defined element-wise in terms of their membership degrees as follows:

$$\begin{aligned}
&\mu_{X_1 \cap X_2}(e) = \min(\mu_{X_1}(e), \mu_{X_2}(e)) \\
&\mu_{X_1 \cup X_2}(e) = \max(\mu_{X_1}(e), \mu_{X_2}(e)) \\
&\mu_{\neg X}(e) = 1 - \mu_X(e)\,,
\end{aligned}$$

for all $X_1, X_2, X \in \widetilde{\mathcal{P}}(E)$. The notion of a *fuzzy relation* can be developed in total analogy to the definition for ordinary sets. Thus, an n-ary fuzzy relation R, for some $n \in \mathbb{N}$, is a fuzzy subset of E^n. In other words, R assigns a membership grade $\mu_R(e_1, \ldots, e_n) \in [0, 1]$ to each n-tuple of elements $e_1, \ldots, e_n \in E$.

In two-valued logic, it is customary to discern between first-order and second-order predicates. First order predicates apply to n-tuples of individuals; thus, the semantical value of an n-ary predicate symbol is an n-ary relation. Second-order predicates, by contrast, apply to properties of individuals, which, in an extensional setting, correspond to subsets of the domain. Consequently, the denotation of a second-order predicate is an n-ary second-order relation, i.e. a subset $R \subseteq \mathcal{P}(E)^n$. The notion of a second-order relation is easily generalized to the situation in fuzzy set theory. In this case, a fuzzy second-order predicate applies to fuzzy properties, which signify fuzzy subsets of the base set E, and every such choice of arguments must be assigned to a membership grade. Consequently, a *fuzzy second-order relation* of arity n, which is suited to model this case, will be defined as a fuzzy subset $R \in \widetilde{\mathcal{P}}(\widetilde{\mathcal{P}}(E)^n)$. In other words, R assigns a membership grade $\mu_R(X_1, \ldots, X_n)$ to each n-tuple of fuzzy set $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, which signifies the extent to which the the relation applies to $(X_1, \ldots, X_n)$. We shall return to these second-order concepts in a minute when introducing fuzzy quantifiers. Before doing that, however, I will briefly relate the fuzzy sets model to the degree-based account of vagueness, and I would also like to sketch the 'success story' of the theory and its major branches of research.

From the perspective of vagueness theory, Zadeh's proposal of fuzzy sets introduces a mathematical model of vague predicates. The membership grades express the degree of *compatibility* of F to the individual e. As in every degree theory, the clear positives can be identified with $\mu_F(e) = 1$ and the clear negatives with $\mu_F(e) = 0$. The intermediate cases where $\mu_F(e)$ is chosen from the open interval $(0, 1)$ permit us to express all those shades of F-ness, which are not adequately captured by the extreme cases $\mu_F(e) \in \{0, 1\}$. For example, we can now choose a fuzzy set $\textsf{\textbf{tall}} \in \widetilde{\mathcal{P}}(E)$ of tall people in some base set E, where $\mu_{\textsf{\textbf{tall}}}(e)$ describes the extent to which the individual e is tall. In practice, it is customary to define fuzzy predicates in terms of the relevant attributes, or dimensions, that they depend on. Hence consider a scale of heights $[0, 300]$, measured in centimetres. We introduce a fuzzy set $\mathrm{TALL} \in \widetilde{\mathcal{P}}([0, 300])$ such that $\mu_{\mathrm{TALL}}(h)$ describes the extent of tallness for a person of height $h \in [0, 300]$. Based on the auxiliary fuzzy set, we then define $\mu_{\textsf{\textbf{tall}}}(e) = \mu_{\mathrm{TALL}}(\textsf{\textbf{height}}(e))$ for all $e \in E$, where the attribute $\textsf{\textbf{height}} : E \longrightarrow [0, 300]$ specifies the height of the individuals in centimetres. Thus fuzzy set theory provides a continuous-valued model of vagueness. However, fuzzy set theory is not limited to the modelling of linguistic vagueness, and it has a very different agenda compared to that of many-valued logics, as pointed out by Zadeh [79, preface, p. xi].

The impressive success of fuzzy sets both in research and applications is probably also due to Zadeh's clear conception of the purpose that fuzziness serves in human reasoning, and the potential of humanistic computing methods which incorporate fuzzy sets and linguistic modelling. Contrasting human intelligence and the 'machine intelligence' achieved by today's computers, Zadeh [79, preface, p. ix] remarks that

> *"The difference in question lies in the ability of the human brain –an ability which present-day digital computers do not possess– to think and reason in imprecise, nonquantitative, fuzzy terms. It is this ability that makes it possible for humans to decipher sloppy handwriting, understand distorted speech, and focus on that information which is relevant to a decision. And it is the lack of this ability that makes even the most sophisticated large scale computers incapable of communicating with humans in natural –rather than artificially constructed– languages."*

The significance of Zadeh's intuitions was demonstrated in 1975 when Mamdani and Assilian presented the first example of a working fuzzy controller [102]. Their groundbreaking experiment demonstrated that it is possible to control a complex system (in this case, a steam engine) by a fuzzy model which is derived from a linguistic, rather than mathematical, description of the control knowledge. The apparent benefits of such fuzzy controllers catapulted fuzzy modelling from academic debate into engineering and commercial products like anti-skid brakes, washing machines, auto-focus cameras etc. Apart from fuzzy control [34, 62, 117], the application of fuzzy modelling techniques was also successful in: approximate reasoning [11, 38, 111] and fuzzy expert systems [94, 112, 195], fuzzy decision making [25, 71, 196], fuzzy design and optimization [95, 197], fuzzy pattern recognition and cluster analysis [10, 116], fuzzy data fusion [15, 39], fuzzy databases [19, 120, 194], fuzzy information retrieval [105, 17, 18] and finally linguistic data summarization [73, 126, 162, 173, 193]. Today, various textbooks on fuzzy set theory are available, e.g. [36, 89, 115]. In addition, there are two collections of the influential publications of L.A. Zadeh [90, 178].

1.5 The case for fuzzy quantification

As mentioned in the section on vagueness theory, linguistic quantifiers like "many" can be vague (or fuzzy, as we now say). In order to clarify some points at the junction of fuzziness and NL quantification, I will make some classificatory distinctions. Hence consider the following examples of quantifying propositions:

I. Every student passed the exam.

II. Every sports car is expensive.

III. Many students passed the exam.

IV. Many sports cars are expensive.

In the first example, the quantifier "every" is applied to the crisp restriction "students" and the crisp scope "(person who) passed the exam". In this case, there is no imprecision whatsoever, and we can clearly decide between the two options "true" and "false". Thus, the quantification results in a two-valued truth value. A quantifier which, given crisp arguments, always results in quantifications which are either clearly true or clearly false, will be called a *precise quantifier*. "every" is an obvious example. In II., the precise quantifier "every" is applied to fuzzy arguments. There is a fuzzy restriction, the fuzzy set of sports cars, and also a fuzzy scope argument, the fuzzy set of expensive things (resolved from the context as denoting the comparison class of "expensive cars"). In this kind of situation, i.e. when at least one argument of a quantifier is genuinely fuzzy, we shall say that there is *fuzziness in the arguments*. The example demonstrates that we must then expect borderline cases and a fuzzy boundary even when the quantifier itself is precise. This suggests a gradual modelling of quantification based on precise quantifiers when there is fuzziness in the arguments. Let us now turn to III., where the quantifier "many" is applied to the choice of crisp arguments known from the first example. Here, we clearly have borderline situations, and the other criteria for vagueness also apply. In order to avoid the possible confusion with the technical term 'fuzzy quantifier' to be introduced later, quantifiers like "many", which exhibit vagueness even for crisp inputs, will be called *approximate quantifiers*. A plausible model of approximate quantifiers should capture their fuzziness and thus profit from the use of fuzzy sets. It is important to keep the cases exemplified by II. and III. clearly separated. In II., we obtain fuzzy quantification results because there is fuzziness in the arguments; the quantifier itself is precise. In III., by contrast, the arguments are crisp. Consequently, the fuzziness of quantification results can also be due to *fuzziness in quantifiers*, which are of the approximate type. My examples II. and III. isolate these factors, thus proving their independence. But of course, these sources can also appear in combination. This is demonstrated by example IV., where an approximate quantifier is applied to fuzzy arguments. Hence we have fuzziness both in the quantifier and in the arguments. As we have seen in II. and III. above, any of these factors can bring about fuzzy quantification results. Consequently, the quantification in IV., where both sources of fuzziness must be considered, also demands a gradual modelling in order to account for these cases. Approximate quantifiers like "many" are actually very common in natural language. The following list presents some more examples of approximate quantifiers expressed by determiners. The corresponding classes of quantifiers are absolute quantifiers, proportional quantifiers, quantifiers of exception, cardinal comparatives and proportional comparatives, respectively.

a. "(absolutely) many", "(absolutely) few", "about ten", "several thousands" . . .
   (approximate specification of the cardinality of a set)

b. "(relatively) many", "(relatively) few", "almost all", "about 40 percent", . . .
   (approximate specification of a proportion of cardinalities)

c. "all except a few", "all except about ten", . . .
   (approximate specification of the allowable number of exceptions)

d. "far more than", "some more than", . . .
   (approximate comparison of cardinalities)

e. "a much larger proportion than", "about the same percentage", . . .
   (approximate comparison of proportions)

When comparing logical and linguistic quantifiers, I introduced the distinction between explicit and implicit quantifiers, which are defined on base sets other than the universe of discourse. In addition to the above 'explicit' cases, approximate quantifiers are also frequently used to express implicit kinds of quantification:

f. "often", "rarely", "recently", "mostly", "almost always", . . .
(approximate temporal specification)

g. "almost everywhere", "hardly anywhere", "partly", . . .
(approximate spatial specification)

h. "usually", "typically", . . .
(dispositional quantification)

i. "in almost every way", "in most respects", . . .
(quantification over properties, ways of doing things)

As mentioned above, there are examples of implicit quantification in dispositional or habitual propositions which do not involve a visible quantifier at all, like the earlier "Slimness is attractive". Zadeh [190, 191] suggests that these cases be treated as (covert) uses of approximate quantifiers like "usually".

It is rather instructive, and also of great importance to my later analysis of fuzzy quantifiers, to relate the above cases I.–IV. to the 'fields of quantification' of Liu and Kerre [99, p. 2], who classify the possible instances of fuzzy quantification into four basic categories. The classification is presented for unrestricted quantification only, i.e. for a unary quantifier Q and a single argument A, as in "There are Q A's", "Q things are A", or $QxA(x)$ in a logical notation. Noticing that both the quantifier Q and predicate A can either be crisp or fuzzy, Liu and Kerre obtain the following table of possible combinations [99, p. 2]:

	A crisp	A fuzzy
Q crisp	I	II
Q fuzzy	III	IV

In this report I am mainly concerned with multi-place quantifiers. For quantifiers which accept several arguments, the classification must be generalized as follows.

Type I:	the quantifier must be precise and all arguments must be crisp;
Type II:	the quantifier must be precise, but fuzzy arguments are admitted;
Type III:	the quantifier is allowed to be fuzzy, but all arguments must be crisp;
Type IV:	both the quantifier and its arguments are allowed to be fuzzy.

Let us now return to my examples introduced at the beginning of this section. It is clear that in the above example I., we have Type I quantification, in case II., a Type II quantification, etc. However, the two classifications are not identical. In my own examples, there is a dichotomy between precise and approximate quantifiers; in particular, approximate quantifiers cannot be precise. This is different in the classification of Liu and Kerre, where the term ‘fuzzy’, which refers to fuzzy sets, includes both the crisp and genuinely fuzzy cases, i.e. the above use of ‘fuzzy’ for quantifiers means: either precise or approximate. Consequently, Type I quantifications are a special case of both Type II and Type III quantifications, which in turn are special cases of Type IV quantifications. Generally speaking, a two-valued modelling is only adequate for Type I quantifications. The most general form, i.e. Type IV quantification, not only includes the remaining cases; it is also most relevant from the perspective of applications. Consequently, achieving a proper treatment of type IV quantifications is the main objective of every model of fuzzy quantification.

1.6 The origins of fuzzy quantification

It was L.A. Zadeh who first brought fuzzy NL quantifiers to scientific attention and who wrote the pioneering papers on the modelling of these quantifiers with methods from fuzzy set theory. In a series of papers starting in the mid-1970’s [180, 181, 182, 183, 184, 185, 186, 187, 188], Zadeh developed the fundamentals of possibility theory[5] [180, 182], which served as the basis for his proposal on natural language semantics, the meaning representation language PRUF [183]. These research efforts were aimed at the development of a theory of approximate reasoning [180, 181, 183], in which the knowledge about the variables is represented in terms of possibility distributions, from which the reasoning process constructs further distributions or linguistic truth values. In these publications Zadeh exposes his first ideas about fuzzy quantifiers, however only in short passages, e.g. [184, p. 166-168], and usually in an exemplary way. In fact, there were no systematic inquiries into the subject before 1983, when Zadeh published the first treatise solely devoted to fuzzy quantifiers. In [188], he introduces the basic distinction of quantifiers of the first and the second kinds (i.e. absolute and proportional), and he develops a framework for modelling such quantifiers in terms of fuzzy numbers. In addition, several generalizations of the familiar notion of cardinality to fuzzy sets are considered, and it is shown how these cardinality measures can be utilized for implementing fuzzy quantification. Finally, Zadeh also makes a proposal for syllogistic reasoning with fuzzy quantifiers. It is this ground-breaking publication, which established fuzzy quantification as a special branch of fuzzy set theory. Some other publications of Zadeh on the subject, which are also influential, comprise: a theory of commonsense knowledge [189], which emphasizes the role of fuzzy NL quantifiers; a further refinement of fuzzy reasoning with fuzzy quantifiers [190]; Zadeh’s quantificational model of dispositions [191]; and the discussion of fuzzy quantifiers in connection with uncertainty management in expert systems [192].

Again, there are some precursors to Zadeh’s work on fuzzy quantification. In *many-valued logic*, the research into quantifiers which accept many-valued arguments was launched in 1939 when J.B. Rosser [129] presented the first treatise on the subject.

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[5] Roughly speaking, possibility theory is concerned with the representation and processing of imprecise information expressed by fuzzy propositions. For example, “Marcel is young” conveys some information about Marcel’s age, which can be described by a possibility distribution.

Other important contributions to quantification in many-valued logics, which antedate Zadeh's work, were made by Rosser/Turquette [130, 131, 132] and Rescher [127, 128]. The treatment of quantifiers in many-valued logic, however, is usually restricted to Type II quantifications only, i.e. generalizations of precise quantifiers applied to many-valued arguments. In principle, some of these methods, notably the proposal of Rescher [127], are also capable of expressing many-valued Type IV quantifications, but the authors were obviously not interested in, or even aware of such cases. To be sure, Rescher [128, p. 201] remarks that he considers the generalization of orthodox $\forall$ and $\exists$ a 'moot question', giving to understand that he has more interesting cases in mind. He envisions quantifications referring to the semantical status of many-valued propositions, which is made accessible from the object language. Rescher's examples (for a three-valued logic) include the following quantifiers:

Quantifier	**Meaning**
$(\exists^I x)Px$	"P has borderline cases"
$(\forall^T x)Px$	"Everything is a clear positive of P"
$(M^T x)Px$	"Most things are clear positives of P"
$(M^I x)Px$	"Most things are borderline cases of P"

The basic idea of accessing the degree of determinacy (i.e. vagueness or fuzziness) through quantification is indeed an interesting one, and I will return to Rescher's examples later. At this point, it is sufficient to observe that the quantifiers $\exists^I, \forall^T, M^T$ and M^I, although applied to three-valued arguments, always result in a two-valued, precise quantification. Thus, Rescher's examples, too, are not an anticipation of Type IV quantifications, which were first considered by Zadeh.

Obviously, natural language quantifiers are also of intrinsic interest to linguists, and there is indeed a linguistic school of 'generalized quantifiers' which emerged at roughly the same time when Zadeh presented his early ideas on fuzzy quantifiers. The linguistic theory, whose beginnings are Barwise [4, 5] and Barwise/Cooper [6], will be discussed at length later on. At this point, it is sufficient to remark that the linguistic model always starts from crisp arguments and precise quantifiers; approximate cases like "many" are either not considered at all [82], or alternatively forced into the precise framework and 'modelled' as precise quantifiers [6, 61]. Barwise and Cooper [6] are well aware of the vagueness inherent to these quantifiers, and suggest an extension of their basic analysis to a three-valued model; however, they do not elaborate this idea further. Zadeh, it appears, is familiar with the linguistic account of NL quantification, which he mentions in the introduction to his 1983 publication [188, p. 149]. Nevertheless, there is no visible influence of the linguistic analysis on his proposal, and his work on fuzzy quantification is certainly original and independent of the linguistic literature. But, linguistic quantifiers are not independent of language; hence it is not clear at this point if the departure from linguistics is really a virtue of Zadeh's approach. To sum up, Zadeh was indeed the first to recognize that quantifying constructions in NL usually express Type IV quantifications, and the basic notion of a fuzzy quantifier he introduces incorporates both fuzziness in quantifiers and in their arguments. His 1983 publication marks the beginning of specialized research into fuzzy quantification and today, there are substantial contributions covering diverse aspects of the subject. Before discussing the technical details of Zadeh's proposal, I would therefore like to explain the main research directions of the evolving field. Having identified the main issues, I will then review Zadeh's basic concept of a fuzzy quantifier and the framework for fuzzy quantification involving such quantifiers that he proposes.

1.7 Issues in fuzzy quantification

From a methodological perspective, we can discern three main issues in fuzzy quantification: interpretation, reasoning and summarization. These three areas differ both in their goals, and in the aspect of fuzzy quantification being investigated.

Interpretation. The most basic problem to be solved is that of precisely describing the meaning of fuzzy quantifiers, i.e. the *modelling problem*. Making available such descriptions is a prerequisite of implementing fuzzy quantifiers on computers, and of using these important NL constructs in men-machine-communication. Obviously, the chosen interpretations must match the expected meaning of the considered NL quantifiers in order to avoid misunderstanding. The computer models should therefore mimick the use of these quantifiers in ordinary language. In order to develop such interpretations, we need a formalism which lets us express the semantics of fuzzy quantifiers with sufficient detail. In this framework, the relationship between linguistic target quantifiers and their mathematical interpretations must then be analysed. In other words, a methodology must be developed which lets us identify the mathematical model of an NL quantifier of interest by describing its expected behaviour. Given a fuzzy quantifying proposition like “Many expensive cars are spacious”, we can then use the models of “most”, “expensive”, “cars” and “spacious” to construct the interpretation of the expression as a whole. As witnessed by the example, which is based on an approximate quantifier “many” applied to the fuzzy argument of “spacious cars”, this methodology must be developed for general Type IV quantifications. Only then will it become possible to model the interesting cases, like the above example and other cases of gradual evaluations. For example, Type IV quantifications are also necessary to rank a collection of cars according to the quality criterion “Few important parts are made from plastic”. Similar criteria are very common in everyday reasoning and typically used when we have to decide among several options. A good deal of the literature on fuzzy quantifiers is concerned with the issue of interpretation, although most of these publications also introduce some prototypical application. Examples comprise the works of Zadeh [188], Ralescu [124] and Yager [170, 176]; see Liu and Kerre [99], Barro et al [3] and Delgado et al [29] for overviews of approaches described in the literature. A more detailed discussion of interpretations for fuzzy quantifiers and the various facets of the ‘modelling problem’ will be given in the next section.

Reasoning Another class of papers is concerned with the manipulation of expressions involving fuzzy quantifiers. Specifically, the goal is to develop methods which deduce further knowledge from a set of facts and rules involving fuzzy quantifiers. Compared to the former issue of interpretation, this reasoning process will usually not demand complete knowledge of the given situation, i.e. there is some uncertainty concerning the exact values of some of the variables. However, even partial, imperfect knowledge of a situation is often sufficient to draw valuable and reliable conclusions; all of us bear witness to this in our everyday problem solving. In computer models of this form of 'approximate reasoning' or 'reasoning under uncertainty', such partial knowledge can be expressed, for example, in terms of facts and rules involving fuzzy predicates and fuzzy quantifiers. Ideally, the axiom schemes and deductive rules (e.g. modus ponens) used by the reasoning system should permit the derivation of new truths (valid formulas) from given ones, and the assumed calculus should also cover the full space of logical consequences of the base knowledge. The classical procedure to achieve this is to define a semantical notion of entailment and then make sure that the calculus parallels semantic entailment, i.e. the calculus should be both correct and complete. However, this route was not taken in Zadeh's pioneering work [188, 190, 191], who develops his theory of approximate reasoning from independent considerations. Important contributions of other authors comprise [35, 123, 143, 152, 153, 167, 168], to name just a few. These approaches are often more particular about the semantical justification of their reasoning schemes. Thöne [152], for example, shows that it is possible to draw precise conclusions in the presence of uncertainty. A survey of the 'state of the art' in reasoning with fuzzy quantifiers is presented in [100].

Summarisation The third main issue in fuzzy quantification is concerned with the generation of quantifying expressions which best describe a given situation. This problematic is different from the generation of quantifying expressions in a fuzzy reasoning system, where the system's limited knowledge of the situation is further elaborated by applying the calculus rules. By contrast, a system for data summarisation is usually supposed to have complete knowledge of the situation of interest, usually stated in terms of elementary facts. The goal is to construct a succinct description of the situation, which captures the important characteristics of the data. In order to express these descriptions, the system is equipped with a repertoire of NL concepts (represented by fuzzy subsets) and of fuzzy quantifiers, which introduce the possible quantities in agreement like "few", "many", "almost all" etc. Typically, the situation to be described comprises a large number of individuals. This is why the extraction of the quantifying description can be viewed as a process of summary generation. The generated descriptions are particularly useful because they can be rephrased in ordinary language, which makes them easily communicable. A typical example of such a summary is "Much sales of components is with a high commission", see [73, p. 31]. As witnessed by the example, the process results in a concise linguistic summary which is often more informative than a list of plain facts. Owing to these advantages, fuzzy quantifiers have become the preferred tool in *linguistic data summarization* [72, 73, 126, 177], building on the basic methodology proposed by Yager [162, 173]. In general there are many possible ways of summarizing a given situation. Research has therefore focussed on the development of heuristic criteria intended to guide summary generation to the most promising solutions. The fundamental requirement on a candidate summary is of course its *truthfulness*, i.e. it should properly describe the situation at hand. Truth or 'validity' is not sufficient to identify the optimal summary, though. Yager [162, 173] therefore adds a measure of informativeness. Subsequent research has identified further validity criteria, which decide upon the quality of the generated summary.

Kacprzyk and Strykowski [73, p. 30], for example, present a system which relies on the following indicators: the degree of imprecision/fuzziness of the summary, its degree of covering the data, the degree of 'appropriateness', and finally the length of the summary. The system has been used for the linguistic summarization of sales data at a computer retailer [72, 73].

1.8 The modelling problem

In this report, I will focus exclusively on the issue of interpretation, or more precisely, the notorious *modelling problem* of accomplishing a plausible interpretation for NL quantifiers with methods from fuzzy set theory. There is good reason for doing that, and much to be gained from a solution which improves the available quantifier interpretations. First of all, the modelling problem is fundamental, i.e. none of the various research directions in fuzzy quantification is really independent of its solution. Thus, it seems methodologically preferable to give priority to the modelling problem, because valid results of approximate reasoning, and a convincing summarization, can only be achieved once the semantics of fuzzy quantifiers is better understood. In addition, solving the modelling problem is also rewarding from a practical point of view. There is a number of applications the implementation of which merely requires a plausible interpretation of fuzzy quantifiers. These applications neither involve reasoning nor summarization. Let me first explain how *reasoning with fuzzy quantifiers* depends on the modelling problem; data summarization and other applications will be considered in turn. In order to carry out reasoning with fuzzy quantifiers, one needs a calculus which specifies the 'admissible moves'. This calculus should parallel the semantics of the logical language, and hence strive to be correct and complete in the ideal case. The semantics of the logical language, however, not only depends on fuzzy set-theoretical models of predicates like "tall"; it also requires an interpretation for the other non-logical symbols and fuzzy quantifiers in particular. This hints at the dependency of reasoning with fuzzy quantifiers on an assumed model of fuzzy quantification, and thus suggests that the issue of interpretation should be elaborated prior to addressing the dependent issue of reasoning. In fact, working out a plausible semantics from which a well-motivated calculus can then be derived, might help to avoid premature proposals on reasoning with fuzzy quantifiers. As mentioned earlier, it must be clear in advance that calculi for NL quantifiers will never be 'perfect', and reasoning with fuzzy quantifiers must always search for a trade-off between correctness (validity of inferences), completeness (coverage of expected inferences), and other factors (see p.11 above). In my view, Zadeh's proposal [188, 190] marks only a beginning in this search for an optimum, because his approach is essentially a variant of syllogistic reasoning, and hence susceptible to similar criticism as the traditional syllogistic logic, (see section 1.1). In addition, it is not anchored in an interpreted language equipped with a formal semantics. These theoretical difficulties notwithstanding, the advances reported in the survey paper of Liu and Kerre [100] foster hope that technical solutions exist which are sufficient for applications. The development of such solutions will certainly profit from an improved understanding of fuzzy quantification. For example, the semantical analysis of NL quantifiers might reveal further structural properties, which can then be cast into new patterns of reasoning.

Now let us investigate how quantifier interpretations, and hence a solution to the modelling problem, affect the results of *linguistic data summarization*. In this area of application, research has focussed on the development of sophisticated search algorithms and ranking criteria, which capture the most relevant quality dimensions for linguistic summaries. Furthermore system prototypes have been implemented which demonstrate the new technology. The exact point where quantifier interpretations enter the scene is, of course, the determination of the validity scores (degree of truth) of the candidate summaries, which is one of the relevant quality dimensions. Consequently, these systems can directly profit from improvements on the interpretation side, because the procedure for calculating degrees of truth can easily be updated by exchanging the model of quantification. Plugging in a new, superior model will improve the 'validity' or 'truth' scores, which results in a better overall ranking of summary candidates. Hence advances in the formal analysis and interpretation of fuzzy quantifiers are crucial for improved reasoning with fuzzy quantifiers and linguistic data summarization. However, there are further areas of application, especially at the crossroads of natural language processing (NLP) and user interfaces, where the potential contribution of fuzzy quantifiers is ever so striking.

1. *Querying of databases and information retrieval (IR) systems.* An improved querying of databases and IR systems can be achieved by integrating certain expressions of natural language into the querying language, however retaining the basic idea of a formal query syntax. For example, imagine a powerful database interface which supports queries involving fuzzy quantifiers and other elements of ordinary language. Prototypes of such interfaces, which permit more natural and convenient ways of querying, have already been developed for SQL databases [22] and for Microsoft ACCESS V.2 [74, 193]. In perspective, this type of database interfaces might prove useful for interactive data discovery [126]. Similar techniques can also be developed for the querying of unstructured data, with obvious applications to information retrieval. Experimental retrieval systems with enhanced querying facilities (including fuzzy quantifiers and other techniques of fuzzy set theory) are described in [56, 18].

2. *Natural language interfaces.* Apart from the above applications based on artificial query languages, models of fuzzy quantification will also gain relevance in the wake of natural language interfaces (NLIs), which permit the users to issue commands in unrestricted natural language. Depending on the application, a complete implementation of such a system must also comprise a model of fuzzy quantification, in order to make sure that quantifiers in the NL queries be interpreted properly. An experimental retrieval system combining NLI technology and query processing with fuzzy quantifiers is described in [55, 54].

3. *Decisionmaking and data fusion.* In a broader context, fuzzy quantifiers can be viewed as a class of linguistic operators for information aggregation and data fusion [59, 15]. Due to their potential for combining evaluations of individual criteria, the use of these operators has already become popular in fuzzy decision support systems, where fuzzy quantifiers serve to implement multi-criteria decisionmaking [27, 67, 75, 170]. An application of aggregation based on fuzzy

quantifiers to process fuzzy temporal rules for mobile robot guidance is described in [109].

For a survey of applications, see also Liu and Kerre [100]. These applications bear witness to the need for solving the modelling problem, in order to make a coherent interpretation of fuzzy NL quantifiers possible. From a more technical point of view, we can divide the overall problem of interpreting fuzzy NL quantifiers into the following stages:

(a) a class of mathematical models or ‘modelling devices’, must be introduced, thus establishing a repository of candidate interpretations for NL quantifiers;

(b) in a second step, the correspondence between NL quantifiers and the available modelling devices must be clarified.

By a *framework for fuzzy quantification*, then, I mean a proposal for solving the modelling problem of NL quantification for a certain class of quantifiers of interest. Thus, in principle, a framework for fuzzy quantification must specify the range of quantificational phenomena it attempts to cover; it must introduce the repository of candidate interpretations; and finally it must explain in formal terms how the natural language expressions of interest find their matching modelling constructs. In practice, there must be some *input* to this process, i.e. a *specification* of the intended NL quantifier, from which the corresponding formal interpretation is then determined. Consequently, a framework for fuzzy quantification is also expected to afford a practical way of specifying the NL quantifiers of interest, and it must provide some kind of *interpretation mechanism*, which associates these specifications (descriptions of NL quantifiers) to matching operational quantifiers (mathematical models of the NL quantifier of interest). The specification medium introduced by the framework should permit a straightforward description in order to be useful in practice. However, it must also be sufficiently expressive in order to catch all quantifiers of interest. Within such a framework, the interpretation mechanism, or *model of fuzzy quantification*, then becomes a mapping which assigns quantifier interpretations to given specifications. Assuming a plausible choice of interpretation mechanism, this general procedure solves the modelling problem because we can now compute quantification results from the descriptions of NL quantifiers combined with the information about their arguments.

1.9 The traditional modelling framework

In his pioneering publication [188], Zadeh develops all the components necessary for establishing a framework for fuzzy quantification, i.e. a solution skeleton to the modelling problem for a certain class of NL quantifiers which must then be instantiated by concrete interpretation mechanisms. The majority of later approaches to fuzzy quantification described in the literature have adopted Zadeh’s ideas. Thus Zadeh’s proposal can rightfully be said to constitute the *traditional framework for fuzzy quantification*. Zadeh [188, 190] embarks on the above strategy for solving the modelling problem. In presenting the framework, he demarcates a class of quantifiers to be treated; he proposes a mathematical model for these quantifiers as well as a system of specifications for such quantifiers; he evolves a general strategy for determining the target operators starting from these descriptions, and he also makes two proposals for concrete models of fuzzy quantification (interpretation mechanisms). We shall consider these components in turn. The subject of study, to begin with, comprises absolute and proportional quantifiers, or quantifiers of the 'first kind' and 'second kind' in Zadeh's terminology [188, p. 149]:

> *"We shall employ the class labels 'fuzzy quantifiers of the first kind' and 'fuzzy quantifiers of the second kind' to refer to absolute and relative counts, respectively, with the understanding that a particular quantifier, e.g.* many*, may be employed in either sense, depending on the context."*

Turning to concrete instances of such quantifiers, Zadeh notes that [188, p. 149]:

> *"Common examples of quantifiers of the first kind are:* several, few, many, not very many, approximately five, close to ten, much larger than ten, a large number, *etc. while those of the second kind are:* most, many, a large fraction, often, once in a while, much of, *etc."*

It should be apparent from these examples that Zadeh adopts a broad notion of quantifiers here, i.e. he considers both quantifiers on the universe of discourse (like "several" or "most") as well as quantifiers defined on other base sets (like "often" and "once in a while"). However, the classificatory distinction of quantifiers as to the agreement or disagreement of their base set with the universe of discourse is unknown to Zadeh. To be sure, he uses the labels 'explicit' vs. 'implicit' quantification, which I utilize to differentiate between these cases, but they bear a different meaning in Zadeh's work. Following Zadeh, a quantifier is only considered implicit if there is no visible quantifying element at all, as in the case of dispositional propositions. By contrast, I include all linguistic constructions which need quantification for their interpretation, but do not involve a quantifier proper in the narrow linguistic sense. In any case, we can assert that Zadeh's approach is concerned with absolute and proportional quantifiers in a wide sense, which also embraces examples like "often", "usually" etc. In a subsequent paper, however, Zadeh identifies quantifiers of the first and second kinds with one-place and two-place quantifiers in general [190, p. 757]:

> *"It is useful to classify fuzzy quantifiers into quantifiers of the first kind, second kind, third kind, etc., depending on the arity of the second-order fuzzy predicate which the quantifier represents."*

In order to eliminate this potential source of confusion, I will generally use the class labels 'absolute' and 'proportional' here, rather than quantifiers of the first and second kinds, and I will use cardinals $n = 1$ or $n = 2$ to denote the arity of the quantifiers, thus discerning the unrestricted and restricted uses. It is apparent from Zadeh's modelling examples in [188] that he, too, is concerned with both cases. The unrestricted use then corresponds to a quantifier restricted by a crisp predicate, as in "Few men are wise". Here, "men" can be viewed as supplying the domain in which the unrestricted statement “There are few wise” can then be interpreted. Resuming, Zadeh’s framework targets at the modelling of the unrestricted and restricted use of absolute and proportional quantifiers. Zadeh also mentions quantifiers of the third kind, exemplified by “many more than” [190, p. 757], or by likelihood ratios and the certainty factors used in expert systems [188, p. 149], but the framework is only developed for the first two kinds.

Now that the scope of Zadeh’s approach to fuzzy quantification has been explained, we will discuss Zadeh’s solution to the modelling problem for the chosen quantifiers. Let us first consider the repository of modelling devices, i.e. the candidate interpretations for NL quantifiers of the indicated types that Zadeh proposes. It must be remarked in advance that Zadeh is not very explicit regarding the difference between fuzzy quantifiers ‘per se’ (i.e. mathematical modelling devices, operational models) and their specifications or representations. In other words, the fundamental notion of a fuzzy quantifier (as an operational model) is only implicit in [188]; in the subsequent publication [190, p. 756/757], Zadeh clearly separated it from representations or specifications of his fuzzy quantifiers. But, what is a fuzzy quantifier? The answer to this question is straightforward from the above characterization of Type IV quantifications: a fuzzy quantifier must accept one or more fuzzy arguments and result in a membership degree which signifies the desired outcome of quantification. Thus, fuzzy quantifiers are essentially fuzzy second-order relations (or membership functions thereof). In his examples, Zadeh develops these notions only for unary quantifiers and quantifiers involving two arguments; he merely hints at the possibility of more general cases [189, p. 149]. I will therefore focus on quantifiers of arities $n = 1$ or $n = 2$ in the following. Hence let $E \neq \varnothing$ be some base set. A *fuzzy one-place quantifier* $\widetilde{Q}$ on E, then, (also called *unary*), is a mapping which associates a gradual quantification result $\widetilde{Q}(X) \in [0, 1]$ to each choice of the fuzzy argument $X \in \widetilde{\mathcal{P}}(E)$. Some examples comprise:

$$\begin{aligned}
&\widetilde{Q}_1(X) = \mu_X(e) \quad \text{for some fixed } e \in E \\
&\widetilde{Q}_2(X) = \sup\{\mu_X(e) : e \in E\} \\
&\widetilde{Q}_3(X) = \mu_{[j]} \quad \text{where } \mu_{[j]} \text{ is the } j\text{-th largest membership grade of } X,\ E \text{ finite} \\
&\widetilde{Q}_4(X) = \frac{\sum_{e \in E} \mu_X(e)}{|E|}, \quad E \text{ finite.}
\end{aligned}$$

The extension to fuzzy two-place quantifiers should be obvious: a *fuzzy two-place quantifier* on E (or *binary quantifier*) is a mapping which associates a gradual quantification result $\widetilde{Q}(X_1, X_2)$ in the unit range to each choice of the fuzzy arguments $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. For some first examples, consider the two-place fuzzy quantifiers defined as follows:

$$\begin{aligned}
\widetilde{Q}_5(X_1, X_2) &= \widetilde{Q}_2(X_1 \cap X_2) \\
\widetilde{Q}_6(X_1, X_2) &= \widetilde{Q}_3(X_1 \cap X_2), \quad E \text{ finite} \\
\widetilde{Q}_7(X_1, X_2) &= \widetilde{Q}_3(X_1 \setminus X_2), \quad E \text{ finite} \\
\widetilde{Q}_8(X_1, X_2) &= \frac{\sum_{e \in E} \min(\mu_{X_1}(e), \mu_{X_2}(e))}{\sum_{e \in E} \mu_{X_1}(e)}, \quad E \text{ finite.}
\end{aligned}$$

The above definition of unary and binary fuzzy quantifiers obviously accounts for fuzzy quantification of the intended kinds. We have one or two arguments, which are allowed to be fuzzy, and the quantifier determines a gradual result. Thus, the proposed concept of fuzzy quantifiers is capable of expressing the desired type IV quantifications for the unrestricted and retricted uses of absolute and proportional quantifiers. In this way, Zadeh's proposal solves the first part of the modelling problem, i.e. providing a suitable class of modelling devices. But, how are these candidate interpretations related to the target class of absolute and proportional quantifiers found in natural languages? For example, are the quantifiers $\widetilde{Q}_1$ to $\widetilde{Q}_8$ introduced above realized in natural language and if so, which linguistic quantifiers do they express? And conversely (this is the more important form of the problem): Is there a systematic way of interpreting NL quantifiers in terms of such fuzzy quantifiers, and how can we justify a particular choice of interpretation? In fact, Thiele [149] has been successful in analyzing this relationship for the 'classical' examples of universal and existential quantifiers. However, an attempt to directly describe this relationship for a broader class of quantifiers would likely be doomed to failure, due to the inherent difficulty in dealing with second-order fuzzy predicates. As explained above, it is more promising to pave a way for describing NL quantifiers and to make provisions for interpretation mechanisms (the particular 'models', or 'approaches' to fuzzy quantification) which decide on the interpretation of these descriptions. Zadeh is well aware of this problematic. In his seminal paper [188] on fuzzy quantifiers, he even skips over the introduction of fuzzy quantifiers as second-order predicates (which he tacitly uses for interpretation, of course), so eager to propose the chosen descriptions and the mechanisms to be used for interpretation. Zadeh's approach rests on two fundamental ideas, one of which relates to the proposed specifications, while the other clarifies the role of the interpretation mechanisms. Zadeh's first idea is concerned with the representation of fuzzy quantifiers. As pointed out in [190, p. 756],

> " . . . *the concept of a fuzzy quantifier is related in an essential way to the concept of cardinality – or, more generally, the concept of measure – of fuzzy sets. More specifically, a fuzzy quantifier may be viewed as a fuzzy characterization of the absolute or relative cardinality of a collection of fuzzy sets.*

Hence Zadeh utilizes that the considered simple quantifiers can be expressed in terms of the cardinality of their argument or on the relative share of two cardinalities: Absolute quantifiers like "about fifty Y_1's are Y_2's" depend on $|Y_1 \cap Y_2|$ while proportional quantifiers like "most Y_1's are Y_2's" depend on $|Y_1 \cap Y_2| / |Y_1|$. In this way, it is possible to replace the second-order notion of a fuzzy quantifier with a first-order representation as a fuzzy subset of the real line (for absolute quantifiers) or of the unit interval (for proportional quantifiers):

$$\widetilde{Q}(Y) = \mu_Q(|Y|) \qquad \text{for absolute quantifiers}$$

$$\widetilde{Q}(Y_1, Y_2) = \mu_Q(|Y_1 \cap Y_2|/|Y_1|) \qquad \text{for proportional quantifiers}$$

Notes

- This reduction is admissible in the crisp case, when the cardinalities $|Y_1 \cap Y_2|$ and $|Y_1|$ are well-defined. We shall see later that Zadeh envisions a similar, cardinality-based evaluation in the fuzzy case as well. However, it is not necessarily so that fuzzy quantification must always reduce to a fuzzy cardinality measure.
- I have followed Zadeh in considering the unrestricted use of absolute quantifiers and the two-place use of proportional quantifiers as fundamental; it should be obvious how to adapt these equalities for restricted absolute and unrestricted proportional quantification.
- In principle, a membership function $\mu_Q : \mathbb{N} \longrightarrow \mathbf{I}$ should be sufficient for specifying absolute quantifiers which refer to cardinal numbers, after all. Some of the later models of fuzzy quantification, however, will demand a specification for arbitrary real numbers.
- In the above equality for proportional quantifiers, the case that $Y_1 = \varnothing$ is silently ignored in the literature. In principle, an additional constant $v_0 \in \mathbf{I}$ would be necessary in order to specify a unique quantification result in this case as well.

To sum up, Zadeh proposes a reduction of second-order fuzzy quantifiers to first-order specifications. In terms of these simplified descriptions, then, an absolute quantifier can be described by a membership function $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$, while a proportional quantifier can be represented by a membership function $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$. A possible choice of μ_Q for the proportional quantifier "almost all" is shown in Fig. 1. In this case, we have a membership function $\mu_{\textbf{almost all}}$ defined by

$$\mu_{\textbf{almost all}}(x) = S(x, 0.7, 0.9) \tag{1}$$

for all $x \in \mathbf{I}$, where S is Zadeh's S-function defined by

$$S(x,\,\alpha,\,\gamma) = \begin{cases} 0 & : \; x \leq \alpha \\ 2 \cdot \left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & : \; \alpha < x \leq \dfrac{\alpha+\gamma}{2} \\ 1 - 2 \cdot \left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & : \; \dfrac{\alpha+\gamma}{2} < x \leq \gamma \\ 1 & : \; x > \gamma \end{cases} \tag{2}$$

for all $x,\,\alpha,\,\gamma \in \mathbf{I}$. Zadeh [188, p. 150] further proposes to view these first-order representations as fuzzy numbers:

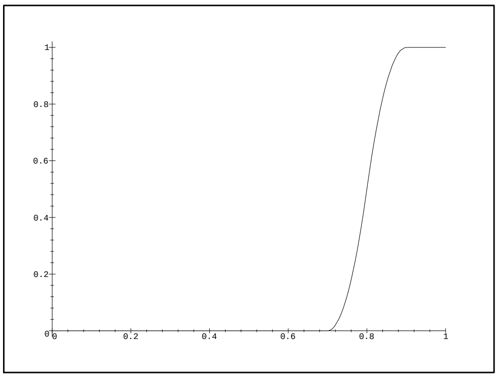

Figure 1: A possible definition of $\mu_{\textsf{\textbf{almost all}}}$

> *“More generally, we shall view a fuzzy quantifier as a fuzzy number which provides a fuzzy characterization of the absolute or relative cardinality of one or more fuzzy or nonfuzzy sets.”*

However, there might be some convexity implications of fuzzy numbers (depending on the chosen notion of fuzzy number, of course) which are violated by artificial quantifiers like “an even number of”. Therefore I will avoid the term “fuzzy number” in connection with Zadeh’s first-order representations. I have decided to use the following unambiguous notation in order to keep the second-order notion of fuzzy quantifiers and their first-order descriptions cleanly separated. The second-order fuzzy quantifiers are symbolized as mappings $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ or $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, i.e. I do not use the μ_Q-notation for them, and they are further labelled by the tilde $\widetilde{\ }$ in order to signify that they refer to Type IV quantifications and thus accept fuzzy arguments. The first order-descriptions by contrast, will always be written in the μ_Q-notation, which then denotes a mapping $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ or $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, depending on the type of the quantifier of interest (i.e. absolute or proportional). Hence $\widetilde{\textsf{\textbf{almost all}}}$ is a second-order fuzzy quantifier $\widetilde{\textsf{\textbf{almost all}}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, while $\mu_{\textsf{\textbf{almost all}}}$ refers to the membership function of its first order description, $\mu_{\textsf{\textbf{almost all}}} : \mathbf{I} \longrightarrow \mathbf{I}$. As to the chosen terminology, I would like to eliminate a possible source of confusion by introducing different terms for $\widetilde{Q}$ (the second-order fuzzy quantifier) and μ_Q (the membership function of the first-order representation). For the sake of making a classificatory distinction, I will reserve the term ‘fuzzy quantifier’ for the second-order construct $\widetilde{Q}$. The first-order descriptions $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ or $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, by contrast, will be called *fuzzy linguistic quantifiers*. It should be pointed out that Zadeh and his followers identify the above cases and generally use ‘fuzzy quantifiers’ and ‘fuzzy linguistic quantifiers’ interchangeably; for clarity, however, I will be particular about my distinction and use ‘fuzzy quantifiers’ and ‘fuzzy linguistic quantifiers’ as technical terms with different meanings as explained above. In addition, when talking about *approaches based on fuzzy linguistic quantifiers*, I refer to those approaches which operate on these μ_Q’s in order to determine the interpretation of NL quantifiers. In this way, I also solve the terminological problem of discerning the traditional account of fuzzy quantification (which rests on first-order μ_Q's or 'fuzzy linguistic quantifiers') from the novel approach exposed here, which introduces a very different framework.

Now that the repository of modelling devices as well as suitable specifications of such quantifiers have been introduced (i.e. unary or binary fuzzy quantifiers as the operations, and fuzzy linguistic quantifiers as representations), it should be pretty obvious how an interpretation mechanism $\mathcal{Z}$ must look like which closes the gap between the first order representations μ_Q and corresponding operational quantifiers $\widetilde{Q} = \mathcal{Z}(\mu_Q)$. In principle, then, the interpretation mechanism is expected to map specifications of absolute quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ to unary or binary quantifiers $\mathcal{Z}^{(1)}_{\mathrm{abs}}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ and $\mathcal{Z}^{(2)}_{\mathrm{abs}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, in order to express the unrestricted and restricted use of the quantifier for a given base set $E \neq \varnothing$. A similar definition is also needed for specifications of proportional quantifiers $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, which should be mapped to unary quantifiers $\mathcal{Z}^{(1)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, for modelling the unrestricted use, and $\mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ to express restricted quantification. Zadeh does not content himself with devising this general framework for fuzzy quantification, however, in which the modelling problem is solved through a reduction to first-order representations. He also takes the next step ahead by showing how to design such interpretation mechanisms. His second fundamental idea on fuzzy quantification, then, is concerned with the functioning of a model of fuzzy quantification. Specifically, Zadeh [188, p. 159] proposes to determine the quantification results from the fuzzy linguistic quantifier μ_Q and the given arguments through a computational metaphor, which reduces fuzzy quantification to a comparison of fuzzy cardinalities or ratios of cardinalities. To this end, he suggests to view the following propositions as semantically equivalent:

$$\text{There are } Q\ A\text{'s} \Leftrightarrow \mathrm{Count}(A) \text{ is } Q$$
$$Q\ A\text{'s are } B\text{'s} \Leftrightarrow \mathrm{Prop}(B|A) \text{ is } Q.$$

Hence in order to evaluate an absolute quantifying statement like "There are about eighty X's", one needs a scalar or fuzzy measure of the cardinality of fuzzy sets, which determines a quantity $\mathrm{Count}(X)$, or $\mathrm{card}(X)$ in my notation. The resulting quantity must then be compared to the fuzzy linguistic quantifier, i.e. to the given μ_Q, and this comparison determines the numerical score of the final quantification result, symbolized $\mathcal{Z}^{(1)}_{\mathrm{abs}}(\mu_Q)(X)$ in my notation. Turning to proportional quantifiers, a proposition like "Almost all X_1's are X_2's" is believed to result in some quantity $\mathrm{Prop}(X_2|X_1)$ denoting the proportion, or relative share of X_2's which are X_1's. Again, this quantity can either be a scalar number or a fuzzy subset in the unit interval $\mathbf{I} = [0, 1]$. By comparing the resulting quantity to the given μ_Q in some way, one then obtains the quantification result, symbolized $\mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2)$ in my notation. It should be apparent from this description that those who adopt Zadeh's ideas have two degrees of freedom for developing approaches to fuzzy quantification: (a) the measure of fuzzy cardinality and relative cardinality, i.e. the definitions to substitute for $\mathrm{card}(X)$ and $\mathrm{Prop}(X_2|X_1)$; and (b), the way in which the comparison of these (absolute or relative) cardinalities is accomplished.

Before turning to the individual approaches which instantiate this framework, and serve to implement fuzzy quantifiers in applications, I would like to comment on a few further differences between Zadeh's terminology and my own classifications. This clarification should suit those readers accustomed to Zadeh's writings on the subject. Zadeh uses the terms 'universe of discourse' and 'base set' in a different way than I do. What I call the 'universe of discourse', i.e. the dedicated base set which supplies the individuals that we can talk about, has no special name in Zadeh's system, and is equated with other base sets, i.e. collections over which a quantification can possibly range (points of time or in space etc.). However, Zadeh uses the term 'universe of discourse', rather than 'base set', to denote all of these collections. In Zadeh's writings [188, p. 159], the term 'base set' is reserved for the qualifying first argument A in a pattern like "Most A's are B's", and it hence corresponds to what I call the 'restriction' of a quantifier. Zadeh has no special term for denoting the second argument B, which I call the 'scope' of the quantifier. To be sure, Zadeh also uses the word 'scope'. In his terminology, however, the scope of a (binary) quantifier is the argument tuple (B, A), where $A, B \in \widetilde{\mathcal{P}}(E)$ are given choices of the variables in the pattern "Q A's are B's", see [188, p. 47].

1.10 A survey of existing approaches

The abstract framework for fuzzy quantification described above introduces specifications and target operations for absolute and proportional quantifiers, and it clarifies the role of interpretation methods. In order to be useful in practice, however, the framework must be populated with concrete proposals for possible interpretation mechanisms, and Zadeh [188] does not miss his opportunity to present two instructive examples. In his first method described in [188], called the *Σ-count approach* in the sequel, Zadeh avails himself of a scalar measure of the cardinality of fuzzy sets and of fractions of such cardinalities. This scalar measure, known as the Σ-count or 'power' of a fuzzy set [31], serves to implement the cardinality comparisons in "$\mathrm{card}(X)$ is Q" and "$\mathrm{Prop}(Y|X)$ is Q", to which Zadeh attempts to reduce all Type IV quantifications. Zadeh's second proposal, referred to as the *FG-count approach* in the sequel, no longer supposes that the cardinality of a fuzzy set can be represented by a single scalar number; by contrast, it now uses a fuzzy subset of the cardinal numbers, determined by the so-called FG-count, in order to describe the cardinality of fuzzy sets, see [188] for details. Other authors have usually adopted Zadeh's classification of absolute and relative quantifiers, and his basic framework for representing and interpreting quantifiers of these types. And most authors share his assumption that Type IV quantifications can be reduced to a comparison of (absolute or relative) cardinalities. Correspondingly, the approaches described in the literature mainly differ in the measure of fuzzy cardinality used and in the way that the required comparison of fuzzy cardinalities is accomplished. The particular methods used for interpretation stem from different considerations and also account for different objectives: In some cases, the new proposals are directly motivated from negative evidence against an earlier technique; the modified method then targets at an improvement for the critical cases. An example is Ralescu's FE-count approach [124]. Other contributions attempt a generalization of existing methods, e.g. by replacing $\min$ and $\max$ with general fuzzy conjunctions and disjunctions [28, 30] or by making use of general implications [3, p. 15, eq. (3)]. Finally, there are also proposals to exchange Zadeh's methods with conceptually different techniques, thus pursuing independent directions within Zadeh's overall framework [170, 172, 174, 176].

Specifically, Yager [165] proposes an approach to fuzzy quantification suited for nondecreasing quantifiers, which is essentially based on the FG-count. In [166], Yager presents a generalization of his method to arbitrary fuzzy conjunctions and conjunctions. In addition, Yager extends his method towards binary proportional quantifiers, thus incorporating importances [171, p.72]. His method, although not introduced in this way, is a generalization of the basic FG-count model (see section A.5). Ralescu [124] attempts to improve upon the FG-count model with a third proposal for interpreting fuzzy quantifiers, known as the *FE-count approach*, where the FG-count is exchanged with the so-called FE-count, another measure for the cardinality of fuzzy sets described by Zadeh [188]. Each of the three basic methods considered so far – i.e. the Σ-count, FG-count and FE-count approaches – adopts its particular notion of fuzzy cardinality, which serves as the basis for performing cardinality comparisons. Thus, these approaches exemplify Zadeh's basic idea of reducing Type IV quantifications to a gradual comparison "$\mathrm{card}(X)$ is Q" or "$\mathrm{Prop}(Y|X)$ is Q". A different approach was proposed by Yager [170], who models fuzzy quantifiers as Ordered Weighted Averaging (OWA) operators, thus reducing fuzzy quantification to a special case of aggregation problem. The basic OWA approach, which is restricted to unary proportional quantifiers based on nondecreasing choices of μ_Q, has later been extended towards nonincreasing and unimodal quantifiers [175]. In addition, Yager has developed several methods for incorporating importances [170, 172, 174]. It should be remarked that the OWA approach, although originally not declared in terms of a cardinality measure, also fits into the general framework because it can be defined from a measure of fuzzy cardinality (FG-count), see note below on page 49. Another reduction to cardinality-based calculations is described in [30]. The three approaches mentioned earlier as well as the OWA approach, have been most important to the theoretical development of fuzzy quantification, and they are most frequently found in applications (in particular the Σ-count and OWA method are very popular). I therefore consider these four methods the *main approaches* to fuzzy quantification, which will be discussed at some more length in the remainder of this chapter. Apart from these well-known methods, there are several minor variants and generalizations which also belong in Zadeh's general framework and share his basic assumptions. Finally, a few independent proposals have been made, which bring in fresh thought and open up some interesting new directions. Let us first consider the approaches in Zadeh's framework before turning to more exotic cases.

To begin with, there are several variants of the OWA approach, which differ in their formalization of two-place quantification, or 'importance qualification' in Yager's terminology. Here, I view the earliest, 1988 method as basic [170]. Two alternative techniques to include weights are described in [172, 174]. A recent proposal of Yager, which is also concerned with importance qualification in an OWA setting, is mentioned in [177]. Yager's models of two-place quantification make essential use of a coefficient derived from the given quantifier, the so-called 'degree of orness' (see section A.4 below for details). Others have tried to develop improved methods for restricted proportional quantification which are also parametrized by the degree of orness. Vila et al [156], for example, propose a convex combination (i.e. linear interpolation) of restricted universal and existential quantification, where the degree of orness is used as the interpolation coefficient. Delgado, Sánchez and Vila [28] propose an integral-based method which incorporates both the FG-count and OWA approaches, depending on the choice of fuzzy conjunction and disjunction.[6] The method is extended towards restricted proportional quantification, again, by employing an aggregation operator parametrized by the degree of orness. The same authors have recently presented a general cardinality-based method which also encompasses the FG-count and OWA approaches as special cases [30]. This proposal certainly marks an important theoretical advance because it explains these techniques from a unifying, cardinality-based perspective. In addition, the 'core' approaches are extended to arbitrary quantifiers (without special assumptions on monotonicity), and a general analysis of relative cardinalities and binary proportional quantification is also provided. Finally, Barro et al [3] consider a generalization of Yager's [171] inclusion approach towards general fuzzy implications (but they do not recommend their method due to some intrinsic problems which it shares with Yager's proposal).

Due to the conceptual vicinity of fuzzy quantification and cardinality assessments, another point of departure is the improvement or generalization of the underlying cardinality measures, from which novel methods for fuzzy quantification can then be derived. The work of Delgado et al [30] mentioned above fits nicely into this category. Historically, the discussion of cardinality measures for fuzzy sets starts with Zadeh's proposal of a fuzzy cardinality in [184], and Blanchard's [14] subsequent research into a suitable notion of fuzzy cardinality. Dubois and Prade [37] improve on Zadeh's original definition of fuzzy cardinality [184] (which may generate non-convex fuzzy cardinalities) and on the FE-count measure (which discerns less structure than possible) by presenting an alternative measure of fuzzy cardinality with improved formal properties. In addition, Dubois and Prade develop a methodology for cardinality comparisons of fuzzy sets, which establishes a meaningful interpretation to statements like "A has more elements than B" when A, B are fuzzy sets. However, they seem to view this as an application of cardinality measures, not as a special kind of fuzzy quantification. Some further publications on the cardinality of fuzzy sets, which I cannot discuss here, have been contributed by Ralescu [125] and Wygralak [161].

Next we shall consider approaches which depart from the general picture framed by Zadeh. Prade [119], to begin with, presents a fuzzy pattern-matching approach to the evaluation of fuzzy quantifiers which can be applied to arbitrary absolute quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ without special monotonicity requirements. The approach does not generate a unique quantification result $\mathcal{Z}(\mu_Q)(X) \in \mathbf{I}$, though, but rather determines interval-valued interpretations $[N, \Pi]$, $N, \Pi \in \mathbf{I}$. As shown by Bosc and Lietard [20, p. 11], the result interval contracts into a unique scalar for monotonic quantifiers; the approach then coincides with the FG-count approach. The contributions of Bosc and

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[6] the method is also described in English in Barro et al [3, p. 19-21].

Lietard [20, 21] are also of interest because they reduce the basic FG-count approach to a Sugeno integral, and the basic OWA approach to a Choquet integral. Thus, these 'core' approaches can be explained by a comprehensive theory of fuzzy measures and integrals. Moreover, the integral-based reformulation is no longer restricted to quantifiers expressed in terms of cardinalities; obviously, it can handle arbitrary fuzzy measures (i.e. nondecreasing unary quantifiers). However, Bosc and Lietard abstract from the problem of supporting nonmonotonic and multiplace quantifiers. Let us now turn to 'independent' approaches, which do not adopt Zadeh's proposal of reducing Type IV quantifications to a comparison of fuzzy cardinalities (actually, the above proposal of Bosc and Lietard also finds its place here). In retrospect, the first such method for fuzzy quantification has been described by Yager [164, p. 196]:

> *"One commonly used approach of accomplishing this is called the substitution approach . . . In this approach, we try to represent a quantified statement by an equivalent logical sentence involving atoms which are instances of the predicate evaluated at the elements in D."*

(Here D is the domain of discourse, i.e. the base set). Yager attributes this method to Suppes 1957 [146]. But, Rescher [128, p. 203] also mentions the method in his standard reference on many-valued logic, and he believes it was introduced earlier by Wittgenstein. In fact, as noted by Bocheński [16, p. 15, p. 349], the method has been rediscovered over the centuries, and its beginnings can be attributed to Albert of Saxony (1316-1390), who first proposed a reduction of existential and universal propositions to disjunctions and conjunctions, cf. [16, (34.07), p. 234]. From a practical perspective, the substitution approach is rather a framework for methods which must still be refined into concrete approaches, because there are various degrees of freedom in instantiating the basic pattern for approximate quantifiers (for a concrete example of a conforming model, see Th-102 on p. 217 below). Another line of independent research was pursued by H. Thiele [147, 148, 149]. Starting from a definition of 'general fuzzy quantifiers', i.e. mappings $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, Thiele develops various concepts useful for classifying such quantifiers. These methods permit him to characterize the general classes of fuzzy universal and existential quantifiers, called T- and S-quantifiers, respectively. Later Thiele also discussed 'median quantifiers' [150]. These are likely not realized in language, however, and they will be of no importance in this sequel. Another specialized class of quantifiers, so-called 'implicational' quantifiers, have been analysed by Hájek and Kohout [69] within their checklist-paradigm for defining fuzzy truth functions.

To sum up, a variety of interpretation mechanisms or more generally, methods for fuzzy quantification, have been proposed which all target at the proper modelling of these quantifiers in conformance with their meaning in ordinary language; a survey of approaches is also given in [3, 29, 99]. However, there is no consensus about the proper choice, i.e. no clear favourite which best answers the linguistic expectations. Thus all models exist in parallel and the Σ-count, OWA and FG-count approaches are often considered in theoretical treatise and in applications (only the FE-count approach is hardly used with a good cause, as we shall see). In other words, no single model has emerged which clearly outperforms the other approaches. Quite the reverse, it appears that each method comes with its own difficulties, and notes on the counter-intuitive behavior of these approaches are scattered over the literature [2, 3, 30, 37, 47, 124, 125, 175]. Specifically, Zadeh's first proposal, the Σ-count approach, was criticized by Yager [175], who presents a counter-example which documents the unwanted 'accumulative' behaviour of the method when there are a lot of small (but non-zero) membership grades. Barro et al [3, p. 13] and Glöckner [47, p. 12-15] have further criticism. And Zadeh himself warns that his approach is not compatible with the formation of antonyms [188, p. 167]. Ralescu [124] criticizes Yager [165] and hence the basic FG-count approach, demonstrating that the method yields implausible interpretations when the quantifier is non-monotonic or unimodal, like the example "a few". Although Yager wisely excludes these cases from consideration [165, p. 639], Ralescu's example is instructive in illustrating the limits of the FG-count model. Arnould and Ralescu [2] also argue against the FG-count approach, referring to Yager [165]. However, the probabilistic model underlying their criticism [2] makes an unusal methodological commitment. Yager's inclusion approach [171], which extends the basic FG-count model towards importance qualification, was criticized by Glöckner [47, p. 21-25], see also section A.5 below. Glöckner also presents a counter-example against the FE-count approach of Ralescu [124], which reveals that the method is unable to model the existential quantifier [47, p. 25/26]. Dubois and Prade [37] put further evidence against the FG-count and FE-count approaches, by criticizing the underlying cardinality measures for fuzzy sets. Glöckner [47, p. 16-20], Delgado, Sánchez and Vila [30, p. 32] and Barro, Bugarín, Cariñena and Díaz-Hermida [3, p. 18] present counter-examples against the three variants of the OWA approach for restricted proportional quantification described in [170], [172] and [174], respectively. Among other points of criticism, these extensions of the OWA approach to importance qualification fail to be monotonic (i.e. inequalities between quantifiers are not preserved), and the approach cannot faithfully represent its target class of operators even for crisp arguments. The quantifier interpolation method of Vila et al [156] fails for similar reasons, as shown by Barro et al [3, p. 19] and Delgado et al [30, p. 32]. Turning to the proposals of Delgado, Sánchez and Vila, their integral-based method [28] shows non-monotonic behaviour, and unexpected results even for crisp arguments, as reported by Barro et al [3, p. 22] referring to restricted proportional quantification. The cardinality-based method presented in [30], which offers a unifying perspective on both the FG-count and OWA approaches, has lost much of its original appeal since Barro et al [3, p. 23] subjected it to a critical inspection. Specifically, the resulting interpretations can show a discontinuous dependency on the membership grades of the involved fuzzy sets, and the approach is neither compatible with negation, antonyms, nor dualisation of quantifiers. Finally, Barro et al [3, p. 15, eq. (3)] discourage the use of their own extension of Yager's inclusion approach; apparently, it only serves to inspect a possible direction of research and demonstrate that it should not be pursued further. In sum, then, we have a scattered picture of rivaling approaches, all of which appear to struggle with their own difficulties.

Let us now discuss the four main approaches at some more length (further technical details and supplementary information can be found in appendix A). In order to provide a common point of reference for the various counter-examples that will be presented, I will commit to a designated scenario. For that purpose, I have chosen

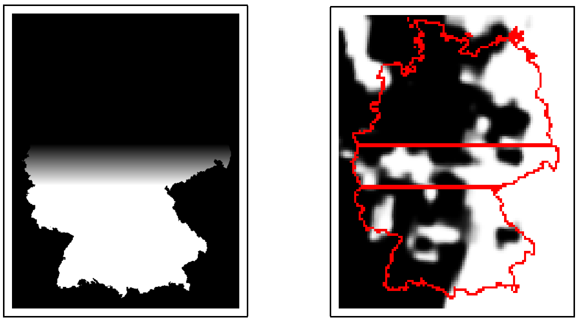

Figure 2: Running example of a base set E, the pixels forming Germany. **Left.** Fuzzy subset X_1: Southern Germany. **Right.** Fuzzy subset X_2: Cloudy.

an example domain of meteorological images which describe ‘cloudiness situations’. These images form part of the document base of an experimental retrieval system for multimedia weather documents [54, 55]. This system had to be capable of ranking satellite images according to accumulative criteria like “Almost all of Southern Germany is cloudy”. Consequently, we will now be concerned with the following *image ranking task*, where the images of interest are first evaluated for their compatibility with a given quantifying criterion, and the computed numerical scores are then used to rearrange the images into a plausible order, which should parallel the perceived quality of match. The fuzzy sets (or images, in this case) involved in this process are depicted in Fig. 2. Here, the base set E comprises the pixel coordinates which form the shape of Germany. There are two images to be considered. X_1 is a fuzzy subset of E which represents Southern Germany. The region of white pixels fully belongs to Southern Germany, and the grey pixels are those which belong to Southern Germany to some degree. X_2 is a fuzzy subset which represents a cloudiness situation. White pixels are fully cloudy, grey pixels are cloudy to some degree. (In the right image, the contours of Germany have been added in order to facilitate interpretation. The lower part fully belongs to Southern Germany, the upper part belongs to Germany, but not to Southern Germany, and the middle part contains the intermediate cases.) The images X_1 (southern Germany) and X_2 (representing a cloudiness situation) constitute the fuzzy arguments to which a binary proportional quantifier $\widetilde{Q}$ is then applied, which determines a quantification result $\widetilde{Q}(X_1, X_2) \in \mathbf{I}$. Given a suitable choice of the quantifier, $\widetilde{Q}(X_1, X_2)$ will express the degree to which “Almost all of southern Germany is cloudy” in the given situation. The simple scenario so defined provides a testbed for approaches to fuzzy quantification, because each method $\mathcal{Z}$ will determine its specific choice of quantifier $\widetilde{Q} = \mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q)$. The specific properties of $\mathcal{Z}$ will express in the numerical scores $\tau = \mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2)$ it assigns to the example images. In many cases, an implausible assignment will also show up in the ranked list of images, which departs from the expected order by relevance.

1.11 The Sigma-count approach

Let us first review the Σ-*count approach* proposed by Zadeh [188], a simple model of fuzzy quantification derived from a scalar cardinality measure for fuzzy sets. The model rests on the computation of Σ-*counts*, also known as the *power* of a fuzzy set, which are defined as the sum of all membership grades, thus

$$\Sigma\text{-Count}(X) = \sum_{e \in E} \mu_X(e)\,.$$

Zadeh's basic idea is that $\Sigma\text{-Count}(X)$ 'may be used as a *numerical summary* of the fuzzy cardinality of a fuzzy set' [184, p. 167], and can hence be substituted for the ordinary cardinality of crisp sets in order to evaluate quantifying expressions. For unrestricted absolute statements "There are Q X's", then, we get the associated score $\mu_Q(\Sigma\text{-Count}(X))$, based on the given $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(E)$. The so-called *relative* Σ-*count* [180] is used for modelling proportional quantification, i.e. the interpretation is based on the relative share of those X_1 that are X_2. For a statement "Q X_1's are X_2's" involving a proportional quantifier, we then obtain the score

$$\tau = \mu_Q\left(\frac{\Sigma\text{-Count}(X_1 \cap X_2)}{\Sigma\text{-Count}(X_1)}\right),$$

where $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. It is apparent from these formulas that the Σ-count approach depends only on very basic arithmetics, which makes it very easy to implement and thus attractive for applications. However, there are serious concerns regarding its ability to catch the actual meaning of NL quantifiers. The first line of criticism is concerned with the known deficiency of the Σ-Count approach to *accumulate* membership grades, see e.g. Ralescu [124, 125] and Yager [175]. Hence suppose $E = \{\mathrm{John}, \mathrm{Lucas}\}$ is a set of persons and **bald** the fuzzy subset defined by

$$\mu_{\textbf{bald}}(\mathrm{John}) = 0.5, \quad \mu_{\textbf{bald}}(\mathrm{Lucas}) = 0.5\,.$$

The Σ-count approach then judges the statement "Exactly one person is bald" as being fully true, which is clearly inacceptable. In general terms, the problem is that a number of smaller membership grades can possibly become indiscernible from a single large membership grade. The images depicted in Fig. 3 further elucidate this mismatch with the expected semantics of quantifiers in ordinary language. In the center result image there is about ten percent cloud coverage of Southern Germany, and the condition "About ten percent of Southern Germany are cloudy" correctly evaluates true in the Σ-count approach. In the right image, however, *all* of Southern Germany is almost not cloudy (i.e. cloudy to the low degree of 10%), which clearly does not mean that ten percent of Southern Germany are cloudy.[7] In fact, Zadeh anticipated these problems, and suggested that 'for some applications, it is necessary to eliminate from the count those elements of F whose grade of membership falls below a specified threshold' [184, p. 167], in order to avoid these unwanted effects. Needless to remark that the

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[7] This is apparent because everyone who claims that 10 percent of Southern Germany be cloudy in the right image, must also be able to tell *which* 10% of Southern Germany is cloudy. However, there is no reasonable choice for that because all pixels which belong to Southern Germany are almost not cloudy.

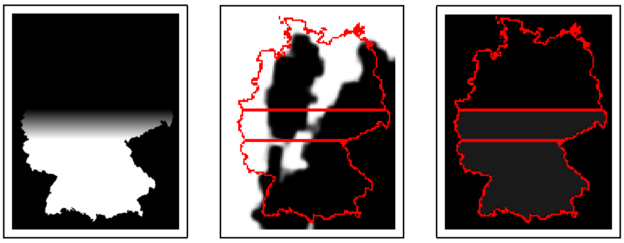

Figure 3: The Σ-count approach accumulates 'small' membership grades in an undesirable way. Results of "About 10 percent of Southern Germany are cloudy". Left: Southern Germany. Result for center image: 1 (plausible). Result for right image: 1 (inadequate).

proposed thresholds run counter to the methodological desideratum of a principled theory which offers explanatory value.

Apart from its accumulative behaviour, the Σ-count approach is subject to further criticism concerning its treatment of non-fuzzy quantifiers. Suppose that the NL quantifier to be modelled is of the precise type, e.g. "more than 30 percent". Such quantifiers require a membership function μ_Q which is also two-valued.[8] But if μ_Q is two-valued, then the Σ-count approach produces a 'degenerate' model in terms of a fuzzy quantifier which only returns the crisp results 0 or 1 (i.e., fully false or fully true). In particular, the resulting interpretations are discontinuous mappings and very sensitive to slight changes in the membership degrees of the fuzzy sets supplied to the quantifier. The example in Fig. 4 illustrates this extreme brittleness of the Σ-count approach. Here, the center image and the right image depict very similar cloudiness situations. However, the results obtained from the Σ-count approach are radically different (0 vs. 1). Obviously, this sensitivity to minor variations in the membership degrees is inacceptable for many real-world applications which depend on sensory data and are thus faced with noise, limited accuracy of sensors, quantization errors etc. It is worth noticing that Zadeh's technical hint of adding thresholds will produce discontinuous behaviour even for genuine fuzzy quantifiers with smooth membership functions. To sum up, the Σ-count approach is appealing at first sight due to its stunningly simple definition, which makes it very easy to implement. However, the known faults of the approach do not encourage its use for applications. Its accumulating effects and brittleness can already be observed in the simple case of one-place quantification. Consequently the Σ-Count approach does not possess a reasonable 'core', which might provide a starting point for a plausible extension to a broader class of quantifiers. This justifies my decision not to develop the Σ-count approach further in the report, because it offers little help for defining models which answer our linguistic expectations.

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[8] As we shall see below in section A.3 which discusses the issue in some more depth, it is not permissible to replace the two-valued mapping μ_Q with a smooth choice of quantifier in order to avoid these effects, because the intended semantics of the quantifier would be lost.

1.12 The OWA approach

Next we shall consider Yager's OWA approach to fuzzy quantification [170, 171], which is based on ordered weighted averaging (OWA) operators. The basic OWA approach is only concerned with proportional quantifiers, which are further required to be 'regular nondecreasing', i.e. $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ satisfies $\mu_Q(0) = 0$, $\mu_Q(1) = 1$ and $\mu_Q(x) \leq \mu_Q(y)$ whenever $x \leq y$. Now let $m = |E|$ be the number of elements in the given domain. Then $\mu_Q(j/m) - \mu_Q((j-1)/m)$ is the quantifier's increment if the number of elements $j-1$ is increased to j, or proportionally from $(j-1)/m$ to j/m. By $\mu_{[j]}(X)$ we denote the j-th greatest membership grade of X (including duplicates), i.e. the j-th element in the ordered sequence of membership values. Intuitively, $\mu_{[j]}(X)$ expresses the degree to which X has at least j elements. Using this notation, Yager's rule for unrestricted proportional quantification then assigns to "Q things are X" the following score,

$$\tau = \sum_{j=1}^{m} (\mu_Q(j/m) - \mu_Q((j-1)/m)) \cdot \mu_{[j]}(X)\,, \tag{3}$$

where $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ must be regular nondecreasing and $X \in \widetilde{\mathcal{P}}(E)$. This 'core' OWA approach is defined for unrestricted (one-place) quantification only. Recognizing the need to incorporate importances and hence support two-place quantification as well, Yager [170, p. 190], [171] suggests to transform the fuzzy arguments X_1, X_2 supplied to a two-place quantifier into a single fuzzy set Z to which the original rule for unary quantification is then applied. A restricted quantifying statement "Q X_1's are X_2's" is then assigned the score of the unrestricted statement "There are Q Z's", which can already be handled by (3). The fuzzy set Z is defined by

$$\mu_Z(e) = \max(\mu_{X_1}(e), 1 - \mathrm{orness}(\mu_Q)) \cdot \mu_{X_2}(e)^{\max(\mu_{X_1}(e), \mathrm{orness}(\mu_Q))}$$

see eq. (250) below for details on $\mathrm{orness}(\mu_Q)$, the so-called 'degree of orness' of μ_Q. This extension to restricted quantification makes it possible to treat cases like "Almost all young are poor", which require two arguments **young**, **poor** $\in \widetilde{\mathcal{P}}(E)$ to be taken into consideration. However, there is negative evidence regarding its linguistic plausibility, because the proposed formula fails to assign plausible models to any 'genuine'

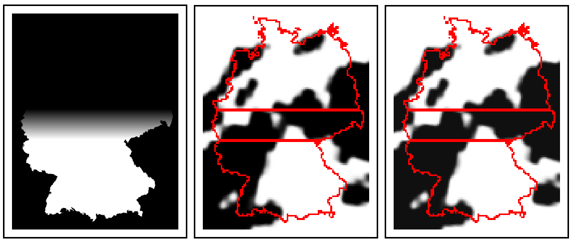

Figure 4: Results of computing "At least 60 percent of Southern Germany are cloudy" with the Σ-count approach. Left: Southern Germany. Result for center image: 0, result for right image: 1.

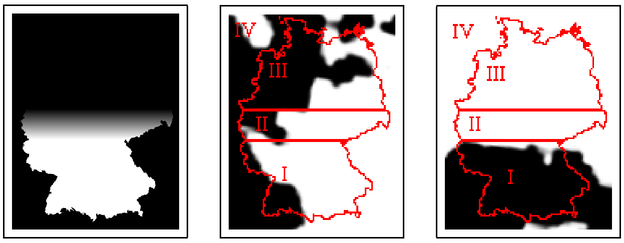

Figure 5: Application of the OWA-approach. Results of “At least 60 percent of Southern Germany are cloudy”. Left: Southern Germany. Result for center image: OWA: 0.1, desirable outcome: 1. Result for right image: OWA: 0.6, desirable: 0.

proportional quantifiers. In fact, the only two cases which are handled successfully are the logical quantifiers “all” and “exists” (see p. 405+ below). Here we shall confine ourselves to a suggestive example, which demonstrates that the theoretical defects are also potentially detrimental to applications. Hence consider the situation depicted in Fig. 5, and the corresponding results that were obtained from OWA operators with importance qualification. In this case, the task was to assess the truth of the proposition “At least 60 percent of Southern Germany are cloudy”, which is obviously valid in the center image, but not in the right image. The OWA approach, though, ranks the right image higher than the center image. As witnessed by Fig. 5, the results of the OWA approach show an undesirable dependency on cloudiness grades in regions III and IV, which do not belong to Southern Germany at all. This kind of behaviour runs counter to the intuitive meaning of the NL quantifier “at least 60 percent”.

Additional difficulties show up once the core OWA approach is extended beyond regular nondecreasing quantifiers. To see this, consider the quantifier “less than 60 percent”, which does not belong to the regular nondecreasing type. In order to make the OWA approach applicable nonetheless, we must resort to equivalent NL paraphrases. For example, we can use “at least 60 percent” and evaluate “Less than 60 percent of the X_1’s are X_2’s” by negating the result of “At least 60 percent of the X_1’s are X_2’s”, which means a reduction to the regular nondecreasing type. Alternatively, we can use “More than 40 percent of the X_1’s are X_2’s” and evaluate “Less than 60 percent of the X_1’s are X_2’s” by computing “More than 40 percent of the X_1’s are not X_2’s”. Intuitively, we should expect the final results of these computations to coincide because they stem from equivalent paraphrases. Yager’s proposal for $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$ treats both cases differently, though. A counter-example is shown in Table 1.12, referring to Fig. 5. Similar remarks apply to the modelling of unimodal and other more complex quantifiers, see p. 408+ for a thorough discussion. To sum up, the basic OWA approach for regular nondecreasing one-place quantifiers appears to be rather well-behaved, at least no counter-examples are known. Thus the ‘core’ approach, which is too limited for important applications of quantifiers, will be considered a potential candidate and starting point for future extensions, which target at covering non-monotonic and two-place quantifiers (importance qualification) as well. However, earlier proposals to affect this extension have failed, and clearly fall short of the interpretation quality

Quantifier	**center image**	**right image**
not at least 60 percent	0.9	0.4
more than 40 percent not	0.4	0
desired result	0	1

Table 3: “Less than 60 percent of Southern Germany are cloudy.” Results of possible renderings when applying the OWA approach to the images in Fig. 5

achieved by the basic method. As was shown by Barro et al [3] and Delgado et al [30, p. 32], this negative evidence also generalizes to other attemps at extending the OWA approach to two-place quantification.

1.13 The FG-count approach

Apart from the Σ-count approach, Zadeh [188] also proposes another model of fuzzy quantification; see also Yager [165] who uses the same basic formulas in the context of rule-based expert systems. The *FG-count approach* tries to improve upon the use of Σ-counts by replacing the former, scalar measure with a fuzzy measure of the cardinality of fuzzy sets. Hence let FG-count(X) denote the fuzzy set of cardinals defined by

$$\mu_{\text{FG-count}(X)}(j) = \mu_{[j]}(X) \tag{4}$$

for all $j \in \mathbb{N}$. (Here and in the following it is convenient to stipulate that $\mu_{[0]}(X) = 1$ and $\mu_{[j]}(X) = 0$ for $j > |E|$). Intuitively, FG-count(X) captures the degree to which X has at least j elements (for all possible j). The FG-count incorporates a richer base of information compared to the Σ-count, which now becomes a summary of the FG-count defined by

$$\Sigma\text{-Count}(X) = \sum_{j=0}^{\infty} \mu_{\text{FG-count}(X)}(j) - 1\,,$$

see Zadeh [188, p. 157].[9] The new measure of fuzzy cardinality can be used to derive a corresponding model of fuzzy quantification. In the ‘basic’ FG-count approach, the score of an absolute quantifying statement “There are Q X’s” becomes

$$\tau = \max\{\min(\mu_Q(j), \mu_{\text{FG-count}(X)}(j)) : j = 0, \dots, |E|\}$$

where $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ must be nondecreasing and $X \in \widetilde{\mathcal{P}}(E)$. The basic FG-count approach so defined only covers nondecreasing quantifiers, and does not incorporate importances. A possible extension towards the ‘hard case’ of two-place, weighted quantification has been described by Yager [171, p.72].[10] A proportional quantifying

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[9] It should be apparent from (3) and (4) that the OWA-approach, too, can be expressed in terms of the fuzzy cardinality measure FG-count(X). This demonstrates that the OWA approach, although not originally introduced in this way, is in fact a cardinality-based model of fuzzy quantification.

[10] Zadeh’s original ‘proposal’ for two-place proportional quantification in the FG-count setting had to be excluded from consideration. The relationship between the above method for two-place quantification and the basic FG-count approach is explained in section A.5.

(a) Southern Germany#1 (b) Southern Germany#2 (c) cloudy

Figure 6: Application of the FG-count approach. Results of “At least 5 percent of Southern Germany are cloudy”. Left: Southern Germany#1. Center image: Southern Germany#2. Right image: Cloudy. Result for Southern Germany#1: 0.55, Result for Southern Germany#2: 0.95. The desired result: 0.

statement “Q X_1’s are X_2’s” is then assigned the score,

$$\begin{aligned}\tau &= \max\left\{\min\left(\mu_Q\left(\frac{\sum_{v\in S}\mu_{X_1}(v)}{\sum_{v\in E}\mu_{X_1}(v)}\right), H_S\right) : S \in \mathcal{P}(E)\right\}\\ H_S &= \min\{\max(1-\mu_{X_1}(v), \mu_{X_2}(v)) : v \in S\}\,,\end{aligned}$$

where $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ is again assumed to be nondecreasing and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. In order to get some impression of the resulting interpretations, we consider the situation depicted in Fig. 6. There are no clouds at all in (the support of) Southern Germany#1, so we should expect that “At least five percent of Southern Germany are cloudy” be completely false. The different result, 0.55, reveals a defect of the proposed model. The example further demonstrates that the resulting operators can be discontinuous: if we replace Southern Germany#1 with the slightly different Southern Germany#2, then the result jumps to 0.95, although there are still no clouds in the region of interest. Hence some interpretations obtained from the two-place formula are very sensitive to noise, which discourages the use of the model for practical applications. Implausible results will also be obtained when quantifications like “Less than 30 percent of Southern Germany are cloudy”, which cannot be interpreted directly due to their monotonicity pattern, are reduced to NL paraphrases involving nondecreasing quantifiers: (a) “It is not the case that at least 30 percent of Southern Germany are cloudy”; and (b) “More than 70 percent of Southern Germany are not cloudy”. These NL paraphrases are equivalent and should therefore be interchangeable. The proposed extension of the FG-count approach to two-place quantification, though, produces incoherent results in both cases. To sum up, the ‘core’ FG-count approach appears to be well-behaved and does not have any overt flaws. But to be useful in practice, it must be extended beyond nondecreasing quantifiers. Most importantly, support for restricted quantification must be added. However, existing attempts at such an extension have not been succesful.

Figure 7: Application of the FE-approach. Results of “The image region X is nonempty”. Left: desired result: 1, FE-count approach: 1. Right: desired result: 1, FE-count approach: 0.5.

1.14 The FE-count approach

Finally we review the FE-count approach of Ralescu [124]. This proposal is rather similar to the FG-count method, but tries to overcome the restriction to nondecreasing quantifiers by replacing the FG-count with a different measure of fuzzy cardinality. The *FE-count* introduced by Zadeh [188] rests on the observation that $\mu_{\text{FG-count}(X)}(j) = \mu_{[j]}(X)$, i.e. the j-th greatest membership grade of the fuzzy set X, only specifies the degree to which X contains *at least* j elements and hence does not fully capture the notion of cardinality. It is therefore changed to $\mu_{\text{FE-count}(X)}(j) = \min(\mu_{[j]}(X), 1 - \mu_{[j+1]}(X))$, which expresses the degree to which X has at least j, but not $j + 1$ elements, i.e. the grade to which X has exactly j elements. The FE-count approach utilizes the new cardinality measure for interpreting absolute quantifying statements “There are Q X’s”. These are now assigned the score

$$\tau = \max\{\min\{\mu_{[j]}(X), 1 - \mu_{[j+1]}(X), \mu_Q(j)\} : j = 0, \ldots, |E|\} ,$$

where $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(E)$. A method for interpreting two-place and proportional quantification in the FE-count approach has not been described in the literature. Let us now consider the example depicted in Fig. 7. The crisp image region shown in left image is clearly nonempty, because it contains several ‘white’ pixels with full degree of membership. The fuzzy image region displayed in the right image is certainly nonempty to the same degree of 1.0, because it embraces the crisp nonempty image region shown on the left. The FE-count approach, however, only rates the left image as nonempty to the desired degree of 1.0. Although the right image displays a larger region, the result drops to 0.5, which is clearly inacceptable. Unlike the FG-count approach it intends to improve upon, the FE-count model therefore shows incoherent behaviour even in the simplest case of one-place quantification based on a nondecreasing, absolute quantifier (in this case, the quantifier $\exists$ defined by $\mu_\exists(0) = 0$, $\mu_\exists(x) = 1$ for all $x > 0$). In other words, there is no reasonable ‘core approach’ worthy of being generalized to a broader class of NL quantifiers. Just like the Σ-count approach, the FE-count proposal must therefore be dropped altogether in the report, because it does not provide a starting point for improved models.

1.15 Chapter summary: The need for a new framework

In this chapter, I have highlighted the characteristics of NL quantifiers. And fuzziness, which enters into quantifying propositions through approximate quantifiers and vague arguments, was shown to be an essential component of linguistic quantification. An evaluation of the main approaches to fuzzy quantification as to their linguistic adequacy came out negative, however, and in the literature, one finds serious criticism on all methods that have been proposed. The typical problems of these approaches are the following. Firstly non-monotonic quantifiers like “about half”, “around twenty” or “an even number of”, appear to be notoriously difficult, and none of the traditional approaches will assign plausible interpretations in this case (this is apparent because only two approaches have a reasonable ‘core’, i.e. the FG-count and OWA methods, which in both cases is restricted to nondecreasing quantifiers). This criticism extends to restricted proportional quantification, as in “Most X_1’s are X_2’s”. Proportional quantifiers in NL are often nondecreasing or nonincreasing in their second argument (“most”, for example, is positive monotonic in X_2), but they are usually lacking special monotonicity properties in their first argument (e.g. “most” is neither nondecreasing nor nonincreasing in X_1). Thus, we expect that traditional approaches, which cannot even handle non-monotonic unary quantifiers, will also fail to determine plausible interpretations for two-place proportional quantifiers. And indeed, the formulas proposed for restricted proportional quantification, as in “Many tall people are lucky”, where both the restriction “tall” and the scope “lucky” are fuzzy, fall short of the linguistic expectations (as witnessed by the above examples for the main approaches and the pointers to the literature for the remaining approaches). This defect must be regarded serious because for applications, proportional quantifiers with importance qualification are usually the most relevant case. Thus, the interpretation methods which evolved from Zadeh’s traditional framework to fuzzy quantification are either faced with counterexamples or much too limited.

In principle it would be possible to experiment further with Zadeh’s working scheme and formulate new proposals within the given set-up. However, there might be structural reasons or other forces intrinsic to Zadeh’s proposal which obstructed progress despite the considerable research efforts that have been made. First of all, it is not only the existing approaches to fuzzy quantification which are too limited – like the FG-count and OWA methods with their restriction to unary monotonic quantifiers. Quite the reverse, the very framework proposed by Zadeh is far too narrow. It artificially restricts attention to absolute and proportional quantifiers, thus disregarding many other types of similar importance to natural language. For example, quantifiers of exception like “all except ten”, cardinal comparatives like “many more than” or “twice as many”, and comparatives on relative cardinalities like “a larger proportion”, should all be covered by a comprehensive theory. Zadeh’s framework, however, is essentially restricted to unary quantifiers and certain binary quantifiers. Consequently, the approaches in this tradition are only declared for these special cases. But, two-place quantification, which marks the limits of what existing approaches can do (or no longer can do), is not the end of the scale concerning the number of arguments that must be handled. Cardinal comparatives, in particular, call for three-place quantification (“Many more X_1’s than X_2’s are X_3’s”), and comparatives on relative cardinalities even demand four-place quantification ("The proportion of X_1's which are X_2's is much greater than the proportion of X_3's which are X_4's"). Finally, the treatment of compound quantifiers ("Many X_1's, X_2's and X_3's are X_4's or X_5's") renders impossible an a priory limit on the arity of quantifiers. In other words, what we need is a theory of fuzzy multi-place quantification, i.e. a generic solution for arbitrary n-place quantifiers. The problem of restricted proportional quantification or 'importance qualification', which posed insurmountable difficulties so far, will then be clarified in passing. It is important to understand that the problem of the traditional framework cannot be solved by simply adding further types of quantifiers. This is because we need an analysis which accounts for arbitrary NL quantifiers, and from the representational perspective, this means that a universal specification medium is required. Zadeh's framework, however, does *not* offer a uniform representation, which suits different kinds of quantifiers. By contrast, each type of quantifier comes with its own specification format (e.g. $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ for absolute quantifiers and $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ for proportional quantifiers). Moreover, each type of quantifier needs its own interpretation method (rule for computing quantification results). For example, Σ-counts are used for interpreting absolute quantifiers while proportional quantifiers demand a different interpretation which involves a fraction of Σ-counts. This makes it tedious to integrate new types of quantifiers. Most importantly, such a piecewise definition of the interpretation method, which is scattered over the considered quantifier types, bears a high risk of incoherence. Because there is no formal link between the separate interpretations, it becomes hard to guarantee the expected systematicity, and to prevent effects of implausibility to enter the overall model. To sum up, the necessary extension to further types of quantifiers would result in a multiplicity of representations and interpretation rules, which is difficult to control and would likely bring about unpredictible interpretations. This lack of a uniform representation, and also the lack of a universal interpretation method which covers all quantifiers, is likely the most serious weakness of Zadeh's proposal.

A generic solution, which incorporates arbitrary types of quantifiers, would certainly be preferable. The traditional framework, however, cannot be extended towards such a solution on formal grounds. This problem is thoroughly intertwined with Zadeh's views on the functioning of interpretation mechanisms. Specifically, the problem can be traced to Zadeh's idea of equating "There are Q X's" and "$\mathrm{card}(X)$ is Q" (for absolute quantifiers), and of proposing a similar rendering of "Q X_1's are X_2's" as "$\mathrm{Prop}(X_2|X_1)$ is Q" in the proportional case. This conception of fuzzy quantification suggests that an interpretation method stipulate definitions of $\mathrm{card}(X)$ and $\mathrm{Prop}(X_2|X_1)$, and further specify how to compare the resulting quantities. It then appears that fuzzy quantification is essentially a matter of discovering the proper cardinality measures and aggregation operators, see Barro et al [3, p. 2]. From a broader perspective, however, this fundamental assumption that fuzzy quantification be reducible to cardinality comparisons is no longer valid. In fact, such reduction is not even possible in the crisp case. As pointed out in the comparison of logical and linguistic quantifiers on p. 14, not all linguistic quantifiers can be reduced to cardinality assessments, because not all (NL) quantifiers are 'quantitative' and treat all elements in the domain alike. This means an a priori limitation for all methods to fuzzy quantification derived from a cardinality measure, which is intrinsic to Zadeh's framework and all evolving approaches. In order to avoid a similar limitation a comprehensive solution to fuzzy quantification must be formulated independently of any cardinality measure. I have already mentioned that the cardinality-based analysis does not combine nicely with quantifiers on infinite base sets, (e.g. mass quantifiers) because most quantifiers of interest, like proportional quantifiers, become non-quantitative in this case [8, p.474]. Consequently, the traditional approaches are usually limited to finite base sets. (It must be remarked, though, that Zadeh [184] discusses mass quantification based on integration rather than element-counting, thus replacing the concept of cardinality with measure-theoretical notions in order to account for such situations). A generic solution to fuzzy quantification should also incorporate infinite base sets. It is likely due to the internal difficulties of existing approaches that applications of fuzzy quantifiers, like fuzzy database querying, tend to support several interpretation methods. Kacprzyk and Zadrożny's fuzzy querying interface to Microsoft Access V.2 [74], for example, supports both the Σ-count and OWA models. And Bosc and Pivert [22], when discussing an extension of the classical SQL language with fuzzy retrieval features, suggest the use of even three methods, i.e. Σ-counts, OWA operators/Choquet integral, and finally FG-counts/Sugeno integral. The burden of selecting the model which works best in the given situation is thus shifted to the user of these systems. However, I am not sure that the users of the database can decide on the proper interpretation when the database designers feel unable to do that. In addition, it is well possible that none of the available options will yield a satisfactory answer to the query. Thus the state of the art in fuzzy quantification does not permit a reliable use of fuzzy quantifiers in applications. The above 'solution' of shifting these problems to the responsibility of the users is not acceptable.

When viewing the development of fuzzy quantification in retrospect, one recognizes that research in this area has lost much of its original impetus: In the beginning, i.e. starting with Zadeh's pioneering 1983 publication, there was rapid progress; it did not take longer than 1988 that the main approaches had entered the scene. Following that came the current period, which is marked by advances in applications like reasoning with fuzzy quantifiers, multi-criteria decisionmaking, fuzzy querying of databases or linguistic data summarization. However, there was a slow-down on the methodical side and no substantial progress in the modelling of fuzzy quantifiers. The seeming stability of this area, and the spread of fuzzy quantifiers into applications, does not imply that the development has settled and reached a theoretically satisfactory analysis. Quite the reverse, the known counter-examples are still unresolved, and the few recent proposals for interpretation methods [28, 29, 30, 156] are also linguistically inconsistent [3, 30]. In my view, it is this lack of linguistic plausibility which hinders the break-through of fuzzy quantifiers into commercial applications. Thus the development of fuzzy quantification appears to be blocked by several factors, and the weakness of existing approaches, as well as the general slow-down of progress, is caused by the way of thinking about fuzzy quantification. This suggests that future research into fuzzy quantification should challenge Zadeh's theoretical skeleton. The precise commitments of his framework should be reexamined and replaced with a new set of basic concepts if necessary, which better reflect the methodical necessities. In other words, a novel framework for fuzzy quantification is needed, which eliminates the representational and interpretational limitations of Zadeh's earlier throw. The new framework should permit a solution of the modelling problem for a representative class of linguistic quantifiers which comprises all examples mentioned so far. To this end, it is necessary to drop the use of first-order representations and the reduction to cardinality comparisons, and install more expedient modelling devices in lieu of these. Pursuing such independent directions might put off the brakes and foster substantial advances in fuzzy quantification.

A novel framework which targets at reliable quantifier interpretations will need to reconcile fuzzy set theory and linguistics. Zadeh himself concedes that his approach be ‘different’ from (i.e. incompatible with) the linguistic model [188, p. 149]. However, it appears that few people have reflected the consequences of this departure from linguistic consensus. In fact, it is not even possible to view the two-valued quantifiers usually considered by linguists [6, 8, 82] as a special case of Zadeh’s fuzzy linguistic quantifiers. The adoption of a very different analysis results not only in poor coverage, though, but also shows up in interpretations which are linguistically inconclusive. This turning away from the competent scientific discipline I consider the worst mistake of existing methods. These approaches are designed from an engineering perspective and intended for use by practitioners; and the solution offered is usually motivated by concepts of fuzzy set theory, rather than derived from linguistic considerations. In particular it appears that none of the authors felt the need for an explicit linguistic validation of their proposals. As witnessed by the numerous counter-examples, the bona fide assumption of linguistic plausibility, implicit in existing work, should definitely be replaced with a more explicit and purposive strategy. Thus, linguistic adequacy must be central to the very design of interpretation methods, and it should be explicitly guaranteed by appropriate means. However few is known about the junctions of fuzzy quantification and linguistic analysis. Fuzzy set theorists neglected the linguistic facets while linguists abstracted from the very fact of fuzziness. This certainly marks a blind spot on the landscape of fuzzy quantification which should be examined in order to trigger progress in fuzzy quantification. In other words, the problem of concrete models must be set back for the moment, in order to render precise what is meant by a plausible interpretation. The explicit formalization of adequacy criteria, then, will make a lasting contribution to our understanding of fuzzy quantification, and establish a solid foundation for a theory of fuzzy NL quantification. In addition, the resulting criteria might sharpen the dispute over interpretation methods and initiate a purposive search for conforming interpretations. As opposed to the bona fide assumption of linguistic adequacy made in earlier work, the candidate models can then be tested rigorously against a catalogue of linguistic desiderata, prior to considering their dissemination into applications.

2 A framework for fuzzy quantification

2.1 Motivation and chapter overview

The main objective of this work is that of developing a conclusive solution to the modelling problem which admits subsequent refinements into a comprehensive theory of fuzzy quantification. In this chapter, we will be concerned with the most fundamental aspects of the modelling problem, i.e. how can linguistic quantifiers be specified in a straightforward way, and how can we systematically establish the matching fuzzy quantifier starting from such a description? In devising a solution to these problems, linguistic plausibility must be the central concern. The reasoning set forth in the introduction gives witness that the traditional framework for fuzzy quantification fails in this respect, and its flaws can partially be attributed to the basic representations μ_Q which are too restrictive and inhomogeneous. Thus, linguistic concerns will likely affect all components of a solution to the modelling problem. The novel framework for fuzzy quantification to be presented in this chapter should be attractive both from the linguistic and fuzzy sets perspective. The framework should be broad enough to cover all phenomena routinely treated by the linguistic theory of quantification; and it should further be structurally compatible to the linguistic analysis, i.e. evolve from established linguistic knowledge. Finally the framework should extend these notions towards approximate quantifiers and fuzzy arguments. It is hoped that the structural affinity to the linguistic analysis will also allow a reuse of the concepts found useful by linguists for linguistic description, which must then be generalized to the fuzzy case.

We have already learned that the traditional framework for fuzzy quantification has not been designed after this list of desiderata. However, the existing approaches to fuzzy quantification and their joint analytic commitments have been similar to such an extent that an induction to the fuzzy sets-based modelling of linguistic quantifers in general is not admissible, the potential of which is not in the least exhausted. Thus my proposal will rest on the hypothesis that fuzzy set theory is a valuable tool for modelling vague NL concepts and also of potential utility to the analysis of linguistic quantifiers. The basic concept is considered worth maintaining and developing into a theory of NL quantification – the difficulties of earlier proposals notwithstanding. Still, the novel approach should pursue a different strategy and derive its force from linguistics because it appears that only fresh thought from this area can give new impetus to research into fuzzy quantification. Recalling the components of such framework identified in the introduction, the envisioned skeleton for a theory of fuzzy quantification must fix the range of quantificational phenomena to be modelled, propose a system of specifications (descriptions) and operations (computational quantifiers) for the considered NL quantifiers, and finally establish the correspondence between NL quantifiers (as expressed by the proposed specifications) and matching computational quantifiers.

The scope of the framework, it has been mentioned, should include all phenomena usually encountered in linguistic discource. More specifically, the framework should cover all (two-valued) generalized quantifiers known from the linguistic theory [6, 8, 82], a concept which will be explained in the first section following this overview. In particular, the new proposal should finally surmount the artificial restrictions of existing approaches, i.e. it should support a broad class of NL quantifiers beyond the absolute and proportional types, which includes multi-place and non-quantitative examples as well as quantifiers on infinite base sets.

Obviously, a powerful notion of fuzzy quantifiers is called for to express type IV quantifications involving approximate quantifiers and fuzzy arguments in this general setting. The view of fuzzy quantifiers as fuzzy second-order predicates which is mentioned in some of Zadeh's works is a good point of departure for the necessary generalization of quantifiers; we only need to make sure that quantifiers of arbitrary arities and defined on arbitrary nonempty base sets are also included.

Thus, we need not give up the operational forms of fuzzy NL quantifiers postulated in fuzzy set theory. The choice of the specification medium, however, is more critical and must avoid the flaws of Zadeh's first-order descriptions. We need a universal specification medium suitable to describe linguistic quantifiers of arbitrary kinds, which is simple but powerful and also useful in practice. The reduced descriptions which replace the former μ_Q's must be strong enough to uniquely identify all quantifiers of interest and to portray all relevant phenomena despite the necessary abstractions. However, the improved specifications should be as easy to use as the membership functions μ_Q known from existing work.

The association between specifications of quantifiers and their matching operational forms is not arbitrary or random in nature, and we should hence strive at analyzing this relationship and formalizing it in mathematical terms. Postulating another ad-hoc rule for interpretation (i.e. mathematical formula, algorithm) will only achieve modest progress, by contrast – should the formula prove defective, like all earlier proposals, not too much of insight has been gained. As a consequence, I suggest that the problem of interpretation be discussed in a more general perspective, and approached in small, theoretically motivated steps. As opposed to existing approaches, which confine themselves to introducing yet another rule for interpreting quantifiers, I thus insist on a clear separation of theoretical analysis (what are plausible models?) and the later identification and implementation of concrete models. Only this explicit a priori analysis will guarantee controllable and reliable results. A candidate model which passes the quality check is worthy of implementation, while all others must be rejected.

To sum up, the framework should actively support the solution of the modelling problem by subdividing the global problem and delegating part of the responsibility to a model of fuzzy quantification (which will then be cast in an algorithm). Naturally such a model cannot include the aspect of specification, which is needed to identify the quantifiers to be modelled. This task is on part of the users who avail themselves of the specification medium to describe the quantifiers of interest. As already mentioned, the framework is expected to guarantee controllable and transparent interpretations and ensure the reliable use of fuzzy quantifiers in applications. To accomplish this, I will first introduce a generic modelling device for describing all possible associations between specifications and operational quantifiers. In accordance with my above considerations, the issue of theoretical analysis, i.e. identifying those models which answer the linguistic expectations on the meaning of fuzzy quantifiers, then becomes strictly separated from the practical issue of implementing quantifiers in these models. This modular approach establishes a level of abstraction suitable for formal inquiries which are no longer constricted with computational details. Due to the fact that the models of fuzzy quantification, which refer to the correspondence between specifications and operational forms, are a certain kind of mappings, the algebraic method can be used. Thus, the plausible choices are characterized by describing their observable behaviour rather than appealing to some ethereal 'nature' or internal structure of such mechanisms. The concept of a 'raw' model will permit a formalization of explicit requirements on plausible models. Unlike ad-hoc interpretation rules, the plausibility criteria that evolve from this procedure will grow into a catalogue capable of refinement: It can be developed incrementally and will not collapse entirely should individual criteria need replacement. It is of course hoped that a careful analysis of coherent interpretations and their formalization into axiomatic requirements will ultimately converge and form a stable body of criteria which identifies the class of plausible models.

2.2 Two-valued quantifiers

The logical quantifiers $\forall$ and $\exists$ can be viewed as abstractions from natural language quantifiers. In the introduction I mentioned Russell's theory of descriptions which attempts an analysis of the singular definite quantifier "the_{sg}" (as in "the author of Waverley") in terms of the logical quantifiers. However, the wealth of linguistic quantifiers was neglected for a long time and the semantical structure of NL quantification was also ill-understood. Thus, R. Montague's discoveries on the proper analysis of noun phrases in (a fragment of) English [107], which constitute the natural locus of quantification, was nothing less than a linguistic sensation. Based on a powerful logical machinery, his intensional logic IL[11], Montague demonstrated that a compositional treatment of quantifiers in natural language is indeed possible, which conforms to Frege's principle (i.e. the meaning of a complex expression is a function of the meaning of its subexpressions). Montague was mainly interested in the syntactico-semantical structure of noun phrases; considering a small number of classical quantifiers was sufficient to demonstrate the points he wanted to make. Still, a comprehensive treatment of NL quantifiers needs to account for a larger repository. Montague's ground-breaking work encouraged further research into various directions including the analysis of a broader class of quantifiers, beyond the simple examples "all" and "some". The linguistic discourse about quantifiers was also influenced by the development of logical systems that admit non-linear, 'branching' modes of quantification, in which the quantifiers operate independently of each others.[12] Hintikka [68] initiated a debate among linguists and philosophers of language claiming that branching quantification is necessary for the semantic description of natural languages. The following example due to Hintikka [68, p. 342] is intended to paraphrase a Henkin-like quantificational structure:

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[11] a variant of intensional type theory which supports the λ-operator; operators for forming the extension and intension of given expressenions, and modal tense operators.

[12] see e.g. Henkin [66] and Walkoe [157] for systems of (two-valued) logic that support partially ordered structures of quantifiers. I will return to the issue of branching quantification on p. 88 below and in Chap. 12, which is exclusively concerned with the issue.

> *"Every writer likes a book of his almost as much as every critic dislikes some book he has reviewed."*

Barwise [5] presented more conclusive examples involving only two quantifiers, like

> *"Most philosophers and most linguists agree with each other about branching quantification"* [5, p. 60].

For that purpose, Barwise needed generalized quantifiers like "most" because the example collapses into linear quantification for the regular choices $\exists$ and $\forall$. Together with R. Cooper, Barwise then prepared a seminal work on such generalized quantifiers, which established the modern linguistic analysis of quantification, i.e. the theory of generalized quantifiers (TGQ). Important contributions to TGQ were also made by J. van Benthem [8, 9], E. Keenan, J. Stavi and L. Moss [81, 82] and many others. The theory of generalized quantifiers evolved rapidly, mainly during the 1980's, and its body of results on linguistic quantification still represents the linguistic consensus on that matter. In fact, the reader is already acquainted with part of the linguistic view of NL quantification. In the introduction, I anticipated some results of TGQ to explain the basic characteristics of linguistic quantifiers and to pinpoint the deficits of existing approaches. Unlike these attempts at fuzzy quantification, the theory of generalized quantifiers rests on a simple but expressive model of (generalized) quantifiers, which captures a wealth of linguistic examples. This basic model of a quantifier is restricted to two-valued (precise) quantifications, but it establishes a uniform representation for all kinds of such quantifiers including unrestricted and restricted quantification, multi-place quantifiers, composite quantifiers, as well as quantitative and non-quantitative examples. There are several (equivalent) ways how the basic concept of a two-valued generalized quantifier can be introduced in formal terms. Hence let me present that definition which best suits my present purposes, and comment on the possible alternatives in due succession.

Definition 1
A two-valued (generalized) quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$*, where* $n \in \mathbb{N}$ *is the arity (number of arguments) of* Q*, and* $\mathbf{2} = \{0, 1\}$ *denotes the set of crisp truth values.*

Notes

- Consequently, a two-valued quantifier Q assigns a crisp quantification result $Q(Y_1, \ldots, Y_n) \in \mathbf{2}$ to each choice of crisp arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Let me point out that the universe E might be any nonempty set of objects; unless otherwise stated, I will always permit base sets of *infinite cardinality* as well. In particular, the domain of quantification need not range over concrete objects. In typical applications of fuzzy quantifiers like fuzzy information retrieval [58] or multi-criteria decision making [170], for example, E becomes a set of search expressions, or the set of premises of a fuzzy IF-THEN rule, respectively.

Let me now comment on the definitional alternatives.

- As a matter of fact, it is not very common to define generalized quantifiers the way I introduced them, i.e. as two-valued mappings. By contrast, many authors assume a slightly different albeit equivalent definition of these generalized quantifiers, which are modelled as n-ary second order relations. The two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ according to my above definition, and the corresponding generalized quantifiers $Q' \in \mathcal{P}(\mathcal{P}(E)^n)$ can obviously be transformed into each other, by utilizing the relationship

  $$(Y_1, \ldots, Y_n) \in Q' \Leftrightarrow Q(Y_1, \ldots, Y_n) = 1$$

  for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. My decision to use the characteristic function rather than the relation itself anticipates the later use of membership functions in the fuzzy case.
  The remaining authors usually prefer a nested representation $Q'' : \mathcal{P}(E)^{n-1} \longrightarrow \mathcal{P}(E)$. This notation highlights the ‘scope’ argument of the quantifier, which accepts the interpretation of the verbal phrase (see note below on p. 62). Again, two-valued quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ as defined by Def. 1, and mappings $Q'' : \mathcal{P}(E)^{n-1} \longrightarrow \mathcal{P}(E)$ are interdefinable, according to the relationship

  $$Y_n \in Q''(Y_1, \ldots, Y_{n-1}) \Leftrightarrow Q(Y_1, \ldots, Y_n) = 1$$

  for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. This style of expressing generalized quantifiers is usually found in work concerned with the relationship between syntactic analysis and semantical interpretation. In particular, the nested representation is adopted in Barwise and Cooper’s foundational work on TGQ. See [45, p. 228+] for an extensive discussion of the alternative perspectives on the basic notion of a generalized quantifier.

- The above definition of two-valued quantifiers includes the case of nullary quantifiers ($n = 0$). These can be identified with the constants 0 and 1. The decision to permit nullary quantifiers proved to be rather convenient in some cases. TGQ, however, rests on a narrower definition and usually restricts attention to quantifiers of arities $n \geq 1$. This is justified from the perspective of natural language description because every NL quantifier must at least offer one slot, which accepts the denotation of the verb phrase (see section 2.8 below for an extended discussion of nullary quantifiers).

- Finally it is common practice in TGQ to permit quantifiers being undefined in some cases. Specifically, it is definite quantifiers like “the” which call for undefined interpretations or other formal devices, in order to account for the possible failure of presuppositions (for example, “the man” only denotes if there is exactly one man in the current domain or discourse context). For our present purposes, though, it is not necessary to incorporate undefined interpretations into the basic concept of a two-valued quantifier, because the issue will be resolved anyway once we proceed to a richer set of truth values. See p. 65 for alternative definitions of singular and plural “the”, and section 2.9 for more details on the issue of undefined interpretation.

The quantifiers that prevail in natural language are of the binary variety, as in "All Y_1's are Y_2's". Let me now introduce some basic examples of such two-place quantifiers.

Definition 2
Let $E \neq \varnothing$ be some base set. We define

$$\begin{aligned}
&\textbf{all}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \subseteq Y_2\\
&\textbf{some}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 \neq \varnothing\\
&\textbf{no}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 = \varnothing\\
&\textbf{at least k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \geq k\\
&\textbf{more than k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| > k\\
&\textbf{at most k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \leq k\\
&\textbf{less than k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| < k\\
&\textbf{exactly k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \setminus Y_2| = k\\
&\textbf{all except k}_E(Y_1,\ Y_2) = 1 \Leftrightarrow |Y_1 \setminus Y_2| \leq k\\
&\textbf{between r and s}_E(Y_1, Y_2) = 1 \Leftrightarrow r \leq |Y_1 \cap Y_2| \leq s
\end{aligned}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, $k, r, s \in \mathbb{N}$.

Notes

- The symbol $|\bullet|$ denotes cardinality. I will usually drop the subscript E when the base set E is understood.
- To give an example how expressions involving these two-valued quantifiers can be evaluated, suppose that

  $$E = \{\text{John, Lucas, Mary}\}$$

  is a set of persons, $\textbf{men} = \{\text{John, Lucas}\} \in \mathcal{P}(E)$ is the set of men in E, and $\textbf{married} = \{\text{Mary, Lucas}\}$ is the set of those persons in E who are married. Then by the above definitions,

  $$\textbf{some}(\textbf{men}, \textbf{married}) = \textbf{some}(\{\text{John, Lucas}\}, \{\text{Mary, Lucas}\}) = 1\,,$$

  but

  $$\textbf{all}(\textbf{men}, \textbf{married}) = \textbf{all}(\{\text{John, Lucas}\}, \{\text{Mary, Lucas}\}) = 0\,.$$

- Natural language quantifiers have a dedicated argument slot occupied by the interpretation of the verbal phrase, e.g. Y_2 in "All Y_1's are Y_2's", or "sleep" in "All men sleep". As remarked in the introduction, this particular argument will be called the *scope* of the quantifier. The scope is the argument exposed in the nested representation of quantifiers presented on page 61. In my "flat" representation, I take the scope to be the n-th (last) argument of an n-place quantifier by convention. For example, in $\textbf{almost all}(Y_1, Y_2)$, the second argument is the scope of the quantifier. The remaining, first argument of a two-place quantifier will again be called its *restriction*.

Of course, it is also possible to fit the usual logical quantifiers $\forall$ and $\exists$ into the TGQ framework, and express these as generalized quantifiers. We then obtain the apparent definitions,

$$\forall(Y) = 1 \Leftrightarrow Y = E \qquad \text{and} \qquad \exists(Y) = 1 \Leftrightarrow Y \neq \varnothing \tag{5}$$

for all $Y \in \mathcal{P}(E)$. It is instructive to consider the relationship between unary $\forall$ and the two-place quantifier "all", in order to clarify the distinction between the restricted and unrestricted use of NL quantifiers, that was already mentioned in the introduction:

- In the case that the NL quantifier, which is implicitly two-place, is employed in a quantifying expression which involves two overt arguments, this corresponds to the restricted use of the quantifier. "All married are men", for example, finds an interpretation in terms of a two-place quantification $\textbf{all}(\textbf{married}, \textbf{men})$. The restricted use of the quantifier corresponds to the natural way in which quantifiers are applied in NL.
- In some cases, it is also possible to model quantifying NL statements by one-place quantification, although the involved NL quantifier regularly takes two arguments. This is the type of quantification that I called unrestricted, because the full domain E is substituted for the unspecified first argument, thus filling in the restriction of the quantifier. The logical quantifier $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ for example, which models the unrestricted use of the NL quantifier "all", can be expressed in terms of two-place **all** by $\forall(Y) = \textbf{all}(E, Y)$ for all $Y \in \mathcal{P}(E)$. The example illustrates how the full domain E provides the missing restriction (first argument), which is required by the two-place base quantifier. In other words, the unrestricted quantification in $\forall(Y)$ means that "All elements of the domain are Y" or simply, "All things are Y".

Clearly the unrestricted use, which is ubiquitous in predicate logic, is of minor importance to the description of natural language. We will see later in section 6.7, p. 181 that two-place, restricted quantification can be reduced to the unrestricted use of quantifiers in certain cases. However, such reduction is only possible for certain types of quantifiers, and it does not generalize to the fuzzy case.

Apart from the above examples of absolute quantifiers which depend on absolute counts, and quantifiers of exception like "all except k", there are many other types of NL quantifiers that deserve interest.[13] The most prominent class is certainly that of proportional quantifiers, which depend on the relative share (or ratio) of Y_1's that are Y_2's. Assuming for convenience that the base set E be finite, we can easily define some generic examples, from which a broad range of NL quantifiers can be derived.

Hence let us make the following stipulations.

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[13] For a more extensive discussion of absolute quantifiers and quantifiers of exception also covering implementation issues, see section 11.8.

Definition 3
Let $E \neq \varnothing$ be some base set. We define

$$\begin{aligned}
&[\textbf{rate} \geq r](Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \geq r\,|Y_1| \\
&[\textbf{rate} > r](Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| > r\,|Y_1| \\
&[\textbf{rate} \leq r](Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \leq r\,|Y_1| \\
&[\textbf{rate} < r](Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| < r\,|Y_1| \\
&[\textbf{rate} = r](Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| = r\,|Y_1| \\
&[r_1 \leq \textbf{rate} \leq r_2] = 1 \Leftrightarrow r_1|Y_1| \leq |Y_1 \cap Y_2| \leq r_2|Y_1|
\end{aligned}$$

for $r \in \mathbf{I}$, $Y_1, Y_2 \in \mathcal{P}(E)$.

In terms of these quantifiers, a statement like “At least 70 percent of the Y_1’s are Y_2’s” can now be modelled by

$$\textbf{at least 70 percent}(Y_1, Y_2) = [\textbf{rate} \geq 0.7](Y_1, Y_2)\,,$$

and “More than 60 percent of the Y_1’s are Y_2’s” can be modelled by

$$\textbf{more than 60 percent}(Y_1, Y_2) = [\textbf{rate} > 0.6](Y_1, Y_2)\,.$$

In particular, the quantifier “most” becomes

$$\textbf{most} = [\textbf{rate} > 0.5]\,. \tag{6}$$

This definition of “most” covers its technical sense, as in “Most Americans voted for Bush”, where one vote can result in an all-or-nothing decision. In other contexts, a gradual modelling would be more appropriate. By instantiating the generic quantifiers $[\textbf{rate} \leq r]$ and $[\textbf{rate} < r]$ in the apparent way, we can also cover natural language quantifiers of the type “at most r percent” and “less than r percent”. Similarly, a statement like “Exactly 10 percent of the Y_1’s are Y_2’s” can now be interpreted as

$$\textbf{exactly 10 percent}(Y_1, Y_2) = [\textbf{rate} = 0.1](Y_1, Y_2)\,.$$

Finally, the parametric quantifier $[r_1 \leq \textbf{rate} \leq r_2]$ is useful for modelling a statement like “Between 10 and 30 percent of the Y_1’s are Y_2’s”, which then becomes

$$\textbf{between 10 and 30 percent}(Y_1, Y_2) = [0.1 \leq \textbf{rate} \leq 0.3](Y_1, Y_2)\,.$$

A more detailed analysis of proportional quantifiers including approximate examples is given below in section 11.9.

Next we shall discuss a couple of further quantifiers taken from different linguistic classes: quantifiers of comparison, definite quantifiers, proper names (which can naturally be modelled through quantification), composite quantifiers, and others. As to quantifiers of comparison, consider a statement like “More Y_1’s than Y_2’s are Y_3’s”. The involved quantifier “more than” can readily be formalized as

$$\textbf{more than}(Y_1, Y_2, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_3| > |Y_2 \cap Y_3|\,.$$

The quantifier “more than” is representative of a class of three-place comparison quantifiers known as cardinal comparatives [82, p. 305]. Despite their obvious practical utility, these quantifiers attracted few interest in existing work on fuzzy quantification, which focussed on the absolute and proportional types. I have decided to pay some more attention to these quantifiers in this report. An extensive discussion of cardinal comparatives including implementation details is presented in section 11.10.

Now turning to the definite type of quantifiers, we must discern the singular and plural usage, and thus introduce different quantifiers $\textbf{the}_{\mathrm{sg}}$ and $\textbf{the}_{\mathrm{pl}}$ in order to model “The Y_1 is Y_2” and “The Y_1’s are Y_2’s”, respectively. We can try a Russellian analysis and force these quantifiers into the two-valued framework,

$$\begin{aligned}\textbf{the}_{\mathrm{sg}}(Y_1, Y_2) = 1 &\Leftrightarrow |Y_1| = 1 \wedge Y_1 \subseteq Y_2\\ \textbf{the}_{\mathrm{pl}}(Y_1, Y_2) = 1 &\Leftrightarrow |Y_1| > 1 \wedge Y_1 \subseteq Y_2\,,\end{aligned}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. As remarked above, it is this type of quantifiers which motivates the incorporation of undefined interpretations, in order to better express the presupposition that Y_1 be a singleton (singular “the”) or contain at least two elements (plural “the”). Denoting ‘undefined’ by $\uparrow$, the quantifiers would typically be written as

$$\textbf{the}_{\mathrm{sg}}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 1 \wedge Y_1 \subseteq Y_2\\ 0 & : \quad |Y_1| = 1 \wedge Y_2 \not\subseteq Y_2\\ \uparrow & : \quad |Y_1| \neq 1 \end{cases}$$

$$\textbf{the}_{\mathrm{pl}}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| > 1 \wedge Y_1 \subseteq Y_2\\ 0 & : \quad |Y_1| > 1 \wedge Y_1 \not\subseteq Y_2\\ \uparrow & : \quad |Y_1| \leq 1 \end{cases}$$

This kind of modelling is easily fitted to the framework for fuzzy quantification developed below, by representing ‘$\uparrow$’ as a third truth value $\frac{1}{2}$, which then means ‘undefined’ or ‘undecided’; see Barwise and Cooper [6, p. 171] for motivation of a three-valued model in the framework of TGQ.

Now advancing to proper names, let us first consider an example which illustrates why proper names can be conceived as quantifiers in the first place. Hence suppose that the given domain E contains some person, say “George”, that we are interested in. We can then model NL statements of the type “George is Y” as a special case of unary quantification, by resorting to the proper name quantifier

$$\textbf{george}(Y) = 1 \Leftrightarrow \mathrm{George} \in Y\,,$$

for all $Y \in \mathcal{P}(E)$. A more detailed discussion of this construction and its generalization to the fuzzy case is presented in section 3.3.

Finally, it is often useful to assume that more complex types of NL expressions be quantifiers as well, which stem from certain syntactic constructions. (Generally, this strategy will aid a compositional interpretation in the sense of Frege’s principle). Examples which involve such composite quantifiers are “All married Y_1’s are Y_2’s”, which results from a construction known as ‘adjectival restriction’, and “Some Y_1’s are Y_2’s or Y_3's", which semantically expresses a union of argument sets. The interpretations of the corresponding quantifiers Q_1, Q_2 are readily stated, viz

$$Q_1(Y_1, Y_2) = \textbf{all}(Y_1 \cap \textbf{married}, Y_2)$$
$$Q_2(Y_1, Y_2, Y_3) = \textbf{some}(Y_1, Y_2 \cup Y_3)$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where **married** $\in \mathcal{P}(E)$ denotes the set of all married people.

As witnessed by the above examples of two-valued quantifiers of various kinds, TGQ has indeed managed to devise a uniform representation, which is capable of expressing a broad range of extensional NL quantifiers. Within this formal framework, linguists and philosophers of language have developed concepts which describe the semantical characteristics of NL quantifiers and apparent relationships between these quantifiers, e.g. negation, antonyms, duality, monotonicity properties, symmetry, having extension, conservativity, adjectival restriction and others (see Gamut [45, pp. 223-256] for explanation. All of these concepts will also be cleanly introduced in a later chapter of this work, and they will serve an important purpose in what follows).

Due to its straightforward analysis of two-valued quantifiers, research in TGQ advanced at a rather fast pace and helped to clarify such questions like the classification of linguistic quantifiers in terms of their formal properties; the search for semantic universals that govern the possible meanings of linguistic terms (NPs), the investigation of algebraic properties like symmetry, circularity etc., and the issue of expressive power; see Gamut [45, p. 223-236] for a survey. The development of TGQ was pursued most actively in the 1980's. Since then, it became an integral part of modern theories of NL semantics like DRT [77, 144] or DQL [33] and others, and can thus be said to represent some kind of linguistic consensus, although there are, of course, some rivaling views, like the analysis of quantifiers in MultiNet [65].

Thus TGQ has had its successes and it was stimulating to linguistic research in many regards. But, there are also some drawbacks. Obviously, TGQ is suited to statically represent the meaning of quantifying propositions. However, knowledge processing involving linguistic quantifiers is difficult on formal grounds, knowing that most quantifiers are not first-order definable. Hence an axiomatization of reasoning schemes for these quantifiers will necessarily be imperfect. In addition, the analysis of linguistic quantifiers expressed by noun phrases that was achieved by TGQ still needs refinement, because it only deals with non-referring terms. In order to interpret referential terms and to resolve anaphora (e.g. anaphoric pronouns like "he"), an extension to some model of discourse is inevitable. The restriction to isolated NPs or sentences must hence be given up in favour of larger units of discourse, or texts. And indeed, there was enormous interest in discourse representation theory [77, 64, 144] once the basic issues of non-referential NPs (extensional generalized quantifiers) had been clarified.

Another peculiarity, which hinders the practical application of TGQ in certain cases, is its lack of vagueness modelling: neither is vagueness permitted in the arguments of a quantifier, nor is the quantifier itself allowed to be vague. This kind of idealization was certainly useful because it permitted research in TGQ to focus on the core issues und achieve rapid progress in these areas. And, the basic need for a treatment of vagueness is acknowledged by the proponents of TGQ, although they refrain from handling vagueness in their own work. For example, Barwise and Cooper have the following remark on the treatment of vagueness in their seminal publication on TGQ [6, p. 163]:

> *"The fixed context assumption is our way of finessing the vagueness of non-logical determiners. We think that a theory of vagueness like that given by Kamp" [78] "for other kinds of basic expressions could be superimposed on our theory. We do not do this here, to keep things manageable".*

In other words, Barwise and Cooper concede the necessity of modelling vagueness, but they consider the vagueness of language a phenomenon that can be isolated from the core problem of analyzing linguistic quantification. The proposed separation of quantificational analysis and vagueness modelling is not admissible on logical grounds, though. This is because the vagueness of quantifiers and their arguments will necessarily express in the representation of linguistic quantifiers and in the concepts used for their classification. A generalized quantifier of TGQ, say, only accepts two-valued arguments and produces two-valued results in return to such arguments. Hence generalized quantifiers are neither suited for processing vague terms which form the quantifiers' arguments, nor are they capable of expressing vague or 'approximate' quantification. If the incorporation of vagueness calls for a different notion of generalized quantifiers, however, then the concepts building on this modelling construct, like antonym, dual, adjectival restriction, quantitativity, conservativity etc. must all be adapted as well and redefined in such a way that they make sense in the more general situation (i.e. in the presence of vagueness).

Barwise and Cooper are not aware of this need because they apparently sympathize with the epistemic view of vagueness. Hence the vagueness of language is essentially a matter of context dependence or lack of knowledge needed for interpretation (e.g. standards of comparison assumed by a speaker). If this view is correct, then the machinery used for semantical interpretation can be kept two-valued, provided that we have some hypothetical "rich context" which fixes the meaning of all context-dependent expressions including vague terms and quantifiers (i.e. under the 'fixed context assumption' of Barwise and Cooper [6, p. 163] mentioned above). Other authors in TGQ, specifically Keenan and Stavi, deny the possibility of modelling such contexts. Because the standard of comparison required for interpreting "many" or "few" is not given, they conclude that such "value judgment dets" cannot be assigned a semantical value at all [82, p. 258]. Barwise and Cooper, by contrast, remain agnostic about the issue of actual context modelling and simply delegate the problem to specialized research.

I have already mentioned that Barwise and Cooper also encourage an extension of their basic framework to a three-valued model [6, p. 171]. Thus, an incorporation of the supervaluationist or three-valued view of vagueness also seems to fit into their methodological picture. However, as we have learnt in the introduction, the epistemic and supervaluationist or three-valued models do not really give a convincing account of Sorites paradoxes and thus, of vagueness. To me, they appear like attempts to lay the 'demon' of vagueness in formal chains, rather than trying to profit from its power, in the same way that natural languages do. The only model of vagueness which accomplishes this is in my view, the continuous-valued model assumed by the degree theories of vagueness. The superiority of this model is also evident from the positivistic argument that only the degree theory of vagueness – as witnessed by its main proponent of fuzzy set theory – has made its way outside academic debate and acquired practical relevance in experimental and commercial software as well as consumer products. Thus I will adopt the fuzzy-sets based model in this work, which is most convincing to me and also successful in practice. In addition, there are proposals for fuzzy-sets based applications which require a plausible model of type IV quantifications. These systems will profit considerably from improved models of linguistic quantifiers based on the fuzzy-set theory.

Now that I opted for a vagueness modelling in the fuzzy sets framework, it is still necessary to account for the issue of context dependence. The problem has not simply gone away because *"linguistic concepts are not only predominantly vague, but their meanings are almost invariably context-dependent as well"* [89, p. 280]. To develop a theory of fuzzy quantification and avoid getting stuck in this complex issue, I will join Barwise and Cooper in assuming a rich context which uniquely determines the meaning of all basic expressions like "tall", "young" (predicates) and "many", "few" (quantifiers), without analysing these contexts any further. In a degree-based setting, this means that "many", "tall" etc. now find a unique interpretation in terms of continuous membership grades.

The resulting gradual model of a quantifier like "many", however, can no longer be represented by a two-valued generalized quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$. And generalized quantifiers are not even useful for interpreting propositions like "All Swedes are blonde", because the fuzzy set **blonde** $\in \widetilde{\mathcal{P}}(E)$ is not an admissible argument of the generalized quantifier **all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$. In other words, we need an extension of two-valued generalized quantifiers to fuzzy generalized quantifiers, which are capable of expressing type IV quantifications.

2.3 Fuzzy quantifiers

In the following, I will present the apparent extension of generalized quantifiers which gives a natural account of approximate quantifiers like "about ten", and further makes all quantifiers applicable to fuzzy arguments like "tall" and "rich". This extension is necessary because NL quantifiers undoubtedly accept this kind of arguments in ordinary language, and because many quantifiers themselves are also vague. Thus, the proposal of generalized quantifiers suitable for type IV quantifications, which I will now make, gives a more realistic picture of linguistic quantifiers compared to the limitations and idealizations of the original concept assumed by the Theory of Generalized Quantifiers.

Definition 4
An n-ary fuzzy quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

Notes

- A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ hence assigns to each n-tuple of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ an interpretation $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$, which is allowed to be gradual.

- The above definition corresponds to Zadeh's [190, pp. 756] alternative view of fuzzy quantifiers as fuzzy second-order predicates. These are modelled as mappings in order to simplify notation. It should be pointed out, though, that unlike Zadeh, I explicitly permit arbitrary arities $n \in \mathbb{N}$. In a similar way, my proposal generalizes Thiele's [149] notion of a (unary) 'general fuzzy quantifier' $Q : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, which it extends beyond the simplest case $n = 1$.

- The generic concept of fuzzy quantifiers assumed here (originally dubbed *fuzzy determiners*), was originally introduced in [46, p. 6]. For the reasons explained in the introduction, I have now switched to the familiar term 'fuzzy quantifier', though, which probably sounds less awkward to non-linguists.

- Given *crisp* arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, and a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, the quantifying expression $\widetilde{Q}(Y_1, \ldots, Y_n)$ should be well-defined as well. Thus, I will assume that ordinary subsets be viewed as a special case of fuzzy subsets, and it is hence understood that $\mathcal{P}(E) \subseteq \widetilde{\mathcal{P}}(E)$, where $\widetilde{\mathcal{P}}(E)$ is again the set of fuzzy subsets of E. Note that this subsumption relationship does not hold if one identifies fuzzy subsets and their membership functions, i.e. if one stipulates that $\widetilde{\mathcal{P}}(E) = \mathbf{I}^E$, where $\mathbf{I}^E$ denotes the set of mappings $f : E \longrightarrow \mathbf{I}$. This is also the reason why I do not enforce this identification in my notation. However, I assume that the appropriate transformations (e.g. from a crisp subset $A \subseteq E$ to its characteristic function $\chi_A \in \mathbf{2}^E \subseteq \mathbf{I}^E$) are carried out and for the sake of readability, these will be suppressed in the notation.

As opposed to two-valued quantifiers, fuzzy quantifiers accept fuzzy input, and hence render it possible to evaluate quantifying statements like "$\widetilde{Q}$ young people are rich". This amounts to the computation of the quantification result $\widetilde{Q}(\textbf{young}, \textbf{rich}) \in \mathbf{I}$, where **young**, **rich** $\in \widetilde{\mathcal{P}}(E)$ are the fuzzy subsets of "young" or "rich" people, respectively. In addition, the admissible outputs of fuzzy quantifiers are free to assume intermediate results in the unit range, and no longer tied to the crisp choices $\{0, 1\}$. Fuzzy quantifiers are therefore capable of expressing all possible shades of quantification results, thus achieving a more natural modelling of the borderline area of an approximate quantifier. Summarizing, the above definition simply spells out the obvious extensions, which are necessary for incorporating fuzzy arguments and approximate quantifiers into the basic notion of a generalized quantifier.

Let us now consider examples of fuzzy quantifiers. To this end, I stipulate some ad-hoc definitions for the unary quantifiers $\widetilde{\forall}, \widetilde{\exists} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ and the binary quantifiers

$\widetilde{\textsf{all}}, \widetilde{\textsf{some}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, viz

$$\widetilde{\forall}(X) = \inf\{\mu_X(e) : e \in E\} \tag{7}$$

$$\widetilde{\exists}(X) = \sup\{\mu_X(e) : e \in E\} \tag{8}$$

$$\widetilde{\textsf{all}}(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\} \tag{9}$$

$$\widetilde{\textsf{some}}(X_1, X_2) = \sup\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\} \tag{10}$$

for all $X, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. These simple quantifiers are readily employed for modelling fuzzy NL quantification. Hence let us assume for the moment that $\widetilde{\textsf{all}}$ provide the correct interpretation of the natural language quantifier "all". A statement like "All young are poor" can then be evaluated by computing the quantification result of $\widetilde{\textsf{all}}(\textsf{young}, \textsf{poor})$, where $\textsf{young}, \textsf{poor} \in \widetilde{\mathcal{P}}(E)$ are the fuzzy denotations of "young" and "poor". Hence let $E = \{\text{Joan, Lucas, Mary}\}$ and suppose that $\textsf{young}, \textsf{rich} \in \widetilde{\mathcal{P}}(E)$ are defined by

$$\mu_{\textsf{young}}(e) = \begin{cases} 1 & : \; e = \text{Joan} \\ 0.7 & : \; e = \text{Lucas} \\ 0.2 & : \; e = \text{Mary} \end{cases}$$

$$\mu_{\textsf{rich}}(e) = \begin{cases} 0.9 & : \; e = \text{Joan} \\ 0.8 & : \; e = \text{Lucas} \\ 0.6 & : \; e = \text{Mary} \end{cases}$$

for all $e \in E$. In Zadeh's succinct notation for fuzzy sets, i.e.

$$X = \sum_{e \in E} f(e)/e$$

symbolizes the fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ with $\mu_X = f$, these fuzzy sets can be expressed thus:

$$\begin{aligned} \textsf{young} &= 1/\text{Joan} + 0.7/\text{Lucas} + 0.2/\text{Mary} \\ \textsf{rich} &= 0.9/\text{Joan} + 0.8/\text{Lucas} + 0.6/\text{Mary}\,. \end{aligned}$$

Referring to this choice of fuzzy arguments, the gradual quantification result of "All young are poor" then becomes

$$\begin{aligned} \widetilde{\textsf{all}}(\textsf{young}, \textsf{poor}) &= \min\{\max(1 - 1, 0.9), \max(1 - 0.7, 0.8), \max(1 - 0.2, 0.6)\} \\ &= \min\{0.9, 0.8, 0.8\} \\ &= 0.8\,. \end{aligned}$$

2.4 The dilemma of fuzzy quantifiers

In the previous section, I have augmented the original generalized quantifiers of TGQ by fuzzy inputs and gradual quantification results, in order to meet the demands of real-world language. The straightforward notion of fuzzy quantifiers that I proposed establishes a class of operators which incorporate the fuzziness of language. Thus, I solved one part of the modelling problem by introducing the required operational quantifiers. The modelling problem has another facet, though, that will be wrestled with in this section. Specifically, we still need to elucidate the relationship between the proposed modelling constructs (fuzzy quantifiers) and the NL quantifiers of interest. Thus, how can we: (a), determine a well-motivated choice of fuzzy quantifier which properly models a given NL quantifier and (b), defend this choice against all possible alternatives, which might appear equally plausible at first sight?

We all have strong intuitions about the proper definition of linguistic quantifiers on crisp subsets of E. For example, the definition of $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ stated in Def. 2 is certainly uncontroversial. When turning to the fuzzy case, however, a plenty of possible options suddenly enters the scene (only to mention the wealth of conceivable fuzzy conjunctions that exist even in the simple propositional case). The resulting set of competing interpretations must hence be pruned to some unique preferred choice, which should be clearly distinguished from the remaining alternatives. Unfortunately, there are only few intuitions in the fuzzy case, which might guide us to the optimal model, i.e. to that choice of fuzzy quantifier which best corresponds to the natural language quantifier of interest. At least, it becomes rather difficult to assess the plausibility of such correspondence assertions, and to check how the the particular selection made measures up with the competing alternatives.

In order to demonstrate this problem, let us consider the simple universal quantifier $\forall$, and make an attempt to extrapolate its meaning to the fuzzy case. In (7), I already made a proposal for a fuzzy analogue of this quantifier. However, what about the following alternative stipulations, which only constitute a small sample of the possible options:

$$\widetilde{\forall}'(X) = \frac{\sum\limits_{e \in E} \mu_X(e)}{|E|}$$

$$\widetilde{\forall}''(X) = \begin{cases} 1 & : \quad X = E \\ 0 & : \quad \text{else} \end{cases}$$

$$\widetilde{\forall}'''(X) = \inf \left\{ \prod_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ \ldots, \ a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$. Some of these quantifiers apparently do not qualify as plausible models of $\forall$ in the fuzzy case, while others are not so easy to assess. The first alternative $\widetilde{\forall}'$ for example can be rejected without further ado, because it fails to coincide with the two-valued quantifier $\forall$ on crisp arguments. The second example also violates some intuitive postulates. For example, producing crisp outcomes only does not seem very natural for a fuzzy quantifier. Moreover, the fuzzy quantifier $\widetilde{\forall}''$ shows an undesirable discontinuity in the membership grades $\mu_X(e)$, which makes it too brittle for applications. Additional evidence for rejecting this choice of quantifier stems from Thiele's analysis of fuzzy universal quantifiers, see [147, 148, 149] and section 4.16 below. As to the third example, there are no a priori concerns against the use of $\widetilde{\forall}'''$

as a fuzzy model of the universal quantifier. In fact, both my original definition of $\widetilde{\forall}$ and the alternative $\widetilde{\forall}'''$ comply with Thiele's analysis, and must hence be considered plausible choices of fuzzy universal quantifiers.

The examples demonstrate that for elementary quantifiers, some basic intuitions persist when admitting fuzzy arguments. However, these are not necessarily strong enough to establish a unique preferred choice. The case-by-case verification that a proposed fuzzy quantifier $\widetilde{Q}$ make a plausible model of the given NL quantifier, soon becomes too tedious when considering a larger number of quantifiers. Moreover, the manual procedure makes it very difficult for the stipulated correspondences to achieve the desired systematicity and consistency. The quest for coherence in particular, even suggests that such direct correspondence assertions be avoided altogether. A generic solution, by contrast, should achieve an adequate modelling of any novel and unanticipated natural language examples, that the resulting theory of fuzzy quantifiers might later be confronted with. This is especially important because we cannot trust that there will always be sufficient intuitions to determine a plausible interpretation for a given quantifier. In unclear cases which lack apparent modelling clues, a generic solution might prove invaluable for establishing the desired correspondences.

This scarcity of intuitions concerning the proper modelling of NL quantifiers in the fuzzy case (or the presence of spurious intuitions that misguide us into the wrong conclusions), becomes evident once we move from the simplest example $\forall$ to other common quantifiers like "at least 10 percent", which is representative of the proportional type. Given a finite base set E, we can readily define a corresponding two-valued quantifier **at least 10 percent** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ by **at least 10 percent** $= [$**rate** $\geq 0.1]$. The target quantifier then takes the more explicit form,

$$\textbf{at least 10 percent}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1 \cap Y_2| \geq |Y_1|/10 \\ 0 & : \quad \text{else} \end{cases} \qquad (11)$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, which also reveals the dependency of the quantifier on the cardinality $|\bullet|$ of crisp sets. However, it is not that easy to provide a straightforward definition of a corresponding fuzzy quantifier $\widetilde{\textbf{at least 10 percent}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$. This is because X_1, X_2 in

$$\widetilde{\textbf{at least 10 percent}}(X_1, X_2)$$

are fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. The familiar cardinality measure for ordinary sets, which made possible the above definition (11) of the two-valued quantifier, is not applicable to these fuzzy arguments, and consequently, it cannot be used to define the fuzzy quantifier. And there is no reliable and uncontroversial notion of the cardinality of fuzzy sets, which avails us with a substitute for $|\bullet|$ in the fuzzy case. It is of course possible to make ad-hoc stipulations, as in the earlier example of the logical quantifiers. Thus we might subscribe to the Σ-count approach, arguing that

$$\widetilde{\textbf{at least 10 percent}}(X_1, X_2) = \begin{cases} 1 & : \quad \Sigma\text{-Count}(Y_1 \cap Y_2) \geq \Sigma\text{-Count}(Y_1)/10 \\ 0 & : \quad \text{else} \end{cases}$$

be the proper fuzzy analogue of “at least 10 percent”. This proposal is representative of those modelling attempts which simply replace the crisp cardinality in (11) with some generalized cardinality for fuzzy sets. The resulting formulas might look appealing at first sight, because they bear a superficial resemblance to the straightforward expressions which describe the basic quantifier in the crisp case. However, these suggestive formalizations easily lure us into models with latent flaws, and the quality of the computed results becomes hardly predictable. The proposed interpretation of “at least 10 percent” for example, can show abrupt changes in response to slight variations of the membership grades. Moreover the earlier criticism of the Σ-count approach exemplified by Fig. 3, also applies to the quantifier “at least 10 percent”.

These difficulties in establishing a conclusive interpretation of “at least 10 percent” hint at an intriguing problem because the class of fuzzy quantifiers is certainly rich enough to contain the intended model – it is only we cannot identify it. Thus it appears that we have run into some kind of dilemma. This *dilemma of fuzzy quantification* shows up in the awkward situation that we either do not have the expressive power available, which is necessary to model the quantifiers of interest, or that we do have the required expressive power, but cannot control the tool that offers it:

- The simple *two-valued quantifiers* of TGQ fail to give a natural account of approximate quantification. However, it is usually easy to define quantifiers which fit into the two-valued framework;
- The proposed *fuzzy quantifiers* achieve a natural modelling of a broad range of NL quantifers including the approximate variety, but we are unable to locate the proper model within the wealth of available choices.

It is this struggle between expressiveness and ease of specification which makes the problem of establishing correspondences that notorious. In order to check whether the dilemma is inescapable, let us now recall the above distinction between the two ways how fuzziness interacts with NL quantification, which shaped the proposed definition of fuzzy quantifiers. According to the classification of Liu and Kerre, fuzziness can enter into a quantifier either through its inputs (when supplied with fuzzy arguments), or it can be inherent to an approximate quantifier itself, which then produces gradual outputs even when the arguments are crisp. In order to capture both phenomena the proposed fuzzy quantifiers had to account for

a. continuous quantification results, in order to express approximate quantifiers; and also

b. the fuzziness of NL concepts inserted into the argument slots, which necessitates the quantifier to accept (and cope with) fuzzy arguments.

At this point, it is instructive to notice that “at least 10 percent”, the quantifier which I used to demonstrate the dilemma, is representative of the precise, non-approximate variety. Thus the difficulty of establishing correspondences is not an artifact introduced by approximate quantifiers. As a matter of fact, the above problems experienced with fuzzy quantifiers are all concerned with the second aspect of also letting fuzzy sets fill the argument slots of a quantifier. Thus the problem of establishing correspondences only arises if one tries to jump into type IV quantifications and treat both sources of fuzziness at a time. This suggests an obvious middle course: why worry about the details of processing fuzzy arguments when it is only the precise description of an NL quantifier that is at issue? In fact, the seeming dilemma can be resolved nicely if we recall my analysis of a 'framework for fuzzy quantification' explained in the introduction. For its range of considered NL quantifiers, such a framework must introduce a class of operational quantifiers, of specifications for such quantifiers, and a description of the relationship between the two. What the 'dilemma of fuzzy quantifiers' really shows is that fuzzy quantifiers, albeit useful as quantifying operations, are not suitable for specification purposes. Nor are two-valued generalized quantifiers of TGQ, which are unable to express the important type III quantifications (i.e. approximate quantifiers). The apparent conclusion is that a practical specification medium must be developed, which avoids the fallacies of both two-valued and fuzzy quantifiers. In order to simplify the specification process, and establish a system of coherent interpretations I therefore suggest the obvious middle course between two-valued quantifiers and fuzzy quantifiers, thus separating the 'hard' part, that of handling fuzziness in the arguments, from the less delicate part, that of supporting gradual quantifications. The resulting intermediary representations, called semi-fuzzy quantifiers, will be discussed in the next section.

2.5 Semi-fuzzy quantifiers

In the previous chapter, we experienced substantial difficulty establishing a model of "at least 10 percent" as a fuzzy quantifier. The unclear behaviour of the linguistic prototype given fuzzy arguments forced me to proclaim the 'dilemma of fuzzy quantifiers', i.e. there is a basic conflict between expressive power and ease of specification. Fuzzy quantifiers, one the one hand, are simply too rich a set of operators to investigate the relevant questions directly and establish the desired correspondences in a straightforward way. Quite the reverse, it even appears that the proposed fuzzy quantifiers are expressive to such an extent, that they probably raise more issues than they answer. The generalized quantifiers of TGQ, on the other hand, are unable to capture approximate quantifiers due to their assumption of bivalent quantifications. Attributing the main source of difficulty to the presence of fuzzy arguments then shaped the idea of taking the obvious 'middle course' between two-valued quantifiers and fuzzy quantifiers. I will hence narrow the scope to quantification based on crisp arguments (as is done in TGQ) but resulting in gradual outputs (similar to fuzzy quantifiers). It is of course hoped that setting aside the intricacies of fuzzy arguments will avail us with a practical specification medium for natural language quantifiers, which also improves our point of departure for solving the puzzles of fuzzy quantification. Let me now define a suitable notion of quantifier to express such type III quantifications.

Definition 5
An n-ary semi-fuzzy quantifier *on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

- Q hence assigns to each n-tuple of crisp subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ a gradual interpretation $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$.
- The concept of semi-fuzzy quantifiers (originally dubbed 'fuzzy pre-determiners') has been introduced in [46, p. 7]. The later change of terminology is due to similar considerations as in the case of fuzzy determiners, i.e. as a courtesy to common practice outside linguistics.

The proposed semi-fuzzy quantifiers are half-way between two-valued quantifiers and fuzzy quantifiers because they have crisp input and fuzzy (gradual) output. By supporting quantification results in the continuous range $\mathbf{I} = [0, 1]$, semi-fuzzy quantifiers accommodate that characteristic of fuzzy quantifiers, which is critical to the modelling of approximate quantification. But, we need to avoid that peculiarity which made it that difficult to define fuzzy quantifiers and to defend the chosen interpretation against competing alternatives. Thus, semi-fuzzy quantifiers accept crisp arguments only. This eliminates the need to specify the intended responses for arbitrary fuzzy arguments. The concept of semi-fuzzy quantifiers (i.e. crisp inputs, gradual quantifications) is rich enough to embed all two-valued quantifiers of TGQ, because the set of all mappings $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ comprises the two-valued quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \{0, 1\}$. Thus, the methodical goal of incorporating two-valued generalized quantifiers into the new class of operators has indeed been achieved. The embedding of two-valued quantifiers permits the reuse of all interpretations for NL quantifiers that are known from TGQ. In perspective, it also lets us profit from the existing body of knowledge on NL quantification, that has already been gathered by linguists.

The limitation to crisp inputs has another consequence, which I consider one of its main virtues. Due to the fact that all arguments are known to be crisp, it is easy to establish the cardinality of the involved sets. Consequently, semi-fuzzy quantifiers can be expressed in terms of the cardinality of their arguments or Boolean combinations thereof. Consider "at least 10 percent", for example. The definition for crisp arguments is certainly uncontroversal, and it is easy to understand how the quantification "at least 10 percent of the Y_1's are Y_2's" must be expressed in terms of $|Y_1|$ and $|Y_1 \cap Y_2|$ given crisp arguments, see (11). It cannot be overestimated that the problem of specifying the corresponding fuzzy quantifier now vanishes, because there is no well-established substitute for $|\bullet|$ in the fuzzy case.

The specification in terms of semi-fuzzy quantifiers is uniform across quantifier types. For example, both absolute quantifiers like "about ten" and proportional quantifiers like "about ten percent" turn into binary quantifiers $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$, which abstract from the common pattern "Q Y_1's are Y_2's" (see also examples below). Semi-fuzzy quantifiers are not only sufficiently expressive to capture all quantifiers in the sense of TGQ, though. They also enrich these with a natural account of approximate quantification through a fuzzy sets model, without sacrificing the conceptual simplicity of the original generalized quantifiers and their descriptive utility. Resuming, the proposed semi-fuzzy quantifiers offer a practical specification medium for a broad range of linguistic quantifiers.

Let us now consider some examples of semi-fuzzy quantifiers. First of all, the two-valued quantifiers introduced in Def. 2 and Def. 3, as well as the derived models of NL quantifiers presented above (i.e. "at least k", "all except k", "at least p percent" etc.), all qualify as instances of semi-fuzzy quantifiers, because the notion of a semi-fuzzy quantifier conveniently embeds the original two-valued quantifiers of TGQ. Hence let me turn to examples of semi-fuzzy quantifiers proper, i.e. approximate quantifiers like "almost all". There are proposals for defining many of such quantifiers in the traditional framework, i.e. in terms of a membership function $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ or $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$. These membership functions are easily translated into corresponding semi-fuzzy quantifiers. To see how this conversion looks like in practice, consider the proportional quantifiers "almost all". In the introduction, I proposed a modelling of "almost all" in terms of the membership function $\mu_{\textbf{almost all}}$ given by eq. (1). In order to define a corresponding semi-fuzzy quantifier, we only need to explicitly state the argument structure, which is suppressed in the membership function $\mu_{\textbf{almost all}}$. The semi-fuzzy quantifier **almost all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ then becomes

$$\textbf{almost all}(Y_1, Y_2) = \begin{cases} \mu_{\text{almost all}}(\frac{|Y_1 \cap Y_2|}{|Y_1|}) & : \quad Y_1 \neq \varnothing \\ 1 & : \quad \text{else} \end{cases} \tag{12}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. We can get a rough picture of how the quantifier behaves from Fig. 1, which displays the plot of the membership function. My decision to stipulate that $\textbf{almost all}(\varnothing, Y) = 1$ is justified by the observation that we also have $\textbf{all}(\varnothing, Y) = 1$, and "almost all" should express a weaker condition than "all", which demands that we let $\textbf{almost all}(\varnothing, Y) = 1$ as well.

In order to catch a larger class of NL quantifiers, it is convenient to define parametric quantifiers from which concrete instances such as "many", "almost all", "often", "almost everywhere" etc. can be derived. For purpose of illustration, let us consider two parametrized examples.

> $\textbf{abs many}_{\rho,\tau} : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$
> "Many X_1's are X_2 compared to an absolute expected value $\rho \in \mathbb{R}$ with sharpness parameter $\tau \in [0, \rho]$"; and
>
> $\textbf{rel many}_{\rho,\tau} : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$
> "The proportion of X_1's that are X_2 is large compared to an expected proportion $\rho \in [0, 1]$ with sharpness parameter $\tau \in [0, \rho]$".

Formally, we define these generic quantifiers as follows (assuming an arbitrary finite domain $E \neq \varnothing$):

$$\textbf{abs many}_{\rho,\tau}(Y_1, Y_2) = S(|Y_1 \cap Y_2|, \rho - \tau, \rho + \tau)$$

$$\textbf{rel many}_{\rho,\tau}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 0 \\ S(\frac{|Y_1 \cap Y_2|}{|Y_1|}, \rho - \tau, \rho + \tau) & : \quad |Y_1| \neq 0 \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where S again denotes Zadeh's S-function [188, pp. 183+], see also eq. (2) above. The proposed definitions of $\textbf{abs many}_{\rho,\tau}$ and $\textbf{rel many}_{\rho,\tau}$

cover various meanings of “many” and similar quantifying expressions such as “often”, “relatively often”, etc. In particular the proposed modelling of “almost all” in terms of equality (12), is just an instance of the general pattern captured by **rel many** and can now be expressed as

$$\textbf{almost all} = \textbf{rel many}_{0.8,\,0.1}.$$

The choice of parameters obviously depends on the quantifier to be modelled, but it is also application-specific. In addition, other options for modelling a given NL quantifier, e.g. in terms of trapezoidal membership functions, might well be equally plausible. I will return to the issue of context-dependence in the chapter summary, which also discusses the problem of constructing plausible membership functions among the wealth of possible choices.

Apart from the generic examples, it is also worthwhile considering the following quantifier (which is unparametrized):

> **as many as possible** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$
> which might be interpreted as denoting “The relative share of X_1’s that are X_2”.

We can give a (very rough) account of the quantifier by defining

$$\textbf{as many as possible}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 0 \\ \frac{|Y_1 \cap Y_2|}{|Y_1|} & : \quad |Y_1| \neq 0 \end{cases} \tag{13}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. Thus “the more, the better”. The above stipulation for empty Y_1, i.e. **as many as possible**$(\varnothing, Y) = 1$, is again motivated by the observation that intuitively, “all” poses a stronger condition, whence **as many as possible**$(\varnothing, Y) \geq$ **all**$(\varnothing, Y) = 1$. We shall see some applications of this quantifier later in the report, which demonstrate that the model, albeit very simplistic, can still be useful. The chosen interpretation of “as many as possible” corresponds to Yager’s ‘pure averaging’ quantifier Q_{mean} described in [170, p. 187].

The above examples of semi-fuzzy quantifiers demonstrate that the proposed concept is indeed useful for specifying natural language quantifiers. The quantifying operators which I described in terms of semi-fuzzy quantifiers show a very clear input-output behaviour. These quantifiers are simple functions of crisp cardinalities. Thus, the relationship between the arguments of the quantifier and the resulting interpretation is easily grasped. It is clear in advance that the specification of a particular quantifier is strongly context-dependent. However, the clarity of semi-fuzzy quantifiers makes our linguistic intuitions applicable again, thus simplifying the identification of a plausible model. We can thus match the available quantifiers with the intended interpretation of the target NL quantifier in the given situation. Unlike fuzzy quantifiers, the new descriptions are easily communicable and thus encourage a constructive dispute about the interpretation of a quantifier in the application of interest.

The diversity of possible linguistic quantifiers is not at all accounted for by the traditional approaches to fuzzy quantification and their μ_Q-based representations. Semi-fuzzy quantifiers, by contrast, are capable of expressing a wealth of possible examples including quantifiers of exception, cardinal comparatives, proper names, definite quantifiers etc. the importance of which has long been recognized by linguists, but escaped the attention of fuzzy set theorists.

This extended coverage is possible because the quantificational structure, i.e. the dependency of quantifiers on a particular combination of the arguments like the absolute count, relative share, difference of cardinalities etc., is considered part of the specifications and thus explicitly encoded by each semi-fuzzy quantifier as an integral part of its definition. Existing approaches, by contrast, isolate this kind of argument structure from a skeleton specification (fuzzy number); the dependency on absolute counts, a ratio of cardinalities etc. must then be imposed by the interpretation mechanism. It is this difference in the assumed responsibility for argument structure which unlike the traditional μ_Q's, makes semi-fuzzy quantifiers a universal specification medium for extensional quantifiers.

In technical terms, semi-fuzzy quantifiers are capable of modelling both quantitative and non-quantitative types of quantifiers, while the traditional approach is limited to the quantitative variety; semi-fuzzy quantifiers make no assumptions on the finiteness of the domain, while previous approaches have usually limited themselves to the finite case; and only semi-fuzzy quantifiers have been designed with fuzzy multi-place quantification in mind and can thus express linguistic quantifiers regardless the number of arguments involved (for details concerning the coverage of quantificational phenomena, see the chapter summary). Semi-fuzzy quantifiers easily embed all two-valued generalized quantifiers in the sense of TGQ, which are enriched by a wealth of approximate types. Existing approaches, however, fail to embed the 'classical' generalized quantifiers, and thus fall behind the standards that have already been set in linguistics.

From a strategic point of view, the affinity of semi-fuzzy quantifiers to TGQ promises a straightforward generalization of the existing, linguistically motivated concepts, which might also contribute to a deeper understanding of fuzzy quantification. These benefits of semi-fuzzy quantifiers will not only foster the development of the theoretical background, but also open new fields of future application. Being a generic specification medium for natural language quantifiers, semi-fuzzy quantifiers not only cover the basic quantifiers for technical applications (like importance aggregation). Unlike the specialized representations of previous approaches, they target at the full range of quantifiers met in dialogue systems, machine translation, and other applications of natural language processing.

2.6 Quantifier fuzzification mechanisms

At this point, the proposed framework comprises fuzzy quantifiers and semi-fuzzy quantifiers. The former are the apparent abstraction of two-valued generalized quantifiers, which also embraces the known classes of operators studied in previous work on fuzzy quantification. However, the expressive power required to describe type IV quantifications is rather obstructive to establishing a well-justified interpretation for given NL quantifiers, because the expected behaviour of a quantifier is typically not very clear when supplied with fuzzy arguments. Thus, we need fuzzy quantifiers as target operations but they make poor a specification medium. Semi-fuzzy quantifiers, by contrast, no longer enforce a commitment to any details beyond the obvious characteristics of the target NL quantifier. With semi-fuzzy quantifiers, we are actually freed from the difficulties caused by fuzzy quantifiers because on the side of the inputs, everything is kept simple and crisp. This effectively avoids the original dilemma of having to specify precise outputs when the arguments of the quantifier are loaded with fuzziness. As a consequence, resorting to semi-fuzzy quantifiers makes the problem of specification much more feasible, due to their virtues as reduced descriptions. Based on semi-fuzzy quantifiers, it hence becomes a straightforward task to describe the intended meaning of given linguistic quantifiers like "at least 10 percent". As witnessed by these modelling examples for prototypical quantifiers, establishing an auxiliary layer of specifications indeed payed off.

Needless to say that semi-fuzzy quantifiers, although useful as descriptions of the quantifiers of interest, are not suitable for interpreting type IV quantifications. For example, we are not yet able to assign a meaning to NL statements like "Almost all rich people are tall", even when we have managed to precisely describe "almost all" in terms of a semi-fuzzy quantifier. These sacrifice the processing of fuzzy arguments like "tall" or "rich" in order to restrict the specifications to the clear cases. Compared to fuzzy quantifiers, we thus have the opposite pattern: semi-fuzzy quantifiers are good for specification purposes, but they make a poor choice of operation medium, and cannot handle the range of possible arguments. Hence both semi-fuzzy and fuzzy quantifiers are necessary components of the evolving framework, and these two concepts will serve complementary functions in a theory of fuzzy quantification. While semi-fuzzy quantifiers provide the specification medium, the original fuzzy quantifiers are still needed as the target operators which actually compute quantification results from given instantiations of fuzzy arguments. Only the particular choice of fuzzy quantifiers, although hard to establish, will achieve the desired completeness of interpretation. In other words, none of the two functions is overly useful as long as the specification and operation layers remain isolated.

This is were models enter the scene, because in my setting, a model of fuzzy quantification is nothing but an explicit description of the relationship between specifications (semi-fuzzy quantifiers) and corresponding operations (fuzzy quantifiers). The formal definition of these models, for which I will coin the term 'quantifier fuzzification mechanism' or QFM for short, is in fact quite simple, and it should now be really obvious from the above considerations:

Definition 6

A quantifier fuzzification mechanism *(QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ of the same arity $n \in \mathbb{N}$ and on the same base set E.*[14]

QFMs are the missing component which closes the gap between the specification and interpretation layers, thus completing the new framework for fuzzy quantification. The

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[14] I avoid to call $\mathcal{F}$ a 'function' or 'mapping' because the collection of semi-fuzzy quantifiers might form a proper class.

proposed framework now lets us specify the intended quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ by describing its behaviour for crisp arguments, determine the matching operation $\mathcal{F}(Q)$, and then apply this operation to interpret quantifying statements, by computing type IV quantifications $\mathcal{F}(Q)(X_1, \dots, X_n)$ for given fuzzy sets $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Consider universal quantification, for example. Starting from the two-valued quantifier $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ defined by (5) and a given QFM $\mathcal{F}$, we automatically obtain the fuzzy analogue $\mathcal{F}(\forall) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$. Examples of prototypical QFMs which can be substituted for $\mathcal{F}$ will be presented in Chap. 7 to Chap. 10 below. In all of these models, we obtain $\mathcal{F}(\forall)(X) = \widetilde{\forall}(X) = \inf\{\mu_X(e) : e \in E\}$, i.e. the fuzzy universal quantifier coincides with my original proposal (7). Having available such a mechanism, we can also apply it to "at least 10 percent" as defined in Def. 3. This solves the above problem of determining the proper definition of "at least 10 percent" as a fuzzy quantifier, which is now given a unique answer *relative to* $\mathcal{F}$, namely $\mathcal{F}($**at least 10 percent**$)$. Given a plausible choice of $\mathcal{F}$, we can simply fetch the desired model of "at least 10 percent", without ever having to rack our brains on the proper specification of the quantifier for fuzzy arguments.

It is probably safe to assume that the coherence and systematicity observed in natural language will reflect in correspondence assertions which are coherent and systematic to a similar degree. In other words, the plausible models of natural language quantification are subjected to strong regularities, which clearly separate these models from meaningless or inconsistent examples. It is this hidden structure of plausible models that must eventually be uncovered and distilled into explicit criteria for 'good' or preferred choices, which answer our intuitive expectations. At this point, it should be remarked that the sole purpose of QFMs is that of introducing variables which range over arbitrary correspondence assignments. Thus, the definition of candidate models *per se* is separated from the subsequent task of characterizing those interpretations which are linguistically valid. Starting from this full class of unrestricted models, which both surrounds all meaningful examples as well as a bunch of incoherent assignments, we can now try and identify the plausible choices by formalizing the intuitive criteria for semantical soundness.

It is the merit of QFMs to bring about a theoretical framework in which such adequacy criteria can be investigated with all the formal rigor which is necessary for turning our linguistic intuitions into precise definitions. Compared to the 'manual' procedure of defining interpretations on a case-by-case basis, the QFM approach eliminates the need to justify each individual choice of quantifier. This is because the fuzzification mechanism *itself* (as opposed to the instances of fuzzification) can now be subjected to a critical evaluation. Provided a sufficient catalogue of criteria, the margin for interpreting individual quantifiers, will be narrowed down to the plausible choices. This axiomatic procedure eliminates the risk of covert inconsistencies which is notorious to manual correspondence assertions or to a splitting of interpretations for each type of quantifers, as is done in the traditional framework.

2.7 The quantification framework assumption

In the last section, I completed the presentation of the new framework for fuzzy quantification, which now comprises the triple of specifications (semi-fuzzy quantifiers), operations (fuzzy quantifiers) and correspondence assignments modelled by QFMs. I also explained how the proposed framework answers the dilemma of fuzzy quantifiers. Every NL quantifier of interest can now be specified in the convenient format offered by semi-fuzzy quantifiers, and we only need to apply the chosen QFM to fetch the final interpretation. In this way, we are no longer forced to specify a precise numerical interpretation of the NL quantifier when there is fuzziness in the arguments: The processing of such arguments is now delegated to the QFM. The proposed notion of unrestricted QFMs will later be constrained to a class of intended choices.

In the given framework, developing further the theory of fuzzy quantification thus boils down to tracing out and formalizing the intuitive conditions on plausible models; compiling a generating system of basic axioms which entail the full catalogue of quality criteria; and finally to the identification and implementation of practical models which satisfy these requirements. This methodology will permit us to assess the phenomenon of fuzzy NL quantification, to give a comprehensive account of its various aspects, and finally to combine these pieces into the first theory of fuzzy quantification elaborated with formal rigor.

In view of such perspectives, it is worthwile reflecting the commitments made by the novel framework. In order to clarify the descriptive capacity that may be expected of the proposed approach, we must trace out its inherent assumptions, which decide upon the expressiveness of the framework and also demarcate its theoretical limits. In fact, there is an implicit assumption here, which silently crept in when I introduced semi-fuzzy quantifiers and quantifier fuzzification mechanisms. This condition can be rendered succinctly in terms of the following construction of underlying semi-fuzzy quantifiers:

Definition 7
Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *be a fuzzy quantifier. The* underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n)\,,$$

for all n*-tuples of* crisp *subsets* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. Hence* $\mathcal{U}(\widetilde{Q})$ *simply 'forgets' that* $\widetilde{Q}$ *can be applied to fuzzy sets, and only considers its behaviour on crisp arguments.*

Taking benefit of this concept, we can now express the following *quantification framework assumption* that must be fulfilled in order to put the proposed framework into operation:

> **Quantification framework assumption (QFA):**
> *If two base quantifiers of interest (i.e. NL quantifiers to be defined directly) have distinct interpretations* $\widetilde{Q} \neq \widetilde{Q}'$ *as fuzzy quantifiers, then they are already distinct on crisp arguments, i.e.* $\mathcal{U}(\widetilde{Q}) \neq \mathcal{U}(\widetilde{Q}')$.

The QFA condition ensures the applicability of the QFM framework because we can then represent $\widetilde{Q}, \widetilde{Q}'$ by $Q = \mathcal{U}(\widetilde{Q})$ and $Q' = \mathcal{U}(\widetilde{Q}')$, without compromising the existence of a QFM $\mathcal{F}$ which takes Q to $\widetilde{Q} = \mathcal{F}(Q)$ and Q' to $\widetilde{Q}' = \mathcal{F}(Q')$. If the QFA is violated by $\widetilde{Q}$ and $\widetilde{Q}'$, however, then it is impossible for any QFM to separate the quantifiers, because $\mathcal{U}(\widetilde{Q}) = \mathcal{U}(\widetilde{Q}')$ entails that the same interpretation $\mathcal{F}(\mathcal{U}(\widetilde{Q})) = \mathcal{F}(\mathcal{U}(\widetilde{Q}'))$ be assigned to both quantifiers.

The QFA expresses the linguistic postulate (universal principle) that there is no pair of base quantifiers in any natural language which coincide on two-valued arguments, but differ for fuzzy arguments. It should be pointed out that the linguistic theory of quantification, TGQ, depends on the very same assumption, because it restricts attention to two-valued arguments only. In doing this, it is silently presumed that crisp arguments will reveal sufficient structure to discuss the phenomenon of interest. Starting from its basic concept of a generalized quantifier, TGQ has conducted research into various facets of NL quantification, and gathered a solid stock of knowledge on such diverse issues as semantic universals, algebraic properties, expressive power, logical definability, and also on purely linguistic matters like ‘there-insertion’ or ‘negative polarity’ items.[15] It is remarkable that all these findings stem from a simplifying model of NL quantifiers which trusts to idealized, crisp inputs. Most of these results have undoubtedly something to say about NL quantification in general, and will likely persist in the fuzzy case. Judging from this perspective, the impressive body of knowledge that we owe to TGQ makes a bold point that the framework assumption be useful, because it demonstrates that the phenomena of linguistic interest can also be reproduced in the test-tube of crisp arguments.

At first sight, the fundamental assumption underlying the quantification framework appears uncritical, because it makes so elementary a requirement. However, there is good reason why I restricted the condition to some notion of ‘base quantifiers’, rather than expressing it unconditionally. Natural language is open-ended, and it is indeed possible to construct a few examples which must be treated with different methods. Thus, it all depends on the notion of base or ‘simplex’ quantifier $\widetilde{Q}$, which must be directly expressed in terms of semi-fuzzy quantifiers, i.e. as $\widetilde{Q} = \mathcal{F}(Q)$.

If the total of conceivable linguistic quantifiers are declared base quantifiers, then it is clear in advance that the QFA will be violated. This is because a continuous-valued logic cannot validate all axioms of Boolean algebra on formal grounds. Natural languages, however, permit us to express arbitrary Boolean combinations of a quantifier’s arguments. Starting from a pair of Boolean expressions which are equivalent in the two-valued, but not in the fuzzy case, we can thus construct a pair of quantifiers whose argument structure parallels these Boolean expressions. These quantifiers will coincide for crisp arguments, but not necessarily for fuzzy arguments, thus violating the QFA criterion. A simple example is $\widetilde{Q}(X) = \widetilde{\forall}(X \cup \neg X)$, or “Everything is X or not X”, versus $\widetilde{Q}'(X) = 1$ *const*, which paraphrases a constantly true proposition like “Everything belongs to the base set”, which by cylindrical extension (i.e. adding a vacuous variable) becomes a unary quantifier. The standard choice of fuzzy universal quanti-

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[15] see e.g. [6] (universals), [8] (algebraic properties), [81] (expressiveness), [9] (definability) and [70] (linguistic issues).

fier (infimum) and the standard set of connectives then results in $\widetilde{Q} \neq \widetilde{Q}'$, although $\mathcal{U}(\widetilde{Q}) = \mathcal{U}(\widetilde{Q}') = 1$ coincide.

It hence appears that we can treat a (large!) fragment of extensional NL quantifiers as base quantifiers, which are modelled directly in terms of a semi-fuzzy quantifier, i.e. $\widetilde{Q} = \mathcal{F}(Q)$. However, certain cases will escape this procedure. The interpretation of these non-simplex quantifiers must be constructed from the interpretation of other base quantifiers, i.e. they do not permit a reduction to crisp arguments and must hence be analyzed at the level of fuzzy quantifiers. To demonstrate which examples are base quantifiers and which are not, let us reconsider the many-valued quantifiers of Rescher [128, p. 199]. Identifying all intermediate truth values $z \in (0, 1)$ with the third truth value of Rescher's original definition, Rescher's quantifiers $\exists^I$, $\forall^T$, M^T and M^I already mentioned on page 27 in the introduction, can readily be extended to unary fuzzy quantifiers defined for all fuzzy arguments. These quantifiers, which make certain aspects of the many-valued interpretation accessible from the object language, exemplify the possibility of counter-examples that cannot be regarded as base quantifiers. Rescher's quantifiers (or apparent derivations) can also be realized linguistically, as shown by the following examples from ordinary English:

1. Many borderline-cases of bald men use hair restorers.
2. All clear cases of insanity are kept under strict observation.
3. All stationary patients are clear cases of dementia.

In order to demonstrate the problems associated with the modelling of Rescher's quantifiers, let us consider the quantifier $\widetilde{\exists}^I : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ defined by $\widetilde{\exists}^I(X) = 1$ if $\mu_X(e) \in (0, 1)$ for some $e \in E$, and $\widetilde{\exists}^I(X) = 0$ otherwise. The type IV quantification $\widetilde{\exists}^I(P)$ then expresses the linguistic pattern "There are borderline cases of P's". In order to check the validity of the QFA criterion in this case, let us suppose that $\widetilde{Q} = \widetilde{\exists}^I$, and the constantly false quantifier $\widetilde{Q}'(X) = 0$ which corresponds to a false proposition like "The base set is empty", both be base quantifiers. We then observe that $\mathcal{U}(\widetilde{\exists}^I) = \mathcal{U}(\widetilde{Q}') = 0$ is constantly false because for crisp arguments, one generally finds no borderline cases. Hence, there are two distinct quantifiers $\widetilde{Q} = \widetilde{\exists}^I$ and $\widetilde{Q}'$ expressible in NL which coincide for crisp arguments. This demonstrates that $\widetilde{\exists}^I$ cannot be considered a base quantifier, because the resulting semi-fuzzy quantifier coincides with the constantly false quantifier, but the expected interpretation $\widetilde{\exists}^I$ is clearly different from the constantly false fuzzy quantifier, thus contradicting the QFA criterion. Moreover, linguists have postulated a universal principle of 'variety' observed by all natural languages, which states that (restricting attention to crisp arguments, as is always done in linguistics), such 'trivial' or non-informative quantifiers like $\mathcal{U}(\widetilde{\exists}^I) = 0$, although available in mathematical models, never occur as basic, non-constructed quantifiers of a natural language [8, p. 452]. This provides further evidence that the representation $Q = \mathcal{U}(\widetilde{\exists}^I)$ must be rejected on linguistic grounds, i.e. $\widetilde{\exists}^I$ must be considered a constructed, rather than simplex quantifier. Obviously, this does not mean that $\widetilde{\exists}^I$ cannot be handled by the framework at all, it just means that it cannot be expressed directly in the form $\widetilde{\exists}^I = \mathcal{F}(Q)$, as stated in the QFA. The possible reduction of $\widetilde{\exists}^I$ to the interpretation of a base quantifier is obvious from its decomposition into "There are R's", where the proposition R abbreviates "borderline case of P". Thus, "borderline case of" is now viewed as part of the argument and no longer attached to the quantifier – an analysis which is equally plausible. Let $h : \mathbf{I} \longrightarrow \mathbf{I}$ be the mapping defined by $h(0) = h(1) = 0$ and $h(z) = 1$ otherwise. Then R can be computed from P by applying the 'linguistic hedge' h, i.e. $\mu_R(e) = h(\mu_P(e))$ for all $e \in E$. We can then express $\widetilde{\exists}^I$ in terms of the existential quantifier $\mathcal{F}(\exists)$, i.e. $\widetilde{\exists}^I(P) = \mathcal{F}(\exists)(R)$, assuming that $\mathcal{F}(\exists)$ coincides with $\exists$ in the crisp case.

Returning to the question of identifying non-simplex quantifiers, it appears that quantifying constructions which explicitly refer to the fuzziness, vagueness or determinacy of their arguments violate the QFA and must hence be viewed as constructed quantifiers. In particular, linguistic hedges can only be applied to the arguments of a fuzzy quantifier rather than a semi-fuzzy one, because modifications of membership grades are meaningless for crisp arguments. Thus quantifiers constructed from hedging are *not* base quantifiers. Such exceptions notwithstanding, the positive examples of base quantifiers clearly prevail in natural languages. They include lexicalized quantifiers like determiners ("most", "many"), discontinuous determiners ("more ... than"), and certain syntactically complex determiners like ("more than half"). A further clarification of the notions of a base quantifier versus constructed quantifier would certainly be desirable, although the majority of linguistic cases clearly belong to the non-constructed type which can be represented in terms of a semi-fuzzy quantifier.

2.8 A note on nullary quantifiers

In closing the chapter, I would like to add two technical remarks concerning the status of nullary quantifiers (this section) and the treatment of undefined quantifications in the proposed framework (next section). The reader will probably have noticed that the above definitions of semi-fuzzy and fuzzy quantifiers admit the case of quantifiers with arity $n = 0$. Here I want to briefly remark on these nullary quantifiers and defend my decision to include the nullary cases into the base definitions of quantifying operators. Let me note in advance that nullary quantifiers are artificial constructs (or boundary cases of quantifiers) which are not observed in natural language. This is because every NL quantifier must at least support one argument slot, which accepts the interpretation of the verbal phrase. For a minimal example, consider "Joan sleeps", where "Joan" is mapped to a one-place quantifier **joan** : $\mathcal{P}(E) \longrightarrow \mathbf{2}$, to which the extension **sleep** $\in \mathcal{P}(E)$ of the verb "sleep" is then inserted. The above example hence translates into **joan**(**sleeps**), which determines the resulting truth value from the known identity of "Joan" and the known set of sleeping people. This minimal example illustrates that the slot for the interpretation of the verb phrase is mandatory in natural language, which excludes the possibility of nullary quantification. Nevertheless, it can be pretty useful for theoretical investigations to have such boundary cases available (for example, they naturally arise from repeated argument insertions, to be discussed below in section 4.10). I have hence decided to define semi-fuzzy and fuzzy quantifiers accordingly. As we shall learn below in Def. 24, admitting quantifiers of arity $n = 0$ will indeed contribute to the succinct presentation of my axiom system for plausible models, which will refer to nullary quantifiers in its first and most fundamental axiom (Z-1).

In order to better understand what nullary quantifiers are, and also in order to justify my later notation for nullary quantification, we need to recall some very basic mathematics. If A is some set and $n \in \mathbb{N}$, then A^n is commonly taken to denote set of mappings from $n = \{0, \ldots, n-1\}$ to A, i.e. $A^n = A^{\{0, \ldots, n-1\}}$. Consequently, $A^0 = A^\varnothing = \{\varnothing\}$. A nullary semi-fuzzy quantifier $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$, i.e. $Q : \{\varnothing\} \longrightarrow \mathbf{I}$, is therefore uniquely determined by the value $Q(\varnothing) \in \mathbf{I}$ which it assumes for the *empty tuple*, written as $\varnothing$. This provides evidence of a one-to-one correspondence between nullary semi-fuzzy quantifiers $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ and elements $Q(\varnothing) = y \in \mathbf{I}$ in the unit interval. Acknowledging these dependencies, I will at times refer to nullary quantifiers as 'constant quantifiers'. To give an example, the quantifier $Q : \{\varnothing\} \longrightarrow \mathbf{I}$ defined by $Q(\varnothing) = \frac{1}{2}$, is a nullary semi-fuzzy quantifier, and $\textbf{true}, \textbf{false} : \{\varnothing\} \longrightarrow \mathbf{I}$, defined by $\textbf{true}(\varnothing) = 1$ and $\textbf{false}(\varnothing) = 0$, are of course also constant semi-fuzzy quantifiers.

Similar considerations apply to nullary *fuzzy* quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^0 \longrightarrow \mathbf{I}$. In this case, we obtain that $\widetilde{\mathcal{P}}(E)^0 = \widetilde{\mathcal{P}}(E)^\varnothing = \{\varnothing\}$, which again substantiates a one-to-one correspondence between nullary or 'constant' fuzzy quantifiers $\widetilde{Q} : \{\varnothing\} \longrightarrow \mathbf{I}$, and elements $\widetilde{Q}(\varnothing) \in \mathbf{I}$. In particular, the set $\mathbf{I}^{(\mathcal{P}(E)^0)} = \mathbf{I}^{\{\varnothing\}}$ of constant semi-fuzzy quantifiers over E coincides with the set of constant fuzzy quantifiers $\mathbf{I}^{(\widetilde{\mathcal{P}}(E)^0)} = \mathbf{I}^{\{\varnothing\}}$, for any considered base set $E \neq \varnothing$.

2.9 A note on undefined interpretation

In TGQ, the concept of two-valued quantifiers is often slightly extended, by also accounting for the possibility of undefined interpretations. It is hence admitted that certain quantifiers, like the definite quantifier "the" already mentioned above, might fail to denote in some cases. This suggests that the original definition of two-place quantifiers be weakened to partial (incompletely defined) mappings, which are allowed to be undefined at certain inputs. Alternatively, one might resort to a three-valued modelling, and hence represent the 'undefined' or 'undecided' case by a third truth value, as suggested by Barwise and Cooper [6, p. 171]. In this report, I have decided not to incorporate this extension to undefined interpretations into the base concept of two-valued quantifiers. In addition, the abstractions of semi-fuzzy and fuzzy quantifiers, which will be central to this report, are both modelled as total mappings as well. This decision to keep undefined interpretations out of the 'pure' base definition of two-valued quantifiers, rests upon the observation that the essential regularities of NL quantifiers can already be expressed at the simplified level of total, two-valued quantifiers, as witnessed by influential publications like [8, 9], which explicitly restrict attention to these well-behaved cases.[16] Other authors like Keenan & Stavi [82, p. 277] and Hamm [61] avoid undefined interpretations altogether and even stick to the Russelian analysis of definite quantifiers. The remaining literature only touches upon the fact that an extension to partial or three-valued quantifiers would be more adequate. The formal apparatus of

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[16] Obviously, this does not entail that all NL quantifiers be two-valued and total.

TGQ, however, is fully developed only for the two-valued, total case, in order to avoid the intricacies introduced by partial or three-valued mappings. In particular, the issue of how the refined quantifiers behave when supplied with the resulting three-valued inputs, is not elaborated in TGQ.[17] In other words, the definitions and theorems usually idealize from the 'real-world dirt' of undecided quantification results, and confine themselves to the simple case of total, two-valued quantifiers. From the perspective of fuzzy quantification, the proposed refinement into three-valued quantifiers is still too coarse, and it is rather held that a further refinement to the continuous-valued case be necessary, in order to achieve a more natural modelling of the non-idealized, approximate quantifiers that are typical of real languages. The refined descriptions of quantifiers that I made possible by introducing semi-fuzzy and fuzzy quantifiers, permit us to express all possible shades of a quantification result, which might tend to the crisp outcomes 0 (false) or 1 (true) only to some degree. In particular, the issue of undefined or undecided results neatly resolves into this more general proposal, because the improved descriptions also permit a quantifier to assume the undecided result (represented by $\frac{1}{2}$). As a by-product of this continuous-valued refinement and the subsequent adaptation of all relevant concepts of TGQ, I will also achieve a thorough development of the basic notions of TGQ for the three-valued subcase of quantifiers that assume their values in $\{0, \frac{1}{2}, 1\}$. In other words, semi-fuzzy quantifiers are rich enough to embed the extensions to three-valued quantifiers, and the corresponding fuzzy quantifiers can easily handle any three-valued arguments. To sum up, my analysis would not profit from supporting undefined interpretations, and the motivating cases for partial quantifiers are already accounted for by the continuous-valued model.

2.10 Chapter summary

In this chapter, I introduced a theoretical skeleton for models of fuzzy quantification which permits a coherent interpretation of a broad class of NL quantifiers. Recalling the characteristics of linguistic quantifiers discussed in the introduction, the framework was expected to support general multi-place quantifiers, non-quantitative examples and the like. The most useful departure for devising such a framework I found to be the theory of generalized quantifiers TGQ, which offers a suitable notion of (crisp) generalized quantifiers that only needs to be extended towards fuzzy quantification. By committing to the basic analysis of TGQ, the later translation of its conceptual apparatus to fuzzy quantifiers will become a straightforward task.

In the introduction, a framework for fuzzy quantification was described as comprising four parts: a description of its scope, which fixes the range of considered quantifiers; a definition of operational target quantifiers; a definition of specifications for such quantifiers; and finally interpretations which connect the specifications of quantifiers to their operational forms. Starting from the linguistic analysis, my goal was that of covering all quantifiers known to the linguistic theory (TGQ), however admitting approximate quantifiers and fuzziness in the arguments of a quantifier. To this

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[17] Barwise and Cooper [6, p. 171] merely suggest that Kleene's three-valued logic might provide the blueprint for this refinement. Interestingly, all standard models of fuzzy quantification developed below will comply with this strategy, see p. 166.

end, I first defined the suitable notion of fuzzy quantifiers which accounts for these considerations. In order to develop practical specifications, it was useful to adopt the fuzzification pattern described in the appendix. The framework thus introduces a system of intermediate representations which are justified by the assumption that certain variables (in this case, the arguments of the quantifier) be crisp, and a fuzzification mechanism for fuzzy quantifiers which translates these simplified descriptions to their operational forms, i.e. to fuzzy quantifiers which admit fuzziness in all variables.

The concrete proposal of semi-fuzzy quantifiers that I made to avail us with such descriptions rests on a clear separation between approximate quantifiers (fuzziness in the quantifier itself) and the problem of fuzzy arguments, the processing of which will be delegated to the particular models of fuzzy quantification. Thus, semi-fuzzy quantifiers are justified by the separate dimensions of fuzzy quantification identified by Liu and Kerre [99, p. 2], i.e. rows vs. columns in the table of 'fields of quantification' shown on p. 25. The plenty of advantages discussed in the previous sections make a strong point that semi-fuzzy quantifiers be practical base representations for NL quantifiers. They practically avoid the 'dilemma of fuzzy quantification' by taking an intermediate course between two-valued quantifiers (which are too weak) and fuzzy quantifiers (which are too expressive for specification purposes). Moreover the dependency of semi-fuzzy quantifiers on crisp arguments admits suggestive definitions based on crisp cardinalities. The proposed notion of semi-fuzzy quantifiers accomplishes the goal of embedding the quantifiers known to linguistics, and it also establishes a universal description format which suits all types of quantifiers. Absolute quantifiers like "about ten" and proportional examples like "most", for example, both map to binary quantifiers $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$. Semi-fuzzy quantifiers therefore achieve the desired independence of any particular types of quantifiers like absolute, proportional etc. Compared to fuzzy quantifiers, the definition of their semi-fuzzy cousins in terms of the cardinality of crisp sets eliminates a multiplicity of choices that were artifacts of an over-expressive modelling device. The compatibility of semi-fuzzy quantifiers with the familiar cardinality measure is a practical benefit for semi-fuzzy quantifiers, the relevance of which can hardly be overvalued.

The separation of specification and operational forms which is intrinsic to my framework then necessitates the stipulation of correspondences between given specifications and their associated target quantifiers. It is clear in advance that a satisfying account of fuzzy quantification can only be given if this passage from specification medium to operation medium be systematic and internally coherent. There should be clear rules for determining a target quantifier from its specification to avoid any ad-hoc choices. In order to be able to discuss distinct systems of such rules, these rule systems, which associate fuzzy quantifiers with given base descriptions, must become an explicit part of my model. I therefore postulated the concept of fuzzification mechanisms to express these correspondences, which encapsulate all details concerning the mapping from specifications to target quantifiers.

By the quantification framework, I thus mean the triple of semi-fuzzy quantifiers (specification medium), fuzzy quantifiers (operation medium) and QFMs (map of correspondences). The scope of the framework spans both the generalized quantifiers of TGQ and the Type IV quantifications of fuzzy set theory. The framework thus includes a variety of extensional quantifiers which are permitted to be multi-place and non-quantitative (i.e. not necessarily definable in terms of cardinalities). In addition, there is no a priory limitation to finite domains.

When discussing the quantification framework assumption, I discussed a few 'exotic' quantifiers which cannot be reduced to a semi-fuzzy quantifier because they then become meaningless. The prime example are Rescher's quantifiers or more generally, complex quantifiers built from a hedging construction. In the following, I would like to comment on further quantificational phenomena not accounted for by my proposal; all of these are also not handled by the linguistic theory and of course, they cannot be treated by existing work on fuzzy quantification. The necessary extensions to cover these cases are sometimes obvious from the usual treatment of these phenomena in logic and linguistics. However, I must concentrate on the core issues of fuzzy NL quantification in this work, and the phenomena not treated do not seem to be intrinsically connected to linguistic quantification as such.

Higher-order quantifiers, to begin with, are not accounted for by the proposed framework. As remarked by Bocheński [16, (48.21)], the distinction between first order propositions (which refer to individuals) and propositions of second order or higher (which refer to sets of individuals, sets of such sets etc.) originates with Russell [134, p. 236+]. The corresponding higher-order logics, and even the full theory of finite types [134, 26, 1], were found to be valuable tools for linguistic description. To keep things simple, however, my current framework only formalizes first-order quantification. Most linguistic quantifiers are first-order but there are also examples of higher-order quantification. As remarked by Keenan and Stavi, these express in 'higher-order' NPs like "two sets of dishes" or "a pride of lions", whose denotations *"arguably include properties of collections of individuals rather than of individuals"* [82, p. 256].

The basic framework further abstracts from so-called *branching quantification*, in which the linear sequence of nested quantifiers is replaced with partial dependencies, thus admitting several quantifiers to operate in parallel and independently of each other [66, 157]. As already mentioned, the relevance of branching quantification to the analysis of language has been a matter of linguistic debate [122, 68, 5, 158], and I take sides with Barwise that branching quantification is indeed necessary to describe certain reciprocal constructions. Thus, I will show in the later Chap. 12 how the basic framework can be extended towards branching quantification. This is also a first step towards fuzzy higher-order quantification because branching quantification, as noted by Barwise, is just a human language technique to conceal the use of certain higher-order constructions.

The quantifiers of TGQ are *extensional*, i.e. they operate on sets of individuals (extensions) rather than 'meanings' or 'intensions' (usually represented as mappings from states of affairs into extensions), so there is no indexing by some parameter which corresponds to states of affairs. As pointed out by Barwise and Cooper [6, p. 203], they deliberately restricted themselves to an extensional fragment of English, striving to highlight the central issues. For pretty much the same reasons, the generalizations of semi-fuzzy and fuzzy quantifiers that I proposed were also chosen to be extensional. The following characterization of extensional determiners due to Keenan and Stavi also extends to the more general types of fuzzy quantifications:

> *"To say that a det d is extensional is to say, for example, that whenever the doc[t]ors and the lawyers are the same individuals then d* doctors *and d* lawyers *have the same properties, e.g., d* doctors attended the meeting *necessarily has the same truth value as d* lawyers attended the meeting*. Thus* every *is extensional, since if the doctors and the lawyers are the same then, every doctor attended iff every lawyer did"* [82, p. 257].

This characterization results in a logic built 'on top' of these quantifiers[18] to be extensional as well, according to the usual 'principle of extensionality' [45, p. 5],

$$\chi \leftrightarrow \chi' \models \varphi \leftrightarrow [\chi'/\chi]\varphi$$

i.e. substitutivity of expressions with identical reference. Following Frege [43], there is a fundamental difference between the sense ('Sinn' in Frege's terms, i.e. meaning, intension) of natural language expressions and their reference ('Bedeutung' in Frege's terms, i.e. extension). Specifically, as shown by Quine [121], there are certain positions in linguistic expressions called 'opaque contexts' in which the substitution of coextensional terms is no longer admissible but rather alters the overall meaning. Thus a complete model of linguistic quantification will likely have to deal with intensional arguments. And the quantifier itself might also be intensional, like "all alleged" [8, p. 448] or "an undisclosed number of" [82, p. 257]:

> *"It is not hard to imagine a situation in which the doctors and the lawyers are the same and but [sic]* an undisclosed number of lawyers attended the meeting *is true and* an undisclosed number of doctors attended the meeting *is false. (Imagine a meeting of medical personnel. The chairman announces the number of doctors in attendance but not the number of lawyers)"* [82, p. 257].

Thus, the principle of extensionality fails in this case. However, *"the restriction to extensional models looks more severe than it actually is. In fact there are not many really intensional NPs or determiners"*, see Gamut [45, p. 228]. In addition, it is relatively clear how to integrate intensional features into a given system of logic. The standard method of accomplishing this was developed by S. Kripke [93, 92], i.e. one must add a primitive notion of a possible state of affairs (or 'possible world' in modal logic), an accessibility relation on states, and define the semantics in terms of these states and their accessibility. This basic approach underlies Montague's intensional logics [106, 107], for example. Should there evolve a need to treat intensional quantifiers as well, an analogous extension of a logic with generalized quantifiers and the corresponding semantical concepts would be possible. For reasons of simplicity and readibility, though, I will not pursue this direction in the report.

Finally the Theory of Generalized Quantifiers, and my proposed extensions as well, specifically target at 'count NPs' [82, p. 256] or 'discrete quantifiers' [8, p. 448] like "a lot of" or "many", rather than so-called 'mass quantifiers' or 'continuous quantifiers'

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[18] such a logic with generalized quantifiers has been developed by Barwise and Feferman [7, Ch. 2].

like “much” or “little”. This peculiarity gains some weight as soon as we broaden the range of NL quantifiers towards temporal or spatial quantification. In this case, the underlying regions in time and space are possibly best modelled as atomless masses. Hence a comprehensive model will need to support some form of ‘mass quantification’. The present work tries to provide a good starting point for developing a theory of mass quantification, by developing all of its concepts for base sets of both finite and infinite cardinality. It is not the goal of this report to develop a theory of fuzzy mass quantification, however. To this end, it would certainly be necessary to incorporate additional elements from the theory of fuzzy measures.

Now that the scope of the framework and its possible extensions have been clarified, we shall contemplate the separation of responsibilities inherent to the framework, in order to identify the directions to be pursued in the remainder of the report. In the introduction, I briefly discussed the sources of imperfection in NL, and specifically, I explained the difference between vagueness (fuzziness) and context-dependence. This classificatory distinction, as well as the ubiquity of both sources of imperfection, is also recognized in fuzzy set theory:

> *“Linguistic concepts are not only predominantly vague, but their meanings are almost invariably context-dependent as well”,*

see Klir and Yuan [89, p. 280]. The meanings of linguistic operations, and quantifiers in particular, make no exception in this respect. Therefore the meaning of linguistic quantifiers can only be fixed once the relevant context is known (e.g. for a given application). However, as pointed out by Klir and Yuan [89, p. 281],

> *“The problem of constructing membership functions that adequately capture the meanings of linguistic terms employed in a particular application, as well as the problem of determining meanings of associated operations on the linguistic terms, are not problems of fuzzy set theory* per se.”

In a way, this problem has something to say about natural language and its relationship to the proposed modelling devices, not about the formal models themselves. (Not surprisingly, then, the problem is also shared both by semi-fuzzy quantifiers and fuzzy linguistic quantifiers μ_Q). Roughly speaking, the less precise or specific the representations of linguistic quantifiers, the more uniquely can they be established. Obviously (as reflected by the QFA), the representations must also be powerful enough to uniquely identify the quantifiers of interest. Thus, a compromise must be sought, and semi-fuzzy quantifiers achieve a reasonable trade-off between nonspecificity and expressive power. Concerning the division of responsibility, it is clear in advance that the particular quantifiers to be modelled involve ‘particulars’ and are thus, idiosyncratic. Consequently the specifications of the relevant quantifiers cannot be determined by a general theory, they must rather be fixed as part of designing an application. It is up to a software engineer or domain expert to select an appropriate system of such base descriptions which capture the meaning of the target quantifiers in the given application. The methodology for interpreting these specifications of quantifiers, however, is independent of the particular decisions regarding the intended specification. By contrast, it is supposed to ensure that *all* semi-fuzzy quantifiers be consistently generalised to corresponding fuzzy quantifiers. In order to develop the theory of fuzzy quantification without getting stuck in context-dependence, it was necessary to introduce the universal specification medium, which encapsulates the context-dependence and other idiosyncratic aspects of linguistic quantification. The theory of quantification, by contrast, must explain how such specifications be interpreted systematically. It is here that I expect observable regularities which should be laid open by subsequent research and distilled into formal criteria. In the remainder of the report, then, I will be concerned with this complex problem. The QFMs which I introduced as some kind of place-holder for the possible correspondence assignments are a prerequisite of formalizing the underlying dependencies between specifications and target operations. By making the correspondence assertions first-class citizens of my theoretical framework, it becomes possible to develop the desired formal criteria which let us assess the perceived quality of an interpretation. By adding more and more such criteria, we can subsequently prune the class of considered models, until the hidden regularities are eventually uncovered that discern intended models from implausible assignments. This process, which allows an incremental refinement, will culminate in a systematical description of plausible models called 'determiner fuzzification schemes', which I will present in the next chapter. In any case, a plausible model defined on the proposed specifications is expected to correctly extrapolate the meaning of simple specifications to the general case of a fuzzy quantifier supplied with fuzzy arguments.

Thus we can further the theory of fuzzy quantification by clearly isolating the context-dependent factors. Concerning a fuzzy sets-based modelling in general, Klir and Yuan have the following remark [89, p. 281]:

> *"while context dependency is not essential for developing theoretical resources for representing and processing linguistic concepts, it is crucial for applications of these tools to real-world problems. That is, a prerequisite to each application of fuzzy set theory are meanings of relevant linguistic concepts expressed in terms of appropriate fuzzy sets as well as meanings of relevant operations on fuzzy sets."*

This difficulty of establishing a precise interpretation is clearly a matter of language and cannot be attributed to the modelling devices chosen for formalization. The problem must therefore be approached by a more general methodology of assessing the meaning of linguistic terms:

> *"These problems belong to the general problem of* knowledge acquisition *within the underlying framework of fuzzy set theory. That is, fuzzy set theory provides a framework within which the process of knowledge acquisition takes place and in which the elicited knowledge can efficiently be represented",*

see Klir and Yuan [89, p. 281]. The application designer who decides on the interpretation of the supported quantifiers should therefore resort to methods for knowledge acquisition which are suitable for constructing fuzzy sets. There are several techniques for determining membership grades which are also applicable to the membership functions of fuzzy quantifiers. A survey of these techniques including direct methods, indirect methods, and approaches based on artifical neural networks (ANNs), is presented in Klir and Yuan [89, Chap. 10, p. 280-301], who also give pointers into the specialized literature.

These methods for acquiring membership functions of fuzzy sets (and thus, for defining semi-fuzzy quantifiers), merely offer some kind of engineering solution however, i.e. they target at a technical answer to the problem, but do not account for any empirical findings on the cognitive representation of linguistic quantifiers. It might hence be interesting to take a look into the psychological literature. Indeed, there are a few empirical studies on the context dependence observed in the meaning of approximate quantifiers. Some promising research into these directions has been carried out by S.E. Newstead and his co-workers [114, 113], who report their findings on the shift in a quantifier's meaning relative to a change in external factors like size of the participating objects, total number of objects etc. The list of considered quantifiers comprises "a few", "few", "several", "many", and "lots of". The empirical basis on the context-dependence of approximate quantifiers is still very scarce, though, and it is not yet very clear how a general understanding of context-dependence might evolve from such isolated pieces of information.

So far, we have been concerned with the possible support for the specification process (i.e. determining a matching choice of semi-fuzzy quantifier), considering both the generic methods for the acquisition of membership functions that are already available today, and potential guidelines derived from psychological experiments that might shape future proposals. Another topic of interest is the *classification of semi-fuzzy quantifiers* into the main classes like absolute, proportional, cardinal comparative etc., which parallel the corresponding types of quantifiers in languages. However, I cannot pursue these interesting directions further in this report because the very complex problem of interpreting the given specifications consistently will absorb my full and undivided attention. Regardless of the specification chosen for a quantifier of interest, we cannot proceed unless we have a plausible interpretation mechanism for carrying out the desired type IV quantifications. The next chapter will hence be concerned with defining a class of such plausible models.

3 The axiomatic class of plausible models

3.1 Motivation and chapter overview

In this chapter, an attempt is made to formalize the precise requirements that must be obeyed by ‘reasonable’ approaches to fuzzy quantification, and hence to suggest an axiomatic class of plausible models. This formalization will build on the quantification framework established in the previous chapter. The suggested framework provides a rich domain of approaches to fuzzy quantification, which is broad enough to contain the models of interest. The ‘raw’ approaches (and potential models) in this class will now be subjected to further study and selection. The primary goal is that of tayloring these totally unrestricted QFMs to a class of true *models* of fuzzy quantification, and hence make explicit the intuitive expectations on plausible interpretations of fuzzy quantifiers. Emphasis will be placed on a sound methodology and uncompromising formal rigor, which targets at a solid axiomatic foundation of fuzzy natural language quantification.

From a methodical perspective, it seems advantageous to decompose the superordinate goal of achieving plausible interpretations of fuzzy quantifiers and separate the following two factors, which both decide upon the success of the overall endeavour. In order to adequately model a given NL quantifier, we need

1. a plausible choice of base descriptions (i.e. suitable semi-fuzzy quantifiers), which is due to the programmer or application designer; and
2. a plausible translation into corresponding fuzzy quantifiers, which is under the responsibility of the fuzzification mechanism.

While the first aspect is considered application-specific and not easily susceptible to systematic study, the second aspect of plausible translation is at the heart of the report, and it is now time to fully enter into this topic. The present chapter is exclusively devoted to the analysis of those generic aspects of fuzzy quantification, which seemingly control the transfer from the base representation of semi-fuzzy quantifiers into the target operators of full-fledged fuzzy quantifiers, and hence potentially resolve the problem of adequate translation, and guide the development of a systematic, well-founded solution. Roughly speaking, I expect a model of fuzzy quantification to be internally coherent and in conformance to linguistic considerations. In addition, these properties should be ascertained for arbitrary types of quantifiers, in order to avoid the tedious justification of plausible interpretations on an individual case basis.

In order to achieve these goals with the desired generality, I will pursue a strategy which is essentially algebraic: rather than making any claims on ‘the meaning’ of a natural language quantifier, and its corresponding model as a fuzzy quantifier, I hence assume that most (if not all) important aspects of the meaning of a quantifier express in terms of its observable behaviour. The considered model of fuzzy quantification can then be required to transport that behaviour which shows up at the base level, to the generalized situation of fuzzy quantifiers. In other words, the reduction of fuzzy quantification to quantifier fuzzification mechanisms, permits me to express linguistic adequacy requirements on 'intended' approaches to fuzzy quantification in terms of preservation and homomorphism properties of the corresponding fuzzification mappings. The resulting axioms are mainly of the 'preservation' or 'compatibility' (homomorphism) type, i.e. I require that

- $\mathcal{F}$ maps particular (simple) semi-fuzzy quantifiers to their obvious fuzzy analogues;
- $\mathcal{F}$ preserves relevant properties and relationships of quantifiers;
- $\mathcal{F}$ commutes with certain operations for building new quantifiers from given ones.[19]

For example, if P is a property of semi-fuzzy quantifiers, and P' is the corresponding property of fuzzy quantifiers, we can ask if a QFM $\mathcal{F}$ preserves P in the sense that whenever $P(Q)$ holds for a semi-fuzzy quantifier Q, we also have $P'(\mathcal{F}(Q))$, i.e. the corresponding property holds for the fuzzy quantifier $\mathcal{F}(Q)$ associated with Q. Likewise, if C is a construction on semi-fuzzy quantifiers (which builds a new semi-fuzzy quantifier $C(Q)$, given Q), and C' is the corresponding construction on fuzzy quantifiers, we can ask if $\mathcal{F}$ is compatible with the construction, in the sense that we always have $\mathcal{F}(C(Q)) = C'(\mathcal{F}(Q))$, i.e. it does not matter whether we first apply the construction and then 'fuzzify' using $\mathcal{F}$, or whether we first apply $\mathcal{F}$ and then perform the construction.

The particular properties that will be considered, and those relationships between quantifiers which bear linguistic relevance, will usually be adopted from the linguistic theory of natural language quantification, TGQ. This seems beneficial in order to answer the intuitive expectations on the interpretation of linguistic quantifiers. It is here that the conceptual superiority of the proposed framework over existing approaches becomes evident, which is much better suited for implementing the algebraic approach. To be specific, the membership functions $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ or $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ that existing approaches build on, do not have direct counterparts in NL description and hence lack a straightforward and unambiguous semantical interpretation. This severely impedes an application of the available linguistic criteria, which might assess the adequacy of existing approaches from the natural language perspective. In addition, the assumed membership functions do not provide a uniform representation for describing the core behaviour of the target quantifiers. A possible attempt to develop formal plausibility criteria on the level of the μ_Q, would inevitably need to discern the heterogeneous base formats. Consequently, a given linguistic criterion would typically need to be fitted to all relevant types of base representations (i.e. absolute quantifiers, proportional quantifiers, and possibly additional kinds), and hence split up into a bunch of individual cases, rather than finding a unifying formalization.

By contrast, the quantification framework pursued here rests on the uniform base representation offered by semi-fuzzy quantifiers, which simply express the behaviour of the target NL quantifier on crisp arguments, and hence come with an obvious and

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[19] in the sense that it does not matter whether we first apply the operation and then apply $\mathcal{F}$, or whether we first apply $\mathcal{F}$ and then apply the (fuzzy analogue of the) considered operation.

clear cut semantics. In particular, the proposed framework is supreme over existing approaches, because both its source and target representations (semi-fuzzy and fuzzy quantifiers) are sufficiently close to the notion of two-valued quantifiers assumed by the linguistic theory of quantification, in order to allow a straightforward adaptation of the concepts of TGQ which describe important aspects of quantifiers. In the report, we will find ample evidence that it is usually easy to reformulate these properties of quantifiers, relationships between quantifiers, and constructions on quantifiers into corresponding properties, relationships and constructions on semi-fuzzy quantifiers and fuzzy quantifiers. Embarking on the algebraic approach, I will then require that those aspects of the meaning of semi-fuzzy quantifiers which express in these criteria be preserved when applying $\mathcal{F}$. As we shall see, TGQ knows about most important aspects of a quantifier's behaviour and successfully forges these into precise definitions. In particular, the proposed models which satisfy the adopted criteria, will indeed respond to a great deal of intuitive adequacy concerns.

I should emphasize, though, that only a very modest number of these criteria will be required explicitly, and directly show up in the defining axioms for plausible moodels. This is because the axiom system should not only fulfill the superordinate goal, and hence precisely describe the intended approaches to fuzzy quantification. By contrast, the axioms should also account for certain design principles, which shape their appearance and guide their specific compilation into the total system. In particular, the conditions should be mutually independent, and hence capture independent aspects or dimensions in the space of models. This requirement of independence is essential from a methodical point of view because it avoids any redundancy in proofs, which might arise from superfluous conditions. In order to meet this criterion, the proposed axiom system must be minimal, i.e. irreducible to a smaller axiom set. Of course, the system must still cover the full set of desired plausibility criteria, which then become part of its deductive hull.

Let me now enter into particulars and introduce the properties, constructions and other required concepts which culminate into the proposal of a concrete axiom system, the so-called ‘DFS axioms’. The rigid axiomatic foundation which is achieved by the suggested axioms, is of particular importance in the context of fuzzy logic where – even in the case of the propositional connectives – a large number of alternative interpretations have been proposed. The DFS axioms, then, will provide a criterion to discern well-founded approaches to fuzzy quantification from ad-hoc ones.

3.2 Correct generalization

Perhaps the most elementary condition on a quantifier fuzzification mechanism is that it properly generalizes the original semi-fuzzy quantifier. We can express this succinctly if we recall the notion of an underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q})$ which simply restricts the fuzzy quantifier $\widetilde{Q}$ to crisp arguments, see Def. 7. It is natural to assume that a model of fuzzy quantification satisfies

$$\mathcal{U}(\mathcal{F}(Q)) = Q\,, \tag{14}$$

for all choices of semi-fuzzy quantifier Q, which means that $\mathcal{F}(Q)$ properly generalizes Q in the sense that

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$$

whenever all arguments are crisp, i.e. provided that $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Concerning the status of this requirement, 'correct generalization' is rather fundamental and should not be confused with the later linguistic postulates which express concepts of TGQ. By contrast, the requirement of 'correct generalization' is necessary to ensure the internal coherence of the QFM, and decides upon its capacity as a fuzzification mechanism. This is because it requires the 'downward compatibility' of the model with respect to to the original specification in terms of crisp arguments, and hence enforces the basic success condition of the fuzzification pattern, which underlies the quantification framework. Due to this enabling role for the proposed framework, the above equality (14) will constitute my first requirement imposed on plausible models of fuzzy quantification. I should note in advance, however, that the corresponding criterion (Z-1) in the final axiom system (stated in Def. 24) will be restricted to the case of nullary and monadic quantifiers ($n \leq 1$). This serves the purpose of keeping the axioms simple. When taken together, these axioms will of course entail the original criterion, and hence guarantee that equality (14) holds for quantifiers of arbitrary arities $n \in \mathbb{N}$.

Let us now consider some examples involving such quantifiers of arities $n \leq 1$. As concerns monadic (one-place) quantifiers, let us assume that $E \neq \varnothing$ is a given domain of persons. We shall further assume that there are some men among the persons in E, but no children. Hence the extensions of "men" and "children" in E, $\textbf{men}, \textbf{children} \in \mathcal{P}(E)$, satisfy $\textbf{men} \neq \varnothing$ and $\textbf{children} = \varnothing$, respectively. Now consider the existential quantifier $\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$, which determines the quantification results of $\exists(\textbf{men}) = 1$ and $\exists(\textbf{children}) = 0$. We certainly expect the fuzzification process to preserve these interpretations, and application of the associated fuzzy quantifier $\mathcal{F}(\exists) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ should result in $\mathcal{F}(\exists)(\textbf{men}) = 1$ and $\mathcal{F}(\exists)(\textbf{children}) = 0$.

Let us now turn to the case of nullary quantifiers ($n = 0$). It has already been remarked in section 2.8 that the sets of nullary semi-fuzzy quantifiers and nullary fuzzy quantifiers coincide, regardless of the chosen base set $E \neq \varnothing$. Hence for every nullary semi-fuzzy $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$, the considered quantifier *itself* already qualifies as a fuzzy quantifier, and there is nothing to generalize or to fuzzify. It is hence essential that these quantifiers be mapped to themselves by every plausible choice of a QFM. Now returning to the above equality (14), it is easily observed that this is precisely what $\mathcal{U}(\mathcal{F}(Q)) = Q$ states in the case of nullary Q. This is because both the nullary semi-fuzzy quantifier Q, as well as the nullary fuzzy quantifier $\mathcal{F}(Q)$, are uniquely determined by their value obtained at the empty tuple $\varnothing$. By Def. 7, then, we obtain from $\mathcal{U}(\mathcal{F}(Q)) = Q$ and the fact that the empty tuple is crisp that these values coincide, i.e. $\mathcal{F}(Q)(\varnothing) = \mathcal{U}(()\mathcal{F}(Q))(\varnothing) = Q(\varnothing)$. Hence Q and $\mathcal{F}(Q)$ coincide, as desired. To give an example, consider the nullary quantifier $\textbf{true}$ defined by $\textbf{true}(\varnothing) = 1$. A conforming choice of $\mathcal{F}$ should result in $\mathcal{F}(\textbf{true}) = \textbf{true}$, in order to catch the semantics of the constant quantifier.

This completes the discussion of the first requirement which must be fulfilled by all models of fuzzy quantification.

3.3 Membership assessment

Membership assessment (crisp or fuzzy) can be modelled through quantification. For an element e of the given base set, we can define a two-valued quantifier π_e which checks if e is present in its argument. Similarly, we can define a fuzzy quantifier $\widetilde{\pi}_e$ which returns the degree to which e is contained in its argument. It is natural to require that the crisp quantifier π_e be mapped to $\widetilde{\pi}_e$, which plays the same role in the fuzzy case.

In order to express this precisely on the formal level, I will now introduce this special type of semi-fuzzy quantifiers (and a corresponding type of fuzzy quantifiers) that allow us to treat membership assessments as a special case of quantification. Let us first recall the concept of a *characteristic function* and agree on the following notation.

Definition 8

Suppose E is a given set and $A \in \mathcal{P}(E)$ a (crisp) subset of E. By $\chi_A : E \longrightarrow \mathbf{2}$ we denote the characteristic function *of A, i.e. the mapping defined by*

$$\chi_A(e) = \begin{cases} 1 & : \quad e \in A \\ 0 & : \quad e \notin A \end{cases}$$

for all $e \in E$.

Building on this concept, I suggest the following definition of projection quantifiers, which covers the case of crisp arguments:

Definition 9

Suppose E is a base set and $e \in E$. The projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ *is defined by*

$$\pi_e(Y) = \chi_Y(e)\,,$$

for all $Y \in \mathcal{P}(E)$.

Notes

- The use of the term ‘projection quantifier’ is motivated by the observation that we can view π_e as the projection of the ‘E-tuple’ $\chi_A : E \longrightarrow \mathbf{2}$ onto that component $\chi_A(e)$, which is indexed by e.
- To present an example, suppose $E = \{\mathrm{Joan}, \mathrm{Lucas}, \mathrm{Mary}\}$ is a set of persons and **married** $\in \mathcal{P}(E)$ the subset of married persons in E. Then **joan** $= \pi_{\mathrm{Joan}} : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is a projection quantifier, and

$$\begin{aligned}\textbf{joan}(\textbf{married}) &= \pi_{\mathrm{Joan}}(\textbf{married}) \\ &= \begin{cases} 1 & : \quad \mathrm{Joan} \in \textbf{married} \\ 0 & : \quad \mathrm{Joan} \notin \textbf{married} \end{cases}\end{aligned}$$

> The example illustrates that π_{Joan} can indeed be used to evaluate statements of the type "Is Joan X?", where X is a crisp predicate. In particular, it is suitable for modelling the interpretation of the proper name "Joan" as a two-valued quantifier (provided that the individual Joan is contained in the considered domain). See [6, p. 164+] for a linguistic case that proper names be treated as a special type of NL quantifiers.

A corresponding definition of fuzzy projection quantifiers is straightforward.

Definition 10
Let a base set E be given and $e \in E$. The fuzzy projection quantifier $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is defined by*

$$\widetilde{\pi}_e(X) = \mu_X(e)$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Notes

- Again, I have coined the term 'fuzzy projection quantifier' because the mapping $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ can be viewed as the projection of the 'E-tuple' $\mu_X : E \longrightarrow \mathbf{I}$ onto its component $\mu_X(e)$, which is indexed by e.
- Fuzzy projection quantifiers provide the missing class of operators, which are suited to model gradual membership assessments like "To which grade is Joan X?" through fuzzy quantification. For example, we simply need to evaluate $\widetilde{\pi}_{\mathrm{Joan}}(\textbf{tall})$ in order to assess the grade to which Joan is tall, and we can compute $\widetilde{\pi}_{\mathrm{Joan}}(\textbf{rich})$ to determine $\mu_{\textbf{rich}}(\mathrm{Joan})$, the degree to which Joan is rich. In particular, the fuzzy projection quantifier $\widetilde{\pi}_{\mathrm{Joan}} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is suitable for interpreting the proper name "Joan" as a fuzzy quantifier (again under the presupposition that the individual Joan be present in the considered base set).

It is apparent from the above reasoning that crisp and fuzzy projection quantifiers play the same basic role of crisp/fuzzy membership assessment. Intuitively, a plausible model of fuzzy quantification should be compatible with the fundamental operation of membership assessment, and hence assign to each crisp projection quantifier π_e its obvious fuzzy counterpart, the corresponding fuzzy projection quantifier $\widetilde{\pi}_e$. In other words, we expect that

$$\mathcal{F}(\pi_e) = \widetilde{\pi}_e \,,$$

regardless of the chosen base set $E \neq \varnothing$ and considered element $e \in E$.
This makes the second requirement to be imposed on all models of fuzzy quantification.

3.4 The induced propositional logic

The linguistic theory of quantification knows a various constructions on natural language quantifiers which involve the use of a Boolean connective (like negation) or corresponding set-theoretic operation (e.g. complementation). Examples comprise the negation of quantifiers, antonyms, duals, intersections and unions in the arguments of a quantifier, conjunctions and disjunctions of quantifiers, and others.[20] In order to generalize these constructions on two-valued quantifiers to the case of semi-fuzzy and fuzzy quantifiers, the Boolean connectives must be replaced with suitable continuous-valued counterparts. Similarly, the move from two-valued quantifiers to fuzzy quantifiers necessitates the use of set-theoretic operations on fuzzy sets, and hence forces us to select a particular fuzzy complement, intersection and union, in terms of which the generalized constructions will then be expressed. Of course, one could simply resort to the standard choices, like standard negation $1 - x$, conjunction $\min$ and disjunction $\max$, but this would make too strong a commitment and foreclose the theoretical analysis of broader classes of models, which rely on general fuzzy negations, conjunctions and disjunctions.

The intent to cover such general models as well, hence forces us to allow for general fuzzy set operations (rather than the standard choices), which are then used to define the constructions on quantifiers. We must then permit the choice of connectives to depend on the considered QFM, and hence fit these constructions to the model. In addition to associating this canonical choice of fuzzy operations to the QFM of interest, we must further justify the particular selection made, which should be well-motivated.

Again, it would be tedious and obstructive to formal analysis, if the required decisions were made on an individual case basis. As in the case of QFMs, the intended formal treatment necessitates the development of a general solution which controls the transfer from propositional functions to corresponding fuzzy truth functions under a given QFM. This construction must be fully general and hence applicable to arbitrary QFMs.

In order to implement this approach, and hence devise a construction parametrized by the QFM, which assigns canonical choices of 'induced' fuzzy truth functions and induced fuzzy set operations to the corresponding crisp concepts, we need a natural embedding of propositional functions into semi-fuzzy quantifiers, to which the QFM of interest can then be applied. In turn, we then need an inverse construction which determines the target fuzzy truth function from the resulting fuzzy quantifier. The required operations on fuzzy sets can then be defined from the resulting fuzzy connectives in the apparent way, i.e. by applying the fuzzy connectives to the observed membership grades.

I have investigated two alternative schemes which rest on distinct embeddings and corresponding inverse constructions, that are motivated from independent considerations. We shall see later in theorem Th-8 that in all intended models, both constructions establish the same canonical choice of induced truth functions This indicates that the construction of induced fuzzy truth functions, which I introduce now, indeed results in the appropriate choice of connectives and fuzzy set operations for the given QFM (more details on the alternative construction can be found below in section 4.4).

In order to establish the link between logical connectives and quantifiers, we first observe that $\mathbf{2}^n \cong \mathcal{P}(\{1, \ldots, n\})$, using the bijection $\eta : \mathbf{2}^n \longrightarrow \mathcal{P}(\{1, \ldots, n\})$ de-

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[20] All mentioned constructions will be formally defined and discussed later on.

fined by

$$\eta(x_1, \ldots, x_n) = \{k \in \{1, \ldots, n\} : x_k = 1\}\,, \tag{15}$$

for all $x_1, \ldots, x_n \in \mathbf{2}$. We can use an analogous construction in the fuzzy case. We then have $\mathbf{I}^n \cong \widetilde{\mathcal{P}}(\{1, \ldots, n\})$, based on the bijection $\widetilde{\eta} : \mathbf{I}^n \longrightarrow \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ defined by

$$\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(k) = x_k\,, \tag{16}$$

for all $x_1, \ldots, x_n \in \mathbf{I}$ and $k \in \{1, \ldots, n\}$. These bijections can be utilized for a translation between semi-fuzzy truth functions (i.e. mappings $f : \mathbf{2}^n \longrightarrow \mathbf{I}$) and corresponding semi-fuzzy quantifiers $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$, which implements the embedding of propositional functions into semi-fuzzy quantifiers, and similarly the translation from fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ into fuzzy truth functions $\widetilde{f} : \mathbf{I}^n \longrightarrow \mathbf{I}$, which implements the required inverse embedding.

Definition 11

Suppose $\mathcal{F}$ is a QFM and $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ is a mapping (i.e. a 'semi-fuzzy truth function') for some $n \in \mathbb{N}$. The semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ is defined by

$$Q_f(Y) = f(\eta^{-1}(Y))$$

for all $Y \in \mathcal{P}(\{1, \ldots, n\})$. In terms of Q_f, the induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ *is then defined by*

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n))\,,$$

for all $x_1, \ldots, x_n \in \mathbf{I}$.

Notes

- If $f : \mathbf{2}^0 \longrightarrow \mathbf{I}$ is a nullary semi-fuzzy truth function (i.e., a constant), then $Q_f : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$ turns into a nullary semi-fuzzy quantifier on the universe $E = \{\varnothing\}$, noticing that $\mathcal{P}(\varnothing) = \mathcal{P}(\{\varnothing\})^0 = \{\varnothing\}$. Hence in this case, the fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^0 \longrightarrow \mathbf{I}$ can be expressed as $\widetilde{\mathcal{F}}(f)(\varnothing) = \mathcal{F}(c)(\varnothing)$, where $c : \mathcal{P}(\{\varnothing\})^0 \longrightarrow \mathbf{I}$ is the constant $c(\varnothing) = f(\varnothing)$.
- We shall not impose restrictions on the induced connectives directly; these will be entailed by the remaining axioms.
- Whenever $\mathcal{F}$ is clear from the context, we shall abbreviate $\widetilde{\mathcal{F}}(f)$ as $\widetilde{f}$. For example, the induced disjunction will be written $\widetilde{\vee}$.

Induced operations on fuzzy sets, i.e. fuzzy complement $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$, fuzzy intersection $\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ and fuzzy union $\widetilde{\cup} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$, can be defined element-wise in terms of the induced negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$, conjunction $\widetilde{\wedge}$:

$\mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$ or disjunction $\widetilde{\vee} : \mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$, respectively. For example, the induced complement $\widetilde{\neg}\, X \in \widetilde{\mathcal{P}}(E)$ of $X \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_{\widetilde{\neg}\, X}(e) = \widetilde{\neg}\, \mu_X(e)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$ and $e \in E$. In the following, I will assume that an arbitary but fixed choice of these connectives and fuzzy set operations is given (usually provided by a QFM).

We shall now consider those constructions on semi-fuzzy and fuzzy quantifiers which either involve a fuzzy truth function (like negation), or a corresponding fuzzy set-theoretic operation (like complementation). The considered constructions therefore depend on the underlying fuzzy truth function or operation on fuzzy sets that has been chosen as a generalisation of the corresponding crisp operation. By introducing a canonical construction of fuzzy truth functions that are induced by a QFM, and by then defining the induced operations on fuzzy sets based on these truth functions in the apparent way, I have solved the problem of selecting a suitable fuzzy negation, fuzzy disjunction etc. from the endless choices of possible fuzzy negations, disjunctions etc. that could be considered for each QFM of interest. It is then assumed that each model of fuzzy quantification should be 'self-consistent', in the sense of being compatible with its own set of induced connectives.

3.5 The Aristotelian square

Based on the induced fuzzy negation and complement, we can now express important constructions on quantifiers like negation, formation of antonyms, and dualisation. These constructions are well-known from logics and linguistics because they express on the linguistic surface, and it is hence essential for models of fuzzy quantification to be compatible with these constructions (even though this requirement proved to be notoriously difficult for previous approaches to fuzzy quantification). Let us now consider those constructions in turn that depend on the induced negation or complementation in some way; constructions that build on other truth functions or set-theoretic operations will be considered later on.

In analogy to the external negation $\neg Q$ fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ based on two-valued negation $\neg : \mathbf{2} \longrightarrow \mathbf{2}$ of a two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ in TGQ [6, p. 186] and [45, p. 236], we shall first introduce the external negation of (semi-) fuzzy quantifiers. In natural language, this operation corresponds to the negation of a whole sentence, rather than negation of the noun phrase.

Definition 12
The external negation $\widetilde{\neg}\, Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$(\widetilde{\neg}\, Q)(Y_1, \ldots, Y_n) = \widetilde{\neg}(Q(Y_1, \ldots, Y_n))\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. The definition of* $\widetilde{\neg}\, \widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *in the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is analogous.*

For example, “no” is the negation of “some”. Hence

$$\textbf{no}(\textbf{women}, \textbf{men}) = \widetilde{\neg}\, \textbf{some}(\textbf{women}, \textbf{men})\,,$$

which formally expresses that “No women are men” can be paraphrased as “It is not the case that some women are men”.[21]
In addition to the external negation $\neg Q$ of a two-valued quantifier, TGQ discerns another type of negation, which corresponds to the *antonym* or *internal negation* $Q\neg$ of a two-valued quantifier [6, p. 186] and [45, p. 237]. Here I prefer the term ‘internal complementation’ (rather than ‘internal negation’) because the construction involves the complementation of one of the argument sets.

Definition 13
Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. The* antonym $Q\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of* Q *is defined by*

$$Q\neg(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. The antonym* $\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined analogously, based on the given fuzzy complement* $\widetilde{\neg}$*.*

Notes

- For example, “no” is the antonym of “all”. Hence

  $$\textbf{no}(\textbf{women}, \textbf{men}) = \textbf{all}(\textbf{women}, \neg\textbf{men})\,,$$

  which formally captures that “No women are men” can be paraphrased as “All women are not men”.

- The antonym is constructed by (crisp or fuzzy) complementation in the *last* argument of the quantifier. This conforms to linguistics expectations because by convention, I have used the last argument to accept the interpretation of the verbal phrase; e.g. the NL expression “No women are men” is interpreted by inserting the interpretation **men** $\in \mathcal{P}(E)$ of the verbal phrase “are men” into the second argument slot. Compatibility with internal complementation should not be artificially restricted to the n-th argument, though, and the particular indexing of the arguments should be inessential to the outcome of fuzzy quantification. Plausible approaches to fuzzy quantification should hence respect complementation in all arguments. By means of permutations of arguments, to be discussed in section 4.5, we will be able to reduce the general compatibility condition to the base condition on complementation in the last argument.

- Zadeh [188, p. 165] has proposed a notion of antonymy for proportional quantifiers, which is defined on their representations in terms of membership functions $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$.

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[21] Here and in the following examples I will assume that the considered negation $\widetilde{\neg}$ satisfies $\widetilde{\neg}\,0 = 1$ and $\widetilde{\neg}\,1 = 0$. This property of the induced negation will be ensured by the axioms stated at the end of the chapter.

TGQ also knows the concept of the *dual* of a two-valued quantifier (written as $Q\widetilde{\square}$ in my notation), which is the antonym of the negation of a quantifier (or equivalently, the negation of the antonym) [6, p. 197], [45, p. 238]. Hence

Definition 14
The dual $Q\widetilde{\square} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $n > 0$ *is defined by*

$$Q\widetilde{\square}(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *The dual* $\widetilde{Q}\widetilde{\square} = \widetilde{\neg}\, \widetilde{Q}\widetilde{\neg}$ *of a fuzzy quantifier* $\widetilde{Q}$ *is defined analogously.*

For example, "some" is the dual of "all". Again resorting to crisp concepts like "men" and "mammals", we may hence assert that

$$\textbf{all}(\textbf{men}, \textbf{mammals}) = \widetilde{\neg}\, \textbf{some}(\textbf{men}, \neg\textbf{mammals})\,.$$

The latter relationship ensures that "All men are mammals" can be paraphrased into "It is not the case that some men are not mammals".

Acknowledging the significance of these constructions to the description of natural language, we expect plausible models of fuzzy quantification to be homomorphic with respect to these constructions on quantifiers. Hence $\mathcal{F}(\textbf{no})$ should be the negation of $\mathcal{F}(\textbf{some})$, $\mathcal{F}(\textbf{no})$ should be the antonym of $\mathcal{F}(\textbf{all})$ and $\mathcal{F}(\textbf{some})$ should be the dual of $\mathcal{F}(\textbf{all})$. Concerning external negation, this will ensure, for example, that "No rich are young" can be paraphrased as "It is not the case that some rich are young", which is justified because $\mathcal{F}(\textbf{no})(\textbf{rich}, \textbf{young}) = \widetilde{\neg}\, \mathcal{F}(\textbf{some})(\textbf{rich}, \textbf{young})$. As to internal complementation, we obtain that $\mathcal{F}(\textbf{no})(\textbf{rich}, \textbf{young}) = \mathcal{F}(\textbf{all})(\textbf{rich}, \widetilde{\neg}\, \textbf{young})$, and hence "No rich are young" can also be phrased as "All rich are not young". Finally in the case of the dual, we conclude from $\mathcal{F}(\textbf{all})(\textbf{rich}, \textbf{old}) = \widetilde{\neg}\, \mathcal{F}(\textbf{some})(\textbf{rich}, \widetilde{\neg}\, \textbf{old})$ that "All rich are old" means the same as "It is not the case that some rich are not old".

The interdependencies of external negation, internal complementation (formation of antonyms) and dualisation are summarized in the *Aristotelian square*.[22] The Aristotelian square of the quantifier "all" is displayed in Fig. 8.

The Aristotelian square expresses in graphical form that the operators $(\bullet)$ (identity), $\widetilde{\neg}(\bullet)$ (external negation), $(\bullet)\neg$ (antonym) and $(\bullet)\widetilde{\square}$ (dualisation) constitute a Klein group structure on semi-fuzzy quantifiers:

$\circ$	$(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)\neg$	$(\bullet)\widetilde{\square}$
$(\bullet)$	$(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)\neg$	$(\bullet)\widetilde{\square}$
$\widetilde{\neg}(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)$	$(\bullet)\widetilde{\square}$	$(\bullet)\neg$
$(\bullet)\neg$	$(\bullet)\neg$	$(\bullet)\widetilde{\square}$	$(\bullet)$	$\widetilde{\neg}(\bullet)$
$(\bullet)\widetilde{\square}$	$(\bullet)\widetilde{\square}$	$(\bullet)\neg$	$\widetilde{\neg}(\bullet)$	$(\bullet)$

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[22] Sometimes also referred to as the *square of opposition*, see e.g. Gamut [45, p.238].

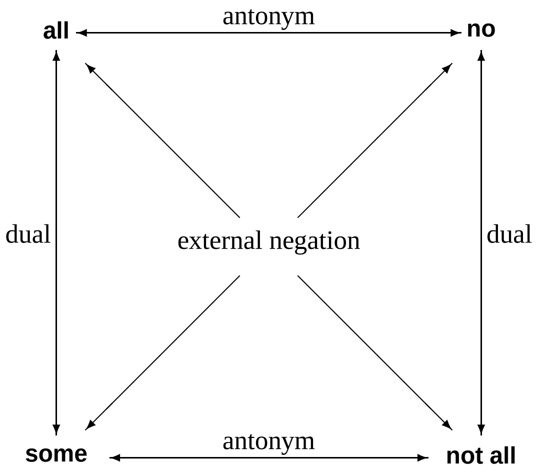

Figure 8: The Aristotelian square of **all**

Or in the case of fuzzy quantifiers,

$\circ$	$(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)\widetilde{\neg}$	$(\bullet)\widetilde{\square}$
$(\bullet)$	$(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)\widetilde{\neg}$	$(\bullet)\widetilde{\square}$
$\widetilde{\neg}(\bullet)$	$\widetilde{\neg}(\bullet)$	$(\bullet)$	$(\bullet)\widetilde{\square}$	$(\bullet)\widetilde{\neg}$
$(\bullet)\widetilde{\neg}$	$(\bullet)\widetilde{\neg}$	$(\bullet)\widetilde{\square}$	$(\bullet)$	$\widetilde{\neg}(\bullet)$
$(\bullet)\widetilde{\square}$	$(\bullet)\widetilde{\square}$	$(\bullet)\widetilde{\neg}$	$\widetilde{\neg}(\bullet)$	$(\bullet)$

The square can be properly set up for arbitrary semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ provided that $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ is involutive and satisfies $\widetilde{\neg} 1 = 0$, $\widetilde{\neg} 0 = 1$. These conditions will of course be entailed by the DFS axioms stated below.

The requirement that $\mathcal{F}$ be compatible with external negation, internal complementation (formation of antonyms) and dualisation, can then be summarized into the condition that the considered model of fuzzy quantification preserve Aristotelian squares (for more details, see section 4.7).

In order to warrant this, there is no need to explicitly require the compliance of $\mathcal{F}$ with all three constructions involved in the square. As will later be shown in Th-11 and Th-12, the targeted conformance to these constructions can be distilled into a single representative requirement, that of preserving duals of quantifiers. It is hence sufficient to demand that

$$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$.
The latter condition comprises the third requirement, which governs plausible choices of models.

3.6 Unions of argument sets

Apart from complementation, we can also perform other set-theoretic operations on the arguments of a quantifier. Due to De Morgan's law, and the known compatiblity of the considered models to complementation, which has been ensured by the previous requirement, it is sufficient to consider Boolean and fuzzy unions of arguments, in order to ensure that the given model of fuzzy quantification fully preserves the Boolean argument structure that can be expressed in NL. Let us hence introduce the construction which builds new quantifiers from given ones by means of forming the union of arguments:

Definition 15
Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. We define the semi-fuzzy quantifier* $Q\cup : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ *by*

$$Q\cup(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E)$*. In the case of fuzzy quantifiers,* $\widetilde{Q}\widetilde{\cup}$ *is defined analogously, based on the given fuzzy set operation* $\widetilde{\cup}$*.*

Notes

- To see how this construction expresses on the level of NL surface, consider the example that "Most men drink or smoke", and its model as a quantifying expression, $\textbf{most}(\textbf{men}, \textbf{drink} \cup \textbf{smoke})$, where I have assumed for the sake of argument that the extensions $\textbf{smoke}, \textbf{drink} \in \mathcal{P}(E)$ of "smoke" and "drink" be crisp. The above quantifying expression can be decomposed into the constructed quantifier "Most Y_1 are Y_2 or Y_3", i.e. $Q' = \textbf{most}\cup$, which is applied to the argument triple $(\textbf{men}, \textbf{drink}, \textbf{smoke})$.
- The construction is also underlying the definition of the two-place NL quantifier "all" in terms of the monadic universal quantifier $\forall$ known from logics, because $\textbf{all}(Y_1, Y_2) = \forall((\neg Y_1) \cup Y_2)$.
- The dual construction of intersections in arguments can be defined along the same lines, see section 4.9.

In order to ensure the full preservation of Boolean argument structure, conforming models of fuzzy quantification can be expected to comply with the above construction of unions in arguments. I hence require that $\mathcal{F}$ preserves the union of arguments, which is formally captured by the equality

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup} \tag{17}$$

which must be valid for arbitrary semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$.
Hence consider the example that "Most rich are old or lucky", whichs is modelled by

the fuzzy quantifying expression $\mathcal{F}(\textbf{most})(\textbf{rich}, \textbf{old}\widetilde{\cup}\textbf{lucky})$ built from the fuzzy extensions $\textbf{rich}, \textbf{old}, \textbf{lucky} \in \widetilde{\mathcal{P}}(E)$ of "rich", "old", and "lucky" persons, respectively. The compliance of $\mathcal{F}$ with equality (17) then ensures that the involved composite quantifier $\mathcal{F}(\textbf{most})\widetilde{\cup}$ can be properly represented by its underlying semi-fuzzy quantifier $Q' = \textbf{most}\cup$. Hence it does not matter whether "Many rich are old or lucky" is computed by evaluating $\mathcal{F}(\textbf{many})(\textbf{rich}, \textbf{old}\widetilde{\cup}\textbf{lucky})$ or by resorting to the decomposition $\mathcal{F}(Q')(\textbf{rich}, \textbf{old}, \textbf{lucky})$ based on $Q'(Y_1, Y_2, Y_3) = \textbf{many}(Y_1, Y_2 \cup Y_3)$.
As mentioned above, the requirement (17) imposed on unions of arguments can be combined with other conditions like compatibility with internal complementation, in order to achieve full preservation of Boolean argument structure. In particular, this will ensure that $\mathcal{F}(\textbf{all})(X_1, X_2) = \mathcal{F}(\forall)((\neg X_1) \cup X_2)$, i.e. the two-place NL quantifier "all" can be determined from the monadic universal quantifier $\forall$ even when there is fuzziness in the arguments.

This completes the discussion of Boolean argument structure, and the corresponding condition of preserving unions in arguments, which formalizes the fourth requirement on models of fuzzy quantification.

3.7 Monotonicity in the arguments

It is essential for a model of fuzzy quantification to elicit the expected entailment relationships. The fifth requirement on plausible models, which is introduced now, captures those entailments that stem from monotonicity properties of the involved quantifiers. In order to express the relevant monotonicity properties of semi-fuzzy and fuzzy quantifiers, let us first recall the definition of the fuzzy inclusion relation.

Definition 16 (Fuzzy inclusion relation)
Suppose E is some set and $X_1,\ X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. We say that X_1 is contained in X_2 (in symbols, $X_1 \subseteq X_2$) if

$$\mu_{X_1}(e) \leq \mu_{X_2}(e)$$

for all $e \in E$.

Based on this concept, I can now state precisely what it means for a (semi-)fuzzy quantifier to be monotonic in one of its arguments.

Definition 17 (Monotonicity in the i-th argument)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is said to be nondecreasing in its i-th argument, *$i \in \{1,\ \ldots, n\}$, if*

$$Q(Y_1, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

whenever the involved arguments $Y_1, \ldots, Y_n, Y_i' \in \mathcal{P}(E)$ satisfy $Y_i \subseteq Y_i'$. Q is said to be nonincreasing in the i-th argument *if under the same conditions, it always holds that*

$$Q(Y_1, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)\,.$$

The corresponding definitions for fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *are entirely analogous. In this case, the arguments range over* $\widetilde{\mathcal{P}}(E)$*, and '*$\subseteq$*' is the fuzzy inclusion relation.*

Notes

- To present an example, "all" is nonincreasing in the first argument and nondecreasing in the second argument. From the former property, we then know that

  $$\textbf{all}(\textbf{married} \cap \textbf{men}, \textbf{have\_children}) \geq \textbf{all}(\textbf{men}, \textbf{have\_children}),$$

  which expresses that "All married men have children" makes a weaker requirement than "All men have children". As to the latter property, we may conclude from the nondecreasing monotonicity of "all" in its second argument that

  $$\textbf{all}(\textbf{men}, \textbf{have\_daughters}) \leq \textbf{all}(\textbf{men}, \textbf{have\_children}) .$$

  Consequently "All men have daughters" poses a stronger condition than "All men have children".

- In TGQ, nondecreasing monotonicity in the scope argument (i.e. in the argument slot which accepts the interpretation of the verb phrase) is usually termed *upward monotonicity*, while a quantifier which is nonincreasing in this position is called *downward monotonic*, see e.g. [45, p. 232+].[23] Recalling my convention of reserving the last argument for denotation of the verb phrase, this means that positive monotonic quantifiers correspond to those quantifiers that are nondecreasing in the n-th argument, and downward monotonicity captures those quantifiers that are nonincreasing in their n-th argument. For example, "all" is upward monotonic and "no" is downward monotonic. Let me remark that the property of upward/downward monotonicity is rather typical of NL quantifiers.

- In the common case of two-place quantification, TGQ has also coined special terms for monotonicity in the first argument. In TGQ, those two-place quantifiers that are nondecreasing in their restriction are dubbed *persistent*, while those that are nonincreasing in the first argument are called *antipersistent*, see e.g. [6, p. 193] and [45, p. 242+]. Simple examples are "some" (persistent) and "all" (antipersistent). Compared to upward/downward monotonicity, there are fewer instances of NL quantifiers which are persistent or antipersistent [45, p. 243]. For example, proportional quantifiers like "most" typically lack both persistence and antipersistence.

- Zadeh [188, p. 164] has coined the similar notions of *monotone nondecreasing* and *monotone nonincreasing* quantifiers, which refer to the membership functions $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ used to define the proportional type.

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[23] Barwise and Cooper [6, p. 184+], however, refer to positive and monotonic kinds as monotone increasing and monotone decreasing, respectively.

Let me now substantiate the close relationship between monotonicity properties of a quantifier and valid patterns of reasoning, and show how the monotonicity type of a quantifier constrains the inventory of admissible syllogisms. Hence consider the quantifier "more than ten", which is nondecreasing in its first argument. It then holds that

$$\begin{aligned}&\textbf{more than ten}(\textbf{married} \cap \textbf{men}, \textbf{have\_children})\\ &\leq \textbf{more than ten}(\textbf{men}, \textbf{have\_children})\,.\end{aligned}$$

It is this inequality which justifies the syllogism

More than ten married men have children
All married men are men
More than ten men have children

Other quantifiers like "some" can be substituted for "more than ten" here, which are also nondecreasing in the first argument. In the case of a quantifier which is nonincreasing in the first argument, like "no" or "all", the above pattern is no longer valid, and must be replaced with a pattern which fits the new quantifier type. In this case, we obtain the converse inequality

$$\textbf{no}(\textbf{men}, \textbf{have\_children}) \leq \textbf{no}(\textbf{married} \cap \textbf{men}, \textbf{have\_children})$$

and a corresponding pattern of reasoning,

No men have children
All married men are men
No married men have children

As concerns monotonicity in the second argument, let us consider the quantifier "most" which is nondecreasing in its second argument. We then obtain that

$$\textbf{most}(\textbf{men}, \textbf{married}) \leq \textbf{most}(\textbf{men}, \textbf{married} \cup \textbf{divorced})\,,$$

i.e. "Most men are married" expresses a stronger condition than "Most men are married or divorced". Again, the above inequality justifies the pattern of reasoning,

Most men are married
All married are married or divorced
Most men are married or divorced

Let us now turn to the fuzzy case. Even when there is fuzziness in the arguments, we would certainly expect the above patterns of reasoning to remain applicable. In fact, it appears that these patterns describe a characteristic of the quantifier which is independent of the particular choice of arguments, and remains applicable even when the involved concepts are fuzzy. Acknowledging that the admissible choices of syllogisms are justified by the underlying inequality that express the monotonicity properties of the considered quantifiers, this means that we must enforce the *preservation of monotonicity properties* in order to maintain the applicable patterns of reasoning. For example, it should hold in a plausible model of fuzzy quantification that

$$\begin{aligned}&\mathcal{F}(\textbf{more than ten})(\textbf{young} \mathbin{\widetilde{\cap}} \textbf{men}, \textbf{rich})\\ &\leq \mathcal{F}(\textbf{more than ten})(\textbf{young} \mathbin{\widetilde{\cap}} \textbf{persons}, \textbf{rich})\end{aligned}$$

assuming that $\textbf{men} \subseteq \textbf{persons}$. It is this inequality which permits us to conclude from “More than ten young men are rich” to the entailed “More than ten young persons are rich”, regardless of the fuzziness in $\textbf{young}, \textbf{rich} \in \widetilde{\mathcal{P}}(E)$. Similarly, it should hold that

$$\mathcal{F}(\textbf{most})(\textbf{young}, \textbf{very\_poor}) \leq \mathcal{F}(\textbf{most})(\textbf{young}, \textbf{poor})\,,$$

assuming that the fuzzy subsets $\textbf{very\_poor}, \textbf{poor} \in \widetilde{\mathcal{P}}(E)$ satisfy $\textbf{very\_poor} \subseteq \textbf{poor}$. We can then draw from “Most young are very poor” the logical conclusion that “Most young are poor”. These examples illustrate the above observation that the considered entailment relationships will transfer to the general case of fuzzy arguments, provided that the chosen model of quantification $\mathcal{F}$ is known to preserve the underlying monotonicity properties. The relevant adequacy condition on QFMs can then be phrased as follows.

Definition 18 (Preservation of monotonicity in the arguments)
A QFM $\mathcal{F}$ is said to preserve monotonicity in the arguments *if semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which are nondecreasing (nonincreasing) in their i-th argument, $i \in \{1, \ldots, n\}$, are mapped to fuzzy quantifiers $\mathcal{F}(Q)$ which are also nondecreasing (nonincreasing) in their i-th argument.*

Notes

- Hence $\mathcal{F}(\textbf{all})$ should be nonincreasing in the first and nondecreasing in the second argument. The above motivating examples based on $\mathcal{F}(\textbf{more than ten})$ and $\mathcal{F}(\textbf{most})$ are also instances of the general preservation property, of course.
- When combined with the other requirements on models of fuzzy quantification, the preservation condition can be restricted to the case that Q is *nonincreasing in its last argument.* Again, this restriction will serve the purpose of simplifying the axioms system and ensuring its independence. The fifth requirement on plausible models will be stated in terms of *nonincreasing* (rather than nondecreasing) monotonicity for technical purposes; when presented in this way, the criterion facilitated the the proof that $\widetilde{\mathcal{F}}(\neg)$ is a strong negation operator. To provide an example where the restricted property applies directly, consider the two-valued quantifier “no”, which is nonincreasing in its last argument. By the restricted preservation condition, then, $\mathcal{F}(\textbf{no})$ is nonincreasing in its last argument also. For instance, this ensures that

  $$\mathcal{F}(\textbf{no})(\textbf{young}, \textbf{rich}) \leq \mathcal{F}(\textbf{no})(\textbf{young}, \textbf{very\_rich})\,,$$

  and hence “No young are very rich” is entailed by “No young are rich”.

To sum up, every plausible model of fuzzy quantification should preserve the relevant entailment relationships, which are often tied to the monotonicity type of the involved quantifier. It is hence crucial for the theory of fuzzy quantification to consider the underlying monotonicity properties of quantifiers which show up in these entailments. The fifth requirement on models of fuzzy quantification hence enforces a criterion which is essential to the preservation of general monotonicity properties, and in turn ensures that all corresponding patterns of reasoning remain valid in the presence of fuzziness.

3.8 The induced extension principle

The final requirement on plausible models of fuzzy quantification is concerned with the problem of establishing a systematic relationship between the interpretation of quantifiers in different base sets. To this end, a homomorphism condition will be introduced which uses powerset mappings to connect the behaviour of the fuzzification mechanism across domains, which achieves the desired coherence of results.

Definition 19 (Powerset mapping)
To each mapping $f : E \longrightarrow E'$, we associate a mapping $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ (the powerset mapping *of f) which is defined by*

$$\widehat{f}(Y) = \{f(e) : e \in Y\} ,$$

for all $Y \in \mathcal{P}(E)$.

Notes

- Often the same symbol f is used to denote both the original mapping and its extension to powersets. In the present context, however, it is important to discern the base mapping f from its associated powerset mapping, and I will hence use the $\widehat{f}$-notation throughout.
- There is a closely related concept, namely that of the inverse image mapping $f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$ of a given $f : E \longrightarrow E'$, which is (as usual) defined by

  $$f^{-1}(V) = \{e \in E : f(e) \in V\} , \tag{18}$$

  for all $V \in \mathcal{P}(E')$. Often if V is a singleton, i.e. $V = \{v\}$ for some $v \in E'$, I will simply write $f^{-1}(v)$.

The underlying mechanism which transports f to $\widehat{f}$ can be generalized to the case of fuzzy sets; such a mechanism is then called an *extension principle*. Formally, I define (a pretty general class of) extension principles as follows.

Definition 20 (Extension principle)
An extension principle *$\mathcal{E}$ assigns to each mapping $f : E \longrightarrow E'$ a corresponding mapping $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$. For convenience, we shall assume that $E, E' \neq \varnothing$.*

Notes

- Extension principles hence provide the desired mechanism which associates fuzzy powerset mappings $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ with given base mappings $f : E \longrightarrow E'$. I do not impose any a priori restrictions on the well-behavedness of extension principles. These will later result from the axioms on plausible models of fuzzy quantification. A survey on the intuitive expectations on reasonable choices, along with the theorems that these adequacy conditions are fulfilled in the models, is given below in section 4.12. These results and the analysis of standard quantifiers in section 4.16 will further reveal that plausible extension principles can be expressed in terms of existential quantification, and can hence be represented by a possibly infinitary formula built from an underlying s-norm.
- I have excluded the case that $E = \varnothing$ or $E' = \varnothing$ in order to allow a simpler definition of induced extension principles that will be associated with QFMs (to be defined below in Def. 22). After all, the case of base mappings f with an empty domain or range is irrelevant to our present purposes anyway. However, the issue is likely to be judged differently when discussing extension principles in a broader context. Let me hence emphasize that the restriction to nonempty sets is just a matter of convenience and in fact marginal to the targeted understanding of of extension principles. This is because every extension principle (in the sense of the above definition) can readily be completed into a 'full' extension principle, which is also defined in the case of $E = \varnothing$ or $E' = \varnothing$. Hence consider $f : \varnothing \longrightarrow E'$. In this case, we stipulate that

  $$\mathcal{E}(f) = c_{\varnothing} \,, \tag{19}$$

  where $c_{\varnothing} : \widetilde{\mathcal{P}}(\varnothing) \longrightarrow \widetilde{\mathcal{P}}(E')$ is the constant which to $\varnothing \in \widetilde{\mathcal{P}}(\varnothing) = \{\varnothing\}$ assigns the set $c_{\varnothing}(\varnothing) = \varnothing \in \widetilde{\mathcal{P}}(E')$. Let us now address the remaining case that $E' = \varnothing$. Here we simply observe from the definition of functions that the considered $f : E \longrightarrow \varnothing$ only qualifies as a mapping if $E = \varnothing$ as well. The case of $E' = \varnothing$ is hence already covered by equality (19).

The prototypical example of an extension principle has been suggested by Zadeh [180]. Recast in my notation, this 'standard extension principle' is defined as follows.

Definition 21 (Standard extension principle)
Let $f : E \longrightarrow E'$ *a mapping. The* standard extension principle *assigns to* f *the fuzzy powerset mapping* $\hat{\hat{f}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ *defined by*

$$\mu_{\hat{\hat{f}}(X)}(y) = \sup\{\mu_X(e) : e \in f^{-1}(y)\}$$

for all $y \in E'$.

Note. The extension principle can also be generalized to n-ary mappings $f : E_1 \times \cdots \times E_n \longrightarrow E'$ which it takes to n-ary fuzzy powerset mappings $\hat{\hat{f}} : \widetilde{\mathcal{P}}(E_1) \times \cdots \times$

$\widetilde{\mathcal{P}}(E_n) \longrightarrow \widetilde{\mathcal{P}}(E')$, defined by

$$\mu_{\hat{\hat{f}}(X_1,\ldots,X_n)}(y) = \sup\{\min_{i=1}^{n} \mu_{X_i}(e_i) : (e_1,\ldots,e_n) \in f^{-1}(y)\}\,,$$

for all $X_1 \in \widetilde{\mathcal{P}}(E_1),\ldots,X_n \in \widetilde{\mathcal{P}}(E_n)$ and $y \in E'$; see Yager [169] for details. This generalized version of the extension principle will be of no importance for the following. However, it is apparent that the general extension principles $\mathcal{E}$ considered here can be generalized to n-place mappings along the same lines.
With each QFM, we can associate a corresponding extension principle through a canonical construction.

Definition 22
Every QFM $\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E \longrightarrow E'$ (where $E,\ E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ defined by

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$; or more succinctly:

$$\mu_{\widehat{\mathcal{F}}f(X)}(e') = \mathcal{F}(\pi_{e'} \circ \widehat{f})(X)$$

The equivalence of the first and second form is obvious from my definition of projection quantifiers, see Def. 9). The powerset functions $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ and the corresponding fuzzy powerset functions $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ obtained from the induced extension principle are important in our context because they can be applied to the argument sets of semi-fuzzy quantifiers (crisp case, $\widehat{f}$) and fuzzy quantifiers (fuzzy case, using $\widehat{\mathcal{F}}(f)$).

We hence require that every ‘reasonable’ choice of $\mathcal{F}$ be compatible with its induced extension principle in the following sense.
Suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $f_1,\ldots,f_n : E' \longrightarrow E$ are given mappings, $E' \neq \varnothing$. We can construct a semi-fuzzy quantifier $Q' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by composing Q with the powerset mappings $\widehat{f}_i,\ldots,\widehat{f_n}$, i.e.

$$Q'(Y_1,\ldots,Y_n) = Q(\widehat{f}_1(Y_1),\ \ldots,\ \widehat{f}_n(Y_n))\,,$$

for all $Y_1,\ldots,Y_n \in \mathcal{P}(E)$. This can be expressed more compactly if we recall the concept of *product mapping*. If $g_1,\ \ldots,\ g_m : A \longrightarrow B$ are any mappings, then $\mathop{\times}\limits_{i=1}^{n} g_i : A^n \longrightarrow B^n$ is defined by

$$(\mathop{\times}\limits_{i=1}^{n} g_i)(x_1,\ \ldots,\ x_n) = (g_1(x_1),\ \ldots,\ g_n(x_n))\,.$$

By using the product, the above definition of Q' then becomes $Q' = Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i$, because

$$(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i)(Y_1,\ldots,Y_n) = Q(\widehat{f}_1(Y_1),\ldots,\widehat{f_n}(Y_n)) \qquad (20)$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E')$; '$\circ$' denotes functional composition. By utilizing the induced extension principle $\widehat{\mathcal{F}}$ of a QFM, we can perform a similar construction on fuzzy quantifiers, thus composing $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ with $\widehat{\mathcal{F}}(f_1), \ldots, \widehat{\mathcal{F}}(f_n)$ to form the fuzzy quantifier $\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$ defined by

$$(\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i))(X_1, \ldots, X_n) = \widetilde{Q}(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n)) \,,$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$. We require that a plausible choice of QFM $\mathcal{F}$ comply with this construction, and hence impose the following homomorphism condition with respect to the application of (crisp or fuzzy) powerset functions:

Definition 23 (Compatibility to functional application)
Let $\mathcal{F}$ be a given QFM. We say that $\mathcal{F}$ is compatible with functional application *if the equality*

$$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) \tag{21}$$

is valid for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and mappings $f_1, \ldots, f_n : E' \longrightarrow E$ with domain $E' \neq \varnothing$.

Notes

- I have chosen the term 'functional application', because the constructed quantifier is obtained from applying the extended functions $f_1, \ldots, f_n$ to the corresponding arguments.
- Let me remark in advance that it is not possible to span the intended class of models if the induced powerset mappings $\widehat{\mathcal{F}}(f_i)$ are replaced with the standard choice of fuzzy powerset mappings $\hat{\hat{f}}_i$, that are obtained from the standard extension principle. As will be shown below in theorem Th-33, the use of the standard extension principle would restrict the admissible models of fuzzy quantification to those that induce the standard disjunction $\widetilde{\mathcal{F}}(\vee) = \max$. I considered this too restrictive because I wanted to have models for arbitrary t- and s-norms. It was the intent to cover such general models that necessitated the development of the induced extension principle and its thorough use in the theory of fuzzy quantification, because all reference to the standard extension principle must be avoided in order to obviate a restriction to the limited class of 'standard models'. (Of course, the standard models are the preferred choice in most applications, but it deepens the knowledge of fuzzy quantification if other models can also be studied).

As mentioned above, the induced extension principle $\widehat{\mathcal{F}}$ enables us to construct fuzzy quantifiers $Q' = Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ on a base set E from a given fuzzy quantifier Q on *another* base set E'. It is this cross-domain characteristic which makes the compliance condition expressed by (21) especially important to the theory of fuzzy quantification. In fact, the required compatibility with functional application will constitute the *only* criterion in the proposed axiom system, which controls the behaviour of $\mathcal{F}$ on different base sets E, E'. For example, if $\beta : E \longrightarrow E'$ is a bijection, $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, and $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{\beta}(Y_1),\ \ldots,\ \widehat{\beta}(Y_n))$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, then

$$\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(\beta)(X_1),\ \ldots,\ \widehat{\mathcal{F}}(\beta)(X_n))$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and

$$\mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(\widehat{\mathcal{F}}(\beta^{-1})(X_1),\ \ldots,\ \widehat{\mathcal{F}}(\beta^{-1})(X_n))$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$, which shows that $\mathcal{F}$ may not depend on any particular properties of elements of a base set E.
The sixth and last requirement that will be imposed on models of fuzzy quantification, viz being compatible with functional application, hence enforces a coherent behaviour of $\mathcal{F}$ across domains.

3.9 Determiner fuzzification schemes: the DFS axioms

The requirements on plausible models of fuzzy quantification can be summarized into the following axiom system. These conditions, which comprise the 'DFS axioms', achieve the definition of a well-motivated target class that I strived for. The individual models in the class will be called a 'determiner fuzzification scheme' or DFS for short. The term 'determiner' has been preferred to 'quantifier' in order to avoid a possible confusion of quantifier fuzzification mechanisms, which span the required background of potential models on which the adequacy conditions can be defined; and determiner fuzzification schemes, which identify the subclass of intended models.

Definition 24 (Determiner fuzzification schemes)
A QFM $\mathcal{F}$ is called a determiner fuzzification scheme *(DFS) if the following conditions are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if* $Q = \pi_e$ *for some* $e \in E$	(Z-2)
Dualisation	$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}$ $\quad n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}$ $\quad n > 0$	(Z-4)
Preservation of monotonicity	*If* Q *is nonincreasing in the* n*-th arg, then* $\mathcal{F}(Q)$ *is nonincreasing the in* n*-th arg,* $n > 0$	(Z-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1,\ \ldots,\ f_n : E' \longrightarrow E$*,* $E' \neq \varnothing$*.*	(Z-6)

Notes

- These conditions simply combine the six requirements developed above, which capture important expectations on models of fuzzy quantification. Some of the requirements have been weakened as much as possible, in order to simplify proofs that a given candidate QFM $\mathcal{F}$ is indeed a model of the theory, and in order to facilitate the proof that the axioms are independent. It is these considerations which motivated the restriction of 'correct generalisation' (Z-1) to quantifiers of arities $n \leq 1$, and the restriction of the monotonicity requirement to quantifiers that are nonincreasing in their last argument. The 'full' requirements are of course entailed by the weakened versions that I included into the axiom system; see theorems Th-2 and Th-16 in the next chapter.

- I have taken pains to avoid the use of the standard connectives and the standard extension principle in the DFS axioms (in favor of the *induced* connectives and induced extension principle of a DFS) because the requirement of being compatible to the standard choices would have excluded all models which induce $\widetilde{\mathcal{F}}(\vee) \neq \max$ and $\widetilde{\mathcal{F}}(\wedge) \neq \min$. (This claim will be formally established in the later theorem Th-47, which states that substituting the standard connectives and standard extension principle into the above definition, provides a characterization of the 'standard models' of fuzzy quantification). By referring to the induced constructions only, one possible pitfall has been circumvented, which would have excluded all non-standard models from consideration.

- The original presentation of determiner fuzzification schemes in [46] was based on a set of nine DFS axioms. These axioms contained some subtle interdependencies, though, which motivated my later effort to condense the original axioms into the equivalent axiom system presented above. See [48] for details on both axiom systems and the proof of their equivalence.

At first sight, the axioms might give a condensed and rather abstract impression. In addition, the precise compilation of the requirements might appear arbitrary to some degree, although the individual criteria are certainly straightforward and supported by ample motivation (see explanations above). In order to better judge the proposed axiom system and appreciate the particular compilation of requirements, let me briefly recall the goals, and especially the design principles, which guided the search for a solid axiomatic foundation and ensure that it is both intuitively appealing, and beneficial from a methodical perspective.

It was the pivotal objective when phrasing and selecting the requirements, to compile an *independent* axiom system which covers the essential adequacy criteria from the perspective of linguistics and fuzzy logic. In view of the goal of obtaining an independent system, it was of course not possible to include all conceivable semantic postulates directly into the axiom set, thus compromising its minimality and irreducibility. In order to achieve the desired independence, the linguistic relevance of the individual axioms had to give priority to the superordinate goal. In other words, the axioms in isolation are *not* necessarily expected to formalize semantical requirements, and need not directly originate from linguistics. By contrast, it is only the axiom system as a whole which is required to entail all properties of interest, and thus achieve a formalization of plausible models. When compiling the DFS axioms, I started from a large number of conceivable adequacy properties, all of which catch some aspect of fuzzy quantification (these semantical postulates will be exposed in the following chapters). I then tried to single out a generating core of requirements, and subsequently eliminated redundancies and interdependences. The final set of axioms as presented above, hence results from a process of simplification and compression. Let me now attest that this process indeed distilled an axiom system with the desired properties and that the primary design goal, that of independence, has hence been achieved. The proposed DFS axioms therefore accomplish a minimal characterisation of the target class of models, and are not reducible to a smaller axiom set.

Theorem 1
The DFS axioms (Z-1)–(Z-6) *are independent, i.e. none of the conditions is entailed by the remaining conditions.*

This makes an important result on the proposed axiomatic foundation, because the independence of the axioms substantiates that the suggested criteria capture distinct aspects which become visible in the behaviour of approaches to fuzzy quantification. Consequently, each of these criteria spans its own dimension, along which the space of possible models can be analyzed.

Although linguistic significance of the *individual* axioms was not the prime concern, and only expected to emerge from the axioms acting in concert, the actual requirements which I compiled into the DFS axioms, do not come off too badly in this respect. In fact, most of these requirements imposed on legal choices of models directly originate from logical or linguistic considerations, like modelling of proper names (Z-2), modelling of duals (Z-3), modelling of unions (Z-4), and the preservation of important entailment relationships (Z-5). The remaining axioms, i.e. the basic requirement of ‘correct generalization’ (Z-1), and the more subtle desideratum of ‘functional application’ (Z-6), are not directly motivated by linguistic considerations, but rather enforce the internal and cross-domain coherence of all admissible models. It should be apparent from this capacity of the latter conditions, that both are essential to the modelling of fuzzy quantification in NL. To sum up, the proposed axiomatic foundation of fuzzy quantification rests on a compilation of formal requirements which are well-motivated, linguistically significant, and mutually independent. It should have become clear from the explanation of the individual requirements that the imposed conditions are necessary for plausible models, and hence do not prune any useful approaches to fuzzy quantification. However, it is not possible to assess from a brief glance at the axioms, if their deductive hull indeed covers all properties of linguistic relevance. This issue of completeness of the axioms, which is concerned with their capacity of answering the important linguistic expectations, can only be judged by making explicit the total of intuitive assumptions on plausible models, by developing formal criteria which assess the properties of interest, and by evaluating these criteria on the suggested class of models. It is apparent that the discussion of these topics requires considerable effort, and the investigation of all questions regarding the completeness or coverage of the axioms, has hence been detached into the subsequent three chapters.

3.10 Chapter summary

The quantification framework that I developed in the previous chapter, introduced a rich class of approaches to fuzzy quantification, which become instances of quantifier fuzzification mechanisms. The definition of these ‘raw’ and totally unrestricted approaches has been shaped in such a way that (apart from many odd examples), it certainly also catches the more interesting cases, i.e. those of plausible models for fuzzy quantification. The framework hence opened the required space of base objects, which can now be subjected to further study. In the present chapter, the next logical step was then undertaken, and I made an effort to identify the subclass of plausible models within the proposed framework. In order to accomplish this identification, I decided to pursue a strategy which is essentially algebraic, and hence characterize the plausible models in terms of those structural aspects which express in observable properties, rather than directly referring to the internal structure of the desired models, or proposing and justifying individual choices of prototypical models (as opposed to embarking on a generic solution).

- I have therefore considered several *properties* of semi-fuzzy quantifiers and fuzzy quantifiers which capture their distinct behavioural dimensions. The properties of interest sometimes originate from logic, but my prime source of these properties is the logico-linguistic Theory of Generalized Quantifiers, which has focused on those criteria which are essential from a linguistic perspective, and best express the intuitive semantical expectations. Having formalized such properties, one can then assert that a ‘reasonable’ approach of fuzzy quantification should preserve the important properties of arbitrary quantifiers (for example, monotonicity properties).
- In addition to preserving linguistic properties of quantifiers, I was also interested in gaining a system which preserves important *relationships* between quantifiers. The prime example are functional relationships between quantifiers which are established by certain *constructions* (like dualisation or negation). Compatibility with such constructions corresponds to the well-known mathematical concept of a homomorphism (structure-preserving mapping).

The basic trust of this algebraic procedure is that all important aspects of plausible models indeed become visible in some way (rather than being confined to some ‘ethereal nature’ of the model), and can hence be described and actually enforced, by formal criteria which constrain the admissible choices of models. It is hence assumed that demanding the preservation of sufficiently many behavioural dimensions, which express in properties, relationships and certain constructions, will eventually achieve the desired characterisation of plausible models. This solution to the superordinate goal of identifying the models should obey an important constraint, though. It is probably not wise to turn all adequacy criteria that come to mind into conditions imposed on admissible models, which would likely result in an axiom set too clumsy for succinct proofs, and hence not conducive to rapid progress of the theory. By contrast, I preferred to distill a reduced system of core requirements on the models, with the capacity of generating the full set of conditions, which then show up in the deductive hull. To be precise, I even demanded the axiom system to be independent, i.e. irreducible to a smaller axiom set. This minimimality of the system helps avoid any redundancy in proofs, and also offers the theoretical advantage that the intuitive concerns on adequate models become separated into distinct formal dimensions, each of which reflects an independent aspect of the meaning of quantifiers.

In the chapter, I explained various intuitive requirements on the models, which were then compiled into 'DFS axioms', the suggested axiomatic framework for plausible models of fuzzy quantification. The development and selection of these criteria was steered by the design principles motivated above, and hence aimed at a succinct and redundancy-free characterization of the intended models. The resulting system is composed of well-motivated requirements which indeed achieve the desired separation of semantical dimensions, as witnessed by the proven independence of the axioms. Concerning the individual conditions that govern legal choices of models, the relevance of the six axioms to fuzzy NL quantification can be sketched as follows.

- The first condition of 'correct generalisation' (Z-1) is mandatory to ensure the internal coherence of the quantification framework. By demanding that the resulting fuzzy quantifiers properly generalize the base descriptions of semi-fuzzy quantifiers which they are supposed to extend, and hence coincide with the base quantifier on crisp arguments, this condition enforces the success condition of the fuzzification pattern, which underlies the use of a fuzzification mechanism.

- The semantical postulate (Z-2) which demands the proper interpretation of projection quantifiers, makes explicit the relationship of crisp/fuzzy membership assessments and quantification. This requirement is essential from a linguistic perspective because it warrants the intended modelling of proper names like "Joan", which are generally viewed as a special type of quantifiers in TGQ.

- The requirement of compatibility with dualisation (Z-3) is also essential from a logical and linguistic standpoint, because it enforces the full compliance of the models not only with the formation of duals, but also with the important constructions of external negation and formation of antonyms.

- The condition on 'internal joins' (Z-4), which demands the conformance of the models to unions of arguments, is also anchored in linguistics, because it achieves a compositional interpretation of NL examples like "Most men drink or smoke". Apart from its linguistic motivation, the condition is of key relevance to the DFS axioms, because it makes the only requirement which mediates the behaviour of the considered QFM for quantifiers of different arities. In fact, it is this condition which achieves the desired cross-arity coherence of the model, and thus contributes to the proper modelling of fuzzy multi-place quantification.

- The requirement of 'monotonicity in arguments' (Z-5) assumes responsibility of the monotonicity type of a base quantifier, which should translate to the fuzzy case. This is especially important in order to preserve the valid entailments which pertain to these monotonicity properties, and which reveal important expectations on the meaning of matching fuzzy quantifiers.

- The final criterion (Z-6) which demands the compliance with functional application, is not directly inspired from linguistics. Noticing that all other axioms refer to quantifiers on a single base set E only, it was necessary to add a requirement which links the behaviour of $\mathcal{F}$ across domains. The criterion of functional application is solely concerned with this issue of cross-domain coherence, which it traces back to the obvious desideratum of compositionality with respect to powerset mappings.

It should be apparent from this summary that the DFS axioms are straightforward, and grounded either in linguistics, or in apparent demands for coherence. In addition, the axioms are *consistent*. We shall experience in the later chapters Chap. 7–Chap. 10 that the axioms admit rich and interesting classes of models.

When formalizing the axioms, I have taken great care to avoid any reference to the standard connectives and the standard extension principle of fuzzy logic. The rationale behind that becomes clear from the later theorem Th-47, which states that the class of models shrinks down to those 'standard' choices which induce $\min$ and $\max$, as soon as the standard connectives and extension principle enter into the axioms. In order to maintain the possibility of 'non-standard models' which induce general fuzzy conjunctions other than $\min$, I therefore developed corresponding 'induced' constructions, and hence associated a canonical choice of fuzzy truth functions and general extension principle with each QFM of interest. Those constructions on quantifiers which depend on a fuzzy truth function or the extension principle, then become parametrized by these induced constructions. This strategy targets at a coherent or 'self-consistent' system which is compatible with its own induced constructions. It must be admitted, though, that the study of these general models is still in its infant stage, and no examples of the general type, beyond the known examples of standard models, have been discovered so far. It is hence an open problem whether models exist which do not induce the standard extension principle or the standard choice of fuzzy connectives, and future research should be directed at this issue.

In this chapter, I have presented a core axiom system for approaches to fuzzy quantification, and I also supplied ample evidence in favour of the proposed axioms, and explained their relevance through convincing natural language examples. However, an axiom set cannot be appropriately judged when looking at the axioms in isolation; and it is crucial to the full understanding of the DFS axioms and their associated models, that the properties of the axiom set *as a whole* be investigated, by exploring the space of logical entailments. Specifically, we should try and make explicit all intuitive expectations on the models, and verify that the total of resulting criteria are valid in the suggested models. This will permit a well-founded decision upon the completeness of the axioms, and ascertain their precise capacity of explaining our linguistic expectations on the interpretation of fuzzy quantifiers in NL.

4 Semantic properties of the models

4.1 Motivation and chapter overview

In the previous chapter, an independent system of condititions has been developed, which capture important requirements on the desired models of fuzzy quantification. It is a natural consequence of the independence of the proposed axiom system, i.e. minimality/irreducibility, that the axioms are rather condensed. Due to this succinct presentation of the plain requirements on plausible models it is not clear at this point whether the axioms are also complete, and capture all of the intuitive expectations that one would like to see incorporated into a plausible model of fuzzy quantification. Of course, there are some tradeoffs here, because some of the desiderata might become mutually inconsistent, when they are displaced from their Boolean origins into a fuzzy framework. In addition, it must be carefully decided upon the core requirements, from which practical concerns and other optional criteria must be clearly separated. For example, considerations on robustness or continuity of the models should not enter into the core axiom system, because discontinuous models, which occur as boundary cases, can well be of interest to theoretical investigations, although they certainly do not qualify for practical application due to their extreme brittleness. Keeping these general factors in mind, I will now take care of the completeness issue. The following chapters will provide ample evidence that the models of the theory are well-behaved in a variety of ways, and hence redeem the promise of providing a satisfying account of fuzzy quantification. As opposed to the previous chapter, which focused on a few representative postulates for reasonable choices of models, we will now be concerned with identifying a larger body of semantical criteria that control the behaviour of plausible approaches to fuzzy quantification. A semantical constraint thus expresses an observable regularity, which should be expected of all models of fuzzy quantification. Most of these criteria originate from the Theory of Generalized Quantifiers and hence reflect linguistic considerations on the models. Typical examples of these conditions are of the homomorphism kind, and require the compatibility of a QFM with constructions of linguistic relevance, like forming negations and antonyms, restricting an argument by an adjective, etc. Due to the fact that TGQ relies on a two-valued notion of quantifiers, the involved concepts must generally be fitted to semi-fuzzy and fuzzy quantifiers. However, the required changes will be apparent in all cases, and the resulting generalized concepts are absolutely straightforward. Apart from adopting concepts from TGQ, some other postulates will also originate from logics, or capture important theoretical considerations. This type of requirement is concerned with the precise interpretation of the standard logical quantifiers; with the expected behaviour of 'induced' concepts like the induced extension principle; the interrelation between unary and multi-place quantification, and others. Finally, some of the plausibility criteria are targeted at the issues raised by the presence of fuzzy sets, and thus have something to say about our understanding of fuzziness in natural language. In each individual case, I have usually tried to set the concept in context and explain its relevance to the theory of fuzzy quantification. Following this brief motivation, I then develop the precise definition, by proposing a criterion which formalizes the semantical postulate. Having introduced the required concepts, the models are then related to the proposed criterion, and judged from their compliance with the given formal requirement.

In order to organize the wealth of results, the material has been split into three subsequent chapters. The present chapter starts by investigating those requirements that I consider absolutely basic, and mandatory for all models without ifs and buts. It will be shown that all of these criteria express universal properties of a DFS, and are indeed valid in arbitrary models of the theory. The subsequent chapter, by contrast, will focus on those concepts that relate several models, and hence require more homogeneity on the model's side. It is there that I consider several ways of granulating the models into natural subclasses, which provide the necessary context of models that fullfill minimal requirements on structural similarity. The last chapter in this series, finally turns to special postulates which cannot be demanded of arbitrary models for various reasons. For example, some possible criteria will conflict with other desirable properties and must hence be considered optional; other characteristics are necessary for practical applications but irrelevant in theoretical contexts; and some conceivable postulates are even inconsistent with the core axioms. The latter examples are valuable in particular, because they demarcate the maximal set of adequacy properties that can be expected of optimal models, by shaping the area beyond these theoretical limits.

4.2 Correct generalisation

Let us start with the most basic requirement on any type of fuzzification mechanism, that of *correct generalisation*. It is essential to the use of the fuzzification pattern that the generalized model which results from the fuzzification mechanism consistently extends the crisp base model from which it is built. In the case of quantifier fuzzification mechanisms, we would hence like the fuzzy quantifiers $\mathcal{F}(Q)$ that result from the mechanism to coincide with the original semi-fuzzy quantifiers Q on arbitrary crisp arguments. I have already shown on p. 96 how this requirement can be succinctly expressed in terms of the 'underlying semi-fuzzy quantifiers' $\mathcal{U}(\widetilde{Q})$, and it then becomes $\mathcal{U}(\mathcal{F}(Q)) = Q$. Acknowledging its importance, correct generalisation has been made an integral part of the DFS axioms, and I have required in (Z-1) that $\mathcal{U}(\mathcal{F}(Q)) = Q$ be valid for all nullary or unary quantifiers The restriction to quantifiers of arities $n \leq 1$ helped to simplify the axiom system, and hence to shorten the required proofs that certain QFMs of interest are indeed models of the theory. Let us now overcome the restriction to $n \leq 1$ and establish that $\mathcal{F}(Q)$ consistently extends Q, regardless of the arity $n \in \mathbb{N}$ of the considered quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Theorem 2
Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier. Then $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. for all crisp *subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,*

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)\,.$$

For example, if E is a set of persons, and **women**, **married** $\in \mathcal{P}(E)$ are the crisp sets of "women" and "married persons" in E, then

$$\mathcal{F}(\textbf{some})(\textbf{women}, \textbf{married}) = \textbf{some}(\textbf{women}, \textbf{married})\,,$$

i.e. the ‘fuzzy some’ obtained by applying $\mathcal{F}$ coincides with the (original) ‘crisp some’ whenever the latter is defined, which is of course highly desirable.

4.3 Properties of the induced truth functions

Let us now turn to the fuzzy truth functions induced by a DFS, which constitute the propositional part of its induced logic. The first theorem is concerned with the identity truth function $\mathrm{id}_{\mathbf{2}} : \mathbf{2} \longrightarrow \mathbf{2}$ defined by $\mathrm{id}_{\mathbf{2}}(0) = 0$, $\mathrm{id}_{\mathbf{2}}(1) = 1$.

Theorem 3 (Identity truth function)
In every DFS $\mathcal{F}$, $\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}}) = \mathrm{id}_{\mathbf{I}}$.

The identity truth function is hence generalised to the fuzzy identity truth function $\mathrm{id}_{\mathbf{I}} : \mathbf{I} \longrightarrow \mathbf{I}$ which maps every gradual truth value $x \in \mathbf{I}$ to itself, which is quite satisfactory.

As for negation, the standard choice in fuzzy logic is certainly $\neg : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\neg x = 1 - x$ for all $x \in \mathbf{I}$. The essential properties of this prototypical choice of negation are captured by the following definition, which guides reasonable choices of negation operators.

Definition 25 (Strong negation)
$\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *if it satisfies*

a. $\widetilde{\neg}\, 0 = 1$ *(boundary condition)*

b. $\widetilde{\neg}\, x_1 \geq \widetilde{\neg}\, x_2$ *for all* $x_1, x_2 \in \mathbf{I}$ *such that* $x_1 < x_2$ *(i.e.* $\widetilde{\neg}$ *is monotonically nonincreasing)*

c. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e.* $\widetilde{\neg}$ *is involutive).*

Notes

- Whenever the standard negation $\neg x = 1 - x$ is being assumed, I will drop the ‘tilde’-notation. Hence the standard fuzzy complement is denoted $\neg X$, where $\mu_{\neg X}(e) = 1 - \mu_X(e)$. Similarly, the external negation of a (semi-) fuzzy quantifier with respect to the standard negation is written $\neg Q$, and the antonym of a fuzzy quantifier with respect to the standard fuzzy complement is written as $\widetilde{Q}\neg$.
- Apart from the standard choice, further examples of these negation operators are e.g. Sugeno’s λ-complements [145].

As witnessed by the following theorem, the fuzzy negation induced by a DFS $\mathcal{F}$ is plausible in the sense of belonging to the class of strong negation operators.

Theorem 4 (Negation)
In every DFS $\mathcal{F}$, $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ is a strong negation operator.

With conjunction, there are several common choices in fuzzy logic (although the standard is certainly $\wedge = \min$). All of these belong to the class of t-norms, which capture the expectations on reasonable conjunction operators, cf. Schweizer/Sklar [139], Klir/Yuan [89, p. 61+].

Definition 26 (t-norms)
$\widetilde{\wedge} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is called a t-norm if it satisfies the following conditions.

a. $x \mathbin{\widetilde{\wedge}} 0 = 0$, *for all* $x \in \mathbf{I}$

b. $x \mathbin{\widetilde{\wedge}} 1 = x$, *for all* $x \in \mathbf{I}$

c. $x_1 \mathbin{\widetilde{\wedge}} x_2 = x_2 \mathbin{\widetilde{\wedge}} x_1$ *for all* $x_1,\, x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \le x_1'$, *then* $x_1 \widetilde{\wedge} x_2 \le x_1' \widetilde{\wedge} x_2$, *for all* $x_1,\, x_1',\, x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\wedge}$ *is monotonically nondecreasing)*

e. $(x_1 \mathbin{\widetilde{\wedge}} x_2) \mathbin{\widetilde{\wedge}} x_3 = x_1 \mathbin{\widetilde{\wedge}} (x_2 \mathbin{\widetilde{\wedge}} x_3)$, *for all* $x_1,\, x_2,\, x_3 \in \mathbf{I}$ *(associativity).*

The prototypical t-norm is the standard fuzzy conjunction $\wedge = \min$ (it is the largest t-norm). Other well-known examples of t-norms are the *algebraic product* $\widetilde{\wedge}_a$ and the *bounded product* $\widetilde{\wedge}_b$, which are defined by

$$\begin{aligned} x_1 \mathbin{\widetilde{\wedge}_a} x_2 &= x_1 \cdot x_2 \\ x_1 \mathbin{\widetilde{\wedge}_b} x_2 &= \max(0,\, x_1 + x_2 - 1) \end{aligned}$$

for all $x_1,\, x_2 \in \mathbf{I}$. For a less well-known examplar, consider $\widetilde{\wedge}_m$, defined by

$$x_1 \mathbin{\widetilde{\wedge}_m} x_2 = \begin{cases} 2x_1x_2 & : \ \max(x_1,\, x_2) < \frac{1}{2} \\ 1 + 2x_1x_2 - x_1 - x_2 & : \ \min(x_1,\, x_2) \ge \frac{1}{2} \\ \min(x_1,\, x_2) & : \ \text{else} \end{cases} \tag{22}$$

for all $x_1,\, x_2 \in \mathbf{I}$, see [46, p. 30]. Of course, there are many other t-norms; for a recent discussion of advanced topics and pointers to other publications, see e.g. [86, 104].
Let us now state that the fuzzy conjunction induced by a DFS is 'reasonable', in the sense of belonging to the class of t-norms.

Theorem 5 (Conjunction)
In every DFS $\mathcal{F}$, $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ is a t-norm.

The dual concept of t-norm is that of an s-norm or t-co-norm, which expresses the essential properties of fuzzy disjunction operators.

Definition 27 (s-norms)
$\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is called an s-norm if it satisfies the following conditions.

a. $x \mathbin{\widetilde{\vee}} 1 = 1$, *for all* $x \in \mathbf{I}$

b. $x \mathbin{\widetilde{\vee}} 0 = x$, *for all* $x \in \mathbf{I}$

c. $x_1 \mathbin{\widetilde{\vee}} x_2 = x_2 \mathbin{\widetilde{\vee}} x_1$ *for all* $x_1,\, x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \le x_1'$*, then* $x_1 \widetilde{\vee} x_2 \le x_1' \widetilde{\vee} x_2$*, for all* $x_1,\, x_1',\, x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\vee}$ *is monotonically nondecreasing)*

e. $(x_1 \mathbin{\widetilde{\vee}} x_2) \mathbin{\widetilde{\vee}} x_3 = x_1 \mathbin{\widetilde{\vee}} (x_2 \mathbin{\widetilde{\vee}} x_3)$*, for all* $x_1,\, x_2,\, x_3 \in \mathbf{I}$ *(associativity).*

Examples of s-norms are the standard choice $\vee = \max$ (it is the smallest s-norm), the algebraic sum $\widetilde{\vee}_a$ and the bounded sum $\widetilde{\vee}_b$ defined by

$$\begin{aligned} x_1 \mathbin{\widetilde{\vee}_a} x_2 &= x_1 + x_2 - x_1 \cdot x_2 \\ x_1 \mathbin{\widetilde{\vee}_b} x_2 &= \min(1,\, x_1 + x_2) \end{aligned}$$

for all $x_1, x_2 \in \mathbf{I}$.

Theorem 6 (Disjunction)
In every DFS, $x_1 \mathbin{\widetilde{\vee}} x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, x_2)$*, i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$*.*

Hence the fuzzy disjunction induced by a DFS is also plausible, and it is definable in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$. A similar point can be made about the other two-place logical connectives, see [46, p. 29 and p. 32]. For example, the induced implication of a DFS can be expressed in terms of $\widetilde{\vee}$ and $\widetilde{\neg}$ (and hence also in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$):

Theorem 7 (Implication)
In every DFS, $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2$.

The only connectives which are not apparently reducible to $\widetilde{\wedge}$ and $\widetilde{\neg}$, because their construction involves a subtle dependency between variables, are the antivalence xor and the equivalence $\leftrightarrow$. I will discuss these connectives later because they require more effort (see remarks on p. 165 and p. 176).

4.4 A different view of the induced propositional logic

The definition of induced fuzzy truth functions presented in Def. 11 is not the only possible choice, and I have already mentioned that an alternative construction has been developed, which is equally straightforward. In fact, the first publication on DFS theory [46] relied on that construction which has now become 'alternative'. It was only the desire to obtain an independent axioms system, which later guided the decision in [48] to introduce a new construction. This new construction has now become standard, because it indeed helped to eliminate some subtle interdependencies in the initial definition of the DFS axioms. In the following, I will explain the principles that underly the first construction of induced connectives, thus demonstrating that it is plausible as well. I will then go on and show that the 'new' and 'old' constructions coincide in every DFS, which makes an additional point that these models are well-motivated. Hence both constructions provide different views of the induced propositional logic from their specific perspective.

The alternative construction of induced truth function, which I define now, rests on the observation that (a) the set of crisp truth values $\mathbf{2} = \{0, 1\}$ and the powerset $\mathcal{P}(\{*\})$ are isomorphic; and (b) the set of continuous truth values $\mathbf{I} = [0, 1]$ and the fuzzy powerset $\widetilde{\mathcal{P}}(\{*\})$ are also isomorphic, where $\{*\}$ is an arbitrary singleton set, e.g. $\{*\} = \{\varnothing\}$. Suitable mappings which establish these isomorphisms are the apparent bijections (a) $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{2}$; and (b) $\widetilde{\pi}_* : \widetilde{\mathcal{P}}(\{*\}) \longrightarrow \mathbf{I}$. The basic idea is to utilize the former bijection for a transfer of the original truth function to a semi-fuzzy quantifier, to which the considered QFM $\mathcal{F}$ can be applied. Now utilizing the latter bijection, the resulting fuzzy quantifier can then be transformed into the desired fuzzy truth function. The alternative construction can hence be described as follows.

Definition 28
Let a mapping $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ be given. We can view f as a semi-fuzzy quantifier $f^* : \mathcal{P}(\{*\})^n \longrightarrow \mathbf{I}$ by defining

$$f^*(X_1, \ldots, X_n) = f(\pi_*(X_1), \ \ldots, \ \pi_*(X_n)) \,.$$

By applying the considered QFM $\mathcal{F}$, f^* is generalized to a fuzzy quantifier $\mathcal{F}(f^*) : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \mathbf{I}$, from which we obtain a fuzzy truth function $\widetilde{\widetilde{\mathcal{F}}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$,

$$\widetilde{\widetilde{\mathcal{F}}}(f)(x_1, \ \ldots x_n) = \mathcal{F}(f^*)(\widetilde{\pi}_*^{-1}(x_1), \ \ldots, \ \widetilde{\pi}_*^{-1}(x_n))$$

for all $x_1, \ \ldots, \ x_n \in \mathbf{I}$.

More details on this construction can be found in [46]. As already remarked above, it results in the same canonical choice of fuzzy truth functions, if the considered $\mathcal{F}$ is a DFS.

Theorem 8
In every DFS, $\widetilde{\mathcal{F}} = \widetilde{\widetilde{\mathcal{F}}}$.

$\widetilde{\mathcal{F}}$ can be distinct from $\widetilde{\widetilde{\mathcal{F}}}$ if $\mathcal{F}$ is not a DFS, though, and indeed the construction used in $\widetilde{\mathcal{F}}$ provided better support for developing DFS theory further and extracting an independent system of axioms. The alternative definition $\widetilde{\widetilde{\mathcal{F}}}$ was less suited to accomplish this task because the construction of $\widetilde{\widetilde{\mathcal{F}}}(\wedge)$ and $\widetilde{\widetilde{\mathcal{F}}}(\vee)$ involves multi-place quantification ($n = 2$), while the computation of $\widetilde{\mathcal{F}}(\wedge)$ and $\widetilde{\mathcal{F}}(\vee)$ is based on one-place quantification. It is this simplification which made it possible to formalize DFS theory in terms of independent conditions and develop the current set of DFS axioms (Z-1)–(Z-6).

I will now discuss homomorphism properties of every DFS with respect to operations on the argument sets.

4.5 Argument permutations

Definition 29 (Argument permutations)
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $\beta : \{1, \ \ldots, \ n\} \longrightarrow \{1, \ \ldots, \ n\}$

a permutation. By $Q\beta : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the semi-fuzzy quantifier defined by*

$$Q\beta(Y_1, \dots, Y_n) = Q(Y_{\beta(1)}, \, \dots, \, Y_{\beta(n)})\,,$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$*. In the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*, the quantifier* $\widetilde{Q}\beta : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined analogously.*

It is well-known that every permutation can be decomposed into a composition of transpositions. These express a single permutation step which simply swaps two selected elements in a given finite sequence:

Definition 30 (Transpositions)
For all $n \in \mathbb{N}$ *(*$n > 0$*) and* $i, \, j \in \{1, \, \dots, \, n\}$*, the* transposition $\tau_{i,\,j} : \{1, \, \dots, \, n\} \longrightarrow \{1, \, \dots, \, n\}$ *is defined by*

$$\tau_{i,\,j}(k) = \begin{cases} i & : \; k = j \\ j & : \; k = i \\ k & : \; \textit{else} \end{cases}$$

for all $k \in \{1, \, \dots, \, n\}$*. I further stipulate a succinct notation for* $\tau_{i,\,n}$*, which is abbreviated by* $\tau_i = \tau_{i,\,n}$*. The transposition* τ_i *has the effect of exchanging positions* i *and* n*.*

Notes

- The restricted class of transpositions τ_i, $i \in \{1, \, \dots, \, n\}$, is still capable of composing arbitrary permutations, because every transposition $\tau_{i,\,j}$ can be expressed as $\tau_{i,\,j} = \tau_i \circ \tau_j \circ \tau_i$.
- In the case of these simple transpositions, $Q\tau_i(X_1, \dots, X_n)$ becomes

  $$Q\tau_i(X_1, \dots, X_n) = Q(X_1, \, \dots, \, X_{i-1}, \, X_n, \, X_{i+1}, \, \dots, \, X_{n-1}, \, X_i)\,.$$

Due to the conceptual and notational simplicity of the transpositions τ_i, these offer the preferred representation of permutations that will be used throughout the report.
The importance of argument permutations/transpositions is witnessed by the following examples.

1. There is a meaning of **only** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ where **only** $=$ **all**τ_1, i.e.

   $$\textbf{only}(X_1, \, X_2) = \textbf{all}(X_2, \, X_1)$$

   for all $X_1, \, X_2 \in \mathcal{P}(E)$. For example, if E is a set of persons, **men** $\in \mathcal{P}(E)$ the set of those $v \in E$ which are men, and **smokers** $\in \mathcal{P}(E)$ is the set[24] of persons who are smokers, then

   $$\textbf{only}(\textbf{men}, \, \textbf{smokers}) = \textbf{all}\tau_1(\textbf{men}, \, \textbf{smokers}) = \textbf{all}(\textbf{smokers}, \, \textbf{men})\,,$$

   i.e. the meaning of “Only men are smokers” coincides with that of “All smokers are men”.

---

[24] we shall assume that this set be crisp for the sake of argument

2. Argument transpositions render it possible to express symmetry properties of quantifiers. For example, the quantifier $\textbf{some} : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is symmetrical in its arguments, which can be stated as

   $$\textbf{some} = \textbf{some}\tau_1 \,.$$

   This warrants that in the above domain of persons/smokers,

   $$\textbf{some}(\textbf{men}, \textbf{smokers}) = \textbf{some}\tau_1(\textbf{men}, \textbf{smokers}) = \textbf{some}(\textbf{smokers}, \textbf{men}) \,,$$

   which expresses our intuition that the meanings of “Some men are smokers” and “Some smokers are men” coincide.

As stated by the following theorem, DFS theory ensures that all considered models of fuzzy quantification commute with argument transpositions, and hence guarantees that the corresponding constructions based on fuzzy quantifiers and fuzzy arguments still exhibit the desired semantics.

Theorem 9
Every DFS $\mathcal{F}$ is compatible with argument transpositions, i.e. whenever a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of $i \in \{1, \ldots, n\}$ are given, then $\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i$.

Recalling that permutations can be expressed as a sequence of transpositions, Th-9 actually ensures that $\mathcal{F}$ commutes with arbitrary permutations of the arguments of a quantifier. In particular, symmetry properties of a quantifier Q carry over to its fuzzified analogue $\mathcal{F}(Q)$. For example, it holds in every DFS that $\mathcal{F}(\textbf{some}) = \mathcal{F}(\textbf{some})\tau_1$ and hence

$$\mathcal{F}(\textbf{some})(\textbf{rich}, \textbf{young}) = \mathcal{F}(\textbf{some})(\textbf{young}, \textbf{rich}) \,,$$

i.e. the meaning of “Some rich people are young” and “Some young people are rich” coincide. Of course, we also obtain that

$$\mathcal{F}(\textbf{only})(\textbf{old}, \textbf{rich}) = \mathcal{F}(\textbf{all})\tau_1(\textbf{old}, \textbf{rich}) = \mathcal{F}(\textbf{all})(\textbf{rich}, \textbf{old}) \,,$$

i.e. “Only old people are rich” means that “All rich people are old”, where the extensions $\textbf{old}, \textbf{rich} \in \widetilde{\mathcal{P}}(E)$ of old and rich people are now assumed to be fuzzy.

4.6 Cylindrical extensions

Let us now consider another property related to the argument structure of quantifiers. In the case that an argument is ‘redundant’ or ‘vacuous’, by having no effect on the actual outcome of a semi-fuzzy quantifier Q, we would certainly expect that the argument will also have no effect on the quantification results obtained from the associated fuzzy quantifier $\mathcal{F}(Q)$. This intuitive criterion can be formalized as follows.

Definition 31
Suppose $\mathcal{F}$ is a QFM. We say that $\mathcal{F}$ is compatible with cylindrical extensions *if the following condition holds for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Whenever $n' \in \mathbb{N}$, $n' \geq n$; $i_1, \dots, i_n \in \{1, \dots, n'\}$ such that $1 \leq i_1 < i_2 < \cdots < i_n \leq n'$, and $Q' : \mathcal{P}(E)^{n'} \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1, \dots, Y_{n'}) = Q(Y_{i_1}, \dots, Y_{i_n})$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$, then

$$\mathcal{F}(Q')(X_1, \dots, X_{n'}) = \mathcal{F}(Q)(X_{i_1}, \dots, X_{i_n}) \,,$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

This property of being compatible with cylindrical extensions is very fundamental. It simply states that vacuous argument positions of a quantifier can be eliminated. For example, if $Q' : \mathcal{P}(E)^4 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and if there exists a semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ such that $Q'(Y_1, Y_2, Y_3, Y_4) = Q(Y_3)$ for all $Y_1, \dots, Y_4 \in \mathcal{P}(E)$, then we know that Q' does not really depend on all arguments; it is apparent that the choice of Y_1, Y_2 and Y_4 has no effect on the quantification result. It is hence straightforward to require that $\mathcal{F}(Q')(X_1, X_2, X_3, X_4) = \mathcal{F}(Q)(X_3)$ for all $X_1, \dots, X_4 \in \widetilde{\mathcal{P}}(E)$, i.e. $\mathcal{F}(Q')$ is also independent of X_1, X_2, X_4, and it can be computed from $\mathcal{F}(Q)$.

Theorem 10

Every DFS $\mathcal{F}$ is compatible with cylindrical extensions.

Hence every model of DFS theory fulfills a property which is vital to every plausible model of multi-place quantification.

4.7 Negation and antonyms

We have considered so far the elementary adequacy properties of QFMs under the semantical constructions of permuting and vacuously extending arguments. The compatibility with these constructions is essential for the approach to be internally consistent albeit rather formal in nature. I now turn to properties that also express on the linguistic surface. I will first consider those constructions that involve a negation step, which can either be applied inside the quantifying expression (internal complementation of arguments, antonyms), or applied from 'outside' to the quantifying expression as a whole (external negation). The compatibility of the models of DFS theory to these semantical constructions is in each case ensured by their known compatibility to dualisation (Z-3) and by the elementary properties underlying the framework that were discussed above.

Theorem 11
Every DFS $\mathcal{F}$ is compatible with the formation of antonyms. Hence if $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$, then $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\widetilde{\neg}$.

The theorem guarantees e.g. that $\mathcal{F}(\textbf{all})(\textbf{rich}, \widetilde{\neg}\,\textbf{lucky}) = \mathcal{F}(\textbf{no})(\textbf{rich}, \textbf{lucky})$. Let us notice that by Th-9, the theorem generalises to arbitrary argument positions. Hence every DFS is fully compatible to the complementation of arguments.

Theorem 12
Every DFS $\mathcal{F}$ is compatible with the negation of quantifiers. Hence if $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, then $\mathcal{F}(\widetilde{\neg}\, Q) = \widetilde{\neg}\, \mathcal{F}(Q)$.

For example, the theorem ensures that

$$\mathcal{F}(\textbf{at most 10})(\textbf{young}, \textbf{rich}) = \widetilde{\neg}\, \mathcal{F}(\textbf{more than 10})(\textbf{young}, \textbf{rich})$$

because $\textbf{at most 10} = \widetilde{\neg}\, \textbf{more than 10}$, i.e. “at most 10” is the negation of “more than 10”.

We can summarize (Z-3), Th-11 and Th-12 as ensuring that every DFS preserves Aristotelian squares.[25] I.e. if $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an arbitrary semi-fuzzy quantifier, then the square

$$\begin{tikzcd}
Q \arrow[rr, leftrightarrow, "\text{antonym}"] \arrow[dd, leftrightarrow, "\text{dual}"'] & & Q\neg \arrow[dd, leftrightarrow, "\text{dual}"] \\
& \text{external negation} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
\widetilde{\neg}\, Q\neg \arrow[rr, leftrightarrow, "\text{antonym}"] & & \widetilde{\neg}\, Q
\end{tikzcd}$$

is transported by $\mathcal{F}$ to the corresponding square on fuzzy quantifiers,

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[25] Because all relations displayed in the Aristotelian square are bidirectional, it is presumed that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ be involutive. However, we already know from Th-4 that $\widetilde{\neg}$ is even a strong negation operator.

$$\begin{tikzcd}
\mathcal{F}(Q) \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[d, leftrightarrow, "\text{dual}"'] \arrow[dr, leftrightarrow] & \mathcal{F}(Q\neg) \arrow[d, leftrightarrow, "\text{dual}"] \\
\mathcal{F}(\widetilde{\neg}\, Q\neg) \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[ur, leftrightarrow, "\text{external negation}" description] & \mathcal{F}(\widetilde{\neg}\, Q)
\end{tikzcd}$$

where the indicated relations of antonymy, negation and duality are still valid, i.e. which coincides with

$$\begin{tikzcd}
\mathcal{F}(Q) \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[d, leftrightarrow, "\text{dual}"'] \arrow[dr, leftrightarrow] & \mathcal{F}(Q)\widetilde{\neg} \arrow[d, leftrightarrow, "\text{dual}"] \\
\widetilde{\neg}\, \mathcal{F}(Q)\widetilde{\neg} \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[ur, leftrightarrow, "\text{external negation}" description] & \widetilde{\neg}\, \mathcal{F}(Q)
\end{tikzcd}$$

Every DFS also commutes with the *Piaget group of transformations*, the relevance of which stems from empirical findings in developmental psychology. (For a note on the importance of the Piaget group of transformations to fuzzy logic, see Dubois & Prade [36, p. 158+]). The Piaget group corresponds to the formation of

(identity) $\mathrm{I}(Q) = Q$
(negation) $\mathrm{N}(Q) = \widetilde{\neg}\, Q$
(reciprocity) $\mathrm{R}(Q) = Q\tau_1\neg\tau_1 \ldots \tau_n\neg\tau_n$
(correlativity) $\mathrm{C}(Q) = \widetilde{\neg}\, \mathrm{R}(Q)\,,$

where $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier (analogously for fuzzy quantifiers).[26] This is obvious from the definability of I, N, R, C in terms of external negation, internal complementation and argument transpositions, with which every DFS is compatible by Th-12, Th-11 and Th-9.

4.8 Symmetrical difference

We already know from Th-11 that every DFS is compatible with argument-wise complementation. Let us now establish that $\mathcal{F}$ respects even more fine-grained application of the negation operator.

Definition 32

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier ($n > 0$) and $A \in \mathcal{P}(E)$ a crisp subset of E. By $Q\triangle A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the semi-fuzzy quantifier defined by

$$Q\triangle A(X_1, \ldots, X_n) = Q(X_1,\ \ldots,\ X_{n-1},\ X_n\triangle A)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$, where $\triangle$ denotes the symmetrical set difference. For fuzzy quantifiers $\widetilde{Q}$, we define $\widetilde{Q}\widetilde{\triangle}A$ analogously, where the fuzzy symmetrical difference $X_1 \widetilde{\triangle} X_2 \in \widetilde{\mathcal{P}}(E)$ is defined by $\mu_{X_1 \widetilde{\triangle} X_2}(e) = \mu_{X_1}(e) \mathbin{\widetilde{\mathrm{xor}}} \mu_{X_2}(e)$ for all $e \in E$.

By using the symmetrical difference, we can negate the membership values of only *part* of the elements of an argument set (namely of those which are contained in A). Hence if $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, then

$$Q\triangle A(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, Z)$$

where

$$\chi_Z(e) = \begin{cases} \neg\chi_{Y_n}(e) & : \quad e \in A \\ \chi_{Y_n}(e) & : \quad e \notin A \end{cases}$$

for all $Y_1, \ldots, Y_n, A \in \mathcal{P}(E)$ and $e \in E$. Likewise if $Q : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy quantifier and $A \in \mathcal{P}(E)$ is crisp, then

$$Q\widetilde{\triangle}A(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{n-1}, Z)$$

where

$$\mu_Z(e) = \begin{cases} \widetilde{\neg}\,\mu_{X_n}(e) & : \quad e \in A \\ \mu_{X_n}(e) & : \quad e \notin A \end{cases}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $e \in E$. Let us now state that every DFS is compatible even with this more fine-structured type of negation.

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[26] The operator-based definition of $\mathrm{R}(Q)$ might look complicated. Provided a choice of arguments $X_1, \ldots, X_n \in \mathcal{P}(E)$, this expression becomes

$$\mathrm{R}(Q)(X_1, \ldots, X_n) = Q(\neg X_1, \ldots, \neg X_n)\,.$$

Theorem 13
Suppose $\mathcal{F}$ is a DFS. Then for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ ($n > 0$) and every crisp subset $A \in \mathcal{P}(E)$, $\mathcal{F}(Q\triangle A) = \mathcal{F}(Q) \,\widetilde{\triangle}\, A$.

4.9 Intersections of arguments

The DFS axiom (Z-4) explicitly requires that a plausible model of fuzzy quantification be compatible with unions of arguments. In addition, we have already seen in Th-11 that the models of DFS theory preserve antonymy, and are hence compatible with the complementation of arguments as well. Now I will turn to the construction of intersecting arguments which has not been considered so far. It is again convenient to introduce an operator-based notation.

Definition 33 (Internal meets)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$. The semi-fuzzy quantifier $Q\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by

$$Q\cap(X_1, \ldots, X_{n+1}) = Q(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1}),$$

for all $X_1, \ldots, X_{n+1} \in \mathcal{P}(E)$. In the case of a fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\widetilde{Q}\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously, based on the induced intersection $\widetilde{\cap}$.

The compatibility of a QFM with intersections in the arguments of a quantifier is then expressed in the apparent way, i.e. $\mathcal{F}$ should satisfy $\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\cap}$ for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$. Knowing that every DFS commutes with unions of arguments, it is not surprising that we also obtain a positive result in the dual case of intersections:

Theorem 14
Every DFS $\mathcal{F}$ is compatible with the intersection of arguments.

For example, $\mathcal{F}(\textbf{some}) = \mathcal{F}(\exists)\widetilde{\cap}$, because the two-place quantifier **some** can be expressed as $\textbf{some} = \exists\cap$.

Let us also notice that from Th-9, every DFS is then in fact known to commute with intersections in arbitrary argument positions. For example, consider the NL statement "Most of the young and rich are tall". In order to interpret this statement and construct its meaning from the fuzzy extensions $\textbf{young}, \textbf{rich}$ and $\textbf{tall} \in \widetilde{\mathcal{P}}(E)$, we use a decomposition into the quantifier Q' defined by $Q'(Y_1, Y_2, Y_3) = \textbf{most}(Y_1 \cap Y_2, Y_3)$, which expresses "Most Y_1's and Y_2's are Y_3's". It is then reasonable to expect that the above statement can be interpreted by applying the resulting fuzzy quantifier $\mathcal{F}(Q')$ to the arguments $\textbf{young}, \textbf{rich}$ and **tall**. This is exactly where the considered property applies, and we conclude that indeed $\mathcal{F}(Q')(\textbf{young}, \textbf{rich}, \textbf{tall}) = \mathcal{F}(\textbf{many})(\textbf{young} \,\widetilde{\cap}\, \textbf{rich}, \textbf{tall})$, as desired.

4.10 Argument insertion

Let us now consider the operation of inserting an argument into a quantifier. By isolating the insertion of a single argument, we obtain the following construction on semi-fuzzy quantifiers:

Definition 34 (Argument insertion)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier of arity* $n > 0$*, and* $A \in \mathcal{P}(E)$*. By* $Q \triangleleft A : \mathcal{P}(E)^{n-1} \longrightarrow \mathbf{I}$ *we denote the semi-fuzzy quantifier defined by*

$$Q \triangleleft A(X_1, \ldots, X_{n-1}) = Q(X_1, \ldots, X_{n-1}, A)$$

for all $X_1, \ldots, X_{n-1} \in \mathcal{P}(E)$*. (An analogous definition of* $\widetilde{Q} \triangleleft A$ *is assumed for fuzzy quantifiers).*

Note. As usual, attention is placed on the last argument. The insertion of other arguments can be modelled by a preceeding argument transposition. For example, we can insert A into the second argument slot by constructing $Q\tau_2 \triangleleft A$. The insertion of multiple arguments can then be decomposed into a sequence of single argument insertions. The significance of argument insertion mainly stems from its ability to model an important natural language construction known as *adjectival restriction*, see e.g. [8, p. 448] and [45, p. 247]. To present an example, an NL sentence like "Many married men have children" can be interpreted by inserting the arguments **married**, **have_children** $\in \mathcal{P}(E)$, which represent the extensions of the NL concepts "married" and "have children", respectively, into the composite quantifier "Many married X's are Y's". The involved composite quantifier is then said to be constructed from the base quantifier "many" by 'adjectival restriction', in this case based on the crisp adjective "married". To see how the above construction of argument insertion supports the modelling of adjectival restriction, let us simply notice that the construction of the composite quantifier Q' corresponds to an intersection with the denotation of the adjective, in this case $Q'(Y_1, Y_2) = \textbf{many}(Y_1 \cap \textbf{married}, Y_2)$. Adjectival restriction can hence be decomposed into the constructions of intersecting argument sets and the insertion of constant arguments like "married". In terms of the relevant operators on quantifiers, we can then express the composite quantifier Q' as

$$Q' = Q\tau_i \cap \triangleleft A\tau_i \,.$$

This has the desired effect of restricting the i-th argument of the semi-fuzzy quantifier Q to a considered subset $A \in \mathcal{P}(E)$, which usually results from the interpretation of a crisp extensional adjective like "married".

The concept of argument insertion is also important because it allows an incremental interpretation of complex expressions in the sense of Frege's *compositionality principle* (see e.g. Gamut [45, p.140]), which states that the meaning of a complex expression is a function of the meanings of its subexpressions. To provide an example (which again illustrates the mechanism of restricting an argument), let us consider the sentence "Most male persons are married". Suppose that E is a base set, and

$\textbf{person}, \textbf{male}, \textbf{married} \in \mathcal{P}(E)$ are the extensions of “person”, “male” and “married” in E, respectively. Informally, we can think of this sentence as being interpreted in the following steps.[27]

$$\begin{aligned} &Q = \text{Most } X_1\text{'s are } X_2\text{'s} \\ &Q' = \text{Most male } Y_1\text{'s are } Y_2\text{'s} \\ &Q'' = \text{Most male persons are } Z\text{'s} \\ &Q''' = \text{Most male persons are married} \end{aligned}$$

Ever since R. Montague published his influential work on ‘the proper treatment of quantification in ordinary English’ [107] (reprinted in [154]), formal linguists would typically model the incremental interpretation process in some variant of the typed λ-calculus. This might look roughly[28] as follows:

$$\begin{aligned} &Q = \lambda X_1 X_2.\textbf{most}(X_1, X_2) = \textbf{most} \\ &Q' = \lambda Y_1 Y_2.\textbf{most}(Y_1 \cap \textbf{male}, Y_2) \\ &Q'' = \lambda Z.Q'(\textbf{person}, Z) \\ &Q''' = Q''(\textbf{married})\,. \end{aligned}$$

By means of the operators that were defined on semi-fuzzy quantifiers, I can recast this

$$\begin{aligned} &Q = \textbf{most} \\ &Q' = \textbf{most}\tau_1 \cap\!\triangleleft \textbf{male}\tau_1 \\ &Q'' = Q'\tau_1 \triangleleft \textbf{person} \\ &Q''' = Q'' \triangleleft \textbf{married}\,. \end{aligned}$$

The following theorem establishes the compatibility of a DFS to the insertion of arguments like “married”.

Theorem 15

Every $\mathcal{F}$ is compatible with argument insertions, i.e. $\mathcal{F}(Q \triangleleft A) = \mathcal{F}(Q) \triangleleft A$ for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and all crisp subsets $A \in \mathcal{P}(E)$.

The theorem hence states that every DFS commutes with the insertion of *crisp* arguments. It is limited in scope to the case of crisp arguments because, given a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a *fuzzy* subset $A \in \widetilde{\mathcal{P}}(E)$, $Q \triangleleft A$ would be undefined. (Q is a *semi*-fuzzy quantifier and hence defined on crisp arguments only!) Nevertheless, the axiom ensures that

$$\mathcal{F}(\textbf{all}\tau_1 \cap\!\triangleleft \textbf{married}\tau_1)(\textbf{men}, \textbf{lucky}) = \mathcal{F}(\textbf{all})(\textbf{married} \mathbin{\widetilde{\cap}} \textbf{men}, \textbf{lucky})$$

---

[27] This decomposition only serves illustrative purposes. A formal linguist is likely to prefer a different decomposition of the above sentence.

[28] For convenience, I will use a variant of the typed λ-calculus which offers product types. In monadic type theory, the expressions might look somewhat different.

provided that **married** $\in \mathcal{P}(E)$ is crisp, i.e. we obtain the same result when applying $\mathcal{F}$ to the composite quantifier

$$Q = \text{all married } Y_1\text{'s are } Y_2$$

and then evaluating $\mathcal{F}(Q)(\textbf{men}, \textbf{lucky})$, or when first applying $\mathcal{F}$ to **all** and then inserting the extensions of "married men" and "lucky". The theorem is also useful because it generally ensures that *boundary conditions* (with respect to the crisp case) are valid. For example, it is apparent from the above theorem that

$$x_1 \mathbin{\widetilde{\wedge}} 1 = x_1$$

for all $x_1 \in \mathbf{I}$, which is one of the defining conditions of t-norms.

I will return later to the problem that adjectival restriction with a fuzzy adjective cannot be modelled directly simply because fuzzy arguments cannot be inserted into a semi-fuzzy quantifier. It is apparent from this simple fact, that a construction different from $Q \triangleleft A$ is needed to handle fuzzy adjectival restriction. It will be shown later in section 6.8, how the insertion of fuzzy arguments into semi-fuzzy quantifiers can be modelled in the framework of DFS theory. The latter construction will then permit a compositional interpretation of fuzzy adjectival restriction as well.

4.11 Monotonicity properties

In this section, I will discuss monotonicity properties of quantifiers, i.e. semantical characteristics that can be expressed through a comparison of gradual quantification results under the natural order '$\leq$'. Such comparisons are of obvious relevance to logic because small results (close to 'false') generally reflect stronger conditions, while a weakening of conditions expresses in larger membership grades (close to 'true').

Theorem 16
Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Then Q is monotonically nondecreasing (nonincreasing) in its i-th argument ($i \leq n$) if and only if $\mathcal{F}(Q)$ is monotonically nondecreasing (nonincreasing) in its i-th argument.

For example, **some** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is monotonically nondecreasing in both arguments. By the theorem, then, $\mathcal{F}(\textbf{some}) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ is nondecreasing in both arguments also. In particular,

$$\mathcal{F}(\textbf{some})(\textbf{young\_men}, \textbf{very\_tall}) \leq \mathcal{F}(\textbf{some})(\textbf{men}, \textbf{tall})\,,$$

i.e. "Some young men are very tall" entails "Some men are tall", if **young_men** $\subseteq$ **men** and **very_tall** $\subseteq$ **tall**.

So far, we have only considered global monotonicity properties, i.e. monotonicity properties which hold unconditionally and for arbitrary choices of argument sets. In some cases, it can also be instructive to consider monotonicity properties which hold only locally, in a specified range of argument sets.

Definition 35
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $U,\ V \in \mathcal{P}(E)^n$ *are given. We say that* Q is locally nondecreasing in the range $(U,\ V)$ *if for all* $X_1, \ldots, X_n, X_1', \ldots, X_n' \in \mathcal{P}(E)$ *such that* $U_i \subseteq X_i \subseteq X_i' \subseteq V_i$ *(*$i = 1,\ \ldots,\ n$*), we have* $Q(X_1, \ldots, X_n) \leq Q(X_1',\ \ldots,\ X_n')$*. We will say that* Q *is locally nonincreasing in the range* $(U,\ V)$ *if under the same conditions,* $Q(X_1, \ldots, X_n) \leq Q(X_1',\ \ldots,\ X_n')$*. On fuzzy quantifiers, local monotonicity is defined analogously, but* $X_1, \ldots, X_n, X_1', \ldots, X_n'$ *are taken from* $\widetilde{\mathcal{P}}(E)$*, and '*$\subseteq$*' is the fuzzy inclusion relation.*

To present an example, consider the proportional quantifier **more than 10 percent** $=$ [**rate** > 0.1], which is neither nonincreasing nor nondecreasing in its first argument. Nevertheless, some characteristics of the quantifier express themselves in its local monotonicity properties. For example, suppose that $A, B \in \mathcal{P}(E)$ are subsets of E, where A is assumed to be nonempty. Then **more than 10 percent** is locally nonincreasing in the range $((A, B),\ (A \cup \neg B, B))$, and it is locally nondecreasing in the range $((A, B),\ (A \cup B, B))$.
A frequently observed case of local monotonicity properties is that of a quantifier which is locally constant in some range (U, V), i.e. both nondecreasing and nonincreasing. Hence let again $E \neq \varnothing$ be some finite base set, $e \in E$ an arbitrary element of E, and $r \in (0, 1]$. Then, [**rate** $\geq r$] is locally constant in the range (U, V), where $U = (\{e\}, \varnothing)$ and $V = (E, \varnothing)$. If $\{e\} \subseteq X_1 \subseteq E$ and $\varnothing \subseteq X_2 \subseteq \varnothing$, i.e. $X_2 = \varnothing$, we always have

$$\frac{|X_1 \cap \varnothing|}{|X_1|} = \frac{|\varnothing|}{|X_1|} = \frac{0}{|X_1|} = 0\,,$$

and hence

$$[\textbf{rate} \geq r](X_1, \varnothing) = 0\,.$$

It is natural to require that $\mathcal{F}$ preserves such local monotonicity properties, i.e. if $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is locally nondecreasing (nonincreasing) in some range (U, V), then we expect $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ to be nondecreasing (nonincreasing) in that range as well.
Let us now state that every DFS preserves monotonicity properties of semi-fuzzy quantifiers even if these hold only locally, i.e. all considered models of fuzzy quantification comply with this requirement:

Theorem 17
Suppose $\mathcal{F}$ *is a DFS,* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a semi-fuzzy quantifier and* $U,\ V \in \mathcal{P}(E)^n$*. Then* Q *is locally nondecreasing (nonincreasing) in the range* $(U,\ V)$ *if and only if* $\mathcal{F}(Q)$ *is locally nondecreasing (nonincreasing) in the range* $(U,\ V)$*.*

The theorem hence ensures that those characteristics of a quantifier which become visible through its local monotonicity properties be preserved when applying a DFS. Hence in the second example above, we obtain that $\mathcal{F}([\textbf{rate} \geq r])(X_1, \varnothing) = 0$ for all fuzzy subsets with nonempty core, which is quite satisfying.

The models can also be shown to be *monotonic* in the sense of preserving inequalities between quantifiers. Let us firstly define a partial order $\leq$ on (semi-)fuzzy quantifiers.

Definition 36
Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers. Let us write* $Q \leq Q'$ *if for all* $X_1, \ldots, X_n \in \mathcal{P}(E)$, $Q(X_1, \ldots, X_n) \leq Q'(X_1, \ldots, X_n)$. *On fuzzy quantifiers, we define* $\leq$ *analogously, where* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

For example, $[\textbf{rate} > 0.5] \leq [\textbf{rate} > 0.2]$, which reflects our intuition that "More than 50 percent of the Y_1's are Y_2's" is a stronger condition than "More than 20 percent of the Y_1's are Y_2's".

Theorem 18
Suppose $\mathcal{F}$ *is a DFS and* $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers. Then* $Q \leq Q'$ *if and only if* $\mathcal{F}(Q) \leq \mathcal{F}(Q')$.

The theorem ensures that inequalities between quantifiers carry over to the corresponding fuzzy quantifiers. Hence

$$\mathcal{F}(\textbf{more than 50 percent})(\textbf{blonde}, \textbf{tall}) \leq \mathcal{F}(\textbf{more than 20 percent})(\textbf{blonde}, \textbf{tall}),$$

as desired.
Let me also give another example, which illustrates the utility of cylindrical extensions. We firstly observe that by Def. 36, $Q \leq Q'$ is defined only if Q and Q' have the same arity n. But it may be useful to express inequalities also in the case of quantifiers involving a different number of arguments. For example, $\textbf{all}\cap : \mathcal{P}(E)^3 \longrightarrow \mathbf{2}$ can be said to be smaller than $\textbf{all} : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ in the sense that

$$\textbf{all}\cap(Y_1, Y_2, Y_3) = \textbf{all}(Y_1, Y_2 \cap Y_3) \leq \textbf{all}(Y_1, Y_2),$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$. Let us now establish that this kind of inequality is preserved by every DFS $\mathcal{F}$. We define the following cylindrical extension $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{2}$ of $\textbf{all} : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$, viz

$$Q(Y_1, Y_2, Y_3) = \textbf{all}(Y_1, Y_2),$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$. This derived semi-fuzzy quantifier Q has the same arity as $\textbf{all}\cap$, and hence the above theorem Th-18 is now applicable. We can further utilize the earlier theorem Th-10 and conclude that

$$\begin{aligned} \mathcal{F}(\textbf{all})(X_1, X_2) &= \mathcal{F}(Q)(X_1, X_2, X_3) && \text{by Th-10} \\ &\leq \mathcal{F}(\textbf{all}\cap)(X_1, X_2, X_3) && \text{by Th-18} \\ &= \mathcal{F}(\textbf{all})(X_1, X_2 \,\widetilde{\cap}\, X_3) \end{aligned}$$

for all $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$. In particular, if $\textbf{old}, \textbf{bald}, \textbf{rich} \in \widetilde{\mathcal{P}}(E)$ are the (fuzzy) extensions of "old", "bald" and "rich", respectively, in our example universe E, then

$$\mathcal{F}(\textbf{all})(\textbf{old}, \textbf{bald} \,\widetilde{\cap}\, \textbf{rich}) \leq \mathcal{F}(\textbf{all})(\textbf{old}, \textbf{bald}),$$

i.e. "All old are bald and rich" expresses a stronger condition than "All old are bald".

Definition 37
Suppose $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers and* $U, V \in \mathcal{P}(E)^n$. *We say that* Q_1 *is (not necessarily strictly) smaller than* Q_2 *in the range* (U, V)*, in symbols:* $Q_1 \leq_{(U,V)} Q_2$*, if for all* $X_1, \ldots, X_n \in \mathcal{P}(E)$ *such that* $U_1 \subseteq X_1 \subseteq V_1, \ldots, U_n \subseteq X_n \subseteq V_n$*,*

$$Q_1(X_1, \ldots, X_n) \leq Q_2(X_1, \ldots, X_n) \,.$$

On semi-fuzzy quantifiers, $Q_1 \leq_{(U,V)} Q_2$ *is defined analogously, but* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*, and '*$\subseteq$*' denotes the fuzzy inclusion relation.*

For example, the two-place quantifier “all” is smaller than “some” whenever the first argument is nonempty, i.e.

$$\textbf{all} \leq_{((\{e\},E\}),\,(E,E))} \textbf{some}\,,$$

for all $e \in E$.
As I now state, every model preserves inequalities between quantifiers even if these hold only locally.

Theorem 19
Suppose $\mathcal{F}$ *is a DFS,* $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $U, V \in \mathcal{P}(E)^n$*. Then*

$$Q_1 \leq_{(U,V)} Q_2 \Leftrightarrow \mathcal{F}(Q_1) \leq_{(U,V)} \mathcal{F}(Q_2) \,.$$

The theorem hence ensures that local inequalities, like those observed in the case of “some” and “all”, are preserved when applying a DFS. In particular, if **tall**, **lucky** $\in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E and **tall** has nonempty support, then

$$\textbf{all}(\textbf{tall}, \textbf{lucky}) \leq \textbf{some}(\textbf{tall}, \textbf{lucky}) \,,$$

as desired.

4.12 Properties of the induced extension principle

Now I turn to the formal properties of the induced extension principle of $\mathcal{F}$. This analysis substantiates that the induced extension principle achieves a plausible assignment of fuzzy powerset mappings to given base mappings. Building on this foundation, it becomes possible to establish that all considered models of fuzzy quantification preserve the important semantical properties of quantitativity and extensionality, which are closely tied to the induced extension principle. In addition, every DFS can be shown to be contextual, and hence captures some intuitive requirements on the intended effects of fuzziness in a quantifier's arguments.
Let us now state the formal results that underly the later theorems of linguistic relevance.

Theorem 20
Suppose $\mathcal{F}$ *is a DFS and* $\widehat{\mathcal{F}}$ *the extension principle induced by* $\mathcal{F}$*. Then for all* $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ *(where* $E \neq \varnothing$, $E' \neq \varnothing$, $E'' \neq \varnothing$*),*

a. $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$

b. $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$

Note. A reader familiar with category theory will recognize this as the statement that $\widehat{\mathcal{F}}$ is a covariant functor from the category of non-empty sets to the category of fuzzy power sets, provided that on objects E, we define $\widehat{\mathcal{F}}(E) = \widetilde{\mathcal{P}}(E)$.[29]It should be apparent from the remarks on p. 111 how to dispense with the restriction to non-empty sets if so desired. $\widehat{\mathcal{F}}$ is a faithful functor because $f \neq g$ implies $\widehat{\mathcal{F}}(f)|_{\mathcal{P}(E)} = \widehat{f} \neq \widehat{g} = \widehat{\mathcal{F}}(g)|_{\mathcal{P}(E)}$, i.e. $\widehat{\mathcal{F}}(f) \neq \widehat{\mathcal{F}}(g)$. $\widehat{\mathcal{F}}$ is also injective on objects; $E \neq E'$ implies $\widetilde{\mathcal{P}}(E) \neq \widetilde{\mathcal{P}}(E')$.

The induced extension principles of all DFSes coincide on injective mappings with the apparent plausible definition:

Theorem 21

Suppose $\mathcal{F}$ is a DFS and $f : E \longrightarrow E'$ is an injection. Then for all $X \in \widetilde{\mathcal{P}}(E)$, $e \in E'$,

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e) = \begin{cases} \mu_X(f^{-1}(e)) & : \quad e \in \operatorname{Im} f \\ 0 & : \quad e \notin \operatorname{Im} f \end{cases}$$

This result on the extension of injective mappings turned out to be rather useful and in the following, it will be used repeatedly.

Next I will establish that a DFS is compatible with exactly one extension principle. Knowing this might facilitate the proof that a given DFS induces a certain extension principle.

Theorem 22

Suppose $\mathcal{F}$ is a DFS and $\mathcal{E}$ an extension principle such that for every semi-fuzzy quantifier $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ and all $f_1 : E \longrightarrow E', \ \ldots, \ f_n : E \longrightarrow E'$, $\mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i}) = \mathcal{F}(Q) \circ \overset{n}{\underset{i=1}{\times}} \mathcal{E}(f_i)$. Then $\widehat{\mathcal{F}} = \mathcal{E}$.

The extension principle $\widehat{\mathcal{F}}$ of a DFS $\mathcal{F}$ is uniquely determined by the fuzzy existential quantifiers $\mathcal{F}(\exists) = \mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ induced by $\mathcal{F}$.

Theorem 23

Suppose $\mathcal{F}$ is a given DFS. For every mapping $f : E \longrightarrow E'$ and all $e' \in E'$, $\mu_{\widehat{\mathcal{F}}(f)(\bullet)}(e') = \mathcal{F}(\exists)\widetilde{\cap}\triangleleft f^{-1}(e')$.

The converse can also be shown: the fuzzy existential quantifiers obtained from a DFS $\mathcal{F}$ are uniquely determined by its extension principle $\widehat{\mathcal{F}}$.

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[29] By the category of fuzzy power sets I mean the category in which the objects are fuzzy power sets $\widetilde{\mathcal{P}}(E)$, the morphisms are mappings $f : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ which to each fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ assign a fuzzy subset $f(X) \in \widetilde{\mathcal{P}}(E')$, and $\circ$ is the usual composition of functions.

Theorem 24
Let a DFS $\mathcal{F}$ *be given. If* $E \neq \varnothing$ *and* $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$*, then* $\mathcal{F}(\exists) = \widetilde{\pi}_\varnothing \circ \widehat{\mathcal{F}}(!)$ *,* *where* $! : E \longrightarrow \{\varnothing\}$ *is the mapping defined by* $!(x) = \varnothing$ *for all* $x \in E$*.*

This completes my analysis of the formal characteristics underlying the extension principles that are induced by the models of DFS theory. Some applications in which this analysis supported the proof of semantical properties of the models are stated below in sections 4.13 (quantitativity), 4.14 (extensionality) and 4.15 (contextuality).

The analysis of the extension principle presented here is complemented with the results on the interpretation of the standard quantifiers, which are presented below in section 4.16. This is because the induced extension principle is closely related to the interpretation of existential quantifiers. Its precise structure becomes apparent once we combine the above theorem Th-23 with the later explicit formula for existential quantification in the models, see Th-32.

4.13 Quantitativity

Many quantifiers of interest, like "almost all", "most" etc., do not depend on any particular characteristics of the elements in the base set. It is not the specific choice of elements that determines the quantification result, but only the quantitative aspects. In the finite case, such quantifiers can be defined from the cardinalities of the arguments (or cardinalities of Boolean combinations). Other quantifiers, like "John" (proper names) or "Most married X_1's are X_2's" (adjectival restriction), though, are closer tied to the domain and its particular elements. Cardinality information about combinations of the arguments is not sufficient to determine the semantical interpretation of these quantifiers in any given situation, and these quantifiers are therefore called 'non-quantitative'.[30] In TGQ, the notion of quantitativity is commonly given an elegant and convincing definition in terms of automorphism invariance, which traces back to Mostowski [108]. This formalisation is easily adopted to the case of semi-fuzzy quantifiers and even fuzzy quantifiers:

Definition 38 (Quantitative semi-fuzzy quantifier)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is called* quantitative *if for all automorphisms*[31] $\beta : E \longrightarrow E$ *and all* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*,*

$$Q(Y_1, \ldots, Y_n) = Q(\widehat{\beta}(Y_1),\ \ldots,\ \widehat{\beta}(Y_n))\,.$$

Similarly

Definition 39 (Quantitative fuzzy quantifier)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is said to be* quantitative *if for all automorphisms* $\beta : E \longrightarrow E$ *and all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,*

$$\widetilde{Q}(X_1, \ldots, X_n) = \widetilde{Q}(\hat{\hat{\beta}}(X_1),\ \ldots,\ \hat{\hat{\beta}}(X_n))\,,$$

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[30] Non-quantitative quantifiers are sometimes also dubbed 'qualitative'.

[31] i.e. bijections of E into itself

where $\hat{\hat{\beta}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ *is obtained by applying the standard extension principle.*

By Th-21, the induced extension principles of all DFSes coincide on injective mappings. Therefore, the explicit mention of the standard extension principle in the above definition does *not* tie its applicability to any particular choice of extension principle.

The definition in terms of automorphism invariance formalizes the expectation that a quantitative quantifier cannot rely on any specific properties of the individual elements gathered in the arguments. By contrast, the quantification results must remain invariant when the elements are consistently renamed or exchanged with others, provided that distinct elements are kept separate. To give an example, consider the base set $E = \{\mathrm{John}, \mathrm{Lucas}, \mathrm{Mary}\}$ and the automorphism β defined by $\beta(\mathrm{John}) = \mathrm{Lucas}$, $\beta(\mathrm{Lucas}) = \mathrm{Mary}$ and $\beta(\mathrm{Mary}) = \mathrm{John}$. In the case of the quantifier "all", which is quantitative, we then obtain that

$$\begin{aligned}&\textbf{all}(\{\mathrm{John}\}, \{\mathrm{John}, \mathrm{Lucas}\})\\ &= 1\\ &= \textbf{all}(\{\mathrm{Lucas}\}, \{\mathrm{Lucas}, \mathrm{Mary}\})\\ &= \textbf{all}(\widehat{\beta}(\{\mathrm{John}\}), \widehat{\beta}(\{\mathrm{John}, \mathrm{Lucas}\}))\,,\end{aligned}$$

as expected. The example also witnesses that the quantifier **John** $= \pi_{\mathrm{John}}$ is non-quantitative, because

$$\pi_{\mathrm{John}}(\{\mathrm{John}\}) = 1 \neq 0 = \pi_{\mathrm{John}}(\{\mathrm{Lucas}\}) = \pi_{\mathrm{John}}(\widehat{\beta}(\{\mathrm{John}\}))\,.$$

As stated by the following theorem, the quantitativity aspect of quantifiers is recognized by every model of DFS theory:

Theorem 25
Suppose $\mathcal{F}$ *is a DFS. For all semi-fuzzy quantifiers* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, Q *is quantitative if and only if* $\mathcal{F}(Q)$ *is quantitative.*

For example, the quantitative quantifiers **all**, **some** and **at least k** are mapped to quantitative fuzzy quantifiers $\mathcal{F}($**all**$)$, $\mathcal{F}($**some**$)$ and $\mathcal{F}($**at least k**$)$, respectively. On the other hand, the non-quantitative projection quantifier **john** $= \pi_{\mathrm{John}}$ is mapped to the fuzzy projection quantifier $\mathcal{F}($**john**$) = \widetilde{\pi}_{\mathrm{John}}$, which is also non-quantitative.

4.14 Extensionality

One of the characteristic properties of natural language quantifiers that has been discovered by TGQ is that of *having extension*: if $E \subseteq E'$ are base sets, the interpretation of the quantifier of interest in E is $Q_E : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, and its interpretation in E' is $Q_{E'} : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$, then

$$Q_E(Y_1, \ldots, Y_n) = Q_{E'}(Y_1, \ldots, Y_n)\,, \tag{23}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$, see e.g. [8, p. 453], [45, p. 250]. Having extension is a pervasive phenomenon. The reader may wish to check that all two-valued or semi-fuzzy quantifiers introduced so far possess this property.[32] For example, suppose E is a set of men and that **married**, **have_children** $\in \mathcal{P}(E)$ are subsets of E. Further suppose that we extend E to a larger base set E' which, in addition to men, also contains, say, their shoes. We should then expect that

$$\textbf{most}_E(\textbf{married}, \textbf{have\_children}) = \textbf{most}_{E'}(\textbf{married}, \textbf{have\_children}),$$

because the shoes we have added to E are neither men, nor do they have children. The cross-domain property of having extension expresses some kind of context insensitivity: given $Y_1, \dots, Y_n \in \mathcal{P}(E)$, we can add an arbitrary number of objects to our original domain E without altering the quantification result. Alternatively, we can drop elements of E which are irrelevant to all argument sets (i.e. not contained in the union of $Y_1, \dots, Y_n$). In other words: if a quantifier has extension, then

$$Q_E(Y_1, \dots, Y_n) = Q_{E_{\min}}(Y_1, \dots, Y_n),$$

where $E_{\min} = Y_1 \cup \dots \cup Y_n$, i.e. $Q_E(Y_1, \dots, Y_n)$ depends only on those elements $e \in E$ which are contained in at least one of the Y_i's; the choice of the total domain E has no impact on the quantification result as long as it is large enough to contain the argument sets $Y_1, \dots, Y_n$ of interest. Having extension is hence a robustness property of NL quantifiers with respect to the choice of the full domain E, which is often to some degree arbitrary. An analogous definition of having extension for fuzzy quantifiers is easily obtained from (23); in this case, the property must hold for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.[33,34] It is natural to require that the fuzzy quantifiers corresponding to given semi-fuzzy quantifiers which have extension also possess this property.

Definition 40 (Extensionality)
A QFM $\mathcal{F}$ is said to be extensional *if it preserves extension, i.e. if each pair of semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $Q' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ such that $E \subseteq E'$ and $Q'|_{\mathcal{P}(E)^n} = Q$, i.e. $Q(X_1, \dots, X_n) = Q'(X_1, \dots, X_n)$ for all $X_1, \dots, X_n \in \mathcal{P}(E)$, is mapped to fuzzy quantifiers $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{F}(Q') : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$ with $\mathcal{F}(Q')|_{\widetilde{\mathcal{P}}(E)^n} = \mathcal{F}(Q)$, i.e. $\mathcal{F}(Q)(X_1, \dots, X_n) = \mathcal{F}(Q')(X_1, \dots, X_n)$, for all X_1, ..., $X_n \in \widetilde{\mathcal{P}}(E)$.*

Theorem 26
Every DFS $\mathcal{F}$ is extensional.

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[32] With the possible exception of "many" in its absolute sense. However, absolute "many" can be modelled by a parametrized family of quantifiers that have extension, where the choice of the parameter is made from the context.

[33] Here we must view $\widetilde{\mathcal{P}}(E)$ as a subset of $\widetilde{\mathcal{P}}(E')$, with the obvious embedding $\widetilde{\mathcal{P}}(E) \ni X \mapsto X' \in \widetilde{\mathcal{P}}(E')$, where

$$\mu_{X'}(e) = \begin{cases} \mu_X(e) & : \; e \in E \\ 0 & : \; e \notin E \end{cases}$$

for all $e \in E'$.

[34] The reader is warned not to confuse this definition of fuzzy quantifiers that 'have extension' with the totally unrelated concept of an *extensional* fuzzy quantifier, introduced by Thiele [149].

This is apparent from (Z-6) and Th-21.
The role of this theorem to the semantics of fuzzy quantifiers becomes clear if the crisp concepts **married** and **have_children** in the above example are substituted with fuzzy replacements. Hence consider the NL statement "Few young are rich". Just as in the crisp example, we certainly expect

$$\mathcal{F}(\textbf{few}_E)(\textbf{young}, \textbf{rich})$$

to be invariant under the precise choice of domain, too, as long as it is large enough to contain the fuzzy sets **young**, **rich** of interest (to be precise, the support of the union of the fuzzy arguments). For example, adding other individuals (like the above shoes) to the domain, which are fully irrelevant to the arguments, should not affect the computed semantic value. By the above theorem, then, every considered model of fuzzy quantification is known to comply with these adequacy considerations.

4.15 Contextuality

The preservation of locally observed properties, like local monotonicity in Th-17 and Th-19, can be explained in terms of a fundamental property which underlies all models of DFS theory.

Unlike most other adequacy criteria discussed so far, this property, called 'contextuality', is not borrowed from logic or linguistics. By contrast, it captures an important aspect of the semantics of fuzzy quantification, which is directly related to the way we perceive fuzziness. In order to formalize contextuality, I first need to recall some familiar notions of fuzzy set theory. Hence let $X \in \widetilde{\mathcal{P}}(E)$ be a given fuzzy subset. The *support* and the *core* of X, in symbols: $\text{spp}(X) \in \mathcal{P}(E)$ and $\text{core}(X) \in \mathcal{P}(E)$, are defined by

$$\text{spp}(X) = \{e \in E : \mu_X(e) > 0\} \tag{24}$$

$$\text{core}(X) = \{e \in E : \mu_X(e) = 1\} . \tag{25}$$

In other words, $\text{spp}(X)$ contains all elements which potentially belong to X and $\text{core}(X)$ contains all elements which fully belong to X. The interpretation of a fuzzy subset X is hence ambiguous only with respect to crisp subsets Y in the *context range*

$$\text{cxt}(X) = \{Y \in \mathcal{P}(E) : \text{core}(X) \subseteq Y \subseteq \text{spp}(Y)\} . \tag{26}$$

For example, let the base set $E = \{a, b, c\}$ be given and suppose that $X \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset

$$\mu_X(e) = \begin{cases} 1 & : \quad x = a \text{ or } x = b \\ \frac{1}{2} & : \quad x = c \end{cases} \tag{27}$$

In this case, the corresponding context range becomes

$$\text{cxt}(X) = \{Y : \{a, b\} \subseteq Y \subseteq \{a, b, c\}\} = \{\{a, b\}, \{a, b, c\}\} .$$

Now let us consider the existential quantifier $\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Because $\exists(\{a, b\}) = \exists(\{a, b, c\}) = 1$, we know that $\exists(Y) = 1$ for all crisp subsets in the context range of X. We hence expect that $\mathcal{F}(\exists)(X) = 1$, simply because the crisp quantification result is always equal to one, regardless of whether we assume that $c \in X$ or that $c \notin X$. Abstracting from the example, we obtain the following apparent definition of contextual equality, relative to a given context range.

Definition 41 (Contextually equal)
Assume that $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *are given. We say that* Q *and* Q' *are* contextually equal *relative to* $(X_1, \ldots, X_n)$*, in symbols:* $Q \sim_{(X_1,\ldots,X_n)} Q'$*, if and only if*

$$Q|_{\mathrm{cxt}(X_1)\times\cdots\times\mathrm{cxt}(X_n)} = Q'|_{\mathrm{cxt}(X_1)\times\cdots\times\mathrm{cxt}(X_n)} \,,$$

i.e.

$$Q(Y_1, \ldots, Y_n) = Q'(Y_1, \ldots, Y_n)$$

for all $Y_1 \in \mathrm{cxt}(X_1), \ldots, Y_n \in \mathrm{cxt}(X_n)$*.*

It is apparent that for each $E \neq \varnothing$, $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, the resulting relation of contextual equality $\sim_{(X_1,\ldots,X_n)}$ is an equivalence relation on the set of all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Definition 42
A QFM $\mathcal{F}$ *is said to be* contextual *if for all* $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and every choice of fuzzy argument sets* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,*

$$Q \sim_{(X_1,\ldots,X_n)} Q'$$

entails that

$$\mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(X_1, \ldots, X_n) \,.$$

As illustrated by the otivating example, it is highly desirable that a QFM satisfies this very elementary albeit fundamental adequacy condition. And indeed, every DFS can be shown to elicit the desired property:

Theorem 27
Every DFS $\mathcal{F}$ *is contextual.*

To give an example of how contextuality is useful in establishing properties of QFMs, consider the following theorem which generalizes Th-15.

Theorem 28
Suppose $\mathcal{F}$ *is a contextual QFM which is compatible with cylindrical extensions. Then* $\mathcal{F}$ *is compatible with argument insertions, i.e.* $\mathcal{F}(Q \triangleleft A) = \mathcal{F}(Q) \triangleleft A$ *for all semi-fuzzy quantifiers* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *and all crisp subsets* $A \in \mathcal{P}(E)$*.*

Other applications of contextuality are presented below in Chap. 6, which substantiate the significance of the novel concept to the modelling of fuzzy quantification.

4.16 Semantics of the standard quantifiers

We have already seen in Th-23 and Th-24 that the existential quantifier can be expressed in terms of the induced extension principle and vice versa. In the following, I would like to present some further results on the interpretation of the universal and existential quantifiers in the considered models of fuzzy quantification.

Let me first recall Thiele's analysis of fuzzy universal and existential quantification [147, 148, 149]. Thiele has developed an axiomatic characterisation of these dual types of quantifiers and their expected semantics in the presence of fuzziness, which culminates in the proposed definitions of T-quantifiers and S-quantifiers. Thiele also develops representation theorems for T- and S-quantifiers which make explicit how these can be decomposed into possibly infinitary formulas involving t- or s-norms. These theorems, which will be presented below, have proven invaluable for establishing the results on the interpretation of the standard quantifiers in DFS theory.

Thiele's definition of T-quantifiers captures the essential requirements on fuzzy universal quantifiers. Making use of the concepts developed here, the proposed definition can be expressed as follows.

Definition 43 (T-quantifiers)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is called a* T-quantifier *if* $\widetilde{Q}$ *satisfies the following axioms:*

a. *For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$, $\widetilde{Q}(X \cup \neg\{e\}) = \mu_X(e)$*;*

b. *For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$, $\widetilde{Q}(X \cap \neg\{e\}) = 0$*;*

c. $\widetilde{Q}$ *is nondecreasing, i.e. for all* $X, X' \in \widetilde{\mathcal{P}}(E)$ *such that* $X \subseteq X'$*, it holds that* $\widetilde{Q}(X) \leq \widetilde{Q}(X')$*;*

d. $\widetilde{Q}$ *is quantitative, i.e. for every automorphism (permutation)* $\beta : E \longrightarrow E$, $\widetilde{Q} \circ \hat{\hat{\beta}} = \widetilde{Q}$.

Note. In the above definition, $\cap$ is the standard fuzzy intersection based on $\min$, and $\cup$ is the standard fuzzy union based on $\max$. However, all fuzzy intersections based on t-norms and all fuzzy unions based on s-norms will give the same results, because one of the arguments is a crisp subset of E.
There is a close relationship between T-quantifiers and t-norms. Following Thiele, I define the connective $\widetilde{\wedge}_{\widetilde{Q}}$ which corresponds to the T-quantifier.

Definition 44
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is a T-quantifier and* $|E| > 1$. $\widetilde{\wedge}_{\widetilde{Q}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$x_1 \,\widetilde{\wedge}_{\widetilde{Q}}\, x_2 = \widetilde{Q}(X)$$

for all $x_1, x_2 \in \mathbf{I}$, *where* $X \in \widetilde{\mathcal{P}}(E)$ *is defined by*

$$\mu_X(e) = \begin{cases} x_1 & : \quad e = e_1 \\ x_2 & : \quad e = e_2 \\ 1 & : \quad \textit{else} \end{cases} \tag{28}$$

where $e_1 \neq e_2$, $e_1, e_2 \in E$ *are two arbitrary distinct elements of* E.

Note. It is evident by the quantitativity of T-quantifiers that $x_1 \mathbin{\widetilde{\wedge}_{\widetilde{Q}}} x_2$ does not depend on the choice of $e_1, e_2 \in E$. It is Thiele's merit of having shown that T-quantifiers are exactly those quantifiers that can be decomposed into the following (possibly infinitary) construction based on an underlying t-norm:

Theorem 29 (Characterisation of T-quantifiers)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is a T-quantifier where* $|E| > 1$. *Then* $\widetilde{\wedge}_{\widetilde{Q}}$ *is a* t*-norm, and*

$$\widetilde{Q}(X) = \inf \left\{ \mathop{\widetilde{\wedge}_{\widetilde{Q}}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1,\ \ldots, a_m\} \in \mathcal{P}(E) \textit{ finite, } a_i \neq a_j \textit{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(See Thiele [147, Th-8.1, p.47])

Building upon Thiele's characterisation theorem for T-quantifiers, I was able to improve upon previous work on the interpretation of the standard quantifiers in DFS theory [46, Th-26, p.42] and show that the fuzzy universal quantifiers $\mathcal{F}(\forall)$ induced by a DFS are plausible in the sense of belonging to the class of T-quantifiers:

Theorem 30 (Universal quantifiers in DFSes)

Suppose $\mathcal{F}$ *is a DFS and* $E \neq \varnothing$ *is a given base set. Then* $\mathcal{F}(\forall) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is a T-quantifier constructed from the induced* t*-norm of* $\mathcal{F}$*, i.e.* $\mathcal{F}(\forall)$ *is defined by*

$$\mathcal{F}(\forall)(X) = \inf \left\{ \mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1,\ \ldots, a_m\} \in \mathcal{P}(E) \textit{ finite, } a_i \neq a_j \textit{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Thiele has also introduced the dual concept of S-quantifiers, which formalize the semantical requirements on reasonable fuzzy existential quantifiers. These are defined as follows (again adapted to my notation):

Definition 45 (S-quantifier)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is called an* S-quantifier *if* $\widetilde{Q}$ *satisfies the following axioms:*

a. *For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$, $\widetilde{Q}(X \cup \{e\}) = 1$;

b. For all $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$, $\widetilde{Q}(X \cap \{e\}) = \mu_X(e)$;

c. $\widetilde{Q}$ *is nondecreasing, i.e. for all* $X, X' \in \widetilde{\mathcal{P}}(E)$ *such that* $X \subseteq X'$, *it holds that* $\widetilde{Q}(X) \le \widetilde{Q}(X')$;

d. $\widetilde{Q}$ *is quantitative, i.e. for every automorphism (permutation)* $\beta : E \longrightarrow E$, $\widetilde{Q} \circ \hat{\hat{\beta}} = \widetilde{Q}$.

Again, it is possible to define a connective, denoted $\widetilde{\vee}_{\widetilde{Q}}$, which will play a special role in characterising the class of S-quantifiers.

Definition 46
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is an S-quantifier and* $|E| > 1$. $\widetilde{\vee}_{\widetilde{Q}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$x_1 \,\widetilde{\vee}_{\widetilde{Q}}\, x_2 = \widetilde{Q}(X)$$

for all $x_1, x_2 \in \mathbf{I}$, *where* $X \in \widetilde{\mathcal{P}}(E)$ *is defined by*

$$\mu_X(e) = \begin{cases} x_1 & : \quad e = e_1 \\ x_2 & : \quad e = e_2 \\ 0 & : \quad \textit{else} \end{cases} \tag{29}$$

and $e_1 \neq e_2$, $e_1, e_2 \in E$ *are two arbitrary distinct elements of* E.

Note. Again, the independence of $\widetilde{\vee}_{\widetilde{Q}}$ on the chosen elements $e_1, e_2 \in E$ is apparent from the quantitativity of S-quantifiers.
We notice the dual characterisation theorem for S-quantifiers that has been proven by Thiele:

Theorem 31 (Characterisation of S-quantifiers)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is an S-quantifier where* $|E| > 1$. *Then* $\widetilde{\vee}_{\widetilde{Q}}$ *is an* s*-norm, and*

$$\widetilde{Q}(X) = \sup \left\{ \mathop{\widetilde{\vee}_{\widetilde{Q}}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ \ldots, \ a_m\} \in \mathcal{P}(E) \textit{ finite, } a_i \neq a_j \textit{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(See Thiele [147, Th-8.2, p.48])

Note. Some properties of s-norm aggregation of infinite collections in the form expressed by the theorem (and hence, as expressed by S-quantifiers) have been studied by Rovatti and Fantuzzi [133] who view S-quantifiers as a special type of non-additive functionals. Based on the characterisation of S-quantifiers, a theorem dual to Th-30 can be proven for existential quantifiers.

Theorem 32

Consider a DFS $\mathcal{F}$ and a base set $E \neq \varnothing$. Then $\mathcal{F}(\exists) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is an S-quantifier constructed from the induced s-norm of $\mathcal{F}$, i.e. $\mathcal{F}(\exists)$ is defined by

$$\mathcal{F}(\exists)(X) = \sup \left\{ \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \, \ldots, \, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Let me remark that if E is finite, i.e. $E = \{e_1, \, \ldots, \, e_m\}$ where the e_i are pairwise distinct, then the expressions presented in the above theorems can be simplified into

$$\mathcal{F}(\forall)(X) = \mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_X(e_i) \,,$$
$$\mathcal{F}(\exists)(X) = \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(e_i) \,.$$

Hence the fuzzy universal (existential) quantifiers of $\mathcal{F}$ are reasonable in the sense that the important relationship between $\forall$ and $\wedge$ ($\exists$ and $\vee$, resp.), which holds in the finite case, is preserved by the DFS. In particular, the above theorems show that in every DFS, the fuzzy existential and fuzzy universal quantifiers are uniquely determined by the induced fuzzy disjunction and conjunction.

Theorem 33

Suppose $\mathcal{F}$ is a DFS, $\widehat{\mathcal{F}}$ its induced extension principle and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$.

a. $\widehat{\mathcal{F}}$ is uniquely determined by $\widetilde{\vee}$, in the way described by Th-23 and Th-32, i.e.

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \sup \{ \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \, \ldots, \, a_m\} \in f^{-1}(e') \text{ finite, } a_i \neq a_j \text{ if } i \neq j \, \}$$

for all $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $e' \in E'$, where $E, E' \neq \varnothing$.

b. $\widetilde{\vee}$ is uniquely determined by $\widehat{\mathcal{F}}$, viz. $x_1 \widetilde{\vee} x_2 = (\widetilde{\pi}_\varnothing \circ \widehat{\mathcal{F}}(!))(X)$ for all $x_1, \, x_2 \in \mathbf{I}$, where $X \in \widetilde{\mathcal{P}}(\{1, \, 2\})$ is defined by $\mu_X(1) = x_1$ and $\mu_X(2) = x_2$, and $!$ is the unique mapping $! : \{1, \, 2\} \longrightarrow \{\varnothing\}$.

In particular, if $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$ is the standard extension principle, then $\widetilde{\vee} = \max$. Because I did not want QFMs in which $\widetilde{\vee} \neq \max$ to be a priori excluded from consideration, it was not possible to state (Z-6) in terms of the standard extension principle. Acknowledging this constraining role, it was hence necessary to introduce general extension principles along with the construction of induced extension principles, which selects an appropriate choice of such general extension principle for each given $\mathcal{F}$. Only by developing the theory of fuzzy quantification in this way it was possible to leave open the chance for models based on different choices of fuzzy disjunction, like bounded sum or algebraic sum, and also the extreme case of drastic sum.

4.17 Fuzzy inverse images

We shall now return to a more theoretically oriented construction which is nevertheless worthwhile investigating, because it contributes to a theorem (in the subsequent section) which elucidates the semantic relationship between unary (i.e. one-place) and general multi-place quantification, that is underlying all models of DFS theory. The required construction is that of *fuzzy inverse images*, a notion closely related to the extension principle.

In the case of crisp sets, I assume the usual definition of inverse images which has already been stated in the previous chapter, see equality (18). Generalising this concept, every QFM $\mathcal{F}$ induces fuzzy inverse images by means of the following construction.

Definition 47
Suppose $\mathcal{F}$ is a QFM and $f : E \longrightarrow E'$ is some mapping. $\mathcal{F}$ induces a fuzzy inverse image mapping *$\widehat{\mathcal{F}}^{-1}(f) : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ which to each $Y \in \widetilde{\mathcal{P}}(E')$ assigns the fuzzy subset $\widehat{\mathcal{F}}^{-1}(f)$ defined by*

$$\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mathcal{F}(\chi_{f^{-1}(\bullet)}(e))(Y)\,,$$

for all $e \in E$.

If $\mathcal{F}$ is a DFS, then its induced fuzzy inverse images coincide with the apparent 'reasonable' definition:

Theorem 34
Suppose $\mathcal{F}$ is a DFS, $f : E \longrightarrow E'$ is a mapping and $Y \in \widetilde{\mathcal{P}}(E')$. Then for all $e \in E$, $\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mu_Y(f(e))$.

Hence $\mathcal{F}$ not only induces a plausible extension principle, but also induces a reasonable choice of the reverse construction.

4.18 The semantics of fuzzy multi-place quantification

In this section, I will uncover the internal structure of fuzzy multi-place quantification and elucidate its semantical grounding into one-place quantification. The key tool for analysing multi-place quantification will be provided by formalizing the *reduction* of an n-place quantifier to a corresponding unary quantifier.[35]

In order to understand how n-place quantifiers can be reduced to one-place quantification, let us recall that for arbitrary sets A, B, C, it always holds that $A^{B\times C} \cong (A^B)^C$, a relationship commonly known as 'currying'. We then have

$$\mathcal{P}(E)^n \cong (\mathbf{2}^E)^n \cong \mathbf{2}^{E\times n} \cong \mathcal{P}(E \times n) \cong \mathcal{P}(E \times \{1, \ldots, n\})\,,$$

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[35] In an earlier report [48], I used the terms 'unary' and 'monadic' interchangeably. Here I am more consistent about that and now stick to the standard terminology, i.e. a quantifier is *unary* if it accepts one argument; and it is *monadic* if each occurrence of the quantifier binds one variable at a time.

and similarly

$$\widetilde{\mathcal{P}}(E)^n \cong (\mathbf{I}^E)^n \cong \mathbf{I}^{E\times n} \cong \widetilde{\mathcal{P}}(E \times n) \cong \widetilde{\mathcal{P}}(E \times \{1, \dots, n\}) \,,$$

where n abbreviates $\{0, \dots, n-1\}$ as usual. For convenience, I have replaced $E \times n$ by $E \times \{1, \dots, n\}$, which better suits the convention of numbering the arguments of an n-place quantifier from 1 to n (rather than from 0 to $n-1$). This suggests that by exploiting the bijection $\mathcal{P}(E)^n \cong \mathcal{P}(E \times \{1, \dots, n\})$, n-place quantification as expressed by some $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ can be replaced by one-place quantification using a one-place quantifier $\langle Q\rangle : \mathcal{P}(E \times \{1, \dots, n\}) \longrightarrow \mathbf{I}$, and that conversely $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ can always be recovered from $\mathcal{F}(\langle Q\rangle) : \widetilde{\mathcal{P}}(E \times \{1, \dots, n\}) \longrightarrow \mathbf{I}$. To establish this result, we need some formal machinery. For a given domain E and $n \in \mathbb{N}$, I will abbreviate $E_n = E \times \{1, \dots, n\}$. This will provide a concise notation for the base sets of the resulting unary quantifiers. For $n = 0$, we obtain the empty product $E_0 = \varnothing$.

Definition 48 ($\imath_i^{n,E}$)
Let E be a given set, $n \in \mathbb{N} \setminus \{0\}$ and $i \in \{1, \dots, n\}$. By $\imath_i^{n,E} : E \longrightarrow E_n$ we denote the inclusion defined by

$$\imath_i^{n,E}(e) = (e, i) \,,$$

for all $e \in E$.

Note. The crisp extension (powerset mapping, see Def. 19) of $\imath_i^{n,E} : E \longrightarrow E_n$ will be denoted by $\widehat{\imath}_i^{n,E} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E_n)$. The inverse image mapping of $\imath_i^{E,n}$ will be denoted $(\imath_i^{E,n})^{-1} : \mathcal{P}(E_n) \longrightarrow \mathcal{P}(E)$, see equality (18).
We can use these injections to define the unary quantifier $\langle Q\rangle$ of interest.

Definition 49
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be an n-place semi-fuzzy quantifier, where $n > 0$. Then $\langle Q\rangle : \mathcal{P}(E_n) \longrightarrow \mathbf{I}$ is defined by

$$\langle Q\rangle(X) = Q((\imath_1^{n,E})^{-1}(X), \dots, (\imath_n^{n,E})^{-1}(X))$$

for all $X \in \mathcal{P}(E_n)$.

For fuzzy quantifiers, $\langle \widetilde{Q}\rangle$ is defined similarly, using the fuzzy inverse image mapping $(\hat{\imath}_i^{n,E})^{-1} : \widetilde{\mathcal{P}}(E_n) \longrightarrow \widetilde{\mathcal{P}}(E)$ of $\imath_i^{n,E}$:

Definition 50
Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ be a fuzzy quantifier, $n > 0$. The fuzzy quantifier $\langle \widetilde{Q}\rangle : \widetilde{\mathcal{P}}(E_n) \longrightarrow \mathbf{I}$ is defined by

$$\langle \widetilde{Q}\rangle(X) = \widetilde{Q}((\hat{\imath}_1^{n,E})^{-1}(X), \dots, (\hat{\imath}_n^{n,E})^{-1}(X)) \,,$$

for all $X \in \widetilde{\mathcal{P}}(E_n)$.

I now wish to establish the relationship between $\langle Q\rangle$ and Q (semi-fuzzy case) and $\langle\widetilde{Q}\rangle$ and $\widetilde{Q}$ (fuzzy case). Let us first introduce a concise notation for iterated unions of a quantifier's arguments:

Definition 51
Suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$ and $k \in \mathbb{N} \setminus \{0\}$. The semi-fuzzy quantifier $Q\cup^k : \mathcal{P}(E)^{n+k-1} \longrightarrow \mathbf{I}$ is inductively defined as follows:

a. $Q\cup^1 = Q$;

b. $Q\cup^k = Q\cup^{k-1}\cup$ *if* $k > 1$.

For fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\widetilde{Q}\widetilde{\cup}^k : \widetilde{\mathcal{P}}(E)^{n+k-1} \longrightarrow \mathbf{I}$ is defined analogously.

Theorem 35
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ where $n > 0$,

$$Q = \langle Q\rangle\cup^n \circ \mathop{\times}_{i=1}^{n} \widehat{\imath}_i^{\,n,E} \,.$$

Note. This demonstrates that n-place crisp (or 'semi-fuzzy') quantification, where $n > 0$, can always be reduced to one-place quantification. If we allowed for empty base sets, this would also go through for $n = 0$; I only had to exclude this case because $E_0 = \varnothing$, i.e. in this case we have $\langle Q\rangle : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which does not qualify as a semi-fuzzy quantifier because the base set is empty.
A theorem analogous to Th-35 can also be proven in the fuzzy case:

Theorem 36
Suppose $\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ has $x \mathbin{\widetilde{\vee}} 0 = 0 \mathbin{\widetilde{\vee}} x = x$ for all $x \in \mathbf{I}$, and $\widetilde{\cup}$ is the fuzzy union element-wise defined in terms of $\widetilde{\vee}$. For every fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $n > 0$,

$$\widetilde{Q} = \langle\widetilde{Q}\rangle\widetilde{\cup}^n \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\imath}}_i^{\,n,E} \,.$$

Note. The theorem shows that it is possible to reduce n-place quantification to one-place quantification in the fuzzy case as well. Nullary quantifiers ($n = 0$) had to be excluded for same reason as in Th-35.
The central fact which links these results to QFMs is the following:

Theorem 37
Suppose $\mathcal{F}$ is a QFM with the following properties:

a. $x \mathbin{\widetilde{\vee}} 0 = 0 \mathbin{\widetilde{\vee}} x = x$ *for all* $x \in \mathbf{I}$;

b. *for all semi-fuzzy quantifiers* $Q : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *where* $n > 0$,

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup};$$

c. $\mathcal{F}$ *satisfies* (Z-6) *(functional application);*

d. *If* E, E' *are nonempty sets and* $f : E \longrightarrow E'$ *is an injective mapping, then* $\widehat{\mathcal{F}}(f) = \hat{\hat{f}}$, *i.e.* $\mathcal{F}$ *coincides with the standard extension principle on injections.*

Then for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$,

$$\mathcal{F}(\langle Q \rangle) = \langle \mathcal{F}(Q) \rangle .$$

In particular, every DFS commutes with $\langle \bullet \rangle$, and hence permits the reduction of multi-place quantification to one-place quantification. In other words, the behaviour of $\mathcal{F}$ on general multi-place quantifiers of arbitrary arities $n > 0$ is already implicit in the definition of $\mathcal{F}$ for one-place quantifiers, because the quantification results of multi-place quantification are completely determined by the behaviour of $\mathcal{F}$ in the unary case. In fact, we can actually *define* $\mathcal{F}(Q)$ in terms of $\mathcal{F}(\langle Q \rangle)$, by utilizing Th-35 and Th-36. The semantical relationship discovered here, and the associated constructions of $\langle Q \rangle$ and $\langle \widetilde{Q} \rangle$, hence allow us to *generate* a full-fledged QFM from its core of an underlying 'unary QFM'. In perspective, this would permit us to restrict attention to the behaviour of $\mathcal{F}$ on one-place quantifiers, and to reformulate the DFS axioms (Z-1)–(Z-6) into axioms imposed on the unary base mechanism which underlies $\mathcal{F}$. However, the resulting axioms are likely to become more abstract than the current DFS axioms, and I shall hence not pursue this idea further here.

From a practical standpoint, the techniques developed in this section have already proven useful for establishing the equivalence of the alternative constructions of induced fuzzy truth functions in all models of the theory.

4.19 Chapter summary

In the chapter, I first explained the general objectives which motivate the research into semantical postulates for fuzzy quantification. Roughly speaking, the investigation of this topic is essential, because 'reasonable' approaches which conform to the essential postulates will achieve linguistic adequacy, logical coherence, and also offer a natural account of fuzziness. Acknowledging the significance of these quality criteria to the modelling of fuzzy quantification, I hence researched into a diversity of such requirements, forged these into precisely defined criteria on QFMs, and established in the theorems, that arbitrary models of fuzzy quantification fulfill these criteria, provided the models comply with the proposed axiom system (Z-1)–(Z-6). Consequently, every model which rests on this axiomatic foundation is known to be well-behaved in a variety of ways. This substantiates the claim that the proposed axiom system, albeit compressed in appearence, indeed covers a wide range of intuitive expectations on the models, which includes the essential requirements.

As to specific adequacy conditions, I first considered the property of correct generalisation, which is directly ensured by condition (Z-1) for quantifiers of arity $n \leq 1$. By combining the latter condition with the remaining axioms, I was then able to establish the 'full' property which now covers unrestricted quantifiers of arbitrary arities. Knowing that the requirement of 'correct generalisation' holds unconditionally in every DFS, is of special importance to the proposed framework, because it provides a post-hoc justification for applying the fuzzification pattern, and hence bears witness that the proposed framework is well-constructed.

Before turning to quantifiers, I then considered the propositional fragment of the models, which is determined by the construction of induced truth functions. As to the simpler case of unary connectives, I first proved that the identity truth function maps to the corresponding identity on the unit interval, which is of course the only acceptable choice for the continuous-valued case. Now considering negation, the models were shown to translate the original crisp negation into a strong negation operator, in conformance with the usual abstraction of reasonable negation operators that is made in fuzzy set theory. Next I investigated the two-place connectives, thus substantiating that these are assigned a reasonable semantics as well. With conjunction, the intuitive expectations on plausible choices are captured by the familiar concept of a t-norm, and indeed all induced conjunctions of the models can be shown to belong to this class. As to disjunction, it is the notion of an s-norm which constitutes the class of or-like connectives, and it was shown that in all considered models, the disjunction translates into the $\widetilde{\neg}$-dual s-norm of the induced conjunction, which is highly desirable. In addition, the induced implication can be expressed in terms of the induced disjunction and negation, following the usual pattern.

This completes my summary of induced truth functions, which are all obtained by instantiating the generic scheme defined in Def. 11. From a methodological standpoint, however, it is not only interesting to analyse these individual instances, but also to analyse the construction of induced connectives itself, and to explore its possible alternatives. By comparing the fuzzy truth functions determined from these schemes, I then gathered more evidence that the obtained choice of connectives is indeed canonical, because it is shared by two independent constructions.

Following this discussion of their propositional structure, attention was then shifted to the quantifiers themselves, and the models were analyzed for their compliance with certain operations on the quantifiers' arguments. Some of these operations underly the linguistic description of NL quantification and are familiar from TGQ, while others are concerned with coherence, and disclose some important properties which are both elementary but easily overlooked, because they are only too self-evident and usually go without saying.

First of all, I considered a construction which permutes argument positions. I started by defining the base operation on a given quantifier, which simply alters the order of the arguments in the way specified by the permutation. The generic argument permutation scheme was then restricted to a special type of elementary transpositions, into which every permutation can be decomposed. I also presented an example in which the construction is directly applied to construct an NL quantifier, and another example which illustrates the utility of the considered construction to express symmetry properties of quantifiers. Because these symmetries reveal an important aspect of the meaning of quantifiers, it is essential that plausible approaches to fuzzy quantification be compatible with the underlying construction. The proposed axiom system is strong enough to ensure that all conforming models be homomorphic to argument transpositions (and hence also to argument permutations). In particular, these models will recognize the symmetry pattern of a given base quantifier, and extrapolate its symmetry properties to the fuzzy case.

Having discussed the issue of argument symmetry, I turned to cylindrical extensions, a construction which augments a given quantifier by vacuous argument positions. This construction is mainly required to fit a number of given quantifiers to a certain joint construction, which requires that all of the involved quantifiers refer to a common argument list. After stating the formal definition of cylindrical extensions, I addressed the question whether the considered models of fuzzy quantification comply with this construction. The results of this study again came out positive. In fact, if an argument of a semi-fuzzy quantifier has no effect on the computed outcomes, then it will also have no effect on the quantification results of the corresponding fuzzy quantifier.

Following the discussion of this coherence requirement, the various types of negation were investigated, in order to assess the degree to which the models preserve Boolean structure. The analysis of the models on this background, revealed the full compliance of a DFS with the formation of antonyms (complementation of arguments) and the external negation of quantifiers (where the negation operator is applied to the quantifying expression as a whole). Combining this with known conformance to dualisation, which is ensured by (Z-3), it is then apparent that every DFS preserves Aristotelian squares of quantifiers. These visualize the interrelations between the various types of negation, which can be applied to a quantifier. The relevance of the Aristotelian squares stems from their close relationship to the Piaget group of transformations, which catches some findings of developmental psychology. Apart from the block-wise type of negation/complementation considered so far, I then discussed a more fine-grained type of negation, which can be modelled by the symmetrical difference of arguments with given crisp sets. In this way, only part of a fuzzy argument can be complemented, while the remaining portion is left unchanged. This fine-structured type of negation does not pose problems to the considered models, though, which can handle it unconditionally, just like the simple negation and complementation.

Having considered the Boolean structure which expresses in the various types of negation, attention was then shifted to the structure imposed by unions and intersections. As to the formation of unions, it is explicitly required by (Z-4) that all models comply with this construction when it occurs in the last argument. By utilizing the earlier result on argument permutation, it is then immediate that every DFS conforms to unions in arbitrary argument position. Now recalling the compliance of the models to complementation, and making use of De Morgan's law, it then came out that every model also conforms to intersections in arbitrary argument positions. In this sense, the models are fully compatible with Boolean argument structure.

In addition, the models were shown to comply with another construction of relevance to natural language, the insertion of arguments into a quantifier. To be precise, the relevance of argument insertions stems from their contribution to an important NL construction known as adjectival restriction, which can be reduced to intersections and argument insertion. In order to discuss this construction, I first introduced an operator-based notation for describing an elementary insertion step. I then proved that every DFS complies with the insertion operation. This demonstrates in particular, that all models comply with adjectival restriction by a crisp adjective (like "married"). The discussion of fuzzy argument insertion has been delayed to the later section 6.8, because it requires special attention. Due to the simple fact that a fuzzy argument cannot be inserted into a semi-fuzzy quantifier, it is necessary to handle fuzzy argument insertion and fuzzy adjectival restriction with a rather different approach.

Acknowledging the importance of monotonicity properties, which constrain the valid conclusions, a special case of monotonicity condition has been included into the set of DFS axioms. The condition enforces that legal models preserve the nonincreasing monotonicity of a quantifier in its last argument, see (Z-5). This artificial restriction of the criterion was motivated by the desire to keep the axiom system as succinct as possible, in order to simplify later proofs which depend on the axioms. However, it is the unrestricted preservation of monotonicity properties of any kind, that one would expect of a reasonable approch to fuzzy quantification. I hence reviewed the monotonicity issue in some depth, and also studied some more general ways in which it shows up in NL quantfiers. To begin with, I now established that monotonicity properties in arguments are preserved by all considered models, regardless of the position of the argument and its monotonicity type (nondecreasing/nonincreasing). I then refined this result, by directing attention towards monotonicity properties which hold only locally, a notion which I defined in terms of closed ranges of crisp sets. The proposed relativization of monotonicity properties to any desired choice of argument ranges, is not critical to the models. In fact, every DFS can be shown to preserve any type of monotonicity properties, even if they hold only locally. Apart from the preservation of monotonicity properties, which pertain to the argument structure of a quantifier, it was also the monotonicity of the fuzzification mechanism itself which received attention. Intuitively, we would expect that the models preserve inequalities between quantifiers, and indeed every DFS was shown to be monotonic in this sense. Again, it is possible make a refinement into local inequalities between quantifiers, which hold only in a given range of arguments, and to prove the compliance of the models with the novel relativized concept.

The subsequent analysis of the induced extension principle also revealed some interesting findings. First of all, it was shown that the induced extension principle is functorial, and hence permits a decomposition of powerset mappings whenever convenient. Attention was then drawn to the fact that on injective mappings, there is only one reasonable definition of the corresponding powerset mapping. In turn, it was shown that all induced extension principles coincide on injective mappings with the apparent plausible definition. Some additional results on the induced extension principle have also been proven. For example, it is only possible for one extension principle at a time, to be compatible with the construction of functional application, which underlies condition (Z-6). Most importantly, the covert relationship between the induced extension principle and existential quantification was now disclosed. In fact, the induced powerset mappings can always be expressed in terms of existential quantification and conversely, existential quantification can always be reduced to an application of the extension principle.

Having presented these general results on the abstract behaviour of the extension principle, I got into details of important semantical properties, which are closely tied to the extension principle. First and foremost, I have generalized the well-known concept of quantitativity to semi-fuzzy and fuzzy quantifiers, adopting the usual modelling of quantitativity in terms of automorphism invariance. The subsequent analysis of the models under the aim of quantitativity revealed that the quantitative type of semi-fuzzy quantifiers is mapped to quantitative counterparts, while non-quantitative examples remain non-quantitative under the fuzzification. This makes a very satisfying result, which demonstrates the suitability of DFSes to model both types of quantifiers, and hence sets the theory in stark contrast to existing approaches, which only know of a limited fragment of the quantitative type.

In addition, the abstract findings on the interpretation of fuzzy powerset mappings, also fostered research into one of the central linguistic requirements. This criterion on the models has been taylored to a characteristic of linguistic quantifiers known as 'having extension', which expresses an important insensitivity property of NL quantification with respect to the precise choice of the domain. Intuitively, the domain simply constitutes a background for quantification, but its total extent should be inessential. Having explained this property and its significance to the modelling of NL semantics, I proposed the definition of an extensional QFM, which captures the precise requirements on those choices of $\mathcal{F}$, which preserve the property of having extension. It was then immediate from the previous result concerning the natural extension of injective mappings, that every model of the theory is indeed extensional. Hence all models of fuzzy quantification comply with one of the key requirements from a linguistic perspective, and properly model a typical and possibly universal aspect of quantifiers in natural language.

I also introduced the novel concept of contextuality, which is concerned with our intuitive understanding of fuzziness, and the relationship between fuzzy sets and crisp sets that live on the same domain. The concept attempts to formalize the straightforward observation that it is not the clear cases with crisp membership grades, that pose problems for interpretation. Quite the reverse, the extra effort in supporting fuzziness is only caused by the undecided or unclear cases with gradual membership, which force us to make additional assumptions in order to ensure a plausible interpretation. Now utilizing the familiar notions of core and support (envelope) of a fuzzy set, it was shown that the collection of these unclear cases can be resolved into a closed range of crisp sets, which precisely describes the ambiguous cases. The fuzzy set is hence viewed as providing a context for interpretation, which clearly delimits the alternatives that must be considered for fuzzy quantification, and hence permits local computations. Based on these preparations, I then introduced the postulate for contextuality, which forces all plausible models to separate clear cases from unclear cases, and strictly confine all ambiguity to the context ranges of the fuzzy arguments. Subsequently, all models of the theory were shown to comply with this criterion. Hence all cases outside the context range are considered irrelevant to the quantification result, because they escape the conceivable range of compatible interpretations of the arguments. Although contextuality makes a very elementary requirement, we shall experience later that it conflicts with another desideratum, that of preserving general convexity properties of quantifiers. However, contextuality clearly outweighs the other considerations, because it captures so elementary an aspect of fuzzy quantification that it is hard to imagine how a violation of this principle could ever be acceptable.

The logical aspects, and in particular the prototypical quantifiers $\forall$ and $\exists$, also received their due attention. The goal of this investigation was to disclose the precise interpretation rules for the universal and existential quantifiers. In particular, I wanted to show that in accordance with intuitive expectations, one could relate universal quantification and conjunction, as well as existential quantification and disjunction. It was further hoped that the research into the standard quantifiers might contribute to the development of an explicit formula which precisely describes the induced extension principle. In order to accomplish these goals, I started from successful work on the standard quantifiers that was described in the literature, i.e. from Thiele's modelling of the logical quantifiers through so-called T- and S-quantifiers. It was then easily shown that the universal and existential quantifiers induced by a DFS, indeed belong to these classes of plausible choices. I then applied Thiele's decomposition theorems and hence obtained the final representation of these quantifiers, which is now explicitly constructed from the induced t-norm/s-norm of the model. The achieved decomposition reveals in particular, that the important relationship between conjunction and universal quantification (disjunction and existential quantification), which holds in the crisp case, is preserved when applying a DFS. By substituting the new representation into the earlier formulas which compute the extension principle from the existential quantifier, I finally obtained a new representation of the induced extension principle. Specifically, the abstract definition of the induced extension principle stated in Def. 22 was broken down into an explicit formula, which reveals its precise internal structure.

Finally, I treated two more theoretically oriented topics. I first reviewed the notion of inverse images (the converse of powerset mappings). Unlike extension principles, there is only one natural generalisation of inverse images to the fuzzy case. Assuming the apparent construction of induced fuzzy inverse images for a given $\mathcal{F}$, we should then expect that the model of fuzzy quantification induce the above natural choice of inverse images. Indeed, all models of fuzzy quantification were shown to be plausible in this respect. The latter result then contributed to the solution of the last problem considered in this chapter, which was concerned with the reducibility of multi-place quantification to a special kind of one-place (unary) quantification in the general case. By exploiting an apparent Currying relationship, I managed to reduce multi-place semi-fuzzy quantifiers and multi-place fuzzy quantifiers to 'simple' unary quantifiers. I then developed the precise conditions that a QFM $\mathcal{F}$ must fulfill in order to be compatible with these reductions. It happens that these conditions are satisfied by every DFS, which completes the proof that in the considered models, multi-place quantification can indeed be reduced to a specifically constructed, unary quantifier, supplied with a single argument which results from a canonical construction.

5 Special subclasses of models

5.1 Motivation and chapter overview

The last chapter has shed some light on the semantical properties shared by arbitrary models of the theory. However, no attempt has been made so far to structure the total of models into natural subclasses and to investigate the specific properties of their models, which might not be shared by other classes of DFSes. Apart from this research into the subclasses themselves, it is also the precise relationships between the subclasses that raise interest. Furthermore the identification of subclasses provides a local context in which those concepts can be developed, that require a minumum degree of homogeneity on the models' side. For example, the structure of the models must be sufficiently similar, in order to permit their reasonable comparison. Most importantly, the relative homogeneity of models in the same class, renders possible the definition of certain constructions on the given subclasses. For example, the models gathered in a class will be sufficiently homogeneous to permit a construction which combines a given collection of models into new models. Finally, the research into subclasses also serves to identify a class of standard models for fuzzy quantification, which best meet our intuitive expectations.

In the chapter, the models are first grouped by their induced negation. I then propose a scheme for model transformations, which is capable of fitting a given QFM to different choices of negation operators. It will come out of this investigation that it is possible to switch from any choice of induced negation to any other choice of induced negation. These results justify the subsequent practice to shape the theory to the $\neg$-DFSes, which induce the standard negation. Following this, the $\neg$-DFSes will be further grouped by their induced disjunction $\widetilde{\vee}$, thus forming the the classes of $\widetilde{\vee}$-DFSes. These classes will be sufficiently homogeneous to allow the development of the model aggregation scheme, which fuses a given collection of $\widetilde{\vee}$-DFSes into a new model. I then formalize the intuitive notion of specificity, i.e. the degree to which a model commits to crisp quantification results, which installs a natural order for the intended comparisons. I will also develop the necessary concepts to locate the extreme cases in terms of specificity. Attention will then be shifted to a different classification of the models, which will now be grouped both by their induced negation and disjunction. I will then define conjunctions and disjunctions of quantifiers, and present some first results concerning the compatibility of a restricted class of models with these novel constructions. I will also describe the precise interpretation of equivalence and antivalence/xor in the considered type of models. Finally I will turn to the issue of identifying the class of standard models for fuzzy quantification within the proposed framework. Intuitively, the standard models should conform to the standard concepts of fuzzy set theory, like the standard negation $1 - x$, the standard conjunction and disjunction $\min$ and $\max$, the standard extension principle etc. These expectations on standard models are easily forged into a precise definition of the target class. Acknowledging the distinguished position of the resulting standard DFSes, it is worthwhile developing an axiomatisation. I will hence adapt the original axiom system in such a way that it uniquely identifies the standard type of models.

5.2 Models which induce the standard negation

To begin with, let us classify the models according to their induced negation.

Definition 52
Let $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *be a strong negation operator. A DFS* $\mathcal{F}$ *is called a* $\widetilde{\neg}$*-DFS if its induced negation coincides with* $\widetilde{\neg}$*, i.e.* $\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}$*. In particular, we will call* $\mathcal{F}$ *a* $\neg$*-DFS if it induces the standard negation* $\neg x = 1 - x$*.*

In the following I will establish that without loss of generality, one can restrict attention to $\neg$-DFSes. To achieve this, a mechanism is needed which allows us to transform a given DFS $\mathcal{F}$ into another. A suitable model transformation scheme is defined as follows.

Definition 53 (Model transformation scheme)
Suppose $\mathcal{F}$ *is a DFS and* $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ *a bijection. For every semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*, we define*

$$\mathcal{F}^\sigma(Q)(X_1, \ldots, X_n) = \sigma^{-1}\mathcal{F}(\sigma Q)(\sigma X_1,\ \ldots,\ \sigma X_n)\,,$$

where σQ *abbreviates* $\sigma \circ Q$*, and* $\sigma X_i \in \widetilde{\mathcal{P}}(E)$ *is the fuzzy subset with* $\mu_{\sigma X_i} = \sigma \circ \mu_{X_i}$*.*

Let us now establish that every proper instantiation of the transformation scheme indeed constructs a new model of fuzzy quantification:

Theorem 38
If $\mathcal{F}$ *is a DFS and* $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ *an increasing bijection, then* $\mathcal{F}^\sigma$ *is a DFS.*

It is well-known [89, Th-3.7] that for every strong negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ there is a monotonically increasing bijection $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ such that $\widetilde{\neg}\, x = \sigma^{-1}(1 - \sigma(x))$ for all $x \in \mathbf{I}$. The mapping σ is called the *generator* of $\widetilde{\neg}$.

Theorem 39
Suppose $\mathcal{F}$ *is a* $\widetilde{\neg}$*-DFS and* $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ *is the generator of* $\widetilde{\neg}$*. Then* $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$ *is a* $\neg$*-DFS and* $\mathcal{F} = \mathcal{F}'^{\sigma}$*.*

The theorem states that the model transformation accomplishes a bidirectional translation between the models which is capable of adapting the induced negation.

The transformation scheme hence achieves a universal translation property with respect to strong negation operators: every model can be fitted to any choice of negation operator and vice versa. Put differently, each class of $\widetilde{\neg}$-DFSes is representative of the full class of models, because the remaining classes of $\widetilde{\neg}'$-DFSes (based on different choices of the negation $\neg'$) can be generated from the given source class. In particular, no models of interest are lost if we focus on $\neg$-DFSes only, i.e. to those models which induce the standard negation. Due to the universal translation property, the definitions of novel concepts and constructions on the models can now be restricted to $\neg$-DFSes whenever convenient.

In the following, the models based on the standard negation are further grouped according to their induced disjunction.

Definition 54
A $\neg$-DFS $\mathcal{F}$ which induces a fuzzy disjunction $\widetilde{\vee}$ is called a $\widetilde{\vee}$-DFS.

The benefit of introducing these classes of models is that $\widetilde{\vee}$-DFSes are sufficiently homogeneous for a straightforward definition of constructions that act on collections of these models. In particular, it now becomes possible to introduce the following model aggregation scheme, which combines a collection of $\widetilde{\vee}$-DFSes into a new $\widetilde{\vee}$-DFS, in accordance with a given aggregation operator Ψ.

Theorem 40 (Model aggregation scheme)
Suppose $\mathcal{J}$ is a non-empty index set and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes. Further suppose that $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ satisfies the following conditions:

- *a.* *If $f \in \mathbf{I}^{\mathcal{J}}$ is constant, i.e. if there is a $c \in \mathbf{I}$ such that $f(j) = c$ for all $j \in \mathcal{J}$, then $\Psi(f) = c$.*
- *b.* *$\Psi(1 - f) = 1 - \Psi(f)$, where $1 - f \in \mathbf{I}^{\mathcal{J}}$ is point-wise defined by $(1 - f)(j) = 1 - f(j)$, for all $j \in \mathcal{J}$.*
- *c.* *Ψ is monotonically increasing, i.e. if $f(j) \leq g(j)$ for all $j \in \mathcal{J}$, then $\Psi(f) \leq \Psi(g)$.*

If we define $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ by

$$\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](Q)(X_1, \dots, X_n) = \Psi((\mathcal{F}_j(Q)(X_1, \dots, X_n))_{j\in\mathcal{J}})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, then $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ is a $\widetilde{\vee}$-DFS.

In particular, convex combinations (e.g., arithmetic mean) and stable symmetric sums [141] of $\widetilde{\vee}$-DFSes are again $\widetilde{\vee}$-DFSes.

The $\neg$-DFSes can be partially ordered by 'specificity' or 'fuzziness', in the sense of closeness to $\frac{1}{2}$. We define a partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ by

$$x \preceq_c y \Leftrightarrow y \leq x \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} \leq x \leq y\,, \tag{30}$$

for all $x, y \in \mathbf{I}$. $\preceq_c$ is Mukaidono's ambiguity relation, see [110]. This basic definition of $\preceq_c$ for scalars can be extended to the case of DFSes in the obvious way:

Definition 55
Suppose $\mathcal{F}$, $\mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is consistently less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, if for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(Q)(X_1, \dots, X_n) \preceq_c \mathcal{F}'(Q)(X_1, \dots, X_n)$.*

We now wish to establish the existence of consistently least specific $\widetilde{\vee}$-DFSes. As it turns out, the greatest lower specificity bound of a collection of $\widetilde{\vee}$-DFSes can be expressed using the fuzzy median, defined as follows.

Definition 56
The fuzzy median $\mathrm{med}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{med}_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & : \quad \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & : \quad \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

The plot of $\mathrm{med}_{\frac{1}{2}}$ is displayed in Fig. 9.

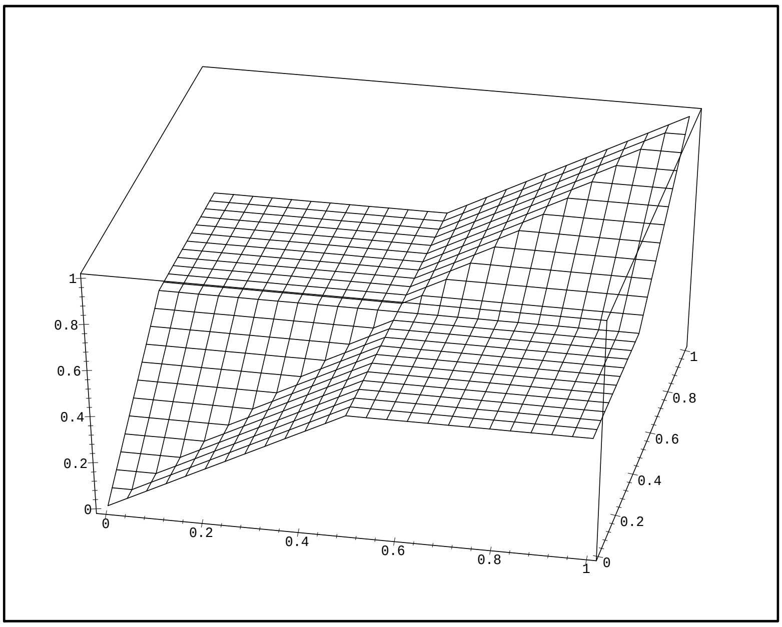

Figure 9: The fuzzy median $\mathrm{med}_{\frac{1}{2}}$

$\mathrm{med}_{\frac{1}{2}}$ is an associative mean operator [15] and the only stable (i.e. idempotent) associative symmetric sum [141].
The fuzzy median can be generalised to an operator $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments. Firstly, because it is associative, idempotent and commutative, $\mathrm{med}_{\frac{1}{2}}$ can be generalized to arbitrary *finite* sets of arguments (just apply $\mathrm{med}_{\frac{1}{2}}$ in any order). Noting that for all finite $X = \{x_1, \ldots, x_n\} \subseteq \mathbf{I}$, $n \geq 2$, it holds

that

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{med}_{\frac{1}{2}} (\min X,\ \max X)\,,$$

the proper definition of $\mathrm{m}_{\frac{1}{2}} X$ in the case $n = 0,\ n = 1$ becomes

$$\begin{aligned} \mathrm{m}_{\frac{1}{2}} \varnothing &= \mathrm{med}_{\frac{1}{2}}(\min \varnothing,\ \max \varnothing) = \mathrm{med}_{\frac{1}{2}}(1,\, 0) = \tfrac{1}{2}\,, \\ \mathrm{m}_{\frac{1}{2}}\{u\} &= \mathrm{med}_{\frac{1}{2}}(\min\{u\},\ \max\{u\}) = \mathrm{med}_{\frac{1}{2}}(u,\, u) = u \end{aligned}$$

I now extend $\mathrm{m}_{\frac{1}{2}}$ to arbitrary subsets $X \subseteq \mathbf{I}$ as follows.

Definition 57
The generalised fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{med}_{\frac{1}{2}}(\inf X,\ \sup X)\,, \qquad \textit{for all } X \in \mathcal{P}(\mathbf{I}).$$

Note. This definition is obviously compatible with the above considerations on the proper definition of $\mathrm{m}_{\frac{1}{2}} X$ for finite subsets of **I**. Based on the generalized fuzzy median, I can now state the desired theorem on the existence and representation of lower specificity bounds on given collections of $\widetilde{\vee}$-DFSes.

Theorem 41
Suppose that $\widetilde{\vee}$ *is an* s*-norm, and* $\mathbb{F}$ *a non-empty collection of* $\widetilde{\vee}$*-DFSes* $\mathcal{F} \in \mathbb{F}$*. Then there exists a greatest lower specificity bound on* $\mathbb{F}$*, i.e. a* $\widetilde{\vee}$*-DFS* $\mathcal{F}_{\mathrm{glb}}$ *such that* $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}$ *for all* $\mathcal{F} \in \mathbb{F}$ *(i.e.* $\mathcal{F}_{\mathrm{glb}}$ *is a lower specificity bound), and for all other lower specificity bounds* $\mathcal{F}'$*,* $\mathcal{F}' \preceq_c \mathcal{F}_{\mathrm{glb}}$*.*
$\mathcal{F}_{\mathrm{glb}}$ *is defined by*

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}\,,$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*.*

In particular, the theorem asserts the existence of a least specific $\widetilde{\vee}$-DFSes, i.e. whenever $\widetilde{\vee}$ is an s-norm such that $\widetilde{\vee}$-DFSes exist, then there exists a least specific $\widetilde{\vee}$-DFS (just apply the above theorem to the collection of all $\widetilde{\vee}$-DFSes).
As concerns the converse issue of most specific models, i.e. least upper bounds with respect to $\preceq_c$, the following definition of ‘specificity consistency’ turns out to provide the key concept:

Definition 58
Suppose $\widetilde{\vee}$ *is an* s*-norm and* $\mathbb{F}$ *is a non-empty collection of* $\widetilde{\vee}$*-DFSes* $\mathcal{F} \in \mathbb{F}$*.* $\mathbb{F}$ *is called* specificity consistent *if for all* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*, either* $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ *or* $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$*, where*

$$R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}\,.$$

Based on this definition of specificity consistency, I can now express the exact conditions under which a collection of $\widetilde{\vee}$-DFSes has a least upper specificity bound, and provide an explicit description of the resulting bound in those cases where it exists.

Theorem 42
Suppose $\widetilde{\vee}$ is an s-norm and $\mathbb{F}$ is a non-empty collection of $\widetilde{\vee}$-DFSes $\mathcal{F} \in \mathbb{F}$.

a. $\mathbb{F}$ has upper specificity bounds exactly if $\mathbb{F}$ is specificity consistent.

b. If $\mathbb{F}$ is specificity consistent, then its least upper specificity bound is the $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{lub}}$ defined by

$$\mathcal{F}_{\mathrm{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}$.

5.3 Models which induce the standard disjunction

The models of fuzzy quantification can also be grouped according to their induced negation and disjunction:

Definition 59
A DFS $\mathcal{F}$ such that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is called a $(\widetilde{\neg}, \widetilde{\vee})$-DFS.

The $(\widetilde{\neg}, \max)$-DFSes in particular, comprise those models that induce the standard disjunction and an arbitrary strong negation operator. These are the models for which I can now present a theorem concerning the interpretation of conjunctions and disjunctions of quantifiers. In order to express these results, it is first necessary to introduce the relevant constructions, which build conjunctions and disjunctions from given quantifiers.

In the Theory of Generalized Quantifiers there are constructions $Q \wedge Q'$ and $Q \vee Q'$ of forming the conjunction (disjunction) of two-valued quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$, see e.g. [6, p. 194], [45, p. 234]. These constructions are easily generalized to (semi)-fuzzy quantifiers.

Definition 60
Suppose $\widetilde{\wedge}, \widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ are given (usually the connectives induced by an assumed QFM). For all semi-fuzzy quantifiers Q, $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the conjunction $Q \mathbin{\widetilde{\wedge}} Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the disjunction $Q \mathbin{\widetilde{\vee}} Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of Q and Q' are defined by

$$(Q \mathbin{\widetilde{\wedge}} Q')(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n) \mathbin{\widetilde{\wedge}} Q'(X_1, \ldots, X_n)$$
$$(Q \mathbin{\widetilde{\vee}} Q')(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n) \mathbin{\widetilde{\vee}} Q'(X_1, \ldots, X_n)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$. For fuzzy quantifiers, $\widetilde{Q} \mathbin{\widetilde{\wedge}} \widetilde{Q}'$ and $\widetilde{Q} \mathbin{\widetilde{\vee}} \widetilde{Q}'$ are defined analogously.

Note. Conjunctions and disjunctions of (semi-)fuzzy quantifiers of different arities can be formed through cylindrical extensions, i.e. by adding vacuous arguments, see section 4.6.
As to the interpretation of these conjunctions and disjunctions of quantifiers, I have the following result which is valid for $(\widetilde{\neg}, \max)$-DFSes.

Theorem 43
Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

a. $\mathcal{F}(Q \wedge Q') \leq \mathcal{F}(Q) \wedge \mathcal{F}(Q')$

b. $\mathcal{F}(Q \vee Q') \geq \mathcal{F}(Q) \vee \mathcal{F}(Q')$.

Notes

- Let me emphasize that the theorem does *not* state the compatibility of $(\widetilde{\neg}, \max)$-DFSes to conjunction and disjunction, it simply establishes bounds on the quantification results. See section 6.4 for a discussion why the inequalities in the above theorem cannot be replaced with equalities on formal grounds.
- Because the theorem refers to the standard fuzzy conjunction and disjunction, the constructions on quantifiers have been written $Q \wedge Q'$ and $Q \vee Q'$, omitting the 'tilde' notation for fuzzy connectives. Similarly, the standard fuzzy intersection and standard fuzzy union will be written $X \cap Y$ and $X \cup Y$, resp., where $\mu_{X \cap Y}(e) = \min(\mu_X(e), \mu_Y(e))$ and $\mu_{X \cup Y}(e) = \max(\mu_X(e), \mu_Y(e))$. The same conventions are stipulated for intersections $\widetilde{Q}\cap$ and unions $\widetilde{Q}\cup$ of the arguments of a fuzzy quantifier, as well as for duals $Q\square$ of semi-fuzzy quantifiers or $\widetilde{Q}\square$ of fuzzy quantifiers, based on the standard negation.

I have not made any claims yet concerning the interpretation of $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow)$ and $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor})$ in a given DFS $\mathcal{F}$, and indeed, there are currently no results available for the full class of models. In the special case of $(\widetilde{\neg}, \max)$-DFSes, though, these connectives are tied to the following interpretation.

Theorem 44
Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $x_1, x_2 \in \mathbf{I}$,

a. $x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 = (x_1 \wedge x_2) \vee (\widetilde{\neg}\, x_1 \wedge \widetilde{\neg}\, x_2)$

b. $x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 = (x_1 \wedge \widetilde{\neg}\, x_2) \vee (\widetilde{\neg}\, x_1 \wedge x_2)\,.$

5.4 The standard models of fuzzy quantification

The most restricted subclass of models we will consider – and the best-behaved – is that of *standard DFSes*, i.e. the class of those models which comply with the standard operations of fuzzy set theory ($\min$, $\max$ etc.). Due to the supreme adequacy properties shown by these models, and their conformance to the established core of fuzzy set theory, it is suggested that this type of DFSes be considered the standard models of fuzzy quantification. Formally, I define standard DFSes as follows.

Definition 61
By a standard DFS *we denote a* $(\neg, \max)$*-DFS.*

Building on the earlier theorems, the fuzzy truth functions induced by a standard DFS can now be summarized as follows.

Theorem 45 (Truth functions in standard DFSes)
In every standard DFS $\mathcal{F}$*,*

$$\begin{aligned}
\widetilde{\neg}\, x_1 &= 1 - x_1 \\
x_1 \mathbin{\widetilde{\vee}} x_2 &= \max(x_1,\, x_2) \\
x_1 \mathbin{\widetilde{\wedge}} x_2 &= \min(x_1,\, x_2) \\
x_1 \mathbin{\widetilde{\rightarrow}} x_2 &= \max(1 - x_1,\, x_2) \\
x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 &= \max(\min(x_1,\, x_2),\ \min(1 - x_1,\, 1 - x_2)) \\
x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 &= \max(\min(x_1,\, 1 - x_2),\ \min(1 - x_1,\, x_2))
\end{aligned}$$

The standard DFSes therefore induce the standard connectives of fuzzy logic, i.e. the propositional fragment of a standard model coincides with the well-known K-standard sequence logic of Dienes [32]. In particular, Kleene's three-valued logic [85, p. 344] is obtained when restricting to the three-valued fragment.[36] From the above theorem Th-33, then, standard DFSes are also known to induce the standard extension principle. As concerns the standard quantifiers, we obtain the familiar choices as well For example, it is apparent from theorems Th-30 and Th-32 that

$$\begin{aligned}
\mathcal{F}(\exists)(X) &= \sup\{\mu_X(e) : e \in E\} \\
\mathcal{F}(\forall)(X) &= \inf\{\mu_X(e) : e \in E\}
\end{aligned}$$

for all $X,\, X_1,\, X_2 \in \widetilde{\mathcal{P}}(E)$. This is quite satisfactory.

Let me further emphasize that the standard models represent a boundary case of DFSes because they induce the smallest fuzzy existential quantifiers, the smallest extension principle, and the largest fuzzy universal quantifiers. This is immediate from Th-32, Th-23 and Th-30, respectively, keeping in mind that $\min$ is the largest t-norm and $\max$ the smallest s-norm.

Finally, we consider the interpretation of two-valued quantifiers in the standard models. Interestingly, there is absolutely no freedom concerning the interpretation assigned to these quantifiers, which is fully determined by the requirements imposed on the standard models:

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[36] Some readers might prefer a different choice of the implication operator, namely $x_1 \widetilde{\rightarrow} x_2 = \min(1,\, 1 - x_1 + x_2)$. However, it is clear that every QFM with the highly desirable property of preserving Aristotelian squares will also preserve the interdefinability of the propositional connectives, and therefore differ from Łukasiewicz logic.

Theorem 46
All standard DFSes coincide on two-valued quantifiers. Hence if $\mathcal{F}, \mathcal{F}'$ are standard DFSes and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued quantifier, then $\mathcal{F}(Q) = \mathcal{F}'(Q)$.

This result establishes a link between the standard models and other fuzzification mechanisms described in the literature. This is because the examples of standard models introduced in [46] are known to coincide with the fuzzification mechanism of Gaines [44] on two-valued quantifiers. We can then conclude from the above theorem that all standard models are indeed compatible with this mechanism. Standard DFSes are more powerful in scope, though, and consistently generalize the Gainesian mechanism to arbitrary semi-fuzzy truth functions and semi-fuzzy quantifiers (see section 10.9 below for a more detailed discussion of the Gainesian fuzzification mechanism). In terms of the classification of quantification types introduced by Kerre and Liu [99, p. 2], the theorem asserts that all 'Type II' quantifications in the sense of the proposed classification are assigned the very same interpretation across all standard models. In other words, the axioms for standard models of fuzzy quantification are strong enough to identify a unique admissible interpretation for arbitrary Type-II quantifications.

5.5 Axiomatisation of the standard models

In this section, I will achieve an axiomatisation of the standard models of fuzzy quantification, by presenting a system of conditions on the considered fuzzification mechanism $\mathcal{F}$ which precisely characterise the class of standard DFSes. It should be apparent from the previous remarks which changes to the base axioms (Z-1)–(Z-6) are necessary to effect a reduction to the standard models. These considerations are summarized in the following adapted system of conditions on $\mathcal{F}$:

Definition 62
Consider a QFM $\mathcal{F}$. For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, we stipulate the following conditions.

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(S-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(S-2)
Dualisation	$\mathcal{F}(Q\square) = \mathcal{F}(Q)\square \quad n > 0$	(S-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\cup \quad n > 0$	(S-4)
Preservation of monotonicity	*If* Q *is nonincreasing in* n*-th arg, then* $\mathcal{F}(Q)$ *is nonincreasing in* n*-th arg,* $n > 0$	(S-5)
Functional application	$\mathcal{F}(Q \circ \underset{i=1}{\overset{n}{\times}} \widehat{f_i}) = \mathcal{F}(Q) \circ \underset{i=1}{\overset{n}{\times}} \hat{\hat{f}}_i$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$*,* $E' \neq \varnothing$*.*	(S-6)

Note. Let me briefly remark on the differences compared to the original DFS axioms (Z-1)–(Z-6). Firstly in (S-3), the induced negation/complement has been replaced with

the standard choice of fuzzy negation/complement. In (S-4), the induced fuzzy union has been replaced with the standard fuzzy union. Finally, (S-6) is built upon the standard extension principle, rather than requiring the compatibility of $\mathcal{F}$ with its induced extension principle.
As stated by the next theorem, these conditions indeed capture the precise requirements for $\mathcal{F}$ to be a standard DFS.

Theorem 47 *The conditions* (S-1)–(S-6) *are necessary and sufficient for* $\mathcal{F}$ *to be a standard DFS.*
(Proof: D.1, p.437+)

Note. In Chap. 7 below, I will identify an important model and show that it is outstanding among the approaches to fuzzy quantification. We shall then see in Def. 97 how the axiomatic characterization of standard DFSes can be further developed into an axiomatization of the distinguished model.

5.6 Chapter summary

Building on the discussion of general properties in the last chapter, I have now made an attempt to group the models into natural subclasses, and to investigate their specific properties and mutual relationships. In particular, the goal was to develop those concepts that require a certain homogeneity on part of the models. For example, certain aggregating constructions, but also comparisons between DFSes, are only possible if the considered models show a similar structure, and hence assume a certain notion of compatibility. In the chapter, I first grouped the models according to their induced negation, thus forming the classes of $\widetilde{\neg}$-DFSes. This classification of the models mainly served to investigate tranformations which permit a transfer between the subclasses, and hence elucidate their precise relationship. In particular, I have proposed the model transformation scheme, which achieves a bidirectional translation between different classes of $\widetilde{\neg}$-DFSes. By instantiating the transformation scheme, a given model of fuzzy quantification can be adapted to any desired choice of induced negation. In this sense, all negation operators are universal to DFSes. Utilizing this universal translation property, we can now restrict attention to a single representative choice, which will be assumed the prominent example, $\neg x = 1 - x$, and the corresponding class of ¬-DFSes which induce the standard negation. Hence the analysis of model transformations culminated in the identification of a single representative class, to which all other classes can be reduced. This makes an important result which might speed up research into fuzzy quantification, because it is sufficient to define all novel concepts on the representative class of ¬-DFSes only, and the intricacies of providing a fully general account can effectively be avoided. By applying the model transformation scheme, the resulting concepts can easily be extended to other types of models if so desired.

Keeping the standard negation fixed, I subsequently refined the original granulation of the models, and further grouped the ¬-DFSes by their induced negation, thus shaping the classes of $\widetilde{\vee}$-DFSes (which induce the s-norm $\widetilde{\vee}$ and the standard negation). I then introduced a number of concepts which live on $\widetilde{\vee}$-DFSes. It is the refinement into the new classes which rendered this development possible, because the behaviour of the models has now become sufficiently coherent. In particular, the relative homogeneity of the models allowed me to define a new construction on the models, that of model aggregation. To this end, I introduced the model aggregation scheme, along with a formalization of the precise conditions which constrain the admissible aggregation operators. This investigation revealed that the $\widetilde{\vee}$-DFSes are closed under an important class of operations which comprises stable symmetric sums, convex combinations, and other important aggregation operators. A concrete example has also been given of how the scheme is applied, viz. the construction of the greatest lower specificity bound from a collection of given models.

Apart from providing the required substrate of classes for introducing novel constructions, the refinement into $\widetilde{\vee}$-DFSes also shows suitable granularity for discussing the specificity of models. This is because the members of the new classes are sufficiently similar, that the models within the classes can now be related to each other in an intuitive sense. Due to the symmetry of DFSes with respect to negation, it is not useful to relate the models by the usual order $\leq$. I therefore adopted Mukaidono's ambiguity relation, and turned it into the specificity order $\preceq_c$ imposed on the models. Roughly speaking, the models are then compared by the degree to which they commit to the two-valued poles. The investigation of this aspect of the models is of particular importance, because one is often interested in obtaining as specific results as possible. Unlike $\leq$, the proposed relation $\preceq_c$ is not a total order, i.e. some choices of models might still fail to be comparable. The models are then considered inconsistent, because there are situations in which they commit to opposite crisp outcomes. However, even in this case of mutually inconsistent models, the $\widetilde{\vee}$-DFSes have been shaped in such a way that it is at least possible to construct a joint lower bound on the models, a closure property which is not valid for unrestricted DFSes. In the chapter, I have developed the formal machinery required for discussing specificity issues. In particular, I was able to precisely identify the boundary cases in terms of specificity. To this end, I first introduced the generalized fuzzy median, which is essential for describing the structure of lower bounds. In fact, it was shown that by instantiating the model aggregation scheme with the generalized fuzzy median, it is possible to effectively construct the greatest lower specificity bound for every given collection of $\widetilde{\vee}$-DFSes. As concerns the converse issue of upper specificity bounds, we required the notion of 'specificity consistency' of a given collection of models. Such a collection is considered specificity consistent if all models are mutually consistent in the sense explained above. Put differently, the collection is specificity consistent if all models are comparable under $\preceq_c$. It came out of my investigation that the notion of specificity consistency is of crucial relevance to the theory of specificity, because it is this criterion which decides upon the existence of least upper specificity bounds. Finally, the precise structure of the upper bounds has been identified.

Following this discussion of specificity, I then focused on two constructions not yet addressed in the previous chapter, that of forming conjunctions and disjunctions of quantifiers. Some first results on the interpretation of conjunctions and disjunctions in DFSes have then been presented, which cover an important subclass of DFSes and all standard models. In particular, I have investigated the compatibility of $(\widetilde{\neg}, \max)$-

DFSes with these constructions, which tie the induced disjunction to the standard choice, and group the conforming models by their induced negation. For this type of models, I was able state inequalities, which express upper and lower bounds on the possible outcome of the conjunction and disjunction. As we shall see in the next chapter, this already makes the strongest result which is possible in the general case, because most DFSes fail to comply with conjunctions and disjunctions in the precise sense. In particular, this applies to all standard DFSes. Apart from researching conjunctions and disjunctions of quantifiers, I also cast a look at the interpretation of the equivalence and antivalence/xor truth functions in models of the assumed type. Again, it will only become obvious in the next chapter, why these choices of truth function pose special difficulties.

Finally, a distinguished subclass of the models has been investigated, that of standard DFSes. In these models, the fuzzy connectives, logical quantifiers, and the extension principle are all tied to their standard interpretation in fuzzy logic. It is this conformance to the standard body of fuzzy set theory which lets me consider this type of DFSes the standard models for fuzzy quantification, which can be defined in the assumed framework. In the chapter, I first presented a formal definition of the models, which makes the extra requirement that the given $\mathcal{F}$ induce the standard negation and standard disjunction. It was then shown that this simple condition is sufficient to achieve the desired standard interpretation of all constructions in the DFS framework. Among other things, it will emerge from this analysis that the propositional part of each standard model coincide with the K-standard sequence logic of Dienes, and that the three-valued fragment coincide with well-known Kleene's logic. The latter observation in particular, will be of key relevance to the later chapters of the report because it guides research into a constructive principle for models of the theory. In addition, I have shown that all standard DFSes coincide on two-valued quantifiers (and by (Z-1), of course also on two-valued arguments). This indicates that all standard models consistently extend the Gainesian fuzzification mechanism, proposed in [44], from two-valued propositional functions to the complex case of multi-place quantification based on arbitrary semi-fuzzy quantifiers. After reviewing these properties of the standard models, I then developed an axiomatization of standard DFSes in terms of a modification of the original axiom system (Z-1)–(Z-6). This effort is justified by the special role of the standard type among all models of the theory. The required changes to the axioms are rather obvious, and indeed have the desired effect of identifying the target class of standard models.

6 Further semantical properties and theoretical adequacy limits

6.1 Motivation and chapter overview

The previous chapters were devoted to the study of plausible models for fuzzy quantification, and developed an analysis of the generic properties shown by all models, as well as special concepts like the specificity order and the model aggregation scheme, which had to be defined on natural subgroups of the models. In this chapter, I will now take care of the optional characteristics of a DFS, which prove useful only in certain situations, and I will also discuss the problematic cases, which fit less nicely into the proposed framework. Typically, these difficulties are not tied to the framework, though, and merely witness that the considered properties pose problems to fuzzy quantification in general.

As to the optional criteria, there are several reasons that might prohibit turning a considered adequacy criterion, appealing as it might be, into a general postulate imposed on all plausible models. First of all, some of the criteria express practical concerns, and would exclude cases of theoretical interest when required for arbitrary models. The desideratum on continuity or smoothness in particular, would compromise the closure of the models with respect to specificity bounds. In addition, some of the properties conflict with other properties which are also intuitively appealing. This situation is of course familiar in fuzzy set theory, due to the known fact that it is impossible to define a Boolean algebra on the set of continuous truth values in $[0, 1]$. In particular, it is well-known that the desiderata of idempotence/distributivity and compliance with the law of contradiction, exclude each other in the fuzzy framework. It is hence not surprising that these conflicts reach beyond the propositional part and also show up in the realm of fuzzy quantification. In the following, we will learn about a few such properties, which are desirable in certain situations, but conflict with other desirable characteristics. Depending on the relative importance of these adequacy criteria, I will then resort to different strategies in order to cope with this kind of situation.

- In those cases where one of the properties clearly outweighs the other, and must be considered a mandatory constraint on plausible models, the other desideratum must be dropped altogether, or fitted to the conflict situation, by weakening the imposed condition to those core situations which are still compatible with the superordinate requirement.
- In those cases where the considerations are of equal weight, it is probably best to regard both conditions as optional features of approaches to fuzzy quantification. It is then the application at hand, which decides upon that pattern of properties which best meet its specific goals.

In the chapter, I first consider the optional type of conditions, which do not raise any conflicts with fundamental postulates. I start by discussing two criteria that capture different aspects of smoothness or continuity, and hence take into account the chief robustness concerns on the models, which arise from an application perspective. Following that, some regularities will be investigated, which describe the way in which a model handles the unspecificity observed in the inputs. Intuitively, one would expect that the model's outputs cannot become more specific when there is less information in the inputs. These concerns regarding the propagation of fuzziness in the model will be formalized in terms of the specificity order $\preceq_c$ introduced in the last chapter. Both in the case of continuity and propagation fuzziness there is good reason not to impose these conditions in general, and they hence constitute important cases of optional conditions.

Having studied these issues in some depth, I then review the construction introduced in the last chapter, of forming the conjunction and disjunction of quantifiers that share the same arity. It is here that I will substantiate the claim that standard models cannot comply with this construction on formal grounds, and compatibility with conjunctions and disjunctions is indeed reserved to rather special types of models (if any). Similar results will be obtained in the subsequent analysis of those quantifiers which show duplications of variables in their defining equation. Again, an incompatibility with certain types of models will be detected, which also includes the standard models. It is hence best to consider these conditions optional, and reserve both the conformity to conjunctions/disjunctions, and the compliance with multiple occurrences of variables, to special types of situations (e.g. theoretical treatise). Following the discussion of these rather abstract criteria, which once again evoke the known conflict between idempotence/distributivity and the law of contradiction in fuzzy logic, I will then turn attention to three conditions of strong linguistic relevance. First of all, I will consider a generalized form of monotonicity which shows up in so-called convex quantifiers, and investigate the potential of the models to preserve such unimodal, bell-shaped, trapezoid etc. monotonicity patterns of quantifiers. I will then discuss one of the central properties of NL quantifiers which has attracted much interest in the literature on TGQ, that of conservativity. In particular, I will pursue two alternative generalizations of conservativity to the fuzzy case, one of which is universally valid, and the other one totally inconsistent, even under assumptions considerably weaker than the proposed DFS axioms. Finally I review the construction of argument insertion, and the related topic of adjectival restriction. To this end, I will elaborate an apparent idea how a compositional treatment of fuzzy argument insertion, and hence of adjectival restriction by fuzzy adjectives, can be achieved in the given framework. Acknowledging the structuring role of Frege's compositionality principle to the formalization of natural language semantics, this will prepare the later specification of those models which are optimally plausible from a linguist's standpoint.

6.2 Continuity conditions

Firstly I introduce two adequacy criteria concerned with distinct aspects of the 'smoothness' or 'continuity' of a DFS. These conditions are essential for the models to be *practical* because it is extremely important for applications that the results of a DFS be stable under slight changes in the inputs. These 'changes' can either occur in the fuzzy argument sets (e.g. due to noise), or they can affect the semi-fuzzy quantifier. For example, if a person A has a slightly different interpretation of quantifier Q compared to person B, then we still want them to understand each other, and the quantification results obtained from the two models of the target quantifier should be very similar in such cases.

In order to express the robustness criterion with respect to slight changes in the fuzzy arguments, a metric on fuzzy subsets is needed, which serves as a numerical quantity of the similarity of the arguments. For all base sets $E \neq \varnothing$ and all $n \in \mathbb{N}$, we define the metric $d : \widetilde{\mathcal{P}}(E)^n \times \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) = \max_{i=1}^{n} \sup\{|\mu_{X_i}(e) - \mu_{X'_i}(e)| : e \in E\}\,, \tag{31}$$

for all $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$. Based on this metric, we can now express the desired criterion for continuity *in arguments*.

Definition 63

We say that a QFM $\mathcal{F}$ is arg-continuous *if and only if $\mathcal{F}$ maps all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to continuous fuzzy quantifiers $\mathcal{F}(Q)$, i.e. for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\mathcal{F}(Q)(X_1, \ldots, X_n), \mathcal{F}(Q)(X'_1, \ldots, X'_n)) < \varepsilon$ for all $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ with $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$.*

Arg-continuity means that a small change in the membership grades $\mu_{X_i}(e)$ of the argument sets does not change $\mathcal{F}(Q)(X_1, \ldots, X_n)$ drastically; it hence expresses an important robustness condition with respect to noise.

The second robustness criterion is intended to capture the idea that slight changes in a semi-fuzzy quantifier should not cause the quantification results to change drastically. To introduce this criterion, we must first define suitable distance measures for semi-fuzzy quantifiers and for fuzzy quantifiers. Hence for all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$d(Q, Q') = \sup\{|Q(Y_1, \ldots, Y_n) - Q'(Y_1, \ldots, Y_n)| : Y_1, \ldots, Y_n \in \mathcal{P}(E)\}\,, \tag{32}$$

and similarly for all fuzzy quantifiers $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$,

$$d(\widetilde{Q}, \widetilde{Q}') = \sup\{|\widetilde{Q}(X_1, \ldots, X_n) - \widetilde{Q}'(X_1, \ldots, X_n)| : X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\}\,. \tag{33}$$

Definition 64

We say that a QFM $\mathcal{F}$ is Q-continuous *if and only if for each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mathcal{F}(Q), \mathcal{F}(Q')) < \varepsilon$ whenever $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ satisfies $d(Q, Q') < \delta$.*

Q-continuity captures an important aspect of robustness with respect to imperfect knowledge about the precise definition of a quantifier; i.e. slightly different definitions of Q will produce similar quantification results.

Both conditions are crucial to the utility of a DFS and must be possessed by every practical model. They are not part of the DFS axioms because I wanted to have models for general t-norms (including the discontinuous variety).

6.3 Propagation of fuzziness

The next two criteria are concerned with the 'propagation of fuzziness', i.e. the way in which the amount of imprecision in the model's inputs affects changes of the model's outputs. To this end, let us recall the partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ defined by equality (30). We can extend $\preceq_c$ to fuzzy sets $X \in \widetilde{\mathcal{P}}(E)$, semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ as follows:

$$\begin{aligned}
X \preceq_c X' &\iff \mu_X(e) \preceq_c \mu_{X'}(e) && \text{for all } e \in E;\\
Q \preceq_c Q' &\iff Q(Y_1, \dots, Y_n) \preceq_c Q'(Y_1, \dots, Y_n) && \text{for all } Y_1, \dots, Y_n \in \mathcal{P}(E);\\
\widetilde{Q} \preceq_c \widetilde{Q}' &\iff \widetilde{Q}(X_1, \dots, X_n) \preceq_c \widetilde{Q}'(X_1, \dots, X_n) && \text{for all } X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)\,.
\end{aligned}$$

Intuitively, we expect that the quantification results become less specific whenever the quantifier or the argument sets become less specific: the fuzzier the input, the fuzzier the output. For example, consider a base set $E = \{\mathrm{Joan}, \mathrm{Lucas}, \mathrm{Mary}\}$ and fuzzy subsets $\textbf{lucky}, \textbf{lucky}' \in \widetilde{\mathcal{P}}(E)$ defined by

$$\begin{aligned}
\textbf{lucky} &= 1/\mathrm{Joan} + 0.8/\mathrm{Lucas} + 0.2/\mathrm{Mary},\\
\textbf{lucky}' &= 0.6/\mathrm{Joan} + 0.5/\mathrm{Lucas} + 0.4/\mathrm{Mary}\,.
\end{aligned}$$

Then $\textbf{lucky}' \preceq_c \textbf{lucky}$, i.e. the former interpretation of "lucky" is less committed to the possible crisp decisions. We should hence expect e.g. that $\mathcal{F}(\textbf{most})(\textbf{rich}, \textbf{lucky}') \preceq_c \mathcal{F}(\textbf{most})(\textbf{rich}, \textbf{lucky})$ as well, and hence the quantification results based on $\textbf{lucky}'$ should be less decided than those computed from the fuzzy subset **lucky**, which bears more specific information.

Definition 65
Let a QFM $\mathcal{F}$ be given.

a. *We say that $\mathcal{F}$* propagates fuzziness in arguments *if and only if the following property is satisfied for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n, X'_1, \dots, X'_n$: If $X_i \preceq_c X'_i$ for all $i = 1, \dots, n$, then $\mathcal{F}(Q)(X_1, \dots, X_n) \preceq_c \mathcal{F}(Q)(X'_1, \dots, X'_n)$.*

b. *We say that $\mathcal{F}$* propagates fuzziness in quantifiers *if and only if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.*

Notes

- Both conditions are certainly natural to require, and I consider them as desirable but optional. A more thorough discussion of propagation of fuzziness and its tradeoffs can be found in Chap. 8, 245.
- The intuitive expectation that the output cannot get more detailed when the input gets fuzzier, is satisfied by the standard connectives $\neg x = 1 - x$, $\wedge = \min$ and $\vee = \max$. Hence the standard models of the proposed system make good candidates for propagation of fuzziness. We shall see below in Th-131 that both conditions are possible but in fact optional for standard models, and also that the

conditions are mutually independent. This indicates that propagation of fuzziness in quantifiers and propagation of fuzziness in arguments open distinct semantical dimensions, in the full space of models for fuzzy quantification.

- Let me also remark that the standard conjunction and disjunction are not the only choice of t- and s-norms that propagate fuzziness in their arguments. In fact, the t-norm $\widetilde{\wedge}_m$ defined by equality (22) apparently also propagates fuzziness in its arguments, a property which transfers to the dual s-norm $\widetilde{\vee}_m$ of $\widetilde{\wedge}_m$ under the standard negation. The corresponding class of $\widetilde{\vee}_m$-DFSes might then witness the existence of nonstandard models that propagate fuzziness. However, I am currently lacking any evidence on the existence of models in this formal class.

6.4 Conjunctions and disjunctions of quantifiers

Let us now review the construction introduced in Def. 60, which builds a new quantifier from a conjunction or disjunction of given ones. In the last chapter, I presented a result on the interpretation of such conjunctions and disjunctions in $(\widetilde{\neg}, \max)$-DFSes. However, I was only able to establish an inequality, rather than the full compatibility of the models with these constructions. The goal of my present investigation is to gain some understanding, why I was unable to prove a better compliance result for $(\widetilde{\neg}, \max)$-DFSes, and strengthen these inequalities into precise equalities. As we shall now learn, the attempts to improve upon the original weak result are bound to fail, because the t-norm $\min$, which is induced by $(\widetilde{\neg}, \max)$-DFSes, violates the law of contradiction.

Theorem 48
Suppose a DFS $\mathcal{F}$ has the property that

$$\mathcal{F}(Q \mathbin{\widetilde{\wedge}} Q') = \mathcal{F}(Q) \mathbin{\widetilde{\wedge}} \mathcal{F}(Q') \tag{34}$$

for all semi-fuzzy quantifiers $Q, \ Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, where $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$. Then $\widetilde{\wedge}$ satisfies $x \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, x = 0$ for all $x \in \mathbf{I}$.

A DFS $\mathcal{F}$ which is homomorphic with respect to conjunctions (or equivalently, disjunctions) of quantifiers therefore induces a t-norm which respects the law of contradiction. This is clearly unacceptable since it would exclude many interesting t-norms; in particular, the standard choice $\widetilde{\mathcal{F}}(\wedge) = \min$. I have therefore *not* required in general that a DFS be homomorphic with respect to conjunctions/disjunctions of quantifiers.

6.5 Multiple occurrences of variables

The reader will certainly have noticed our special treatment of the propositional connectives $\leftrightarrow$ and xor. The difficulties in proving properties of the models with respect to these connectives are caused by the fact that the definition of $\leftrightarrow, \mathrm{xor} : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$

involves *multiple occurrences* of propositional variables, viz.

$$\begin{aligned} x_1 \leftrightarrow x_2 &= (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2) \\ &= (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \\ x_1 \operatorname{xor} x_2 &= (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2) \\ &= (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)\,, \end{aligned}$$

for all $x_1, x_2 \in \{0, 1\}$. A brief glance at the DFS axioms reveals that there is *no* axiom which describes this case of duplicated variables. In order to be able to discuss the issue on the formal level, let us introduce the following construction which permits multiple occurrences of a variable in the definition of a quantifier.

Definition 66
Suppose $Q : \mathcal{P}(E)^m \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $\xi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ is a mapping. By $Q\xi : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the semi-fuzzy quantifier defined by $Q\xi(Y_1, \ldots, Y_n) = Q(Y_{\xi(1)}, \ldots, Y_{\xi(n)})$, for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We use an analogous definition for fuzzy quantifiers.

The interesting case is that of a non-injective ξ, which inserts the same variable in two (or more) argument positions of the original quantifier Q.

Theorem 49
Suppose $\mathcal{F}$ is a DFS which is compatible with the duplication of variables, i.e. whenever $Q : \mathcal{P}(E)^m \longrightarrow \mathbf{I}$ and $\xi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ for some $n \in \mathbb{N}$, then $\mathcal{F}(Q\xi) = \mathcal{F}(Q)\xi$. Then the induced conjunction $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ satisfies $x \,\widetilde{\wedge}\, \widetilde{\neg}\, x = 0$ for all $x \in \mathbf{I}$.

Again, I consider this too restrictive and therefore have *not* required that $\mathcal{F}$ be homomorphic with respect to the duplication of variables.

6.6 Convex quantifiers

Apart from the monotonicity type of quantifiers, TGQ has also developed more sophisticated concepts which describe the characteristic shape of a given quantifier. The following definition of convex quantifiers covers unimodal, bell-shaped, trapezoidal and other generic examples.

Definition 67
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier such that $n > 0$. Q is said to be convex in its i-th argument, *where $i \in \{1, \ldots, n\}$, if*

$$Q(X_1, \ldots, X_n) \geq \min(\quad \begin{aligned} &Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\ &Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) \end{aligned}$$

whenever $X_1, \ldots, X_n, X_i', X_i'' \in \mathcal{P}(E)$ and $X_i' \subseteq X_i \subseteq X_i''$.
Convexity of a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ in the i-th argument is defined analogously, where $X_1, \ldots, X_n, X_i', X_i'' \in \widetilde{\mathcal{P}}(E)$, and '$\subseteq$' is the fuzzy inclusion relation.

Note. In TGQ, those quantifiers that I call ‘convex’ are usually dubbed ‘continuous’, see e.g. [151] and [45, Def. 16, p. 250]. I have decided to change terminology in order to avoid the possible ambiguity of ‘continuous’, which could also mean ‘smooth’. To present an example, the absolute quantifier “between 10 and 20” is convex in both arguments, and the proportional quantifier “about 30 percent” is convex in the second argument. Some well-known properties of convex quantifiers (in the sense of TGQ) also carry over to semi-fuzzy and fuzzy quantifiers.

Theorem 50 (Conjunctions of convex semi-fuzzy quantifiers)
Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers of arity* $n > 0$ *which are convex in the* i*-th argument, where* $i \in \{1, \dots, n\}$*. Then the semi-fuzzy quantifier* $Q \wedge Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*, defined by*

$$(Q \wedge Q')(X_1, \dots, X_n) = \min(Q(X_1, \dots, X_n), Q'(X_1, \dots, X_n))$$

for all $X_1, \dots, X_n \in \mathcal{P}(E)$*, is also convex in the* i*-th argument.*

Note. The theorem states that conjunctions of convex semi-fuzzy quantifiers are convex (provided the standard fuzzy conjunction $\wedge = \min$ is chosen).
A similar point can be made about fuzzy quantifiers.

Theorem 51 (Conjunctions of convex fuzzy quantifiers)
Suppose $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *are fuzzy quantifiers of arity* $n > 0$ *which are convex in the* i*-th argument, where* $i \in \{1, \dots, n\}$*. Then the fuzzy quantifier* $\widetilde{Q} \wedge \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*, defined by*

$$(\widetilde{Q} \wedge \widetilde{Q}')(X_1, \dots, X_n) = \min(\widetilde{Q}(X_1, \dots, X_n), \widetilde{Q}'(X_1, \dots, X_n))$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$*, is also convex in the* i*-th argument.*

Let us also observe that every convex semi-fuzzy quantifier can be decomposed into a conjunction of a nonincreasing and a nondecreasing semi-fuzzy quantifier:

Theorem 52 (Decomposition of convex semi-fuzzy quantifiers)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is convex in its* i*-th argument,* $i \in \{1, \dots, n\}$*, if and only if* Q *is the conjunction of a nondecreasing and a nonincreasing semi-fuzzy quantifier, i.e. if there exist* $Q^+, Q^- : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *such that* Q^+ *is nondecreasing in its* i*-th argument;* Q^- *is nonincreasing in its* i*-th argument, and* $Q = Q^+ \wedge Q^-$*.*

Again, a similar point can be made about fuzzy quantifiers:

Theorem 53 (Decomposition of convex fuzzy quantifiers)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is convex in its* i*-th argument,* $i \in \{1, \dots, n\}$*, if and only if* $\widetilde{Q}$ *is the conjunction of a nondecreasing and a nonincreasing fuzzy quantifier, i.e. if there exist* $\widetilde{Q}^+, \widetilde{Q}^- : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *such that* $\widetilde{Q}^+$ *is nondecreasing in its* i*-th argument;* $\widetilde{Q}^-$ *is nonincreasing in its* i*-th argument, and* $\widetilde{Q} = \widetilde{Q}^+ \wedge \widetilde{Q}^-$*.*

We say that a QFM $\mathcal{F}$ preserves convexity if convexity of a quantifier in its arguments is preserved when applying $\mathcal{F}$.

Definition 68

A QFM $\mathcal{F}$ is said to preserve convexity of n-ary quantifiers, *where $n \in \mathbb{N} \setminus \{0\}$, if and only if every n-ary semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which is convex in its i-th argument is mapped to a fuzzy quantifier $\mathcal{F}(Q)$ which is also convex in its i-th argument. $\mathcal{F}$ is said to* preserve convexity *if $\mathcal{F}$ preserves the convexity of n-ary quantifiers for all $n > 0$.*

As we shall now see, preservation of convexity is a plausibility criterion which in its strong form conflicts with other desirable properties. Let us first notice that contextuality of a QFM excludes preservation of convexity.

Theorem 54

Suppose $\mathcal{F}$ is a contextual QFM with the following properties: for every base set $E \neq \varnothing$,

a. the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;

b. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;

c. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists $e \in E$ such that $\mu_X(e) < 1$, then $\mathcal{F}(\sim\forall)(X) > 0$, where $\sim\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by

$$(\sim\forall)(Y) = \begin{cases} 1 & : \quad X \neq E \\ 0 & : \quad X = E \end{cases}$$

Then $\mathcal{F}$ does not preserve convexity of one-place quantifiers $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on finite base sets $E \neq \varnothing$. In particular, no DFS preserves convexity.

This means that even if we restrict to the simple case of one-place quantifiers on finite base sets, there is still no QFM $\mathcal{F}$ which conforms to the very elementary semantical postulates stated in the theorem, and at the same time preserves convexity under the simplifying assumptions.

Because contextuality is a rather fundamental condition, it seems better to weaken the requirements on the preservation of convexity, rather than compromising contextuality or the other elementary conditions.

Noticing that the common examples of convex NL quantifiers are typically of the quantitative kind, it is straightforward to weaken the requirement of preserving convexity to quantitative convex quantifiers (see Def. 38 and Def. 39 for the assumed definitions of quantitativity). Unfortunately, this weakening is insufficient yet and the targeted class of convex quantifiers still too broad, as witnessed by the following theorem.

Theorem 55

Suppose $\mathcal{F}$ is a contextual QFM which is compatible with cylindrical extensions and satisfies the following properties: for all base sets $E \neq \varnothing$,

a. *the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;*

b. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;*

c. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists and there exists some $e \in E$ such that $\mu_X(e') = 0$ for all $e' \in E \backslash \{e\}$ and $\mu_X(e) < 1$, then $\mathcal{F}(\sim\exists)(X) > 0$, where $\sim\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by*

$$(\sim\exists)(Y) = \begin{cases} 1 & : \quad X = \varnothing \\ 0 & : \quad X \neq \varnothing \end{cases}$$

Then $\mathcal{F}$ does not preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$ even on finite base sets. In particular, no DFS preserves the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$.

This leaves open the possibility that certain models will preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n = 1$. For simplicity, we shall investigate this preservation property for the case of finite domains only.

Definition 69
A QFM $\mathcal{F}$ is said to weakly preserve convexity *if $\mathcal{F}$ preserves the convexity of quantitative one-place quantifiers on finite domains.*

In this case, we get positive results on the existence of models that satisfy the specified criterion, and hence weakly preserve convexity. An example of the conforming DFS $\mathcal{M}_{CX}$ will be given below in Def. 96.

Let is now turn to a special case of two-place quantification. In general, we have negative results concerning the preservation of convexity for quantifiers of arity $n > 1$, even if these are quantitative (see Th-55). However it is possible for a DFS to preserve convexity properties of two-place quantifiers in the case of absolute two-place quantifiers, which are of obvious interest to natural language interpretation.

Theorem 56
Suppose $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is an absolute quantifier on a finite base set, i.e. there exists a quantitative one-place quantifier $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ such that $Q = Q'\cap$. If a DFS $\mathcal{F}$ has the property of weakly preserving convexity and Q is convex in its arguments, then $\mathcal{F}(Q)$ is also convex in its arguments.

Although the proportional type is not covered, the theorem demonstrates that weak preservation of convexity is strong enough a condition to ensure a proper interpretation of many NL quantifiers of interest, e.g. **between 10 and 20**, **about 50** and others.

6.7 Conservativity

One of the pervasive properties of NL quantifiers is *conservativity* [82, p. 275, eq. (40)], [9, p. 445], [8, p. 452], [45, pp. 245-249].

Definition 70 (Conservativity)
We shall call $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ conservative *if*

$$Q(X_1, X_2) = Q(X_1, X_1 \cap X_2)$$

for all $X_1, X_2 \in \mathcal{P}(E)$*.*

All two-valued or semi-fuzzy quantifiers introduced so far are conservative.[37] In particular, all proportional quantifiers are conservative by definition, see Def. 166. To give an example, if E is a set of persons, **married** $\in \mathcal{P}(E)$ is the subset of married persons, and **have_children** $\in \mathcal{P}(E)$ is the set of persons who have children, then the conservative semi-fuzzy quantifier **almost all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ satisfies

$$\begin{aligned}&\textbf{almost all}(\textbf{married}, \textbf{have\_children})\\ &= \textbf{almost all}(\textbf{married}, \textbf{married} \cap \textbf{have\_children})\end{aligned}$$

i.e. the meanings of “Almost all married persons have children” and “Almost all married persons are married persons who have children” coincide. Like having extension, conservativity expresses an aspect of context insensitivity: if an element of the domain is irrelevant to the restriction (first argument) of a two-place quantifier, then it does not affect the quantification result at all. For example, every conservative $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ apparently satisfies $Q(X_1, X_2 \cap X_1) = Q(X_1, X_2 \cup \neg X_1)$. Hence in the case that $e \notin X_1$ for a given element e of the base set, it does not matter whether $e \in X_2$ or $e \notin X_2$. A corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ should at least possess the following property of *weak conservativity*:

Definition 71 (Weak conservativity)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ *is said to be* weakly conservative *if*

$$\widetilde{Q}(X_1, X_2) = \widetilde{Q}(X_1, \mathrm{spp}(X_1) \cap X_2)\,,$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$*, where* $\mathrm{spp}(X_1)$ *is the support of* X_1*, see* (24)*.*

This definition is sufficiently strong to capture the context insensitivity aspect of conservativity: an element $e \in E$ which is irrelevant to the restriction of the quantifier, i.e. $\mu_{X_1}(e) = 0$, has no effect on the quantification result, which is independent of $\mu_{X_2}(e)$.

Theorem 57
Every DFS $\mathcal{F}$ weakly preserves conservativity, i.e. if $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *is conservative, then* $\mathcal{F}(Q)$ *is weakly conservative.*

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[37] with the only exception of “only”.

Definition 72 (Strong conservativity)
Let us say that a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ *is* strongly conservative *if*

$$\widetilde{Q}(X_1, X_2) = \widetilde{Q}(X_1, X_1 \mathbin{\widetilde{\cap}} X_2) \qquad \textit{for all } X_1, X_2 \in \widetilde{\mathcal{P}}(E).$$

In addition to the context insensitivity aspect, strong conservativity also reflects the definition of crisp conservativity in terms of intersection with the first argument.

Theorem 58
Assume the QFM $\mathcal{F}$ satisfies the following conditions: (a) $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$; (b) $\widetilde{\neg}$ is a strong negation operator; (c) $\widetilde{\wedge}$ is a t-norm; (d) $\mathcal{F}$ is compatible with internal meets, see Def. 33; (e) $\mathcal{F}$ is compatible with dualisation. Then $\mathcal{F}$ does not strongly preserve conservativity, i.e. there are conservative $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ such that $\mathcal{F}(Q)$ is not strongly conservative. In particular, no DFS strongly preserves conservativity.

Hence strong preservation of conservativity cannot be ensured in a fuzzy framework, even under assumptions which are much weaker than the DFS axioms. However, the weak form of preserving conservativity already covers the most important aspects from an adequacy perspective, and it should be sufficient for most practical concerns that the considered models comply with conservativity in this sense.

It is instructive to see how conservativity interacts with the property of having extension, in case a quantifier satisfies both. Then for all crisp $Y_1, Y_2 \in \mathcal{P}(E)$, $Y_1 \neq \varnothing$,

$$\begin{aligned} Q_E(Y_1, Y_2) &= Q_E(Y_1, Y_1 \cap Y_2) \\ &= Q_{E'}(E', Y_1 \cap Y_2) \\ &= Q'_{E'}(Y_1 \cap Y_2)\,, \end{aligned}$$

where $E' = Y_1$, and $Q'_{E'} : \mathcal{P}(E') \longrightarrow \mathbf{I}$ is the unrestricted form of $Q_{E'}$ defined by $Q'_{E'}(Z) = Q_{E'}(E', Z)$ for all $Z \in \mathcal{P}(E')$. This example demonstrates that in the crisp case, restricted quantification based on a quantifier which is conservative and has extension can be reduced to unrestricted (one-place) quantification on another domain (supplied by the first argument). The example thus explains why one-place, unrestricted quantification is important although natural language quantifiers are typically at least two-place. Of course, such a reduction is not possible in the fuzzy case because a fuzzy subset $Y_1 \in \widetilde{\mathcal{P}}(E)$ cannot serve as a domain (we have only admitted crisp base sets E).

6.8 Fuzzy argument insertion

In our comments on argument insertion (see p. 136) we have remarked that adjectival restriction with fuzzy adjectives cannot be modelled directly: if $A \in \widetilde{\mathcal{P}}(E)$ is a *fuzzy* subset of E, then only $\mathcal{F}(Q)\triangleleft A$ is defined, but not $Q\triangleleft A$. However, one can ask if $\mathcal{F}(Q)\triangleleft A$ can be represented by a semi-fuzzy quantifier Q', i.e. if there is a Q' such that

$$\mathcal{F}(Q)\triangleleft A = \mathcal{F}(Q')\,. \tag{35}$$

The obvious choice of Q' is the following.

Definition 73
Suppose $\mathcal{F}$ is a QFM, $Q : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $A \in \widetilde{\mathcal{P}}(E)$. Then $Q \mathbin{\widetilde{\triangleleft}} A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$Q \mathbin{\widetilde{\triangleleft}} A = \mathcal{U}(\mathcal{F}(Q) \triangleleft A)\,,$$

i.e. $Q \mathbin{\widetilde{\triangleleft}} A(Y_1, \ldots, Y_n) = \mathcal{F}(Q)(Y_1, \ldots, Y_n, A)$ for all crisp $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Notes

- $Q \mathbin{\widetilde{\triangleleft}} A$ is written with the 'tilde' notation $\widetilde{\triangleleft}$ in order to distinguish it from $Q \triangleleft A$ and emphasise that it depends on the chosen QFM $\mathcal{F}$.
- as already noted in [46, p. 54], $Q' = Q \mathbin{\widetilde{\triangleleft}} A$ is the only choice of Q' which possibly satisfies (35), because any Q' which satisfies $\mathcal{F}(Q') = \mathcal{F}(Q) \triangleleft A$ also satisfies

  $$Q' = \mathcal{U}(\mathcal{F}(Q')) = \mathcal{U}(\mathcal{F}(Q) \triangleleft A) = Q \mathbin{\widetilde{\triangleleft}} A\,,$$

  which is apparent from Th-2.

Unfortunately, $Q \mathbin{\widetilde{\triangleleft}} A$ is not guaranteed to fulfill (35) in a QFM (not even in a DFS). Let us hence turn this equality into an adequacy condition which ensures that $Q \mathbin{\widetilde{\triangleleft}} A$ conveys the intended meaning in a given QFM $\mathcal{F}$:

Definition 74
Suppose $\mathcal{F}$ is a QFM. We say that $\mathcal{F}$ is compatible with fuzzy argument insertion *if for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and every $A \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} A) = \mathcal{F}(Q) \triangleleft A$.*

The main application of this property in natural language is that of adjectival restriction of a quantifier by means of a fuzzy adjective. For example, suppose E is a set of people, and **lucky** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of those people in E who are lucky. Further suppose that **almost all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier which models "almost all". Finally, suppose the DFS $\mathcal{F}$ is chosen as the model of fuzzy quantification. We can then construct the semi-fuzzy quantifier $Q' =$ **almost all**$\cap \mathbin{\widetilde{\triangleleft}}$ **lucky**. If $\mathcal{F}$ is compatible with fuzzy argument insertion, then the semi-fuzzy quantifier Q' is guaranteed to adequately model the composite expression "Almost all X_1's are lucky X_2's", because

$$\mathcal{F}(Q')(X_1, X_2) = \mathcal{F}(\textbf{almost all})(X_1, X_2 \mathbin{\widetilde{\cap}} \textbf{lucky})$$

for all fuzzy arguments $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, which (relative to $\mathcal{F}$) is the proper expression for interpreting "Almost all X_1's are lucky X_2's" in the fuzzy case. Compatibility with fuzzy argument insertion is a very restrictive adequacy condition. We shall present the unique standard DFS which fulfills this condition on p. 210.

6.9 Chapter summary

The results presented in this chapter complete the series of investigations related to the semantical properties of approaches to fuzzy quantification, which ultimately decide upon the suitability of these approaches for modelling natural language quantification. While the first chapter in the series was devoted to generic properties shown by every DFS, and the next chapter devoted to specific constructions and concepts that live on natural subgroups of the models, the present chapter finally addressed those cases of properties which cannot be assumed of all prototypical models. Roughly speaking, there are two joint features which link the discussed properties, and motivated my decision to bundle them into this chapter.

- First, these properties are typically too strong from a theoretical perspective. To be specific, some of the properties compromise the desired closure of the models under certain constructions. The properties also exclude whole classes of models which violate application requirements, but deserve theoretical interest nonetheless. Consider turning the criterion of arg-continuity into a mandatory requirement on all models, for example. This would exclude the boundary case of those models which induce the smallest t-norm, because the drastic product is known to be discontinuous. Hence there is good reason for not including this condition into the list of absolute requirements which must be obeyed by all admissible models. These methodical considerations nonwithstanding, it certainly pays off to formalize the continuity requirement, in order to justify the selection of robust models which respond to the demands of practical applications.
- The second type of properties comprises the problematic cases, which result in a partial inconsistency with some desired models, or even an absolute inconsistency with all acceptable models. In the first case (e.g. duplication of variables), it is possibly best to regard the additional requirement as optional, while in the latter case of total inconsistency, the strategy suggested itself of weakening the conflicting condition, and elaborating that core requirement which is still compatible with the given axiomatic foundation.

As to specific results of the chapter, I first took care of the practical concern of robustness or stability against small changes in the inputs. In other words, all practical models should absorb slight variations due to noise, quantization errors etc., which are typical of real-world applications. It was pointed out that there are two kinds of input to the fuzzification mechanism, viz. the fuzzy quantifier and its arguments. Hence two facets of robustness must be distinguished, (a) robustness against variation in the quantifier, and (b), robustness against variation in the arguments. In both cases, the variation observed can either be non-systematic (random or resulting from imprecision) or systematic, in which case it reflects the varying interpretations of quantifiers and NL concepts across language users. Regardless of the type of source which causes the variation, it is necessary for robust system behaviour that at least the minimal requirement of continuity be satisfied, which must be required for both types of sources. These general ideas shaped the subsequent definition of models which are

- Q-continuous, i.e. continuous in the base quantifiers, in order to ensure the robustness of the quantification results under slight changes in the quantifier; and
- arg-continuous, which ensures the robustness of the quantification results under slight changes in interpretation of NL concepts, i.e. the resulting fuzzy quantifiers must be continuous in their arguments.

Due to the omnipresence of noise and random factors in any type of real-world applications, and due to the inevitable differences in the way that people conceive language, it is essential for any practical model to obey these continuity conditions. Nonetheless, the robustness criteria should only be considered optional. As explained above, this will keep a number of interesting albeit discontinuous examples in the full class of models, which evolve as boundary cases.

Following the discussion of these robustness issues, I then investigated the compatibility of the models with the specificity order $\preceq_c$. Roughly speaking, it seems reasonable to assume that whenever either the quantifier or the argument sets become fuzzier, the computed quantification results should become fuzzier, too. In order to express this condition in terms of the specificity order, the basic definition of $\preceq_c$ as a relation on scalars, was extended to fuzzy arguments and semi-fuzzy as well as fuzzy quantifiers in the apparent way. This made it possible to state precisely what it means for a QFM to propagate fuzziness in its inputs, i.e. the quantifiers and their arguments. As concerns examples of models which propagate fuzziness, I first recalled that the standard choice of fuzzy connectives $\min$, $\max$ and $1 - x$ are known to be compatible with $\preceq_c$. Hence the standard DFSes bear the potential of including the new type of models which propagate fuzziness (although other choices beyond standard models are also conceivable). As we shall see in the next chapter, the standard DFSes redeem this promise, and indeed provide a rich source of examples with this property. Not all standard models share these properties, though, which will be evidenced in the subsequent Chap. 8. It is also there that the reasons become clear, why the criteria of propagating fuzziness in quantifiers and arguments should not be considered mandatory, because other demands on the models could outweigh these conditions in certain types of applications.

Turning to the problematic cases, I first investigated a compositionality requirement with respect to conjunctions and disjunctions of quantifiers. These constructions have already been introduced in the previous chapter, which also presented a theorem concerning the compliance of the models with this construction. However, the theorem Th-43 only covered a special case of models, that of $(\widetilde{\neg}, \max)$-DFSes. In addition, the theorem did not claim the exact conformity of the models to this construction; by contrast, it only established upper and lower bounds. In the present chapter, we now learnt that these results cannot be improved from inequalities into precise equalities, at least not in the general case. To be specific, it was shown that the compliance of a model with conjunctions and disjunctions, entails that the induced t-norm satisfy the law of contradiction. For example, the bounded product would make a possible choice of the induced conjunction. Concerning the standard models, the novel result strengthens the previous theorem Th-43, and shows that the presented bounds are based on inequalities proper. This is because the standard conjunction $\min$ does not satisfy the law of contradiction, and hence standard models do not match with conjunctions and disjunctions. In order not to rule out the standard choices, it is therefore necessary to accept a possible violation of compositionality in these cases.

Similar considerations apply in the case of another construction, which builds a new quantifier from a given one by unifying variables in different argument positions. This construction was motivated by the examples of the equivalence (biimplication) and antivalence (xor) connectives, which require multiple occurrences of variables in their definition. In order to develop the formal analysis of such duplications of variables, I proposed a formal definition which captures rather general manipulations in the variables, including the known constructions of permutations and cylindrical extensions. Apart from these unproblematic cases, the new construction can also model the desired multiple occurrences of variables. Based on this formalization, I was then able to show that every model which combines with multiple occurrences of variables, induces a t-norm that respects the law of contradiction. Hence by the same reasoning as in the last example, the compliance with duplications of variables cannot be expected of general models, because this would exclude the prioritized class of standard choices.

Having considered these rather abstract criteria, I again approached the Theory of Generalized Quantifiers, and explored the models under their compliance with three concepts of relevance to linguistic description. To begin with, TGQ knows a more general property that can also be expressed in terms of inequalities, which abstract from simple, unidirectional monotonicity conditions. The conforming quantifiers are dubbed ‘continuous’ in TGQ, but I have decided to change terminology here and call these quantifiers *convex* in the considered argument, in order to avoid a possible confusion with continuity in its smoothness sense. Roughly speaking, the convexity condition no longer requires that the dependency of the quantifier on the argument of interest be uniformly increasing or decreasing (as in the unproblematic case of monotonicity properties), but also allows more general dependencies that are bell-shaped, trapezoidal etc. Special cases include unimodal quantifiers [176, p. 131], which have a unique peak of unity membership. For example, “about ten” can be modelled as a unimodal quantifier. In the chapter, I first pursued the apparent generalisation of the convexity property to semi-fuzzy and fuzzy quantifiers. I then established the closure property of convex quantifiers under the standard conjunction, and I further proved that every convex quantifier can be decomposed into a conjunction of a nonincreasing and a nondecreasing quantifier. Both results apply to semi-fuzzy as well as fuzzy quantifiers, and generalize analogous properties of two-valued convex (continuous) quantifiers that are known from TGQ. Having dwelled upon the notion of convexity, I then discussed the intuitive expectation that the convex shape of quantifiers be preserved when applying a QFM. It came out of this investigation, though, that the preservation of convexity causes severe problems for fuzzy quantification, and there is clear negative evidence against it in the general case. To be specific, it is impossible for a QFM to preserve general convexity properties even under assumptions considerably weaker than the DFS axioms. This indicates that it is the preservation condition which must be weakened, rather than the definition of plausible models. In practice, it would be acceptable for a model of fuzzy quantification to deform the convex shape of those formal quantifiers which do not actually express in natural language, provided that the model behave as desired in all cases of urgent relevance from the linguistic perspective. I hence observed that most (probably all) convex quantifiers of NL are of the quantitative kind, and adjusted the condition of propagating convexity accordingly. However, this refinement did not remedy the core problem, and forced me to consider another reformulation of the criterion, which still covers the frequent case of convex absolute quantifiers, like the NL examples “about ten”, “between ten and twenty” etc. This property of weakly preserving convexity distills those aspects of the convexity requirements that are compatible with the more elementary base assumptions, and can hence be achieved in a fuzzy framework. I will present a positive example which demonstrates that the property can indeed be fulfilled in Chap. 7. However, the property is still demanding even in its weak form, and certainly confined to a very small fragment of the models.

Following the discussion of convexity properties, I turned to the definition and analysis of *conservative* quantifiers. Conservativity makes one of the pervasive and perhaps even universal properties of simple NL quantifiers, which catches hold of an important aspect of context insensitivity. Basically, a quantifier is considered conservative if it is the restriction of the quantifier which decides upon its interpretation, and not the chosen domain. For example, the truth of “All men are mortal” depends on the set of considered men only, and their specific properties; while the remaining elements of the base set, which fall outside the considered subdomain of ‘men’, have no effect at all on the quantification results. The original notion of conservativity familiar from TGQ is readily extended to a definition for arbitrary semi-fuzzy quantifiers, and Def. 70 assumes the trivial adaptations without mention. The most appropriate generalization of conservativity to full-fledged fuzzy quantifiers is less obvious, though. In the chapter, I pursued two different definitions which extend the original concept to the case of fuzzy arguments.

a. The proposed definition of weak conservativity merely targets at domain insensitivity, which comprises the key aspect of conservativity anyway. The quantification results of weakly conservative quantifiers should hence be independent of those elements which are totally irrelevant to the restriction of the quantifier. This conception of conservativity combines easily with the proposed framework. Based on this notion of conservative fuzzy quantifiers, it can then be shown that all models of fuzzy quantification weakly preserve conservativity, and hence map conservative semi-fuzzy quantifiers to fuzzy quantifiers which are conservative in the weak sense.

b. The alternative proposal of strong conservativity is obtained when the crisp intersection in the original definition of conservativity is replaced with the induced fuzzy intersection $\widetilde{\cap}$, and the scope of the new definition is then shifted to fuzzy quantifiers and fuzzy arguments. Apart from the context insensitivity aspect, the resulting strong form of conservativity also seizes the definition of crisp conservativity in terms of an intersection with the first argument. Concerning the compliance of this alternative notion with the models of fuzzy quantification, the chapter provided negative evidence, though. As witnessed by Th-58, strong conservativity conflicts with fundamental assumptions on plausible choices of models, and hence cannot be ensured in a fuzzy framework.

As indicated by these findings, it is the weak sense of conservativity that is adequate in the presence of fuzzy arguments, and the definition of two-valued conservativity in terms of a conjunction of restriction and scope, must be considered a contingent property which is idiosyncratic to the simple Boolean case.

Finally, I discussed the construction of fuzzy argument insertion. In order to motivate this criterion, I briefly reviewed adjectival restriction based on crisp adjectives like "married", which can be modelled by crisp argument insertions. From a linguistic point of view, it is crucial to ensure a compositional interpretation of adjectival restriction, in accordance with Frege's compositionality principle. Hence the interpretation of a composite expression should be in functional dependence on the meanings of its subexpressions. When applied to the current situation of adjectival restriction this means that the target quantifier $\widetilde{Q}'$ which results from adjectival restriction of Q by A should be representable in terms of a semi-fuzzy quantifier Q', which hence satisfies $\widetilde{Q}' = \mathcal{F}(Q')$, and Q' should show a functional dependency on the original representation Q and the interpretation of the adjective A, i.e. $Q = c(Q, A)$, where c is the construction for adjectival restriction. In the crisp case, these goals were of course accomplished by letting $c(Q, A) = Q\tau_i \cap \triangleleft A \tau_i$; this restricts the i-th argument of Q to the crisp denotation A of the given adjective. In the important case of fuzzy adjectives, the strategy suggests itself to decompose fuzzy adjectival restriction in an analogous way, and hence resolve it into an intersection, argument transpositions, and the crucial step of fuzzy argument insertion. However, fuzzy arguments cannot be inserted into the assumed base representations of semi-fuzzy quantifiers. Consequently the target construction cannot be modelled directly, and a different approach is necessary in order to implement fuzzy argument insertion and achieve the desired compositional interpretation in the general case. To this end, I observed that there are narrow constraints on the allowable choices of $Q \mathbin{\widetilde{\triangleleft}} A$, which is intended to model the quantifier which results from inserting the fuzzy argument A into the last argument slot of the semi-fuzzy quantifier Q. In fact, it is clear from theorem Th-2 that there is at most one conceivable choice of semi-fuzzy quantifier $Q' = Q \mathbin{\widetilde{\triangleleft}} A$ which possibly satisfies $\mathcal{F}(Q') = \mathcal{F}(Q) \triangleleft A$, and I was able to trace this quantifier to the explicit formula $Q' = \mathcal{U}(\mathcal{F}(Q) \triangleleft A)$. These considerations prune the allowable alternatives and leave over a unique definition for $Q \mathbin{\widetilde{\triangleleft}} A$, which must result from the construction of Q' described above. However, it is not a matter of course that this choice of $Q' = Q \mathbin{\widetilde{\triangleleft}} A$ properly represents $\mathcal{F}(Q) \triangleleft A$ in terms of an underlying semi-fuzzy quantifier. Quite the reverse, we must explicitly require this and enforce the equality $\mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} A) = \mathcal{F}(Q) \triangleleft A$, which makes the defining criterion for compatibility with fuzzy argument insertion. A QFM compliant with this construction then permits an explicit representation of intermediate quantifiers which result from the insertion of a fuzzy argument into one of the argument slots, because it offers compatible base descriptions $Q' = Q \mathbin{\widetilde{\triangleleft}} A$ on the level of semi-fuzzy quantifiers. We shall experience in the next chapter that fuzzy argument insertion poses an extremely demanding requirement. A prototypical example which fulfills the requirement will then be presented, but it will also be substantiated that this particular model already exhausts all positive choices among the standard DFSes, and hence constitutes the unique standard model which complies with fuzzy argument insertion.

7 The class of models defined in terms of three-valued cuts and fuzzy-median aggregation

7.1 Motivation and chapter overview

By stating the DFS axioms, I have made explicit the linguistic expectations on models of fuzzy quantification. In the following chapters, I will now turn attention from the discussion of generic semantic postulates to the models themselves, which instantiate the axiomatic framework. From a theoretical position, this shift to concrete models is necessary because these testify the consistency of the proposed axioms. In addition, the analysis of these examples and the identification of their internal structure, might disclose the fundamental principles that underly fuzzy quantification. Needless to say that the development of practical models and their subsequent computer implementation, is absolutely mandatory in order to make the theory useful for applications. The following chapters are devoted to the investigation of increasingly broader classes of models, each of which has its relative merits to the theory of fuzzy quantification. These classes will be defined in terms of parametrized mechanisms which build a variety of models from a generic constructive principle. For each considered classes, I will hence start by introducing the parametrized base mechanism, which spans a 'raw' class of totally unrestricted fuzzification mechanisms. By imposing conditions on the allowable choices of parameters, the unrestricted class of QFMs can then be pruned to the subclass of well-behaved models, which can be defined from the base mechanism.

In order to implement this basic strategy, we must first locate a suitable starting point from which to abstract the required constructive principle. As we shall see in a minute, the proven techniques of fuzzy logic, and in particular α-cuts which resolve a fuzzy concept into layers of crisp computations, are not suited to define models of the theory. The failure of α-cuts to define models, is apparently caused by their lack of symmetry with respect to complementation. It is this observation which pointed my attention to the richer descriptions offered by three-valued sets, because these permit the definition of a three-valued cutting mechanism which overcomes the asymmetry problem. In order to ensure a consistent processing of the resulting three-valued sets, I first develop a constructive principle for Kleene's three-valued logic, which is known to underly all standard models. The basic mechanism will then be fitted to the case of semi-fuzzy quantifiers, which can be accomplished by fuzzy median aggregation. In this way, we obtain a fuzzy quantification result $Q_\gamma(X_1, \ldots, X_n) \in \mathbf{I}$ for each choice of the cutting parameter γ. The results spread over the cut levels must then be aggregated by applying an aggregation operator $\mathcal{B}$. This completes the base construction of fuzzification mechanisms in terms of three-valued cuts and fuzzy-median aggregation, which opens the raw class of $\mathcal{M}_\mathcal{B}$-QFMs. Embarking on the general strategy described above, I will then develop a characterisation of the class of $\mathcal{M}_\mathcal{B}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping $\mathcal{B}$. A simplified construction based on aggregation mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ will now be apparent from this characterisation, which reduces some of the redundancy in the original description. Building on this improved representation, I then introduce some prototypical $\mathcal{M}_\mathcal{B}$-DFSes, which also comprise the most important exemplars. Following the presentation of these examples, I then develop the formal apparatus, which is necessary to assess the characteristic properties of the individual models. In some cases, it also comes out that a considered property is shared by all $\mathcal{M}_{\mathcal{B}}$-DFSes. In addition, the available tools for analysing $\mathcal{M}_{\mathcal{B}}$-DFSes will help to identify the extreme cases of models in terms of specificity. Finally, I will disclose that one of the proposed models combines unique semantical properties. In order to acknowledge its distinguished position, the remainder of the chapter will be dedicated to a thorough discussion of this prime example, which is foremost among all standard models of fuzzy quantification.

7.2 Alpha-cuts are not suited to define models

In the proposed framework, an instance of fuzzy quantification can be desribed in terms of the following data, (a) the given semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, and (b), a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. The problem to be solved is that of making the quantifier applicable to the fuzzy arguments, although it is (directly) defined for crisp arguments only. One idea that comes to mind is that of resorting to the familiar concept of α-cuts and strict α-cuts, which resolve the fuzzy arguments into layers of crisp arguments which are organized by the cutting parameter α. For each considered layer, the resulting crisp arguments can then be supplied to the semi-fuzzy quantifier, which determines a corresponding quantification result for each cut level. In order to formalize this basic approach, let me first recall the usual definition of α-cuts and strict α-cuts:

Definition 75
Let E be a given set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\alpha \in \mathbf{I}$. By $X_{\geq\alpha} \in \mathcal{P}(E)$ we denote the α-cut

$$X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\} .$$

Definition 76
Let $X \in \widetilde{\mathcal{P}}(E)$ be given and $\alpha \in \mathbf{I}$. By $X_{>\alpha} \in \mathcal{P}(E)$ we denote the strict α-cut

$$X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\} .$$

The effect of α-cutting relative to $\mu_X(e)$ is displayed in Fig. 10.
α-cuts and strict α-cuts are linked by negation, as witnessed by the following equalities. For all fuzzy subsets $X \in \widetilde{\mathcal{P}}(E)$ and $\alpha \in \mathbf{I}$,

$$(\neg X)_{\geq\alpha} = \neg(X_{>\neg\alpha}) \tag{36}$$

$$(\neg X)_{>\alpha} = \neg(X_{\geq\neg\alpha}) . \tag{37}$$

It is apparent from these equalities that neither α-cuts nor strict α-cuts are compatible with complementation.
We can now seize the above suggestion of evaluating the quantifier Q for the crisp arguments obtained at each cut level $\alpha \in \mathbf{I}$. The quantification results $f(\alpha) = Q((X_1)_{\geq\alpha}, \dots, (X_n)_{\geq\alpha})$ obtained at the cut levels $\alpha \in \mathbf{I}$ must then be combined into a single scalar result. This process is delegated to a subsequent aggregation step.

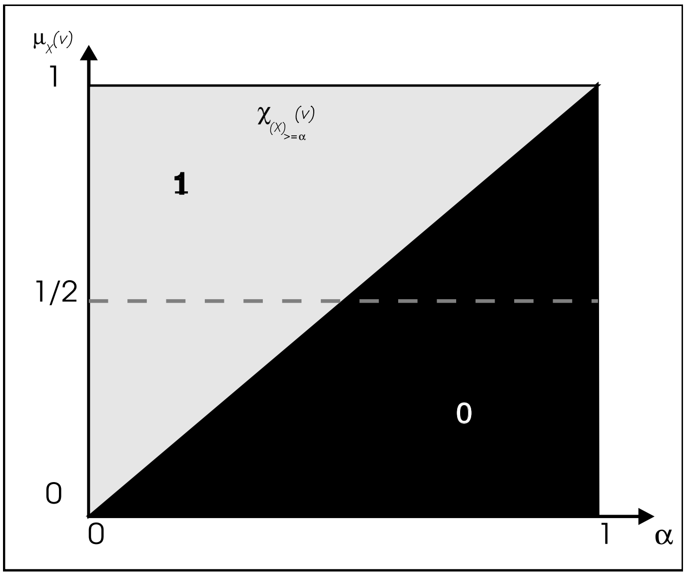

Figure 10: α-cut as a function of α and $\mu_X(v)$

Probably the first choice of aggregation operation which comes to mind is that of integration. Let us hence consider the following attempt to define a QFM based on α-cuts and integration:

$$\mathcal{A}(Q)(X_1, \ldots, X_n) = \int_0^1 Q((X_1)_{\geq\alpha}, \ldots, (X_n)_{\geq\alpha})\, d\alpha\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We should first notice that this is really only an ‘attempt’ to define a QFM, because the above integral may not exist (I have not made any assumptions on the ‘well-behavedness’ of Q, like measurability). Hence $\mathcal{A}$ is only partially defined.[38] However, even if we allow an arbitrary completion of $\mathcal{A}$ into a fully defined fuzzification mechanism, the resulting QFM $\mathcal{A}$ still fails to be a DFS. This is because $\mathcal{A}$ is subject to an additional flaw, apart from its ‘definition gaps’. It is easily observed that $\mathcal{A}$ does not comply with the construction of internal complementation, and hence violates a semantical postulate which is known from Th-11 to be mandatory for all intended models.

From a broader perspective, the failure of the above attempt to define a model by integration over α-cuts is just an instance of a more general fact, and cannot be attributed to the choice of the integral which served as the aggregation operator. By contrast, it can be shown that it is impossible in general, to define models of the DFS axioms from layers of α-cuts. This is because there exists no choice of aggregation operator which makes a DFS of the proposed base construction:

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[38] To see this, consider a non-measurable function $f : \mathbf{I} \longrightarrow \mathbf{I}$ and let $E = \mathbf{I}$. Define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by $Q(X) = f(\inf X)$ for all $X \in \mathcal{P}(\mathbf{I})$. $\mathcal{A}$ is undefined on the fuzzy subset $X^* \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by $\mu_{X^*}(e) = e$ for all $e \in \mathbf{I}$, because $Q(X^*{}_{\geq\alpha}) = f(\alpha)$.

Theorem 59 (Nonexistence of alpha-cut based DFSes)
Define $\mathcal{C}$, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I} \quad \mapsto \quad \mathcal{C}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$ *by*

$$\mu_{\mathcal{C}(Q)(X_1,\dots,X_n)}(\alpha) = Q((X_1)_{\geq\alpha},\ \dots,\ (X_n)_{\geq\alpha})$$

for all $\alpha \in \mathbf{I}$*. There exists no* $\mathcal{S} : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ *such that* $\mathcal{F} = \mathcal{S} \circ \mathcal{C}$ *satisfies the following conditions:*

- $\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}}) = \mathrm{id}_{\mathbf{I}}$*, i.e. the identity truth function is mapped to its proper fuzzy analogue;*
- $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator;*
- $\mathcal{F}$ *is compatible with internal complementation.*

In particular, there exists no $\mathcal{S} : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ *such that* $\mathcal{F} = \mathcal{S} \circ \mathcal{C}$ *is a DFS.*

Note. The second part of the theorem is again apparent from Th-11. This failure of the α-cut based approach to define plausible models of fuzzy quantification is caused by their lack of symmetry with respect to complementation, see (36) and (37). Due to this negative evidence concerning the potential utility of α-cuts to decompose fuzzy quantification into several layers of quantification involving crisp arguments, I will now consider a different but related mechanism. The well-known *resolution principle* has successfully been applied in other contexts for the transfer of crisp concepts into corresponding concepts for fuzzy sets. Adapted to the present situation of semi-fuzzy quantifiers supplied with fuzzy arguments, the resolution principle becomes:

Definition 77 (Resolution principle)
Let $f : \mathcal{P}(E) \longrightarrow V$ *be a mapping. By* $\mathcal{R}(f)$ *we denote the mapping* $\mathcal{R}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(V)$ *which to each* $X \in \widetilde{\mathcal{P}}(E)$ *associates the fuzzy set* $\mathcal{R}(f)(X)$ *defined by*

$$\mu_{\mathcal{R}(f)(X)}(v) = \sup\{\alpha \in \mathbf{I} : f(X_{\geq\alpha}) = v\}$$

for all $v \in V$*. We then say that* $\mathcal{R}(f)$ *is obtained from* f *by applying the the resolution principle.*

Note. I have stated the resolution principle for one-place functions only. Because $\mathcal{P}(E)^n \cong \mathcal{P}(E \times n)$ and $\widetilde{\mathcal{P}}(E)^n \cong \widetilde{\mathcal{P}}(E \times n)$, this generalises to n-place functions $f : \mathcal{P}(E)^n \longrightarrow V$ in the obvious way, i.e. by component-wise α-cutting.
By applying the resolution principle, then, a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is mapped to that $\mathcal{R}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$ which to each choice of $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ assigns the fuzzy subset $\mathcal{R}(Q)(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by

$$\mu_{\mathcal{R}(Q)(X_1,\dots,X_n)}(v) = \sup\{\alpha \in \mathbf{I} : Q((X_1)_{\geq\alpha},\ \dots,\ (X_n)_{\geq\alpha}) = v\}\,,$$

for all $v \in \mathbf{I}$.
Upon closer inspection, the resolution principle comes out as a special but important case of the general α-cut based approach discussed first, which has already been shown to be doomed to failure. As a corollary to the above theorem on α-cut based attempts to define DFSes, we hence obtain:

Theorem 60 (Nonexistence of models based on the resolution principle)
It is not possible to define models of the theory based on the resolution principle, i.e. no $\mathcal{S} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *exists which makes* $\mathcal{F} = \mathcal{S} \circ \mathcal{R}$ *a DFS.*

By the α*-cut representation* of a choice of fuzzy arguments $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, I denote the fuzzy set $\mathrm{RES}(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(\mathcal{P}(E))^n$ defined by

$$\mu_{\mathrm{RES}(X_1,\ldots,X_n)}((Y_1, \ldots, Y_n)) = \sup\{\alpha \in \mathbf{I} : (X_1)_{\geq\alpha} = Y_1,\ \ldots,\ (X_n)_{\geq\alpha} = Y_n\}\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Of course, DFSes can be defined based on this α-cut representation of $X_1, \ldots, X_n$ (at least if DFSes exist at all, which will be proven below). But this is a trivial statement, since $(X_1, \ldots, X_n)$ can be recovered from its α-cut representation.

7.3 From two-valued logic to Kleene's logic, and beyond

As witnessed by Th-45, the $\{0,\ \frac{1}{2},\ 1\}$-valued portion of the logic induced by a standard model coincides with Kleene's three-valued logic (see [85, p. 344]; also described e.g. in [87, p. 29]). This observation provides a suitable starting point for defining models of the theory. Kleene's logic can be constructed from two-valued logic by the following mechanism of generalizing a propositional function $f : \mathbf{2}^n \longrightarrow \mathbf{2}$ to the three-valued $\breve{f} : \{0,\ \frac{1}{2},\ 1\}^n \longrightarrow \{0,\ \frac{1}{2},\ 1\}$.

Suppose that $x_1, \ldots, x_n \in \{0,\ \frac{1}{2},\ 1\}$ are given. Associate to each x_i a set $\mathcal{T}(y_i) \in \mathcal{P}(\mathbf{2})$ as follows:

$$\mathcal{T}(y_i) = \begin{cases} \{0\} & : \quad x_i = 0 \\ \{0, 1\} & : \quad x_i = \frac{1}{2} \\ \{1\} & : \quad x_i = 1 \end{cases}$$

The set $\mathcal{T}(y_i)$ collects the alternative interpretations of y_i as two-valued truth values: 0 and 1 are non-ambiguous and correspond to unique interpretations $\{0\}$ and $\{1\}$, respectively, while the $\frac{1}{2}$ represents absence of knowledge, and is hence ambiguous between all possible choices in $\{0, 1\}$. In terms of these sets of alternatives, we then define $\breve{f} : \{0,\ \frac{1}{2},\ 1\}^n \longrightarrow \{0,\ \frac{1}{2},\ 1\}$ by

$$\breve{f}(x_1, \ldots, x_n) = \begin{cases} 1 & : \quad f(y_1, \ldots, y_n) = 1 \text{ for all } y_i \in \mathcal{T}(y_i),\ i = 1, \ldots, n \\ 0 & : \quad f(y_1, \ldots, y_n) = 0 \text{ for all } y_i \in \mathcal{T}(y_i),\ i = 1, \ldots, n \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

for all $x_1, \ldots, x_n \in \{0,\ \frac{1}{2},\ 1\}$.
In this definition, the 'indeterminate' value $\frac{1}{2}$ is treated by considering both alternatives $0, 1$. The truth-values 0 and 1 do not induce any indeterminacy. Let us now recall the definition of the generalized fuzzy median, see Def. 57. By using $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, I can now state the above definition more compactly, viz

$$\breve{f}(x_1, \ldots, x_n) = \mathrm{m}_{\frac{1}{2}}\{f(y_1, \ldots, y_n) : y_i \in \mathcal{T}(y_i), i = 1, \ldots, n\}\,. \qquad (38)$$

This median-based definition has the advantage that it can also be applied to functions $f : \mathbf{2}^n \longrightarrow \mathbf{I}$, which it maps to $\breve{f} : \{0, \frac{1}{2}, 1\}^n \longrightarrow \mathbf{I}$, and thus takes us beyond Kleene's logic.

7.4 Adaptation of the mechanism to quantifiers

In this sequel, we are more interested in quantifiers rather than propositional functions. Hence let us adapt the above mechanism to the case of three-valued quantifiers, which are then translated into quantifiers that accept three-valued arguments. To this end, I must first introduce some concepts related to three-valued subsets, which are necessary to express the definition of the target mechanism.
Three-valued subsets of a given base set E will be modelled in a way analogous to fuzzy subsets, i.e. we shall assume that each three-valued subset X of E is uniquely characterised by its membership function $\nu_X : E \longrightarrow \{0, \frac{1}{2}, 1\}$ (I use the symbol ν_X rather than μ_X in order make unambiguous that the membership function is three-valued). The collection of three-valued subsets of a given E will be denoted $\breve{\mathcal{P}}(E)$; we shall assume that $\breve{\mathcal{P}}(E)$ is a set, and clearly we have $\breve{\mathcal{P}}(E) \cong \{0, \frac{1}{2}, 1\}^E$. As in the case of fuzzy subsets, it might be convenient to identify three-valued subsets and their membership functions, i.e. to stipulate $\breve{\mathcal{P}}(E) = \{0, \frac{1}{2}, 1\}^E$. However, I will again not enforce this identification.

I will assume that each crisp subset $X \in \mathcal{P}(E)$ can be viewed as a three-valued subset of E, and that each three-valued subset X of E can be viewed as a fuzzy subset of E. Hence at times, the same symbol X will be used to denote a particular crisp subset of E, as well as the corresponding three-valued and fuzzy subsets. If one chooses to identify membership functions and three-valued/fuzzy subsets, then the crisp subset X is distinct from its representation as a three-valued or fuzzy subset, which corresponds to its characteristic function χ_X. In this case, it is understood that the appropriate transformations (i.e., using characteristic function) are performed whenever X is substituted for a three-valued or fuzzy subset.

Let us now recall the construction of Kleene's logic presented in the previous section, which was based on the representation of three-valued $x \in \{0, \frac{1}{2}, 1\}$ by corresponding ambiguity ranges $\mathcal{T}(x) \in \mathcal{P}(\mathbf{2})$. This basic construction is easily adapted to three-valued subsets. In order to express the 'ambiguity' or 'indeterminacy' which originates from occurrences of the membership grade $\frac{1}{2}$, a closed range of crisp subsets of E will then be associated with each three-valued subset $X \in \breve{\mathcal{P}}(E)$. This range, denoted $\mathcal{T}(X)$, is intended to capture the alternative interpretations of the three-valued set X in terms of compatible two-valued sets.

Definition 78
Suppose E is some set and $X \in \breve{\mathcal{P}}(E)$ is a three-valued subset of E. We associate with X crisp subsets $X^{\min}, X^{\max} \in \mathcal{P}(E)$, defined by

$$X^{\min} = \{e \in E : \nu_X(e) = 1\}$$
$$X^{\max} = \{e \in E : \nu_X(e) \geq \tfrac{1}{2}\} .$$

Based on $X^{\min}$ and $X^{\max}$, we associate with X the range of crisp sets $\mathcal{T}(X) \subseteq \mathcal{P}(E)$ defined by

$$\mathcal{T}(X) = \{Y \in \mathcal{P}(E) : X^{\min} \subseteq Y \subseteq X^{\max}\} .$$

Notes

- If $e \in E$ is such that $\nu_X(e) = 1$, then all $Y \in \mathcal{T}(X)$ contain e; if $e \in E$ is such that $\nu_X(e) = 0$, then no $Y \in \mathcal{T}(X)$ contains e, and for those $e \in E$ with $\nu_X = \frac{1}{2}$, $\mathcal{T}(X)$ contains all combinations of the alternatives $e \in Y$, $e \notin Y$.
- The pairs $(X^{\min}, X^{\max})$, $X^{\min} \subseteq X^{\max}$ form a representation of the three-valued sets X because every X can be recovered from $(X^{\min}, X^{\max})$ in the apparent way, viz

  $$\nu_X(e) = \begin{cases} 1 & : \quad e \in X^{\min} \\ 0 & : \quad e \notin X^{\max} \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

  for all $e \in E$.

Based on this concept, which resolves each three-valued subset into its ambiguity range of compatible crisp subsets, I can now modify the above mechanism for extending two-valued propositional functions to three-valued arguments, and fit its core idea to the case of three-valued quantifiers. It is then convenient to abbreviate

$$\mathcal{T}(X_1, \dots, X_n) = \mathcal{T}(X_1) \times \dots \times \mathcal{T}(X_n) .$$

I can then define

$$\begin{aligned} &\breve{Q}(X_1, \dots, X_n) \\ &= \begin{cases} 1 & : \quad Q(Y_1, \dots, Y_n) = 1 \text{ for all } (Y_1, \dots, Y_n) \in \mathcal{T}(X_1, \dots, X_n) \\ 0 & : \quad Q(Y_1, \dots, Y_n) = 0 \text{ for all } (Y_1, \dots, Y_n) \in \mathcal{T}(X_1, \dots, X_n) \\ \frac{1}{2} & : \quad \text{else} \end{cases} \end{aligned}$$

for all $X_1, \dots, X_n \in \breve{\mathcal{P}}(E)$. This construction extends the three-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ to a quantifier $\breve{Q} : \breve{\mathcal{P}}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$, which now accepts three-valued arguments. Again, we can profit from the generalized fuzzy median $\mathrm{m}_{\frac{1}{2}}$ and rephrase this according to the same idea which underlies equality (38). Stated this way, the mechanism can be applied to arbitrary semi-fuzzy quantifiers as well, which produce continuous rather than three-valued outputs. A given semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is then mapped to a quantifier $\breve{Q} : \breve{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, which generalizes its base quantifier Q to the case of three-valued arguments:

Definition 79
To all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, we associate a corresponding $\breve{Q} : \breve{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ defined by

$$\breve{Q}(X_1, \dots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}(X_1), \dots, Y_n \in \mathcal{T}(X_n)\}$$

for all $X_1, \dots, X_n \in \breve{\mathcal{P}}(E)$.

The resulting $\breve{Q}$ now accepts three-valued arguments. In the general case, though, we must deal with unrestricted fuzzy arguments which are not tied to membership grades in $\{0, \frac{1}{2}, 1\}$. In the following, I hence suggest a reduction of fuzzy quantification to the case of three-valued arguments.

7.5 Three-valued cuts of fuzzy subsets

We are now able to evaluate quantifying statements which involve three-valued arguments. In order to further generalize this construction and support arbitrary fuzzy arguments, I propose the use of a cutting mechanism which reduces the fuzzy arguments of the quantifier to corresponding three-valued arguments, and control this reduction by a cutting parameter $\gamma \in \mathbf{I}$. We know from Th-59 that α-cuts (i.e. simple two-valued cuts) are not suited to effect the reduction, due to their lack of symmetry with respect to complementation. However, it is apparent how a reduction to three-valued sets (rather than the crisp sets of α-cuts) can be performed in a symmetrical fashion, and hence achieves the desired compliance with negation. Let me first state the definition for individual membership grades in the unit interval, which are cut to membership grades in $\{0, \frac{1}{2}, 1\}$ which correspond to 'false', 'unknown' and 'true', respectively.

Definition 80
For every $x \in \mathbf{I}$ and $\gamma \in \mathbf{I}$, the three-valued cut of x at γ is defined by

$$t_\gamma(x) = \begin{cases} 1 & : \quad x \geq \frac{1}{2} + \frac{1}{2}\gamma \\ \frac{1}{2} & : \quad \frac{1}{2} - \frac{1}{2}\gamma < x < \frac{1}{2} + \frac{1}{2}\gamma \\ 0 & : \quad x \leq \frac{1}{2} - \frac{1}{2}\gamma \end{cases}$$

if $\gamma > 0$, and

$$t_0(x) = \begin{cases} 1 & : \quad x > \frac{1}{2} \\ \frac{1}{2} & : \quad x = \frac{1}{2} \\ 0 & : \quad x < \frac{1}{2} \end{cases}$$

in the case that $\gamma = 0$.

Note. The cutting parameter γ can be conceived of as a degree of 'cautiousness' because the larger γ becomes, the closer $t_\gamma(x)$ will approach the 'undecided' result of $\frac{1}{2}$. Hence $t_{\gamma'}(x) \preceq_c t_\gamma(x)$ whenever $\gamma \leq \gamma'$.
The three-valued cut mechanism for scalars can be extended to three-valued cuts of fuzzy subsets, by applying it element-wise to the membership functions:

Definition 81
Suppose E is some set and $X \in \widetilde{\mathcal{P}}(E)$ a subset of E. The three-valued cut of X at $\gamma \in \mathbf{I}$ is the three-valued subset $\mathrm{T}_\gamma(X) \in \breve{\mathcal{P}}\,(E)$ defined by

$$\nu_{\mathrm{T}_\gamma(X)}(e) = t_\gamma(\mu_X(e))\,,$$

for all $e \in E$.

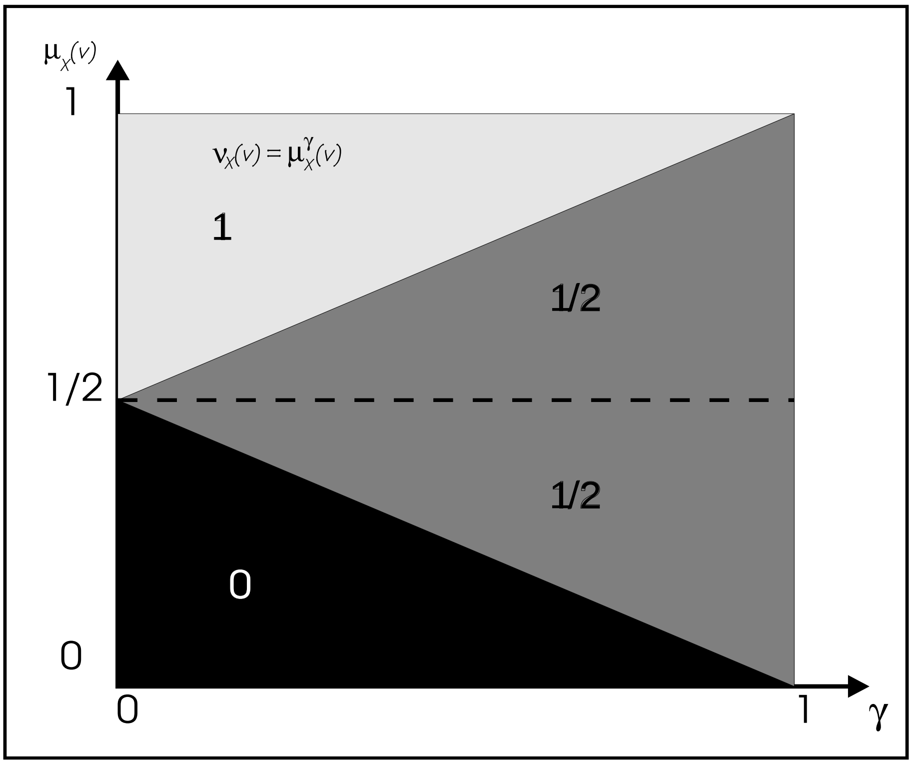

Figure 11: Three-valued cut as a function of γ and $\mu_X(v)$

Note. The effect of three-valued cutting is displayed in Fig. 11.
Usually the three-valued cut will not be applied in isolation. By contrast, it will be composed with the construction introduced in Def. 78, and hence resolved into a pair of crisp sets $((\mathrm{T}_\gamma(X))^{\min}, (\mathrm{T}_\gamma(X))^{\max})$, or alternatively into the crisp range $\mathcal{T}(\mathrm{T}_\gamma(X))$ determined by the three-valued set which results from the cut operation. Due to this practice, which generally combines the cut operation with the subsequent resolution, it is convenient to introduce the succinct notation $X_\gamma^{\min}$ and $X_\gamma^{\max}$ for the above pair of crisp sets, and also introduce an abbreviation $\mathcal{T}_\gamma(X)$ for the cut range $\mathcal{T}(\mathrm{T}_\gamma(X))$ which represents the three-valued cut at the 'cautiousness level' $\gamma \in \mathbf{I}$ by a set of crisp alternatives.

Definition 82

Let E be some set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\gamma \in \mathbf{I}$. $X_\gamma^{\min}, X_\gamma^{\max} \in \mathcal{P}(E)$

and $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ *are defined by*

$$\begin{aligned} X_\gamma^{\min} &= (\mathrm{T}_\gamma(X))^{\min} \\ X_\gamma^{\max} &= (\mathrm{T}_\gamma(X))^{\max} \\ \mathcal{T}_\gamma(X) &= \mathcal{T}(\mathrm{T}_\gamma(X)) = \{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\}\,. \end{aligned}$$

Hence if $\gamma > 0$*, then*

$$\begin{aligned} X_\gamma^{\min} &= X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} \\ X_\gamma^{\max} &= X_{> \frac{1}{2} - \frac{1}{2}\gamma}\,, \end{aligned}$$

and in the case that $\gamma = 0$*,*

$$\begin{aligned} X_0^{\min} &= X_{> \frac{1}{2}} \\ X_0^{\max} &= X_{\geq \frac{1}{2}}\,. \end{aligned}$$

Notes

- Let us recall that the three-valued set which results from the three-valued cut of X at $\gamma \in \mathbf{I}$ can be represented by the pair $(X_\gamma^{\min},\, X_\gamma^{\max})$, and further notice that both $X_\gamma^{\min}$ and $X_\gamma^{\max}$ are defined by α-cuts. Hence every three-valued cut can be represented by a pair of α-cuts, which in turn also determines the corresponding range of crisp sets, $\mathcal{T}_\gamma(X)$.

- The γ-indexed family $(X_\gamma^{\min},\, X_\gamma^{\max})_{\gamma \in \mathbf{I}}$ faithfully represents X because X can recovered from this representation by means of

  $$\mu_X(e) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sup\{\gamma \in \mathbf{I} : e \in X_\gamma^{\min}\} & : \quad e \in X_0^{\min} \\ \frac{1}{2} - \frac{1}{2}\sup\{\gamma \in \mathbf{I} : e \notin X_\gamma^{\max}\} & : \quad e \notin X_0^{\min} \end{cases}$$

  for all $e \in E$. Hence every fuzzy subset X can be resolved into a family of pairs $(X_\gamma^{\min}, X_\gamma^{\max})_{\gamma \in \mathbf{I}}$ which represent the three-valued cuts of X at all choices of $\gamma \in \mathbf{I}$.

- In turn, X is also faithfully represented by the family of cut ranges $(\mathcal{T}_\gamma(X))_{\gamma \in \mathbf{I}}$, because $X_\gamma^{\min}$ and $X_\gamma^{\max}$ can be recovered from these ranges by forming the intersection $X_\gamma^{\min} = \cap\, \mathcal{T}_\gamma(X)$ and the union $X_\gamma^{\max} = \cup\, \mathcal{T}_\gamma(X)$. Hence when resorting to the cut ranges, we also achieve the resolution property for three-valued cuts.

- Abusing language, I will usually identify the pair of crisp sets $(X_\gamma^{\min}, X_\gamma^{\max})$, and the cut range $\mathcal{T}_\gamma(X)$ with the three-valued cut $\mathrm{T}_\gamma(X)$, and refer to both as ‘the three-valued cut of X at γ’ although in a precise sense, the three-valued cut is only represented by these modelling devices.

It is also instructive to observe how the choice of the 'cautiousness parameter' γ effects the results of the cutting operation on a given fuzzy subset X. If $\gamma = 0$, the set of indeterminates (i.e. of those $e \in E$ such that $e \in X_\gamma^{\max} \setminus X_\gamma^{\min}$) contains only those $e \in E$ with $\mu_X(e) = \frac{1}{2}$; all other elements of E are mapped to the closest truth value in $\{0, 1\}$. As γ increases, the set of indeterminates is increasing. For $\gamma = 1$, then, the level of maximal cautiousness is reached where all elements of E except those with $\mu_X(e) \in \{0, 1\}$ are interpreted as indeterminates (see also Fig. 11).

Some obvious properties of $X_\gamma^{\min}$ and $X_\gamma^{\max}$ are summarized in the following theorem.

Theorem 61 *Suppose that E is a given set and $X, X' \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. Then for all $\gamma \in \mathbf{I}$,*

a. $(\neg X)_\gamma^{\min} = \neg(X_\gamma^{\max})$ *and* $(\neg X)_\gamma^{\max} = \neg(X_\gamma^{\min})$*;*

b. $(X \cap X')_\gamma^{\min} = X_\gamma^{\min} \cap X'^{\min}_\gamma$ *and* $(X \cap X')_\gamma^{\max} = X_\gamma^{\max} \cap X'^{\max}_\gamma$*;*

c. $(X \cup X')_\gamma^{\min} = X_\gamma^{\min} \cup X'^{\min}_\gamma$ *and* $(X \cup X')_\gamma^{\max} = X_\gamma^{\max} \cup X'^{\max}_\gamma$*.*

(Proof: D.2, p.442+)

This demonstrates that the three-valued cutting mechanism is compatible with all Boolean operations on the arguments of a quantifier, i.e.

$$\begin{aligned}
\mathcal{T}_\gamma((\neg X)) &= \{\neg Y : Y \in \mathcal{T}_\gamma(X)\} \\
\mathcal{T}_\gamma((X \cap X')) &= \{Y \cap Y' : Y \in \mathcal{T}_\gamma(X), Y' \in \mathcal{T}_\gamma(X')\} \\
\mathcal{T}_\gamma((X \cup X')) &= \{Y \cup Y' : Y \in \mathcal{T}_\gamma(X), Y' \in \mathcal{T}_\gamma(X')\} \,,
\end{aligned}$$

for all $X, X' \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$.

7.6 The fuzzy-median based aggregation mechanism

In this section we can put the pieces together and combine the interpretation mechanism for three-valued arguments, which takes Q to $\breve{Q}$, and the three-valued cutting operation. The three-valued cuts at a given level γ will then be applied to reduce given choices of fuzzy arguments to the simpler situation of three-valued arguments, which can already be handled by $\breve{Q}$. Hence in order to evaluate a quantifying statement based on a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ at a given level of cautiousness γ, we apply the above mechanism to Q, yielding $\breve{Q} : \breve{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, and then insert the three-valued cuts of the fuzzy argument sets at γ. In fact, both mechanisms, i.e. quantifier generalisation and three-valued cut, are exclusively used in this specific combination. It is hence justified to introduce a shorthand notation for the composition of the two mechanisms, and subsequently treat the resulting combination as an atomic construction which can be used to define models. This approach is summarized in the following definition.

Definition 83
For every $\gamma \in \mathbf{I}$, we denote by $(\bullet)_\gamma$ the QFM defined by

$$Q_\gamma(X_1, \dots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Notes

- It should be apparent how the definition of $Q_\gamma(X_1, \dots, X_n)$ evolves from the composition of $\breve{Q}$ introduced in Def. 79, and the three valued cuts defined by Def. 81.
- Considering the resulting construction in total, the basic idea that can be identified is that of treating the crisp ranges which correspond to the three-valued cuts as providing a number of alternatives to be checked. Hence in order to evaluate a semi-fuzzy quantifier Q at a certain cut level γ, we have to consider all choices of $Q(Y_1, \dots, Y_n)$, where $Y_i \in \mathcal{T}_\gamma(X_i)$. The set of alternative results obtained in this way must then be aggregated to a single result in the unit interval, which is achieved by applying the generalized fuzzy median.
- If the base set E is finite, then the number of *distinct* three-valued cuts of $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ is finite as well. In this case, $f(\gamma) = Q_\gamma(X_1, \dots, X_n)$ is a step function, i.e. piece-wise constant in γ, except for a finite number of discontinuities. It is this property which makes the concept of three-valued cuts and the definition of $(\bullet)_\gamma$ still suited for efficient implementation: results need to be computed only for the (usually small) number of distinct three-valued cuts. More details on the computational aspect can be found in Chap. 11.

The next theorem presents a result concerning the monotonicity properties of $f(\gamma) = Q_\gamma(X_1, \dots, X_n)$, when viewed as a function of $\gamma \in \mathbf{I}$.

Theorem 62
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

a. If $Q_0(X_1, \dots, X_n) > \frac{1}{2}$, then $Q_\gamma(X_1, \dots, X_n)$ is monotonically nonincreasing in γ and $Q_\gamma(X_1, \dots, X_n) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

b. If $Q_0(X_1, \dots, X_n) = \frac{1}{2}$, then $Q_\gamma(X_1, \dots, X_n) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. If $Q_0(X_1, \dots, X_n) < \frac{1}{2}$, then $Q_\gamma(X_1, \dots, X_n)$ is monotonically nondecreasing in γ and $Q_\gamma(X_1, \dots, X_n) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

Note. These monotonicity properties reflect the intuition that as soon as one becomes more cautious (i.e. the ‘level of cautiousness’ γ increases), the results obtained will become less specific, in the sense of being closer to $\frac{1}{2}$.
Owing to this fuzzy-median based aggregation mechanism, we are now able to interpret fuzzy quantifiers for any fixed choice of the cutting parameter $\gamma \in \mathbf{I}$. However,

none of the QFMs $(\bullet)_\gamma$ is a DFS yet, because the required information is spread over various cut levels, and the fuzzy median suppresses too much structure at each isolated cut level. Nonetheless, these QFMs prove useful in defining models of fuzzy quantification. We simply need to abstract from individual cut levels, and simultaneously consider the results obtained at all levels of cautiousness γ.

7.7 The unrestricted class of $\mathcal{M}_\mathcal{B}$-QFMs

In order to implement this basic idea of supplying the assumed QFM with the total amount of information, which is spread across cautiousness levels, we consider the γ-indexed family $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$. Unlike the individual QFMs $(\bullet)_\gamma$, which fail to be a DFS because they only see one choice of γ at a time, the *collection* of their results $Q_\gamma(X_1, \ldots, X_n)$, obtained for all choices of $\gamma \in \mathbf{I}$, contains the desired information across cut levels. In order to construct fuzzification mechanisms which have a chance of being a DFS, it is hence suggested to apply an aggregation operator to these γ-indexed results. This can be done e.g. by means of integration:

Definition 84
By $\mathcal{M}$ we denote the QFM defined by

$$\mathcal{M}(Q)(X_1, \ldots, X_n) = \int_0^1 Q_\gamma(X_1, \ldots, X_n)\, d\gamma$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Note. The integral is known to exist because the integrands $f(\gamma) = Q_\gamma(X_1, \ldots, X_n)$ are bounded and monotonic by Th-62.
Let me now state that the fuzzification mechanism so defined indeed qualifies as a model of fuzzy quantification.

Theorem 63
$\mathcal{M}$ is a standard DFS.

Notes

- $\mathcal{M}$ is the first DFS that I discovered, and subsequently implemented in order to support quantifying queries in an experimental retrieval system for multimedia weather documents [53].
- As we shall see below, $\mathcal{M}$ is also a ‘practical’ model because it satisfies both continuity requirements that have been defined for QFMs. Consequently, the quantification results obtained from $\mathcal{M}$ are robust against variation or noise in the fuzzy arguments $X_1, \ldots, X_n$ and in the chosen quantifier Q.
- For the implementation of quantifiers in $\mathcal{M}$ see Chap. 11.

The integral, which was used in the definition of $\mathcal{M}$, is not the only possible way of abstracting from the cutting parameter γ. In the following, I will hence generalize this example and consider a broader range of aggregation operators. I will use the symbol $\mathcal{B}$ for such aggregation operators, and the resulting QFM will be denoted $\mathcal{M}_{\mathcal{B}}$. Before introducing the desired construction of QFMs, let us first identify the precise domain on which these aggregation operators can act.

Definition 85

$\mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^-$ *and* $\mathbb{B} \subseteq \mathbf{I}^{\mathbf{I}}$ *are defined by*

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) > \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [\tfrac{1}{2}, 1] \text{ and } f \text{ nonincreasing } \}\\
\mathbb{B}^{\frac{1}{2}} &= \{f \in \mathbf{I}^{\mathbf{I}} : f(x) = \tfrac{1}{2} \text{ for all } x \in \mathbf{I}\ \}\\
\mathbb{B}^- &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) < \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [0, \tfrac{1}{2}] \text{ and } f \text{ nondecreasing } \}\\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^- \,.
\end{aligned}$$

Note. In the following, we shall denote by $\mathrm{c}_a : \mathbf{I} \longrightarrow \mathbf{I}$ the constant mapping

$$\mathrm{c}_a(x) = a \,, \tag{39}$$

for all $a, x \in \mathbf{I}$. Using this notation, apparently $\mathbb{B}^{\frac{1}{2}} = \{\mathrm{c}_{\frac{1}{2}}\}$.

In terms of these abbreviations, we can now express the following theorem, which is actually a corollary of Th-62:

Theorem 64

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. If $Q_0(X_1, \dots, X_n) > \frac{1}{2}$, *then* $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}^+$;

b. If $Q_0(X_1, \dots, X_n) = \frac{1}{2}$, *then* $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}^{\frac{1}{2}}$ *(i.e. constantly* $\frac{1}{2}$*);*

c. If $Q_0(X_1, \dots, X_n) < \frac{1}{2}$, *then* $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}^-$.

In particular, the theorem substantiates that regardless of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, it always holds that

$$(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B} \,.$$

Hence $\mathbb{B}$ is large enough to embed all mappings $f(\gamma) = Q_\gamma(X_1, \dots, X_n)$ which can arise from a possible quantification instance, as given by Q and a choice of $X_1, \dots, X_n$. In other words, it contains the full range of operands that must possibly be accepted by the assumed aggregation operator $\mathcal{B}$. Conversely, the set $\mathbb{B}$ is also exhausted by the possible range of $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$. Hence for each $f \in \mathbb{B}$ there exist choices of Q and $X_1, \dots, X_n$ such that $f = (Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$, as I will now state:

Theorem 65
Suppose f is some mapping $f \in \mathbb{B}$. Define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = f(\inf Y) \tag{Th-65.a.i}$$

for all $Y \in \mathcal{P}(\mathbf{I})$ and let $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ be the fuzzy subset with membership function

$$\mu_X(z) = \tfrac{1}{2} + \tfrac{1}{2}z \tag{Th-65.a.ii}$$

for all $z \in \mathbf{I}$. Then

$$Q_\gamma(X) = f(\gamma)$$

for all $\gamma \in \mathbf{I}$.

Note. Combining this with the above observations, we now have evidence that $\mathbb{B}$ is the minimal set of mappings $f : \mathbf{I} \longrightarrow \mathbf{I}$ which embeds all possible operands of the assumed aggregation mapping. Hence $\mathbb{B}$ precisely characterises the domain of suitable aggregation operators.

Let us now return to the original idea of abstracting from the above definition of $\mathcal{M}$, and formulating a generic construction of QFMs, by aggregating the results of $Q_\gamma(X_1, \ldots, X_n)$ obtained for all choices of the cautiousness parameter. We know from the last theorem that a corresponding aggregation operator must be defined on $\mathbb{B}$, because $(Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}} \in \mathbb{B}$, and no smaller set $A \subseteq \mathbf{I}^{\mathbf{I}}$ will suffice.
Hence consider an aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$. By composing $\mathcal{B}$ with the γ-indexed cut mechanism, we can now construct a QFM, denoted $\mathcal{M}_\mathcal{B}$, which corresponds to the operator $\mathcal{B}$. This basic suggestion can be formalized as follows.

Definition 86
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. The QFM $\mathcal{M}_\mathcal{B}$ is defined by

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}})\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{M}_\mathcal{B}$-QFMs I mean the class of all QFMs $\mathcal{M}_\mathcal{B}$ defined in this way. Because no conditions whatsoever were imposed on the aggregation mapping $\mathcal{B}$, it is apparent that the resulting fuzzification mechanisms constitute a ‘raw’, totally unrestricted class of $\mathcal{M}_\mathcal{B}$-QFMs, which apart from the well-behaved models, also contains many QFMs which violate the axiomatic requirements. In order to shrink down this ‘raw’ class to the reasonable cases of $\mathcal{M}_\mathcal{B}$-DFSes and hence identify its plausible models, I will now address the problem of formalizing the required conditions on admissible choices of the aggregation mapping.

7.8 Characterisation of $\mathcal{M}_\mathcal{B}$-DFSes

In this section, I will identify the class of well-behaved models among the ‘raw’ $\mathcal{M}_\mathcal{B}$-QFMs, based on properties that are visible in the aggregation mapping. To this end, I first introduce some constructions on $\mathbb{B}$. These will permit us to capture the relevant behaviour of the aggregation mapping in a system of conditions imposed on $\mathcal{B}$. In turn, the proposed conditions are then shown to be necessary and sufficient for $\mathcal{M}_{\mathcal{B}}$ to be a DFS, and thus precisely characterise the admissible choices of $\mathcal{B}$. To begin with, let me now define the required constructions on the domain of the aggregation operator.

Definition 87

Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e., either nondecreasing or nonincreasing). The mappings* $f^\flat$, $f^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$ *are defined by:*

$$f^\sharp = \begin{cases} \lim\limits_{y \searrow x} f(y) & : \quad x < 1 \\ f(1) & : \quad x = 1 \end{cases}$$

$$f^\flat = \begin{cases} \lim\limits_{y \nearrow x} f(y) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases}$$

for all $x \in \mathbf{I}$.

Notes

- Let me remark that $f^\sharp$ and $f^\flat$ are well-defined, i.e. the limits in the above expressions exist, regardless of f. This is because f is known to be bounded and monotonic.
- It is further apparent that if $f \in \mathbb{B}$, then $f^\sharp \in \mathbb{B}$ and $f^\flat \in \mathbb{B}$.

I will now introduce several coefficients which describe certain aspects of a mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Definition 88 *Suppose that* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e., either nondecreasing or nonincreasing). We define*

$$f_0^* = \lim_{\gamma \searrow 0} f(\gamma) \tag{40}$$

$$f_*^0 = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{41}$$

$$f_*^{\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \tag{42}$$

$$f_1^* = \lim_{\gamma \nearrow 1} f(\gamma) \tag{43}$$

$$f_*^{1\uparrow} = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\}\,. \tag{44}$$

As usual, let us stipulate that $\sup \varnothing = 0$ *and* $\inf \varnothing = 1$.

Notes

- All limits in the definition of these coefficients exist due to the assumed monotonicity of $f : \mathbf{I} \longrightarrow \mathbf{I}$.

- The conditions imposed on legal choices of $\mathcal{B}$ will only utilize one of these coefficients, $f_*^{\frac{1}{2}}$. The remaining coefficients will permit the subsequent definition of example models.

Based on these concepts, I can now state several axioms governing the behaviour of reasonable choices of $\mathcal{B}$.

Definition 89 *Suppose that an aggregation mapping $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathcal{B}$, we define the following conditions on $\mathcal{B}$:*

$$\mathcal{B}(f) = f(0) \quad \textit{if } f \textit{ is constant, i.e. } f(x) = f(0) \textit{ for all } x \in \mathbf{I} \tag{B-1}$$

$$\mathcal{B}(1 - f) = 1 - \mathcal{B}(f) \tag{B-2}$$

$$\textit{If } \widehat{f}(\mathbf{I}) \subseteq \{0, \tfrac{1}{2}, 1\}\textit{, then} \tag{B-3}$$

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases}$$

$$\mathcal{B}(f^\sharp) = \mathcal{B}(f^\flat) \tag{B-4}$$

$$\textit{If } f \leq g\textit{, then } \mathcal{B}(f) \leq \mathcal{B}(g) \tag{B-5}$$

Note. Let me briefly comment on the meaning of the individual conditions. (B-1) states that $\mathcal{B}$ preserves constants. In particular, all $\mathcal{M}_\mathcal{B}$-QFMs such that $\mathcal{B}$ satisfies (B-1) coincide on three-valued argument sets. (B-2) expresses that $\mathcal{B}$ is compatible with the standard negation $1 - x$. (B-3) ensures that all conforming $\mathcal{M}_\mathcal{B}$-QFMs coincide on three-valued quantifiers. (B-4) expresses some kind of insensitivity property of $\mathcal{B}$, which turns out to be crucial with respect to $\mathcal{M}_\mathcal{B}$ satisfying *functional application* (Z-6). Finally, (B-5) expresses that $\mathcal{B}$ is monotonic, i.e. application of $\mathcal{B}$ preserves inequalities.
Taken as a whole, the proposed system of conditions on $\mathcal{B}$ precisely captures the requirements on reasonable choices of $\mathcal{B}$. To see this, we first notice that every $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies (B-1) to (B-5) makes $\mathcal{M}_\mathcal{B}$ a standard DFS:

Theorem 66
The conditions (B-1)–(B-5) *on $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*

Next I consider the converse issue that every choice of $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ which makes $\mathcal{M}_\mathcal{B}$ a DFS, in fact satisfies the proposed set of conditions. Again, I have a positive result here.

Theorem 67
The conditions (B-1)–(B-5) *on $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

Combining this with Th-66, it is easily seen that

Theorem 68
Every $\mathcal{M}_{\mathcal{B}}$-DFS is a standard DFS.

Let us further notice that

Theorem 69
The conditions (B-1)–(B-5) *are independent.*

Hence none of these conditions can be expressed in terms of the remaining conditions. In other words, I have succeeded in formalizing a minimal set of conditions, and the conditions indeed *separate* the distinct factors that guide reasonable choices of $\mathcal{B}$. In addition, the mutual independence of the conditions avoids redundant effort in proofs.

7.9 A simplified construction based on mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$

In the following, a simplified representation of $\mathcal{M}_{\mathcal{B}}$-DFSes will be developed, which facilitates the presentation and investigation of specific models. To this end, let us first observe that $\mathcal{B}$ is completely determined by its behaviour on $\mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$, provided that it satisfies (B-2). This is apparent because for every $f \in \mathbb{B}^-$, we obtain that $1 - f \in \mathbb{B}^+$ and hence $\mathcal{B}(f) = 1 - \mathcal{B}(1 - f)$. Noticing that $\mathrm{c}_{\frac{1}{2}} = 1 - \mathrm{c}_{\frac{1}{2}}$, we can further conclude that in every $\mathcal{B}$ which satisfies (B-2), it holds that

$$\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = 1 - \mathcal{B}(1 - \mathrm{c}_{\frac{1}{2}}) = 1 - \mathcal{B}(\mathrm{c}_{\frac{1}{2}})\,,$$

i.e. $\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$. We can hence strengthen the above remark, and state that $\mathcal{B}$ is even fully determined by its behaviour on $\mathbb{B}^+$ only. If we further assume that $\mathcal{B}$ satisfies (B-5), then we also know that $\mathcal{B}(f) \geq \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$ for all $f \in \mathbb{B}^+$. This means that we can restrict the range of possible $\mathcal{B}(f)$ to the upper half of the unit interval, $[\frac{1}{2}, 1]$. In order to develop the simplified constructions, let us now recall from Th-67 that the considered conditions (B-2) and (B-5) are necessary for $\mathcal{M}_{\mathcal{B}}$ to be a DFS. Hence no models of interest are lost if we focus on those $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfy (B-2) and result in $\mathcal{B}(f) \geq \frac{1}{2}$ for all $f \in \mathbb{B}^+$. In this case, I can give a more concise description of the models:

Definition 90
By $\mathbb{H} \subseteq \mathbf{I}^{\mathbf{I}}$ we denote the set of nonincreasing mappings $f : \mathbf{I} \longrightarrow \mathbf{I}$, $f \neq \mathrm{c}_0$, i.e.

$$\mathbb{H} = \{f \in \mathbf{I}^{\mathbf{I}} : \ f \textit{ nonincreasing and } f(0) > 0\,\}\,.$$

We can then associate with each $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ a $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ as follows:

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f - 1) & : \ f \in \mathbb{B}^+ \\ \frac{1}{2} & : \ f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1 - 2f) & : \ f \in \mathbb{B}^- \end{cases} \tag{45}$$

It is apparent that each $\mathcal{B}$ constructed in this way again satisfies (B-2) and results in $\mathcal{B}(f) \geq \frac{1}{2}$ for all $f \in \mathbb{B}^+$. Conversely, all suitable choices of $\mathcal{B}$ can be represented in terms of this new construction, as I will now state.

Theorem 70
Consider $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ and suppose that $\mathcal{B}$ satisfies (B-2) *and results in $\mathcal{B}(f) \geq \frac{1}{2}$ for all $f \in \mathbb{B}^+$. Then $\mathcal{B}$ can be defined in terms of a mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to equality* (45). *$\mathcal{B}'$ is defined by*

$$\mathcal{B}'(f) = 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1\,. \tag{46}$$

By the above reasoning, these conditions are satisfied by every $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ which fulfills (B-2) and (B-5); in particular, every choice of $\mathcal{B}$ which makes $\mathcal{M}_\mathcal{B}$ a DFS can be represented in this simplified format. We can hence restrict attention on properties of mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$, and still cover all of the desired models.

7.10 Characterisation of the models in terms of conditions on $\mathcal{B}'$

Let us now characterise the $\mathcal{M}_\mathcal{B}$-models in terms of conditions on the core aggregation mapping $\mathcal{B}'$, which are hence fitted to the simplified construction.

Definition 91 *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathbb{H}$, we define the following conditions on $\mathcal{B}'$:*

$$\mathcal{B}'(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{C-1}$$

$$\text{If } f(\mathbf{I}) \subseteq \{0,1\}, \text{ then } \mathcal{B}'(f) = f_*^0, \tag{C-2}$$

$$\mathcal{B}'(f^\sharp) = \mathcal{B}'(f^\flat) \quad \text{if } \widehat{f}((0,1]) \neq \{0\} \tag{C-3}$$

$$\text{If } f \leq g, \text{ then } \mathcal{B}'(f) \leq \mathcal{B}'(g) \tag{C-4}$$

These conditions on $\mathcal{B}'$ are straightforward from the known conditions (B-1)–(B-5) that must be imposed on the corresponding $\mathcal{B}$. The following results on the (C-1)–(C-4) are therefore apparent from the previous results on the ‘B-conditions’:

Theorem 71
The conditions (C-1)–(C-4) *on $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*

Theorem 72
The conditions (C-1)–(C-4) *on $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

Theorem 73
The conditions (C-1)–(C-4) *are independent.*

Note. In the above theorems on the ‘C-conditions’, it is understood that $\mathcal{M}_\mathcal{B}$ is constructed from $\mathcal{B}'$ according to (45) and Def. 86.

To sum up, the simplified construction of the relevant $\mathcal{M}_{\mathcal{B}}$-QFMs in terms of a mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ also lends itself to a succinct characterisation of the target models. Based on this characterisation, it has now become easy to verify that a 'DFS candidate' $\mathcal{M}_{\mathcal{B}}$ indeed qualifies as one of the intended models, by performing a few simple checks on the aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ or $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$. In fact, knowing the precise contraints that govern admissible choices of $\mathcal{B}'$, also guided my search into examples of models.

7.11 Examples of $\mathcal{M}_{\mathcal{B}}$-models

In the following, I will present some examples of $\mathcal{M}_{\mathcal{B}}$-QFMs. For convenience, the new construction based on $\mathcal{B}'$ will be utilized, in order to simplify the presentation of these models and achieve succinct definitions. Let us first reconsider the model $\mathcal{M}$, which motivated the generalization to the class of $\mathcal{M}_{\mathcal{B}}$-QFMs, and fit its definition into the new format. In terms of the improved notation which is now available, the original definition of $\mathcal{M}$ can be rephrased as follows.

Theorem 74
$\mathcal{M}$ is the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'_{\int}(f) = \int_0^1 f(x)\,dx\,,$$

for all $f \in \mathbb{H}$.

Now turning to new examples, I first introduce the following model $\mathcal{M}_U$.

Definition 92
By $\mathcal{M}_U$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'{}_U(f) = \max(f_*^{1\uparrow}, f_1^*)$$

for all $f \in \mathbb{H}$, where the coefficients f_1^* and $f_*^{1\uparrow}$ are defined by equalities (43) *and* (44)*, resp.*

As I now state, $\mathcal{M}_U$ is indeed a plausible model. In fact, this can be asserted for a broader type of $\mathcal{M}_{\mathcal{B}}$-QFMs, of which $\mathcal{M}_U$ is only a special example:

Theorem 75
Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^{1\uparrow} \oplus f_1^*\,, \tag{Th-75.a}$$

for all $f \in \mathbb{H}$, where the coefficients $f_*^{1\uparrow}$ and f_1^* are defined by (44) *and* (43)*, respectively. Further suppose that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'$ according to equality* (45)*, and that $\mathcal{M}_{\mathcal{B}}$ is the QFM defined in terms of $\mathcal{B}$ according to Def. 86. The QFM $\mathcal{M}_{\mathcal{B}}$ is a standard DFS.*

In particular, $\mathcal{M}_U$ is a standard DFS. We will see later on that $\mathcal{M}_U$ plays a special role among $\mathcal{M}_\mathcal{B}$-type, by establishing a lower bound on these models with respect to the specificity order.
In [46, Def. 44, p. 63], another example $\mathcal{M}^*$ was presented, which is defined as follows.

Definition 93
By $\mathcal{M}^*$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by

$$\mathcal{B}^{*\prime}(f) = f_*^0 \cdot f_0^* \,,$$

for all $f \in \mathbb{H}$, where the coefficients f_*^0 and f_0^* are defined by (41) *and* (40)*, resp.*

The following model, based on a similar construction, will be of some interest to the later analysis of the class of $\mathcal{M}_\mathcal{B}$-DFSes.

Definition 94
By $\mathcal{M}_S$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by

$$\mathcal{B}'{}_S(f) = \min(f_*^0, f_0^*)$$

for all $f \in \mathbb{H}$, using the same abbreviations as for $\mathcal{M}^*$.

Abstracting from the underlying construction, we may assert the following.

Theorem 76
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the QFM $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (45) *and Def. 86. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_S$ is a standard DFS. Like $\mathcal{M}_U$, the model $\mathcal{M}_S$ is also of special relevance to the $\mathcal{M}_\mathcal{B}$-type. We shall see later that it plays the opposite role, and constitutes an upper bound on the models under the specificity order.
In [46, Def. 45, p. 64], a further type of model $\mathcal{M}_*$ was presented.

Definition 95
By $\mathcal{M}_*$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by

$$\mathcal{B}'_*(f) = \sup\{x \cdot f(x) : x \in \mathbf{I}\} \,,$$

for all $f \in \mathbb{H}$.

By slightly altering this construction, I discovered a model which is now known to be of key relevance to DFS theory.

Definition 96
By $\mathcal{M}_{CX}$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'_{CX}(f) = \sup\{\min(x, f(x)) : x \in \mathbf{I}\}$$

for all $f \in \mathbb{H}$.

By abstracting from the constructive principle underlying these examples, we obtain the following more general result:

Theorem 77
Suppose $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a continuous t-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} \qquad \text{(Th-77.a)}$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'$ according to equality (45)*. Then the QFM $\mathcal{M}_{\mathcal{B}}$, defined by Def. 86, is a standard DFS.*

Note. In particular, $\mathcal{M}_{CX}$ and $\mathcal{M}_*$ are standard models.
As remarked above, $\mathcal{M}_{CX}$ is a DFS with unique properties, and a separate section has therefore been devoted to its discussion. Before focusing on this particular model in sect. 7.13, however, it is first necessary to develop the criteria on $\mathcal{B}$ or $\mathcal{B}'$ which permit us to check whether a given DFS $\mathcal{M}_{\mathcal{B}}$ satisfies an adequacy property of interest. For example, it would be helpful if we could perform specificity comparisons solely based on the known aggregation mappings $\mathcal{B}'$; if we could test $\mathcal{M}_{\mathcal{B}}$ for Q-continuity and arg-continuity by looking at $\mathcal{B}'$ only, etc. In turn, these conditions will then permit a deeper investigation of $\mathcal{M}_{CX}$ and other prominent models of the theory.

7.12 Properties of $\mathcal{M}_{\mathcal{B}}$-models

Let us now take a closer look at $\mathcal{M}_{\mathcal{B}}$-DFSes. The present goal is to expose the internal structure of these models and relate it to the properties of interest, like the continuity requirements or propagation of fuzziness. I am also interested in identifying properties shared by all $\mathcal{M}_{\mathcal{B}}$-DFSes, which hence unveil some characteristic aspects of this class of models. In addition, it is useful to locate models with special properties, and in particular the boundary cases with respect to the specificity order. Knowing these extreme cases delimits the space of possible models, which is opened by the proposed base construction. These special examples hence provide some insight into the behaviour that must be expected by other models of interest, which are known to be in-between the extreme poles.
To begin with, I state a theorem which simplifies the comparison of $\mathcal{M}_{\mathcal{B}}$-DFSes with respect to specificity:

Theorem 78
Suppose $\mathcal{B}'_1, \mathcal{B}'_2 : \mathbb{H} \longrightarrow \mathbf{I}$ are given. Further suppose that $\mathcal{B}_1, \mathcal{B}_2 \in$ BB are the mappings associated with $\mathcal{B}'_1$ and $\mathcal{B}'_2$, respectively, according to equality (45)*, and $\mathcal{M}_{\mathcal{B}_1}$, $\mathcal{M}_{\mathcal{B}_2}$ are the corresponding QFMs defined by Def. 86. Then $\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2}$ if and only if $\mathcal{B}'_1 \leq \mathcal{B}'_2$.*

Based on the above theorem and the examples of models introduced so far, it is now easily shown that $\preceq_c$ is a genuine partial order on $\mathcal{M}_{\mathcal{B}}$-DFSes.

Theorem 79
$\preceq_c$ is not a total order on $\mathcal{M}_{\mathcal{B}}$-DFSes.

Note. In particular, the standard DFSes are only partially ordered by $\preceq_c$.
In practice, this means that we cannot simply compile a list of the example models (or other models) ordered by specificity, because some of these might fail to be comparable under $\preceq_c$. However, it is still possible to investigate extreme cases of $\mathcal{M}_{\mathcal{B}}$-DFSes with respect to the specificity order. For example, it is not hard to prove that the DFS $\mathcal{M}_U$ defined by Def. 92 represents a boundary case of $\mathcal{M}_{\mathcal{B}}$-DFSes in terms of specificity:

Theorem 80
$\mathcal{M}_U$ is the least specific $\mathcal{M}_{\mathcal{B}}$-DFS.

Let us now consider the question of the existence of most specific $\mathcal{M}_{\mathcal{B}}$-DFSes. We first observe that

Theorem 81
All $\mathcal{M}_{\mathcal{B}}$-DFSes are specificity consistent.

It is then immediate from Th-42 that there exists a least upper specificity bound $\mathcal{F}_{\mathrm{lub}}$ on every nonempty collection of $\mathcal{M}_{\mathcal{B}}$-models. As I will now show, $\mathcal{F}_{\mathrm{lub}}$ is in fact an $\mathcal{M}_{\mathcal{B}}$-DFS.

Theorem 82
Let $\mathbb{F}$ be a given nonempty collection of $\mathcal{M}_{\mathcal{B}}$-DFSes. The least upper specificity bound $\mathcal{F}_{\mathrm{lub}}$ of $\mathbb{F}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS.

Hence arbitrary collections of $\mathcal{M}_{\mathcal{B}}$-QFMs have a least upper specificity bound, which is again an $\mathcal{M}_{\mathcal{B}}$-DFS. If we start from the collection of *all* $\mathcal{M}_{\mathcal{B}}$-DFSes, then we obtain the model $\mathcal{M}_S$ defined in Def. 94, which represents the other extreme case of $\mathcal{M}_{\mathcal{B}}$-DFS in terms of specificity:

Theorem 83
$\mathcal{M}_S$ is the most specific $\mathcal{M}_{\mathcal{B}}$-DFS.

Next I will discuss the two continuity requirements that were identified in sect. 6.2, and develop the required criteria for deciding upon the Q-continuity and arg-continuity of a given $\mathcal{M}_{\mathcal{B}}$-DFS. Let us first make some general observations how the continuity conditions are related to $(\bullet)_\gamma$.

Theorem 84
Let $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given. Then $d(Q_\gamma, Q'_\gamma) \leq d(Q, Q')$ for all $\gamma \in \mathbf{I}$.

We can use this inequality to formulate a condition on $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ which is necessary and sufficient for $\mathcal{M}_{\mathcal{B}}$ to be Q-continuous. To this end, we first define a metric $d : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbf{I}$ by

$$d(f, g) = \sup\{|f(\gamma) - g(\gamma)| : \gamma \in \mathbf{I}\} \tag{47}$$

for all $f, g \in \mathbb{H}$.

Theorem 85
Suppose $\mathcal{M}_{\mathcal{B}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS and $\mathcal{B}'$ is the corresponding mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$. $\mathcal{M}_{\mathcal{B}}$ is Q-continuous if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < \delta$.

In the case of continuity in arguments, we need a different distance measure on $\mathbb{H}$. Hence let us define $d' : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbf{I}$ by

$$d'(f, g) = \sup\{\inf\{\gamma' : \gamma' \in \mathbf{I}, \max(f(\gamma'), g(\gamma')) \leq \min(f(\gamma), g(\gamma))\} - \gamma : \gamma \in \mathbf{I}\}\,, \tag{48}$$

for all $f, g \in \mathbb{H}$. Let us notice that d' is only a 'pseudo-metric'. It is symmetric and satisfies the triangle inequality, but fails to be a metric in the strict sense because $d'(f, g) = 0$ does not entail that $f = g$. However, those f, g with $d'(f, g) = 0$ are treated alike by reasonable choices of $\mathcal{B}'$. It is this fact which justifies the use of d' in lieu of a metric, which permits a successful reduction of arg-continuity to a criterion decidable from the aggregation mapping. The next theorem established this desired criterion, and hence shows how the arg-continuous $\mathcal{M}_{\mathcal{B}}$ models can be characterised in terms of the underlying mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$.

Theorem 86
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies (C-2), (C-3) *and* (C-4). *Further suppose that $\mathcal{M}_{\mathcal{B}}$ is defined in terms of $\mathcal{B}'$ according to* (45) *and Def. 86. Then the following conditions are equivalent:*

a. $\mathcal{M}_{\mathcal{B}}$ is arg-continuous.

b. for all $f \in \mathbb{H}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $d'(f, g) < \delta$.

Sometimes the following sufficient condition is simpler to check.

Theorem 87
Let $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (C-2), (C-3) *and* (C-4). *If for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon$ whenever $f \leq g$ and $d'(f, g) < \delta$, then $\mathcal{M}_{\mathcal{B}}$ is arg-continuous.*

Both theorems have proven useful for deciding the continuity issue in the case of the example models. In the following, I present the results of this investigation, which establish or reject the properties of Q-continuity and arg-continuity for the examples of $\mathcal{M}_{\mathcal{B}}$-models.

Theorem 88
Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ *is an* s*-norm and* $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by* (Th-75.a)*. Further suppose that* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is defined in terms of* $\mathcal{B}'$ *according to equality* (45)*, and that* $\mathcal{M}_\mathcal{B}$ *is the QFM defined in terms of* $\mathcal{B}$ *according to Def. 86. The QFM* $\mathcal{M}_\mathcal{B}$ *is neither Q-continuous nor arg-continuous.*

In particular, $\mathcal{M}_U$ violates both continuity conditions.

Theorem 89
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by* $\mathcal{B}'(f) = f_*^0 \odot f_0^*$ *for all* $f \in \mathbb{H}$*, where* $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ *is a* t*-norm. Further suppose that the QFM* $\mathcal{M}_\mathcal{B}$ *is defined in terms of* $\mathcal{B}'$ *according to* (45) *and Def. 86. Then* $\mathcal{M}_\mathcal{B}$ *is neither Q-continuous nor arg-continuous.*

In particular, $\mathcal{M}_S$ and $\mathcal{M}^*$ fail at both continuity conditions. These results illustrate that $\mathcal{M}_U$ and $\mathcal{M}_S$ are only of theoretical interest, because they represent extreme cases in terms of specificity. Due to their discontinuity, these models are not suited for applications. The DFS $\mathcal{M}^*$ is also impractical.

Having considered these boundary cases, I will now discuss practical models. In order to establish that $\mathcal{M}$ is arg-continuous, it is useful to know how the distance measures d and d' are related. To this end, I introduce the mapping $(\bullet)^\Diamond : \mathbb{H} \longrightarrow \mathbb{H}$ defined by

$$f^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : f(\gamma) < v\} \tag{49}$$

for all $f \in \mathbb{H}$ and $v \in \mathbf{I}$. It is easily checked that indeed $f^\Diamond \in \mathbb{H}$ whenever $f \in \mathbb{H}$. Let us now utilize this concept to unveil the relationship between d and d'.

Theorem 90
For all $f, g \in \mathbb{H}$*,* $d'(f, g) = d(f^\Diamond, g^\Diamond)$*.*

Building on the above theorem, it is then easy to show that the model $\mathcal{M}$ satisfies both continuity conditions:

Theorem 91
$\mathcal{M}$ *is both Q-continuous and arg-continuous.*

This proves that $\mathcal{M}$ is indeed practical, and hence a model worth considering for use in applications that need fuzzy quantifiers.

As concerns the $\mathcal{M}_{CX}$-type, I have the following positive result:

Theorem 92
Let $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ *be a uniformly continuous* t*-norm, i.e. for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $|x_1 \odot y_1 - x_2 \odot y_2| < \varepsilon$ *whenever* $x_1, x_2, y_1, y_2 \in \mathbf{I}$ *satisfy* $\|(x_1, y_1) - (x_2, y_2)\| < \delta$*. Further suppose that* $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by equality* (Th-77.a)*, and define the QFM* $\mathcal{M}_\mathcal{B}$ *in terms of* $\mathcal{B}'$ *according to* (45) *and Def. 86 as usual. Then* $\mathcal{M}_\mathcal{B}$ *is both Q-continuous and arg-continuous.*

In particular, the model $\mathcal{M}_{CX}$ which exhibits the best theoretical properties (see section 7.13 below), is indeed a good choice for applications, because it satisfies both continuity conditions. The theorem also shows that $\mathcal{M}_*$, presented on page 209, is a practical DFS.

This completes the discussion of the continuity conditions. Next we consider the requirement of propagating fuzziness in quantifiers and arguments, see Def. 65. It can be shown that all $\mathcal{M}_\mathcal{B}$-models conform to both facets of propagating fuzziness.

Theorem 93
Every $\mathcal{M}_\mathcal{B}$-DFS propagates fuzziness in quantifiers.

Theorem 94
Every $\mathcal{M}_\mathcal{B}$-DFS propagates fuzziness in arguments.

The above theorems indicate that the $\mathcal{M}_\mathcal{B}$-DFSes constitute a subclass of the standard models which is especially well-behaved. We shall see in later chapters that other types of models can violate the requirement of propagating fuzziness. The two aspects of propagating fuzziness then become independent attributes that may or may not be fulfilled by a model of interest.

Finally let us turn attention to the behaviour of $\mathcal{M}_\mathcal{B}$-DFSes for three-valued quantifiers and for three-valued arguments. As it turns out, there is no room for variation in these cases, and the results of all $\mathcal{M}_\mathcal{B}$-models are completely determined from the underlying construction based on three-valued cuts and the fuzzy median.

Theorem 95
All $\mathcal{M}_\mathcal{B}$-DFSes coincide on three-valued arguments, i.e. in the case that all arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ involved in the quantification have membership grades $\mu_{X_i}(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$.

Note. This is different from general standard DFSes which by Th-2, are guaranteed to coincide for two-valued arguments only.
A similar result can be established for three-valued quantifiers.

Theorem 96
All $\mathcal{M}_\mathcal{B}$-DFSes coincide on three-valued semi-fuzzy quantifiers, i.e. in the case that Q assumes results in the restricted range $\{0, \frac{1}{2}, 1\}$ only, and can hence be expressed as $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$.

Again, this is different from general standard DFSes, which are guaranteed to coincide only for two-valued quantifiers, see Th-46.

7.13 A standard model with unique properties: $\mathcal{M}_{CX}$

We already know from the previous section that the model $\mathcal{M}_{CX}$ is both Q-continuous and arg-continuous. It is hence a practical model and suited for applications. In addition, $\mathcal{M}_{CX}$ propagates fuzziness in quantifiers and in arguments, just like every $\mathcal{M}_{\mathcal{B}}$-DFS. However, these results provide no clue yet, what *exactly* it is that makes $\mathcal{M}_{CX}$ so special. The present section is devoted exclusively to this issue and will substantiate that $\mathcal{M}_{CX}$ is indeed unique among $\mathcal{M}_{\mathcal{B}}$-DFSes. In fact, the particular qualities held by this model are distinguished even among the full class of standard models of fuzzy quantification.

In section 4.10, I have introduced the construction of fuzzy argument insertion, and I explained that only those models which conform to this construction can adequately represent adjectival restriction by a fuzzy adjective, as in "Most X's are lucky Y's". Compatibility with fuzzy argument insertion is therefore highly desirable from a linguistic standpoint, because it ensures that this general type of adjectival restriction can be interpreted in a compositional way. As concerns fuzzy argument insertion, the following positive result has been proven for $\mathcal{M}_{CX}$:

Theorem 97
The DFS $\mathcal{M}_{CX}$ is compatible with fuzzy argument insertion.

It is this property held by the model, which explains the distinguished role of $\mathcal{M}_{CX}$. In fact, it can be shown that $\mathcal{M}_{CX}$ is the unique standard DFS which complies with this adequacy condition:

Theorem 98
$\mathcal{M}_{CX}$ is the only standard DFS which is compatible with fuzzy argument insertion.

Hence $\mathcal{M}_{CX}$ is foremost among the standard models. Due to the special importance of $\mathcal{M}_{CX}$, it is probably worthwhile to attempt an axiomatization of the model, in terms of a system of conditions which uniquely characterise $\mathcal{M}_{CX}$. In fact, the desired axiomatization is rather obvious, because we only need to augment the previous axiomatization of the standard models, which has been presented in Def. 62, by the requirement of being compatible with fuzzy argument insertion:

Definition 97
Consider a QFM $\mathcal{F}$. For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, I introduce the

following conditions.

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(M-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(M-2)
Dualisation	$\mathcal{F}(Q\square) = \mathcal{F}(Q)\square \quad n > 0$	(M-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\cup \quad n > 0$	(M-4)
Preservation of monotonicity	*If* Q *is nonincreasing in* n*-th arg, then* $\mathcal{F}(Q)$ *is nonincreasing in* n*-th arg,* $n > 0$	(M-5)
Functional application	$\mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i}) = \mathcal{F}(Q) \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(M-6)
Fuzzy argument insertion	$\mathcal{F}(Q \,\widetilde{\triangleleft}\, A) = \mathcal{F}(Q) \triangleleft A$ *for all* $A \in \widetilde{\mathcal{P}}(E)$, $n > 0$	(M-7)

We can then assert that the above conditions (M-1)–(M-7) indeed achieve an axiomatisation of the model $\mathcal{M}_{CX}$.

Theorem 99 *$\mathcal{M}_{CX}$ is the unique QFM which satisfies* (M-1)–(M-7).
(Proof: D.3, p.443+)

Note. By the above reasoning, this theorem is really straightforward and an immediate consequence of Th-47 and Th-98.
As witnessed by the theorem, we have succeeded to uniquely identify $\mathcal{M}_{CX}$ in the full space of possible QFMs, solely based on its observable behaviour. This was possible because the structure of $\mathcal{M}_{CX}$ shows up in the distinguishing property (M-7), which is sufficient to characterise $\mathcal{M}_{CX}$ within the class of standard models. Because the system (M-1)–(M-7) also comprises a minimal set of conditions, this makes a rather satisfying result from a theoretical point of view.

In addition to being compatible with fuzzy argument insertion, $\mathcal{M}_{CX}$ also exhibits a number of other remarkable characteristics. At present, no other models are known which possess any of these distinguished properties. I hence suspect that the other properties, too, are unique to $\mathcal{M}_{CX}$, but there is no proof of this yet, and more research should be directed to these issues.
Let us first consider the preservation of convexity properties.

Theorem 100
The DFS $\mathcal{M}_{CX}$ weakly preserves convexity.

Theorem 101
Suppose $\mathcal{M}_{\mathcal{B}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS. If $\mathcal{M}_{\mathcal{B}}$ weakly preserves convexity, then $\mathcal{M}_{CX} \preceq_c \mathcal{M}_{\mathcal{B}}$.

The above theorems substantiate that $\mathcal{M}_{CX}$ is the least specific $\mathcal{M}_{\mathcal{B}}$-DFS which weakly preserves convexity. As remarked above, there is no evidence yet that other models exist which fulfill this criterion, and it might hence express another distinctive property of $\mathcal{M}_{CX}$.

Let us further observe that $\mathcal{M}_{CX}$ is a concrete implementation of a so-called 'substitution approach' to fuzzy quantification [164], i.e. the fuzzy quantifier is modelled by constructing an equivalent logical formula. This is apparent if we rewrite $\mathcal{M}_{CX}$ as follows.

Theorem 102
For every $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,*

$$\begin{aligned}&\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)\\&= \sup\{\widetilde{Q}^L_{V,\,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\\&= \inf\{\widetilde{Q}^U_{V,\,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\end{aligned}$$

where

$$\widetilde{Q}^L_{V,\,W}(X_1, \ldots, X_n) \tag{50}$$

$$= \min(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), L(Q, V, W)) \tag{51}$$

$$\widetilde{Q}^U_{V,\,W}(X_1, \ldots, X_n) \tag{52}$$

$$= \max(1 - \widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), U(Q, V, W)) \tag{53}$$

$$\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) \tag{54}$$

$$= \min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) \tag{55}$$

$$L(Q, V, W) \tag{56}$$

$$= \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \textit{all } i\} \tag{57}$$

$$U(Q, V, W) \tag{58}$$

$$= \sup\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \textit{all } i\}\,. \tag{59}$$

Notes

- To see how this is related to the substitution approach, let us notice that in the common case of a finite base set, $\inf$ and $\sup$ reduce to the logical connectives $\wedge = \max$ and $\vee = \min$ as usual. The infimum and supremum are only needed to express the infinitary conjunctions and disjunctions that are required for quantification on infinite domains. In addition, it was necessary to allow occurrences of continuous-valued constants $L(Q, V, W), U(Q, V, W) \in \mathbf{I}$ in the resulting formulas because the fuzzification mechanism is applied to general semi-fuzzy quantifiers, not only to two-valued quantifiers.
- From a different point of view, the above representation of $\mathcal{M}_{CX}$ demonstrates that the model can be defined independently of the cut ranges and the median-based aggregation mechanism. By contrast, it can be rephrased into a compact form, which reduces fuzzy quantification in $\mathcal{M}_{CX}$ to the evaluation of a Boolean

formula involving fuzzy connectives and continuous-valued propositional variables.

Before discussing any implications of the above reformulation of $\mathcal{M}_{CX}$ in terms of the substitution approach, I will first give an example of the proposed construction, in order to facilitate understanding of the above reformulation of $\mathcal{M}_{CX}$, and to elucidate the structure of the constructed formula and the function of the involved coefficients. Hence let us consider a simple two-element base set $E = \{a, b\}$ and a semi-fuzzy quantifier defined on E, say

$$Q(Y) = \begin{cases} 0 & : \quad Y = \varnothing \\ \frac{2}{3} & : \quad Y = \{b\} \\ 1 & : \quad Y = \{a\} \vee Y = \{a, b\} \end{cases}$$

for all $Y \in \mathcal{P}(\{a, b\})$. Let us now construct the quantification formula step by step, in order to gain some understanding how it is built from the base quantifier. Hence let a fuzzy argument set $X \in \widetilde{\mathcal{P}}(\{a, b\})$ be given. It is convenient to abbreviate $p = \mu_X(a)$, $q = \mu_X(b)$. The following table lists all possible choices of $V, W \in \mathcal{P}(\{a, b\})$ with $V \subseteq W$, along with the corresponding expression $\widetilde{\widetilde{\Xi}}_{V,W}(X)$ which expresses the compatibility of X with the closed range of subsets $\{Y \in \mathcal{P}(E) : V \subseteq Y \subseteq W\}$, and the lower and upper bounds $L(Q, V, W)$ and $U(Q, V, W)$ on the quantification results achieved by Q in this given range of subsets.

V	W	$\widetilde{\widetilde{\Xi}}_{V,W}(X)$	$L(Q, V, W)$	$U(Q, V, W)$
$\varnothing$	$\varnothing$	$\neg p \wedge \neg q$	0	0
$\varnothing$	$\{a\}$	$\neg q$	0	1
$\varnothing$	$\{b\}$	$\neg p$	0	$\frac{2}{3}$
$\varnothing$	$\{a, b\}$	1	0	1
$\{a\}$	$\{a\}$	$p \wedge \neg q$	1	1
$\{a\}$	$\{a, b\}$	p	1	1
$\{b\}$	$\{b\}$	$\neg p \wedge q$	$\frac{2}{3}$	$\frac{2}{3}$
$\{b\}$	$\{a, b\}$	q	$\frac{2}{3}$	1
$\{a, b\}$	$\{a, b\}$	$p \wedge q$	1	1

Based on this table, we can then fill in the formulas which define the upper and lower bound quantifiers $\widetilde{Q}^U_{V,\,W}(X)$ and $\widetilde{Q}^L_{V,\,W}(X)$:

V	W	$\widetilde{Q}^L_{V,\,W}(X)$	$\widetilde{Q}^U_{V,\,W}(X)$
$\varnothing$	$\varnothing$	$\neg p \wedge \neg q \wedge 0$	$p \vee q \vee 0$
$\varnothing$	$\{a\}$	$\neg q \wedge 0$	$q \vee 1$
$\varnothing$	$\{b\}$	$\neg p \wedge 0$	$p \vee \frac{2}{3}$
$\varnothing$	$\{a, b\}$	$1 \wedge 0$	$0 \vee 1$
$\{a\}$	$\{a\}$	$p \wedge \neg q \wedge 1$	$\neg p \vee q \vee 1$
$\{a\}$	$\{a, b\}$	$p \wedge 1$	$\neg p \vee 1$
$\{b\}$	$\{b\}$	$\neg p \wedge q \wedge \frac{2}{3}$	$p \vee \neg q \vee \frac{2}{3}$
$\{b\}$	$\{a, b\}$	$q \wedge \frac{2}{3}$	$\neg q \vee 1$
$\{a, b\}$	$\{a, b\}$	$p \wedge q \wedge 1$	$\neg p \vee \neg q \vee 1$

Recalling the first construction of the quantification formula described in Th-102, which resorts to the lower bound quantifiers $\widetilde{Q}^{L}_{V,\,W}$, we can now build the desired formula which reduces $\mathcal{M}_{CX}(Q)(X)$ to a propositional expression involving continuous-valued variables:

$$\begin{aligned}\mathcal{M}_{CX}(Q)(X) = \max\{&\neg p \wedge \neg q \wedge 0, \neg q \wedge 0, \neg p \wedge 0, 1 \wedge 0, p \wedge \neg q \wedge 1, p \wedge 1,\\ &\neg p \wedge q \wedge \tfrac{2}{3}, q \wedge \tfrac{2}{3}, p \wedge q \wedge 1\}\,.\end{aligned}$$

After some element-wise simplification, this becomes

$$\mathcal{M}_{CX}(Q)(X) = \max\{0, 0, 0, 0, p \wedge \neg q, p, \neg p \wedge q \wedge \tfrac{2}{3}, q \wedge \tfrac{2}{3}, p \wedge q\}\,.$$

By eliminating all occurrences of the identity 0 of $\max$, and by utilizing the law of absorption, the above expression can then be reduced to

$$\mathcal{M}_{CX}(Q)(X) = p \vee (q \wedge \tfrac{2}{3})\,.$$

Of course, we could also have applied the second construction of the quantification formula proposed in Th-102, which is based on the upper bound quantifiers $\widetilde{Q}^{U}_{V,\,W}$. In this case, too, we start by building the 'raw' formula which expresses $\mathcal{M}_{CX}(Q)(X)$, now based on the second construction. We then obtain the conjunction,

$$\begin{aligned}\mathcal{M}_{CX}(Q)(X) = \min\{&p \vee q \vee 0, q \vee 1, p \vee \tfrac{2}{3}, 0 \vee 1, \neg p \vee q \vee 1, \neg p \vee 1,\\ &p \vee \neg q \vee \tfrac{2}{3}, \neg q \vee 1, \neg p \vee \neg q \vee 1\}\,.\end{aligned}$$

Now effecting some element-wise simplification, the raw formula becomes

$$\mathcal{M}_{CX}(Q)(X) = \min\{p \vee q, 1, p \vee \tfrac{2}{3}, 1, 1, 1, p \vee \neg q \vee \tfrac{2}{3}, 1, 1\}\,.$$

Again by removing occurrences of the identity 1 of $\min$, and by taking benefit of the law of absorption in order to eliminate some of the subexpressions, we can further simplify the latter result into

$$\mathcal{M}_{CX}(Q)(X) = (p \vee q) \wedge (p \vee \tfrac{2}{3})\,.$$

Now recalling the law of distributivity, we can extract the variable p from the two disjunctions, and hence rewrite the above equation as

$$\mathcal{M}_{CX}(Q)(X) = p \vee (q \wedge \tfrac{2}{3})\,.$$

Hence the result of the second construction indeed coincides with that obtained from the first construction, as claimed by the theorem. This completes the example, which demonstrated the utility of the substitution formulas to the modelling of fuzzy quantification, and I will now discuss some further observations which are apparent from the reformulation of $\mathcal{M}_{CX}$.

The above representation of $\mathcal{M}_{CX}$, which is made explicit by Th-102, also reveals the robustness of this model against variation in the quantifier and its arguments. This is important because in practice, there often remains some uncertainty concerning the precise choice of numeric membership grades which best model the target NL quantifier and the arguments (i.e. the target NL concepts which are to be modelled by suitable fuzzy sets).

- Let us first observe from the above representation that $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)$ is not only arg-continuous (i.e. a smooth function of the membership grades $\mu_{X_i}(e)$ attained by the arguments), but indeed extremely stable against slight changes of the membership grades. This is because for all $i \in \{1, \ \ldots, n\}$ and $e \in E$,

  $$\left| \frac{\partial \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)}{\partial \mu_{X_i}(e)} \right| \leq 1$$

  whenever the partial derivative is defined, which is obvious from the above representation. A change in the arguments which does not exceed a given Δ hence cannot change the quantification results by more than Δ. This indicates that with $\mathcal{M}_{CX}$, the precise choice of membership grades (which might be to some degree arbitrary), is rather uncritical.

- A similar remark can also be made concerning the robustness of the quantification results $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)$ against variation in the chosen quantifier. In this case, we observe that for all $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$,

  $$\left| \frac{\partial \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)}{\partial Q(Y_1, \ldots, Y_n)} \right| \leq 1$$

  whenever the partial derivative is defined, which is again obvious from the above representation. In fact, a change of $Q(Y_1, \ldots, Y_n)$ by some $\Delta > 0$ can maximally change the quantification results by Δ. Hence the precise choice of the semi-fuzzy quantifier is uncritical with $\mathcal{M}_{CX}$ as well.

This means that the imprecision in the choice of numeric membership grades for quantifier and arguments is not amplified in any way when applying $\mathcal{M}_{CX}$, but either kept at its original level, or even suppressed.

Summarizing, I have managed to relate the model $\mathcal{M}_{CX}$ to Yager's suggestion of modelling fuzzy quantification by constructing a suitable logical formula [164], known as the 'substitution approach'. In addition, I have discussed some important implications that are apparent from the representation of $\mathcal{M}_{CX}$ in terms of a fuzzy propositional formula. In the following, I will continue along these lines, of connecting the model to existing work on fuzzy quantification. To this end, I first recall the definition of the Sugeno integral, which is very closely tied to the model.

Definition 98
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing semi-fuzzy quantifier and $X \in \widetilde{\mathcal{P}}(E)$. The Sugeno integral $(S) \int X \, dQ$ *is defined by*

$$(S) \int X \, dQ = \sup\{\min(\alpha, Q(X_{\geq \alpha})) : \alpha \in \mathbf{I}\} \, .$$

Let us now observe that $\mathcal{M}_{CX}$ properly generalizes the Sugeno integral:

Theorem 103

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is nondecreasing. Then for all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$(S) \int X \, dQ = \mathcal{M}_{CX}(Q)(X) \, .$$

Hence $\mathcal{M}_{CX}(Q)$ coincides with the well-known Sugeno integral whenever the latter is defined. Unlike the integral, though, $\mathcal{M}_{CX}$ is defined for arbitrary semi-fuzzy quantifiers, and hence extends the Sugeno integral from monotonic measures to the 'hard' cases of unrestricted multiplace and non-monotonic quantifiers.
In order to relate this result with previous approaches to fuzzy quantification, I Hence let us formally define the quantity $\mu_{[j]}(X)$ already mentioned in the introduction.

Definition 99

Let a finite base set $E \neq \varnothing$ *of cardinality* $|E| = m$ *be given. For a fuzzy subset* $X \in \widetilde{\mathcal{P}}(E)$*, we denote by* $\mu_{[j]}(X) \in \mathbf{I}$*,* $j = 1, \ldots, m$*, the* j*-th largest membership value of* X *(including duplicates).*
More formally, consider an ordering of the elements of E *such that* $E = \{e_1, \ldots, e_m\}$ *and* $\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)$*. Then define* $\mu_{[j]}(X) = \mu_X(e_j)$*. It is apparent that the results do not depend on the chosen ordering if ambiguities exist.*
Let us further stipulate that $\mu_{[0]}(X) = 1$ *and that* $\mu_{[j]}(X) = 0$ *whenever* $j > m$*.*

As a corollary to the above theorem, we then obtain (cf. [20]):

Theorem 104

Suppose $E \neq \varnothing$ *is a finite base set,* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is a nondecreasing mapping and* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is defined by* $Q(Y) = q(|Y|)$ *for all* $Y \in \mathcal{P}(E)$*. Then for all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$\mathcal{M}_{CX}(Q)(X) = \max\{\min(q(j), \mu_{[j]}(X)) : 0 \leq j \leq |E|\} \, .$$

Notes

- Hence $\mathcal{M}_{CX}$ consistently generalises the basic FG-count approach of [165, 188], which is restricted to quantitative and nondecreasing one-place quantifiers.
- From a computational point of view, the theorem shows that in the special case of nondecreasing quantitative unary quantifiers on finite domains, $\mathcal{M}_{CX}(Q)(X)$ can be determined from a fuzzy cardinality measure on X, namely from the well-known FG-count(X). As will be shown later in Th-245, it is possible to generalize this result to arbitrary quantitative unary quantifiers on finite domains. In this generic case, though, the FG-count must be replaced with a different measure of fuzzy cardinality, the fuzzy interval cardinality $\|X\|_{\text{iv}}$, which will be introduced below in Def. 162.

In closing this section on $\mathcal{M}_{CX}$, I finally discuss two additional properties of the model which make it particular suited for real-world applications. The first property is concerned with an aspect of parsimony. Typical applications of fuzzy logic do not fully exploit the available range of continuous truth values. In many situations, it is sufficient to restrict attention to a modest number of truth values/membership grades. For example, a repertoire of five possible grades

$$\Upsilon = \{0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 1\} \tag{60}$$

might be adequate, which correspond to the ordinal scale

$$\text{‘false’, ‘quite false’, ‘undecided’, ‘quite true’, ‘true’}\,. \tag{61}$$

The model $\mathcal{M}_{CX}$ combines well with such ordinal scales of truth values, and hence supports a parsimoneous use of membership grades. This is because $\mathcal{M}_{CX}$ does not ‘invent’ any new truth values, provided that the restricted scale is closed under negation:

Theorem 105

Let $\Upsilon \subset \mathbf{I}$ be a given set with the following properties:

- Υ *is finite;*
- *if* $\upsilon \in \Upsilon$*, then* $1 - \upsilon \in \Upsilon$*;*
- $\{0, 1\} \subseteq \Upsilon$.

Further suppose that $Q : \mathcal{P}(E)^n \longrightarrow \Upsilon$ is a semi-fuzzy quantifier with quantification results in Υ and that $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are Υ-valued fuzzy subsets of E, i.e. based on membership functions $\mu_{X_i} : E \longrightarrow \Upsilon$, $i = 1, \ldots, n$. Then it also holds that $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) \in \Upsilon$.

Hence in the above example of the five element set Υ defined by (60), fuzzy quantification based on $\mathcal{M}_{CX}$ will always yield results in $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, provided that both the semi-fuzzy quantifier and its arguments assume values in this restricted range only.

So far, we have considered a restriction to finite scales of membership grades, which is possible in many applications and reduces the complexity of the model. The achieved parsimony limits the set of available truth values but does not alter the basic modelling style, which is essentially numerical and heavily relies on the use of numbers to express non-numerical, symbolic natural language concepts. In applications which emphasize the linguistic aspects of fuzzy quantification, it might be more natural to depart from numerical modelling altogether, and express the membership grades and gradual truth values in terms of ordinal scales, like the one presented in (61). Due to the fact that the computational procedures of fuzzy modelling rely on the use of numerical membership grades, the question then arises of how these ordinal scales should be embedded into the continuous range of truth values in $[0, 1]$, and hence be interpreted in terms of numerical membership grades. In practice, there is seldom perfect knowledge concerning the precise choice of numeric membership grades which best represent the given ordinal scale. It is hence essential to a good model of fuzzy information processing, and in particular of fuzzy quantification, that a certain variation in the choice of membership grades, which may result from this uncertainty concerning the precise embedding into numerical representations, be absorbed by the interpretation process. As stated by the next theorem, $\mathcal{M}_{CX}$ is robust with respect to such variation in the numeric membership grades, as long as these changes are symmetrical with respect to negation. Recalling the model transformation scheme introduced in Def. 53, we can express this property as follows.

Theorem 106
Suppose $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping which satisfies the following requirements:

- *σ is a bijection;*
- *σ is nondecreasing;*
- *σ is symmetrical with respect to negation, i.e. $\sigma(1-x) = 1-\sigma(x)$ for all $x \in \mathbf{I}$.*

Then ${\mathcal{M}_{CX}}^{\sigma} = \mathcal{M}_{CX}$.

The theorem states that if the scale of continuous truth values is deformed in a consistent way, then the results obtained from fuzzy quantification based on $\mathcal{M}_{CX}$ will be subjected to exactly the same deformation, and automatically fit into the adapted scale.

Let us return to the above example on finite truth values, which is also suited to demonstrate how the property expressed by the theorem is related to the original problem, that of absorbing the uncertainty concerning the interpretation of ordinal truth values in terms of numerical assignments. In the example, I have chosen to interpret the ordinal scale presented in (61) in terms of the numbers $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. However, it might be more appropriate to model 'quite true' by $\frac{4}{5}$ instead of the earlier $\frac{3}{4}$. The requirements stated in the theorem then guide us to adapt 'quite false' accordingly, which receives the new interpretation of $\frac{1}{5}$. In terms of the re-interpretation mapping σ, these decisions can be expressed as

$$\sigma(\tfrac{3}{4}) = \tfrac{4}{5}, \qquad\qquad \sigma(\tfrac{1}{4}) = \tfrac{1}{5}\,. \tag{62}$$

Let us now consider an arbitrary extension of σ to a 'full' mapping $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$, which conforms to the requirements of Th-106 and fulfills the criteria stated in (62). We then also know that $\sigma(0) = 0$, $\sigma(\frac{1}{2}) = \frac{1}{2}$ and $\sigma(1) = 1$. Hence by applying σ, the original interpretation of the ordinal scale (61) in terms of $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, is consistently transformed into an alternative interpretation of the scale, which is now based on $\{0, \frac{1}{5}, \frac{1}{2}, \frac{4}{5}, 1\}$. Combined with the earlier result Th-105, the new theorem Th-106 then ensure that the results obtained from $\mathcal{M}_{CX}$ automatically adapt to the new interpretation of the ordinal scale, provided that both the semi-fuzzy quantifier and the fuzzy argument sets are fitted to the new interpretation by applying σ. Hence suppose that we had $\mathcal{M}_{CX}(\textbf{many})(\textbf{rich}, \textbf{lucky}) = \frac{3}{4}$ prior to the re-interpretation, which corresponds to the ordinal result 'quite true'. Now we fit **rich**, **lucky** to the alternative interpretation by applying σ, hence obtaining the new arguments σ**rich**, σ**lucky**. The quantifier must also be adapted to the new interpretation, and hence translates into $\sigma\textsf{\textbf{many}}$. Let us now instantiate the theorem by the given data; we then obtain that

$$\mathcal{M}_{CX}{}^{\sigma}(\textsf{\textbf{many}})(\textsf{\textbf{rich}}, \textsf{\textbf{lucky}}) = \mathcal{M}_{CX}(\textsf{\textbf{many}})(\textsf{\textbf{rich}}, \textsf{\textbf{lucky}})\,. \tag{63}$$

Now expanding the left hand member of the equality according to Def. 53, the above equality (63) becomes

$$\sigma^{-1}\mathcal{M}_{CX}(\sigma\textsf{\textbf{many}})(\sigma\textsf{\textbf{rich}}, \sigma\textsf{\textbf{lucky}}) = \mathcal{M}_{CX}(\textsf{\textbf{many}})(\textsf{\textbf{rich}}, \textsf{\textbf{lucky}})\,,$$

which finally can be rewritten as

$$\mathcal{M}_{CX}(\sigma\textsf{\textbf{many}})(\sigma\textsf{\textbf{rich}}, \sigma\textsf{\textbf{lucky}}) = \sigma(\mathcal{M}_{CX}(\textsf{\textbf{many}})(\textsf{\textbf{rich}}, \textsf{\textbf{lucky}}))\,.$$

Hence the quantification result obtained from the re-interpretation of the ordinal scale (left-hand member of the equation) can be computed from the quantification result based on the original interpretation of the ordinal scale, simply by applying the re-interpretation mapping σ which fits the result into the new system (right-hand member of equation). In the above example, then, the equation ensures that the previous result of

$$\mathcal{M}_{CX}(\textsf{\textbf{many}})(\textsf{\textbf{rich}}, \textsf{\textbf{lucky}}) = \tfrac{3}{4}\,,$$

which corresponds to ‘quite true’ under the original interpretation, now translates into

$$\mathcal{M}_{CX}(\sigma\textsf{\textbf{many}})(\sigma\textsf{\textbf{rich}}, \sigma\textsf{\textbf{lucky}}) = \tfrac{4}{5}\,,$$

which is the proper representation of ‘quite true’ in the adapted system.
These remarks complete the discussion of the last example, which was intended to demonstrate the interactions between the two properties that were identified for $\mathcal{M}_{CX}$. The example indicates that when taken together, these conditions ensure that the model of fuzzy quantification be suited to handle ordinal scales of truth values, provided that these are closed under negation. This is because the first condition (expressed in Th-105) permits a restriction to the finite set of numerical membership grades that interpret the ordinal values. In particular, it is possible to translate the numerical quantification result back into an ordinal result in the assumed base scale. In turn, the second condition (expressed in Th-106) asserts that the precise numerical interpretation of the ordinal base scale is inessential, because the re-translation of the result into the ordinal scale is invariant under the chosen assignment of numerical membership grades.

To sum up, I have formalized two interesting properties of QFMs which take care of some typical concerns that arise in real-world applications: the first property of the model frees the application from handling the full range of continuous membership grades, and permits the application to decide itself upon the desired granularity of discerned membership grades. The second condition acknowledges the problem that a precise choice of numerical membership grades, which is enforced by the underlying numeric model, is typically hard to justify from an application point of view, and possibly even arbitrary to some degree. It is therefore mandatory that the model of fuzzy quantification be robust against this type of uncertainty, and hence absorbs the resulting variation in possible choices of numeric membership grades. As opposed to the desideratum of smoothness in quantifiers and arguments, which captures the case of non-systematic small variations, the second property suggested here captures the remaining case of coherent variations, which are allowed to be arbitrarily large. Apart from formalizing these properties and explaining their relevance to applications, I have also demonstrated by an example, that the properties are most useful when taken together, because the model is then suited to handle ordinal scales of truth values, regardless of the chosen numerical interpretation. As witnessed by the above theorems Th-105 and Th-106, in which the two criteria are developed, the model $\mathcal{M}_{CX}$ shows both of these adequacy properties, which makes an even bolder point for preferring this particular model in applications.

7.14 Chapter summary

This chapter was devoted to the quest of a constructive principle for models of fuzzy quantification. The primary objective was to establish a class of unrestricted QFMs from a uniform base construction, which provides a rich source of plausible models. We can then impose conditions on the 'raw' QFMs, which translate our expectations on admissible models into requirements on the underlying construction, and hence precisely locate the subclass of intended models within the total of unrestricted QFMs that can be built from this construction.

In order to identify such a constructive principle suited to span an interesting class of models, I have first investigated the potential utility of proven mechanisms of fuzzy set theory to define models of fuzzy quantification. These techniques have furthered the development of fuzzy set theory because they support a generalization of two-valued, crisp concepts into corresponding concepts that live on fuzzy sets. This 'automatic' transfer from a crisp concept to a corresponding concept on fuzzy sets is possible because these techniques achieve a uniform reduction of the fuzzy analogue to computations based on the original crisp concept. Acknowledging the significance of these techniques to fuzzy logic in general, I have reviewed the well-known α-cut reductions, and the resolution principle in particular, which decompose the considered construction on fuzzy sets into layers of crisp computations. Their known utility to other parts of fuzzy logic notwithstanding, the formal analysis of these techniques revealed that they offer little help when it comes to defining models of fuzzy quantification. I have also tracked the reason why these common tools of fuzzy set theory cannot be used to define DFSes, and explained this failure in terms of their lack of symmetry with respect to complementation. It was this observation which guided the search of the base construction into a reformulation of the cutting mechanism, which should be done in a way compatible with complementation. Quite obviously, this requirement cannot be satisfied by modified two-valued cuts.

I hence decided to utilize a three-valued cutting mechanism, which overcomes the asymmetry problem. A resolution property similar to that of α-cuts has been proven for the proposed mechanism, which demonstrates that it can faithfully represent a given fuzzy subset in terms of the information spread over its three-valued cuts. The mechanism is hence comparable in expressive power to the known α-cuts. In principle, the novel representation in terms of three-valued sets still lends itself to implementation, because every three-valued cut can be represented by a pair of α-cuts. However, a direct reduction into layers of crisp computations is not possible with three-valued cuts, because these result in three-valued rather than crisp subsets. It was therefore suggested to further resolve three-valued sets into closed ranges of crisp sets, on which the necessary computations can be effected. The individual results collected for all choices of crisp sets in the considered ranges must then be aggregated into the final outcome of the interpretation process. Observing that the targeted class of standard models embeds Kleene's three-valued logic, it seemed advantageous to develop the aggregation procedure in such a way, that the resulting mechanism implements a constructive principle for Kleene's logic. This basic approach was then fitted to the case of three-valued quantifiers rather than propositional functions. Finally it was shown how the necessary generalization to semi-fuzzy quantifiers, and hence from three-valued to continuous-valued quantification results, can be accomplished by fuzzy median aggregation. These considerations culminated in the definition of the QFMs $(\bullet)_\gamma$. Given a semi-fuzzy quantifier Q and a choice of fuzzy arguments $X_1, \dots, X_n$, the construction of $Q_\gamma(X_1, \dots, X_n)$ first collects all quantification results $Q(Y_1, \dots, Y_n)$ obtained from choices of crisp sets $Y_1, \dots, Y_n$ in the cut ranges. It then applies the generalized fuzzy median and combines these alternatives into the quantification result $Q_\gamma(X_1, \dots, X_n) \in \mathbf{I}$, which represents the interpretation of the fuzzy quantifying expression at the given 'level of cautiousness' $\gamma \in \mathbf{I}$. In this way, we obtain an interpretation result $Q_\gamma(X_1, \dots, X_n)$ for all choices of $\gamma \in \mathbf{I}$. However, none of the QFMs $(\bullet)_\gamma$ is a DFS itself. Therefore a second aggregation step is necessary which considers all local results of $Q_\gamma(X_1, \dots, X_n)$ in parallel and hence exploits the total information spread over the cut levels γ, in order to build models from this construction. This second aggregation stage is delegated to an aggregation operator $\mathcal{B}$, which abstracts from the cutting parameter $\gamma \in \mathbf{I}$ and hence determines the final quantification result $\mathcal{M}_\mathcal{B}(Q)(X_1, \dots, X_n) = \mathcal{B}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}})$. This makes a generic base construction for QFMs, parametrized by the chosen aggregation operator $\mathcal{B}$, which spans the unrestricted class of $\mathcal{M}_\mathcal{B}$-QFMs. I have also given a first example, the DFS $\mathcal{M}$, in order to demonstrate that the chosen approach succeeds in defining plausible models, and hence deserves further investigation. Following that, some formal machinery has been developed, which extracts the relevant features that govern the behaviour of all 'reasonable' aggregation mappings $\mathcal{B}$. In terms of these concepts, I then managed to express an independent system of necessary and sufficient conditions which capture the precise requirements on $\mathcal{B}$ that make $\mathcal{M}_\mathcal{B}$ a DFS. It also came out of this analysis that all $\mathcal{M}_\mathcal{B}$-DFSes are indeed standard models.

This knowledge of the necessary conditions on $\mathcal{B}$ which hold whenever $\mathcal{M}_\mathcal{B}$ is a plausible model, also rendered possible a simplification of the base construction, because it now emerged that every conforming choice of $\mathcal{B}$ is fully determined by its behaviour on the restricted domain, $\mathbb{B}^+$. The elimination of this redundance still contained in $\mathcal{B}$, then guided a reduction into simplified aggregation mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$, which capture the core behaviour of the original mapping. Starting from a given $\mathcal{B}'$, the 'full' aggregation mapping $\mathcal{B}$ can then be recovered by means of a simple transformation. The mappings $\mathcal{B}'$ are useful because they better support the definition of models, which can typically be stated in a more compact form. In addition, no models of interest are lost under the simplified representation, which was proven to cover all plausible models. Taking benefit of the simplified representation, I have then introduced some prominent examples of $\mathcal{M}_{\mathcal{B}}$-DFSes, which instantiate the proposed base construction. Some of these models also occupy a special position in the broader classes of models which will be introduced in subsequent chapters, or even to the full class of standard models.

Now that we have several models available, the question arises of how these models are interrelated, and which particular model should be preferred in a given application. In order to decide upon these issues, it is necessary to assess the characteristic properties of given $\mathcal{M}_{\mathcal{B}}$-DFSes, which help to locate them within the full range of models, e.g. by relating the models to the extreme poles. This kind of analysis also provides the necessary information for a comparison of models according to semantical or practical considerations. This helps to identify that choice of model which best suits a given application. Following these lines, I have developed the required criteria for checking an $\mathcal{M}_{\mathcal{B}}$ model for its special semantical characteristics, but also for more practical concerns, like different aspects of robustness. In order to permit a simple and quick assessment of these properties, the corresponding criteria have generally been reduced to elementary tests, which can be readily decided from the aggregation mappings. For example, I have shown how to simplify a comparison of the models by specificity, which can effectively be reduced to an inequality on the level of $\mathcal{B}'$. From this elementary criterion, it was then easily proven that $\preceq_c$ is not a linear order on $\mathcal{M}_{\mathcal{B}}$-DFSes, i.e. we cannot compile a list of such models sorted by specificity. However, it was possible to identify the extreme cases of $\mathcal{M}_{\mathcal{B}}$-models in terms of specificity. According to this analysis, the results that must be expected from a given $\mathcal{M}_{\mathcal{B}}$ model are in between the least specific bound determined by $\mathcal{M}_U$, and the upper specificity bound established by $\mathcal{M}_S$. In particular, all $\mathcal{M}_{\mathcal{B}}$-DFSes are specificity consistent, and hence rather homogeneous compared to the broader classes of models that will be considered in later chapters of the report.

Following this discussion of specificity issues, I turned to the two facets of continuity which govern the behaviour of practical $\mathcal{M}_{\mathcal{B}}$-DFSes. Based on the proposed distance measures on $\mathcal{B}'$, I was able to express precise conditions on $\mathcal{B}'$ which ensure that $\mathcal{M}_{\mathcal{B}}$ be Q-continuous and arg-continuous, respectively, and hence exhibit some minimum stability against small fluctuations in their inputs. Having analysed this precondition of practical models, I then addressed the issue of propagating fuzziness. This investigation revealed that $\mathcal{M}_{\mathcal{B}}$-DFSes are quite homogeneous in this respect, too, because all of these models propagate fuzziness both in quantifiers and arguments. Hence less specific input cannot result in more specific output when resorting to these models of fuzzy quantification, in conformity with our intuitive expectations. The relative homogeneity of the $\mathcal{M}_{\mathcal{B}}$ models also expresses in their uniform behaviour when fed with three-valued inputs. In fact, all $\mathcal{M}_{\mathcal{B}}$-DFSes can be shown to coincide on three-valued quantifiers, and also on three-valued arguments (given an arbitrary quantifier). This is different from general standard models which coincide on two-valued quantifiers and two-valued arguments only.

The above criteria for extracting the characteristic features of $\mathcal{M}_{\mathcal{B}}$-DFSes have then been applied to the example models, in order to assess their particular properties. This investigation of the example models revealed that the models $\mathcal{M}_U$ and $\mathcal{M}_S$, which represent the extreme poles in terms of specificity, are of few utility to applications. These models violate both continuity conditions, and hence lack the minumum robustness requirements that must be assured for applications. Consequently, these models are of theoretical interest only, due to their distinguished position in the full class of models. The models $\mathcal{M}$ and $\mathcal{M}_{CX}$, by contrast, were shown to exhibit the desired robustness, because they conform to both continuity requirements. This makes these models a good choice for application. In fact, the model $\mathcal{M}$ is the first example of a DFS, which was used for the implementation of fuzzy quantifiers in a prototypical application [53]. As concerns the DFS $\mathcal{M}_{CX}$, it soon emerged from my investigation that the model combines several astonishing properties which make it foremost among all standard models of fuzzy quantification. A separate section has hence been devoted to the model $\mathcal{M}_{CX}$, in order to acknowledge its unique position, and to detail its characteristic properties.

The distinguishing property of $\mathcal{M}_{CX}$, which lets the model surpass all standard models, is that $\mathcal{M}_{CX}$ complies with fuzzy argument insertion, and hence supports a compositional interpretation of fuzzy adjectival restriction, as in "Many young rich are lucky". Knowing that all other standard models fall short of this property, it hence captures a genuine characteristic of $\mathcal{M}_{CX}$. This observation made it possible to bring forward an axiomatization in terms of a minimal set of conditions which uniquely identify the model based on its observable behaviour. In fact, it was sufficient to augment the core axiom system for standard models by the requirement of fuzzy argument insertion in order to achieve this result.

Apart from its prime characteristic, the compliance with fuzzy argument insertion, the model combines various additional qualities which might also be distinctive, because no other models are currently known which share these characteristics. To begin with, $\mathcal{M}_{CX}$ is the only known model which preserves convexity properties of quantifiers (to the maximal degree possible for a DFS). In addition, it bounds by specificity all models with this property, should further examples of such models exist. The model can also be related to existing research on fuzzy quantification in several ways. It hence seizes some recurrent themes which are essential to the modelling of fuzzy quantifiers, and develops these into a consistent account of fuzzy quantification in its generality. In particular, $\mathcal{M}_{CX}$ can be shown to implement the so-called 'substitution approach' [164]; the model hence expresses fuzzy quantification in terms of a propositional formula involving the standard fuzzy connectives, and continuous-valued propositional variables. In order to elucidate this alternative representation of $\mathcal{M}_{CX}$, I have presented a detailed example which explains in a step-by-step fashion how the substitution formula is constructed from the coefficients $L(Q, V, W)$ and $U(Q, V, W)$ sampled from the quantifier, and the compatibility measure $\widetilde{\Xi}_{V,W}(X)$ which judges the degree to which the fuzzy set X belongs to a range of crisp sets $\{Y \in \mathcal{P}(E) : V \subseteq Y \subseteq W\}$. The new representation also contributes to the development of algorithms, because the resulting formulas can guide the interpretation of fuzzy quantifiers in the model. In fact, some of the evaluation formulas presented in Chap. 11 have been developed from this representation. We will then see that the formulas for $\mathcal{M}_{CX}$ can be simplified further in the case of quantitative one-place quantifiers. Based on a suitable measure of fuzzy cardinality, it will then be possible to compute the quantification results of $\mathcal{M}_{CX}$ directly from cardinality information. Some other theorems are also apparent from the alternative representation of the model. First of all, it is apparent from this representation that $\mathcal{M}_{CX}$ is not only continuous in quantifiers and their arguments, but indeed achieves extreme robustness against fluctuations in the input, because all involved gradients are tied to the range $[-1, 1]$. This means that the imprecision in the choice of numeric membership grades for quantifier and arguments is not amplified in any way when applying $\mathcal{M}_{CX}$, but either kept at its original level, or even suppressed. Hence the precise choices of numerical membership grades, which are necessary to fit the NL quantifier and its arguments into the fuzzy framework, are rather uncritical with the model. The representation of $\mathcal{M}_{CX}$ in terms of the substitution formulas also facilitated the proof that $\mathcal{M}_{CX}$ consistently extends the well-known Sugeno integral to arbitrary n-place quantifiers, and hence overcomes its restriction to monotonic measures. Recalling the known relationship between the Sugeno integral and the FG-count approach, it then came out that the model also generalizes the FG-count approach to the 'hard' cases of multi-place and non-monotonic quantification.

Some further insensitivity properties of $\mathcal{M}_{CX}$ have also been proven from this representation, which answer some typical needs that arise when fuzzy sets meet real-world applications. Firstly, it is often preferable to restrict the set of allowable membership grades, in order to reduce the complexity of the system, but also to better fit fuzzy techniques to the application, which now decides itself upon the desired granularity of the model. The interpretations of fuzzy quantifiers must then be compatible with the restriction of admissible truth values, and when supplied with conforming input, also assume results from the allowable choices. The second property is concerned with the notorious problem of determining that choice of numerical membership grades which best represent the target NL concepts. Apart from small, non-systematic fluctuations in the inputs, which $\mathcal{M}_{CX}$ absorbs due to its continuity and uncritical shape, the model should also withstand larger coherent changes, in order to optimally suppress any variation which may result from the uncertainty concerning the proper numerical assignments. Taken together, both conditions ensure that the model can handle ordinal scales of membership. I have presented a detailed example which explains how the conjunction of these properties makes the model immune against the numerical interpretation of the ordinal base scale. Knowing that the model $\mathcal{M}_{CX}$ combines both properties, it can hence be applied for fuzzy quantification with ordinal grades of membership. To sum up, I have shown that the DFS $\mathcal{M}_{CX}$ combines unique semantical properties, and hence constitutes the best model of fuzzy quantification from a linguistic perspective. It is this distinguished agreement with linguistic expectations which makes it the preferred choice for all applications that need to implement natural language semantics. Apart from its linguistic adequacy, the model also has practical virtues, notably its particular robustness against both random and systematic variation in the model's inputs. Finally, the model supports a parsimoneous modelling style, because it permits a restriction to ordinal scales of truth values.

8 The class of models defined in terms of upper and lower bounds on three-valued cuts

8.1 Motivation and chapter overview

In the previous chapter, a rich class of standard models for fuzzy quantification has been introduced, which comply with the axiomatic postulates for fuzzy natural language quantification. In addition, a distinguished model has been identified, which is the preferred choice for all applications that emphasize the natural language aspect of fuzzy quantifiers, and need to capture their intuitive semantics. This positive result on the model $\mathcal{M}_{CX}$ might suggest that we can stop the exploration of further models, now that the universal, perfect case of a standard DFS has been discovered. Things are more complicated, though, because there are situations in which different requirements on the models prevail over the desideratum of linguistic adequacy, which are not necessarily satisfied by the model $\mathcal{M}_{CX}$, nor by any other choice of an $\mathcal{M}_{\mathcal{B}}$-DFS. Such demands may arise, for example, in non-linguistic applications of information aggregation, data fusion, and multi-criteria decisionmaking. These applications usually involve the computation of a result ranking, or a weighting of alternatives, which in turn permits the selection of the best choice. In this kind of situation, it might then become the prime concern to ensure a fine-grained differentiation of the available options, and the model of fuzzy quantification must hence be as discriminating as possible.

As we shall learn below, the known models of the $\mathcal{M}_{\mathcal{B}}$-type are disadvantageous in this respect, however. Surprisingly, the reason for their suboptimal discriminating power is buried in the fact that every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness. There is an apparent trade-off here because propagation of fuzziness undoubtedly captures one of our expectations on plausible models – it is simply hard to understand that the results should become more specific when the inputs (quantifier or argument) get fuzzier. However, there is a price to be paid for the propagation of fuzziness: as the input becomes less specific, the result of an $\mathcal{M}_{\mathcal{B}}$-DFS is likely to attain the least specific value of $\frac{1}{2}$, see Th-121 and Th-127 below. In some applications, it might hence be preferable to sacrifice the propagation of fuzziness, in favour of a model with enhanced discriminatory force, which still remains useful when the input is overly fuzzy. This makes the first point for considering models beyond the $\mathcal{M}_{\mathcal{B}}$-type, in order to better serve such non-linguistic, special-purpose applications. Secondly, the investigation of additional models is also necessary in order to better relate existing work on fuzzy quantification to the axiomatic solution embarked upon here. The results of the previous chapter substantiate that the Sugeno integral and hence the ‘basic’ FG-count approach can be fitted into the proposed framework; they can be consistently generalized to the ‘hard’ cases of fuzzy quantification involving multi-place, non-quantitative and/or non-monotonic quantifiers. By generalizing the base construction for the $\mathcal{M}_{\mathcal{B}}$ models, and hence opening a richer class of models, it might be possible to prove a similar result for the Choquet integral and hence the ‘basic’ OWA approach. Thirdly, the study of a richer class of models is of great theoretical interest, because it also aims at a better understanding of the full class of standard models. A generalization of the constructive principle might separate those aspects of the models which are idiosyncratic to the chosen base mechanism, from those that express genuine facts about fuzzy quantification. Specifically, I would like to study some prototypical standard models which fail to propagate fuzziness in quantifiers and/or arguments, and gain some insight into the structure and properties of these models, which are rather distinct from $\mathcal{M}_{\mathcal{B}}$-DFSes. The bottom line is that broadening the considered class of models will also broaden the knowledge about fuzzy quantification in general.

In order to achieve the desired generalization of $\mathcal{M}_{\mathcal{B}}$-type models, the use of the fuzzy median in the definition of $Q_\gamma(X_1, \ldots, X_n)$ will be replaced with a more general construction. We get an idea of how to proceed if we simply expand the definition of the generalized fuzzy median and rewrite $Q_\gamma(X_1, \ldots, X_n)$ as

$$\begin{aligned} Q_\gamma(X_1, \ldots, X_n) \;\; &= \;\; \mathrm{med}_{\frac{1}{2}}(\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}, \\ &\qquad\quad \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\})\,. \end{aligned} \tag{64}$$

(This reformulation is justified by Def. 57 and Def. 83). The fuzzy median can then be replaced with other connectives, e.g. the arithmetic mean $(x + y)/2$. If we view $\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ and $\inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ as mappings that depend on γ, then we can even eliminate the pointwise application of the connective and define more ‘holistic’ mechanisms.

In the subsequent chapter, I will seize this basic suggestion, and implement a corresponding extension of $\mathcal{M}_{\mathcal{B}}$-DFSes. Following the usual scheme of developing classes of models, I will first motivate the new base construction, which spans the unrestricted class of $\mathcal{F}_\xi$-QFMs. Again, the full class of raw mechanisms will then be shrunk to the reasonable cases, by characterizing the $\mathcal{F}_\xi$-DFSes in terms of conditions imposed on the aggregation mapping ξ. After presenting examples of models, I will then develop the necessary formal machinery, to assess the relevant properties of $\mathcal{F}_\xi$-DFSes. Finally, the resulting techniques will be tested on the example models. As will emerge from this analysis, the novel class of models is rich enough to accomplish the three objectives which necessitated the quest for additional models. First of all, the two criteria of propagating fuzziness will no longer be tied to all models, and now become independent attributes which may or may not be possessed by the considered examples. The second goal will also be achieved, by identifying a prototypical model in the new class, which generalizes the Choquet integral. Finally, the introduction of the new class will also have a number of theoretical ramifications, and hence sheds some light into the unknown of fuzzy quantification.

8.2 The unrestricted class of $\mathcal{F}_\xi$-QFMs

As mentioned above, it was decided to build the new class from a generalisation of the construction for $\mathcal{M}_{\mathcal{B}}$-type models, which is already understood. Hence let us reconsider the definition of $Q_\gamma(X_1, \ldots, X_n)$, which underlies the construction of the known models. As demonstrated by equality (64) above, the application of the fuzzy median connective $\mathrm{med}_{\frac{1}{2}}$ can be separated from a prior computation of the supremum and infimum $\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ and $\inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$, respectively. By abstracting from the cutting parameter γ in these expressions, we arrive at the following definition of upper and lower bound mappings, which delimit the possible quantification results in the cut ranges:

Definition 100
Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and fuzzy arguments* $X_1, \dots, X_n$ *be given. I define the upper bound mapping* $\top_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ *and the lower bound mapping* $\bot_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ *by*

$$\begin{aligned}\top_{Q,X_1,\dots,X_n}(\gamma) &= \sup\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\}\\ \bot_{Q,X_1,\dots,X_n}(\gamma) &= \inf\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\}\,.\end{aligned}$$

The following properties of $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$ are apparent:

Theorem 107
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ *a choice of fuzzy arguments. Then*

1. $\top_{Q,X_1,\dots,X_n}$ *is monotonically nondecreasing;*
2. $\bot_{Q,X_1,\dots,X_n}$ *is monotonically nonincreasing;*
3. $\bot_{Q,X_1,\dots,X_n} \leq \top_{Q,X_1,\dots,X_n}$.

We can hence define the domain $\mathbb{T}$ of aggregation operators $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which combine the results of $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$ as follows.

Definition 101
$\mathbb{T} \subseteq \mathbf{I}^{\mathbf{I}} \times \mathbf{I}^{\mathbf{I}}$ *is defined by*

$$\mathbb{T} = \{(\top,\bot) : \ \top : \mathbf{I} \longrightarrow \mathbf{I} \textit{ nondecreasing, } \bot : \mathbf{I} \longrightarrow \mathbf{I} \textit{ nonincreasing, } \bot \leq \top\}\,.$$

It is apparent from Th-107 that $(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) \in \mathbb{T}$, regardless of the semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. In addition, it can be shown that $\mathbb{T}$ is the minimal set which embeds all such pairs of mappings.

Theorem 108
Let $(\top,\bot) \in \mathbb{T}$ *be given. We define semi-fuzzy quantifiers* $Q', Q'', Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *by*

$$Q'(Y) = \top(\sup Y') \tag{65}$$

$$Q''(Y) = \bot(\inf Y'') \tag{66}$$

$$Q(Y) = \begin{cases} Q''(Y) & : \ Y' = \varnothing \\ Q'(Y) & : \ \textit{else} \end{cases} \tag{67}$$

where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \tag{68}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \tag{69}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$.
Further suppose that the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ *is defined by*

$$\mu_X(c, z) = \begin{cases} \frac{1}{2} - \frac{1}{2}z & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2}z & : \quad c = 1 \end{cases} \tag{70}$$

for all $(c, z) \in \mathbf{2} \times \mathbf{I}$.
Then $\top = \top_{Q,X}$ *and* $\bot = \bot_{Q,X}$.

Based on the aggregation operator $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, we define a corresponding QFM $\mathcal{F}_\xi$ in the obvious way.

Definition 102
For every mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, *the QFM* $\mathcal{F}_\xi$ *is defined by*

$$\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}), \tag{71}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all fuzzy subsets* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

The class of QFMs defined in this way will be called the class of $\mathcal{F}_\xi$-QFMs. Apparently, it contains a number of models that do not fulfill the DFS axioms. I hence impose five elementary conditions on the aggregation mapping ξ which provide a characterisation of the well-behaved models, i.e. of the class of $\mathcal{F}_\xi$-DFSes.

8.3 Characterisation of $\mathcal{F}_\xi$-models

Definition 103 *For all* $(\top, \bot) \in \mathbb{T}$, *we impose the following conditions on aggregation mappings* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$.

If $\top = \bot$, *then* $\xi(\top, \bot) = \top(0)$ (X-1)

$\xi(1 - \bot, 1 - \top) = 1 - \xi(\top, \bot)$ (X-2)

If $\top = \mathrm{c}_1$ *and* $\bot(\mathbf{I}) \subseteq \{0, 1\}$, *then* $\xi(\top, \bot) = \frac{1}{2} + \frac{1}{2}\bot_*^0$ (X-3)

$\xi(\top^\flat, \bot) = \xi(\top^\sharp, \bot)$ (X-4)

If $(\top', \bot') \in \mathbb{T}$ *such that* $\top \le \top'$ *and* $\bot \le \bot'$, *then* $\xi(\top, \bot) \le \xi(\top', \bot')$ (X-5)

As stated in the following theorems, the conditions imposed on ξ capture exactly the requirements that make $\mathcal{F}_\xi$ a DFS. Let us show first that (X-1) to (X-5) are sufficient for $\mathcal{F}_\xi$ to be a standard model.

Theorem 109
If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5), *then* $\mathcal{F}_\xi$ *is a standard DFS.*

Theorem 110
The conditions (X-1) *to* (X-5) *on* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *are necessary for* $\mathcal{F}_\xi$ *to be a DFS.*

Hence the 'X-conditions' are necessary and sufficient for $\mathcal{F}_\xi$ to be a DFS, and all $\mathcal{F}_\xi$-DFSes are indeed standard DFSes. The criteria can also be shown to be independent. To facilitate the independence proof, I first relate $\mathcal{M}_\mathcal{B}$-QFMs to the broader class of $\mathcal{F}_\xi$-QFMs:

Theorem 111
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is a given aggregation mapping. Then* $\mathcal{M}_\mathcal{B} = \mathcal{F}_\xi$*, where* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is defined by*

$$\xi(\top, \bot) = \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) \tag{72}$$

for all $(\top, \bot) \in \mathbb{T}$*, and* $\mathrm{med}_{\frac{1}{2}}(\top, \bot)$ *abbreviates*

$$\mathrm{med}_{\frac{1}{2}}(\top, \bot)(\gamma) = \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma))\,,$$

for all $\gamma \in \mathbf{I}$.

Hence all $\mathcal{M}_\mathcal{B}$-QFMs are $\mathcal{F}_\xi$-QFMs, and all $\mathcal{M}_\mathcal{B}$-DFSes are $\mathcal{F}_\xi$-DFSes.
Sometimes we should be aware of the relationship between the 'B-conditions' and the 'X-conditions' in the case of $\mathcal{M}_\mathcal{B}$-QFMs. The next theorem helps us to prove that the 'X-conditions' are independent, because the 'B-conditions' have already been shown to be independent in [48]:

Theorem 112
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given and* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is defined by equality* (72)*. Then*

1. (B-1) *is equivalent to* (X-1)*;*
2. (B-2) *is equivalent to* (X-2)*;*
3. *(a)* (B-3) *entails* (X-3)*;*
   (b) the conjunction of (X-2) *and* (X-3) *entails* (B-3)*;*
4. *(a)* (B-4) *entails* (X-4)*;*
   (b) the conjunction of (X-2) *and* (X-4) *entails* (B-4)*;*
5. (B-5) *is equivalent to* (X-5)*.*

Note. The theorem also proved useful in other contexts, e.g. to show that the $\mathcal{M}_\mathcal{B}$-DFSes are exactly those $\mathcal{F}_\xi$-DFSes that propagate fuzziness in both quantifiers and arguments.

Theorem 113
The conditions (X-1) *to* (X-5) *are independent.*

8.4 Examples of $\mathcal{F}_\xi$-models

Let us now give examples of 'genuine' $\mathcal{F}_\xi$-DFSes (i.e. models that go beyond the special case of $\mathcal{M}_\mathcal{B}$-DFSes).

Definition 104
The QFM $\mathcal{F}_{\mathrm{Ch}} = \mathcal{F}_{\xi_{\mathrm{Ch}}}$ *is defined in terms of* $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ *by*

$$\xi_{\mathrm{Ch}}(\top, \bot) = \tfrac{1}{2}\int_0^1 \top(\gamma)\, d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\, d\gamma\,,$$

for all $(\top, \bot) \in \mathbb{T}$.

Note. Both integrals are guaranteed to exist because $\top$ and $\bot$ are monotonic and bounded.

Theorem 114
$\mathcal{F}_{\mathrm{Ch}}$ *is a standard DFS.*

The DFS $\mathcal{F}_{\mathrm{Ch}}$ is of special interest because of its close relationship to the well-known Choquet integral, which is defined as follows.

Definition 105
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is a nondecreasing semi-fuzzy quantifier and* $X \in \widetilde{\mathcal{P}}(E)$*. The* Choquet integral $(Ch)\int X\, dQ$ *is defined by*

$$(Ch)\int X\, dQ = \int_0^1 Q(X_{\geq\alpha})\, d\alpha\,.$$

Theorem 115
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is nondecreasing. Then for all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$(Ch)\int X\, dQ = \mathcal{F}_{\mathrm{Ch}}(Q)(X)\,.$$

Hence $\mathcal{F}_{\mathrm{Ch}}$ coincides with the Choquet integral on fuzzy quantifiers whenever the latter is defined. Recalling the notation $\mu_{[j]}(X)$ introduced in Def. 99, we then obtain the following corollary to the above theorem (cf. [20]):

Theorem 116
Suppose $E \neq \varnothing$ *is a finite base set,* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is a nondecreasing mapping such that* $q(0) = 0$*,* $q(|E|) = 1$*, and* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is defined by* $Q(Y) = q(|Y|)$ *for all* $Y \in \mathcal{P}(E)$*. Then for all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$\mathcal{F}_{\mathrm{Ch}}(Q)(X) = \sum_{j=1}^{|E|}(q(j) - q(j-1)) \cdot \mu_{[j]}(X)\,,$$

i.e. $\mathcal{F}_{\mathrm{Ch}}$ *consistently generalises Yager's OWA approach [170].*

Definition 106
The QFM $\mathcal{F}_S$ is defined in terms of $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ by

$$\xi_S(\top, \bot) = \begin{cases} \min(\top_1^*, \frac{1}{2} + \frac{1}{2}\bot_*^{\leq\frac{1}{2}}) & : \quad \bot(0) > \frac{1}{2} \\ \max(\bot_1^*, \frac{1}{2} - \frac{1}{2}\top_*^{\geq\frac{1}{2}}) & : \quad \top(0) < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$, where the coefficients $f_*^{\leq\frac{1}{2}}, f_*^{\geq\frac{1}{2}} \in \mathbf{I}$ are defined by

$$f_*^{\leq\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \leq \tfrac{1}{2}\} \tag{73}$$

$$f_*^{\geq\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \geq \tfrac{1}{2}\}\,, \tag{74}$$

for all $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 117
$\mathcal{F}_S$ is a standard DFS.

A third model of interest is the following QFM $\mathcal{F}_A$:

Definition 107
The QFM $\mathcal{F}_A$ is defined in terms of $\xi_A : \mathbb{T} \longrightarrow \mathbf{I}$ by

$$\xi_A(\top, \bot) = \begin{cases} \min(\bot_0^*, \frac{1}{2} + \frac{1}{2}\bot_*^0) & : \quad \bot_0^* > \frac{1}{2} \\ \max(\top_0^*, \frac{1}{2} - \frac{1}{2}\top_*^{1\downarrow}) & : \quad \top_0^* < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$.

Theorem 118
$\mathcal{F}_A$ is a standard DFS.

8.5 Properties of the $\mathcal{F}_\xi$-models

Turning to properties of $\mathcal{F}_\xi$-DFSes, I shall first investigate the precise conditions under which an $\mathcal{F}_\xi$-DFS propagates fuzziness in quantifiers and/or in arguments.

Definition 108
We say that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ propagates fuzziness *if and only if*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\top \preceq_c \top'$ and $\bot \preceq_c \bot'$.

Theorem 119
An $\mathcal{F}_\xi$-QFM propagates fuzziness in quantifiers if and only if ξ propagates fuzziness.

If $\mathcal{F}_\xi$ is a DFS, then ξ's propagating fuzziness is equivalent to the following condition, which is much easier to check:

Theorem 120
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-1) *to* (X-5)*. Then ξ propagates fuzziness if and only if*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

for all $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$.

For proofs that a given $\mathcal{F}_\xi$-DFS does not propagate fuzziness in quantifiers, the following necessary condition can be of interest.

Theorem 121
Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a mapping which satisfies (X-1) *to* (X-5)*. If ξ propagates fuzziness, then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $(\top, \bot) \in \mathbb{T}$ such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$.

It is this condition which explains why the results of $\mathcal{M}_\mathcal{B}$-DFSes tend to attain $\frac{1}{2}$ when the input is overly fuzzy. If one really needs different quantification results for $(\top, \bot), (\top', \bot')$ with $\bot(0) \leq \frac{1}{2} \leq \top(0)$ and $\bot'(0) \leq \frac{1}{2} \leq \top'(0)$, one obviously must resort to $\mathcal{F}_\xi$-DFSes that do not propagate fuzziness in quantifiers.
As concerns the examples of $\mathcal{F}_\xi$-models, one can attest the following.

Theorem 122
$\mathcal{F}_{\mathrm{Ch}}$ does not propagate fuzziness in quantifiers.

Hence $\mathcal{F}_{\mathrm{Ch}}$ is a 'genuine' $\mathcal{F}_\xi$-DFS (i.e. not an $\mathcal{M}_\mathcal{B}$-DFS) by Th-93. In particular, this proves that the $\mathcal{F}_\xi$-DFSes indeed form a more general class of models than $\mathcal{M}_\mathcal{B}$-DFSes. For the DFS $\mathcal{F}_S$, we obtain a positive result.

Theorem 123
$\mathcal{F}_S$ propagates fuzziness in quantifiers.

Turning to $\mathcal{F}_A$, we have

Theorem 124
$\mathcal{F}_A$ does not propagate fuzziness in quantifiers.

We can also state the necessary and sufficient conditions on ξ for $\mathcal{F}_\xi$ to propagate fuzziness in arguments. To this end, I first introduce the following property of ξ.

Definition 109
We say that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ propagates unspecificity *if and only if*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ *satisfy* $\top \geq \top'$ *and* $\bot \leq \bot'$*.*

Theorem 125
An $\mathcal{F}_\xi$*-QFM propagates fuzziness in arguments if and only if* ξ *propagates unspecificity.*

If $\mathcal{F}_\xi$ is sufficiently well-behaved (in particular, if $\mathcal{F}_\xi$ is a DFS), it is possible to state the following equivalent condition:

Theorem 126
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2), (X-4) *and* (X-5)*. Then the following conditions are equivalent:*

a. ξ *propagates unspecificity;*

b. for all $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) \geq \frac{1}{2}$, $\xi(\top, \bot) = \xi(\mathrm{c}_1, \bot)$.

I have also established a necessary condition which facilitates the proof that a given $\mathcal{F}_\xi$-DFSes does not propagate fuzziness in arguments:

Theorem 127
If an $\mathcal{F}_\xi$*-DFS propagates fuzziness in arguments, then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $(\top, \bot) \in \mathbb{T}$ *such that* $\top(0) \geq \frac{1}{2} \geq \bot(0)$*.*

For example, we can use this condition to prove that

Theorem 128
$\mathcal{F}_{\mathrm{Ch}}$ *does not propagate fuzziness in arguments.*

As concerns $\mathcal{F}_S$, we have the following result.

Theorem 129
$\mathcal{F}_S$ *does not propagate fuzziness in arguments.*

Note. Hence $\mathcal{F}_S$ is a ‘genuine’ $\mathcal{F}_\xi$-DFS as well, which is apparent from Th-94.
Turning to $\mathcal{F}_A$, which failed to propagate fuzziness in quantifiers, it is easily observed that $\mathcal{F}_A$ still propagates fuzziness in its arguments:

Theorem 130
$\mathcal{F}_A$ *propagates fuzziness in arguments.*

In particular, the conditions of propagating fuzziness in quantifiers and arguments are independent in the case of $\mathcal{F}_\xi$-DFSes, as stated in the following corollary.

Theorem 131
The conditions of propagating fuzziness in quantifiers and in arguments are independent for $\mathcal{F}_\xi$-DFSes.

Finally, we can characterize the subclass of $\mathcal{M}_\mathcal{B}$-DFSes which are exactly those $\mathcal{F}_\xi$-DFSes that propagate fuzziness in both quantifiers and arguments.

Theorem 132
Suppose an $\mathcal{F}_\xi$-DFS propagates fuzziness in both quantifiers and arguments. Then $\mathcal{F}_\xi$ is an $\mathcal{M}_\mathcal{B}$-DFS.

Note. The converse implication is already known from Th-93 and Th-94.
Next I shall investigate the exact conditions under which an $\mathcal{F}_\xi$-QFM is Q-continuous or arg-continuous. To be able to discuss Q-continuous $\mathcal{F}_\xi$-QFMs, we introduce a metric $d : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$. For all nondecreasing mappings $\top, \top' : \mathbf{I} \longrightarrow \mathbf{I}$, I define

$$d(\top, \top') = \sup\{|\top(\gamma) - \top'(\gamma)| : \gamma \in \mathbf{I}\}\,. \tag{75}$$

I proceed similarly for nondecreasing mappings $\bot, \bot' : \mathbf{I} \longrightarrow \mathbf{I}$. In this case,

$$d(\bot, \bot') = \sup\{|\bot(\gamma) - \bot'(\gamma)| : \gamma \in \mathbf{I}\}\,. \tag{76}$$

Finally, I define $d : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$ by

$$d((\top, \bot), (\top', \bot')) = \max(d(\top, \top'), d(\bot, \bot'))\,, \tag{77}$$

for all $(\top, \bot), (\top', \bot') \in \mathbb{T}$. It is apparent that d is indeed a metric. I will utilize d to express a condition on ξ which characterises the Q-continuous $\mathcal{F}_\xi$-QFMs.

Theorem 133
Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (X-5)*. Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi$ is Q-continuous;

b. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ satisfy $d((\top, \bot), (\top', \bot')) < \delta$.

If ξ is sufficiently well-behaved, then the above condition can be simplified to the following criterion, which is easier to check.

Theorem 134
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2) *and* (X-5)*. Then the following conditions are equivalent:*

a. *$\mathcal{F}_\xi$ is Q-continuous;*

b. *for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\xi(\top', \bot) - \xi(\top, \bot) < \varepsilon$ whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ satisfy $d(\top, \top') < \delta$ and $\top \leq \top'$.*

I have the following results for the examples of $\mathcal{F}_\xi$-DFSes.

Theorem 135
$\mathcal{F}_{\mathrm{Ch}}$ is Q-continuous.

Theorem 136
$\mathcal{F}_S$ is not Q-continuous.

Theorem 137
$\mathcal{F}_A$ is not Q-continuous.

As concerns continuity in arguments, I first need to introduce another distance measure $d' : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$, which can be used to characterise the arg-continuous $\mathcal{F}_\xi$-QFMs in terms of conditions on ξ. For all nondecreasing mappings $\top, \top' : \mathbf{I} \longrightarrow \mathbf{I}$, we define

$$d'(\top, \top') = \sup\{\inf\{\gamma' : \min(\top(\gamma'), \top'(\gamma')) \geq \max(\top(\gamma), \top'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\} . \tag{78}$$

Similarly for nonincreasing mappings $\bot, \bot' : \mathbf{I} \longrightarrow \mathbf{I}$,

$$d'(\bot, \bot') = \sup\{\inf\{\gamma' : \max(\bot(\gamma'), \bot'(\gamma')) \leq \min(\bot(\gamma), \bot'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\} . \tag{79}$$

Finally, we define $d' : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$ by

$$d'((\top, \bot), (\top', \bot')) = \max(d'(\top, \top'), d'(\bot, \bot')) , \tag{80}$$

for all $(\top, \bot), (\top', \bot') \in \mathbb{T}$. It is easily checked that d' is a 'pseudo-metric', i.e. it is symmetric and satisfies the triangular inequality, but $d'((\top, \bot), (\top', \bot')) = 0$ does not imply that $(\top, \bot) = (\top', \bot')$. However, d' is a metric modulo $\sharp\flat$, i.e. on the equivalence classes of $(\top, \bot) \sim (\top', \bot') \Leftrightarrow (\top^{\flat\sharp}, \bot^{\flat\sharp}) = (\top'^{\flat\sharp}, \bot'^{\flat\sharp})$. Hence $d'((\top, \bot), (\top', \bot')) = 0$ entails that $(\top, \bot) \sim (\top', \bot')$, i.e. $\xi(\top, \bot) = \xi(\top', \bot')$ whenever ξ satisfies (X-2), (X-4) and (X-5). Based on d', I can now assert the following.

Theorem 138
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2), (X-4) *and* (X-5). *Then the following conditions are equivalent:*

a. *$\mathcal{F}_\xi$ is arg-continuous;*

b. *for all $(\top, \bot) \in \mathbb{T}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top', \bot') \in \mathbb{T}$ satisfies $d'((\top, \bot), (\top', \bot')) < \delta$.*

In some cases, the following sufficient condition can shorten the proof that a given $\mathcal{F}_\xi$ is arg-continuous.

Theorem 139
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2) *and* (X-5) *Then* $\mathcal{F}_\xi$ *is arg-continuous if the following condition holds: For all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\xi(\top', \bot) - \xi(\top, \bot) < \varepsilon$ *whenever* $(\top, \bot), (\top', \bot) \in \mathbb{T}$ *satisfy* $d'(\top, \top') < \delta$ *and* $\top \leq \top'$.

Based on these theorems, it is easy to prove the following.

Theorem 140
$\mathcal{F}_{\mathrm{Ch}}$ *is arg-continuous.*

Theorem 141
$\mathcal{F}_S$ *is not arg-continuous.*

Theorem 142
$\mathcal{F}_A$ *is not arg-continuous.*

Hence $\mathcal{F}_{\mathrm{Ch}}$ is continuous both in quantifiers and arguments; which is important for applications. The second example, $\mathcal{F}_S$, fails at both continuity conditions and is hence not practical. (We will see below that $\mathcal{F}_S$ is of theoretical interest because it represents a boundary case of $\mathcal{F}_\xi$-DFS).
I am also interested in the specificity of $\mathcal{F}_\xi$-DFSes. The following theorem facilitates the proof that a given $\mathcal{F}_\xi$-QFM is less specific than another $\mathcal{F}_\xi$-QFM by relating the specificity order on $\mathcal{F}_\xi$ to the specificity order on ξ:

Theorem 143
Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ *be given mappings. Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$*;*

b. $\xi \preceq_c \xi'$.

In the case of $\mathcal{F}_\xi$-models that propagate fuzziness in quantifiers, it is sufficient to check a simpler condition.

Theorem 144
Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ *be given mappings which satisfy* (X-1) *to* (X-5) *and suppose that* ξ, ξ' *have the additional property that* $\xi(\top, \bot) = \xi'(\top, \bot) = \frac{1}{2}$ *whenever* $(\top, \bot) \in \mathbb{T}$ *with* $\top(0) \geq \frac{1}{2} \geq \bot(0)$. *Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$*;*

b. for all $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) > \frac{1}{2}$, $\xi(\top, \bot) \leq \xi'(\top, \bot)$.

As regards least specific $\mathcal{F}_\xi$-DFSes, we can prove the following:

Theorem 145
$\mathcal{M}_U$ is the least specific $\mathcal{F}_\xi$-DFS.

Turning to the issue of most specific models, I first state a theorem for establishing or rejecting specificity consistency. This is useful because specificity consistency is tightly coupled to the existence of least upper specificity bounds, see Th-42.

Theorem 146
Consider a pair of mappings $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$. The QFMs $\mathcal{F}_\xi$ and $\mathcal{F}_{\xi'}$ are specificity consistent if and only if ξ, ξ' are specificity consistent, i.e. for all $(\top, \bot) \in \mathbb{T}$, either $\{\xi(\top, \bot), \xi'(\top, \bot)\} \subseteq [0, \frac{1}{2}]$ or $\{\xi(\top, \bot), \xi'(\top, \bot)\} \subseteq [\frac{1}{2}, 1]$.

An investigation of a possible most specific $\mathcal{F}_\xi$-DFS reveals the following.

Theorem 147
The class of $\mathcal{F}_\xi$-DFSes is not specificity consistent.

Hence by Th-42, a 'most specific $\mathcal{F}_\xi$-DFS' does not exist. However, we obtain a positive result if we restrict attention to the class of $\mathcal{F}_\xi$-DFSes which propagate fuzziness in quantifiers or arguments. This is apparent from the following observation.

Theorem 148
Suppose $\mathbb{F}$ is a collection of $\mathcal{F}_\xi$-DFSes $\mathcal{F}_\xi \in \mathbb{F}$ with the property that $\xi(\top, \bot) = \frac{1}{2}$ whenever $(\top, \bot) \in \mathbb{T}$ is such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$. Then $\mathbb{F}$ is specificity consistent.

We then have the following corollaries.

Theorem 149
The class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers is specificity consistent.

Theorem 150
The class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in arguments is specificity consistent.

By Th-42, the $\mathcal{F}_\xi$-models that propagate fuzziness in quantifiers have a least upper specificity bound which, as it turns out, also propagates fuzziness in quantifiers.

Theorem 151
$\mathcal{F}_S$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in quantifiers.

Similarly, we can conclude from Th-150 that there is a most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments.

Theorem 152
$\mathcal{F}_A$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments.

8.6 Chapter summary

To sum up, the chapter made an effort to extend the class of known models, and to show that models exist which do not propagate fuzziness. In order to better judge the advances made, let me recall the reasons why I wanted to explore standard models beyond the original class of $\mathcal{M}_{\mathcal{B}}$-DFSes. The first reason was concerned with propagation of fuzziness. $\mathcal{M}_{\mathcal{B}}$-DFSes are particularly well-behaved in this respect because they propagate fuzziness in quantifiers as well as in arguments: the fuzzier the input, the fuzzier the output. In most cases, this is the expected and desirable behaviour because one usually does not want the results to become more precise when there is less precision in the input. However, I have already remarked that applications exist in which the propagation of fuzziness should be sacrificed, in order to prevent the results from attaining the least specific value of $\frac{1}{2}$. This might be appropriate, for example, when the input is overly fuzzy and one still needs a fine-grained result ranking. The extension of the original $\mathcal{M}_{\mathcal{B}}$-type models to a richer class was also necessary for relating the present approach to existing work on fuzzy quantification. In order to embed the Choquet integral and hence the core OWA approach into the axiomatic framework, the simple $\mathcal{M}_{\mathcal{B}}$-models proved to be insufficient, because the Choquet integral does not propagate fuzziness. Finally, it was expected that the study of a broader class would gain new insight into the structure of fuzzy quantification, which might contribute to a complete classification of standard models.

In order to define a suitable class of models, it was necessary to identify a corresponding constructive principle, which can then be instantiated into concrete models. I have hence reviewed those concepts which originally proved useful for defining $\mathcal{M}_{\mathcal{B}}$-DFSes. The construction of these models in terms of three-valued cuts provided a suitable starting point for the generalisation to a broader class of models. To this end, $Q_\gamma(X_1, \ldots, X_n)$ and $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ had to be replaced with a pair of upper and lower bound mappings $(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) \in \mathbb{T}$, along with an aggregation operator $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which maps such pairs into quantification results $\xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n})$. These mappings capture some important aspects of the quantifier and its intended behaviour for the considered fuzzy arguments. However, the important information is scattered across the cut levels and hence a subsequent aggregation step is needed. This is accomplished by applying a mapping ξ, which computes the final quantification result $\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n})$. In order to identify the subclass of well-behaved models within the unrestricted class of resulting $\mathcal{F}_\xi$-QFMs, I have then formalized the precise requirements on ξ that make $\mathcal{F}_\xi$ a DFS, by presenting a system of necessary and sufficient conditions. Due to the fact that these conditions are mutually independent, the proposed system avoids any redundancy in later proofs. I have also developed the full set of criteria required to check whether a given $\mathcal{F}_\xi$-type model propagates fuzziness in quantifiers and/or arguments and hence complies with the intuitive expectation that less detailed input should not result in more specific output; whether it satisfies the continuity requirements and hence shows a certain robustness against noise in the arguments or alternative interpretations of a fuzzy quantifier; and how it compares to other models in terms of specificity. In particular, I have shown that the class of $\mathcal{F}_\xi$-DFSes is broad enough to contain models which are rather different from $\mathcal{M}_{\mathcal{B}}$-DFSes. Among the $\mathcal{F}_\xi$-DFSes, some models neither propagate fuzziness in arguments nor in quantifiers; some models propagate fuzziness in quantifiers, but not in arguments, while others propagate fuzziness in arguments, but not in quantifiers. Of course, there are also models that satisfy both requirements. By investigating the latter class, I was able to locate the known $\mathcal{M}_{\mathcal{B}}$-DFSes within the full class of $\mathcal{F}_{\xi}$-DFSes, and characterize them as precisely those $\mathcal{F}_{\xi}$-DFSes which propagate fuzziness both in quantifiers and arguments.

My analysis of the new $\mathcal{F}_{\xi}$-type models also explains why the models which do not propagate fuzziness have a chance of performing better than those that propagate fuzziness in situations where the inputs are overly fuzzy. This can be mainly attributed to the property described in Th-121 and Th-127: if an $\mathcal{F}_{\xi}$-DFS propagates fuzziness in quantifiers or in arguments, then $\mathcal{F}_{\xi}(Q)(X_1, \ldots, X_n) = \frac{1}{2}$ whenever $\top_{Q,(X_1,\ldots,X_n)} \geq \frac{1}{2}$ and $\bot_{Q,(X_1,\ldots,X_n)} \leq \frac{1}{2}$. Hence there is a certain range in which the results of the given $\mathcal{F}_{\xi}$-DFS are constantly $\frac{1}{2}$, which can be undesirable if one needs a fine-grained result ranking. Because both types of propagating fuzziness cause this kind of behaviour, one must resort to models that fail at both conditions if one needs specific results even when there is a lot of fuzziness in the inputs. The DFS $\mathcal{F}_{\mathrm{Ch}}$ is a promising choice in such situations because it also fulfills the continuity requirements. It is anticipated that $\mathcal{F}_{\mathrm{Ch}}$ will find a number of uses in real-world applications that utilize fuzzy quantifiers. However, $\mathcal{F}_{\mathrm{Ch}}$ is also an interesting model from a theoretical point of view because the model can be shown to embed the Choquet integral, thus generalizing it to the case of non-monotonic and multi-place quantifiers. The chapter hence also succeeds in relating DFS theory with existing work on fuzzy quantification because the Choquet integral is known to embed the core OWA approach.

Concerning theoretical aspects of fuzzy quantification, the chapter proves that there are standard models beyond $\mathcal{M}_{\mathcal{B}}$-DFSes, and it also substantiates the existence of standard DFSes which do not propagate fuzziness in arguments and/or quantifiers. In particular, the conditions of propagating fuzziness in quantifiers and arguments have been shown to be independent for $\mathcal{F}_{\xi}$-type models. This is quite different from the original $\mathcal{M}_{\mathcal{B}}$-DFSes which are rather homogeneous. The mutual similarity of the $\mathcal{M}_{\mathcal{B}}$-models also expresses in their specificity consistency, a property which must also be abandoned when turning to broader classes of models. As witnessed by Th-147, the $\mathcal{F}_{\xi}$ models are not specificity consistent, and hence a 'most specific $\mathcal{F}_{\xi}$-DFS' does not exist. However, there is a most specific $\mathcal{F}_{\xi}$-DFS which propagates fuzziness in quantifiers, viz $\mathcal{F}_S$, and there is also a most specific $\mathcal{F}_{\xi}$-DFS which propagates fuzziness in arguments, viz $\mathcal{F}_A$. Apparently $\mathcal{F}_S$ and $\mathcal{F}_A$ are not practical models because they fail on both continuity conditions, but this seems to be typical for boundary cases with respect to specificity.

As concerns the collection of $\mathcal{F}_{\xi}$-DFSes in its entirety, it is not clear at this stage whether the new models form a 'natural' class with certain distinguished properties. However, the introduction of $\mathcal{F}_{\xi}$-DFSes clearly led to the discovery of relevant models, like $\mathcal{F}_{\mathrm{Ch}}$, which can be expressed in terms of the new construction. Most importantly, it turned out that the upper and lower bound mappings $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ are easy to compute for common quantifiers. This is witnessed, for example, by a successful implementation of absolute and proportional quantifiers in the model $\mathcal{F}_{\mathrm{Ch}}$, which is described in Chap. 11 below. Apart from its theoretical merits, I hence consider the new $\mathcal{F}_\xi$-DFSes a fruitful source of practical models, which will prove useful in future applications.

9 The full class of models defined in terms of three-valued cuts

9.1 Motivation and chapter overview

In this chapter, a further step will be taken to extend the class of known models. By abstracting from the mechanism used to define $\mathcal{F}_\xi$-QFMs, I first introduce the full class of QFMs definable in terms of three-valued cuts: the class of $\mathcal{F}_\Omega$-QFMs. Unlike $\mathcal{F}_\xi$-QFMs, the definition of which is based on the upper and lower bounds on the results obtained for the three-valued cuts and a subsequent aggregation step, the new models are defined directly in terms of the 'raw' result set obtained for the cuts, to which an aggregation mapping Ω is then applied. Hence the new approach captures all models definable in terms of three-valued cuts, and promises to span a general class of models worthwhile investigating. After introducing the surrounding class of $\mathcal{F}_\Omega$-QFMs, the structure of its well-behaved members is then analysed, by making explicit the necessary and sufficient conditions on the aggregation mapping Ω that make $\mathcal{F}_\Omega$ a DFS. In addition, the required theory will be developed that permits us to check interesting properties of $\mathcal{F}_\Omega$-models, e.g. whether a given $\mathcal{F}_\Omega$ propagates fuzziness, and how given $\mathcal{F}_\Omega$-QFMs are related in terms of specificity. It is shown that the new class of models is genuinely broader than $\mathcal{F}_\xi$-DFSes. However, it does not introduce any new 'practical' models because those $\mathcal{F}_\Omega$-DFSes which are Q-continuous, and hence potentially suited for applications, are in fact $\mathcal{F}_\xi$-DFSes. These findings hence provide a justification for $\mathcal{F}_\xi$-QFMs. It is also shown that the full class of standard models which propagate fuzziness both in quantifiers and arguments, is genuinely broader than the class of $\mathcal{M}_\mathcal{B}$-DFSes. But again, all models outside the known range of models fail to be Q-continuous. Apart from investigating these properties, a subclass of $\mathcal{F}_\Omega$-QFMs will also be introduced, the class of $\mathcal{F}_\omega$-QFMs. These QFMs can be expressed in terms of a simpler construction which excludes some of the 'raw' $\mathcal{F}_\Omega$-QFMs. I show how this subclass is related to the full class of $\mathcal{F}_\Omega$-QFMs. Among other things, this investigation reveals that the considered subclass still contains all well-behaved models, and hence the $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFSes coincide. The relevance of $\mathcal{F}_\omega$-QFMs stems from the fact that they can easily be linked to the alternative classes of models introduced later on. In other words, $\mathcal{F}_\omega$-QFMs are needed to establish the link between the models defined in terms of three-valued cuts and those defined in terms of the extension principle. An investigation of $\mathcal{F}_\omega$-QFMs is hence essential to the proof that these classes coincide, which is one of the main contributions to DFS theory made in this chapter.

9.2 The unrestricted class of $\mathcal{F}_\Omega$-QFMs

To begin with, I will now extend the class of $\mathcal{F}_\xi$-QFMs to the full class of QFMs definable in terms of three-valued cuts of the argument sets. Hence let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. In order to spot a starting point for the desired generalization, we re-consider the definition of $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$. Apparently, the upper and lower bound mappings can be decomposed into (a) the three-valued cut mechanism, and (b) a subsequent $\inf/\sup$-based aggregation:

$$\begin{aligned} &\top_{Q,X_1,\ldots,X_n}(\gamma) \\ &= \sup\{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}_\gamma(X_1,\ldots,X_n)\} \\ &= \sup \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \end{aligned} \tag{81}$$

and

$$\begin{aligned} &\bot_{Q,X_1,\ldots,X_n}(\gamma) \\ &= \inf\{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}_\gamma(X_1,\ldots,X_n)\} \\ &= \inf \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \end{aligned} \tag{82}$$

for all $\gamma \in \mathbf{I}$, provided we define $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ as follows.

Definition 110
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$, the mapping $\mathcal{S}_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by

$$\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) = \{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}_\gamma(X_1,\ldots,X_n)\}\,,$$

for all $\gamma \in \mathbf{I}$.

Some basic properties of $\mathcal{S}_{Q,X_1,\ldots,X_n}$ are stated in this theorem.

Theorem 153
Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and choice of fuzzy subsets $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. Then

a. $\mathcal{S}_{Q,X_1,\ldots,X_n}(0) \neq \varnothing$;

b. $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \subseteq \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma')$ *whenever* $\gamma, \gamma' \in \mathbf{I}$ *with* $\gamma \leq \gamma'$.

It is hence apparent that all possible choices of $\mathcal{S}_{Q,X_1,\ldots,X_n}$ are contained in the following set $\mathbb{K}$.

Definition 111
$\mathbb{K} \subseteq \mathcal{P}(\mathbf{I})^{\mathbf{I}}$ *is defined by*

$$\mathbb{K} = \{\mathcal{S} \in \mathcal{P}(\mathbf{I})^{\mathbf{I}} : \mathcal{S}(0) \neq \varnothing \text{ and } \mathcal{S}(\gamma) \subseteq \mathcal{S}(\gamma') \text{ whenever } \gamma \leq \gamma'\}\,.$$

As I will now state, $\mathbb{K}$ is the minimal set which contains all possible choices for $\mathcal{S}_{Q,X_1,\ldots,X_n}$. To this end, I first have to introduce coefficients $s(z) \in \mathbf{I}$ associated with $\mathcal{S} \in \mathbb{K}$, which will play an essential role throughout this chapter.

Definition 112
Consider $\mathcal{S} \in \mathbb{K}$. We associate with $\mathcal{S}$ a mapping $s : \mathbf{I} \longrightarrow \mathbf{I}$ defined by

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in \mathcal{S}(\gamma)\}\,,$$

for all $z \in \mathbf{I}$.

It is convenient to define a notation for the $s(z)$'s obtained from a given quantifier and arguments.

Definition 113
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$*, we denote the mapping* s *obtained from* $\mathbb{S}_{Q,X_1,\dots,X_n}$ *by applying Def. 112 by* $s_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$*. The resulting mapping is hence defined by*

$$s_{Q,X_1,\dots,X_n}(z) = \inf\{\gamma \in \mathbf{I} : z \in \mathbb{S}_{Q,X_1,\dots,X_n}(\gamma)\}\,,$$

for all $z \in \mathbf{I}$*.*

As we shall see later, all $\mathcal{F}_\Omega$-DFSes can be defined in terms of $s_{Q,X_1,\dots,X_n}$.

Theorem 154
Let $S \in \mathbb{K}$ *be given and define* $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *by*

$$Q(Y) = Q_{\inf Y'}(Y'') \tag{83}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$*, where*

$$Y' = \{y \in \mathbf{I} : (0, y) \in Y\} \tag{84}$$

$$Y'' = \{y \in \mathbf{I} : (1, y) \in Y\} \tag{85}$$

and the $Q_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$*,* $z \in \mathbf{I}$ *are defined by*

$$Q_z(Y'') = \begin{cases} z & : & \sup Y'' > s(z) \\ z_0 & : & \textit{else} \end{cases} \tag{86}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$ *if* $z \notin S(s(z))$*, and*

$$Q_z(Y'') = \begin{cases} z & : & \sup Y'' \geq s(z) \\ z_0 & : & \textit{else} \end{cases} \tag{87}$$

in the case that $z \in S(s(z))$*.* z_0 *is an arbitrary element*

$$z_0 \in S(0)\,, \tag{88}$$

which exists by Th-153. Further suppose that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ *is defined by*

$$\mu_X(a, y) = \begin{cases} \frac{1}{2} & : & a = 0 \\ \frac{1}{2} - \frac{1}{2}y & : & a = 1 \end{cases} \tag{89}$$

for all $a \in \mathbf{2}$*,* $y \in \mathbf{I}$*. Then* $\mathbb{S}_{Q,X} = S$*.*

Hence $\mathbb{K}$ is exactly the set of all $S = \mathbb{S}_{Q,X_1,\dots,X_n}$ obtained for arbitrary choices of quantifiers and arguments. In order to obtain a quantification result from $\mathbb{S}_{Q,X_1,\dots,X_n}$, I apply an aggregation operator $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ in the obvious way.

Definition 114
Consider an aggregation operator $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$. *The corresponding QFM* $\mathcal{F}_\Omega$ *is defined by*

$$\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) = \Omega(S_{Q,X_1,\ldots,X_n}) ,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and fuzzy arguments* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{F}_\Omega$-QFMs, I mean the collection of all QFMs defined in this way. As usual, we must impose conditions to shrink the full class of $\mathcal{F}_\Omega$ to its subclass of $\mathcal{F}_\Omega$-DFSes.

9.3 Characterisation of the $\mathcal{F}_\Omega$-models

Definition 115
For all $S \in \mathbb{K}$, *we define* $S^\sharp, S^\flat \in \mathbb{K}$ *as follows.*

$$S^\sharp = \begin{cases} \cap\{S(\gamma') : \gamma' > \gamma\} & : \quad \gamma < 1 \\ \mathbf{I} & : \quad \gamma = 1 \end{cases} \qquad S^\flat = \begin{cases} S(0) & : \quad \gamma = 0 \\ \cup\{S(\gamma') : \gamma' < \gamma\} & : \quad \gamma > 0 \end{cases}$$

for all $\gamma \in \mathbf{I}$.

Note. The definition is slightly asymmetric; I have departed from the usual scheme of defining $S^\sharp(1) = S(1)$ in this case because the present definition of $S^\sharp(1) = \mathbf{I}$ allows more compact conditions on Ω, and eventually for shorter proofs.
I further stipulate a definition of $S \sqsubseteq S'$ which will serve to express a monotonicity condition on Ω.

Definition 116
For all $S, S' \in \mathbb{K}$, *let us say that* $S \sqsubseteq S'$ *if and only if the following two conditions are valid for all* $\gamma \in \mathbf{I}$:

1. *for all* $z \in S(\gamma)$, *there exists* $z' \in S'(\gamma)$ *with* $z' \geq z$;
2. *for all* $z' \in S'(\gamma)$, *there exists* $z \in S(\gamma)$ *with* $z \leq z'$.

It is apparent from this definition that $\sqsubseteq$ is reflexive and transitive, but not necessarily antisymmetric (i.e. $S \sqsubseteq S'$ and $S' \sqsubseteq S$ does not imply that $S = S'$). Hence $\sqsubseteq$ is a preorder.
We are now ready to state the conditions on reasonable choices of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, in analogy to the conditions (B-1)–(B-5) for $\mathcal{M}_\mathcal{B}$-models and to the conditions (X-1)–(X-5) for the $\mathcal{F}_\xi$-type:

Definition 117 *Consider* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$. *We impose the following conditions on* Ω. *For* *all* $S \in \mathbb{K}$*,*

$$\textit{If there exists } a \in \mathbf{I} \textit{ with } S(\gamma) = \{a\} \textit{ for all } \gamma \in \mathbf{I}\textit{, then } \Omega(S) = a. \tag{Ω-1}$$

$$\textit{If } S'(\gamma) = \{1 - z : z \in S(\gamma)\} \textit{ for all } \gamma \in \mathbf{I}\textit{, then } \Omega(S') = 1 - \Omega(S). \tag{Ω-2}$$

$$\textit{If } 1 \in S(0) \textit{ and } S(\gamma) \subseteq \{0, 1\} \textit{ for all } \gamma \in \mathbf{I}\textit{, then } \Omega(S) = \tfrac{1}{2} + \tfrac{1}{2}s(0). \tag{Ω-3}$$

$$\Omega(S) = \Omega(S^\sharp) \tag{Ω-4}$$

$$\textit{If } S' \in \mathbb{K} \textit{ satisfies } S \sqsubseteq S'\textit{, then } \Omega(S) \le \Omega(S'). \tag{Ω-5}$$

Note. The only condition which is slighly different from the usual scheme is (Ω-4). The departure from requiring $\Omega(S^\sharp) = \Omega(S^\flat)$ turned out to shorten the proofs. The latter equality is entailed by the above conditions, however.

Theorem 155
The conditions (Ω-1)–(Ω-5) *on* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *are sufficient for* $\mathcal{F}_\Omega$ *to be a standard DFS.*

In the following, I introduce another construction which elucidates the exact properties of $S \in \mathbb{K}$ that a conforming choice of Ω can rely on.

Definition 118
For all $S \in \mathbb{K}$*, we define* $S^\ddagger \in \mathbb{K}$ *by*

$$S^\ddagger(\gamma) = \{z \in \mathbf{I} : \textit{there exist } z', z'' \in S(\gamma) \textit{ with } z' \le z \le z''\}$$

for all $\gamma \in \mathbf{I}$*.*

Note. It is apparent that indeed $S^\ddagger \in \mathbb{K}$. The effect of applying ‡ to S is that of ‘filling the gaps’ in the interior of S. The resulting $S^\ddagger$ will be a closed, half-open, or open interval.

The importance of this construction with respect to $\mathcal{F}_\Omega$-QFMs stems from the invariance of well-behaved $\mathcal{F}_\Omega$-QFMs with respect to the gap-filling operation:

Theorem 156
Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is a given mapping such that* $\mathcal{F}_\Omega$ *satisfies* (Z-5)*. Then*

$$\Omega(S) = \Omega(S^\ddagger)\,,$$

for all $S \in \mathbb{K}$*.*

This means that a well-behaved choice of Ω may only depend on $\sup S(\gamma)$, $\inf S(\gamma)$, and the knowledge whether $\sup S(\gamma) \in S(\gamma)$ and $\inf S(\gamma) \in S(\gamma)$. Apart from this, the ‘interior structure’ of $S(\gamma)$ is irrelevant to the determination of $\Omega(S)$.

The above gap-filling operation has also proven useful for proving that (Ω-5) is necessary for $\mathcal{F}_\Omega$ to satisfy (Z-5). The other ‘Ω-conditions’ are easily shown to be necessary for $\mathcal{F}_\Omega$ to be a DFS, and require only minor adjustments of the corresponding proofs for $\mathcal{F}_\xi$-QFMs that were presented in [50].

Theorem 157
The conditions (Ω-1)–(Ω-5) *on* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *are necessary for* $\mathcal{F}_\Omega$ *to be a DFS.*

Hence the 'Ω-conditions' are necessary and sufficient for $\mathcal{F}_\Omega$ to be a DFS, and all $\mathcal{F}_\Omega$-DFSes are indeed standard models. In order to prove that the criteria are independent, I relate $\mathcal{F}_\xi$-QFMs to their apparent superclass of $\mathcal{F}_\Omega$-QFMs.

Theorem 158
Consider an aggregation mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$*. Then* $\mathcal{F}_\xi = \mathcal{F}_\Omega$*, where* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is defined by*

$$\Omega(S) = \xi(\top_S, \bot_S)\,, \tag{90}$$

for all $S \in \mathbb{K}$*, and* $(\top_S, \bot_S) \in \mathbb{T}$ *is defined by*

$$\top_S(\gamma) = \sup S(\gamma) \tag{91}$$
$$\bot_S(\gamma) = \inf S(\gamma) \tag{92}$$

for all $\gamma \in \mathbf{I}$.

This is apparent. Hence all $\mathcal{F}_\xi$-QFMs are $\mathcal{F}_\Omega$-QFMs and all $\mathcal{F}_\xi$-DFSes are $\mathcal{F}_\Omega$-DFSes. The next theorem permits to reduce the independence proof of the conditions on Ω to the independence proof of the conditions imposed on ξ.

Theorem 159
Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is given and* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is defined by* (90)*. Then*

a. (X-1) *is equivalent to* (Ω-1)*;*

b. (X-2) *is equivalent to* (Ω-2)*;*

c. (X-3) *is equivalent to* (Ω-3)*;*

d.
   1. *the conjunction of* (X-2), (X-4) *and* (X-5) *implies* (Ω-4)*;*
   2. (Ω-4) *implies* (X-4)*;*

e. (X-5) *is equivalent to* (Ω-5)*.*

Based on this theorem and the known independence of the conditions (X-1)–(X-5), it is now easy to prove the desired result concerning independence.

Theorem 160
The conditions (Ω-1)–(Ω-5) *imposed on* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *are independent.*

9.4 The unrestricted class of $\mathcal{F}_\omega$-QFMs

As has been remarked above, every $\mathcal{F}_\Omega$-DFS can be defined in terms of the mapping $s_{Q,(X_1,\dots,X_n)} : \mathbf{I} \longrightarrow \mathbf{I}$ and this usually makes a simpler representation. It therefore makes sense to introduce the class of QFMs definable in terms of $s_{Q,(X_1,\dots,X_n)} : \mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 161
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments. Then ${s_{Q,X_1,\dots,X_n}}^{-1}(0) \neq \varnothing$, i.e. there exists $z_0 \in \mathbf{I}$ with $s_{Q,X_1,\dots,X_n}(z_0) = 0$.

Hence all possible choices of $s_{Q,X_1,\dots,X_n}$ are contained in the following set $\mathbb{L}$.

Definition 119
$\mathbb{L} \subseteq \mathbf{I}^\mathbf{I}$ is defined by

$$\mathbb{L} = \{s \in \mathbf{I}^\mathbf{I} : s^{-1}(0) \neq \varnothing\} .$$

The following theorem states that $\mathbb{L}$ is the minimum subset of $\mathbf{I}^\mathbf{I}$ which contains all possible mappings $s_{Q,X_1,\dots,X_n}$:

Theorem 162
For all $s \in \mathbb{L}$, let us define $S : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{93}$$

for all $\gamma \in \mathbf{I}$. It is apparent that $S \in \mathbb{K}$. Let us further suppose that $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ is defined by (83) *and that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ is the fuzzy subset defined by* (89)*. Then $s_{Q,X} = s$.*

In order to define quantification results based on $s_{Q,X_1,\dots,X_n}$, we need an aggregation mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. The corresponding QFM $\mathcal{F}_\omega$ is defined in the usual way.

Definition 120
Let a mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given. By $\mathcal{F}_\omega$ we denote the QFM defined by

$$\mathcal{F}_\omega(Q)(X_1, \dots, X_n) = \omega(s_{Q,X_1,\dots,X_n}) ,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

9.5 The classes of $\mathcal{F}_\omega$-models and $\mathcal{F}_\Omega$-models coincide

It is obvious from the definition of $s_{Q,X_1,\dots,X_n}$ in terms of $S_{Q,X_1,\dots,X_n}$ that all $\mathcal{F}_\omega$-QFMs are $\mathcal{F}_\Omega$-QFMs, using the apparent choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$,

$$\Omega(S) = \omega(s) \tag{94}$$

where $s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\}$, see Def. 113. It is then clear from Def. 114 and Def. 120 that

$$\mathcal{F}_\Omega = \mathcal{F}_\omega \,. \tag{95}$$

The converse is not true, i.e. it is not the case that all $\mathcal{F}_\Omega$-QFMs are $\mathcal{F}_\omega$-QFMs. However, if an $\mathcal{F}_\Omega$-QFMs is sufficiently 'well-behaved', then it is also an $\mathcal{F}_\omega$-QFM. In particular, this is the case for an $\mathcal{F}_\Omega$-DFS.

Theorem 163

a. *If* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-4)*, then* $\mathcal{F}_\Omega = \mathcal{F}_\omega$*, provided we define* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *by*

$$\omega(s) = \Omega(S) \tag{96}$$

for all $s \in \mathbb{L}$*, where*

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{97}$$

for all $\gamma \in \mathbf{I}$.

b. *If* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *does not satisfy* (Ω-4)*, then* $\mathcal{F}_\Omega$ *is not an* $\mathcal{F}_\omega$*-QFM.*

Therefore an $\mathcal{F}_\Omega$-QFM is an $\mathcal{F}_\omega$-QFM if and only if it satisfies (Ω-4). Let us recall that by Th-157, (Ω-4) is necessary for $\mathcal{F}_\Omega$ to be a DFS. This means that we do not lose any models of interest if we restrict attention to the class of those $\mathcal{F}_\Omega$-QFMs which satisfy (Ω-4), and can hence be expressed as $\mathcal{F}_\omega$-QFMs.

9.6 Characterisation of the $\mathcal{F}_\omega$-models

It is then convenient to switch from (Ω-1)–(Ω-5) to corresponding conditions on $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. To accomplish this, I first define a preorder $\sqsubseteq \subseteq \mathbb{L} \times \mathbb{L}$, which is needed to express a monotonicity condition.

Definition 121
For all $s, s' \in \mathbb{L}$*,* $s \sqsubseteq s'$ *if and only if the following two conditions hold:*

a. *for all* $z \in \mathbf{I}$, $\inf\{s'(z') : z' \geq z\} \leq s(z)$*;*

b. *for all* $z' \in \mathbf{I}$, $\inf\{s(z) : z \leq z'\} \leq s'(z')$.

In the case of $\mathcal{F}_\omega$-QFMs, I can express the conditions on $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ even more succintly.

Definition 122 *We impose the following conditions on* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$*. For all* $s \in \mathbb{L}$*,*

$$\text{If } s^{-1}([0,1)) = \{a\}, \text{ then } \omega(s) = a. \tag{ω-1}$$

$$\text{If } s'(z) = s(1-z) \text{ for all } z \in \mathbf{I}, \text{ then } \omega(s') = 1 - \omega(s). \tag{ω-2}$$

$$\text{If } s(1) = 0 \text{ and } s^{-1}([0,1)) \subseteq \{0,1\}, \text{ then } \omega(s) = \tfrac{1}{2} + \tfrac{1}{2}s(0). \tag{ω-3}$$

$$\text{If } s' \in \mathbb{L} \text{ with } s \sqsubseteq s', \text{ then } \omega(s) \leq \omega(s'). \tag{ω-4}$$

Theorem 164
Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be given and suppose that* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is defined in terms of* ω *according to* (94)*. Then*

a. Ω *satisfies* (Ω-1) *if and only if* ω *satisfies* (ω-1)*;*

b. Ω *satisfies* (Ω-2) *if and only if* ω *satisfies* (ω-2)*;*

c. Ω *satisfies* (Ω-3) *if and only if* ω *satisfies* (ω-3)*;*

d. Ω *satisfies* (Ω-4)*;*

e. Ω *satisfies* (Ω-5) *if and only if* ω *satisfies* (ω-4)*.*

Due to these relationships, the following theorems are obvious from the corresponding results for Ω.

Theorem 165
The conditions (ω-1)–(ω-4) *are sufficient for* $\mathcal{F}_\omega$ *to be a standard DFS.*

Theorem 166
The conditions (ω-1)–(ω-4) *are necessary for* $\mathcal{F}_\omega$ *to be a DFS.*

Theorem 167
The conditions (ω-1)–(ω-4) *are independent.*

To sum up, $\mathcal{F}_\omega$-DFSes comprise all $\mathcal{F}_\Omega$-DFSes, they are usually easier to define, and simpler conditions (ω-1)–(ω-4) have to be checked. However, the monotonicity condition (ω-4) on ω is somewhat more complicated compared to the monotonicity condition (Ω-5) on Ω. In the following, I hence introduce a simpler preorder $\trianglelefteq$ for expressing monotonicity, which when combined with an additional condition can replace $\sqsubseteq$ and the corresponding monotonicity condition (ω-4). $\trianglelefteq$ is defined as follows.

Definition 123
For all $s, s' \in \mathbb{L}$, $s \trianglelefteq s'$ *if and only if the following two conditions hold:*

a. for all $z \in \mathbf{I}$*, there exists* $z' \geq z$ *with* $s'(z') \leq s(z)$*;*

b. for all $z' \in \mathbf{I}$*, there exists* $z \leq z$ *with* $s(z) \leq s'(z')$*.*

In order to state the additional condition, it is necessary to introduce a construction on $s \in \mathbb{L}$ which corresponds to the gap-filling operation $S^{\ddagger}$ defined on $S \in \mathbb{K}$.

Definition 124
For all $s \in \mathbb{L}$, $s^{\ddagger} : \mathbf{I} \longrightarrow \mathbf{I}$ is defined by

$$s^{\ddagger}(z) = \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z'') : z'' \geq z\}) \,,$$

for all $z \in \mathbf{I}$.

Some basic properties of ‡ are the following.

Theorem 168
Let $s \in \mathbb{L}$ be given. Then

- *a.* $s^{\ddagger} \leq s$;
- *b.* $s^{\ddagger} \in \mathbb{L}$;
- *c.* $s^{\ddagger}$ *is concave, i.e.* $s^{\ddagger}(z_2) \leq \max(s^{\ddagger}(z_1), s^{\ddagger}(z_3))$ *whenever* $z_1 \leq z_2 \leq z_3$.

I have needed this concavification construction for the proof that ω's satisfying (ω-4) entails that Ω defined by (94) satisfies (Ω-5). However, it will also play its role in defining examples of $\mathcal{F}_\omega$ models, see Def. 125 and Def. 128. The connection between ‡ and monotonic behaviour of ω becomes visible in the next theorem, which facilitates the proof that a given ω satisfies (ω-4), by reducing it to the ‡-invariance of ω, and its monotonicity with respect to the simplified preorder $\trianglelefteq$.

Theorem 169

For all $\omega : \mathbb{L} \longrightarrow \mathbf{I}$, the monotonicity condition (ω-4) is equivalent to the conjunction of the following two conditions:

- *a. for all $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$, it holds that $\omega(s) \leq \omega(s')$;*
- *b. for all $s \in \mathbb{L}$, $\omega(s^{\ddagger}) = \omega(s)$.*

9.7 Examples of $\mathcal{F}_\Omega$-models

I will now present four examples of 'genuine' $\mathcal{F}_\omega$-models, i.e. of $\mathcal{F}_\omega$-DFSes which do not belong to the class of $\mathcal{F}_\xi$-DFSes. To this end, it is necessary to introduce some coefficients defined in terms of a given $s \in \mathbb{L}$.

Definition 125
For all $s \in \mathbb{L}$, the coefficients $s_*^{\top,0}, s_*^{\perp,0}, {}_{,1}^{\top,*}{}_{,1}^{\perp,*} s_*^{\leq\frac{1}{2}}, s_*^{\geq\frac{1}{2}} \in \mathbf{I}$ are defined by

$$s_*^{\top,0} = \sup s^{\ddagger^{-1}}(0) \tag{98}$$

$$s_*^{\perp,0} = \inf s^{\ddagger^{-1}}(0) \tag{99}$$

$$s_1^{\top,*} = \sup s^{-1}([0,1)) \tag{100}$$

$$s_1^{\perp,*} = \inf s^{-1}([0,1)) \tag{101}$$

$$s_*^{\leq\frac{1}{2}} = \inf\{s(z) : z \leq \tfrac{1}{2}\} \tag{102}$$

$$s_*^{\geq\frac{1}{2}} = \inf\{s(z) : z \geq \tfrac{1}{2}\}\,. \tag{103}$$

Based on these coefficients, I now define the examples of $\mathcal{F}_\omega$-models.

Definition 126
By $\omega_M : \mathbb{L} \longrightarrow \mathbf{I}$ we denote the following mapping,

$$\omega_M(s) = \begin{cases} \min(s_*^{\perp,0}, \frac{1}{2} + \frac{1}{2} s_*^{\leq\frac{1}{2}}) & : \quad s_*^{\perp,0} > \frac{1}{2} \\ \max(s_*^{\top,0}, \frac{1}{2} - \frac{1}{2} s_*^{\geq\frac{1}{2}}) & : \quad s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. The QFM $\mathcal{F}_M$ is defined in terms of ω_M according to Def. 120, i.e. $\mathcal{F}_M = \mathcal{F}_{\omega_M}$.

Let us first notice that the QFM $\mathcal{F}_M$ so defined is indeed a DFS.

Theorem 170
$\mathcal{F}_M$ is a standard DFS.

Let me also remark that $\mathcal{F}_M$ is indeed a ‘genuine’ $\mathcal{F}_\omega$-DFS.

Theorem 171
$\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_M = \mathcal{F}_\xi$.

In particular, this proves that the $\mathcal{F}_\omega$-DFSes are really more general than $\mathcal{F}_\xi$-DFSes, i.e. the $\mathcal{F}_\xi$-DFSes form a proper subclass of the $\mathcal{F}_\omega$-DFSes.

Definition 127
By $\omega_P : \mathbb{L} \longrightarrow \mathbf{I}$ we denote the mapping defined by

$$\omega_P(s) = \begin{cases} \min(s_1^{\top,*}, \frac{1}{2} + \frac{1}{2} s_*^{\leq\frac{1}{2}}) & : \quad s_*^{\perp,0} > \frac{1}{2} \\ \max(s_1^{\perp,*}, \frac{1}{2} - \frac{1}{2} s_*^{\geq\frac{1}{2}}) & : \quad s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. *We define the QFM* $\mathcal{F}_P$ *in terms of* ω_P *according to Def. 120, i.e.* $\mathcal{F}_P = \mathcal{F}_{\omega_P}$.

Theorem 172
$\mathcal{F}_P$ *is a standard DFS.*

Let us also observe that $\mathcal{F}_P$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 173
$\mathcal{F}_P$ *is not an* $\mathcal{F}_\xi$*-DFS, i.e. there exists no* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *such that* $\mathcal{F}_P = \mathcal{F}_\xi$.

It is possible to obtain an even more specific DFS by slightly changing the definition of $\mathcal{F}_P$.

Definition 128
By $\omega_Z : \mathbb{L} \longrightarrow \mathbf{I}$ *we denote the mapping defined by*

$$\omega_Z(s) = \begin{cases} \min(s_1^{\top,*}, \frac{1}{2} + \frac{1}{2} s_*^{\leq\frac{1}{2}}) & : \quad s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1] \\ \max(s_1^{\perp,*}, \frac{1}{2} - \frac{1}{2} s_*^{\geq\frac{1}{2}}) & : \quad s^{\ddagger^{-1}}(0) \subseteq [0, \frac{1}{2}] \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. *We define the QFM* $\mathcal{F}_Z$ *in terms of* ω_Z *according to Def. 120, i.e.* $\mathcal{F}_Z = \mathcal{F}_{\omega_Z}$.

Note. It has been shown in [51, p. 43+] that ω_Z is well-defined.

Theorem 174
$\mathcal{F}_Z$ *is a standard DFS.*

Again, it is easily shown that $\mathcal{F}_Z$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 175
$\mathcal{F}_Z$ *is not an* $\mathcal{F}_\xi$*-DFS, i.e. there exists no* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *such that* $\mathcal{F}_Z = \mathcal{F}_\xi$.

Definition 129
By $\omega_R : \mathbb{L} \longrightarrow \mathbf{I}$ *we denote the mapping defined by*

$$\omega_R(s) = \begin{cases} \min(s_*^{\perp,0}, \frac{1}{2} + \frac{1}{2} s(0)) & : \quad s_*^{\perp,0} > \frac{1}{2} \\ \max(s_*^{\top,0}, \frac{1}{2} - \frac{1}{2} s(1)) & : \quad s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. *We define the QFM* $\mathcal{F}_R$ *in terms of* ω_R *according to Def. 120, i.e.* $\mathcal{F}_R = \mathcal{F}_{\omega_R}$.

Theorem 176
$\mathcal{F}_R$ is a standard DFS.

Again, it can be asserted that $\mathcal{F}_R$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 177
$\mathcal{F}_R$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_R = \mathcal{F}_\xi$.

9.8 Properties of $\mathcal{F}_\Omega$-models

Now that the defining conditions of $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFS have been established and examples of the new classes of models have been given, I turn to additional properties like propagation of fuzziness. Usually I state the corresponding conditions both for the representation in terms of $\mathcal{F}_\Omega$ and in terms of $\mathcal{F}_\omega$. This provides maximum flexibility in later proofs whether a model at hand does or does not possess these properties.

Definition 130
For all $S, S' \in \mathbb{K}$, we say that S is fuzzier (less crisp) than S', in symbols: $S \preceq_c S'$, if and only if the following conditions are satisfied for all $\gamma \in \mathbf{I}$.

$$\text{for all } z' \in S'(\gamma)\text{, there exists } z \in S(\gamma) \text{ such that } z \preceq_c z'; \tag{104}$$

$$\text{for all } z \in S(\gamma)\text{, there exists } z' \in S'(\gamma) \text{ such that } z \preceq_c z'. \tag{105}$$

Definition 131
Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given. We say that Ω propagates fuzziness *if and only if*

$$\Omega(S) \preceq_c \Omega(S')$$

whenever $S, S' \in \mathbb{K}$ satisfy $S \preceq_c S'$.

Theorem 178
For all $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers if and only if Ω propagates fuzziness.

The following condition permits a simplified check if a given Ω propagates fuzziness.

Theorem 179
Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-1)–(Ω-5). *Then Ω propagates fuzziness if and only if*

$$\Omega(S) = \Omega(S^\ddagger \cap [\tfrac{1}{2}, 1])\,,$$

for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$.

Definition 132
Let $S, S' \in \mathbb{K}$ be given. We say that S is less specific than S', in symbols $S \Subset S'$, if and only if

$$S(\gamma) \supseteq S'(\gamma)$$

for all $\gamma \in \mathbf{I}$.

Definition 133

Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *be given. We say that* Ω propagates unspecificity *if and only if*

$$\Omega(S) \preceq_c \Omega(S')$$

for every choice of $S, S' \in \mathbb{K}$ *with* $S \Subset S'$.

Theorem 180

For all $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ *propagates fuzziness in arguments if and only if* Ω *propagates unspecificity.*

The above criterion for Ω propagating unspecificity can be simplified as follows.

Theorem 181

Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-1), (Ω-2), (Ω-4) *and* (Ω-5). *Then the following conditions are equivalent:*

a. Ω *propagates unspecificity;*

b. for all $s \in \mathbb{K}$ *with* $S(0) \subseteq [\frac{1}{2}, 1]$, *it holds that* $\Omega(S) = \Omega(S')$, *where* $S' \in \mathbb{K}$ *is defined by*

$$S'(\gamma) = \begin{cases} [z_*, 1] & : \quad z_* \in S(\gamma) \\ (z_*, 1] & : \quad z_* \notin S(\gamma) \end{cases} \tag{106}$$

for all $\gamma \in \mathbf{I}$, *and where* $z_* = z_*(\gamma)$ *abbreviates*

$$z_* = \inf S(\gamma)\,. \tag{107}$$

Definition 134

For all $s, s' \in \mathbb{L}$, *we say that* s *is fuzzier (less crisp) than* s', *in symbols* $s \preceq_c s'$, *if and only if*

for all $z \in \mathbf{I}$, *there exists* $z' \in \mathbf{I}$ *with* $z \preceq_c z'$ *and* $s'(z') \leq s(z)$; (108)

for all $z' \in \mathbf{I}$, *there exists* $z \in \mathbf{I}$ *with* $z \preceq_c z'$ *and* $s(z) \leq s'(z')$. (109)

Definition 135

A mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is said to* propagate fuzziness *if and only if* $\omega(s) \preceq_c \omega(s')$ *for all choices of* $s, s' \in \mathbb{L}$ *with* $s \preceq_c s'$.

Theorem 182

Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is* ‡*-invariant, i.e.* $\omega(s^\ddagger) = \omega(s)$ *for all* $s \in \mathbb{L}$. *Then* $\mathcal{F}_\omega$ *propagates fuzziness in quantifiers if and only if* ω *propagates fuzziness.*

If ω is well-behaved, then we can further simplify the condition that must be tested for establishing or rejecting that ω propagate fuzziness.

Theorem 183
Suppose that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *satisfies* (ω-1)–(ω-4). ω *propagates fuzziness if and only if for all* $s \in \mathbb{L}$ *with* $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$*, it holds that* $\omega(s) = \omega(s')$*, where*

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{110}$$

for all $z \in \mathbf{I}$.

Definition 136
A mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is said to* propagate unspecificity *if and only if* $\omega(s) \preceq_c \omega(s')$ *whenever* $s, s' \in \mathbb{L}$ *satisfy* $s \leq s'$.

Theorem 184
Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be a given mapping. Then* $\mathcal{F}_\omega$ *propagates fuzziness in arguments if and only if* ω *propagates unspecificity.*

Again, it is possible to simplify the condition imposed on ω.

Theorem 185
Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *satisfies* (ω-1), (ω-2) *and* (ω-4). *Then the following conditions are equivalent.*

a. ω *propagates unspecificity;*

b. for all $s \in \mathbb{L}$ *with* $s^{-1} \cap [\frac{1}{2}, 1] \neq \varnothing$*, it holds that* $\omega(s) = \omega(s')$*, where* $s' \in \mathbb{L}$ *is defined by*

$$s'(z) = \inf\{s(z') : z' \leq z\} \tag{111}$$

for all $z \in \mathbf{I}$.

Now let us apply these criteria to the examples of $\mathcal{F}_\omega$-models.

Theorem 186
$\mathcal{F}_M$ *propagates fuzziness in quantifiers.*

Theorem 187
$\mathcal{F}_M$ *propagates fuzziness in arguments.*

Let us recall from Th-171 that $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, in particular not an $\mathcal{M}_\mathcal{B}$-DFS. Hence the class of $\mathcal{M}_\mathcal{B}$-DFSes, which propagate fuzziness in both arguments and quantifiers, does not include all standard models with this property. $\mathcal{F}_M$ is a counterexample which demonstrates that the class of standard DFSes which propagate fuzziness both in quantifiers and arguments is genuinely broader than the class of $\mathcal{M}_\mathcal{B}$-models.

Theorem 188
$\mathcal{F}_P$ *propagates fuzziness in quantifiers.*

Theorem 189
$\mathcal{F}_P$ does not propagate fuzziness in arguments.

Theorem 190
$\mathcal{F}_Z$ propagates fuzziness in quantifiers.

Theorem 191
$\mathcal{F}_Z$ does not propagate fuzziness in arguments.

As concerns $\mathcal{F}_R$, we obtain the following results.

Theorem 192
$\mathcal{F}_R$ does not propagate fuzziness in quantifiers.

Theorem 193
$\mathcal{F}_R$ propagates fuzziness in arguments.

Hence there are $\mathcal{F}_\omega$-DFSes beyond $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers, but not in arguments. In particular, the class of standard models that propagate fuzziness in quantifiers but not in arguments is genuinely broader than the class of $\mathcal{F}_\xi$-DFSes with this property. I will show later that the class of $\mathcal{F}_\omega$-DFSes with this property is still specificity consistent and investigate its least upper specificity bound.

Definition 137
A collection $\mathbb{\Omega}$ of mappings $\Omega \in \mathbb{\Omega}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is called specificity consistent *if and only if for all $S \in \mathbb{K}$, either $\{\Omega(S) : \Omega \in \mathbb{K}\} \subseteq [\frac{1}{2}, 1]$ or $\{\Omega(S) : \Omega \in \mathbb{K}\} \subseteq [0, \frac{1}{2}]$.*

Theorem 194
Suppose $\mathbb{\Omega}$ is a collection of mappings $\Omega \in \mathbb{\Omega}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ and let $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathbb{\Omega}\}$ be the corresponding collection of QFMs. Then $\mathbb{F}$ is specificity consistent if and only if $\mathbb{\Omega}$ is specificity consistent.

Theorem 195
Suppose $\mathbb{\Omega}$ is a collection of mappings $\Omega \in \mathbb{\Omega}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ which satisfy (Ω-5)*, and let $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathbb{\Omega}\}$ be the corresponding collection of DFSes. Further suppose that every $\Omega \in \mathbb{\Omega}$ has the additional property that $\Omega(S) = \frac{1}{2}$ for all $S \in \mathbb{K}$ with $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\mathbb{F}$ is specificity consistent.*

Definition 138
We say that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is fuzzier (less crisp) than $\Omega' : \mathbb{K} \longrightarrow \mathbf{I}$, in symbols: $\Omega \preceq_c \Omega'$, if and only if $\Omega(S) \preceq_c \Omega'(S)$ for all $S \in \mathbb{K}$.

Theorem 196
Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given mappings and let $\mathcal{F}_\Omega$, $\mathcal{F}_{\Omega'}$ be the corresponding QFMs defined by Def. 114. Then $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ if and only if $\Omega \preceq_c \Omega'$.

This criterion for comparing specificity can be further simplified in the frequent case that some basic assumptions can be made on Ω, Ω'.

Theorem 197
Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given mappings which satisfy (Ω-2) and (Ω-5). Further suppose that $\Omega(S) = \frac{1}{2} = \Omega'(S)$ whenever $S \in \mathbb{K}$ has $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\Omega \preceq_c \Omega'$ if and only if $\Omega(S) \leq \Omega'(S)$ for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$.

Similar criteria can be established in the case of mappings $\omega : \mathbb{L} \longrightarrow \mathbf{I}$.

Definition 139
A collection $\mathbb{\omega}$ of mappings $\omega \in \mathbb{\omega}$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is called specificity consistent if and only if for all $s \in \mathbb{L}$, either $\{\omega(s) : \omega \in \mathbb{L}\} \subseteq [\frac{1}{2}, 1]$ or $\{\omega(s) : \omega \in \mathbb{L}\} \subseteq [0, \frac{1}{2}]$.

Theorem 198
Suppose $\mathbb{\omega}$ is a collection of mappings $\omega \in \mathbb{\omega}$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$, and let $\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \mathbb{\omega}\}$ be the corresponding collection of QFMs. Then $\mathbb{F}$ is specificity consistent if and only if $\mathbb{\omega}$ is specificity consistent.

Theorem 199
Suppose $\mathbb{\omega}$ is a collection of mappings $\omega \in \mathbb{\omega}$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ which satisfy (ω-1)–(ω-4), and let $\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \mathbb{\omega}\}$ be the corresponding collection of DFSes. Further suppose that every $\omega \in \mathbb{\omega}$ has the additional property that $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\mathbb{F}$ is specificity consistent.

The following theorems show that the above property is possessed both by $\mathcal{F}_\omega$-DFSes that propagate fuzziness in quantifiers and by those that propagate fuzziness in arguments:

Theorem 200
Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1)–(ω-4) and suppose that the corresponding DFS $\mathcal{F}_\omega$ propagates fuzziness in quantifiers. Then $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.

In particular,

Theorem 201
The collection of $\mathcal{F}_\omega$-DFSes that propagate fuzziness in quantifiers is specificity consistent.

Theorem 202
Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1)–(ω-4) and suppose that the corresponding DFS $\mathcal{F}_\omega$ propagates fuzziness in arguments. Then $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.

Therefore

Theorem 203
The collection of $\mathcal{F}_\omega$-DFSes that propagate fuzziness in arguments is specificity consistent.

Definition 140
We say that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is fuzzier (less crisp) than $\omega' : \mathbb{L} \longrightarrow \mathbf{I}$, in symbols: $\omega \preceq_c \omega'$, if and only if $\omega(s) \preceq_c \omega'(s)$ for all $s \in \mathbb{L}$.

Theorem 204
Let $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ be given mappings and let $\mathcal{F}_\omega$, $\mathcal{F}_{\omega'}$ be the corresponding QFMs defined by Def. 120. Then $\mathcal{F}_\omega \preceq_c \mathcal{F}_{\omega'}$ if and only if $\omega \preceq_c \omega'$.

Again, it is possible to simplify the condition in typical situations.

Theorem 205
Let $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ be given mappings which satisfy (ω-2) and (ω-4). Further suppose that $\omega(s) = \frac{1}{2} = \omega'(s)$ whenever $s \in \mathbb{L}$ satisfies $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\omega \preceq_c \omega'$ if and only if $\omega(s) \leq \omega'(s)$ for all $s \in \mathbb{L}$ with $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$.

The precondition of the theorem is e.g. satisfied by the models that propagate fuzziness. Based on this simplified criterion, it is now easy to prove the following results concerning specificity bounds.

Theorem 206
$\mathcal{F}_Z$ is the most specific $\mathcal{F}_\omega$-DFS that propagates fuzziness in quantifiers.

Theorem 207
$\mathcal{F}_R$ is the most specific $\mathcal{F}_\omega$-DFS that propagates fuzziness in arguments.

Theorem 208
$\mathcal{F}_M$ is the most specific $\mathcal{F}_\omega$-DFS that propagates fuzziness both in quantifiers and arguments.

As concerns the issue of identifying the least specific model, we obtain the following result which confirms the special role of $\mathcal{M}_U$.

Theorem 209
$\mathcal{M}_U$ is the least specific $\mathcal{F}_\omega$-DFS.

Finally let us consider continuity properties of $\mathcal{F}_\Omega$-models. This investigation will help me to relate the new class of DFSes to its subclass of $\mathcal{F}_\xi$-models. To this end, I introduce the following operation $^\square$.

Definition 141
For all $S \in \mathbb{K}$, $S^{\square} \in \mathbb{K}$ is defined by

$$S^{\square}(\gamma) = [\inf S(\gamma), \sup S(\gamma)]$$

for all $\gamma \in \mathbf{I}$.

Note. It is apparent from Def. 111 that indeed $S^{\square} \in \mathbb{K}$.

Theorem 210
For all $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ is an $\mathcal{F}_\xi$-QFM if and only if Ω is $^{\square}$-invariant, i.e. $\Omega(S) = \Omega(S^{\square})$ for all $S \in \mathbb{K}$.

Utilizing this relationship, the following theorem is straightforward.

Theorem 211
Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be an ‡-invariant mapping. If $\mathcal{F}_\Omega$ is Q-continuous, then it is an $\mathcal{F}_\xi$-QFM, i.e. there exists $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_\Omega = \mathcal{F}_\xi$. In particular, the theorem is applicable to all $\mathcal{F}_\Omega$-models.

Hence all $\mathcal{F}_\Omega$-models that are interesting from a practical perspective are already contained in the class of $\mathcal{F}_\xi$-QFMs.

9.9 Chapter summary

Summarizing, this chapter was devoted to the search for a more general type of models. In order to ensure that the new models subsume the known $\mathcal{F}_\xi$-DFSes, which form the broadest class of standard models developed in the previous chapters, it was considered best to start from the underlying mechanism that was used to define ξ, and to pursue an apparent generalization. I hence observed that the mappings $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ used to define $\mathcal{F}_\xi$ can be decomposed into subsequent application of the three-valued cut mechanism (which generates an ambiguity set of alternative interpretations for each cut level) followed by an aggregation step based on the infimum or supremum. In order to abstract from the concepts used to define $\mathcal{F}_\xi$, and to capture the full class of standard models that depend on three-valued cuts, it was straightforward to drop the $\sup/\inf$-based aggregation step and to start an investigation of those models that can be defined in terms of the 'raw' information obtained at the cut levels, i.e. in terms of the result sets $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ which represent the ambiguity range of all possible interpretations of Q given the three-valued cuts of $X_1, \ldots, X_n$ at the cut levels γ. This generalization results in a new class of models genuinely broader than $\mathcal{F}_\xi$-DFSes, the *full* class of models definable in terms of three-valued cuts.

In developing the theory of these models, I first identified the precise range of possible mappings $S = \mathcal{S}_{Q,X_1,\ldots,X_n}$ that can result from a choice of quantifier Q and fuzzy arguments $X_1, \ldots, X_n$. The resulting set $\mathbb{K}$ provides the proper domain to define aggregation operators $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, from which QFMs can then be constructed in the apparent way, i.e. $\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) = \Omega(\mathcal{S}_{Q,X_1,\ldots,X_n})$.

After introducing $\mathcal{F}_\Omega$-QFMs, I developed all formal machinery required to express the precise conditions on Ω that make $\mathcal{F}_\Omega$ a DFS. In particular, I have characterised the class of $\mathcal{F}_\Omega$-DFSes in terms of a set of necessary and sufficient conditions, and I have shown that these conditions are independent. This analysis also reveals that all $\mathcal{F}_\Omega$-models are in fact standard DFSes, and hence fulfill the expectations on standard models of fuzzy quantification. In addition, the known class of $\mathcal{F}_\xi$-QFMs has been related to its apparent superclass of $\mathcal{F}_\Omega$-QFMs.
I then focused on an apparent subclass of $\mathcal{F}_\Omega$-QFMs, the class of $\mathcal{F}_\omega$-QFMs. These are obtained by defining coefficients $s_{Q,X_1,\ldots,X_n}(z) = \inf\{\gamma : z \in S_{Q,X_1,\ldots,X_n}(\gamma)$ which extract an important characteristic of the result sets $S_{Q,X_1,\ldots,X_n}(\gamma)$. Introducing this construction offers the advantage that we no longer need to work with *sets* of results, as it was the case with the $S_{Q,X_1,\ldots,X_n}(\gamma)$, which are subsets of the unit interval. By contrast, we can now focus on scalars $s_{Q,X_1,\ldots,X_n}$ in the unit range, and a subsequent aggregation by applying the chosen $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. Among other things, this greatly simplifies the definition of models, and hence all examples of $\mathcal{F}_\Omega$-DFSes were presented in this succinct format.
Noticing that the new coefficients $s_{Q,X_1,\ldots,X_n}$ are functions of $S_{Q,X_1,\ldots,X_n}$ which suppress some of the original information, the question then arises if some of the $\mathcal{F}_\Omega$-models are lost under the new construction. To resolve this issue whether the $\mathcal{F}_\omega$-DFSes form a subclass proper, and to gain some insight into their structure, I have introduced the concepts required to characterise adequate choices of ω. Building on these definitions, a set of independent conditions that precisely describe the $\mathcal{F}_\omega$-DFSes in terms of necessary and sufficient criteria on ω has been developed. In addition, the $\mathcal{F}_\omega$-QFMs have been related to their superclass of $\mathcal{F}_\Omega$-QFMs. This analysis revealed that the move from $\mathcal{F}_\Omega$-QFMs to $\mathcal{F}_\omega$-QFMs does not result in any loss of intended models, i.e. the classes of $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFSes coincide.
Turning to examples of $\mathcal{F}_\Omega$-DFSes (or synonymously, $\mathcal{F}_\omega$-DFSes), the simplified format was utilized to define the four $\mathcal{F}_\omega$-models $\mathcal{F}_M$, $\mathcal{F}_P$, $\mathcal{F}_Z$ and $\mathcal{F}_R$, all of which were shown to be ‘genuine’ members which go beyond the class of $\mathcal{F}_\xi$-QFMs. In order to gain more knowledge of these models, and to locate them precisely within the class of $\mathcal{F}_\omega$-DFSes, the full set of conditions on Ω and ω was then developed, that are required to investigate the characteristic properties of DFSes.

To this end, I first extended the specificity order to the case of $S \preceq_c S'$ and $s \preceq_c s'$. This allowed me to reduce $\mathcal{F}_\Omega$'s propagating fuzziness in quantifiers to the requirement that Ω propagate fuzziness, i.e. $S \preceq_c S'$ entails $\Omega(S) \preceq_c \Omega(S')$. Based on a different relation $S \Subset S'$ defined on $\mathbb{K}$, it was then possible to define a condition of propagating unspecificity on Ω, and to prove that $\mathcal{F}_\Omega$ propagates fuzziness in arguments if and only if Ω propagates unspecificity. In addition, I have shown that both the condition of propagating fuzziness and the condition of propagating unspecificity can be further simplified if the considered Ω is well-behaved (in particular if $\mathcal{F}_\Omega$ is a DFS). In this common case, a very elementary test on Ω is sufficient for detecting or rejecting these properties. All of these results have also been transferred to $\mathcal{F}_\omega$-QFMs, and hence turned into corresponding conditions on ω. After developing the formal apparatus required to investigate propagation of fuzziness in $\mathcal{F}_\Omega$- and $\mathcal{F}_\omega$-QFMs, the issue of most specific and least specific models was then discussed in some depth. Acknowledging its relevance to the existence of most specific models, I first extended the notion of specificity consistency to collections $\mathbb{\Omega}$ of aggregation mappings Ω and proved that the resulting criterion on $\mathbb{\Omega}$ precisely captures specificity consistency of the class of QFMs $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathbb{\Omega}\}$. Hence the question whether $\mathbb{F}$ has a least upper specificity bound can be decided by looking at the aggregation mappings in $\mathbb{\Omega}$. I have also shown how the criterion can be simplified in common situations. Following this, the question was addressed how a specificity comparison $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ can be reformulated as a condition $\Omega \preceq_c \Omega'$ imposed on the aggregation mappings. Again, the condition for $\Omega \preceq_c \Omega'$ can be reduced to a very simple check in many typical situations. All of the above concepts and theorems were then adapted to $\mathcal{F}_\omega$-QFMs, in order to provide similar support for specificity comparison in those cases where the models of interest are defined in terms of an aggregation mapping ω.

Based on these preparations, it was easy to prove some results concerning propagation of fuzziness that elucidate the structure of the class of $\mathcal{F}_\omega$-DFSes, and that relate the examples of $\mathcal{F}_\omega$-DFSes to the class as a whole. First of all, the full class of $\mathcal{F}_\omega$-DFSes is not specificity consistent (because its subclass of $\mathcal{F}_\xi$-DFSes is known to violate specificity consistency), and hence a 'most specific $\mathcal{F}_\omega$-DFS' ceases to exist. However, the class of models that propagate fuzziness in quantifiers was shown to be specificity consistent, and the most specific $\mathcal{F}_\omega$-DFS with this property was also identified, and turned out to be $\mathcal{F}_Z$. Recalling that $\mathcal{F}_Z$ is not an $\mathcal{F}_\xi$-QFM, this demonstrates that the class of $\mathcal{F}_\omega$-DFSes which propagate fuzziness in quantifiers is an extension proper of the class of $\mathcal{F}_\xi$-DFSes with the same behaviour. Turning to propagation of fuzziness in quantifiers, it was possible to prove a similar result. The corresponding class of $\mathcal{F}_\omega$-models was shown to be specificity consistent, and $\mathcal{F}_R$ was established to be the most specific $\mathcal{F}_\omega$-DFS with this property. Again, I conclude from the fact that $\mathcal{F}_R$ is a 'genuine' $\mathcal{F}_\omega$-DFS that the $\mathcal{F}_\omega$-DFSes contain models which propagate fuzziness in arguments beyond those already known from the study of $\mathcal{F}_\xi$-DFSes. I then investigated those standard models that propagate fuzziness both in quantifiers and arguments. The model $\mathcal{F}_M$ was shown to be the most specific $\mathcal{F}_\omega$-DFS with these properties. The class of $\mathcal{F}_\xi$-DFSes that fulfill both conditions is known to coincide with the class of $\mathcal{M}_\mathcal{B}$-DFSes. Because $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, this proves that there are standard models beyond the $\mathcal{M}_\mathcal{B}$-type which propagate fuzziness both in quantifiers and arguments.
The problem of identifying the greatest lower specificity bound has also been addressed. In fact, the least specific $\mathcal{F}_\omega$-DFS was proven to coincide with one of the $\mathcal{M}_\mathcal{B}$-models, namely $\mathcal{M}_U$, which was already known to be the least specific $\mathcal{M}_\mathcal{B}$- and $\mathcal{F}_\xi$-DFS.

Finally, I have addressed the continuity issue. It is indispensible for applications that the chosen QFM be robust against slight variations in the chosen quantifier and in its arguments, which might e.g. result from noise. In addition, both continuity conditions are desirable in order to account for imperfect knowledge of the precise interpretation of the involved NL quantifier and NL concepts in terms of numeric membership grades. Based on an auxiliary filling construction $S^\square$, it was then shown that every $\mathcal{F}_\Omega$-QFM which is continuous in quantifiers is in fact an $\mathcal{F}_\xi$-QFM. The class of $\mathcal{F}_\omega$-DFSes which are Q-continuous therefore collapses into the class of Q-continuous $\mathcal{F}_\xi$-DFSes, and those Q-continuous $\mathcal{F}_\omega$-DFSes which propagate fuzziness in quantifiers and arguments collapse into the class of $\mathcal{M}_{\mathcal{B}}$-DFSes. This proves that all *practical* models are already contained in the class of $\mathcal{F}_{\xi}$-DFSes, and those practical models which propagate fuzziness both in quantifiers and arguments are contained in the class of $\mathcal{M}_{\mathcal{B}}$-DFSes. This justifies the development and thorough analysis of these simpler classes in the previous chapters, because every model of practical interest will belong to one of these classes. It can hence be expressed through constructions simpler than those used to define $\mathcal{F}_{\Omega}$- and $\mathcal{F}_{\omega}$-QFMs, which in turn permit a simpler check of the relevant formal properties, like being a DFS, propagation of fuzziness, specificity comparisons, and continuity, and which suggest simple algorithms for implementing quantifiers in the model.

10 The class of models based on the extension principle

10.1 Motivation and chapter overview

In this chapter, an attempt is made to define models from independent considerations, and to establish a new class of fuzzification mechanisms not constructed from three-valued cuts. Starting from a straightforward definition of argument similarity, I first introduce the full class of QFMs defined in terms of the similarity measure, the class of $\mathcal{F}_\psi$-QFMs. It encloses the interesting subclass of $\mathcal{F}_\varphi$-QFMs, i.e. the class of models defined through the standard extension principle (which serves to aggregate similarity grades). The necessary and sufficient conditions are then developed, which the aggregation mappings must satisfy in order to make the corresponding fuzzification mechanism a DFS. Based on this analysis, it becomes possible to prove the main result of this chapter, which states that the classes of $\mathcal{F}_\omega$-DFSes and $\mathcal{F}_\psi$-DFSes/$\mathcal{F}_\varphi$-DFSes coincide. Because the same class of models is obtained from independent considerations, this provides evidence that it indeed represents a natural class of standard models of fuzzy quantification.

10.2 A similarity measure on fuzzy arguments

To begin with, the similarity grade $\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n)$ of the fuzzy arguments $(X_1, \dots, X_n)$ to a choice of crisp arguments $(Y_1, \dots, Y_n)$ can be defined as follows.

Definition 142

Let $E \neq \varnothing$ be some base set and $Y \in \mathcal{P}(E)$. The mapping $\Xi_Y : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is defined by

$$\Xi_Y(X) = \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\})$$

for all $X \in \widetilde{\mathcal{P}}(E)$. For n-tuples of arguments $Y_1, \dots, Y_n \in \mathcal{P}(E)$, we define $\Xi^{(n)}_{Y_1,\dots,Y_n} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$\Xi^{(n)}_{Y_1,\dots,Y_n}(X_1, \dots, X_n) = \bigwedge_{i=1}^{n} \Xi_{Y_i}(X_i)$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Whenever n is clear from context, we shall omit the superscript and write $\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n)$ instead of $\Xi^{(n)}_{Y_1,\dots,Y_n}(X_1, \dots, X_n)$.

At times, it will be convenient to use the following abbreviation. Let us recall the fuzzy equivalence connective $\leftrightarrow : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ defined by

$$x \leftrightarrow y = (x \wedge y) \vee (\neg x \wedge \neg y)$$

for all $x, y \in \mathbf{I}$. In the case that $y \in \{0, 1\}$, this apparently becomes

$$x \leftrightarrow y = \begin{cases} x & : \quad y = 1 \\ \neg x & : \quad y = 0 \end{cases}$$

Now consider a base set $E \neq \varnothing$ and let $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$. I make use of the $\leftrightarrow$-connective to define $\delta_{X,Y} : E \longrightarrow \mathbf{I}$ by

$$\delta_{X,Y}(e) = (\mu_X(e) \leftrightarrow \chi_Y(e)) = \begin{cases} \mu_X(e) & : \quad e \in Y \\ 1 - \mu_X(e) & : \quad e \notin Y \end{cases} \tag{112}$$

for all $e \in E$. In terms of $\delta_{Y,E}$, we can now conveniently reformulate $\Xi_Y(X)$. In particular, we can express $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, by

$$\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \inf\{\delta_{X_i,Y_i}(e) : e \in E, i = 1, \ldots, n\}\,. \tag{113}$$

In order to illustrate the proposed definition of argument similarity, let us now consider an example, which also profits from the more succinct alternative notation. Hence let a two-element base set $E = \{a, b\}$ be given, and suppose that $X \in \widetilde{\mathcal{P}}(E)$ is defined by $\mu_X(a) = \frac{1}{3}$, $\mu_X(b) = \frac{3}{4}$. Then

$$\begin{aligned}
\Xi_\varnothing(X) &= \min\{\delta_{X,\varnothing}(a),\ \delta_{X,\varnothing}(b)\} \\
&= \min\{1 - \mu_X(a), 1 - \mu_X(b)\} \\
&= \min\{\tfrac{2}{3}, \tfrac{1}{4}\} \\
&= \tfrac{1}{4} \\
\Xi_{\{a\}}(X) &= \min\{\delta_{X,\{a\}}(a),\ \delta_{X,\{a\}}(b)\} \\
&= \min\{\mu_X(a), 1 - \mu_X(b)\} \\
&= \min\{\tfrac{1}{3}, \tfrac{1}{4}\} \\
&= \tfrac{1}{4} \\
\Xi_{\{b\}}(X) &= \min\{\delta_{X,\{b\}}(a),\ \delta_{X,\{b\}}(b)\} \\
&= \min\{1 - \mu_X(a), \mu_X(b)\} \\
&= \min\{\tfrac{2}{3}, \tfrac{3}{4}\} \\
&= \tfrac{2}{3} \\
\Xi_{\{a,\,b\}}(X) &= \min\{\delta_{X,\{a,\,b\}}(a), \delta_{X,\{a,\,b\}}(b)\} \\
&= \min\{\mu_X(a), \mu_X(b)\} \\
&= \min\{\tfrac{1}{3}, \tfrac{3}{4}\} \\
&= \tfrac{1}{3}\,.
\end{aligned}$$

Next I define the set of all compatibility grades which corresponds to a given choice of fuzzy arguments $X_1, \ldots, X_n$.

Definition 143

Let $E \neq \varnothing$ be a given base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $n \geq 0$. Then $D^{(n)}_{X_1,\ldots,X_n} \in \mathcal{P}(\mathbf{I})$ is defined by

$$D^{(n)}_{X_1,\ldots,X_n} = \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : Y_1, \ldots, Y_n \in \mathcal{P}(E)\}\,.$$

Whenever this causes no ambiguity, the superscript (n) will be omitted, thus abbreviating $D_{X_1,\ldots,X_n} = D^{(n)}_{X_1,\ldots,X_n}$.

Note. The superscript is only needed to discern $D_{\varnothing}^{(0)}$ (which corresponds to the empty tuple) from $D_{\varnothing}^{(1)}$ (which corresponds to the empty set).

Definition 144
By $\mathbb{D} \subseteq \mathcal{P}(\mathcal{P}(\mathbf{I}))$ we denote the set of all $D \in \mathcal{P}(\mathbf{I})$ with the following properties:

1. *$D \cap [\frac{1}{2}, 1] = \{r_+\}$ for some $r_+ = r_+(D) \in [\frac{1}{2}, 1]$;*
2. *for all $D' \subseteq D$ with $D' \neq \varnothing$, $\inf D' \in D$;*
3. *if $r_+ > \frac{1}{2}$, then $\sup D \setminus \{r_+\} = 1 - r_+$.*

Theorem 212
Suppose $E \neq \varnothing$ is some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. Then $D_{X_1,\ldots,X_n} \in \mathbb{D}$.

Hence $\mathbb{D}$ is large enough to contain all $D_{X_1,\ldots,X_n}$. As we shall see later in Th-215, $\mathbb{D}$ is indeed the smallest possible subset of $\mathcal{P}(\mathcal{P}(\mathbf{I}))$ which contains all $D_{X_1,\ldots,X_n}$. (The theorem has been delayed because it then becomes a corollary).

10.3 The unrestricted class of $\mathcal{F}_\psi$-QFMs

In order to define the new class of fuzzification mechanisms, I now relate the similarity information expressed by $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)$ to the behaviour of a quantifier on its arguments.

Definition 145
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then $A_{Q,X_1,\ldots,X_n}^{(n)} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by

$$A_{Q,X_1,\ldots,X_n}^{(n)}(z) = \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : (Y_1, \ldots, Y_n) \in Q^{-1}(z)\}$$

for all $z \in \mathbf{I}$. When n is clear from context, I usually omit the superscript (n), thus abbreviating $A_{Q,X_1,\ldots,X_n} = A_{Q,X_1,\ldots,X_n}^{(n)}$.

Note. Again, the superscript is only needed to eliminate the ambiguity between $A_{Q,\varnothing}^{(0)}$, where Q is a nullary quantifier and $\varnothing$ the empty tuple, and $A_{Q,\varnothing}^{(1)}$, where Q is a one-place quantifier and $\varnothing$ is the empty argument set.
Next I will describe the range of all possible $A_{Q,X_1,\ldots,X_n}$.

Definition 146
By $\mathbb{A}$ we denote the set of all mappings $A : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ with the following properties:

a. $\cup\{A(z) : z \in \mathbf{I}\} \in \mathbb{D}$;

b. for all $z, z' \in \mathbf{I}$, $\sup A(z) > \frac{1}{2}$ and $\sup A(z') > \frac{1}{2}$ entails that $z = z'$.

In the following, $D(A)$ denotes the set

$$D(A) = \cup\{A(z) : z \in \mathbf{I}\}\,. \tag{114}$$

In addition, r_+ abbreviates $r_+(A) = r_+(D(A))$. It is then apparent from Def. 146.a and Def. 144 that there exists $z_+ = z_+(A) \in \mathbf{I}$ with

$$r_+ \in A(z_+)\,. \tag{115}$$

In the following, z_+ is assumed to be an arbitrary but fixed choice of $z_+ \in \mathbf{I}$ which satisfies (115) for a considered $A \in \mathbb{A}$.

Theorem 213
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. *Then* $\mathcal{A}_{Q,X_1,\ldots,X_n} \in \mathbb{A}$.

Hence $\mathbb{A}$ contains all $\mathcal{A}_{Q,X_1,\ldots,X_n}$.

Theorem 214
Let $A \in \mathbb{A}$ *be given and* $D(A) = \cup\{A(z) : z \in \mathbf{I}\}$.

a. *If* $D(A) = \{1\}$, *then* $A = \mathcal{A}^{(0)}_{Q,\varnothing}$, *where* $Q : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ *is the constant quantifier* $Q(\varnothing) = z_+$.

b. *If* $D(A) \neq \{1\}$, *then we can choose some mapping* $\zeta : D(A) \longrightarrow \mathbf{I}$ *with*

$$r \in A(\zeta(r)) \tag{116}$$

for all $r \in D(A)$. *If* $r_+ = r_+(A)$ *equals* $\frac{1}{2}$, *then* $r_+ \in D(A) \cap [0, 1 - r_+]$. *If* $r_+ > \frac{1}{2}$, *then we recall from Def. 146 that* $\sup D(A) \setminus \{r_+\} = 1 - r_+$. *Because* $D(A) \neq \{1\}$ *by assumption, we hence know that there exists*

$$r_- \in D(A) \cap [0, 1 - r_+] \tag{117}$$

and we shall assume an arbitrary choice of r_- *with this property. Based on* r_-, *we define* $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ *by*

$$\mu_X(z, r) = \begin{cases} r & : \ r \in A(z) \setminus \{r_+\} \\ r_- & : \ r \notin A(z) \vee r = r_+ > \frac{1}{2} \\ \frac{1}{2} & : \ r = r_+ = \frac{1}{2} \end{cases} \tag{118}$$

for all $z, r \in \mathbf{I}$. *For all* $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, *we abbreviate*

$$r' = r'(Y) = \Xi_Y(X) \tag{119}$$

$$z' = z'(Y) = \inf\{z \in \mathbf{I} : (z, r') \in Y \textit{ and } r' = r'(Y) \in A(z)\}\,. \tag{120}$$

Based on ζ, *we define* $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *by*

$$Q(Y) = \begin{cases} z' & : \ r' \in A(z') \\ \zeta(r') & : \ r' \notin A(z') \end{cases} \tag{121}$$

for all $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$.
Then $A = \mathcal{A}_{Q,X}$.

We also obtain the following corollary concerning $\mathbb{D}$:

Theorem 215
For all $D \in \mathbb{D}$,

a. *If $D = \{1\}$, then $D = D^{(0)}_{\varnothing}$, where $\varnothing$ is the empty tuple $\varnothing \in \mathcal{P}(\{*\})^0$.*

b. *If $D \neq \{1\}$, then there exists $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ such that $D = D_X$.*

Hence $\mathbb{D}$ is indeed the smallest subset of $\mathcal{P}(\mathcal{P}(\mathbf{I}))$ which contains all $D_{X_1,\ldots,X_n}$.
In order to carry out the desired aggregation, which will turn the compatibility grades into a fuzzification mechanism, I now deploy mappings $\psi : \mathbb{A} \longrightarrow \mathbf{I}$. These can be used to define a QFM in the apparent way, by composing with the $\mathcal{A}_{Q,X_1,\ldots,X_n}$'s:

Definition 147
Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given. The QFM $\mathcal{F}_\psi$ is defined by

$$\mathcal{F}_\psi(Q)(X_1, \ldots, X_n) = \psi(\mathcal{A}_{Q,X_1,\ldots,X_n})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

This definition spans the full class of QFMs definable in terms of argument similarity, and I will now investigate its well-behaved models.

10.4 Characterisation of the $\mathcal{F}_\psi$-models

Let us now tackle the goal of characterising the class of $\mathcal{F}_\psi$-DFSes, by making explicit the structure of all plausible models. To this end, we need some more notation, for expressing the properties expected from legal choices of ψ. As usual, the goal is that of characterising the new class of models in terms of the necessary and sufficient conditions on the aggregation mapping. In order to describe the desired monotonicity properties, I first define a suitable preorder on $\mathbb{A}$.

Definition 148
For all $A, A' \in \mathbb{A}$, we say that $A \sqsubseteq A'$ if and only if the following conditions are satisfied by A, A'.

a. *for all $z \in \mathbf{I}$ and all $r \in A(z)$, there exists $z' \geq z$ with $r \in A'(z')$;*

b. *for all $z' \in \mathbf{I}$ and all $r \in A'(z')$, there exists $z \leq z'$ with $r \in A(z)$.*

Next I introduce a 'cut/fill operator' $\boxplus$ on $\mathbb{A}$. The invariance of ψ with respect to $\boxplus$ turned out to be essential for ψ to satisfy (Z-4).

Definition 149
For all $A \in \mathbb{A}$, $\boxplus A \in \mathbb{A}$ is defined by

$$\boxplus A(z) = [0, \widehat{\boxplus} A(z)]\,, \tag{122}$$

where

$$\widehat{\boxplus}A(z) = \min(\sup A(z), \tfrac{1}{2}) \tag{123}$$

for all $z \in \mathbf{I}$.

Notes

- It is immediate from the definition of $\boxplus A$ that $\boxplus A \in \mathbb{A}$, see Def. 146.
- For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, I abbreviate

$$\boxplus_{Q,X_1,\ldots,X_n} = \boxplus A_{Q,X_1,\ldots,X_n} \tag{124}$$
$$\widehat{\boxplus}_{Q,X_1,\ldots,X_n} = \widehat{\boxplus} A_{Q,X_1,\ldots,X_n} \,. \tag{125}$$

- In the above definition, $\widehat{\boxplus}$ has been used to express $\boxplus$. In fact, both are definable in terms of each other, because conversely

$$\widehat{\boxplus}A(z) = \sup \boxplus A(z)\,, \tag{126}$$

  for all $A \in \mathbb{A}$ and $z \in \mathbf{I}$.

The cut/fill operator $\boxplus$ is of special relevance to the characterisation of $\mathcal{F}_\psi$-DFSes because $\boxplus$-invariance ensures that (Z-6) be valid.
In order to define the conditions on ψ succinctly and to support the corresponding proofs, it is useful to introduce some additional abbreviations. For all $A \in \mathbb{A}$,

$$\mathrm{NV}(A) = \{z \in \mathbf{I} : A(z) \neq \varnothing\} \tag{127}$$
$$\mathrm{VL}(A) = \{z \in \mathbf{I} : A(z) \setminus \{0\} \neq \varnothing\} = \{z \in \mathbf{I} : A(z) \cap (0,1] \neq \varnothing\}\,. \tag{128}$$

I have now introduced all notation required to express the conditions on admissible choices of ψ.

Definition 150 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given. The conditions* (ψ-1)–(ψ-5) *are defined as follows. For all* $A, A' \in \mathbb{A}$,

If $D(A) = \{1\}$, *then* $\psi(A) = z_+$. (ψ-1)

If $A(z) = A'(1-z)$ *for all* $z \in \mathbf{I}$, *then* $\psi(A) = 1 - \psi(A')$. (ψ-2)

If $\mathrm{NV}(A) \subseteq \{0,1\}$ *and* $r_+ \in A(1)$,
then $\psi(A) = 1 - \sup A(0)$. (ψ-3)

If $A \sqsubseteq A'$, *then* $\psi(A) \leq \psi(A')$. (ψ-4)

$\psi(\boxplus A) = \psi(A)$. ($\psi$-5)

Let us first notice that the above set of conditions indeed captures the essential requirements on ψ which ensure that $\mathcal{F}_\psi$ be a DFS. This is straightforward from the following observation on the behaviour of ψ on two-valued quantifiers.

Theorem 216
If $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-2) and (ψ-3), then $\mathcal{F}_\psi$ coincides with all standard DFSes on two-valued quantifiers, i.e. for every standard DFS $\mathcal{F}$ and two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \{0, 1\}$, it holds that $\mathcal{F}_\psi(Q) = \mathcal{F}(Q)$.

Based on this theorem, it is then easy to prove the following result.

Theorem 217
The conditions (ψ-1)–(ψ-5) on $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ are sufficient for $\mathcal{F}_\psi$ to be a standard DFS.

The converse claim is also true, i.e. it can be shown that the underlying mapping $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-1)–(ψ-5) whenever $\mathcal{F}_\psi$ is a DFS.

Theorem 218
The conditions (ψ-1)–(ψ-5) on $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ are necessary for $\mathcal{F}_\psi$ to be a DFS.

In addition, the system of conditions (ψ-1)–(ψ-5) is known to be minimal in the sense that none of the conditions can be expressed in terms of the remaining ones:

Theorem 219
The conditions (ψ-1)–(ψ-5) are independent.

This condition warrants that there is no redundant effort in proofs that a considered $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ makes $\mathcal{F}_\psi$ a standard DFS.

10.5 The classes of $\mathcal{F}_\psi$-models and $\mathcal{F}_\omega$-models coincide

Let us now establish the central result of this chapter, that the new class of $\mathcal{F}_\psi$-DFSes coincides with the full class of models definable in terms of three-valued cuts, i.e. the class of $\mathcal{F}_\Omega$- or synonymously, $\mathcal{F}_\omega$-DFSes. It is here that we need the construction of $\mathcal{F}_\omega$-QFMs, which provides the link between the models defined in terms of three-valued cuts and those defined in terms of argument similarity. To this end, let us now see how $A_{Q,X_1,\ldots,X_n}$ relates to the coefficient $s_{Q,X_1,\ldots,X_n}$ that was used to define $\mathcal{F}_\omega$-QFMs.

Theorem 220
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $z \in \mathbf{I}$,

$$s_{Q,X_1,\ldots,X_n}(z) = s(A_{Q,X_1,\ldots,X_n})(z) \,,$$

where $s(A) \in \mathbb{L}$, $A \in \mathbb{A}$ *is defined by*

$$s(A)(z) = \max(0, 1 - 2 \cdot \sup A(z)) \tag{129}$$

for all $z \in \mathbf{I}$.

As a by-product of this result, we now discover that all $\mathcal{F}_\omega$-QFMs are in fact $\mathcal{F}_\psi$-QFMs. In particular, all $\mathcal{F}_\Omega$- and $\mathcal{F}_\omega$-DFSes constitute a subclass of the new class of $\mathcal{F}_\psi$-models.

Theorem 221
Every $\mathcal{F}_\omega$*-QFM is an* $\mathcal{F}_\psi$*-QFM, i.e. for all* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$, *there exists* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *with* $\mathcal{F}_\omega = \mathcal{F}_\psi$. ψ *is defined by*

$$\psi(A) = \omega(s(A)) \tag{130}$$

for all $A \in \mathbb{A}$.

As to the converse subsumption, let us recall from Th-218 that every choice of ψ which makes $\mathcal{F}_\psi$ a DFS satisfies (ψ-5). As I will now show, this entails that the unrestricted class of $\mathcal{F}_\psi$-QFMS, although considerably broader than the class of $\mathcal{F}_\omega$-QFMs, does not introduce any new DFSes compared to those that already belong to the class of $\mathcal{F}_\omega$-DFSes. To see this, let us notice the following relationship between $s(A)$ and $\widehat{\boxplus}A$.

Theorem 222
Let $A \in \mathbb{A}$ *be given. Then*

$$\widehat{\boxplus}A(z) = \tfrac{1}{2} - \tfrac{1}{2}s(A)(z) \tag{131}$$

and

$$s(A)(z) = 1 - 2\widehat{\boxplus}A(z)\,, \tag{132}$$

for all $z \in \mathbf{I}$.

Based on this relationship, it is then apparent that all $\boxplus$-invariant $\mathcal{F}_\psi$-QFMs are in fact $\mathcal{F}_\omega$-QFMs.

Theorem 223
Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-5). *Then* $\mathcal{F}_\psi$ *is an* $\mathcal{F}_\omega$*-QFM, i.e.*

$$\mathcal{F}_\psi = \mathcal{F}_\omega$$

provided we define

$$\omega(s) = \psi(A_s), \tag{133}$$

for all $s \in \mathbb{L}$, *where*

$$A_s(z) = [0, \tfrac{1}{2} - \tfrac{1}{2}s(z)] \tag{134}$$

for all $z \in \mathbf{I}$. *In particular, all* $\mathcal{F}_\psi$*-DFSes are* $\mathcal{F}_\omega$*-DFSes.*

The next theorem characterizes the precise subclass of $\mathcal{F}_\psi$-QFMs that can be represented as $\mathcal{F}_\omega$-QFMs.

Theorem 224
The $\mathcal{F}_\omega$-QFMs are exactly those $\mathcal{F}_\psi$-QFMs that depend on a mapping $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ which satisfies (ψ-5)*.*

Note. Compared to the previous theorem, this demonstrates that *only* those $\mathcal{F}_\psi$-QFMs can be represented as $\mathcal{F}_\omega$-QFMs, that are defined from $\boxplus$-invariant choices of ψ.
In the following, I will hence assume that (ψ-5) be valid. It is then easily shown how the conditions (ω-1)–(ω-4) imposed on ω relate to the conditions (ψ-1)–(ψ-5) imposed on the corresponding ψ. These dependencies are made explicit in the next theorem.

Theorem 225
Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by (130)*. Then*

a. ω *satisfies* (ω-1) *if and only if* ψ *satisfies* (ψ-1)

b. ω *satisfies* (ω-2) *if and only if* ψ *satisfies* (ψ-2)*;*

c. ω *satisfies* (ω-3) *if and only if* ψ *satisfies* (ψ-3)*;*

d. ω *satisfies* (ω-4) *if and only if* ψ *satisfies* (ψ-4)*;*

e. ψ *satisfies* (ψ-5)*.*

Note. The theorem was of invaluable help for proving Th-219, by reducing the independence proof of the new set of 'ψ-conditions' to the known theorem on the independence of the 'ω-conditions'.

10.6 The unrestricted class of $\mathcal{F}_\varphi$-QFMs

In the following, I will discuss a slight reformulation of the aggregation mechanism which shows that the $\mathcal{F}_\psi$-DFSes coincide with the models defined in terms of the standard extension principle. The discovered class of models is hence theoretically appealing, because it evolves from the fundamental principle that underlies fuzzy set theory.
In order to define the class of those QFMs that depend on the extension principle, let us consider the following basic construction.

Definition 151
For all $A \in \mathbb{A}$, we denote by $f_A : \mathbf{I} \longrightarrow \mathbf{I}$ the mapping defined by

$$f_A(z) = \sup A(z)$$

for all $z \in \mathbf{I}$.

It is apparent from (123) that

$$\widehat{\boxplus}A(z) = \min(f_A(z), \tfrac{1}{2})\,, \tag{135}$$

for all $A \in \mathbb{A}$ and $z \in \mathbf{I}$. In addition, it is obvious from Def. 149 that $\boxplus A$ can be defined in terms of f_A, i.e. there exists g such that $\boxplus A = g(f_A)$ for all $A \in \mathbb{A}$. In turn, I conclude that every ψ which makes $\mathcal{F}_\psi$ a DFS, can be defined in terms of f_A because every such ψ is $\boxplus$-invariant by Th-218, and hence $\psi(A) = \psi(\boxplus A) = \psi(g(f_A))$. In other words, we do not lose any models of interest if we restrict attention to those QFMs that are a function of f_A. (The precise relationship between the resulting classes of models will later be described in Th-228 and Th-229).
I now introduce the constructions necessary to define the new class of QFMs.

Definition 152
Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. By $f_{Q,X_1,\ldots,X_n} = f^{(n)}_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ we denote the mapping defined by

$$f_{Q,X_1,\ldots,X_n} = f_{A_{Q,X_1,\ldots,X_n}}\,,$$

i.e.

$$f_{Q,X_1,\ldots,X_n}(z) = \sup A_{Q,X_1,\ldots,X_n}(z)$$

for all $z \in \mathbf{I}$.

Notes

- Again, the superscript (n) in $f^{(n)}_{Q,X_1,\ldots,X_n}$ is usually omitted when no ambiguity arises.
- $f_{Q,X_1,\ldots,X_n}(z)$ expresses a measure of the maximal similarity of $(X_1, \ldots, X_n)$ to those $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$ which are mapped to $Q(Y_1, \ldots, Y_n) = z$.

Next I describe the range of all possible f_A.

Definition 153
By $\mathbb{X} \in \mathcal{P}(\mathbf{I}^{\mathbf{I}})$ we denote the set of all mappings $f : \mathbf{I} \longrightarrow \mathbf{I}$ with the following properties:

a. $\operatorname{Im} f \cap [\frac{1}{2}, 1] = \{r_+\}$ *for some* $r_+ = r_+(f) \geq \frac{1}{2}$*;*

b. If $z_+ = z_+(f) \in \mathbf{I}$ *is chosen such that* $f(z_+) = r_+$*, then* $f(z) \leq 1 - r_+$ *for all* $z \neq z_+$*.*

Theorem 226
For all $A \in \mathbb{A}$, $f_A \in \mathbb{X}$. In particular, if $Q : \mathcal{P}(E)^n$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, then $f_{Q,X_1,\ldots,X_n} \in \mathbb{X}$.

Theorem 227
For all $f \in \mathbb{X}$, there exists $A \in \mathbb{A}$ with $f = f_A$. In particular, there exist $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ with $f = f_{Q,X_1,\ldots,X_n}$.

Hence $\mathbb{X}$ is indeed the range of all possible f_A and $f_{Q,X_1,\ldots,X_n}$. We can therefore define the class of QFMs computable from $f_{Q,X_1,\ldots,X_n}$, called $\mathcal{F}_\varphi$-QFMs, in the apparent way.

Definition 154
Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given. The QFM $\mathcal{F}_\varphi$ is defined by

$$\mathcal{F}_\varphi(Q)(X_1, \ldots, X_n) = \varphi(f_{Q,X_1,\ldots,X_n}) ,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

The $\mathcal{F}_\varphi$-QFMs comprise the class of those fuzzification mechanisms which can be defined from the argument similarity grades by applying the extension principle. This is because $f_{Q,X_1,\ldots,X_n}$ is obtained from the standard extension principle in the following way. We start from a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. By applying the extension principle, we obtain $\hat{\hat{Q}} : \widetilde{\mathcal{P}}(\mathcal{P}(E)) \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$. Hence for a given $V \in \widetilde{\mathcal{P}}(\mathcal{P}(E))$, $\hat{\hat{Q}}(V) \in \widetilde{\mathcal{P}}(\mathbf{I})$ is the fuzzy subset defined by

$$\mu_{\hat{\hat{Q}}(V)}(z) = \sup\{\mu_V(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in Q^{-1}(z)\}$$

for all $z \in \mathbf{I}$, see Def. 21. Given a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we now express $V = V_{X_1,\ldots,X_n}$ in terms of argument similarity, viz

$$\mu_V(Y_1, \ldots, Y_n) = \Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) ,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. It is then apparent from Def. 152 that

$$f_{Q,X_1,\ldots,X_n}(z) = \mu_{\hat{\hat{Q}}(V)}(z)$$

for all $z \in \mathbf{I}$. Because $V = V_{X_1,\ldots,X_n}$ represents argument similarity, $\hat{\hat{Q}}$ is obtained from Q by applying the standard extension principle, $f_{Q,X_1,\ldots,X_n}$ is defined by composing $\hat{\hat{Q}}$ and $V_{X_1,\ldots,X_n}$, and $\mathcal{F}_\varphi(Q)(X_1, \ldots, X_n) = \varphi(f_{Q,X_1,\ldots,X_n})$ is a function of $f_{Q,X_1,\ldots,X_n}$, this proves the claim that every $\mathcal{F}_\varphi$ is defined from the argument similarity grades by applying the extension principle. Noticing that no additional assumptions were made in defining $f_{Q,X_1,\ldots,X_n}$, which merely composes similarity assessment and the extended $\hat{\hat{Q}}$, this demonstrates that the $\mathcal{F}_\varphi$-QFMs are precisely the QFMs definable in terms of argument similarity and the standard extension principle. The $\mathcal{F}_\varphi$-QFMs hence constitute an interesting class of fuzzification mechanisms.

10.7 The classes of $\mathcal{F}_\varphi$-models and $\mathcal{F}_\psi$-models coincide

Before going into the details of the new class of models, and disclosing the structure of its well-behaved models, let us first make two observations, which establish the precise relationship between $\mathcal{F}_\varphi$-QFMs and their apparent superclass of $\mathcal{F}_\psi$-QFMs.

Theorem 228
All $\mathcal{F}_\varphi$-QFMs are $\mathcal{F}_\psi$-QFMs, i.e. $\mathcal{F}_\varphi = \mathcal{F}_\psi$, provided that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined in dependence on $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ by

$$\psi(A) = \varphi(f_A)\,, \tag{136}$$

for all $A \in \mathbb{A}$.

Conversely,

Theorem 229
Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-5). Then $\mathcal{F}_\psi = \mathcal{F}_\varphi$, where $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ is defined by

$$\varphi(f) = \psi(A_f)\,, \tag{137}$$

for all $f \in \mathbb{X}$, and

$$A_f(z) = \begin{cases} [0, f(z)] & : \quad f(z) \leq \frac{1}{2} \\ [0, 1 - f(z)] \cup \{f(z)\} & : \quad f(z) > \frac{1}{2} \end{cases} \tag{138}$$

for all $z \in \mathbf{I}$.

These relationships are straightforward from the structure of the involved base constructions, and my earlier remark on equality (135). Recalling that from Th-218 that the condition (ψ-5) is necessary for $\mathcal{F}_\psi$ to be a DFS, this substantiates that the two classes of $\mathcal{F}_\varphi$-DFSes and $\mathcal{F}_\psi$-DFSes indeed coincide, and merely provide different views on the models definable in terms of the extension principle.

10.8 Characterisation of the $\mathcal{F}_\varphi$-models

I now impose a number of conditions on admissable choices of φ. Firstly let us define a pre-eorder on $\mathbb{X}$, again needed to express a monotonicity condition.

Definition 155
For all $f, f' \in \mathbb{X}$, we write $f \sqsubseteq f'$ if and only if the following conditions are satisfied for f, f'.

a. for all $z \in \mathbf{I}$, $\sup\{f'(z') : z' \geq z\} \geq f(z)$;

b. for all $z' \in \mathbf{I}$, $\sup\{f(z) : z \leq z'\} \geq f'(z')$.

We can now state the conditions that must be obeyed by φ in order to make $\mathcal{F}_\varphi$ a DFS.

Definition 156 *Let* $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *be given. The conditions* (φ-1)–(φ-5) *are defined as follows. For all* $f, f' \in \mathbb{X}$,

$$\text{If } f^{-1}((0,1]) = \{z_+\} \text{ and } f(z_+) = 1, \text{ then then } \varphi(f) = z_+. \tag{φ-1}$$

$$\text{If } f'(z) = f(1-z) \text{ for all } z \in \mathbf{I}, \text{ then } \varphi(f') = 1 - \varphi(f). \tag{φ-2}$$

$$\begin{gathered}\text{If } f^{-1}((0,1]) \subseteq \{0,1\} \text{ and } f(1) \geq \tfrac{1}{2}, \\ \text{then } \varphi(f) = 1 - f(0).\end{gathered} \tag{φ-3}$$

$$\text{If } f \sqsubseteq f', \text{ then } \varphi(f) \leq \varphi(f'). \tag{φ-4}$$

$$\text{If } f'(z) = \min(f(z), \tfrac{1}{2}) \text{ for all } z \in \mathbf{I}, \text{ then } \varphi(f') = \varphi(f). \tag{φ-5}$$

The proof that these conditions describe precisely the intended class of models, becomes feasible once we notice the close relationship between the 'φ-conditions' and corresponding 'ψ-conditions'.

Theorem 230
Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *be given and suppose that* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *is defined by* (136). *Then*

a. φ *satisfies* (φ-1) *if and only if* ψ *satisfies* (ψ-1)*;*

b. φ *satisfies* (φ-2) *if and only if* ψ *satisfies* (ψ-2)*;*

c. φ *satisfies* (φ-3) *if and only if* ψ *satisfies* (ψ-3)*;*

d. φ *satisfies* (φ-4) *if and only if* ψ *satisfies* (ψ-4)*;*

e. φ *satisfies* (φ-5) *if and only if* ψ *satisfies* (ψ-5)*.*

The following theorems are then straightforward from the previous results on $\mathcal{F}_\psi$-QFMs:

Theorem 231
If $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *satisfies* (φ-1)–(φ-5)*, then* $\mathcal{F}_\varphi$ *is a standard DFS.*

Theorem 232
Consider $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$*. If* $\mathcal{F}_\varphi$ *is a DFS, then* φ *satisfies* (φ-1)–(φ-5)*.*

Theorem 233
The conditions (φ-1)–(φ-5) *are independent.*

10.9 An alternative measure of argument similarity

In [46, pp. 66-78], I have made a first attempt to define DFSes in terms of the extension principle. The construction of these models was motivated by the fuzzification mechanism which Gaines [44] proposed as a 'foundation of fuzzy reasoning'. This basic mechanism was then fitted to the purpose of defining DFSes. Because the resulting approach also relies on the extension principle, but utilizes a different notion of argument compatibility, the question arises how this 'Gainesian approach' relates to the $\mathcal{F}_\varphi$-QFMs defined in terms of the extension principle. In order to answer this question, I recall some concepts needed to define the new models.

First let us define the compatibility $\theta(x, y)$ of a gradual truth value $x \in \mathbf{I}$ with a crisp truth value $y \in \mathbf{2} = \{0, 1\}$.

Definition 157

$\theta : \mathbf{I} \times \mathbf{2} \longrightarrow \mathbf{I}$ is defined by

$$\theta(x, y) = \begin{cases} 2x & : \quad x \leq \frac{1}{2}, y = 1 \\ 2 - 2x & : \quad x \geq \frac{1}{2}, y = 0 \\ 1 & : \quad \text{else} \end{cases}$$

for all $x \in \mathbf{I}$, $y \in \{0, 1\}$.

Hence a gradual truth value $x \leq \frac{1}{2}$ is considered fully compatible with 'false' ($y = 0$), but only gradually compatible with 'true' ($y = 1$), and a gradual truth value $x' \geq \frac{1}{2}$ is considered fully compatible with 'true' ($y = 1$), but only gradually compatible with 'false' ($y = 0$). θ can be applied to compare membership grades $\mu_X(e)$ ($X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E) with 'crisp' membership values $\chi_Y(e)$ (i.e. 'Is $e \in Y$?', $Y \in \mathcal{P}(E)$ crisp), where $e \in E$ is some element of the universe. This suggests the following definition of the compatibility $\Theta(X, Y)$ of a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ with a crisp subset $Y \in \mathcal{P}(E)$.

Definition 158

Let E be a nonempty set. The mapping $\Theta = \Theta_E : \widetilde{\mathcal{P}}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by

$$\Theta(X, Y) = \inf\{\theta(\mu_X(e), \chi_Y(e)) : e \in E\},$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$.

Notes

- The compatibility $\Theta(X, Y)$ of a fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ with a crisp set $Y \in \mathcal{P}(E)$ is therefore the minimal degree of element-wise compatibility of the membership function of X and the characteristic function of Y.
- To present an example, let us reconsider the two-element base set $E = \{a, b\}$, and again suppose that $X \in \widetilde{\mathcal{P}}(E)$ is defined by $\mu_X(a) = \frac{1}{3}$, $\mu_X(b) = \frac{3}{4}$. In this

case, the compatibility grades become

$$\begin{aligned}\Theta(X,\,\varnothing) &= \min\{\theta(\mu_X(a),\,\chi_\varnothing(a)),\,\theta(\mu_X(b),\,\chi_\varnothing(b))\}\\ &= \min\{\theta(\tfrac{1}{3},\,0),\,\theta(\tfrac{3}{4},\,0)\}\\ &= \min\{1,\,\tfrac{1}{2}\}\\ &= \tfrac{1}{2}\\ \Theta(X,\,\{a\}) &= \min\{\theta(\tfrac{1}{3},\,1),\,\theta(\tfrac{3}{4},\,0)\}\\ &= \min\{1,\,\tfrac{1}{2}\}\\ &= \tfrac{1}{2}\\ \Theta(X,\,\{b\}) &= \min\{\theta(\tfrac{1}{3},\,0),\,\theta(\tfrac{3}{4},\,1)\}\\ &= \min\{1,\,1\}\\ &= 1\\ \Theta(X,\,\{a,\,b\}) &= \min\{\theta(\tfrac{1}{3},\,1),\,\theta(\tfrac{3}{4},\,1)\}\\ &= \min\{\tfrac{2}{3},\,1\}\\ &= \tfrac{2}{3}\,.\end{aligned}$$

Based on $\Theta(X,\,Y)$, I now define $\widetilde{Q}_z(X_1,\dots,X_n)$, the compatibility of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to the gradual truth value $z \in \mathbf{I}$, given a choice $(X_1,\dots,X_n) \in \widetilde{\mathcal{P}}(E)^n$ of fuzzy argument sets.

Definition 159
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $z \in \mathbf{I}$. The fuzzy quantifier $\widetilde{Q}_z : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{Q}_z(X_1,\dots,X_n) = \sup\{\min_{i=1}^{n} \Theta(X_i,\,Y_i) : Y = (Y_1,\dots,Y_n) \in Q^{-1}(z)\}\,,$$

for all $(X_1,\dots,X_n) \in \widetilde{\mathcal{P}}(E)^n$.

In [46, p. 71], I have argued that the fuzzification mechanism proposed by Gaines can naturally be expressed in terms of $\widetilde{Q}_z$. In addition, three examples were developed which illustrate how DFSes can be defined from $(\widetilde{Q}_z(X_1,\dots,X_n))_{z\in\mathbf{I}}$. However, these models have subsequently been shown to be $\mathcal{M}_\mathcal{B}$-DFSes. The next theorem establishes that all such models, which are defined as a function of $(\widetilde{Q}_z(X_1,\dots,X_n))_{z\in\mathbf{I}}$, are in fact $\mathcal{F}_\varphi$-DFSes:

Theorem 234
Consider a QFM $\mathcal{F}$. Then the following statements are equivalent:

a. $\mathcal{F}$ is an $\mathcal{F}_\varphi$-QFM which satisfies (φ-5);

b. $\mathcal{F}$ is a function of the coefficients $\widetilde{Q}_z(X_1,\dots,X_n)$.

Hence the QFMs defined in terms of $\widetilde{Q}_z$ are exactly the $\mathcal{F}_\varphi$-QFMs which satisfy (φ-5). I conclude from Th-232 that the 'Gainesian' models defined in terms of $\widetilde{Q}_z$ coincide with the models defined in terms of the extension principle, i.e. with the $\mathcal{F}_\varphi$-DFSes.

10.10 Chapter summary

To sum up, this chapter has introduced a different construction of QFMs and developed the corresponding theory, in order to span a new class of models which is interesting for theoretical investigation because of its motivation from independent considerations. This departure from the three-valued cut scheme was necessary because this scheme has now been fully exploited by the introduction of $\mathcal{F}_\Omega$-QFMs. The models $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ defined in a previous publication on DFS theory [46] represent an earlier effort to accomplish the intended departure, which was inspired by the fuzzification mechanism proposed by Gaines [44]. These models, though, were subsequently shown to be $\mathcal{M}_\mathcal{B}$-DFSes, and no systematic attempt was made to extract the mechanism underlying these models and to develop a general class of models. In principle, the 'Gainesian' fuzzification mechanism is a good point of departure, due to its foundation in the extension principle of fuzzy set theory. However, the assumed compatibility measure (of a gradual to a crisp truth value; of a fuzzy subset to a crisp set) was considered somewhat awkward, and raised some concerns that the required definitions and theorems would become more complicated than necessary, to capture the target class of standard models. Consequently, I started by defining a simpler measure which quantizes the similarity of fuzzy subsets to given crisp sets $\Xi_Y(X)$, and corresponding tuples of arguments $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n)$. It was then necessary to introduce the set of similarity grades $D_{X_1,\ldots,X_n}$ that are generated from a choice of fuzzy subsets $X_1,\ldots,X_n$ under the similarity measure, and to characterize its range of possible values, $\mathbb{D}$. After that, the key construction was introduced, which to each potential quantification result assigns the set of similarity grades $A_{Q,X_1,\ldots,X_n}(z)$, which are generated by those choices of crisp $Y_1,\ldots,Y_n$ with $Q(Y_1,\ldots,Y_n) = z$. After characterising the range $\mathbb{A}$ of possible $A_{Q,X_1,\ldots,X_n}$, the class of QFMs definable in terms of argument similarity was introduced in the apparent way, based on aggregation mappings $\psi : \mathbb{A} \longrightarrow \mathbf{I}$. In order to express properties of the mappings ψ that are of relevance to the resulting QFMs $\mathcal{F}_\psi$, the required concepts were then developed, and subsequently applied to analyse the precise conditions on ψ under which the resulting QFM $\mathcal{F}_\psi$ becomes a DFS. The proposed system of conditions (ψ-1)–(ψ-5) was shown to be necessary and sufficient for $\mathcal{F}_\psi$ to be a DFS, and all $\mathcal{F}_\psi$-DFSes were proven to be standard models. In addition the independence of the criteria was established. Next I turned to the issue of relating the new class of $\mathcal{F}_\psi$-DFSes to the known class of $\mathcal{F}_\Omega$/$\mathcal{F}_\omega$-DFSes. It came as a surprise that every $\mathcal{F}_\psi$-DFS is in fact an $\mathcal{F}_\omega$-DFS (and vice versa), i.e. the class of $\mathcal{F}_\psi$-DFSes coincides with the class of $\mathcal{F}_\omega$-DFSes. Noticing that the two classes of models arose from constructions which are conceptually very different and motivated independently, this finding confirms that the $\mathcal{F}_\omega$-DFSes (or synonymously, $\mathcal{F}_\psi$-DFSes) form a natural class of standard models of fuzzy quantification, that might even comprise the full class of standard DFSes. The latter hypothesis calls for the development of analytic tools for a deeper investigation of these models, in order to locate their precise place within the standard models.

The remainder of the chapter was concerned with the class of models defined in terms of the extension principle. To this end, a mapping f_A was derived from each $A \in \mathbb{A}$. By composing these mappings with $A_{Q,X_1,\ldots,X_n}$ I defined the new base construction, that of $f_{Q,X_1,\ldots,X_n}$. For each potential quantification result z, $f_{Q,X_1,\ldots,X_n}(z)$ expressed the maximum similarity of the fuzzy arguments $X_1, \ldots, X_n$ to a choice of crisp arguments $Y_1, \ldots, Y_n \in \widetilde{\mathcal{P}}(E)$ subject to the condition that $Q(Y_1, \ldots, Y_n) = z$. Next the set $\mathbb{X}$ was introduced and shown to describe precisely the set of those mappings f that occur as $f = f_{Q,X_1,\ldots,X_n}$ for a choice of Q and $X_1, \ldots, X_n$. Hence $\mathbb{X}$ is the proper domain of aggregation operators $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ which span the new class of $\mathcal{F}_\varphi$-QFMs in the usual way, i.e. $\mathcal{F}_\varphi(Q)(X_1, \ldots, X_n) = \varphi(f_{Q,X_1,\ldots,X_n})$. I have explained that the resulting fuzzification mechanisms are exactly the QFMs definable in terms of the standard extension principle, which is applied to the similarity grades obtained for the arguments of the quantifier. The move from the base construction $A_{Q,X_1,\ldots,X_n}$ to the new construction $f_{Q,X_1,\ldots,X_n}$ means a great simplification because we now deal with a single scalar $f_{Q,X_1,\ldots,X_n}(z)$ in the unit range, rather than sets of such scalars $A_{Q,X_1,\ldots,X_n}(z)$. It is hence worthwhile studying this subclass of models and elucidating their structure, although no new DFSes are introduced compared to the full class of $\mathcal{F}_\psi$-DFSes. Interestingly, the converse is also true, and in fact no models are *lost* when restricting attention to the subclass of $\mathcal{F}_\varphi$-DFSes. This is because every $\mathcal{F}_\psi$-DFS is known to satisfy (ψ-5), which entails that $\psi(A)$ can be computed from f_A, which underlies the definition of $\mathcal{F}_\varphi$-QFMs. It is hence of particular interest to develop conditions that fit this simpler presentation of $\mathcal{F}_\psi$-DFSes, which is offered by $f_{Q,X_1,\ldots,X_n}$ and aggregation mappings φ. Due to the close relationship between $A_{Q,X_1,\ldots,X_n}$ and the derived $f_{Q,X_1,\ldots,X_n}$, the precise conditions on φ which make $\mathcal{F}_\varphi$ a DFS are apparent from the corresponding conditions (ψ-1)–(ψ-5) imposed on ψ. By adapting these conditions, it was easy to obtain a set of necessary and sufficient conditions (φ-1)–(φ-5) imposed on φ, and to prove that these conditions are independent.

Finally I have reviewed the fuzzification mechanism proposed by Gaines [44] and its reformulation as a base construction for QFMs proposed in [46]. In the course of this investigation, it was proven that all of the resulting QFMs are $\mathcal{F}_\varphi$-QFMs and hence definable in terms of argument similarity and the extension principle. Conversely, all 'reasonable' choices of $\mathcal{F}_\varphi$ where φ satisfies at least (φ-5), can be expressed as 'Gainesian' QFMs, and hence be reduced to a mechanism claimed to provide a 'foundation of fuzzy reasoning' [44].

11 Implementation of quantifiers in the models

11.1 Motivation and chapter overview

The theory of fuzzy quantification presented in this sequel rests on a strict separation of semantical aspects and computational considerations. In the previous chapters, this strategy has proven itself invaluable for developing a general interpretation framework and formalizing the notion of plausible models. A subsequent investigation of various constructive principles then resulted in the identification of concrete classes of such models. These theoretical advances will only be useful in practice, if we succeed in implementing the resulting models. In order to close the gap between the proposed formal analysis and practical applications, I will now describe the required algorithms for evaluating expressions $\mathcal{F}(Q)(X_1, \ldots, X_n)$, where $\mathcal{F}$ is the model of fuzzy quantification, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is the quantifier of interest, and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ is a choice of fuzzy arguments. I will focus only on models suited for real-world use; there is no point covering models of mere theoretical interest. Specifically, every practical model will be supposed to show a certain insensitivity against small changes in the data. In formal terms, then, the model must be both Q-continuous and arg-continuous, see Def. 63 and Def. 64. Some of the models introduced in Chap. 7-10 fall short of these robustness criteria, i.e. these models do not qualify as candidates for implementation. In particular, we need not consider any models beyond the $\mathcal{F}_\xi$-type, recalling from Th-211 that it includes all known 'practical' models. In this chapter, we will hence be concerned with the implementation of $\mathcal{F}_\xi$-DFSes only. Due to the diversity of this type of models, it is not possible to devise a 'generic' implementation which fits all examples. By contrast, I will present various techniques which are useful building-blocks for implementing arbitrary $\mathcal{F}_\xi$-models. These techniques are concerned with common patterns found in all $\mathcal{F}_\xi$-DFSes. For example, consider the joint constructive principle underlying these models, i.e. the quantification result is always computed by applying an aggregation mapping ξ to the upper and lower bounds $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$. The development of algorithms for computing these upper and lower bounds, then, is of obvious utility to the implementation of arbitrary $\mathcal{F}_\xi$-models. Due to its focus on implementation, efficiency considerations will play an important part in this chapter. Quite frequently, the description of a general solution will be followed by discussions of special cases which aim at performance improvements in the most relevant situations. In particular, I will present variants of the algorithms optimized for floating-point and integer arithmetics, in order to suit the demands of different types of applications. The prototypical models $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ will serve to demonstrate the utility of the proposed methods. They will also be used to explain how the available components can be combined into complete implementations.

The chapter is organized as follows. I start with a discussion of simple two-valued quantifiers, which share the same interpretation across all standard models. These quantifiers can be expressed in terms of the membership grades of the arguments and the connectives $\min$, $\max$ and $\neg$. I will also consider a few quantifiers of mathematical rather than linguistic interest, which compare two fuzzy sets by cardinality, which assess their degree of being equal, etc. Another interesting type are unary quantifiers on finite base sets (no longer required to be two-valued). My discussion of these quantifiers will reveal an important relationship between quantitativity and cardinality assessments. A similar analysis is then pursued for the model $\mathcal{M}_{CX}$, which reduces fuzzy one-place quantification to calculations based on a measure of fuzzy cardinality. This analysis will be useful, among other things, to establish the interpretation of quantifiers like “between ten and twenty” or “exactly five” in the standard models.

Having considered these basic examples, I then set out to develop algorithms for more complex quantifiers, in order to cover all cases of linguistic significance. To this end, I first show how the mappings $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$, on which every $\mathcal{F}_\xi$-DFS is based, can be given a finite representation. This is necessary because in practice, one can consider only a limited number of cutting levels, rather than the infinite range of all $\gamma \in \mathbf{I}$. I will show that the desired reduction is always possible if the base set is finite, i.e. every $\mathcal{F}_\xi$-DFS can be reformulated in such a way that it operates on the proposed finite representation. The prototypical models will serve to demonstrate this reformulation, which results in the basic computational procedures for implementing quantifiers in these models. These ‘raw’ procedures are not yet optimized in any way, and some further improvements are necessary in order to make them applicable to domains of realistic size. To this end, I first show that $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$ can be computed from cardinality coefficients sampled from $X_1, \dots, X_n$ and their Boolean combinations. Subsequently, I then show how these coefficients can be efficiently computed from histogram information. By combining these techniques, I finally obtain practical algorithms for implementing quantifiers in the chosen models. These algorithms are surprisingly simple, but sufficiently powerful to cover absolute quantifiers like “at least ten”, quantifiers of exception like “all except ten”, proportional quantifiers like “most” and cardinal comparatives like “more than”. Some examples are also presented along with measured processing times, which demonstrate that these algorithms are pretty fast, and ready for use in practical applications.

11.2 ‘Simple’ quantifiers

Before turning to more complex cases, let us first consider the standard quantifiers. The following results are straightforward from the representation of the universal and existential quantifiers developed in Th-30 and Th-32. It is sufficient to notice that the considered quantifiers are constructed from the logical ones in terms of negation, antonym, or unions/intersections of arguments, to which every model is known to conform by Th-12, Th-11, (Z-4) and Th-14, respectively.

Theorem 235

In every standard DFS $\mathcal{F}$ and for all $E \neq \varnothing$,

$$\begin{aligned}
&\mathcal{F}(\exists)(X) = \sup\{\mu_X(e) : e \in E\}\\
&\mathcal{F}(\forall)(X) = \inf\{\mu_X(e) : e \in E\}\\
&\mathcal{F}(\textbf{all})(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}\\
&\mathcal{F}(\textbf{some})(X_1, X_2) = \sup\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}\\
&\mathcal{F}(\textbf{no})(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), 1 - \mu_{X_2}(e)) : e \in E\}
\end{aligned}$$

for all $X, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$.

These interpretations of the elementary quantifiers are quite satisfactory. Next we shall discuss the NL quantifier “at least k” and its variants. It is convenient to start with the unrestricted version of the quantifier, and to transfer these results in a subsequent step to the (slightly) more complicated case of restricted quantification, which involves two arguments. Hence let us focus on this type of quantifiers:

Definition 160
Suppose $E \neq \varnothing$ is a base set and $k \in \mathbb{N}$. The quantifier $[\geq k] : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by

$$[\geq k](Y) = \begin{cases} 1 & : \quad |Y| \geq k \\ 0 & : \quad \textit{else} \end{cases}$$

for all $Y \in \mathcal{P}(E)$.

$[\geq k]$ is a standard quantifier because in the crisp case, it can be expressed as a Boolean combination of the existential and universal quantifiers. As to the interpretation of $[\geq k]$ in the models, I will first establish generic bounds on the possible quantification results, which are valid in arbitrary DFSes, i.e. not limited in scope to the standard models. Based on the coefficients $\mu_{[j]}(X)$ stipulated in Def. 99, which denote the j-th largest membership grade of the fuzzy subset X, these upper and lower bounds on the interpretation of $[\geq k]$ can be expressed as follows.

Theorem 236
Let $E \neq \varnothing$ be a given finite base set and $k \in \mathbb{N}$. Then in every DFS $\mathcal{F}$,

$$\mu_{[1]}(X) \mathbin{\widetilde{\wedge}} \ldots \mathbin{\widetilde{\wedge}} \mu_{[k]}(X) \leq \mathcal{F}([\geq k])(X) \leq \mu_{[k]}(X) \mathbin{\widetilde{\vee}} \ldots \mathbin{\widetilde{\vee}} \mu_{[m]}(X) \,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, where $m = |E|$.

In the case of a standard model, the upper and lower bounds coincide, because $\widetilde{\wedge}$ and $\widetilde{\vee}$ then become $\min$ and $\max$, respectively. Consequently, the above theorem uniquely determines the interpretation of $[\geq k]$ in the common models (this is apparent from the fact that the $\mu_{[j]}(X)$ form a non-increasing sequence). In addition, it is a rather straightforward task to extend this analysis to base sets of infinite cardinality as well. Summing up, the following result can then be proven for the regular models.

Theorem 237
Suppose that $\mathcal{F}$ is a standard DFS, $E \neq \varnothing$ is a nonempty base set and $k \in \mathbb{N}$. Then

$$\mathcal{F}([\geq k])(X) = \sup\{\alpha \in \mathbf{I} : |X_{\geq \alpha}| \geq k\} \,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$. In particular, if E is finite, then

$$\mathcal{F}([\geq k])(X) = \mu_{[k]}(X) \,.$$

Note. Recalling the notion of FG-count [188], the theorem hence asserts that

$$\mathcal{F}([\geq k])(X) = \mu_{\text{FG-count}(X)}(k)\,.$$

My analysis of $[\geq k]$ also reveals how some derived NL quantifiers are interpreted in the models. By decomposing the two-place quantifier "at least k" into **at least k** $=$ $[\geq k]\cap$, and by further utilizing that all models are compatible with intersections of argument sets, we can now deduce that

$$\mathcal{F}(\textbf{at least k})(X_1,\, X_2) = \sup\{\alpha \in \mathbf{I} : |(X_1 \cap X_2)_{\geq\alpha}| \geq k\}\,,$$

for all $X_1,\, X_2 \in \widetilde{\mathcal{P}}(E)$, which in the finite case again becomes

$$\mathcal{F}(\textbf{at least k})(X_1,\, X_2) = \mu_{[k]}(X_1 \cap X_2)\,.$$

Next we consider some apparent derivations of "at least k": the quantifiers "more than k", "less than k" and "at most k". In terms of the known construction of *external negation*, all of these quantifiers can be reduced to "at least k", i.e.

$$\begin{aligned} \textbf{more than k} &= \textbf{at least k+1} \\ \textbf{less than k} &= 1 - \textbf{at least k} \\ \textbf{at most k} &= 1 - \textbf{more than k} \end{aligned}$$

(this is immediate from the definition of these quantifiers, see Def. 2). Owing to Th-12, these quantifiers can hence be computed from the above interpretation of "at least k" in the apparent ways.

Finally let us consider some examples of 'simple' quantifiers which are not directly available in NL, but still useful to express relationships between fuzzy subsets and to compare these by cardinality. The following definition introduces these quantifiers and also stipulates the corresponding notation.

Definition 161

Let $E \neq \varnothing$ be some base set. For finite E, we define quantifiers $[\textbf{card}\geq]$, $[\textbf{card}>]$, $[\textbf{card}=] : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ by

$$[\textbf{card}\geq](Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| \geq |Y_2| \\ 0 & : \quad \text{else} \end{cases} \tag{139}$$

$$[\textbf{card}>](Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| > |Y_2| \\ 0 & : \quad \text{else} \end{cases} \tag{140}$$

$$[\textbf{card}=](Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = |Y_2| \\ 0 & : \quad \text{else} \end{cases} \tag{141}$$

*for all $Y_1, Y_2 \in \mathcal{P}(E)$. In addition, we define the quantifier **eq** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ by*

$$\textbf{eq}(Y_1, Y_2) = \begin{cases} 1 & : \quad Y_1 = Y_2 \\ 0 & : \quad Y_1 \neq Y_2 \end{cases} \tag{142}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. (In this case, no assumptions on the finiteness of E are necessary).

Notes

- These quantifiers serve the following purposes: $[\textbf{card}\geq](Y_1, Y_2)$ checks if the cardinality of Y_1 is at least as large as the cardinality of Y_2; $[\textbf{card}>](Y_1, Y_2)$ checks if the cardinality of Y_1 exceeds that of Y_2; $[\textbf{card}=](Y_1, Y_2)$ checks if Y_1 and Y_2 have the same cardinality; and $\textbf{eq}(Y_1, Y_2)$ checks if Y_1 and Y_2 are identical.
- Some further quantifiers which capture set-theoretic notions have already been introduced without mentioning their set-theoretic counterparts. In particular, the quantifier $\textbf{all}(Y_1, Y_2)$ checks 'subsethood' (inclusion) of Y_1 in Y_2; $\textbf{no}(Y_1, Y_2)$ checks Y_1 and Y_2 for being disjoint; and $\pi_e(Y)$ checks membership of the element e in a set Y. As pointed out by W.V. Quine, this close relationship between the universal quantifier and subsethood has long been known in logic, and 'figured already in Peirce's 1870 algebra of absolute and relative terms, thus even antedating any coherent logic of the variable itself' [63, p. 355].

It is interesting to consider the corresponding fuzzy quantifiers, which generalize these kinds of comparisons to fuzzy sets:

Theorem 238 *Let $E \neq \varnothing$ be some finite base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Then in every standard DFS $\mathcal{F}$,*

a. $\mathcal{F}([\textbf{card}\geq])(X_1, X_2) = \max\{\min(\mu_{[k]}(X_1), 1-\mu_{[k+1]}(X_2)) : 0 \leq k \leq |E|\}$;

b. $\mathcal{F}([\textbf{card}>])(X_1, X_2) = \max\{\min(\mu_k(X_1), 1-\mu_k(X_2)) : 1 \leq k \leq |E|\}$;

c. $\mathcal{F}([\textbf{card}=])(X_1, X_2) = \max\{\min\{\mu_{[k]}(X_1), 1-\mu_{[k+1]}(X_1), \mu_{[k]}(X_2), 1-\mu_{[k+1]}(X_2)\} : 0 \leq k \leq |E|\}$.

(Proof: D.4, p.443+)

Among other things, the resulting fuzzy quantifiers are useful for evaluating statements like "The number of X_1's which are X_2's is larger than the number of X_3's which are X_4's". The interpretation of this statement can now be calculated thus,

$$\mathcal{F}([\textbf{card}>])(X_1 \cap X_2, X_3 \cap X_4)\,.$$

This is straightforward from Th-238 and Th-14, Th-9.

Theorem 239 *Let $E \neq \varnothing$ be some base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. In every standard DFS $\mathcal{F}$,*

$$\begin{aligned}\mathcal{F}(\textbf{eq})(X_1, X_2) = \min(&\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) :\\ &\qquad \min(\mu_{X_1}(e), \mu_{X_2}(e)) \geq 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e))\},\\ &\inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) :\\ &\qquad 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) > \min(\mu_{X_1}(e), \mu_{X_2}(e))\})\,.\end{aligned}$$

(Proof: D.5, p.452+)

We shall see shortly how further examples of simple quantifiers, like "between r and s", can be implemented in the standard models. This analysis rests on some general observations, which we will now make.

11.3 Direct implementation of special quantifiers in $\mathcal{M}_{CX}$

In this section we shall take a closer look at quantitative (automorphism-invariant) one-place quantifiers. We notice that the quantitative unary quantifiers on finite base sets are exactly those quantifiers that only depend on cardinality information:

Theorem 240
A one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ is quantitative if and only if there exists a mapping $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ such that $Q(Y) = q(|Y|)$, for all $Y \in \mathcal{P}(E)$. q is defined by

$$q(j) = Q(Y_j) \tag{143}$$

for $j \in \{0, \dots, |E|\}$, where $Y_j \in \mathcal{P}(E)$ is an arbitrary subset of cardinality $|Y_j| = j$.

Notes

- In particular, if the quantifier has extension, then there exists $\mu_Q : \mathbb{N} \longrightarrow \mathbf{I}$ such that for all finite base sets $E \neq \varnothing$, $q(j) = \mu_Q(j)$ for all $j \in \{0, \dots, |E|\}$.
- The above result can also be likened to Mostowski's analysis of two-valued automorphism-invariant quantifiers, which can be expressed as $Q(Y) = T(\xi_0, \xi_1)$, where $\xi_0 = |E \setminus Y|$ and $\xi_1 = |Y|$, see Mostowski [108, p. 13]. Assuming a fixed choice of base set (i.e. 'quantifiers restricted to E' in Mostowski's terminology), ξ_0 can obviously be eliminated noticing that $\xi_0 = |E| - \xi_1$.

A similar analysis is also possible for fuzzy quantifiers.

Theorem 241
Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ be a unary fuzzy quantifier on a base set of finite cardinality $|E| = m$.
Then the following are equivalent.

a. $\widetilde{Q}$ is quantitative;

b. There exists a mapping $g : \mathbf{I}^m \longrightarrow \mathbf{I}$ such that

$$\widetilde{Q}(X) = g(\mu_{[1]}(X), \mu_{[2]}(X), \dots, \mu_{[m]}(X)) \tag{144}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

(Proof: D.6, p.454+)

In particular, every quantitative unary fuzzy quantifier can be expressed in terms of FG-count(X), see (4).

Returning to semi-fuzzy quantifiers, let us now consider the properties of the mapping $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ for some special types of quantifiers.

Theorem 242
A quantitative one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set is convex if and only if there exists $j_{\mathrm{pk}} \in \{0, \ldots, m\}$ such that $q(\ell) \leq q(u)$ for all $\ell \leq u \leq j_{\mathrm{pk}}$, and $q(\ell) \geq q(u)$ for all $j_{\mathrm{pk}} \leq \ell \leq u$, where $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is the mapping defined by (143)*.*

Theorem 243
A quantitative one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set is nondecreasing (nonincreasing) if and only if the mapping q defined by (143) *is nondecreasing (nonincreasing).*

Let us now simplify the formulas for $\bot_{Q,X}(\gamma)$ and $\top_{Q,X}(\gamma)$ in the case of quantitative Q, which is also useful for the median-based models because

$$Q_\gamma(X) = \mathrm{med}_{\frac{1}{2}}(\top_{Q,X}(\gamma), \bot_{Q,X}(\gamma))$$

by Th-111. To this end, we define

$$q^{\min}(\ell, u) = \min\{q(k) : \ell \leq k \leq u\} \tag{145}$$

$$q^{\max}(\ell, u) = \max\{q(k) : \ell \leq k \leq u\} \tag{146}$$

for all $0 \leq \ell \leq u \leq |E|$. We can then assert the following.

Theorem 244
For every quantitative one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set, all $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$,

$$\begin{aligned} \bot_{Q,X}(\gamma) &= q^{\min}(\ell, u) \\ \top_{Q,X}(\gamma) &= q^{\max}(\ell, u) \\ Q_\gamma(X) &= \mathrm{med}_{\frac{1}{2}}(q^{\min}(\ell, u), q^{\max}(\ell, u)), \end{aligned}$$

abbreviating $\ell = |X^{\min}_\gamma|$ and $u = |X^{\max}_\gamma|$.

In view of Th-242 and Th-243, some simplifications can be made. In the case that Q is *convex*, the formulas for computing $q^{\min}$ and $q^{\max}$ reduce to

$$q^{\min}(\ell, u) = \min(q(\ell), q(u)), \quad q^{\max}(\ell, u) = \begin{cases} q(\ell) & : \ \ell > j_{\mathrm{pk}} \\ q(u) & : \ u < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \ \ell \leq j_{\mathrm{pk}} \leq u \end{cases} \tag{147}$$

In the frequent situation that Q is *monotonic*, these expressions can be further simplified to

$$q^{\min}(\ell, u) = q(\ell), \quad q^{\max}(\ell, u) = q(u) \qquad \text{if } Q \text{ nondecreasing} \tag{148}$$

$$q^{\min}(\ell, u) = q(u), \quad q^{\max}(\ell, u) = q(\ell) \qquad \text{if } Q \text{ nonincreasing.} \tag{149}$$

Up to this point, I have investigated some properties of semi-fuzzy quantifiers on finite base sets, assuming quantitativity and only one argument for simplicity. Next we consider how fuzzy quantification involving these quantifiers can be effected in practice. It must hence be shown how expressions of the form $\mathcal{F}(Q)(X)$, where Q is a quantitative one-place quantifier on some finite base set $E \neq \varnothing$ and $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy argument, can be implemented in a given model $\mathcal{F}$. In the case of $\mathcal{M}_{CX}$, it is possible to state an explicit formula with fixed structure, which directly computes quantification results for such expressions. Specifically, the interpretation of the considered quantifiers is reduced to a calculation involving only the usual fuzzy propositional connectives $\min$, $\max$, $1 - x$, which are applied to cardinality coefficients determined by the following new notion of *fuzzy interval cardinality*:

Definition 162
For every fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$, the fuzzy interval cardinality $\|X\|_{\mathrm{iv}} \in \widetilde{\mathcal{P}}(\mathbb{N} \times \mathbb{N})$ *is defined by*

$$\mu_{\|X\|_{\mathrm{iv}}}(\ell, u) = \begin{cases} \min(\mu_{[\ell]}(X), 1 - \mu_{[u+1]}(X)) & : \ \ell \leq u \\ 0 & : \ \text{else} \end{cases} \qquad \textit{for all } \ell, u \in \mathbb{N}. \tag{150}$$

Notes

- Intuitively, $\mu_{\|X\|_{\mathrm{iv}}}(\ell, u)$ expresses the degree to which X has between ℓ and u elements. Consequently, it is dubbed a ‘fuzzy interval cardinality’ because it assigns a membership grade to each *interval* of integers, i.e. to every closed range $\ell \leq k \leq u$ of numbers $k \in \mathbb{N}$, where $\ell \leq u$. Existing fuzzy cardinality measures, by contrast, assign a membership grade to each individual integer, but not to ranges of integers.
- It is apparent from (150) that $\|X\|_{\mathrm{iv}}$ can be expressed in terms of the FG-count, noticing that $\mu_{[j]}(X) = \mu_{\text{FG-count}(X)}(j)$. Obviously, this does not mean that a proposal for fuzzy quantification based on $\|X\|_{\mathrm{iv}}$ is just a variant of the FG-count approach.

The relevance of the proposed fuzzy interval cardinality to $\mathcal{M}_{CX}$ is revealed by the following theorem.

Theorem 245
For every quantitative one-place quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set and all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned} \mathcal{M}_{CX}(Q)(X) &= \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell, u), q^{\min}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\} \\ &= \min\{\max(1 - \mu_{\|X\|_{\mathrm{iv}}}(\ell, u), q^{\max}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\} \,. \end{aligned}$$

Notes

- Among other things, the theorem shows that the cardinality-based approach to fuzzy quantification can be recovered in the case of quantitative one-place quantifiers on finite domains, if we rely on $\mathcal{M}_{CX}$ for modelling fuzzy quantification (which is foremost among the models anyway). Let me emphasize that absolutely no assumptions regarding monotonicity or other properties of Q are necessary; the obtained results are guaranteed to be plausible (in the sense formalized by the theory) for arbitrary and totally unrestricted choices of quantifiers as long as Q is quantitative. The fuzzy interval cardinality stipulated above therefore achieves the first formalization of fuzzy cardinality for fuzzy sets, which gives a provably satisfying account of fuzzy quantification. In particular, my approach fully covers Zadeh's quantifiers of the first kind, including all non-monotonic examples.
- As has been remarked above, $\|X\|_{\mathrm{iv}}$ can be expressed in terms of FG-count(X). However, the converse claim is equally true. Specifically, it is instructive to notice that

  $$\mu_{\text{FG-count}(X)}(j) = \mu_{\|X\|_{\mathrm{iv}}}(j, |E|)\,, \qquad \mu_{\text{FE-count}(X)}(j) = \mu_{\|X\|_{\mathrm{iv}}}(j, j)\,.$$

  This explains why with general quantifiers μ_Q, the FG-count approach and the FE-count approach yield reasonable results in some cases (i.e. those in which they coincide with $\mathcal{M}_{CX}$) but fail at others.

Further results on the interpretation of two-valued quantifiers in standard models are easily proven from this representation of $\mathcal{M}_{CX}$, recalling the earlier Th-46. For example, consider the unary convex quantifier $[\geq r : \leq s]$ defined as follows.

Definition 163
Let $E \neq \varnothing$ be some base set and $r, s \in \mathbb{N}$, $r \leq s$. The quantifier $[\geq r : \leq s] : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by

$$[\geq r : \leq s](Y) = \begin{cases} 1 & : \quad r \leq |Y| \leq s \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y \in \mathcal{P}(E)$.

The quantifier is apparently useful for interpreting statements like "Between ten and twenty of the married persons have children". Let us now make explicit the concrete interpretation of $[\geq r : \leq s]$ in the standard models. The following theorem is rather straightforward from the achieved representation of $\mathcal{M}_{CX}$, and the known fact that all standard models coincide on two-valued quantifiers:

Theorem 246
Let $E \neq \varnothing$ be a finite base set and $r, s \in \mathbb{N}$, where $r \leq s$. Further suppose that $\mathcal{F}$ is

a given standard DFS. Then $\mathcal{F}([\geq r : \leq s]) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is the fuzzy quantifier defined by*

$$\mathcal{F}([\geq r : \leq s])(X) = \mu_{\|X\|_{\mathrm{iv}}}(r, s) = \min(\mu_{[r]}(X), 1 - \mu_{[s+1]}(X))$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(Proof: D.7, p.456+)

This analysis of $\mathcal{F}([\geq r : \leq s])$ is readily extended to the convex natural language quantifiers “between r and s” that were introduced in Def. 2. Recalling the operation of intersecting arguments defined in Def. 33, the two-place quantifier “between r and s” becomes **between r and s** $= [\geq r : \leq s]\cap$, i.e.

$$\textbf{between r and s}(Y_1, Y_2) = [\geq r : \leq s](Y_1 \cap Y_2)$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. Now applying Th-14, we directly obtain from the above Th-246 that in every standard DFS,

$$\begin{aligned}\textbf{between r and s}(X_1, X_2) &= \mu_{\|X_1 \cap X_2\|_{\mathrm{iv}}}(r, s) \\ &= \min(\mu_{[r]}(X_1 \cap X_2), 1 - \mu_{[s+1]}(X_1 \cap X_2)) \,,\end{aligned}$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $E \neq \varnothing$ is again assumed to be finite. The latter result also covers the quantifier “exactly k”, which can be expressed as **exactly k** $=$ **between k and k**. Therefore

$$\begin{aligned}\mathcal{F}(\textbf{exactly k})(X_1, X_2) &= \mu_{\|X_1 \cap X_2\|_{\mathrm{iv}}}(k, k) \\ &= \min(\mu_{[k]}(X_1 \cap X_2), 1 - \mu_{[k+1]}(X_1 \cap X_2)) \\ &= \mu_{\text{FE-count}(X_1 \cap X_2)}(k)\end{aligned}$$

in all standard models.

11.4 The core algorithms for general quantifiers

In the previous section, it was shown that $\mathcal{M}_{CX}(Q)(X)$ can be reduced to a fuzzy propositional formula, which is built from constants $q^{\min}(\ell, u), q^{\max}(\ell, u) \in \mathbf{I}$ sampled from the quantifier, and from cardinality coefficients $\mu_{\|X\|_{\mathrm{iv}}}(\ell, u) \in \mathbf{I}$ obtained from the argument.

For the important proportional kind and other two-place quantifiers, however, there is no known reduction of $\mathcal{M}_{CX}(Q)(X_1, X_2)$ to a closed-form expression involving some (relative) notion of fuzzy cardinality. This makes it necessary to develop a more general, iterative procedure for computing quantification results. The need for such a general procedure is even more obvious in the case of the other models. For example, there is no apparent method of directly computing $\mathcal{M}(Q)(X)$ or $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$ from a formula of fixed structure like that presented in Th-245, even in the simplest case of a quantitative one-place quantifier on a finite domain.

In the course of implementing quantifiers in such general $\mathcal{F}_\xi$-DFSes, the first hindrance that we face is this. According to Def. 102, these models are defined by

$$\mathcal{F}_\xi(Q)(X_1, \dots, X_n) = \xi(\top, \bot)\,,$$

where $\top = \top_{Q,X_1,\dots,X_n}$ and $\bot = \bot_{Q,X_1,\dots,X_n}$ are obtained from the three-valued cuts of the argument sets at *all* cutting levels $\gamma \in \mathbf{I}$. The point at issue is that the cutting parameter γ takes its values in a *continuous range*. An implementation on digital computers, however, can only consider a finite sample of relevant cut levels $\Gamma = \{\gamma_0, \dots, \gamma_m\}$, along with the corresponding results of $\top$ and $\bot$ at these levels. In order to overcome this problem, I will now take a closer look at the specific shape of $\top$ and $\bot$ for a certain type of fuzzy sets, which includes all fuzzy arguments on finite base sets and important classes of fuzzy sets defined on base sets of transfinite cardinality. I will show that in this case, the desired reduction to a finite sample of $\top(\gamma)$ and $\bot(\gamma)$ is always possible. Moreover, I will spell out the fundamental computational procedures for $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$, by reformulating these prototypical models in such a way that they operate on the chosen finite sample only. In this way, it becomes possible to compute quantification results of arbitrary quantifiers on finite base sets in the prototype models.

In order to develop the desired finite representation of $\top$ and $\bot$, we need some further notation. Hence let some base set $E \neq \varnothing$ be given (not required to be finite) and further let $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. We shall denote by $A(X_1, \dots, X_n) \in \mathcal{P}(\mathbf{I})$ the set of membership grades assumed by one of the X_i's. Hence

$$A(X_1, \dots, X_n) = \cup\{\operatorname{Im}\mu_{X_i} : i \in \{1, \dots, n\}\}\,, \tag{151}$$

i.e. $A(X_1, \dots, X_n) = \{\mu_{X_i}(e) : e \in E, i \in \{1, \dots, n\}\}$. In dependence on $A(X_1, \dots, X_n)$, we can further define the corresponding set of three-valued cut levels $\Gamma(X_1, \dots, X_n) \in \mathcal{P}(\mathbf{I})$ according to

$$\begin{aligned}\Gamma(X_1, \dots, X_n) = \{2\alpha - 1 &: \alpha \in A(X_1, \dots, X_n) \cap [\tfrac{1}{2}, 1]\}\\ \cup\{1 - 2\alpha &: \alpha \in A(X_1, \dots, X_n) \cap [0, \tfrac{1}{2})\}\\ \cup\{0, 1\}&\,.\end{aligned} \tag{152}$$

Notes

- It is obvious that for finite base sets, $\Gamma(X_1, \dots, X_n)$ is always finite as well.
- It should be pointed out that $\Gamma(X_1, \dots, X_n)$ always includes the boundary cases $\gamma = 0$ and $\gamma = 1$, which I enforced by an explicit union with $\{0, 1\}$ in the defining equation for $\Gamma(X_1, \dots, X_n)$. I decided to incorporate these boundary cases into $\Gamma(X_1, \dots, X_n)$ because knowing that $0 \in \Gamma(X_1, \dots, X_n)$ and $1 \in \Gamma(X_1, \dots, X_n)$ will considerably simplify the presentation of the later algorithms which operate on $\Gamma(X_1, \dots, X_n)$.
- Obviously, $\Gamma(X_1, \dots, X_n)$ can always be decomposed into a union of the components $\Gamma(X_i)$, i.e.

$$\Gamma(X_1, \dots, X_n) = \Gamma(X_1) \cup \dots \cup \Gamma(X_n)\,. \tag{153}$$

The following observation on the behaviour of $\Gamma(\bullet)$ for Boolean combinations will later prove useful when implementing proportional quantifiers:

Theorem 247
Let $E \neq \varnothing$ be a given base set. Then

a. *for all $X \in \widetilde{\mathcal{P}}(E)$, $\Gamma(\neg X) = \Gamma(X)$;*

b. *for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$,*

$$\Gamma(X_1 \cap X_2) \subseteq \Gamma(X_1, X_2)$$

and

$$\Gamma(X_1 \cup X_2) \subseteq \Gamma(X_1, X_2)\,.$$

(Proof: D.8, p.458+)

Now suppose that $\Gamma(X_1, \ldots, X_n)$ is finite and that $\Gamma \supseteq \Gamma(X_1, \ldots, X_n)$ is a finite superset of $\Gamma(X_1, \ldots, X_n)$.[39] Knowing that $\{0, 1\} \subseteq \Gamma(X_1, \ldots, X_n)$, Γ can then be written as $\Gamma = \{\gamma_0, \ldots, \gamma_m\}$ where $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. In dependence on $\gamma_0, \ldots, \gamma_m$, we shall define derived coefficients $\overline{\gamma}_0, \ldots, \overline{\gamma}_{m-1}$ according to

$$\overline{\gamma}_j = \tfrac{\gamma_j + \gamma_{j+1}}{2} \tag{154}$$

for all $j \in \{0, \ldots, m-1\}$. In the following, it will be convenient to introduce a succinct notation for the results of $\top_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j)$ and $\bot_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j)$ that are observed at each $\overline{\gamma}_j$. Let us hence stipulate that

$$\top_j = \top_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j) \tag{155}$$

$$\bot_j = \bot_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j) \tag{156}$$

for $j \in \{0, \ldots, m-1\}$. When discussing $\mathcal{M}_{\mathcal{B}}$-models, it will also be useful to have a shorthand notation for $Q_{\overline{\gamma}_j}(X_1, \ldots, X_n)$. I therefore abbreviate

$$C_j = Q_{\overline{\gamma}_j}(X_1, \ldots, X_n) = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j) \tag{157}$$

for all $j \in \{0, \ldots, m-1\}$, where the equalities are immediate from Th-111, (155) and (156).

Theorem 248
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments such that $\Gamma(X_1, \ldots, X_n)$ is finite. Further let $0 = \gamma_0 < \gamma_1 < \cdots <$

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[39] The use of the finite superset $\Gamma \supseteq \Gamma(X_1, \ldots, X_n)$, rather than $\Gamma(X_1, \ldots, X_n)$, has shown itself more convenient for proving the subsequent theorems.

$\gamma_{m-1} < \gamma_m = 1$ *be given such that* $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X_1, \ldots, X_n)$. *Then for all* $j \in \{0, \ldots, m-1\}$ *and all* $\gamma \in (\gamma_j, \gamma_{j+1})$,

$$\begin{aligned} \top_{Q,X_1,\ldots,X_n}(\gamma) &= \top_j \\ \bot_{Q,X_1,\ldots,X_n}(\gamma) &= \bot_j \\ Q_\gamma(X_1, \ldots, X_n) &= C_j \,. \end{aligned}$$

(Proof: D.9, p.460+)

In other words, $\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ reduce to simple step functions (with a finite number of steps), which are locally constant in the open intervals (γ_j, γ_{j+1}), $j = 0, \ldots, m-1$. In turn, the mapping $Q_\gamma(X_1, \ldots, X_n)$ which underlies the construction of $\mathcal{M}_\mathcal{B}$-QFMs also reduces to a simple step function. The relevance of these observations manifests itself in the next theorem.

Theorem 249
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *be a semi-fuzzy quantifier and* $(\gamma_j)_{j\in\{0,\ldots,m\}}$ *a finite* $\mathbf{I}$*-valued sequence such that*

$$0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1\,.$$

Further suppose that $(\top, \bot), (\top', \bot') \in \mathbb{T}$ *satisfy*

$$\begin{aligned} \top(\gamma) &= \top'(\gamma') \\ \bot(\gamma) &= \bot'(\gamma') \end{aligned}$$

for all $j \in \{0, \ldots, m-1\}$ *and* $\gamma, \gamma' \in (\gamma_j, \gamma_{j+1})$.[40] *Then*

$$\xi(\top, \bot) = \xi(\top', \bot')$$

for every choice of $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *which satisfies* (X-2), (X-4) *and* (X-5).
(Proof: D.10, p.462+)

Note. The theorem states that the results of the step functions $\top$ and $\bot$ at the finite number of interval boundaries are inessential, provided that ξ satisfies (X-2), (X-4) and (X-5). In other words, $\xi(\top, \bot)$ is fully determined by the finite sample of γ_j, $\top_j = \top(\overline{\gamma}_j)$ and $\bot_j = \bot(\overline{\gamma}_j)$, because any choice of $(\top', \bot') \in \mathbb{T}$ with $\top'(\gamma) = \top_j$ and $\bot'(\gamma) = \bot_j$ for all $j \in \{0, \ldots, m-1\}$ and $\gamma \in (\gamma_j, \gamma_{j+1})$ will reproduce the desired result $\xi(\top', \bot') = \xi(\top, \bot)$.

Let us now put this result into context in order to highlight its significance to the models of fuzzy quantification. We know from Th-110 that every $\mathcal{F}_\xi$-DFS is constructed from a choice of $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which satisfies the critical conditions (X-2), (X-4) and (X-5). Now suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are chosen such that $\Gamma(X_1, \ldots, X_n)$ is finite. By combining the

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[40] Hence $\top, \top'$ and $\bot, \bot'$ are pairs of step functions which are allowed to differ only at the boundaries $\{\gamma_j : j \in \{0, \ldots, m\}\}$ of the steps, and which are required to coincide everywhere else.

above theorems Th-248 and Th-249, it then becomes apparent that $\mathcal{F}_\xi(Q)(X_1, \dots, X_n)$ can be computed from the finite number of 3-tuples

$$(\gamma_j, \top_j, \bot_j),$$

which are obtained from an arbitrary sample $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$ with $\{\gamma_0, \dots, \gamma_m\} \supset \Gamma(X_1, \dots, X_n)$. Recalling that $Q_\gamma(X_1, \dots, X_n) = \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\dots,X_n}(\gamma), \bot_{Q,X_1,\dots,X_n}(\gamma))$ and $C_j = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j)$, this demonstrates in particular that $Q_\gamma(X_1, \dots, X_n)$ can be specified with sufficient detail by listing the finite number of pairs (γ_j, C_j), which carry all necessary information to compute the quantification results in a given $\mathcal{M}_\mathcal{B}$-DFS.

Based on the improved analysis of $\top_{Q,X_1,\dots,X_n}$ and $\bot_{Q,X_1,\dots,X_n}$ achieved in the above theorems, the model $\mathcal{F}_{\mathrm{Ch}}$ can now be expressed in a form which lends itself better to computation.

Theorem 250
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments such that $\Gamma(X_1, \dots, X_n)$ is finite. Further suppose that $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, \dots, X_n)$ is given, where $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. Then

$$\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, \dots, X_n) = \tfrac{1}{2} \sum_{j=0}^{m-1} (\gamma_{j+1} - \gamma_j)(\top_j + \bot_j)\,. \tag{158}$$

(Proof: D.11, p.464+)

Next we will consider the prototypical examples of $\mathcal{M}_\mathcal{B}$-DFSes that were chosen for implementation, i.e. $\mathcal{M}$ and $\mathcal{M}_{CX}$. In order to exploit Th-248 for a more computational description of these models as well, we first need some observations on the possible shapes of C_j and corresponding quantification results in the $\mathcal{M}_\mathcal{B}$-type models.

Theorem 251
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments such that $\Gamma(X_1, \dots, X_n)$ is finite. Further let $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, \dots, X_n)$ be given with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. If $C_0 = \frac{1}{2}$, then

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \dots, X_n) = \tfrac{1}{2}$$

in every $\mathcal{M}_\mathcal{B}$-DFS.
(Proof: D.12, p.464+)

In order to achieve a further simplification, let us introduce additional abbreviations

$$J^* = \{j \in \{0, \dots, m-1\} : C_j = \tfrac{1}{2}\} \tag{159}$$

$$j^* = \begin{cases} \min J^* & : \quad J^* \neq \varnothing \\ m & : \quad J^* = \varnothing \end{cases} \tag{160}$$

It is obvious from Def. 85 that for all $f \in \mathbb{B}$, $\gamma' \geq \gamma$ results in $f(\gamma') \preceq_c f(\gamma)$. Hence $f(\gamma) = \frac{1}{2}$ for some $\gamma \in \mathbf{I}$ entails that $f(\gamma') = \frac{1}{2}$ for all $\gamma' > \gamma$ as well. Recalling from Th-64 that $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}$, we conclude that in fact

$$C_j = \tfrac{1}{2} \tag{161}$$

for all $j \geq j^*$. Combining this with some apparent observations on the fuzzy median $\mathrm{med}_{\frac{1}{2}}$, we can now assert the following.

Theorem 252

Suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a given semi-fuzzy quantifier, and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given fuzzy arguments such that $\Gamma(X_1, \ldots, X_n)$ is a finite subset of $\mathbf{I}$. Further assume a choice of $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X_1, \ldots, X_n)$ with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$.

a. If $\bot_0 > \frac{1}{2}$, then

$$C_j = \begin{cases} \bot_j & : \quad j < j^* \\ \frac{1}{2} & : \quad j \geq j^* \end{cases}$$

for all $j \in \{0, \ldots, m-1\}$.

b. If $\top_0 < \frac{1}{2}$, then

$$C_j = \begin{cases} \top_j & : \quad j < j^* \\ \frac{1}{2} & : \quad j \geq j^* \end{cases}$$

for all $j \in \{0, \ldots, m-1\}$.

(Proof: D.13, p.464+)

Based on these preparations, it is now rather easy to decompose $\mathcal{M}(Q)(X_1, \ldots, X_n)$ into a simple weighted summation:

Theorem 253

Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given, and suppose that $\Gamma(X_1, \ldots, X_n)$ is finite. Further assume a choice of $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X_1, \ldots, X_n)$ with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. Then

$$\mathcal{M}(Q)(X_1, \ldots, X_n) = \begin{cases} \left(\sum\limits_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) \bot_j \right) + \frac{1}{2}(1 - \gamma_{j^*}) & : \quad \bot_0 > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot_0 \leq \frac{1}{2} \leq \top_0 \\ \left(\sum\limits_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) \top_j \right) + \frac{1}{2}(1 - \gamma_{j^*}) & : \quad \top_0 < \frac{1}{2} \end{cases}$$

(Proof: D.14, p.466+)

The computation of $\mathcal{M}_{CX}$ is even simpler compared to $\mathcal{F}_{\mathrm{Ch}}$ and $\mathcal{M}$, which require a summation in order to determine the final outcome of quantification from the results obtained at the individual j's. In fact, it is sufficient for calculating the quantification result $\mathcal{M}_{CX}(Q)(X_1, \dots, X_n)$, to simply determine the minimal choice of $j \in \{0, \dots, m-1\}$ which satisfies a certain inequality.

Theorem 254

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments such that $\Gamma(X_1, \dots, X_n)$ is finite. Further let $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, \dots, X_n)$ be a subset of $\mathbf{I}$ with $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$. In the case that $C_0 > \frac{1}{2}$, let us abbreviate

$$B_j = 2\bot_j - 1\,, \tag{162}$$

while in the case that $C_0 < \frac{1}{2}$, we stipulate

$$B_j = 1 - 2\top_j \tag{163}$$

for all $j \in \{0, \dots, m-1\}$. Let us further abbreviate

$$\widehat{J} = \{j \in \{0, \dots, m-1\} : B_j \leq \gamma_{j+1}\} \tag{164}$$

$$\widehat{\jmath} = \min \widehat{J}\,. \tag{165}$$

Then

$$\mathcal{M}_{CX}(Q)(X_1, \dots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2}\max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}}) & : \quad \bot_0 > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot_0 \leq \frac{1}{2} \leq \top_0 \\ \frac{1}{2} - \frac{1}{2}\max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}}) & : \quad \top_0 < \frac{1}{2}\,. \end{cases}$$

(Proof: D.15, p.467+)

To sum up, $\bot_{Q,X_1,\dots,X_n}$, $\top_{Q,X_1,\dots,X_n}$ and $Q_\gamma(X_1, \dots, X_n)$ can be specified with sufficient detail by a finite number of γ_j, $\top_j$, $\bot_j$, from which the quantification results $\mathcal{F}_\xi(Q)(X_1, \dots, X_n)$ of arbitrary $\mathcal{F}_\xi$-DFSes can be computed. For discussing the $\mathcal{M}_\mathcal{B}$-type, it proved convenient to utilize a derived coefficient $C_j = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j)$ instead of $\top_j$ and $\bot_j$. The representation of these C_j's was further simplified based on the known monotonicity properties of $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}$. My analysis of $\top_{Q,X_1,\dots,X_n}$, $\bot_{Q,X_1,\dots,X_n}$ and $Q_\gamma(X_1, \dots, X_n)$ made it possible to develop more explicit formulations of the prototypical models $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$, which lend themselves directly to implementation.

In the remainder of the chapter, I will describe further refinements which improve the efficiency of the raw algorithms presented so far. These efficiency improvements will not require any changes to the computation procedures themselves. By contrast, they rest on the observation that the analysis achieved in theorems Th-250, Th-253 and Th-254 still depends on $\top_j$, $\bot_j$ and/or C_j, and it is these coefficients which now receive careful attention. This will also provide the necessary support for implementing further models of the $\mathcal{F}_\xi$- and $\mathcal{M}_\mathcal{B}$ types, which are all known to be expressible in terms of $\top_j$ and $\bot_j$ by Th-249.

11.5 Refinement for quantitative quantifiers

In this section, we will be concerned with the issue of computing $\top_j = \top_{Q,X_1,\dots,X_n}(\overline{\gamma}_j)$ and $\bot_j = \bot_{Q,X_1,\dots,X_n}(\overline{\gamma}_j)$ efficiently, where $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is assumed to have a finite domain $E \neq \varnothing$.

In principle, Def. 100 permits the direct computation of $\top_j$ and $\bot_j$ based on an exhaustive search of the maximum and minimum[41] of $\mathcal{S}_{Q,X_1,\dots,X_n}(\overline{\gamma}_j)$, i.e. $\top_j = \max S_j$ and $\bot_j = \min S_j$, where

$$S_j = \mathcal{S}_{Q,X_1,\dots,X_n}(\overline{\gamma}_j) = \{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathcal{T}_{\overline{\gamma}_j}(X_1,\dots,X_n)\} \,.$$

However, the exhaustive search of the minimum and maximum becomes impractical for domains of realistic size. In the worst case, the number of elements in S_j can be exponential in the size of the domain. Therefore the effort for computing the minimum and maximum directly soon becomes prohibitive.

In some cases, it is possible to shortcut the exhaustive search. For example, consider a quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which is nondecreasing in all arguments. The coefficients $\top_j$ and $\bot_j$ can then be expressed as $\top_j = Q((X_1)^{\max}_{\overline{\gamma}_j},\dots,(X_n)^{\max}_{\overline{\gamma}_j})$ and $\bot_j = Q((X_1)^{\min}_{\overline{\gamma}_j},\dots,(X_n)^{\min}_{\overline{\gamma}_j})$. It is hence sufficient to consider a single choice of $Y_i \in \mathcal{T}_{\overline{\gamma}_j}(X_i)$, and all other choices of Y_i are known to be irrelevant. In particular, this approach will work for $n = 1$, i.e. in the case of nondecreasing unary quantifiers $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$. It therefore renders possible the efficient evaluation of *fuzzy measures* in the models of interest. In the remaining cases where this a priori simplification of $\top_j$ and $\bot_j$ is not applicable, a careful analysis of other regularities of NL quantifiers is necessary in order to identify possible simplifications, from which an efficient implementation of $\top_j$ and $\bot_j$ can be derived. The particular regularity that will be assumed in the following, is that of *quantitativity*, see Def. 38. Consequently, I will restrict attention to those semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite domain $E \neq \varnothing$ which also exhibit automorphism-invariance, i.e.

$$Q(Y_1,\dots,Y_n) = Q(\widehat{\beta}(Y_1),\dots,\widehat{\beta}(Y_n))$$

for all automorphisms β of E and $Y_1,\dots,Y_n \in \mathcal{P}(E)$. How can we exploit this regularity in order to simplify the computation of $\top_j$ and $\bot_j$? To this end, let us observe that $\top_j$ and $\bot_j$ are computed from S_j, which in turn is computed from the set $\mathcal{T}_{\overline{\gamma}_j}(X_1,\dots,X_n) \subseteq \mathcal{P}(E)^n$ defined by Def. 82. This suggests that we can spare unnecessary work, if we manage to define an equivalence relation on $\mathcal{P}(E)^n$ which identifies argument tuples $(Y_1,\dots,Y_n), (Y'_1,\dots,Y'_n) \in \mathcal{T}_{\overline{\gamma}_j}(X_1,\dots,X_n)$ with

$$Q(Y_1,\dots,Y_n) = Q(Y'_1,\dots,Y'_n) \,.$$

Hence let $\sim \subseteq \mathcal{P}(E)^n \times \mathcal{P}(E)^n$ denote the following relation,

$$(Y_1,\dots,Y_n) \sim (Y'_1,\dots,Y'_n) \quad \Leftrightarrow$$

$$\text{there exists an automorphism } \beta \text{ of } E \text{ s.th. } (Y'_1,\dots,Y'_n) = (\widehat{\beta}(Y_1),\dots,\widehat{\beta}(Y_n)) \,.$$

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[41] The use of $\min$ and $\max$ rather than $\inf$ and $\sup$ is possible because $\mathcal{S}_{Q,X_1,\dots,X_n}(\overline{\gamma}_j)$ is finite. This is immediate from Def. 110 and the finiteness of the base set.

It is apparent that $\sim$ is indeed an equivalence relation on $\mathcal{P}(E)^n$. In addition, we can conclude from $(Y_1, \ldots, Y_n) \sim (Y_1', \ldots, Y_n')$ that $Q(Y_1, \ldots, Y_n) = Q(Y_1', \ldots, Y_n')$, because Q is assumed to be automorphism-invariant. This suggests that we can avoid redundant effort in computing $\top_j$ and $\bot_j$ in the following way. For each $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$, let $(Y_1, \ldots, Y_n)^*$ denote a representative under $\sim$, which is linked to the given $(Y_1, \ldots, Y_n)$ by $(Y_1, \ldots, Y_n) \sim (Y_1, \ldots, Y_n)^*$. Due to the fact that Q is automorphism invariant, we then know that $Q(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)^*$. Now let $\mathcal{T}^*$ denote the set of all representatives for argument tuples in $\mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)$, i.e.

$$\mathcal{T}^* = \{(Y_1, \ldots, Y_n)^* : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)\}\,.$$

Clearly

$$\begin{aligned} S_j &= \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)\} \\ &= \{Q(Y_1, \ldots, Y_n)^* : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)\} \\ &= \{Q(Z_1, \ldots, Z_n) : (Z_1, \ldots, Z_n) \in \mathcal{T}^*\} \end{aligned} \tag{166}$$

and

$$|\mathcal{T}^*| \leq |\mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)|\,.$$

Consequently, the search of the maximum and minimum can now be restricted to the small(er) number of representatives in $\mathcal{T}^*$, and need not exhaust the total collection of alternatives in $\mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)$.

In practice, it is advisable to advance yet another step and eliminate these representatives as well, in favour of a purely numerical scheme which rests on cardinality information only. The promise of such a scheme is that of avoiding any reference to set-based information (like $\mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)$ or $\mathcal{T}^*$), which might be hard to represent in a uniform way, and awkward for processing purposes. By contrast, the cardinal numbers on which the numerical scheme will operate, are easy to represent and permit very fast processing on digital computers. For the development of such a scheme, we can utilize a special regularity observed with all quantitative quantifiers, which establishes a link between quantitativity (automorphism-invariance) and definability of the quantifier in terms of cardinalities sampled from the arguments. For example, consider the quantitative unary quantifiers anticipated in section 11.3. It was shown there that every quantifier of the considered type can be expressed in terms of cardinality information, i.e. there is a choice of $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that $Q(Y) = q(|Y|)$ for all $Y \in \mathcal{P}(E)$. Based on this representation, I was able to develop efficient reformulations of $\top_j = \top_{Q,X}(\overline{\gamma}_j)$ and $\bot_j = \bot_{Q,X}(\overline{\gamma}_j)$, which reduce to simple coefficients $q^{\max}(\ell, u)$ and $q^{\min}(\ell, u)$, respectively.

The goal of the present section is to abstract from this special case and develop a similar representation for arbitrary quantifiers, which are only assumed to be quantitative, and declared on a finite base set. I will show that the previous analysis of unary quantifiers can be broadened to a fully general result concerning quantitative quantifiers on finite base sets. In particular, the computation of $\top_j$, $\bot_j$ and C_j from cardinality information is always possible if Q belongs to this type of quantifier, which obviates the need to operate on the original set-based data in $\mathcal{T}_{\overline{\gamma}_j}(X_1, \ldots, X_n)$. For the main class of NL quantifiers, which are two-place and conservative (like the absolute and proportional kinds), this basic approach will then be further refined, in order to achieve optimal performance for all quantifiers of relevance to applications.

To begin with, it is well-known from TGQ that the quantitative two-valued quantifiers on finite base sets are exactly those quantifiers which can be computed from the cardinalities of the arguments and their Boolean combinations [8, p. 471]. As I will now show, this characterisation of quantitative quantifiers on finite base sets generalizes to semi-fuzzy quantifiers, i.e. the quantitative semi-fuzzy quantifiers on finite domains are precisely those semi-fuzzy quantifiers which solely depend on the cardinality of their arguments (or Boolean combinations thereof). In order to state the theorem, I need some more notation. Hence let a finite base set $E \neq \varnothing$ and $n \in \mathbb{N}$ be given. For every choice of $\ell_1, \ldots, \ell_n \in \{0, 1\}$, we define the Boolean combination $\Phi_{\ell_1,\ldots,\ell_n}$ by

$$\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n) = Y_1^{(\ell_1)} \cap \cdots \cap Y_n^{(\ell_n)} \tag{167}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, where

$$Y^{(\ell)} = \begin{cases} Y & : \ \ell = 1 \\ \neg Y & : \ \ell = 0 \end{cases} \tag{168}$$

for $Y \in \mathcal{P}(E)$ and $\ell \in \{0, 1\}$. Based on these abbreviations, we can now assert the following.

Theorem 255
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ is quantitative if and only if Q can be computed from the cardinalities of its arguments and their Boolean combinations, i.e. there exist Boolean expressions $\Phi_1(Y_1, \ldots, Y_n), \ldots, \Phi_K(Y_1, \ldots, Y_n)$ for some $K \in \mathbb{N}$, and a mapping $q : \{0, \ldots, |E|\}^K \longrightarrow \mathbf{I}$ such that

$$Q(Y_1, \ldots, Y_n) = q(|\Phi_1(Y_1, \ldots, Y_n)|, \ldots, |\Phi_K(Y_1, \ldots, Y_n)|)\,, \tag{169}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. In particular, Q can be expressed as $Q(Y_1, \ldots, Y_n) = q(c)$, where $c : \{0, 1\}^n \longrightarrow \{0, \ldots, |E|\}$ is defined by

$$c_{\ell_1,\ldots,\ell_n} = |\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)|$$

for all $\ell_1, \ldots, \ell_n \in \{0, 1\}$, see (167).
(Proof: D.16, p.470+)

Notes

- For convenience, I will assume that the Boolean expressions $\Phi_i(Y_1, \ldots, Y_n)$ are constructed from the arguments $Y_1, \ldots, Y_n$ by forming unions $\cup$, intersections $\cap$ and complementation $\neg$; of course, a minimal set of operations $\mathrm{op} \in \{\cap, \neg\}$ would be sufficient as well. It is not required that all variables $Y_1, \ldots, Y_n$ actually participate in every considered Φ_i. In particular, trivial combinations like $\Phi_1(Y_1, Y_2, Y_3) = Y_2$ and more complex ones like $\Phi_2(Y_1, Y_2, Y_3) = (Y_1 \cap \neg Y_2) \cup Y_3$ are equally admissible.

- The second part of the theorem establishes a worst-case analysis in terms of complexity. It asserts that for unrestricted quantifiers, the number of Boolean combinations required to express Q in terms of cardinalities, is bounded by 2^n, where n is the arity of the quantifier.

It is instructive to compare the general result of Th-255 to the earlier analysis of quantitative one-place quantifiers that was given in section 11.3. In the new notation, the special case covered there can now be represented in terms of $K = 1$, $\Phi(Y) = Y$, $c = |\Phi(Y)| = |Y|$ and $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$. It is then guaranteed by Th-240 that indeed $Q(Y) = q(c) = q(|Y|)$, provided that q is defined according to (143).

Obviously, things will not remain that simple once we consider a more general type of quantifiers. However, most NL quantifiers will not require the worst-case analysis presented in Th-255. These quantifiers are far from being ‘unrestricted’, and they often permit considerable simplifications due to their regular structure. In particular, most NL quantifiers are known to be two-place and conservative, so it is worthwhile studying this type of quantifiers. Hence let us start by discussing general two-place quantifiers, and subsequently taylor this analysis to the conservative type. It is well-known from TGQ that every quantitative two-valued quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ on a finite base set can be computed from $a = |Y_1 \setminus Y_2|$, $b = |Y_2 \setminus Y_1|$, $c = |Y_1 \cap Y_2|$ and $d = |E \setminus (Y_1 \cup Y_2)|$, where $Y_1, Y_2 \in \mathcal{P}(E)$ are the arguments of Q, c.f. [9, p. 457]. This result obviously transfers to the more general situation of semi-fuzzy quantifiers:

Theorem 256 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a two-place quantifier on a finite base set $E \neq \varnothing$. If Q is quantitative, then $Q(Y_1, Y_2)$ can be expressed in terms of $a = |Y_1 \setminus Y_2|$, $b = |Y_2 \setminus Y_1|$, $c = |Y_1 \cap Y_2|$ and $d = |E \setminus (Y_1 \cup Y_2)|$, for any choice of crisp arguments $Y_1, Y_2 \in \mathcal{P}(E)$.*
(Proof: D.17, p.476+)

Note. The theorem merely instantiates the general framework introduced above for $n = 2$ (actually, it is an apparent corollary to Th-255). Its sole purpose is that of stipulating symbols a, b, c, d which refer to the cardinality coefficients sampled from $\Phi_1(Y_1, Y_2) = Y_1 \setminus Y_2$, $\Phi_2(Y_1, Y_2) = Y_2 \setminus Y_1$, $\Phi_3(Y_1, Y_2) = Y_1 \cap Y_2$, and $\Phi_4(Y_1, Y_2) = E \setminus (Y_1 \cup Y_2)$. Things become more interesting once additional assumptions are imposed on Q. In fact, it has been shown by van Benthem [8, p. 446], [9, p. 458] that in the case of a conservative two-valued quantifier, the coefficients a and c are sufficient for determining the quantification result $Q(Y_1, Y_2)$, which then becomes independent of b and d. Again, this result generalizes to semi-fuzzy quantifiers.

Theorem 257 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a quantitative two-place quantifier on a finite base set $E \neq \varnothing$. If Q is conservative, then $Q(Y_1, Y_2)$ is fully determined by $a = |Y_1 \setminus Y_2|$ and $c = |Y_1 \cap Y_2|$, i.e. there exists $q : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$ such that*

$$Q(Y_1, Y_2) = q(a, c)$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$.
(Proof: D.18, p.476+)

In other words, we can dispense with the coefficients b and d in practice, and limit ourselves to determining a and c. For my purposes, it is beneficial to slightly reformulate this result, and replace the relevant cardinality coefficients a and c by another choice of coefficients $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$. We can then assert the following corollary to the previous theorem.

Theorem 258
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a quantitative two-place quantifier on a finite base set $E \neq \varnothing$. If Q is conservative, then $Q(Y_1, Y_2)$ is fully determined by $|Y_1|$ and $|Y_1 \cap Y_2|$, for all $Y_1, Y_2 \in \mathcal{P}(E)$.
(Proof: D.19, p.477+)

Notes

- Hence every quantitative and conservative quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ on a finite base set can be represented in terms of $K = 2$, $\Phi_1(Y_1, Y_2) = Y_1$, $\Phi_2(Y_1, Y_2) = Y_1 \cap Y_2$, $c_1 = |\Phi_1(Y_1, Y_2)| = |Y_1|$ and $c_2 = |\Phi_2(Y_1, Y_2)| = |Y_1 \cap Y_2|$, based on a suitable choice of $q : \{0, \ldots, |E|\}^K \longrightarrow \mathbf{I}$ with

  $$Q(Y_1, Y_2) = q(c_1, c_2) \tag{170}$$

  for all $Y_1, Y_2 \in \mathcal{P}(E)$.

- In particular, every proportional quantifier belongs to this general class. For example, the proposed definition of the proportional quantifier “almost all” already conforms to the above scheme, cf. (12) and (170). In this case, the mapping q becomes

  $$q(c_1, c_2) = \begin{cases} \mu_{\text{almost all}}(c_2/c_1) & : \quad c_1 > 0 \\ 1 & : \quad \text{else} \end{cases}$$

  for all $c_1, c_2 \in \{0, \ldots, |E|\}$. Based on this choice of q, the quantifier can then be rewritten **almost all**$(Y_1, Y_2) = q(c_1, c_2)$.

The proposed representation of quantifiers by a mapping $q : \{0, \ldots, |E|\}^K \longrightarrow \mathbf{I}$, which is possible for every quantitative Q on a finite base set, provides a suitable point of departure for developing the required algorithms that will implement $\top_j$ and $\bot_j$ more efficiently. Based on these preparations, I will now consider the set $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ defined by Def. 110, from which the coefficients $\top_{Q,X_1,\ldots,X_n}(\gamma) = \sup \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ and $\bot_{Q,X_1,\ldots,X_n}(\gamma) = \inf \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ are calculated, see (81) and (82). Due to these dependencies, a faster method for computing $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ will also speed up the implementation of $\top_{Q,X_1,\ldots,X_n}(\gamma)$ and $\bot_{Q,X_1,\ldots,X_n}(\gamma)$. In order to achieve this improvement, let us observe that $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ can now be computed from q (rather than Q):

$$\begin{aligned}
&\mathcal{S}_{Q,X_1,\dots,X_n}(\gamma)\\
&= \{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)\} && \text{by Def. 82}\\
&= \{q(c_1,\dots,c_K) : (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n),\\
&\qquad c_1 = |\Phi_1(Y_1,\dots,Y_n)|,\dots,c_K = |\Phi_K(Y_1,\dots,Y_n)|\} && \text{by Th-255}\\
&= \{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in \{(|\Phi_1(Y_1,\dots,Y_n)|,\dots,\\
&\qquad\qquad |\Phi_K(Y_1,\dots,Y_n)|) : (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)\}\,,
\end{aligned}$$

i.e.

$$\mathcal{S}_{Q,X_1,\dots,X_n}(\gamma) = \{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_\gamma(X_1,\dots,X_n)\}\,, \qquad (171)$$

where $R_\gamma(X_1,\dots,X_n) = R_\gamma^{\Phi_1,\dots,\Phi_K}(X_1,\dots,X_n) \subseteq \{0,\dots,|E|\}^K$ is defined by

$$\begin{aligned} R_\gamma(X_1,\dots,X_n) = \{(|\Phi_1(Y_1,\dots,Y_n)|,\dots,|\Phi_K(Y_1,\dots,Y_n)|) :\\ (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)\}\,. \end{aligned} \qquad (172)$$

This is quite obvious. By replacing the instances of quantification $Q(Y_1,\dots,Y_n)$ with instances of $q(c_1,\dots,c_K)$, a similar gain in performance is achieved as in (166), where I resorted to equivalence classes. Conceptually, we can view the move to $q(c_1,\dots,c_K)$, $c_r = |\Phi_r(Y_1,\dots,Y_n)|$, as involving the obvious equivalence relation

$$\begin{aligned} (Y_1,\dots,Y_n) \approx (Y'_1,\dots,Y'_n) \Leftrightarrow |\Phi_r(Y_1,\dots,Y_n)| = |\Phi_r(Y'_1,\dots,Y'_n)|\\ \text{for all } r \in \{1,\dots,K\}. \end{aligned}$$

Clearly $(Y_1,\dots,Y_n) \approx (Y'_1,\dots,Y'_n)$ entails that

$$Q(Y_1,\dots,Y_n) = q(c_1,\dots,c_k) = Q(Y'_1,\dots,Y'_n)$$

for all $(Y_1,\dots,Y_n), (Y'_1,\dots,Y'_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)$, because in this case,

$$|\Phi_r(Y_1,\dots,Y_n)| = c_r = |\Phi_r(Y'_1,\dots,Y'_n)|$$

for all $r = 1,\dots,K$. Due to the fact that $\mathcal{S}_{Q,X_1,\dots,X_n}(\gamma)$ is now computed from $R_\gamma(X_1,\dots,X_n)$, it is ensured that only one instance of $q(c_1,\dots,c_K)$ must be computed in this case, rather than treating $Q(Y_1,\dots,Y_n)$ and $Q(Y'_1,\dots,Y'_n)$ separately. Compared to the earlier approach based on representatives under $\sim$, the new method has the benefit of replacing the set-based information (i.e., equivalence classes and their representatives) altogether, and reducing the computation of the set $\mathcal{S}_{Q,X_1,\dots,X_n}(\gamma)$ and of the derived coefficients $\top_{Q,X_1,\dots,X_n}(\gamma)$, $\bot_{Q,X_1,\dots,X_n}(\gamma)$ and $Q_\gamma(X_1,\dots,X_n)$ to simple computations on cardinal numbers. Specifically, these quantities now become

$$\top_{Q,X_1,\dots,X_n}(\gamma) = \max\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_\gamma(X_1,\dots,X_n)\}\,, \qquad (173)$$

$$\bot_{Q,X_1,\dots,X_n}(\gamma) = \min\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_\gamma(X_1,\dots,X_n)\} \qquad (174)$$

and

$$Q_\gamma(X_1,\dots,X_n) = \mathrm{m}_{\frac{1}{2}}\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_\gamma(X_1,\dots,X_n)\} \tag{175}$$

which is straightforward from (171), (81), (82) and Def. 83. In the above expressions, $\min$ and $\max$ have been used, rather than the infimum and supremum required by (81) and (82). This is again possible because the base set E, and consequently the set $S_{Q,X_1,\dots,X_n}(\gamma)$, is known to be finite.

Usually it will be sufficient to consider those choices of $\gamma = \overline{\gamma}_j$ only, that are derived from a given set $\Gamma = \{\gamma_0,\dots,\gamma_m\} \supseteq \Gamma(X_1,\dots,X_n)$ according to (154). The associated coefficients $\top_j$, $\bot_j$ and C_j, which are indexed by $j \in \{0,\dots,m-1\}$ rather than $\overline{\gamma}_j$, now become

$$\begin{aligned}
\top_j &= \max\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_j\} \\
\bot_j &= \min\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_j\} \\
C_j &= \mathrm{med}_{\frac{1}{2}}(\top_j,\bot_j) = \mathrm{m}_{\frac{1}{2}}\{q(c_1,\dots,c_K) : (c_1,\dots,c_K) \in R_j\}\,,
\end{aligned}$$

where

$$R_j = R_{\overline{\gamma}_j}(X_1,\dots,X_n)\,.$$

This is apparent from (155) and (81); (156) and (82); and finally (157) and (64).

Notes

- As indicated by the superscripts, the relation $R_\gamma^{\Phi_1,\dots,\Phi_K}$ depends on the assumed choice of Boolean combinations $\Phi_1,\dots,\Phi_K$ from which the cardinality coefficients $c_j = |\Phi_j(Y_1,\dots,Y_n)|$ are sampled. For the sake of readability, though, the superscript will generally be suppressed whenever $\Phi_1,\dots,\Phi_K$ are clear from the context.

- At this point, it is instructive to return to the simple example of unary quantifiers, i.e. $K = 1$, $\Phi(Y) = Y$, $c = |\Phi(Y)| = |Y|$ under the proposed analysis. In this case, the relation $R_\gamma(X)$ and the associated R_j become

  $$\begin{aligned}
  R_\gamma(X) &= \{k : |X_\gamma^{\min}| \le k \le |X_\gamma^{\max}|\} \\
  R_j &= \{k : \ell(j) \le k \le u(j)\}\,,
  \end{aligned}$$

  where $\ell(j) = |X_{\overline{\gamma}_j}^{\min}|$ and $u(j) = |X_{\overline{\gamma}_j}^{\max}|$.

- I should remark in advance that it is often not necessary to unfold all tuples in the relation R_j and hence consider all quantification results. Quite the reverse, the monotonicity properties of the quantifiers of interest usually permit the restriction to a small number of coefficients derived from R_j. Usually, these coefficients can be directly computed from cardinality data, which achieves an additional cut in processing times compared to an approach which does not utilize such a priori knowledge. An example of the possible simplifications has already been given in section 11.3. To be specific, eq. (147) illustrates how the computation of $\top_{Q,X}$ and $\bot_{Q,X}$ can be simplified in the case of a convex quantifier; it is

> then sufficient to inspect only three cases, i.e. $q(\ell)$, $q(u)$ and $q(j_{\mathrm{pk}})$, rather than the full range $\{\ell, \ell+1, \dots, u-1, u\}$. A further simplification has been demonstrated for monotonic quantifiers. In this case, the computation of the quantities of interest can even be reduced to only two instances, $q(\ell)$ and $q(u)$, that must be evaluated. Comparable savings in effort can be achieved when analysing more complex quantifiers (e.g. proportional) in this way, see 11.9 below.

In the following, I will show that the relation $R_\gamma(X_1, \dots, X_n)$ can always be computed from the upper and lower cardinality bounds $u_r = |(Z_r)_\gamma^{\max}|$ and $\ell_r = |(Z_r)_\gamma^{\min}|$ of Boolean combinations $Z_1 = \Psi_1(X_1, \dots, X_n), \dots, Z_L = \Psi_L(X_1, \dots, X_n)$ of the arguments $X_1, \dots, X_n$. In order to state this as a theorem and develop the formal machinery for carrying out the proofs, we need some further notation. Hence suppose that $E \neq \varnothing$ is some base set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $v \in \{0, 1\}$. We then define the fuzzy subset $X^{\langle v \rangle} \in \widetilde{\mathcal{P}}(E)$ by

$$X^{\langle v \rangle} = \begin{cases} X & : \quad v = 1 \\ \neg X & : \quad v = 0\,, \end{cases} \tag{176}$$

where $\neg X$ is the standard fuzzy complement of X. Using this notation $X^{\langle v \rangle}$, we can conveniently write polynomials of negated and non-negated fuzzy sets, e.g. min-terms like $X_1^{\langle v_1 \rangle} \cap \dots \cap X_n^{\langle v_n \rangle}$ where $v_0, \dots, v_n \in \{0, 1\}$. However, min-terms involving a fixed number of variables are not sufficient for our current purposes, because we are now dealing with fuzzy sets rather than crisp sets, and consequently it is no longer the case that every Boolean combination can be reduced to a disjunction of min-terms (i.e., to disjunctive normal form). Therefore we will consider Boolean expressions of the following form,

$$X_{i_1}^{\langle v_1 \rangle} \cap \dots \cap X_{i_m}^{\langle v_m \rangle}$$

where $0 \leq i_1 < \dots < i_m \leq n$ and $v_j \in \{0, 1\}$ for all $j \in \{1, \dots, m\}$. Compared to the earlier fixed-length min-terms, it has now been made possible that only a selection of the X_i participate in the conjunction, while others are omitted. In the following, I will develop a more compact notation for these expressions. Hence let $\mathbb{V}$ denote the set of mappings,

$$\mathbb{V} = \{V : A \longrightarrow \{0, 1\} | A \subseteq \{1, \dots, n\}\}\,. \tag{177}$$

Abusing notation, I will identify each $V \in \mathbb{V}$ and its graph, i.e. V will be written as $V = \{(i_1, v_1), \dots, (i_m, v_m)\}$, where $1 \leq i_1 < i_2 < \dots < i_m \leq n$ and $v_j = V(i_j) \in \{0, 1\}$ for all $j \in \{1, \dots, m\}$. Based on this representation of V, we can define a corresponding Boolean combination of the fuzzy subsets $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ by

$$Z_V = \Psi_V(X_1, \dots, X_n) = X_{i_1}^{\langle v_1 \rangle} \cap \dots \cap X_{i_m}^{\langle v_m \rangle}\,. \tag{178}$$

In dependence on V, we further abbreviate

$$\ell_V = |(Z_V)_\gamma^{\min}| \tag{179}$$

$$u_V = |(Z_V)_\gamma^{\max}| \tag{180}$$

for all $V \in \mathbb{V}$, where $\gamma \in \mathbf{I}$ is a given choice of the cutting parameter. It is these Boolean combinations $Z_V = \Psi_V(X_1, \ldots, X_n)$ from which the cardinality information expressed by the upper and lower cardinality bounds u_V and ℓ_V will be sampled. However, it is not a trivial task to express the relation $R_\gamma(X_1, \ldots, X_n)$ in terms of the cardinality coefficients ℓ_V and u_V, $V \in \mathbb{V}$. I therefore take an intermediate step, and first show how the cardinalities of arbitrary Boolean combinations of the $(X_i)_\gamma^{\min}$ and $(X_i)_\gamma^{\max}$ can be computed from the known coefficients ℓ_V and u_V, $V \in \mathbb{V}$.

Hence let $E \neq \varnothing$ be some base set and $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy subset of E. For $p \in \{0, *, 1, +, -\}$, we define

$$X^{[p]} = \begin{cases} \neg(X_\gamma^{\max}) & : \quad p = 0 \\ X_\gamma^{\max} \cap \neg X_\gamma^{\min} & : \quad p = * \\ X_\gamma^{\min} & : \quad p = 1 \\ X_\gamma^{\max} & : \quad p = + \\ \neg(X_\gamma^{\min}) & : \quad p = - \end{cases} \tag{181}$$

In order to express the relation $R_\gamma(X_1, \ldots, X_n)$, it is sufficient to know the cardinality of all Boolean combinations ${X_1}^{[p_1]} \cap \cdots \cap {X_n}^{[p_n]}$ where $p \in \{0, *, 1\}$. This will first be ascertained for the most fine-grained choice of $R_\gamma(X_1, \ldots, X_n)$, which is sampled from all min-terms of the variables $Y_1, \ldots, Y_n$:

Theorem 259

Let $E \neq \varnothing$ be a finite base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Further let $\gamma \in \mathbf{I}$ be given. For all $p = (p_1, \ldots, p_n) \in \{0, *, 1\}^n$, we abbreviate

$$D(p) = \{d \in \{0,1\}^n : (d_1, p_1), \ldots, (d_n, p_n) \in \{(0,0), (1,1), (*,0), (*,1)\}\} \tag{182}$$

$$S(p) = X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]} \tag{183}$$

$$c(p) = |S(p)| = |X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]}| \tag{184}$$

We further define

$$\begin{aligned} \Lambda = \{(\lambda_p)_{p \in \{0,*,1\}^n} \mid & \text{ for all } p \in \{0,*,1\}^n,\ \lambda_p : \{0,1\}^n \longrightarrow \{0, \ldots, |E|\} \\ & \text{satisfies} \sum_{d \in \{0,1\}^n} \lambda_p(d) = c(p) \text{ and } \lambda_p(d) = 0 \text{ for all } d \notin D(p)\}\,. \end{aligned} \tag{185}$$

The relation $R = R_\gamma^{\Phi_{0,\ldots,0}, \ldots, \Phi_{1,\ldots,1}}(X_1, \ldots, X_n)$ generated by all min-terms

$$\Phi_{d_1, \ldots, d_n}(Y_1, \ldots, Y_n) = Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$$

and corresponding cardinality coefficients $c_{d_1, \ldots, d_n} = |\Phi_{d_1, \ldots, d_n}(Y_1, \ldots, Y_n)|$ according to (172)*, can then be expressed as*

$$\begin{aligned} R = \{c : \{0,1\}^n \longrightarrow \{0, \ldots, |E|\} \mid & \text{ there exists } (\lambda_p)_{p \in \{0,*,1\}^n} \in \Lambda \\ & \text{such that for all } d \in \{0,1\}^n,\ c_d = \sum_{p \in \{0,*,1\}^n} \lambda_p(d)\}\,. \end{aligned} \tag{186}$$

(Proof: D.20, p.478+)

In other words, R can be computed from the cardinalities of the sets $X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]}$, where $p = (p_1, \ldots, p_n) \in \{0, *, 1\}^n$. This result is important because R is the 'universal' relation assumed in the worst-case analysis of Th-255, which can express every quantitative Q. As shown in the next theorem, the above reduction to the universal relation R is sufficient to compute every other relation $R_\gamma(X_1, \ldots, X_n) = R_\gamma^{\Phi_1, \ldots, \Phi_K}(X_1, \ldots, X_n)$ as well, which results from a choice of Boolean combinations $\Phi_1(Y_1, \ldots, Y_n), \ldots, \Phi_K(Y_1, \ldots, Y_n)$:

Theorem 260

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and suppose that $\Phi'_1(Y_1, \ldots, Y_n)$, $\ldots$, $\Phi'_K(Y_1, \ldots, Y_n)$ are Boolean combinations of the crisp variables $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We shall further abbreviate

$$R = R_\gamma^{\Phi_{0,\ldots,0}, \ldots, \Phi_{1,\ldots,1}}(X_1, \ldots, X_n) \tag{187}$$

$$R' = R_\gamma^{\Phi'_1, \ldots, \Phi'_K}(X_1, \ldots, X_n) \tag{188}$$

where $\Phi_{d_1,\ldots,d_n}(Y_1, \ldots, Y_n) = Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$, see (168)*. In addition, I will use cardinality coefficients $c_{d_1,\ldots,d_n} = |\Phi_{d_1,\ldots,d_n}(Y_1, \ldots, Y_n)|$.*
Then

$$R' = \{(c'_1, \ldots, c'_K) : (c_{0,\ldots,0}, \ldots, c_{1,\ldots,1}) \in R, c'_j = \sum_{d \in D_j} c_d,\ j = 1, \ldots, K\}\,, \tag{189}$$

where

$$D_j = \{d = (d_1, \ldots, d_n) \in \{0,1\}^n : \Phi'_j(Y_1, \ldots, Y_n) \cap \Phi_d(Y_1, \ldots, Y_n) \neq \varnothing\} \tag{190}$$

for all $j \in \{1, \ldots, K\}$.
(Proof: D.21, p.482+)

Hence $\top_j$ and $\bot_j$ can be computed from some choice of R', and R' can be computed from the most fine-grained, or 'universal' relation R, which in turn can be computed from the cardinalities $|X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]}|$, $p_i \in \{0, *, 1\}$. The next goal is that of computing the cardinality of ${X_1}^{[p_1]} \cap \cdots \cap {X_n}^{[p_n]}$ from the cardinality coefficients u_V and ℓ_V, $V \in \mathbb{V}$. In order to accomplish this, I need additional intermediate representations. Rather than fixed-length Boolean combinations of the above type, i.e. ${X_1}^{[p_1]} \cap \cdots \cap {X_n}^{[p_n]}$, I therefore introduce more flexible Boolean combinations which take the following form,

$$X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_m}^{[p_m]}$$

where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $p_j \in \{0, 1, *, +, -\}$ for all $j \in \{1, \ldots, m\}$. The relevant argument positions i_j and corresponding p_j can again be viewed as the graph $P = \{(i_1, p_1), \ldots, (i_m, p_m)\}$ of a mapping $P : A \longrightarrow \{0, 1, *, +, -\}$, where

$A \subseteq \{1, \dots, n\} = \{i_j : j = 1, \dots, m\}$ and $P(i_j) = p_j$. We can therefore capture all intended combinations of the arguments by letting P range over the following collection,

$$\mathbb{P}^* = \{P : A \longrightarrow \{0, 1, *, +, -\} | A \subseteq \{1, \ \dots, n\}\} . \tag{191}$$

Based on a given $P \in \mathbb{P}^*$ and its graph representation $P = \{(i_1, p_1), \dots, (i_m, p_m)\}$, we can now define the associated crisp set $S(P) \in \mathcal{P}(E)$ and its cardinality $c(P)$ by

$$S(P) = X_{i_1}^{[p_1]} \cap \dots \cap X_{i_m}^{[p_m]} \tag{192}$$

$$c(P) = |S(P)| = |X_{i_1}^{[p_1]} \cap \dots \cap X_{i_m}^{[p_m]}| . \tag{193}$$

For reasons that will soon become clear, I also define the following restricted range $\mathbb{P}$, in which P is allowed to assume values in $\{0, 1, +, -\}$ only:

$$\mathbb{P} = \{P : A \longrightarrow \{0, 1, +, -\} | A \subseteq \{1, \ \dots, n\}\} . \tag{194}$$

Compared to $\mathbb{P}^*$, $\mathbb{P}$ hence comprises those mappings $P \in \mathbb{P}^*$ which satisfy $P(i) \neq *$ for all $i \in \operatorname{Dom} P$. The following theorem asserts that the restricted range $\mathbb{P}$ is sufficient for computing $c(P_*)$ for all $P_* \in \mathbb{P}^*$, by presenting a suitable reduction:

Theorem 261

Let $E \neq \varnothing$ be a finite base set and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $P_* \in \mathbb{P}^*$,

$$c(P_*) = \sum_{P \in \mathbb{P}} \sigma(P) \cdot c(P) \, , \tag{195}$$

where the integer-valued mapping $\sigma = \sigma_{P_*} : \mathbb{P} \longrightarrow \mathbb{Z}$ is inductively defined as follows. Given $P_* \in \mathbb{P}$, let

$$i_* = \max\{i \in \operatorname{Dom} P_* : P_*(i) = *\} . \tag{196}$$

In dependence on i_*, we stipulate

a. If $i_* = 0$, then

$$\sigma_{P_*}(P) = \begin{cases} 1 & : \quad P = P_* \\ 0 & : \quad P \neq P_* \end{cases} \tag{197}$$

for all $P \in \mathbb{P}$.

b. If $i_* > 0$, then

$$\sigma_{P_*}(P) = \sigma_{P'}(P) - \sigma_{P''}(P) \tag{198}$$

for all $P \in \mathbb{P}$, where

$$P' = (P \setminus \{(i_*, *)\}) \cup \{(i_*, +)\} \tag{199}$$

$$P'' = (P \setminus \{(i_*, *)\}) \cup \{(i_*, 1)\} . \tag{200}$$

(Proof: D.22, p.483+)

Hence $c(P_*)$ can indeed be computed from the cardinalities $c(P)$, $P \in \mathbb{P}$. Having shown that the case $p = *$ can be eliminated, we shall assume in the following that $p \in \{0, 1, +, -\}$ only, and correspondingly focus on the case that $P \in \mathbb{P}$. It remains to be shown that the cardinalities $c(P) = |X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_m}^{[p_m]}|$ can be computed from the quantities u_V and ℓ_V, $V \in \mathbb{V}$ sampled from $X_1, \ldots, X_n$. In the following, I will introduce some new concepts which simplify the description of the necessary calculations. For $p \in \{0, 1, +, -\}$, we define its 'polarity' $\mathrm{pol}(p) \in \{0, 1\}$ and its 'type', $\mathrm{type}(p) \in \{\max, \min\}$, by

$$\mathrm{pol}(p) = \begin{cases} 1 & : \quad p \in \{1, +\} \\ 0 & : \quad p \in \{0, -\} \end{cases} \tag{201}$$

$$\mathrm{type}(p) = \begin{cases} \max & : \quad p \in \{+, -\} \\ \min & : \quad p \in \{0, 1\} \end{cases} \tag{202}$$

The polarity of p is 0 if $X^{[p]}$ is negated, i.e. in the cases $X^{[0]} = \neg(X_\gamma^{\max})$ and $X^{[-]} = \neg(X_\gamma^{\min})$, while the polarity is one in the non-negated cases $X^{[1]} = X_\gamma^{\min}$ and $X^{[+]} = X_\gamma^{\max}$. The type of p is 'max' if $X^{[p]} = (X^{\langle \mathrm{pol}(p) \rangle})_\gamma^{\max}$ i.e. in the cases $X^{[+]} = X_\gamma^{\max}$ and $X^{[-]} = \neg(X_\gamma^{\min}) = (\neg X)_\gamma^{\max}$, see Th-61. The type of p is 'min' if $X^{[p]} = (X^{\langle \mathrm{pol}(p) \rangle})_\gamma^{\min}$ and hence in the cases $X^{[0]} = \neg(X_\gamma^{\max}) = (\neg X)_\gamma^{\min}$ and $X^{[1]} = X_\gamma^{\min}$. Combining the results for the individual cases, we hence obtain that

$$X^{[p]} = (X^{\langle \mathrm{pol}(p) \rangle})_\gamma^{\mathrm{type}(p)} \tag{203}$$

for all $p \in \{0, 1, +, -\}$. The following theorem completes the intended reduction, by showing how $c(P)$, $P \in \mathbb{P}$ can be computed from the cardinality coefficients ℓ_V and u_V, $V \in \mathbb{V}$.

Theorem 262

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Then for all $P \in \mathbb{P}$,

$$c(P) = \sum_{V \in \mathbb{V}} \zeta_P(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_P(V, \max) \cdot u_V \,, \tag{204}$$

where the integer-valued mapping $\zeta = \zeta_P : \mathbb{V} \times \{\min, \max\} \longrightarrow \mathbb{Z}$ is inductively defined as follows. Recalling the graph representation $P = \{(i_1, p_1), \ldots, (i_m, p_m)\}$ of P, where $p_j = P(i_j)$, let

$$i_* = \max\{i \in \mathrm{Dom}\, P : \mathrm{type}(P(i)) \neq \mathrm{type}(p_m)\} \,. \tag{205}$$

In dependence on i_*, we stipulate

a. If $i_* = 0$, then

$$\zeta_P(V, y) = \begin{cases} 1 & : \quad V = \{(i_1, \mathrm{pol}(p_1)), \ldots, (i_m, \mathrm{pol}(p_m))\}, y = \mathrm{type}(p_m) \\ 0 & : \quad \text{else} \end{cases} \tag{206}$$

for all $V \in \mathbb{V}$ and $y \in \{\min, \max\}$.

b. If $i_* > 0$, then

$$\zeta_P(V,y) = \zeta_{P'}(V,y) - \zeta_{P''}(V,y) \tag{207}$$

for all $V \in \mathbb{V}$ and $y \in \{\min, \max\}$, where

$$\begin{align} P' &= P \setminus \{(i_*, p_*)\} \tag{208}\\ P'' &= P' \cup \{(i_*, p')\} \tag{209}\\ p_* &= P(i_*) \tag{210}\end{align}$$

and

$$p' = \begin{cases} 0 & : \quad p_* = + \\ 1 & : \quad p_* = - \\ + & : \quad p_* = 0 \\ - & : \quad p_* = 1\,. \end{cases} \tag{211}$$

(Proof: D.23, p.486+)

By subsequently applying Th-259 to Th-262, every relation $R_\gamma(X_1, \ldots, X_n)$ can be computed from a choice of cardinality coefficients $\ell_r = |(Z_r)_\gamma^{\min}|$ and $u_r = |(Z_r)_\gamma^{\max}|$, which are sampled from suitable Boolean combinations $Z_r = \Psi_r(X_1, \ldots, X_n)$ of the arguments. All of the theorems are constructive in style, i.e. they also present a specific choice of Boolean combinations and show how the relation $R_\gamma(X_1, \ldots, X_n)$ can be computed from the resulting quantities ℓ_r and u_r.

In order to spell out a procedure which covers all possible $R_\gamma(X_1, \ldots, X_n)$ in full generality, it was of course necessary to make worst-case assumptions in terms of complexity of the quantifiers to be modelled. In other words, the number of Boolean combinations required for the fully general analysis amounts to $|V| = 3^n$. Although the arity n of NL quantifiers is very small in practice (usually $n = 2$ or $n = 3$), this indicates that the generic solution might not be optimal from the performance point of view, and should rather be viewed as a proof of concept, and as a suitable point of departure for developing highly efficient, dedicated solutions for typical classes of quantifiers.

The potential for simplification can be demonstrated nicely for conservative quantifiers, which I will now analyse in some more depth. My goal is to develop a simple and efficient way of computing $R_\gamma(X_1, X_2)$. Hence let $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ be a two-place quantifier on a finite base set which is both quantitative and conservative. As has been shown in Th-258 above, we can describe the quantifier in terms of a mapping $q : \{0, \ldots, |E|\}^2 \longrightarrow \mathbf{I}$, based on an analysis in terms of $K = 2$, $\Phi_1(Y_1, Y_2) = Y_1$, $\Phi_2(Y_1, Y_2) = Y_1 \cap Y_2$, $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$ for all $Y_1, Y_2 \in \mathcal{P}(E)$. The corresponding relation $R_\gamma(X_1, X_2)$, $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, can then be determined from $Z_1 = \Psi_1(X_1, X_2) = X_1$, $Z_2 = \Psi_2(X_1, X_2) = X_1 \cap X_2$ and $Z_3 = \Psi_3(X_1, X_2) = X_1 \cap \neg X_2$ in the following way.

Theorem 263

Let $E \neq \varnothing$ be a finite base set and suppose that $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. Further let $\gamma \in \mathbf{I}$, and suppose that $R_\gamma(X_1, X_2) \subseteq \{0, \ldots, |E|\} \times \{0, \ldots, |E|\}$, is the relation defined by

$$R_\gamma(X_1, X_2) = \{(|Y_1|, |Y_1 \cap Y_2|) : Y_1 \in \mathcal{T}_\gamma(X_1), Y_2 \in \mathcal{T}_\gamma(X_2)\}\,.$$

Then $R_\gamma(X_1, X_2)$ can be computed from $\ell_1 = |(X_1)_\gamma^{\min}|$, $\ell_2 = |(X_1 \cap X_2)_\gamma^{\min}|$, $\ell_3 = |(X_1 \cap \neg X_2)_\gamma^{\min}|$, $u_1 = |(X_1)_\gamma^{\max}|$, $u_2 = |(X_1 \cap X_2)_\gamma^{\max}|$, and $u_3 = |(X_1 \cap \neg X_2)_\gamma^{\max}|$, viz

$$R_\gamma(X_1, X_2) = \{(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(\ell_2, c_1 - u_3) \leq c_2 \leq \min(u_2, c_1 - \ell_3)\}\,.$$

(Proof: D.24, p.490+)

Note. In this case, the 'worst-case analysis' for $n = 2$ results in $3^2 = 9$ combinations of X_1, X_2 that must possibly be considered for computing R, i.e. $X_1 \cap X_2$, $X_1 \cap \neg X_2$, $\neg X_1 \cap X_2$, $\neg X_1 \cap \neg X_2$, X_1, $\neg X_1$, X_2, $\neg X_2$, and E. Of these, $\neg X_1$ and $\neg X_2$ are obviously redundant, because their cardinality bounds can be computed from the cardinality bounds of X_1 and X_2. The full domain E can also be eliminated, because it bears no information about X_1 and X_2. This leaves a total of 6 combinations which are potentially necessary for computing $R_\gamma(X_1, X_2)$. Hence the conservativity of Q shrinks down this set of required combinations by one half, i.e. only the combinations X_1, $X_1 \cap X_2$ and $X_1 \cap \neg X_2$ must actually be considered. Let us now put this theorem into context. Its significance becomes clear once we recall the earlier Th-258, which states that a conservative quantifier can be computed from the cardinality of $\Phi_1(Y_1, Y_2) = Y_1$ and $\Phi_2(Y_1, Y_2) = Y_1 \cap Y_2$. Hence $\top_{Q,X_1,X_2}$ and $\bot_{Q,X_1,X_2}$ can be computed from that choice of $R_\gamma(X_1, X_2)$ specified in the above theorem. The theorem therefore ensures that if Q is conservative, then there exist mappings $q^{\min}, q^{\max} : \{0, \ldots, |E|\}^4 \longrightarrow \mathbf{I}$ such that

$$\top_{Q,X_1,X_2}(\gamma) = q^{\max}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) \qquad (212)$$

$$\bot_{Q,X_1,X_2}(\gamma) = q^{\min}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) \qquad (213)$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$, where the coefficients $\ell_1, \ell_2, \ell_3, u_1, u_2$ and u_3 are defined as above. This is obvious from the Th-263 if we define these mappings as follows,

$$q^{\max}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) = \max\{q(c_1, c_2) : (c_1, c_2) \in R\}$$
$$q^{\min}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) = \min\{q(c_1, c_2) : (c_1, c_2) \in R\}\,,$$

where $R \subseteq \{0, \ldots, |E|\} \times \{0, \ldots, |E|\}$ is the relation

$$R = \{(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(\ell_2, c_1 - u_3) \leq c_2 \leq \min(u_2, c_1 - \ell_3)\}\,.$$

We shall see below in section 11.9 how this analysis of conservative quantifiers can be incorporated into an efficient implementation of proportional quantifiers in the models.

Other classes of quantifiers can be analysed in a similar way. This will be demonstrated below in section 11.10, where the method is extended to cardinal comparatives like "more … than". The proposed scheme of analysing quantitative quantifiers on finite base sets therefore enables me to implement the relevant NL quantifiers in all models of interest. As witnessed by Th-263, for example, the proposed solution of computing $\top_j$ and $\bot_j$ from a relation $R_j = R_{\overline{\gamma}_j}$ on cardinals achieves a very simple description which is also computationally straightforward. In the next section, I will be concerned with the remaining issue of computing the quantities $\ell_r = |(Z_r)^{\min}_{\overline{\gamma}_j}|$ and $u_r = |(Z_r)^{\max}_{\overline{\gamma}_j}|$ efficiently, to which the computation of R_j has been reduced. Upon solving this problem, all necessary ingredients will then be available for implementing quantifiers in the models. As a courtesy to the reader, I will then put all the pieces together and restate the complete algorithms in pseudo-code. In particular, this self-contained presentation is intended as a point of departure for concrete implementations.

11.6 Computation of cardinality bounds

Given Q and $X_1, \ldots, X_n$, we are now able to compute $\top_{Q,X_1,\ldots,X_n}$, $\bot_{Q,X_1,\ldots,X_n}$ and $Q_\gamma(X_1, \ldots, X_n)$ from the upper cardinality bounds $u_r = |(Z_r)^{\max}_\gamma|$ and lower cardinality bounds $\ell_r = |(Z_r)^{\min}_\gamma|$ of Boolean combinations $Z_1, \ldots, Z_L$ of the arguments. I have therefore succeeded in reformulating the computation of $\top_{Q,X_1,\ldots,X_n}$, $\bot_{Q,X_1,\ldots,X_n}$ and $Q_\gamma(X_1, \ldots, X_n)$ in such a way that it depends only on cardinality information expressed by the tuple of coefficients $(\ell_1, \ldots, \ell_L, u_1, \ldots, u_L)$, see Th-259 to Th-262 and eqns. (173), (174), (175), respectively. Once the cardinality bounds $\ell_1, \ldots, \ell_L, u_1, \ldots, u_L$ are known, the subsequent computations can dispense with any direct reference to the fuzzy sets $X_1, \ldots, X_n$, which might be too large and awkward to operate upon. The definition of the cardinality coefficients ℓ_r and u_r, however, still depends on $X_1, \ldots, X_n$, which means that up to this point, I have merely shifted the burden of dealing with the $X_1, \ldots, X_n$ into the computation of these coefficients.

When computing quantification results, it is necessary to calculate the u_r and ℓ_r for every considered choice of the cutting parameter γ that controls the three-valued cuts. Due to the fact that these coefficients must be computed repeatedly, their computation should be implemented as efficiently as possible. Consequently, my above strategy will now be re-iterated, in order to ban any direct reference to $X_1, \ldots, X_n$ from the computation of the cardinality bounds ℓ_r and u_r. Basically, a pre-processing step will be applied which extracts the relevant information from the given fuzzy subsets, and the extracted cardinality information will then be represented in a way which lends itself to efficient processing of the ℓ_r and u_r.

In order to discuss the intermediate representation extracted from the fuzzy sets, and in order to construct a computation procedure which operates upon this representation, it is possible to consider each Z_r in isolation. I will hence focus on one fuzzy set at a time, which for simplicity I denote by X, and which is supposed to be chosen from $\{Z_1, \ldots, Z_L\}$. In addition, I will assume that a sample of cutting levels $\Gamma = \{\gamma_0, \ldots, \gamma_m\}$ be given, equipped with the usual properties, i.e. $0 = \gamma_0 < \gamma_1 < \cdots <$

$\gamma_m = 1$ and $\Gamma(X) \subseteq \Gamma$. It is then convenient to index the results of the cardinality bounds for each $\overline{\gamma}_j$ by j. I will hence denote by $\ell(j)$ and $u(j)$ the lower and upper cardinality bounds of the considered fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ in the j-th iteration, i.e.

$$\ell(j) = |X^{\min}_{\overline{\gamma}_j}| \tag{214}$$

$$u(j) = |X^{\max}_{\overline{\gamma}_j}| \tag{215}$$

for $j \in \{0, \dots, m-1\}$. Rather than computing the coefficients $\ell(j)$ and $u(j)$ in each iteration $j = 0, \dots, m-1$ from scratch, it would be advantageous to devise a procedure which computes the values of these coefficients in the current iteration, i.e. $\ell(j)$ and $u(j)$, from their respective values in the previous iteration, i.e. from $\ell(j-1)$ and $u(j-1)$. This re-use of earlier computations promises a cut in processing times due to the elimination of redundant effort. For the sake of efficiency, this procedure should not refer directly to the fuzzy set X from which the coefficients $\ell(j)$ and $u(j)$ are sampled. By contrast, all computations should rest on cardinality information only. A suitable mechanism which extracts exactly those aspects of a fuzzy subset that characterize its cardinality, is provided by the histogram.[42]

Definition 164 *By the* histogram *of a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ (where E is some finite set), I denote the mapping Hist$_X$: $\mathbf{I} \longrightarrow \mathbb{N}$ defined by*

$$\mathit{Hist}_X(z) = |\mu_X^{-1}(z)| = |\{e \in E : \mu_X(e) = z\}| \tag{216}$$

for all $z \in \mathbf{I}$.

Notes

- It is apparent from the assumed finiteness of E that Hist$_X$ is well-defined, i.e. the term $|\mu_X^{-1}(z)|$ always refers to a finite cardinality.
- The proposed histogram representation can be likened to Rescher's *truth status statistics* ξ_z, which it generalizes from the m-valued to the continuous valued case, see [128, p. 201].

Let us now recall the notation

$$A(X) = \{\mu_X(e) : e \in E\}$$

introduced in (151). The finiteness of E obviously entails that $A(X)$ be a finite set as well. In addition, it is quite clear that $A(X)$ is the support of Hist$_X$, i.e.

$$A(X) = \{z \in \mathbf{I} : \mathrm{Hist}_X(z) \neq 0\} .$$

---

[42] This is apparent because two fuzzy subsets $X \in \widetilde{\mathcal{P}}(E)$, $X' \in \widetilde{\mathcal{P}}(E')$ of finite base sets $E, E' \neq \varnothing$ have the same histograms, i.e. Hist$_X$ = Hist$_{X'}$, if and only if there exists a bijection $\beta : E \longrightarrow E'$ such that $X' = \hat{\hat{\beta}}(X)$ (and hence also $X = \hat{\hat{\beta^{-1}}}(X)$). Consequently, histograms capture exactly the cardinality aspect of fuzzy sets, because two finite sets have the same cardinality if and only if there exists a bijection between them.

In particular, Hist_X has finite support. This is important because having finite support, Hist_X can be uniquely determined from a finite representation. To see this, consider a finite set $A \supseteq A(X)$; because $A(X)$ is finite, such sets are known to exist. We can then restrict Hist_X to the finite sample of $z \in A$, by defining a corresponding mapping $H_A : A \longrightarrow \mathbb{N}$ according to

$$H_A(z) = \mathrm{Hist}_X(z)$$

for all $z \in A$. It is obvious that Hist_X can be recovered from H_A, because $A \supseteq A(X)$ grants that the support of Hist_X is contained in A. Therefore $z \in \mathbf{I} \setminus A$ entails that $\mathrm{Hist}_X(z) = 0$. This proves that knowing H_A is sufficient to determine $\mathrm{Hist}_X(z)$ in the full range of all $z \in \mathbf{I}$, because $\mathrm{Hist}_X(z)$ can be expressed in terms of H_A according to the obvious rule

$$\mathrm{Hist}_X(z) = \begin{cases} H_A(z) & : \quad z \in A \\ 0 & : \quad \text{else} \end{cases} \tag{217}$$

for all $z \in \mathbf{I}$. The benefit gained by this representation is that the 'generating' mapping $H_A : A \longrightarrow \mathbb{N}$ is declared on a finite domain. Therefore H_A can be specified by listing the finite number of pairs in its graph, i.e. as a finite list which comprises all pairs in $\{(z, \mathrm{Hist}_X(z)) : z \in A\}$. In particular, this demonstrates that the histogram of X has a finite specification, and is hence suitable for representation on digital computers (modulo the finite precision of floating point numbers, of course).

Having introduced the histogram-based representation of the considered fuzzy subset X, I will now connect this representation to the given set of three-valued cut levels $\Gamma \supseteq \Gamma(X)$, in order to achieve an incremental computation of the cardinality coefficients. Hence suppose that $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, \dots, X_n)$ is given, where $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$. Let us define $A_\Gamma \in \mathcal{P}(\mathbf{I})$ by

$$A_\Gamma = \{\tfrac{1}{2} + \tfrac{1}{2}\gamma : \gamma \in \Gamma\} \cup \{\tfrac{1}{2} - \tfrac{1}{2}\gamma : \gamma \in \Gamma\}\,. \tag{218}$$

It is apparent from the finiteness of Γ that A_Γ has finite cardinality as well. Furthermore, A_Γ clearly satisfies the desired condition that

$$A_\Gamma \supseteq A_{\Gamma(X)} \supseteq A(X)\,. \tag{219}$$

This is obvious from $\Gamma \supseteq \Gamma(X)$, (151), (152) and (218). By the above reasoning, we hence know that the support of Hist_X is contained in A_Γ, i.e. the histogram of X can be represented by $H_{A_\Gamma} : A_\Gamma \longrightarrow \mathbb{N}$, see (217). For notational convenience, I will not refer to H_{A_Γ} directly. It will be advantageous to split H_{A_Γ} into two mappings H^+ and H^-, which let us access the histogram information in terms of j rather than $\overline{\gamma}_j$. These mappings $H^+, H^- : \{0, \dots, m\} \longrightarrow \mathbf{I}$ are defined by

$$H^+(j) = \mathrm{Hist}_X(\tfrac{1}{2} + \tfrac{1}{2}\gamma_j) = |\{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\}| \tag{220}$$

$$H^-(j) = \mathrm{Hist}_X(\tfrac{1}{2} - \tfrac{1}{2}\gamma_j) = |\{e \in E : \mu_X(e) = \tfrac{1}{2} - \tfrac{1}{2}\gamma_j\}| \tag{221}$$

This is just a convenient representation of H_{A_Γ}, because H^+ and H^- are apparently sampled from H_{A_Γ}, and because the original mapping H_{A_Γ} can be recovered from H^+

and H^- as follows,

$$H_{A_\Gamma}(z) = \begin{cases} H^+(j) & : \quad z = \frac{1}{2} + \frac{1}{2}\gamma_j \\ H^-(j) & : \quad z = \frac{1}{2} - \frac{1}{2}\gamma_j \end{cases}$$

for all $z \in A_\Gamma$. Based on the mappings H^+ and H^-, which must be computed from X in a preprocessing step, it is now possible to formulate an incremental procedure, which computes the current cardinality bounds $\ell(j)$ and $u(j)$ in the j-th iteration from the cardinality bounds $\ell(j-1)$ and $u(j-1)$ computed in the previous iteration, and from the cardinality information stored in $H^+(j)$ and $H^-(j)$.

Theorem 264
Let $E \neq \varnothing$ be a finite base set and $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E. Further let $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X)$ be given such that $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. Then

$$\begin{aligned} \ell(0) &= \sum_{k=1}^{m} H^+(k) \\ u(0) &= \ell(0) + H^+(0)\,, \end{aligned}$$

and

$$\begin{aligned} \ell(j) &= \ell(j-1) - H^+(j) \\ u(j) &= u(j-1) + H^-(j) \end{aligned}$$

for all $j \in \{1, \ldots, m-1\}$.
(Proof: D.25, p.498+)

Obviously, neither the formulas for initialisation nor those for updating refer to the original fuzzy set; it is only the histogram of X that must be known for computing the cardinality bounds. I have therefore succeeded in eliminating the fuzzy set X from the algorithm for computing $\ell(j)$ and $u(j)$. Unlike the fuzzy sets themselves, their histograms all look alike. This greatly facilitates processing, because the histograms of arbitrary fuzzy sets can easily be represented in a uniform way. By storing the histogram in an array, its data can be accessed very efficiently. Due to the fact that the update rules described in Th-264 only involve this lookup operation, i.e. one access to the histogram data in $H^+(j)$ or $H^-(j)$, and one additional arithmetic operation (sum), the proposed method results in a very fast implementation. The presentation of a histogram-based update rule in Th-264 completes my analysis of fuzzy quantification in $\mathcal{F}_\xi$-DFSes and the subsequent discussion of possible efficiency improvements.

From a more general perspective, what I have shown is that $\mathcal{F}_\xi(Q)(X_1, \ldots, X_n)$ can always be expressed as a function

$$\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \Xi(\mathrm{Hist}_{Z_1}, \ldots, \mathrm{Hist}_{Z_L})$$

which depends only on the histograms of Boolean combinations $Z_1, \ldots, Z_L$ of the arguments. However, this should not lure us into *defining* fuzzy quantifiers as functions of histograms in the spirit of Rescher's treatment of quantifiers in many-valued logic, see [127] and [128, pp. 197-206]. This is because the histogram-based analysis rests on the assumption of automorphism-invariance, a property shared by most, but not all, NL quantifiers. When attempting to define linguistic quantifiers directly in terms of $\Xi(\mathrm{Hist}_{Z_1}, \dots, \mathrm{Hist}_{Z_L})$, the correspondence problem discussed in Chap. 2 returns with full force. Consequently, the analysis of certain quantifiers in terms of histograms is only possible in special cases, and clearly too narrow for a comprehensive treatment of fuzzy NL quantification. In addition, it makes no provisions as to the systematicity of interpretation.

11.7 Implementation of unary quantifiers

In this section, I finally present the complete algorithms for quantitative one-place quantifiers on finite base sets. It is true that all the necessary techniques for implementing these quantifiers in the prototypical models $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ have already been described. Nevertheless, it is presumably helpful to collect the results scattered across the previous sections, and show how the different parts of the implementation work in concert.

Hence let $E \neq \varnothing$ be a finite base set and suppose that $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is quantitative. Further let $X \in \widetilde{\mathcal{P}}(E)$ be a given fuzzy argument. We then know from Th-250, Th-253 and Th-254 that $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$, $\mathcal{M}(Q)(X)$ and $\mathcal{M}_{CX}(Q)(X)$ can be computed from quantities $\top_j$, $\bot_j$ sampled from some choice of $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X)$, where $0 = \gamma_0 < \gamma_1 < \dots < \gamma_m = 1$. For convenience, I will assume in the following that $\Gamma = \Gamma(X)$. Recalling (152), (220) and (221), it is then apparent how H^+, H^- as well as γ_j, $j = 0, \dots, m$ can be computed from the fuzzy set X. I will write `H`$^+$`[j]` and `H`$^-$`[j]` to denote $H^+(j)$ and $H^-(j)$, respectively. An $(m+1)$-dimensional array `G` will be used to access the γ_j's, i.e. `G[j]` must be set to γ_j, $j = 0, \dots, m$. In particular, this means that `G[0]` $= 0$ and `G[m]` $= 1$, cf. (152). The resulting algorithms which compute $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$, $\mathcal{M}(Q)(X)$ and $\mathcal{M}_{CX}(Q)(X)$ from these data, are depicted in tables 4, 5 and 6, respectively. Starting with $j = 0$, each of the algorithms enters a main loop which increments j in each iteration. This lets me compute the current value of the cardinality bounds $\ell = |X^{\min}_{\overline{\gamma}_j}|$ and $u = |X^{\max}_{\overline{\gamma}_j}|$ in the j-th iteration from their values in the previous iteration and from the histogram information stored in H^+ and H^- as described in Th-264. The update of these coefficients takes place in the code fragment with the heading 'update clauses'. The quantities $\top_j = \top_{Q,X}(\overline{\gamma}_j)$, $\bot_j = \bot_{Q,X}(\overline{\gamma}_j)$ and $C_j = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j)$ can then be computed from $q^{\min}(\ell, u)$ and $q^{\max}(\ell, u)$, as described in Th-244. Apart from incorporating the update of the cardinality coefficients ℓ and u, the algorithms merely reflect the earlier analysis in Th-250, Th-253 and Th-254.

In these algorithms, I decided to supply the fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ as the input parameter, which directly reflects the expected input-output behaviour (i.e. X is the input, and $\mathcal{F}(Q)(X)$ is the returned output). The histogram information in H$^+$ and H$^-$ and the array `G` which collects the γ_j's, must then be computed from X in the obvious way. However, it also makes sense to consider a variant of these algorithms where the algorithms expect H$^+$, H$^-$ and `G` as their inputs, rather than the fuzzy set X from which

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$
(floating-point arithmetics)

```
INPUT: $X$
Compute G, H$^+$, H$^-$and $m$; // see text
// initialize $\ell$, $u$
$\ell$ := $\sum_{j=1}^{m}$ H$^+$[j];
$u$ := $\ell$ + H$^+$[0];
sum = G[1] * ($q^{\min}$($\ell$,$u$) + $q^{\max}$($\ell$,$u$));
for( j := 1; j<m; j := j + 1 ) {
    // update clauses for $\ell$ and $u$
    $\ell$ := $\ell$ - H$^+$[j];
    $u$ := $u$ + H$^-$[j];
    sum := sum + (G[j+1] - G[j])
                  * ($q^{\min}$($\ell$,$u$) + $q^{\max}$($\ell$,$u$));
    }
return sum / 2;
END
```

Table 4: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{F}_{\mathrm{Ch}}$ (based on floating-point arithmetics)

Algorithm for computing $\mathcal{M}(Q)(X)$
(floating-point arithmetics)

```
INPUT: $X$
Compute G, H$^+$, H$^-$and $m$; // see text
// initialize $\ell$, $u$
$\ell$ := $\sum_{j=1}^{m}$ H$^+$[j];
$u$ := $\ell$ + H$^+$[0];
$\top$ := $q^{\max}$($\ell$,$u$);
$\bot$ := $q^{\min}$($\ell$,$u$);
if( $\bot > \frac{1}{2}$ ) {
    sum := G[1] * $\bot$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell$ and $u$
        $\ell$ := $\ell$ - H$^+$[j];
        $u$ := $u$ + H$^-$[j];
        $\bot$ := $q^{\min}$($\ell$,$u$);
        if( $\bot \leq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\bot$;
        }
    }
else if( $\top < \frac{1}{2}$ ) {
    sum := G[1] * $\top$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell$ and $u$
        $\ell$ := $\ell$ - H$^+$[j];
        $u$ := $u$ + H$^-$[j];
        $\top$ := $q^{\max}$($\ell$,$u$);
        if( $\top \geq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\top$;
        }
    }
else
    { return $\frac{1}{2}$; }
return sum + $\frac{1}{2}$ * (1 - G[j]);
END
```

Table 5: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{M}$ (based on floating-point arithmetics)

Algorithm for computing $\mathcal{M}_{CX}(Q)(X)$
(floating-point arithmetics)

```
INPUT: $X$
Compute G, H$^+$, H$^-$and $m$; // see text
// initialize $\ell$, $u$
$\ell$ := $\sum_{j=1}^{m}$ H$^+$[j];
$u$ := $\ell$ + H$^+$[0];
$\top$ := $q^{\max}(\ell,u)$;
$\bot$ := $q^{\min}(\ell,u)$;
j := 0;
if( $\bot > \frac{1}{2}$ ) {
    B := 2 * $\bot$ - 1;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell$ and $u$
        $\ell$ := $\ell$ - H$^+$[j];
        $u$ := $u$ + H$^-$[j];
        B := 2 * $q^{\min}(\ell,u)$ - 1;
        }
    return $\frac{1}{2}$ + $\frac{1}{2}$ max(B, G[j]);
    }
else if( $\top < \frac{1}{2}$ ) {
    B := 1 - 2 * $\top$;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell$ and $u$
        $\ell$ := $\ell$ - H$^+$[j];
        $u$ := $u$ + H$^-$[j];
        B := 1 - 2 * $q^{\max}(\ell,u)$;
        }
    return $\frac{1}{2}$ - $\frac{1}{2}$ max(B, G[j]);
    }
return $\frac{1}{2}$;
END
```

Table 6: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{M}_{CX}$ (based on floating-point arithmetics)

these coefficients have been sampled. This separation of the histogram extraction from the core algorithms, will abstract from any specifics concerning the representation of the fuzzy set X. This is because the resulting histograms can be represented in a uniform way, which need not be the case for different types of fuzzy subsets and heterogeneous domains. By separating the histogram extraction from the main algorithms, it becomes possible to develop a generic solution which proves useful for implementing quantifiers in arbitrary domains.

The three algorithms presented so far resort to floating-point arithmetics, i.e. the membership grades $\mu_X(e)$ and consequently the γ_j's, are allowed to assume arbitrary values in the unit interval. In some cases, however, it can be advantageous to depart from this scheme, and rather profit from a reformulation in terms of integer arithmetics. Roughly speaking, the integer-based solution will be beneficial if the size of the domain is large compared to the number of supported membership grades. This will be the case e.g. in fuzzy image processing, where the size of the domain (number of image pixels) easily reaches one million and more; and where a coarse discernment of intensities is often sufficient (e.g., one byte or 256 intensity grades per pixel).
Now let us inquire how the basic algorithms can be fitted to integer arithmetics. In this case, $X \in \widetilde{\mathcal{P}}(E)$ is no longer allowed to assume arbitrary membership grades $\mu_X(e) \in$

I. By contrast, there is a fixed choice of $m' \in \mathbb{N} \setminus \{0\}$ such that all membership values of the fuzzy argument set X satisfy

$$\mu_X(e) \in U_{m'} \tag{222}$$

for all $e \in E$, where

$$U_{m'} = \left\{0, \frac{1}{m'}, \dots, \frac{m'-1}{m'}, 1\right\} . \tag{223}$$

If $X \in \widetilde{\mathcal{P}}(E)$ satisfies (222), i.e. X is admissible for the chosen m', then we can represent the histogram of X as an $(m'+1)$-dimensional array $\text{Hist}_X : \{0, \dots, m'\} \longrightarrow \mathbb{N}$, defined by

$$\text{Hist}_X[k] = \left|\{e \in E : \mu_X(e) = \frac{k}{m'}\}\right|$$

for all $k = 0, \dots, m'$. For simplicity, we shall further assume that m' is even, (i.e. $m' = 2m$ for a given $m \in \mathbb{N} \setminus \{0\}$). Then Hist_X can be represented by the following choices of the arrays H^+ and H^-,

$$\begin{aligned} H^+(j) &= \tfrac{1}{2} + j/m' \\ H^-(j) &= \tfrac{1}{2} - j/m' \end{aligned}$$

for $j = 0, \dots, m$. The integer-based implementation therefore rests on the choice of

$$\Gamma = \{j/m : j = 0, \dots, m\}\,, \tag{224}$$

which is known from (222) to include $\Gamma(X)$. Starting from $j \in \{0, \dots, m\}$, we can now compute γ_j according to $\gamma_j = j/m$. Unlike the floating-point implementation, the integer-based reformulation can therefore dispense with an array `G` which explicitly stores the relevant values of γ_j. The integer-based algorithms for implementing unary quantifiers in $\mathcal{F}_{\text{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ are shown in tables 7, 8 and 9, respectively. In fact, these algorithms are rather similar to their floating-point equivalents, because they are also based on Th-250, Th-253 and Th-254; and because they also profit from the incremental update of ℓ and u as described in Th-264. However, some slight optimizations have been made which reflect that Γ is now defined by (224). Unlike the floating-point case, this choice of Γ makes it possible that both $\texttt{H}^+\texttt{[j]} = 0$ and $\texttt{H}^-\texttt{[j]} = 0$ in the current iteration j, which entails that no update of ℓ, u, $q^{\min}(\ell, u)$ and $q^{\max}(\ell, u)$ is necessary (see Th-264 for justification). In order to avoid the vacuous recomputation of ℓ and u in this situation, it pays off to check if the above criterion applies, and consequently to perform an update of $q^{\min}(\ell, u)$ and $q^{\max}(\ell, u)$ only when it is known to be necessary. The variables `ch` ('change') or `nc` ('no change') are used to recognize this situation. `ch` is set to 'true' only if at least one of ℓ, u has changed in the current iteration, which forces $q^{\min}(\ell, u)$ and $q^{\max}(\ell, u)$ to be recomputed. The flag `nc` is set to true only if none of the coefficients ℓ and u has changed in the current iteration, which can be used to control the updating of $q^{\min}(\ell, u)$ and $q^{\max}(\ell, u)$ in a similar way. Apart from these modifications which aim at performance optimization,

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$
(integer arithmetics)

```
INPUT: $X$
// initialise H, $\ell$, $u$
H := Hist$_X$;
$\ell$ := $\sum_{j=1}^{m}$ H[m+j];
$u$ := $\ell$ + H[m];
Q := $q^{\min}(\ell,u)$ + $q^{\max}(\ell,u)$;
sum := Q;
for( j := 1; j<m; j := j + 1 ) {
    ch := false; // "change"
    // update clauses for $\ell$ and $u$
    if( H[m+j] $\neq$ 0 )
        { $\ell$ := $\ell$ - H[m+j]; ch := true; }
    if( H[m-j] $\neq$ 0 )
        { $u$ := $u$ + H[m-j]; ch := true; }
    if( ch )
        // one of $\ell$ or $u$ has changed
        { Q := $q^{\min}(\ell,u)$ + $q^{\max}(\ell,u)$; }
    sum := sum + Q;
    }
return sum / m'; // where m' = 2*m
END
```

Table 7: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{F}_{\mathrm{Ch}}$ (based on integer arithmetics)

the algorithm for $\mathcal{M}_{CX}$ has undergone some further changes. Compared to Th-254, I have altered the recognition of the 'stop condition', which is replaced with a simpler criterion in the final algorithm (see table 9). This modification is apparent from the use of integer arithmetics, because the possible membership grades are now tied to the set of all k/m', $k \in \{0, \ldots, m'\}$, see (222). The reduced number of arithmetic operations necessary to check the modified stopping condition, results in a further speed-up of the algorithm for computing $\mathcal{M}_{CX}(Q)(X)$.
A further simplification of the algorithms for $\mathcal{M}(Q)(X)$ and $\mathcal{M}_{CX}(Q)(X)$ is possible in the frequent case that Q is monotonic. For example, if Q is nondecreasing, then $q^{\min}(\ell, u) = q(\ell)$ and $q^{\max}(\ell, u) = q(u)$, see (148). We can therefore omit the updating of u in the first for-loop and likewise omit the updating of ℓ in the second for-loop of the algorithms, because only the remaining coefficient is actually needed for computing $q^{\min}(\ell, u)$ or $q^{\max}(\ell, u)$ in these cases.

11.8 Implementation of absolute quantifiers and quantifiers of exception

Unary quantifiers, like those discussed in the previous section, usually do not express directly at the NL surface. Nevertheless, the modelling and implementation of these quantifiers proves invaluable for treating important classes of quantifiers in NL: those which depend on an absolute count (i.e., *absolute quantifiers*), and those which specify the admissible exceptions to a general rule (i.e., *quantifiers of exception*).

Definition 165 *For every two-place semi-fuzzy quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$,*

- *Q is called* absolute *if there exists a quantitative one-place quantifier Q' :*

Algorithm for computing $\mathcal{M}(Q)(X)$
(integer arithmetics)

```
INPUT: $X$
// initialise H, $\ell$, $u$
H := Hist$_X$;
$\ell$ := $\sum_{j=1}^{m}$ H[m+j];
$u$ := $\ell$ + H[m];
$\top$ := $q^{\max}(\ell,u)$;
$\bot$ := $q^{\min}(\ell,u)$;
if( $\bot > \frac{1}{2}$ ) {
   sum := $\bot$;
   for( j := 1; j<m; j := j + 1 ) {
      nc:= true; // "no change"
      // update clauses for $\ell$ and $u$
      if( H[m+j] $\neq$ 0 )
         { $\ell$ := $\ell$ - H[m+j]; nc:= false; }
      if( H[m-j] $\neq$ 0 )
         { $u$ := $u$ + H[m-j]; nc:= false; }
      if( nc)
         { sum := sum + $\bot$; continue; }
      // one of $\ell$ or $u$ has changed
      $\bot$ := $q^{\min}(\ell,u)$;
      if( $\bot \leq \frac{1}{2}$ )
         break;
      sum := sum + $\bot$;
      }
   }
else if( $\top < \frac{1}{2}$ ) {
   sum := $\top$;
   for( j := 1; j<m; j := j + 1 ) {
      nc:= true;
      if( H[m+j] $\neq$ 0 )
         { $\ell$ := $\ell$ - H[m+j]; nc:= false; }
      if( H[m-j] $\neq$ 0 )
         { $u$ := $u$ + H[m-j]; nc:= false; }
      if( nc)
         { sum := sum + $\top$; continue; }
      // one of $\ell$ or $u$ has changed
      $\top$ := $q^{\max}(\ell,u)$;
      if( $\top \geq \frac{1}{2}$ )
         break;
      sum := sum + $\top$;
      }
   }
else
   { return $\frac{1}{2}$; }
return (sum + $\frac{1}{2}$*(m-j)) / m;
END
```

Table 8: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{M}$ (based on integer arithmetics)

$\mathcal{P}(E) \longrightarrow \mathbf{I}$ *such that* $Q = Q'\cap$, *i.e.* $Q(Y_1, Y_2) = Q'(X_1 \cap X_2)$ *for all* $Y_1, Y_2 \in \mathcal{P}(E)$.

- *Q is called a* quantifier of exception *if there exists an absolute quantifier* $Q'' : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *such that* $Q = Q''\neg$, *i.e.* $Q(Y_1, Y_2) = Q''(X_1, \neg X_2)$ *for all* $Y_1, Y_2 \in \mathcal{P}(E)$.

Notes

- Absolute quantifiers are of course well-known both in fuzzy set theory and in TGQ; the above definition only serves to describe more precisely the class of quantifiers handled by the algorithms to follow. As already mentioned in the introduction, absolute quantifiers are called ‘quantifiers of the first kind’ in Zadeh’s

Algorithm for computing $\mathcal{M}_{CX}(Q)(X)$
(integer arithmetics)

```
INPUT: $X$
// initialise H, $\ell$, $u$
H := Hist$_X$;
$\ell$ := $\sum_{j=1}^{m}$ H[m+j];
$u$ := $\ell$ + H[m];
$\top$ := $q^{\max}(\ell,u)$;
$\bot$ := $q^{\min}(\ell,u)$;
if( $\bot > \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false; // "change"
        // update clauses for $\ell$ and $u$
        if( H[m+j] $\neq$ 0 )
            { $\ell$ := $\ell$ - H[m+j]; ch := true; }
        if( H[m-j] $\neq$ 0 )
            { $u$ := $u$ + H[m-j]; ch := true; }
        if( ch )
            // one of $\ell$ or $u$ has changed
            { $\bot$ := $q^{\min}(\ell,u)$; }
        if( $\bot \leq$ m+j )
            { return (m+j)/m′; }
        }
    return 1;
    }
else if( $\top < \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false;
        // update clauses for $\ell$ and $u$
        if( H[m+j] $\neq$ 0 )
            { $\ell$ := $\ell$ - H[m+j]; ch := true; }
        if( H[m-j] $\neq$ 0 )
            { $u$ := $u$ + H[m-j]; ch := true; }
        if( ch )
            // one of $\ell$ or $u$ has changed
            { $\top$ := $q^{\max}(\ell,u)$; }
        if( $\top \geq$ m-j )
            { return (m-j)/m′; }
        }
    return 0;
    } return $\frac{1}{2}$;
END
```

Table 9: Algorithm for evaluating quantitative one-place quantifiers in $\mathcal{M}_{CX}$ (based on integer arithmetics)

publications on fuzzy quantifiers, e.g. [188, p. 149]. In TGQ, these quantifiers are normally called ‘absolute’, but Keenan and Moss [81, p. 98] also use the term ‘cardinal determiner’.

- Quantifiers of exception are not mentioned in the literature on fuzzy quantifiers, but these quantifiers are well-known to TGQ, see e.g. [82]. The term ‘exception determiner’ is also common [81, p. 123].

To give one more example of the first type, the two-place quantifier “about 50” is an absolute quantifier. Some examples of quantifiers of exception are presented in Table 10. The DFS axioms ensure that

$$\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q')(X_1 \cap X_2)\,, \tag{225}$$

Quantifier	**Antonym (absolute)**
all	**no**
all except exactly k	**exactly k**
all except about k	**about k**
all except at most k	**at most k**

Table 10: Examples of quantifiers of exception

whenever $Q = Q'\cap$ is an absolute quantifier and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Similarly if $Q = Q'\cap\neg$ is a quantifier of exception, then

$$\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q')(X_1 \cap \neg X_2)\,, \tag{226}$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is the quantitative base quantifier from which Q is constructed. Owing to these reductions, we can therefore use the algorithms for computing $\mathcal{F}(Q')(X)$, $\mathcal{F} \in \{\mathcal{M}_{CX}, \mathcal{M}, \mathcal{F}_{\mathrm{Ch}}\}$ to evaluate absolute quantifiers and quantifiers of exception.

For example, consider the absolute quantifier **about 50** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$. According to the above definition, we can then reduce the absolute quantifier to an intersection

$$\textbf{about 50} = [{\sim}50]\cap\,,$$

where $[{\sim}50] : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a suitable unary quantifier. A possible choice of $[{\sim}50]$ is the following.

$$[{\sim}50](Y) = \mathrm{SZ}(|Y|, 35, 45, 55, 85)$$

for all $Y \in \mathcal{P}(E)$, where ‘SZ’ is Zadeh’s SZ-function [188, p. 184] (which has roughly the shape of a smoothened trapezoid), supposed to be nonzero in the range 35..85 and to reach full membership in the range 45..55. Owing to (225), we can then use the fuzzy quantifier $\mathcal{F}([{\sim}50]) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ in order to interpret “about 50” in the chosen model, e.g. for $\mathcal{F} = \mathcal{M}_{CX}$,

$$\mathcal{M}_{CX}(\textbf{about 50})(X_1, X_2) = \mathcal{M}_{CX}([{\sim}50])(X_1 \cap X_2)\,,$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. On the other hand, consider the quantifier of exception “all except k”, which can be defined by

$$\textbf{all except k}(Y_1, Y_2) = [\leq k](Y_1 \cap \neg Y_2)$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, i.e. **all except k** $= [\leq k]\cap\neg$. Let us now see how $\mathcal{M}_{CX}([\leq k])$ can be used for implementing $\mathcal{M}_{CX}$(**all except k**). Following the generic solution presented in (226), the computation of $\mathcal{M}_{CX}$(**all except k**)(X_1, X_2) then reduces to

$$\begin{aligned}\mathcal{M}_{CX}(\textbf{all except k})(X_1, X_2) &= \mathcal{M}_{CX}([\leq k])(X_1 \cap \neg X_2)\\ &= 1 - \mu_{[k+1]}(X_1 \cap \neg X_2)\,,\end{aligned}$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Similar examples of practical relevance can be treated along the same lines. Noticing that the assumptions which my analysis makes on $\mathcal{F}$ are valid in every DFS, this reduction to the unary case is possible in arbitrary models.

11.9 Implementation of proportional quantifiers

Proportional quantifiers like “almost all”, “most” or “many” are probably the most relevant from an application point of view: in many cases, it is not the absolute numbers, but rather some proportion or ratio of quantities, that raises interest. Hence let me define this basic type of quantifiers formally, which lets us communicate about such proportions:

Definition 166

A two-place semi-fuzzy quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *on a finite base set is called* proportional *if there exist* $v_0 \in \mathbf{I}$, $f : \mathbf{I} \longrightarrow \mathbf{I}$ *such that*

$$Q(Y_1, Y_2) = q(c_1, c_2) = \begin{cases} f(c_2/c_1) & : \quad c_1 \neq 0 \\ v_0 & : \quad \textit{else} \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$*, where* $c_1 = |Y_1|$ *and* $c_2 = |Y_1 \cap Y_2|$.

Notes

- For example, the proposed definition of “almost all” in terms of equality (12) fits into this scheme, where $f(z) = S(x, 0.7, 0.9)$ and $v_0 = 1$.
- Usually f and v_0 can be chosen independently of E, i.e. Q has extension; see section 4.14.
- Let us also observe that all proportional quantifiers are conservative by definition, because they can be expressed in terms of $|Y_1|$ and $|Y_1 \cap Y_2|$; see Th-258.

Due to the fact that all proportional quantifiers are conservative, we already know *in principle* how $\top_{Q,X_1,X_2}$ and $\bot_{Q,X_1,X_2}$ can be computed from q and X_1, X_2. The earlier theorem Th-263 elucidates the precise shape of the relation $R_\gamma(X_1, X_2)$, from which we obtain

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max\{q(c_1, c_2) : (c_1, c_2) \in R_\gamma(X_1, X_2)\} \\ \bot_{Q,X_1,X_2}(\gamma) &= \min\{q(c_1, c_2) : (c_1, c_2) \in R_\gamma(X_1, X_2)\}\,. \end{aligned}$$

In particular, the theorem explains how $R_\gamma(X_1, X_2)$ can be computed from coefficients $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$ which denote the cardinality of $(X_1)_\gamma^{\min}$, $(X_1 \cap X_2)_\gamma^{\min}$, $(X_1 \cap \neg X_2)_\gamma^{\min}$, $(X_1)_\gamma^{\max}$, $(X_1 \cap X_2)_\gamma^{\max}$ and $(X_1 \cap \neg X_2)_\gamma^{\max}$, respectively. Based on this analysis, I have already shown in (212) and (213) that

$$\top_{Q,X_1,X_2}(\gamma) = q^{\max}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3)$$

and

$$\bot_{Q,X_1,X_2}(\gamma) = q^{\min}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3)\,,$$

where $q^{\min}, q^{\max} : \{0, \dots, |E|\}^6 \longrightarrow \mathbf{I}$ are defined by

$$q^{\max}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) = \\ \max\{q(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} \tag{227}$$

and

$$q^{\min}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3) = \\ \min\{q(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\}\,. \tag{228}$$

In practice, we must specify a choice of $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, X_2)$ with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_m = 1$, which makes the formulas for computing quantification results in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ applicable (see Th-250, Th-253 and Th-254). Based on the above analysis, we are then prepared to compute the quantities $\top_j = \top_{Q,X_1,X_2}(\overline{\gamma}_j)$ and $\bot_j = \bot_{Q,X_1,X_2}(\overline{\gamma}_j)$ from the coefficients ℓ_k and u_k obtained at a given cut level $\overline{\gamma}_j$, $j \in \{0, \dots, m-1\}$. This results in the basic computation procedure for proportional quantifiers (or more generally, all conservative quantifiers). The method is not necessarily optimal in terms of efficiency, however. This is because for typical quantifiers, it is not necessary to systematically inspect all pairs $(c_1, c_2) \in R_j$ in order to compute $\top_j$ and $\bot_j$. By contrast, it is often possible to compute $\top_j$ and $\bot_j$ directly from f, v_0 and the given cardinality coefficients.

In the following, I will consider the typical case of a proportional quantifier which is nondecreasing in its second argument. Examples comprise “more than half”, “at least 10 percent”, “almost all”, “many” etc. For these quantifiers, I show how $q^{\min}$ and $q^{\max}$ can be computed more efficiently if one circumvents the explicit computation of the set

$$\{q(c_1, c_2) : \ell_1 \leq c_1 \leq c_2, \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\}\,.$$

By avoiding the direct maximisation/minimisation over all $q(c_1, c_2)$, I shortcut an operation which can become computationally costly. Apart from simplifying the computation of $\top_j$ and $\bot_j$, I also show that some of the coefficients $\ell_1, \ell_2, \ell_3, u_1, u_2$ and u_3 are superfluous for nondecreasing f, i.e. when q, and in turn Q, are nondecreasing in their second argument. This permits a further simplification of the algorithms in some cases, because we need not keep track of the irrelevant coefficients.

Theorem 265 *Let $E \neq \varnothing$ be a finite base set and $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ a proportional quantifier on E, i.e.*

$$Q(Y_1, Y_2) = q(c_1, c_2) = \begin{cases} f(c_2/c_1) & : \ c_1 \neq 0 \\ v_0 & : \ c_1 = 0 \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where $c_1 = |Y_1|$, $c_2 = |Y_1 \cap Y_2|$, $f : \mathbf{I} \longrightarrow \mathbf{I}$ and $v_0 \in \mathbf{I}$. Further suppose that f is nondecreasing, i.e. Q is nondecreasing in its second argument. Then

$$\top_{Q,X_1,X_2}(\gamma) = q^{\max}(\ell_1, \ell_3, u_1, u_2)$$
$$\bot_{Q,X_1,X_2}(\gamma) = q^{\min}(\ell_1, \ell_2, u_1, u_3)$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ *and* $\gamma \in \mathbf{I}$, *where the cardinality coefficients* $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$ *are defined as in Th-263, and* $q^{\min}, q^{\max} : \{0, \dots, |E|\}^4 \longrightarrow \mathbf{I}$ *are piecewise defined as follows.*

1. $\ell_1 > 0$. *Then* $q^{\min}(\ell_1, \ell_2, u_1, u_3) = f(\ell_2/(\ell_2 + u_3))$.
2. $\ell_1 = 0$.
   - *a.* $\ell_2 + u_3 > 0$.
     Then $q^{\min}(\ell_1, \ell_2, u_1, u_3) = \min(v_0,\ f(\ell_2/(\ell_2 + u_3)))$.
   - *b.* $\ell_2 + u_3 = 0$.
     - *i.* $u_1 > 0$.
       Then $q^{\min}(\ell_1, \ell_2, u_1, u_3) = \min(v_0, f(1))$.
     - *ii.* $u_1 = 0$. *Then* $q^{\min}(\ell_1, \ell_2, u_1, u_3) = v_0$.

For $q^{\max}(\ell_1, \ell_3, u_1, u_2)$, *we have:*

1. $\ell_1 > 0$. *Then* $q^{\max}(\ell_1, \ell_3, u_1, u_2) = f(u_2/(u_2 + \ell_3))$.
2. $\ell_1 = 0$.
   - *a.* $u_2 + \ell_3 > 0$.
     Then $q^{\max}(\ell_1, \ell_3, u_1, u_2) = \max(v_0,\ f(u_2/(u_2 + \ell_3)))$.
   - *b.* $u_2 + \ell_3 = 0$.
     - *i.* $u_1 > 0$.
       Then $q^{\max}(\ell_1, \ell_3, u_1, u_2) = \max(v_0,\ f(0))$.
     - *ii.* $u_1 = 0$. *Then* $q^{\max}(\ell_1, \ell_3, u_1, u_2) = v_0$.

(Proof: D.26, p.502+)

Notes

- If $v_0 \leq f(1)$, then $\min(v_0, f(1)) = v_0$, i.e. we need not distinguish 2.b.i and 2.b.ii in the computation of $q^{\min}$.
- If $f(0) \leq v_0$, then 2.b.i and 2.b.ii need not be distinguished in the computation of $q^{\max}$.

I will now develop the above analysis into complete algorithms for computing proportional quantifiers in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$. Hence let a proportional quantifier Q be given, and suppose that Q is specified in terms of the mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$ and constant $v_0 \in \mathbf{I}$. In order to demonstrate the intended optimizations, I will assume that f is non-decreasing. Further let $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ be a given choice of fuzzy arguments. In order to put the framework into action and implement Q in the prototypical models, we must choose some $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, X_2)$ with $0 = \gamma_0 < \gamma_1 < \dots < \gamma_m = 1$, $m \in \mathbb{N} \setminus \{0\}$. For developing the solution based on floating-point arithmetics, it is

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, X_2)$
(floating-point arithmetics)

```
INPUT: $X_1, X_2$
Compute G, m, $H_1^+$, $H_2^+$,$H_3^+$,$H_1^-$, $H_2^-$, $H_3^-$; // see text
// initialize $\ell$, $u$:
for( k := 1; k $\leq 3$; k := k+1 )
    { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k^+$[j]; $u_k$ := $\ell_k$ + $H_k^+$[0]; }
sum := G[1] * ($q^{\min}(\ell_1, \ell_2, u_1, u_3)$ + $q^{\max}(\ell_1, \ell_3, u_1, u_2)$);
for( j := 1; j<m; j := j + 1 ) {
    // update clauses for $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$
    for( k := 1; k $\leq 3$; k := k+1 )
        { $\ell_k$ := $\ell_k$ - $H_k^+$[j]; $u_k$ := $u_k$ + $H_k^-$[j]; }
    sum := sum + (G[j+1] - G[j])
                 * ($q^{\min}(\ell_1, \ell_2, u_1, u_3)$ + $q^{\max}(\ell_1, \ell_3, u_1, u_2)$);
    }
return sum / 2;
END
```

Table 11: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{F}_{\mathrm{Ch}}$ (based on floating-point arithmetics)

again convenient to assume the minimal choice of Γ, i.e. $\Gamma = \Gamma(X_1, X_2)$. The above rule for computing $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ and $q^{\max}(\ell_1, \ell_3, u_1, u_2)$ reduces $\top_j$ and $\bot_j$ to a function of coefficients ℓ_k, u_k, $k \in \{1, 2, 3\}$ sampled from $Z_1 = \Psi_1(X_1, X_2) = X_1$, $Z_2 = \Psi_2(X_1, X_2) = X_1 \cap X_2$ and $Z_3 = \Psi_3(X_1, X_2) = X_1 \cap \neg X_2$. Noticing that $\Gamma(X_1 \cap X_2) \subseteq \Gamma(X_1, X_2)$ and $\Gamma(X_1 \cap \neg X_2) \subseteq \Gamma(X_1, X_2)$ by Th-247, we can represent the histograms Hist_{Z_k}, $k \in \{1, 2, 3\}$ by $(m + 1)$-dimensional arrays H_k^+ and H_k^- defined in accordance with (220) and (221). Currently, my implementation of proportional quantifiers refers to inputs X_1 and X_2, in order to make the algorithms self-contained. Again, it might be advantageous to separate the computation of the histograms H_k^+ and H_k^- from the main algorithms.[43] In a practical implementation, the assumed fuzzy sets input can easily be exchanged with the resulting histograms if so desired. Apart from the histogram arrays H_k^+ and H_k^-, we must further compute the array `G` of cut levels `G[j]` $= \gamma_j$, $j \in \{0, \dots, m\}$, and the cardinal m, which specifies the number of γ_j's. Based on these preparations, the earlier theorem Th-264 then permits an incremental computation of the quantities ℓ_k and u_k, $k \in \{1, 2, 3\}$. In turn, the above theorem Th-265 lets us compute $\top_j = q^{\max}(\ell_1, \ell_3, u_1, u_2)$ and $\bot_j = q^{\min}(\ell_1, \ell_2, u_1, u_3)$ from the current values of the ℓ_k and u_k in the j-th iteration. Finally theorems Th-250, Th-253 and Th-254 show how to compute the final outcome of quantification in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ from the given data. The resulting algorithms which implement proportional quantification in the models $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ are presented in tables 11, 12 and 13, respectively.

Let me point the reader's attention to the organization of the algorithms for $\mathcal{M}$ and $\mathcal{M}_{CX}$. First of all, both algorithms have been split into two main cases, and associated main loops, in order to restrict computation to either $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ or to $q^{\max}(\ell_1, \ell_3, u_1, u_2)$. Of course, it is assumed that $q^{\min}$ and $q^{\max}$ be computed effi-

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[43] In this case, the extraction of the histogram data from X_1 and X_2 is delegated to a preprocessing step, and it is H_k^+ and H_k^- that serve as the input of the algorithms, rather than the fuzzy sets X_1 and X_2. The histograms then provide a uniform representation suited for arbitrary fuzzy sets, which obviates the need for the algorithms to operate on heterogeneous data.

Algorithm for computing $\mathcal{M}(Q)(X_1, X_2)$
(floating-point arithmetics)

```
INPUT: $X_1, X_2$
Compute G, m, $H_1^+$, $H_2^+$,$H_3^+$,$H_1^-$, $H_2^-$, $H_3^-$; // see text
// initialize $\ell$, $u$:
for( k := 1; k $\leq 3$; k := k+1 )
    { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k^+$[j]; $u_k$ := $\ell_k$ + $H_k^+$[0]; }
$\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
if( cq $> \frac{1}{2}$ ) {
    sum := G[1] * $\bot$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell_1, \ell_2, u_1, u_3$
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_2$ := $\ell_2$ - $H_2^+$[j];
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_3$ := $u_3$ + $H_3^-$[j];
        $\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
        if( $\bot \leq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\bot$;
        }
    }
else if( $\top < \frac{1}{2}$ ) {
    sum := G[1] * $\top$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell_1, \ell_3, u_1, u_2$
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_3$ := $\ell_3$ - $H_3^+$[j];
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_2$ := $u_2$ + $H_2^-$[j];
        $\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
        if( $\top \geq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\top$;
        }
    }
else
    { return $\frac{1}{2}$; }
return sum + $\frac{1}{2}$ * (1 - G[j]);
END
```

Table 12: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{M}$ (based on floating-point arithmetics)

ciently, following theorem Th-265. Due to the fact that the quantities $q^{\min}$ and $q^{\max}$ need not be computed simultaneously, some simplifications are possible, i.e. the so-called 'update clauses' which keep track of the ℓ_k and u_k can be restricted to those coefficients that are actually needed for computing $q^{\min}$ (in the first main loop) or $q^{\max}$ (in the alternative main loop). For example, consider the first main loop in the algorithm for $\mathcal{M}_{CX}$ shown in table 12. In this case, the quantification result solely depends on $q^{\min}(\ell_1, \ell_2, u_1, u_3)$, which in turn can be computed from the current values of ℓ_1, ℓ_2, u_1 and u_3 in the j-th iteration. It is therefore sufficient to update the coefficients ℓ_1, ℓ_2, u_1 and u_3 only; while the updating of ℓ_3 and u_2, and also the computation of $q^{\max}(\ell_1, \ell_3, u_1, u_2)$, can be omitted. It is also instructive to compare these algorithms to their counterparts for unary quantifiers. In fact, the basic organisation of the algorithms is identical in both cases; all differences are confined to the update clauses which keep track of different sets of coefficients ℓ_k and u_k.

It should be apparent how the basic algorithms can be fitted to nonincreasing propor-

Algorithm for computing $\mathcal{M}_{CX}(Q)(X_1, X_2)$
(floating-point arithmetics)

```
INPUT: $X_1, X_2$
Compute G, m, $H_1^+$, $H_2^+$,$H_3^+$,$H_1^-$, $H_2^-$, $H_3^-$; // see text
// initialize $\ell$, $u$:
for( k := 1; k $\leq$ 3; k := k+1 )
    { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k^+$[j]; $u_k$ := $\ell_k$ + $H_k^+$[0]; }
$\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
j := 0;
if( $\bot$ > $\frac{1}{2}$) {
    B := 2 * $\bot$ - 1;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell_1, \ell_2, u_1, u_3$
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_2$ := $\ell_2$ - $H_2^+$[j];
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_3$ := $u_3$ + $H_3^-$[j];
        B := 2 * $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ - 1;
        }
    return $\frac{1}{2}$ + $\frac{1}{2}$ max(B, G[j]);
    }
else if( $\top < \frac{1}{2}$ ) {
    B := 1 - 2 * $\top$;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell_1, \ell_3, u_1, u_2$
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_3$ := $\ell_3$ - $H_3^+$[j];
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_2$ := $u_2$ + $H_2^-$[j];
        B := 1 - 2 * $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
        }
    return $\frac{1}{2}$ - $\frac{1}{2}$ max(B, G[j]);
    }
return $\frac{1}{2}$;
END
```

Table 13: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{M}_{CX}$ (based on floating-point arithmetics)

tional quantifiers like “less than 60 percent” as well; suffice it to observe that if Q (and hence f) is nonincreasing, then $Q' = \neg Q$ is also a proportional quantifier, which can be described in terms of $f' = \neg f$ and $v_0' = \neg v_0$. Noticing that the resulting mapping f' is nondecreasing, the above algorithms can be used to compute $\mathcal{F}(Q')(X_1, X_2)$, where $\mathcal{F}$ is the model of interest. According to Th-12, the computation of $\mathcal{F}(Q)(X_1, X_2)$ can therefore be reduced to $\mathcal{F}(Q)(X_1, X_2) = \neg\mathcal{F}(Q')(X_1, X_2)$.

I have stated the above algorithms for a special case often met in practice, in order to demonstrate the potential for optimization. However, these algorithms are easily adapted to general proportional quantifiers, which need not fulfill any monotonicity requirements. To this end, it is sufficient to replace all occurrences of $q^{\max}(\ell_1, \ell_3, u_1, u_2)$ and $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ in the above algorithms with their generic counterparts, i.e. $q^{\max}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3)$ and $q^{\min}(\ell_1, \ell_2, \ell_3, u_1, u_2, u_3)$ defined by (227) and (228). Additional update clauses must then be added, because it is now necessary to keep track of all coefficients ℓ_1, ℓ_2, ℓ_3, u_1, u_2 and u_3.

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, X_2)$
(integer arithmetics)

```
INPUT: $X_1, X_2$
// initialise $H_k$, $\ell$, $u$
$H_1$ := Hist$_{X_1}$;
$H_2$ := Hist$_{X_1 \cap X_2}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2}$;
for( k := 1; k $\leq 3$; k := k+1 )
   { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k$[m+j]; $u_k$ := $\ell_k$ + $H_k$[m]; }
Q := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ + $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
sum := Q;
for( j := 1; j<m; j := j + 1 ) {
   ch := false; // "change"
   // update clauses for $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$
   if( $H_1$[m+j] $\neq$ 0 )
      { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; ch := true; }
   if( $H_2$[m+j] $\neq$ 0 )
      { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; ch := true; }
   if( $H_3$[m+j] $\neq$ 0 )
      { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; ch := true; }
   if( $H_1$[m-j] $\neq$ 0 )
      { $u_1$ := $u_1$ + $H_1$[m-j]; ch := true; }
   if( $H_2$[m-j] $\neq$ 0 )
      { $u_2$ := $u_2$ + $H_2$[m-j]; ch := true; }
   if( $H_3$[m-j] $\neq$ 0 )
      { $u_3$ := $u_3$ + $H_3$[m-j]; ch := true; }
   if( ch )
      // one of the $\ell_k$ or $u_k$ has changed
      { Q := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$ + $q^{\max}(\ell_1, \ell_3, u_1, u_2)$; }
   sum := sum + Q;
   }
return sum / m′; // where m′ = 2*m
END
```

Table 14: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{F}_{\mathrm{Ch}}$ (based on integer arithmetics)

As in the case of unary quantifiers, it also makes sense to develop special versions of the algorithms, which are optimized for integer arithmetics. Hence let Q be a proportional quantifier and suppose that $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are given fuzzy arguments. Further let $m' \in \mathbb{N} \setminus \{0\}$ be given, which specifies the available range of integers. Once m' is chosen, admissible fuzzy arguments are no longer allowed to assume arbitrary membership grades in $\mathbf{I} = [0, 1]$. By contrast, we again require that

$$\mu_{X_i}(e) \in U_{m'}$$

for all $e \in E$ and $i \in \{1, 2\}$, where $U_{m'}$ is defined as in the unary case, cf. (223). Under these assumptions on legal choices of fuzzy arguments, the above algorithms which implement proportional quantifiers in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ can now be adapted in complete analogy to the earlier changes for unary quantifiers, see pp. 323-325. The resulting algorithms for proportional quantification based on integer arithmetics are shown in table 14 ($\mathcal{F}_{\mathrm{Ch}}$), table 15 ($\mathcal{M}$), and table 16 (for $\mathcal{M}_{CX}$). Specifically, the algorithms have again been augmented with flags `ch` (‘change’) or `nc` (‘no change’), which keep track of any changes of the ℓ_k or u_k in the current iteration. Only if `ch` is set to true (or `nc` to false), a recomputation of $q^{\min}$ or $q^{\max}$ becomes necessary; otherwise, the algorithms will directly skip forward to the next iteration.
In practice, the use of these integer-based algorithms should again be considered when processing speed is more important than accuracy, i.e. the domain of quantification is very large, and the loss of precision due to a small choice of m' is acceptable. The final

Algorithm for computing $\mathcal{M}(Q)(X_1, X_2)$
(integer arithmetics)

```
INPUT: $X_1, X_2$
// initialise $H_k$, $\ell$, $u$
$H_1$ := Hist$_{X_1}$;
$H_2$ := Hist$_{X_1 \cap X_2}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2}$;
for( k := 1; k $\leq$ 3; k := k+1 )
    { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k$[m+j]; $u_k$ := $\ell_k$ + $H_k$[m]; }
$\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
if( $\bot > \frac{1}{2}$ ) {
    sum := $\bot$;
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true; // "no change"
        // update clauses for $\ell_1, \ell_2, u_1, u_3$
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; nc:= false; }
        if( $H_2$[m+j] $\neq$ 0 )
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; nc:= false; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; nc:= false; }
        if( $H_3$[m-j] $\neq$ 0 )
            { $u_3$ := $u_3$ + $H_3$[m-j]; nc:= false; }
        if( nc)
            { sum := sum + $\bot$; continue; }
        // one of $\ell_1, \ell_2, u_1, u_3$ has changed
        $\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
        if( $\bot \leq \frac{1}{2}$ ) break;
        sum := sum + $\bot$;
        }
    }
else if( $\top < \frac{1}{2}$ ) {
    sum := $\top$;
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true;
        // update clauses for $\ell_1, \ell_3, u_1, u_2$
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; nc:= false; }
        if( $H_3$[m+j] $\neq$ 0 )
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; nc:= false; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; nc:= false; }
        if( $H_2$[m-j] $\neq$ 0 )
            { $u_2$ := $u_2$ + $H_2$[m-j]; nc:= false; }
        if( nc)
            { sum := sum + $\top$; continue; }
        // one of $\ell_1, \ell_3, u_1, u_2$ has changed
        $\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
        if( $\top \geq \frac{1}{2}$ ) break;
        sum := sum + $\top$;
        }
    }
else
    { return $\frac{1}{2}$; }
return (sum + $\frac{1}{2}$*(m-j)) / m;
END
```

Table 15: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{M}$ (based on integer arithmetics)

decision between the floating-point and integer-based variants should then be based on practical tests, which assess the processing times for typical instances of quantification.

The presented algorithms permit an efficient implementation which will be fast enough for most applications. Due to the inherent complexity of proportional quantification compared to the unary case, the algorithms are necessarily more complicated and computationally demanding than those for simple one-place quantifiers. In some situa-

Algorithm for computing $\mathcal{M}_{CX}(Q)(X_1, X_2)$
(integer arithmetics)

```
INPUT: $X_1, X_2$
// initialise $H_k$, $\ell$, $u$
$H_1$ := Hist$_{X_1}$;
$H_2$ := Hist$_{X_1 \cap X_2}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2}$;
for( k := 1; k $\leq$ 3; k := k+1 )
    { $\ell_k$ := $\sum_{j=1}^{m}$ $H_k$[m+j]; $u_k$ := $\ell_k$ + $H_k$[m]; }
$\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$;
if( $\bot > \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false; // "change"
        // update clauses for $\ell_1, \ell_2, u_1, u_3$
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; ch := true; }
        if( $H_2$[m+j] $\neq$ 0 )
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; ch := true; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; ch := true; }
        if( $H_3$[m-j] $\neq$ 0 )
            { $u_3$ := $u_3$ + $H_3$[m-j]; ch := true; }
        if( ch )
            // one of $\ell_1, \ell_2, u_1, u_3$ has changed
            { $\bot$ := $q^{\min}(\ell_1, \ell_2, u_1, u_3)$; }
        if( $\bot \leq$ m+j )
            { return (m+j)/m'; }
        }
    return 1;
    }
else if( $\top < \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false;
        // update clauses for $\ell_1, \ell_3, u_1, u_2$
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; ch := true; }
        if( $H_3$[m+j] $\neq$ 0 )
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; ch := true; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; ch := true; }
        if( $H_2$[m-j] $\neq$ 0 )
            { $u_2$ := $u_2$ + $H_2$[m-j]; ch := true; }
        if( ch )
            // one of $\ell_1, \ell_3, u_1, u_2$ has changed
            { $\top$ := $q^{\max}(\ell_1, \ell_3, u_1, u_2)$; }
        if( $\top \geq$ m-j )
            { return (m-j)/m'; }
        }
    return 0;
    }
return $\frac{1}{2}$;
END
```

Table 16: Algorithm for evaluating two-place proportional quantifiers in $\mathcal{M}_{CX}$ (based on integer arithmetics)

tions, however, it is possible to shortcut the general solution and delegate proportional quantification to the fast algorithms for unary quantifiers. Specifically, this simplification is possible when the first argument, i.e. the restriction of the quantifier, happens to be a crisp set, as in "Most married men are bald". In this case, the restriction **married men** $\in \mathcal{P}(E)$ can serve as the domain of a derived unary quantifier, as shown by the following theorem.

Theorem 266 *Let* $E \neq \varnothing$ *be a finite base set and suppose that* $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *is a proportional quantifier on* E*. Further let* f *denote the associated mapping* $f : \mathbf{I} \longrightarrow \mathbf{I}$

and v_0 the associated constant, see Def. 166. In the following, we will consider a crisp set $V \in \mathcal{P}(E)$, $V \neq \varnothing$. Then in every DFS $\mathcal{F}$,

$$\mathcal{F}(Q)(V, X) = \mathcal{F}(Q')(X')$$

for all $X \in \widetilde{\mathcal{P}}(E)$, where $X' \in \widetilde{\mathcal{P}}(V)$ is the restriction of X to V, i.e.

$$\mu_{X'}(e) = \mu_X(e) \tag{229}$$

for all $e \in V$; and where $Q' : \mathcal{P}(V) \longrightarrow \mathbf{I}$ is the quantitative unary quantifier defined from $q' : \{0, \ldots, |V|\} \longrightarrow \mathbf{I}$,

$$q'(c) = f(c/|V|) \tag{230}$$

for all $c \in \{0, \ldots, |V|\}$, according to $Q'(Y) = q'(|Y|)$ for all $Y \in \mathcal{P}(V)$.
(Proof: D.27, p.510+)

In other words, the theorem presents a reduction of two-place proportional quantification on the full domain to simple one-place quantification on a restricted domain, which is supplied by the first argument. The reduction is possible whenever the first argument qualifies as a proper domain, i.e. it must be crisp and nonempty. Recognizing this situation promises a boost in performance, because the algorithms for unary quantification become applicable, which are much simpler (and consequently, faster) than the generic algorithms for proportional quantification. Apart from efficiency considerations, the theorem is also interesting from a theoretical perspective because it links unrestricted proportional to unrestricted absolute quantification.

For purposes of illustration, let me briefly discuss the above example "Most married men are bald" in the context of the theorem. Hence suppose that "most" is defined in terms of $f : \mathbf{I} \longrightarrow \mathbf{I}$ and $v_0 \in \mathbf{I}$ according to Def. 166; a possible choice of f and v_0 is: $f(x) = 1$ if $x > \frac{1}{2}$ and $f(x) = 0$ otherwise; and $v_0 = 1$. The mapping $q' : \{0, \ldots, |\textbf{married men}|\} \longrightarrow \mathbf{I}$ then becomes

$$q'(c) = f(c/|\textbf{married men}|)$$

for all $c \in \{0, \ldots, |\textbf{married men}|\}$. For example, if there are ten married men in the considered domain, then

$$q'(c) = \begin{cases} 1 & : & c > 5 \\ 0 & : & c \leq 5 \end{cases} \tag{231}$$

for all $c \in \{0, \ldots, 10\}$. Consequently, the derived quantifier $Q' : \mathcal{P}(\textbf{married men}) \longrightarrow \mathbf{I}$ becomes

$$Q'(Y) = q'(|Y|) = \begin{cases} 1 & : & |Y| > 5 \\ 0 & : & |Y| \leq 5 \end{cases}$$

for all $Y \in \mathcal{P}(E)$. In other words, Q' is the one-place quantifier $[\geq 6]$ defined in Def. 160. Under the assumption that **married men** has ten members, the proportional quantification in "Most married men are bald" therefore reduces to the simpler statement,

$$\mathcal{F}(\textbf{most})(\textbf{married men}, \textbf{bald}) = \mathcal{F}([\geq 6])(\textbf{bald}'),$$

where $\textbf{bald}'$ is the restriction of the fuzzy set **bald** to the new domain **married men**. Finally we utilize the earlier theorem Th-237 and conclude that

$$\mathcal{F}(\textbf{most})(\textbf{married men}, \textbf{bald}) = \mu_{[6]}(\textbf{bald}')$$

in every standard model.

This example completes my discussion of proportional quantification. We are now prepared to evaluate proportional quantifiers efficiently, either based on floating-point or integer arithmetics. The proposed algorithms for proportional quantification in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ cover the 'hard' case of proportional two-place quantification, i.e. the aggregation can be weighted by importances, which are specified by the first argument of the quantifier. In Th-265, I further improved the earlier analysis of conservative quantifiers given in Th-263, and hence achieved a more efficient computation in the frequent case of those proportional quantifiers which are also known to be monotonic. Finally, I considered the special case of unrestricted proportional quantification, i.e. one-place proportional quantification, or proportional quantification restricted by crisp importances. In this case, I described an apparent reduction which renders applicable the algorithms for one-place quantifiers. In those applications which do not need importance qualification, this reduction will result in an additional boost of processing speed.

11.10 Implementation of cardinal comparatives

In this section I will discuss a natural class of quantifiers which has received little attention in the literature on fuzzy quantifiers: that of *cardinal comparatives* [82, p. 305] or *comparative determiners* [81, p. 123]. This class comprises all NL quantifiers that express a comparison of two cardinalities, which are sampled from two restriction arguments (linguistically realized by two noun phrases A, B) and a common scope argument C (linguistically realized by the verb phrase). For example, the quantifier "more than", which underlies the pattern "More A's than B's are C's", is a cardinal comparative. Formally, I define cardinal comparatives as follows.

Definition 167
Let $E \neq \varnothing$ be a finite base set. A three-place quantifier $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{I}$ is called a cardinal comparative *if $Q(Y_1, Y_2, Y_3)$ depends on $|Y_1 \cap Y_3|$ and $|Y_2 \cap Y_3|$ only, i.e. there exists $q : \{0, \ldots, |E|\}^2 \longrightarrow \mathbf{I}$ such that*

$$Q(Y_1, Y_2, Y_3) = q(c_1, c_2) \tag{232}$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$, where $c_1 = |Y_1 \cap Y_3|$ and $c_2 = |Y_2 \cap Y_3|$.

For example, "more than" can be defined as

$$\textbf{more than}(Y_1, Y_2, Y_3) = \begin{cases} 1 & : \quad |Y_1 \cap Y_3| - |Y_2 \cap Y_3| > 0 \\ 0 & : \quad \text{else} \end{cases} \tag{233}$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$. The quantifier can be used to interpret statements like "More men than women own a car", which becomes

$$\textbf{more than}(\textbf{men}, \textbf{women}, \textbf{own a car})\,.$$

We can also model quantifiers of the type "exactly k more than", i.e.

$$\textbf{exactly k more than}(Y_1, Y_2, Y_3) = \begin{cases} 1 & : & |Y_1 \cap Y_3| - |Y_2 \cap Y_3| = k \\ 0 & : & \text{else} \end{cases}$$

and "at least k more than", i.e.

$$\textbf{at least k more than}(Y_1, Y_2, Y_3) = \begin{cases} 1 & : & |Y_1 \cap Y_3| - |Y_2 \cap Y_3| \geq k \\ 0 & : & \text{else} \end{cases}$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$, where $k \in \mathbb{N}$. Special cases include "more than" and "same number of", which can be expressed as

$$\textbf{more than} = \textbf{at least 1 more than}$$
$$\textbf{same number of} = \textbf{exactly 0 more than}\,.$$

In particular, a statement like "The same number of men and women are married" now translates into the quantifying expression **same number of**(**men**, **women**, **married**). Further examples of cardinal comparatives comprise "less than", "twice as many", "at least twice as many", "many more than", etc. Resorting to (232), the comparatives "(exactly) twice as many" and "at least twice as many" can be expressed in terms of the following mappings q_1 and q_2, respectively:

$$q_1(c_1, c_2) = \begin{cases} 1 & : & c_1 = 2\,c_2 \\ 0 & : & \text{else} \end{cases} \qquad \text{for "twice as many"}$$

$$q_2(c_1, c_2) = \begin{cases} 1 & : & c_1 \geq 2\,c_2 \\ 0 & : & \text{else} \end{cases} \qquad \text{for "at least twice as many"}$$

A conceivable definition of "many more than" is

$$\textbf{many more than}(Y_1, Y_2, Y_3) = q(c_1, c_2)$$

for all $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$, where

$$q(c_1, c_2) = \begin{cases} 0 & : & c_1 \leq c_2 + \alpha \\ (c_1 - c_2 - \alpha) \cdot \beta & : & c_2 + \alpha < c_1 \leq c_2 + \alpha + 1/\beta \\ 1 & : & c_2 > c_2 + \alpha + 1/\beta \end{cases}$$

for $c_1, c_2 \in \{0, \dots, |E|\}$. Obviously, suitable choices of $\alpha, \beta \in \mathbb{R}^+$ can only be made from the context. However, the rough shape of the resulting quantifier is easily grasped from q, so it should usually not pose problems to select a quantifier which suits the task at hand.

Now that we are able to define the cardinal comparatives of NL in terms of semi-fuzzy quantifiers, let us consider the relation $R_\gamma^{\Phi_1,\Phi_2}(X_1, X_2, X_3)$, where $\Phi_1(Y_1, Y_2, Y_3) = Y_1 \cap Y_3$ and $\Phi_2(Y_1, Y_2, Y_3) = Y_2 \cap Y_3$. Specifically, it must be shown how to express this relation in terms of quantities ℓ_r, u_r sampled from Boolean combinations of the fuzzy arguments $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$. The simple description of this target relation in terms of four Boolean combinations that are required, is set forth in the next theorem.

Theorem 267
Let $E \neq \varnothing$ *be a finite base set and* $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$*. Further let* Φ_1, Φ_2 *denote the Boolean combinations* $\Phi_1(Y_1, Y_2, Y_3) = Y_1 \cap Y_3$ *and* $\Phi_2(Y_1, Y_2, Y_3) = Y_2 \cap Y_3$ *of the crisp variables* $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$*. Then the relation* $R = R_\gamma^{\Phi_1,\Phi_2}(X_1, X_2, X_3)$ *defined by* (172) *can be computed from the cardinality coefficients* $\ell_r = |(Z_r)_\gamma^{\min}|$ *and* $u_r = |(Z_r)_\gamma^{\max}|$ *sampled from* $Z_1 = \Psi_1(X_1, X_2, X_3) = X_1 \cap X_3$*,* $Z_2 = \Psi_2(X_1, X_2, X_3) = X_2 \cap X_3$*,* $Z_3 = \Psi_3(X_1, X_2, X_3) = X_1 \cap \neg X_2 \cap X_3$ *and* $Z_4 = \Psi_4(X_1, X_2, X_3) = \neg X_1 \cap X_2 \cap X_3$*. In terms of* $\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4 \in \{0, \dots, |E|\}$*,* R *then becomes*

$$R = \{(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\}\} . \tag{234}$$

(Proof: D.28, p.511+)

In practice, it is not necessary to consider all pairs $(c_1, c_2) \in R$, because those cardinal comparatives actually found in NL conform to simple monotonicity patterns. For example, “More Y_1’s than Y_2’s are Y_3’s” is nondecreasing in the first and nonincreasing in the second argument, and the quantifier in “Less Y_1’s than Y_2’s are Y_3’s” is nonincreasing in the first and nondecreasing in the second argument. Here I will restrict to the first of these monotonicity patterns, and study the corresponding quantifiers in some more depth; all results of this investigation are readily transfered to the second monotonicity pattern and I will omit the obvious modifications for the sake of brevity. Let me first link the monotonicity of a cardinal comparative Q to the monotonicity of the mapping q on cardinals.

Theorem 268
Let $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{I}$ *be a cardinal comparative on a finite base set* $E \neq \varnothing$*. Further let* $q : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$ *be the corresponding mapping defined by* (232)*, i.e.* $Q(Y_1, Y_2, Y_3) = q(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|)$ *for all* $Y_1, Y_2, Y_3 \in \mathcal{P}(E)$*. Then the following are equivalent:*

a. Q *is nondecreasing in the first and nonincreasing in its second argument;*

b. q *is nondecreasing in its first and nonincreasing in its second argument.*

(Proof: D.29, p.525+)

Hence the considered monotonicity pattern of cardinal comparatives reflects in a very simple pattern of the mapping q. In turn, knowing these monotonicity properties permits a simplified description of $\top_{Q,X_1,X_2,X_3}$ and $\bot_{Q,X_1,X_2,X_3}$, because the minimum and maximum must then be assumed at the ‘extreme poles’. Consequently, we obtain the following corollary to Th-267 and Th-268.

Theorem 269
Let $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{I}$ *be a cardinal comparative on a finite base set* $E \neq \varnothing$*, and let* q

be the corresponding mapping $q : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$*. If* Q *is nondecreasing in the first and nonincreasing in its second argument, then*

$$\begin{aligned}\top_{Q,X_1,X_2,X_3}(\gamma) &= \max\{q(c_1, \max(c_1 - u_3 + \ell_4, \ell_2)) : \ell_1 \le c_1 \le u_1\}\\ \bot_{Q,X_1,X_2,X_3}(\gamma) &= \min\{q(c_1, \min(c_1 - \ell_3 + u_4, u_2)) : \ell_1 \le c_1 \le u_1\}\end{aligned}$$

for all $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ *and* $\gamma \in \mathbf{I}$*, referring to the cardinality coefficients* ℓ_r *and* u_r*,* $r \in \{1, 2, 3, 4\}$ *defined in Th-267.*
(Proof: D.30, p.527+)

The particular algorithms which implement cardinal comparatives in $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ can easily be developed from this analysis of the considered variety of quantifiers, again utilizing the basic results that were proven for these models in Th-250, Th-253 and Th-254, respectively. In fact, we can proceed in total analogy to the strategy used for absolute and proportional quantifiers. In the general case of unrestricted cardinal comparatives, we can express $\top_{Q,X_1,X_2,X_3}$ and $\bot_{Q,X_1,X_2,X_3}$, for a given choice of $\gamma \in \mathbf{I}$, as

$$\begin{aligned}\top_{Q,X_1,X_2,X_3}(\gamma) &= q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)\\ \bot_{Q,X_1,X_2,X_3}(\gamma) &= q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)\,,\end{aligned}$$

where

$$\begin{aligned}q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4) &= \max\{q(c_1, c_2) : (c_1, c_2) \in R_\gamma(X_1, X_2, X_3)\}\\ q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4) &= \min\{q(c_1, c_2) : (c_1, c_2) \in R_\gamma(X_1, X_2, X_3)\}\,.\end{aligned}$$

These coefficients can readily be computed, because we know the explicit representation of $R_\gamma(X_1, X_2, X_3)$, which was established in Th-267. In the typical case that the quantifier of interest belongs to the monotonicity pattern assumed in Th-269 (or the reverse pattern), a considerable cut in processing times becomes possible. In particular, $\top_{Q,X_1,X_2,X_3}$ and $\bot_{Q,X_1,X_2,X_3}$ can then be expressed in the following simplified way,

$$\begin{aligned}\top_{Q,X_1,X_2,X_3}(\gamma) &= q^{\max}(\ell_1, \ell_2, \ell_4, u_1, u_3)\\ \bot_{Q,X_1,X_2,X_3}(\gamma) &= q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4)\,,\end{aligned}$$

where $q^{\max}$ and $q^{\min}$ now read

$$\begin{aligned}q^{\max}(\ell_1, \ell_2, \ell_4, u_1, u_3) &= \max\{q(c_1, \max(c_1 - u_3 + \ell_4, \ell_2)) : \ell_1 \le c_1 \le u_1\}\\ q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4) &= \min\{q(c_1, \min(c_1 - \ell_3 + u_4, u_2)) : \ell_1 \le c_1 \le u_1\}\,,\end{aligned}$$

see Th-269.

In the algorithms to be presented now, I will consider both the typical monotonic and the general quantifiers, in order to optimally account for both cases. Hence let us recall the analysis of the three prototypical models achieved in theorems Th-250, Th-253 and Th-254, which will now be elaborated into a complete and practical implementation of cardinal comparatives. In order to instantiate the core formulas stated in these theorems, it is again necessary to fix the set of relevant cutting parameters, i.e. to choose

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, X_2, X_3)$
(floating-point arithmetics)

```
INPUT: $X_1, X_2, X_3$
Compute G, m, H$_1^+$, H$_2^+$,H$_3^+$,H$_4^+$,H$_1^-$, H$_2^-$, H$_3^-$,H$_4^-$; // see text
// initialize $\ell$, $u$:
for( r := 1; r $\leq 4$; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ H$_r^+$[j]; $u_r$ := $\ell_r$ + H$_r^+$[0]; }
sum := G[1] * ($q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$ + $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$);
for( j := 1; j<m; j := j + 1 ) {
    // update clauses for $\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4$
    for( r := 1; r $\leq 4$; r := r+1 )
        { $\ell_r$ := $\ell_r$ - H$_r^+$[j]; $u_r$ := $u_r$ + H$_r^-$[j]; }
    sum := sum + (G[j+1] - G[j])
                  * ($q^{\min}(\ell_1, \ell_2, \ell_3, u_4, u_1, u_2, u_3, u_4)$ + $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$);
    }
return sum / 2;
END
```

Table 17: Algorithm for evaluating cardinal comparatives in $\mathcal{F}_{\mathrm{Ch}}$ (based on floating-point arithmetics)

some $\Gamma = \{\gamma_0, \ldots, \gamma_m\}$ such that $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$ and $\Gamma \supseteq \Gamma(X_1, X_2, X_3)$.

When working with floating-point arithmetics, it is advantageous to minimize the set Γ, and hence have $\Gamma = \Gamma(X_1, X_2, X_3)$. Owing to Th-247, we then know that

$$\begin{aligned}
\Gamma(Z_1) &= \Gamma(X_1 \cap X_3) \subseteq \Gamma \\
\Gamma(Z_2) &= \Gamma(X_2 \cap X_3) \subseteq \Gamma \\
\Gamma(Z_3) &= \Gamma(X_1 \cap \neg X_2 \cap X_3) \subseteq \Gamma \\
\Gamma(Z_4) &= \Gamma(\neg X_1 \cap X_2 \cap X_3) \subseteq \Gamma\,,
\end{aligned}$$

where

$$\begin{aligned}
Z_1 &= \Phi_1(X_1, X_2, X_3) = X_1 \cap X_3 \\
Z_2 &= \Phi_2(X_1, X_2, X_3) = X_2 \cap X_3 \\
Z_3 &= \Phi_3(X_1, X_2, X_3) = X_1 \cap \neg X_2 \cap X_3 \\
Z_4 &= \Phi_4(X_1, X_2, X_3) = \neg X_1 \cap X_2 \cap X_3\,,
\end{aligned}$$

as stipulated in Th-267. Consequently the histograms Hist_{Z_r}, $r \in \{1, 2, 3, 4\}$, can be represented by $(m + 1)$-dimensional arrays H_r^+ and H_r^- defined in accordance with (220) and (221). Recalling that the cardinality coefficients ℓ_r, u_r are defined by $\ell_r = \ell_r(j) = |(Z_r)_{\overline{\gamma}_j}^{\min}|$ and $u_r = u_r(j) = |(Z_r)_{\overline{\gamma}_j}^{\max}|$ for $j \in \{0, \ldots, m-1\}$, we can now utilize the incremental rule presented in Th-264 in order to compute these coefficients efficiently. The particular algorithms so obtained, that implement cardinal comparatives based on floating-point arithmetics in the three prototype models, are shown in table 17 ($\mathcal{F}_{\mathrm{Ch}}$), table 18 ($\mathcal{M}$) and table 19 (for $\mathcal{M}_{CX}$), respectively.

These algorithms provide a generic solution for arbitrary cardinal comparatives, because they implement the full variety of these quantifiers without any special requirements on monotonicity properties. However, some tags have been included into the

Algorithm for computing $\mathcal{M}(Q)(X_1, X_2, X_3)$

(floating-point arithmetics)

```
INPUT: $X_1, X_2, X_3$
Compute G, m, H$_1^+$, H$_2^+$,H$_3^+$,H$_4^+$,H$_1^-$, H$_2^-$, H$_3^-$, H$_4^-$; // see text
// initialize $\ell$, $u$:
for( r := 1; r $\leq 4$; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ H$_r^+$[j]; $u_r$ := $\ell_r$ + H$_r^+$[0]; }
$\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
if( cq $> \frac{1}{2}$ ) {
    sum := G[1] * $\bot$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell$ and $u$:
        $\ell_1$ := $\ell_1$ - H$_1^+$[j];
        $\ell_2$ := $\ell_2$ - H$_2^+$[j]; // [*]
        $\ell_3$ := $\ell_3$ - H$_3^+$[j];
        $\ell_4$ := $\ell_4$ - H$_4^+$[j]; // [*]
        $u_1$ := $u_1$ + H$_1^-$[j];
        $u_2$ := $u_2$ + H$_2^-$[j];
        $u_3$ := $u_3$ + H$_3^-$[j]; // [*]
        $u_4$ := $u_4$ + H$_4^-$[j];
        $\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; // [1]
        // $\bot$ := $q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4)$; // [2]
        if( $\bot \leq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\bot$;
        }
    }
else if( $\top < \frac{1}{2}$ ) {
    sum := G[1] * $\top$;
    for( j := 1; j<m; j := j + 1 ) {
        // update clauses for $\ell$ and $u$:
        $\ell_1$ := $\ell_1$ - H$_1^+$[j];
        $\ell_2$ := $\ell_2$ - H$_2^+$[j];
        $\ell_3$ := $\ell_3$ - H$_3^+$[j]; // [+]
        $\ell_4$ := $\ell_4$ - H$_4^+$[j];
        $u_1$ := $u_1$ + H$_1^-$[j];
        $u_2$ := $u_2$ + H$_2^-$[j]; // [*]
        $u_3$ := $u_3$ + H$_3^-$[j];
        $u_4$ := $u_4$ + H$_4^-$[j]; // [*]
        $\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, u_4, u_1, u_2, u_3, u_4)$; // [1]
        // $\top$ := $q^{\max}(\ell_1, \ell_2, u_4, u_1, u_3)$; // [2]
        if( $\top \geq \frac{1}{2}$ )
            break;
        sum := sum + (G[j+1] - G[j]) * $\top$;
        }
    }
else
    { return $\frac{1}{2}$; }
return sum + $\frac{1}{2}$ * (1 - G[j]);
END
```

Table 18: Algorithm for evaluating cardinal comparatives in $\mathcal{M}$ (based on floating-point arithmetics)

Algorithm for computing $\mathcal{M}_{CX}(Q)(X_1, X_2, X_3)$
(floating-point arithmetics)

```
INPUT: $X_1, X_2, X_3$
Compute G, m, $H_1^+$, $H_2^+$,$H_3^+$,$H_4^+$,$H_1^-$, $H_2^-$, $H_3^-$,$H_4^-$; // see text
// initialize $\ell$, $u$:
for( r := 1; r $\leq 4$; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ $H_r^+$[j]; $u_r$ := $\ell_r$ + $H_r^+$[0]; }
$\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
j := 0;
if( $\bot$ > $\frac{1}{2}$) {
    B := 2 * $\bot$ - 1;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell$, $u$:
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_2$ := $\ell_2$ - $H_2^+$[j]; // [*]
        $\ell_3$ := $\ell_3$ - $H_3^+$[j];
        $\ell_4$ := $\ell_4$ - $H_4^+$[j]; // [*]
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_2$ := $u_2$ + $H_2^-$[j];
        $u_3$ := $u_3$ + $H_3^-$[j]; // [*]
        $u_4$ := $u_4$ + $H_4^-$[j];
        B := 2 * $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$ - 1; // [1]
        // B := 2 * $q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4)$ - 1; // [2]
        }
    return $\frac{1}{2}$ + $\frac{1}{2}$ max(B, G[j]);
    }
else if( $\top < \frac{1}{2}$ ) {
    B := 1 - 2 * $\top$;
    while( B > G[j+1] ) {
        j := j + 1;
        // update clauses for $\ell$, $u$:
        $\ell_1$ := $\ell_1$ - $H_1^+$[j];
        $\ell_2$ := $\ell_2$ - $H_2^+$[j];
        $\ell_3$ := $\ell_3$ - $H_3^+$[j]; // [*]
        $\ell_4$ := $\ell_4$ - $H_4^+$[j];
        $u_1$ := $u_1$ + $H_1^-$[j];
        $u_2$ := $u_2$ + $H_2^-$[j]; // [*]
        $u_3$ := $u_3$ + $H_3^-$[j];
        $u_4$ := $u_4$ + $H_4^-$[j]; // [*]
        B := 1 - 2 * $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; // [1]
        // B := 1 - 2 * $q^{\max}(\ell_1, \ell_2, \ell_4, u_1, u_3)$; // [2]
        }
    return $\frac{1}{2}$ - $\frac{1}{2}$ max(B, G[j]);
    }
return $\frac{1}{2}$;
END
```

Table 19: Algorithm for evaluating cardinal comparatives in $\mathcal{M}_{CX}$ (based on floating-point arithmetics)

pseudo-code, which indicate how the algorithms can be taylored to the prominent monotonicity type that was considered in Th-269. In this typical case of a natural language comparative, a more efficient implementation is possible due to the simplified description of $q^{\min}$ and $q^{\max}$ presented above. Consequently, some of the coefficients ℓ_r and u_r need not be tracked, because they take no effect on $q^{\min}$ or $q^{\max}$. In turn, this means that the corresponding 'update clauses' for these quantities can be dropped; i.e. the lines with a trailing `[*]` can be eliminated. In addition, the lines marked with `[1]` should be replaced with those labelled `[2]`, which refer to the simplified formulas for computing $q^{\min}$ and $q^{\max}$ in the monotonic case. These simple changes will boost performance for virtually all cardinal comparatives in natural languages (noticing that quantifiers of the reverse monotonicity type, like "less than", can be treated along the same lines), and result in a practical implementation of cardinal comparatives which is ready for use in the targeted applications of fuzzy quantifiers.

Having presented a solution for floating-point arithmetics, I will now add variants of the above algorithms for integer arithmetics. Hence let Q be a cardinal comparative (not required to be monotonic) and let $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ be given fuzzy arguments. Further suppose that $m' \in \mathbb{N} \setminus \{0\}$ is given, which specifies the available range of integers. The given choice of m' again restricts admissible fuzzy arguments to a finite set of equidistant membership grades in the unit interval, i.e. all legal choices of arguments must satisfy

$$\mu_{X_i}(e) \in U_{m'}$$

for all $e \in E$ and $i \in \{1, 2, 3\}$, where $U_{m'}$ is defined as before, see (223). By pursuing the same strategy that already proved useful for deriving the integer-based implementation of unary quantifiers (see pp. 323-325) and also of proportional quantifiers, I have also developed the corresponding solutions which implement cardinal comparatives based on integer arithmetics. The resulting variants of the above algorithms are shown in table 20 ($\mathcal{F}_{\mathrm{Ch}}$), table 21 ($\mathcal{M}$), and table 22 (for $\mathcal{M}_{CX}$). The listings are again annotated by tags `[*]`, `[1]`, `[2]`, in order to indicate the possible optimizations for monotonic quantifiers. In this case, the update clauses annotated by `[*]` can again be deleted, and the lines marked with `[1]` should be replaced by their simplified variants `[2]`.

Finally I would like to draw the reader's attention to a special case of comparative quantifiers which permits additional simplifications. In many cases, a cardinal comparative $Q(Y_1, Y_2, Y_3)$ can be expressed in terms of the difference $c_1 - c_2 = |Y_1 \cap Y_3| - |Y_2 \cap Y_3|$. For example, this is the case with "more than", cf. (233) above. The simplicity of these quantifiers translates into rules of computation that are equally simple. In particular, we need not know the complete relation

$$R_\gamma^{\Phi_1,\Phi_2}(X_1, X_2, X_3) = \{(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|) : (Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3)\}$$

in order to compute $\top_{Q,X_1,X_2,X_3}$ and $\bot_{Q,X_1,X_2,X_3}$. By contrast, it is sufficient to know the set of differences

$$\{|Y_1 \cap Y_3| - |Y_2 \cap Y_3| : (Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3)\}\,.$$

Algorithm for computing $\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, X_2, X_3)$
(integer arithmetics)

```
INPUT: $X_1, X_2, X_3$
// initialise $H_r$, $\ell$, $u$
$H_1$ := Hist$_{X_1 \cap X_3}$;
$H_2$ := Hist$_{X_2 \cap X_3}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2 \cap X_3}$;
$H_4$ := Hist$_{\neg X_1 \cap X_2 \cap X_3}$;
for( r := 1; r $\leq 4$; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ $H_r$[m+j]; $u_r$ := $\ell_r$ + $H_r$[m]; }
Q := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$ + $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
sum := Q;
for( j := 1; j<m; j := j + 1 ) {
    ch := false; // "change"
    // update clauses for $\ell$ and $u$:
    for( r := 1; r $\leq 4$; r := r+1 ) {
        if( $H_r$[m+j] $\neq$ 0 )
            { $\ell_r$ := $\ell_r$ - $H_r$[m+j]; ch := true;}
        if( $H_r$[m-j] $\neq$ 0 )
            { $u_r$ := $u_r$ + $H_r$[m-j]; ch := true; }
        }
    if( ch )
        // one of the $\ell_r$ or $u_r$ has changed
        { Q := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$ + $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; }
    sum := sum + Q;
    }
return sum / m′; // where m′ = 2*m
END
```

Table 20: Algorithm for evaluating cardinal comparatives in $\mathcal{F}_{\mathrm{Ch}}$ (based on integer arithmetics)

Hence let us describe the latter set directly, which might pay off from the efficiency point of view because all choices of arguments

$$(Y_1, Y_2, Y_3), (Y_1', Y_2', Y_3') \in \mathcal{T}_\gamma(X_1, X_2, X_3)$$

with

$$|Y_1 \cap Y_3| - |Y_2 \cap Y_3| = |Y_1' \cap Y_3'| - |Y_2' \cap Y_3'|$$

now become identified, i.e. it is sufficient to consider one representative only. The new format therefore lets me eliminate some computational redundancy. The next theorem shows how to compute the set of differences:

Theorem 270
Let $E \neq \varnothing$ be a finite base set, $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}&\{c_1 - c_2 : (Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3), c_1 = |Y_1 \cap Y_3|, c_2 = |Y_2 \cap Y_3|\}\\ &= \{d : \ell_3 - u_4 \leq d \leq u_3 - \ell_4\},\end{aligned}$$

referring to the cardinality coefficients ℓ_3, ℓ_4, u_3 and u_4 introduced in Th-267.
(Proof: D.31, p.528+)

The latter theorem has some obvious applications. For example, consider the quantifier “more than” defined by (233). We now observe that the quantifier can be expressed

Algorithm for computing $\mathcal{M}(Q)(X_1, X_2, X_3)$
(integer arithmetics)

```
INPUT: $X_1, X_2, X_3$
// initialise $H_r$, $\ell$, $u$
$H_1$ := Hist$_{X_1 \cap X_3}$;
$H_2$ := Hist$_{X_2 \cap X_3}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2 \cap X_3}$;
$H_4$ := Hist$_{\neg X_1 \cap X_2 \cap X_3}$;
for( r := 1; r $\leq$ 4; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ $H_r$[m+j]; $u_r$ := $\ell_r$ + $H_r$[m]; }
$\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
if( $\bot > \frac{1}{2}$ ) {
    sum := $\bot$;
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true; // "no change"
        // update clauses for $\ell$, $u$:
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; nc:= false; }
        if( $H_2$[m+j] $\neq$ 0 ) // [*]
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; nc:= false; } // [*]
        if( $H_3$[m+j] $\neq$ 0 )
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; nc:= false; }
        if( $H_4$[m+j] $\neq$ 0 ) // [*]
            { $\ell_4$ := $\ell_4$ - $H_4$[m+j]; nc:= false; } // [*]
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; nc:= false; }
        if( $H_2$[m-j] $\neq$ 0 )
            { $u_2$ := $u_2$ + $H_2$[m-j]; nc:= false; }
        if( $H_3$[m-j] $\neq$ 0 ) // [*]
            { $u_3$ := $u_3$ + $H_3$[m-j]; nc:= false; } // [*]
        if( $H_4$[m-j] $\neq$ 0 )
            { $u_4$ := $u_4$ + $H_4$[m-j]; nc:= false; }
        if( nc)
            { sum := sum + $\bot$; continue; }
        // one of the $\ell_r$, $u_r$ has changed
        $\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; // [1]
        // $\bot$ := $q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4)$; // [2]
        if( $\bot \leq \frac{1}{2}$ ) break;
        sum := sum + $\bot$;
        }
    }
else if( $\top < \frac{1}{2}$ ) {
    sum := $\top$;
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true;
        // update clauses for $\ell$ and $u$:
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; nc:= false; }
        if( $H_2$[m+j] $\neq$ 0 )
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; nc:= false; }
        if( $H_3$[m+j] $\neq$ 0 ) // [*]
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; nc:= false; } // [*]
        if( $H_4$[m+j] $\neq$ 0 )
            { $\ell_4$ := $\ell_4$ - $H_4$[m+j]; nc:= false; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; nc:= false; }
        if( $H_2$[m-j] $\neq$ 0 ) // [*]
            { $u_2$ := $u_2$ + $H_2$[m-j]; nc:= false; } // [*]
        if( $H_3$[m-j] $\neq$ 0 )
            { $u_3$ := $u_3$ + $H_3$[m-j]; nc:= false; }
        if( $H_4$[m-j] $\neq$ 0 ) // [*]
            { $u_4$ := $u_4$ + $H_4$[m-j]; nc:= false; } // [*]
        if( nc)
            { sum := sum + $\top$; continue; }
        // one of the $\ell_r, u_r$ has changed
        $\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; // [1]
        // $\top$ := $q^{\max}(\ell_1, \ell_2, \ell_4, u_1, u_3)$; // [2]
        if( $\top \geq \frac{1}{2}$ ) break;
        sum := sum + $\top$;
        }
    }
else
    { return $\frac{1}{2}$; }
return (sum + $\frac{1}{2}$*(m-j)) / m;
END
```

Table 21: Algorithm for evaluating cardinal comparatives in $\mathcal{M}$ (based on integer arithmetics)

Algorithm for computing $\mathcal{M}_{CX}(Q)(X_1, X_2, X_3)$
(integer arithmetics)

```
INPUT: $X_1, X_2, X_3$
// initialise $H_r$, $\ell$, $u$
$H_1$ := Hist$_{X_1 \cap X_3}$;
$H_2$ := Hist$_{X_2 \cap X_3}$;
$H_3$ := Hist$_{X_1 \cap \neg X_2 \cap X_3}$;
$H_4$ := Hist$_{\neg X_1 \cap X_2 \cap X_3}$;
for( r := 1; r $\leq$ 4; r := r+1 )
    { $\ell_r$ := $\sum_{j=1}^{m}$ $H_r$[m+j]; $u_r$ := $\ell_r$ + $H_r$[m]; }
$\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
$\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$;
if( $\bot > \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false; // "change"
        // update clauses for $\ell$ and $u$:
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; ch := true; }
        if( $H_2$[m+j] $\neq$ 0 ) // [*]
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; ch := true; } // [*]
        if( $H_3$[m+j] $\neq$ 0 )
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; ch := true; }
        if( $H_4$[m+j] $\neq$ 0 ) // [*]
            { $\ell_4$ := $\ell_4$ - $H_4$[m+j]; ch := true; } // [*]
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; ch := true; }
        if( $H_2$[m-j] $\neq$ 0 )
            { $u_2$ := $u_2$ + $H_2$[m-j]; ch := true; }
        if( $H_3$[m-j] $\neq$ 0 ) // [*]
            { $u_3$ := $u_3$ + $H_3$[m-j]; ch := true; } // [*]
        if( $H_4$[m-j] $\neq$ 0 )
            { $u_4$ := $u_4$ + $H_4$[m-j]; ch := true; }
        if( ch )
            // one of the $\ell_r, u_r$ has changed
            { $\bot$ := $q^{\min}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; } // [1]
            // { $\bot$ := $q^{\min}(\ell_1, \ell_3, u_1, u_2, u_4)$; } // [2]
        if( $\bot \leq$ m+j )
            { return (m+j)/m'; }
        }
    return 1;
    }
else if( $\top < \frac{1}{2}$ ) {
    for( j := 1; j<m; j := j + 1 ) {
        ch := false;
        // update clauses for $\ell$ and $u$:
        if( $H_1$[m+j] $\neq$ 0 )
            { $\ell_1$ := $\ell_1$ - $H_1$[m+j]; ch := true; }
        if( $H_2$[m+j] $\neq$ 0 )
            { $\ell_2$ := $\ell_2$ - $H_2$[m+j]; ch := true; }
        if( $H_3$[m+j] $\neq$ 0 ) // [*]
            { $\ell_3$ := $\ell_3$ - $H_3$[m+j]; ch := true; } // [*]
        if( $H_4$[m+j] $\neq$ 0 )
            { $\ell_4$ := $\ell_4$ - $H_4$[m+j]; ch := true; }
        if( $H_1$[m-j] $\neq$ 0 )
            { $u_1$ := $u_1$ + $H_1$[m-j]; ch := true; }
        if( $H_2$[m-j] $\neq$ 0 ) // [*]
            { $u_2$ := $u_2$ + $H_2$[m-j]; ch := true; } // [*]
        if( $H_3$[m-j] $\neq$ 0 )
            { $u_3$ := $u_3$ + $H_3$[m-j]; ch := true; }
        if( $H_4$[m-j] $\neq$ 0 ) // [*]
            { $u_4$ := $u_4$ + $H_4$[m-j]; ch := true; } // [*]
        if( ch )
            // one of the $\ell_r, u_r$ has changed
            { $\top$ := $q^{\max}(\ell_1, \ell_2, \ell_3, \ell_4, u_1, u_2, u_3, u_4)$; } // [1]
            // { $\top$ := $q^{\max}(\ell_1, \ell_2, \ell_4, u_1, u_3)$; } // [2]
        if( $\top \geq$ m-j )
            { return (m-j)/m'; }
        }
    return 0;
    }
return $\frac{1}{2}$;
END
```

Table 22: Algorithm for evaluating cardinal comparatives in $\mathcal{M}_{CX}$ (based on integer arithmetics)

directly in terms of $d = c_1 - c_2$, i.e.

$$\textbf{more than}(Y_1, Y_2, Y_3) = q(d)\,,$$

where $q : \mathbb{Z} \longrightarrow \{0, 1\}$ is the mapping

$$q(d) = \begin{cases} 1 & : \quad d > 0 \\ 0 & : \quad d \leq 0 \end{cases}$$

for all $d \in \mathbb{Z}$. Let us further notice that q is nondecreasing in d. We therefore conclude from the above theorem that the upper and lower bound mappings $\top_{\textbf{more than}, X_1, X_2, X_3}$ and $\bot_{\textbf{more than}, X_1, X_2, X_3}$ reduce to the following simple form,

$$\begin{aligned} \top_{\textbf{more than}, X_1, X_2, X_3}(\gamma) &= q(u_3 - \ell_4) = \begin{cases} 1 & : \quad u_3 - \ell_4 > 0 \\ 0 & : \quad u_3 - \ell_4 \leq 0 \end{cases} \\ \bot_{\textbf{more than}, X_1, X_2, X_3}(\gamma) &= q(\ell_3 - u_4) = \begin{cases} 1 & : \quad \ell_3 - u_4 > 0 \\ 0 & : \quad \ell_3 - u_4 \leq 0 \end{cases} \end{aligned}$$

for all $\gamma \in \mathbf{I}$. Substituting these expressions for $q^{\max}$ and $q^{\min}$ in the above algorithms will speed up the implementation of the comparative "more than". Other cardinal comparatives like "less than", "at least k more than" and "exactly k more than" can be treated analogously.

This completes my discussion of cardinal comparatives. I have presented a formal definition of these quantifiers which abstracts from the NL examples, and I have shown how the resulting quantifiers can be implemented in the proposed framework. Some possible optimizations of the algorithms have also been described, which become admissible when the quantifier exhibits the typical monotonicity pattern.

In presenting the formal definition of cardinal comparatives, and showing how they can be implemented, I have augmented the classes of fuzzy quantifiers known to fuzzy set theory. The new class is an interesting one, because comparisons of cardinalities are not only frequently seen in NL, but also of obvious relevance to applications.

11.11 Chapter summary and application examples

In this chapter, I have shown that the new, 'nice' models put forth in this sequel, can also compete with the old, semantically imperfect ones when it comes to efficient implementation. In order to provide the theory with the required computational backing, I have developed the basic techniques for implementing quantifiers in arbitrary models of the $\mathcal{F}_\xi$-type.[44] Among the models, I have chosen three prototypical examples which serve to demonstrate these techniques. The model $\mathcal{M}_{CX}$ is particularly important due to its outstanding formal properties and its relationships with the 'basic' FG-count approach and the Sugeno integral. The model $\mathcal{F}_{\mathrm{Ch}}$ is mainly interesting because it extends the 'basic' OWA approach and the Choquet integral; in addition, it is a typical representative of a genuine $\mathcal{F}_\xi$-DFS which does not share the properties of the

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[44] The possible extension to $\mathcal{F}_\Omega$-DFSes would be pointless due to their discontinuous behaviour.

$\mathcal{M}_{\mathcal{B}}$-models. The model $\mathcal{M}$ was chosen because its behaviour is somewhat in between $\mathcal{F}_{\mathrm{Ch}}$ and $\mathcal{M}_{CX}$: the former model also uses integration to abstract from the cut levels, however based on simple averaging rather than fuzzy median aggregation, while the latter model does not rely on integration, but also uses the median-based aggregation scheme of $\mathcal{M}_{\mathcal{B}}$-DFSes. In addition, $\mathcal{M}$ is the model used in the experimental retrieval system for weather documents, from which the examples in the present work are taken [46, 52]. All of the prototype models are particularly robust, which further recommends them for applications. By presenting the complete algorithms for implementing the relevant kinds of quantifiers in these models, I therefore provided a solution for fuzzy quantification, which is ready for integration into applications. Despite the complexity of the prototype models and the aggregation mechanisms of $\mathcal{F}_{\xi}$- and $\mathcal{M}_{\mathcal{B}}$-DFSes, the resulting algorithms are remarkably simple and clearly structured. This nourishes my trust that the examples will also provide a blueprint for implementing further models if so desired. In particular, the transfer of the prototypical algorithms to novel models is greatly facilitated by the modular design of my solution, which is organized in a number of individual components, from which I then built the implementations.

The first component which contributes to these implementations is concerned with the continuous range of the cutting parameter: it is practically impossible to consider all choices of γ in the unit interval when computing a quantification result. Fortunately, I was able to prove that in the relevant cases, it is possible to restrict to a finite sample of cutting levels, which fully determines the outcome of quantification. In particular, such a restriction is always possible for finite base sets, and the sample of cut-levels then assumes a very simple form. Having eliminated the first hindrance of continuous cut ranges, I reconsidered the prototypical models and solved the subsequent problem of computing quantification results in these models from the finite representation. Obviously, this step must be re-iterated for all models to be implemented. Indeed, expressing a model of interest in terms of the finite representation is the chief task when adapting the above algorithms. However, we know in advance from Th-249 that such a reformulation is always possible, so this should usually not cause problems in practice.

As soon as the models are expressed in terms of the finite sample of $\top_j$'s and $\bot_j$'s, this new description can be used to compute quantification results (because everything has now turned finite). However, some further refinements are necessary in order to turn these 'raw' solutions into efficient algorithms which not only compute the correct results, but are also useful in practice. Specifically, the reformulation of the models in terms of $\top_j$ and $\bot_j$ will only achieve sufficient performance if the quantities $\top_j$ and $\bot_j$ can also be calculated efficiently. Computing these in a direct or 'brute force' way from the three-valued cuts, thus paralleling the definition of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ in Def. 100, is not a viable approach, because the total cut range $\mathcal{T}_\gamma(X_1,\ldots,X_n)$ will grow beyond computational limits for domains of realistic size. Since my reformulation of the models makes intensive use of $\top_j$ and $\bot_j$, I therefore decided to replace the direct computation of $\top_j$ and $\bot_j$ from the three-valued cuts with a smarter technique based on cardinality information.

The three-valued cutting mechanism, which is essentially supervaluationist, considers all possible alternatives and thus involves a large number of calculations at each cut level. The actual quantification result, however, usually depends only on a small sample of these alternatives. In order to eliminate these redundant computations, I took a closer look at the quantitative variety of quantifiers, which comprise almost all examples of relevance to technical applications. I first reviewed a well-known result of TGQ which states that every quantitative two-valued quantifier can be expressed in terms of the cardinalities of its arguments and their Boolean combinations. I then observed that a corresponding claim can be proven for semi-fuzzy quantifiers: these are also quantitative if and only if they are reducible to cardinalities of Boolean combinations, which are supplied to the defining mapping $q : \{0, \ldots, |E|\}^K \longrightarrow \mathbf{I}$ of the quantifier, see Th-255. Starting from the apparent equality (171), I then showed in the sequence of theorems Th-259 to Th-262 that the quantification result $\mathcal{F}_\xi(Q)(X_1, \ldots, X_n)$ of an $\mathcal{F}_\xi$-model can always be computed from pairs of upper and lower cardinality indices u_r and ℓ_r, which are sampled from Boolean combinations of the arguments. In other words, my analysis of a quantitative semi-fuzzy quantifier Q can be generalized to the corresponding fuzzy quantifier $\mathcal{F}_\xi(Q)$ in a constructive fashion, which precisely reveals how $\top_j = \top_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j)$ and $\bot_j = \bot_{Q,X_1,\ldots,X_n}(\overline{\gamma}_j)$ can be calculated from the defining mapping $q : \{0, \ldots, |E|\}^K \longrightarrow \mathbf{I}$ and from the known cardinality coefficients u_r and ℓ_r. Some optimizations have then been added in order to speed up processing of the important types of quantifiers. In Th-263, I first improved the analysis of conservative quantifiers, which also covers the proportional type. In the later Th-267, I then refined the general analysis for cardinal comparatives, which are the prime case of three-place quantifiers. Taken together, these improvements in the analysis of quantitative quantifiers establish the second component needed for efficient implementation. In particular, the novel representation of quantitative quantifiers which circumvents the direct computation of the three-valued cuts, makes it possible to calculate quantification results from cardinality information represented by pairs of coefficients ℓ_r and u_r, and it is no longer necessary to check all alternatives in the total cut range $\mathcal{T}_\gamma(X_1, \ldots, X_n)$.

Finally I identified another matter worthy of optimization: having reduced the computation of $\top_j$ and $\bot_j$ to cardinality indices u_r and ℓ_r, it now became essential that these coefficients be computed as efficiently as possible. I therefore had to develop a fast mechanism for calculating the required cardinality information, in order to combine the other performance improvements to a complete solution. In technical terms, it remained to be shown how the upper and lower cardinality indices $u_r = u_r(j) = |(Z_r)_\gamma^{\max}|$ and $\ell_r = \ell_r(j) = |(Z_r)_\gamma^{\min}|$ can be calculated with the least computational effort, given a choice of $\gamma = \overline{\gamma}_j$, i.e. in the j-th iteration (in the following, I will drop the subscript r for simplicity). The obvious goal was that of avoiding the explicit computation of the sets $Z_\gamma^{\min}$ and $Z_\gamma^{\max}$, as well as the subsequent computation of their element counts $|Z_\gamma^{\min}|$ and $|Z_\gamma^{\max}|$, which determine the current coefficients $\ell(j)$ and $u(j)$. In order to shortcut this process and eliminate the redundancy of repeated element counting, I pursued the idea of precomputing the cardinalities of the layers $\{e \in E : \mu_Z(e) = \alpha\}$ for all relevant choices of the membership grade α. The resulting pairs of membership grades and associated cardinalities can then be neatly organized in the histogram of Z. Starting from these data, it was a straightforward task to develop incremental update rules for the coefficients u and ℓ, which compute the upper and lower cardinalities associated with the three-valued cut from the histogram of the involved fuzzy subset. Surprisingly, it is sufficient to look up two coefficients from the given histogram, in order to keep track of ℓ and u and compute their new values in the current iteration from their known values in the previous iteration.

These basic techniques for implementing quantifiers in $\mathcal{F}_\xi$-DFSes have then been combined into complete algorithms for implementing the relevant types of NL quantifiers in the prototype models. Due to my extensive preparations, it was rather easy to compose these algorithms from the available components, i.e. the reformulation of the models in terms of $\top_j$ and $\bot_j$; the reformulation of $\top_j$ and $\bot_j$ in terms of the defining mapping q and the relevant cardinality indices; and finally the computation of the cardinality coefficients from precomputed histograms. Specifically, I presented the complete algorithms for implementing absolute and proportional quantifiers in the prototype models. Beyond these familiar types of quantifiers, I also covered quantifiers of exception like “all except k” and cardinal comparatives like “much more than”, which received few attention in the existing literature on fuzzy quantifiers, in spite of their apparent utility to applications. It is worth noticing that the proposed algorithms for cardinal comparatives comprise the first systematic and complete implementation of an important class of quantifiers ‘of the third kind’ in the sense of Zadeh [190, p. 757].

The successful implementation of such diverse types of quantifiers in the chosen models demonstrates that the proposed approach to fuzzy quantification is not only theoretically appealing, but also viable in practice. In particular, I have shown that the featured models $\mathcal{F}_{\mathrm{Ch}}$, $\mathcal{M}$ and $\mathcal{M}_{CX}$ are computational, by describing efficient algorithms which implement the relevant types of quantifiers in these models. In each case, I presented two variants of the algorithms: the default version which uses floating-point arithmetics, and an alternative version taylored to integer arithmetics. The latter version should be considered for application when a substantial gain in performance outweighs the resulting loss of accuracy. Roughly speaking, the integer-based solution will yield the maximum benefit if the considered integer range is small compared to the size of the domain. Operations on intensity images with millions of pixels and a limit to eight bit accuracy are a prime example. Due to their clear structure, the proposed algorithms are easily fitted to additional classes of quantifiers, although the prominent cases should be covered by the examples. In addition, the algorithms can also provide a paradigm for implementing other types of models.

In closing this chapter, I would like to discuss two examples of fuzzy quantification. These examples are mainly intended to provide some first evidence that the proposed algorithms are indeed useful and efficient. In addition, it is hoped that these examples will also be of interest as such, and point attention to some promising areas for the use of fuzzy quantifiers in future applications.

The first example highlights the benefits of fuzzy quantifiers for the weighted querying of retrieval systems. In information retrieval, one is typically faced with a large number of unstructured documents or document representatives (database records). In order to better visualize the search results, it can be helpful to rank the documents according to some meaningful criterion, and to present them sorted by their ranking. Apart from separating the relevant documents (which are shown first) from the irrelevant documents (which are shown very late or even cut off), this ranking procedure can also highlight the degree to which the filtering criterion applies, i.e. beyond the computed order, the numerical relevance grades themselves and their distribution can also be of interest. In this process, fuzzy quantification comes to the fore in at least two ways. First of all, fuzzy quantifiers are valuable aggregation operators, which naturally incorporate importance qualification. In this case, we have a number of subcriteria; these comprise the domain of quantification, E. Each criterion $e \in E$ has an associated importance $\mu_W(e) \in \mathbf{I}$ and an associated grade of applicability $\mu_{A,d}(e)$ with respect to the considered document d. These grades of importance can be organized into fuzzy subsets $W, A_d \in \widetilde{\mathcal{P}}(E)$. Based on a semi-fuzzy quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$, e.g. $Q = \textbf{almost all}$, we can now express the aggregation score ϱ_d, which corresponds to the NL statement

"Q important criteria apply to the document d".

The ranking criterion then becomes

$$\varrho_d = \widetilde{Q}(W, A_d)\,,$$

where $\widetilde{Q} = \mathcal{F}(Q)$ for the chosen DFS $\mathcal{F}$. The documents $d \in D$ of the document base D are then sorted according to their relevance grade ϱ_d and presented in decreasing order of relevance (most relevant first). For example, we might determine the documents to which most relevant criteria $e \in E$ apply, by computing the ranking according to

$$\varrho_d = \widetilde{\textbf{most}}(W, A_d)\,.$$

In this case, the fuzzy quantifier $\widetilde{\textbf{most}} = \mathcal{F}(\textbf{most})$ is independent of the assumed standard model, see (6) and Th-46.

This basic approach has been utilized for improving retrieval performance in a meta-search engine for bibliographic databases, the 'retrieval assistant' [58]. In this system, the possible combinations of search terms are no longer limited to Boolean operators; apart from the Boolean connectives, the retrieval assistant also supports the approximate quantifiers "almost all", "many", "a few" and others. In addition, all terms can be weighted by their importance with respect to the aggregation.

Similar techniques have been employed by Bordogna & Pasi [18], in order to support more powerful queries to information retrieval systems which also include fuzzy quantification. Further examples which demonstrate the benefits of fuzzy quantification to the querying of information systems are [22, 74, 76, 126]. It should be emphasized, however, that all of these existing applications resort to the Σ-count or OWA approaches, and therefore do not profit from the improvements of fuzzy quantification achieved in this report.

The proposed method for weighted aggregation with fuzzy quantifiers is not restricted to textual documents and classical information retrieval, though. The following example starts from a meaningful query to an image database, which involves the use of fuzzy quantification. When executing the query, the processing of the quantifier is delegated to the model $\mathcal{M}$ presented in Chap. 7. The document base assumed in the example again consists of digitized satellite images, from which corresponding images of cloudiness situations have been extracted. The goal is to rank these cloudiness situations according to a criterion like "Q of Southern Germany is cloudy", where Q is a quantifier like "more than 20 percent" or "as much as possible". In the example, we have a uniform criterion, i.e. the fuzzy predicate "cloudy", which is applied repeatedly and evaluated for all pixels in the image of cloudiness grades. The global criterion therefore expresses some kind of accumulation, which ranges over the spatial region of Southern Germany (see Fig. 2). The criterion results in a single scalar evaluation for each image. Consequently, the accumulation process totally eliminates the spatial dimension; it results in the numerical score ϱ_d which determines the relevance of each image under the query.

Specifically, I will now consider the case that $Q =$ **as much as possible**, which results in the ranking criterion

"As much as possible of Southern Germany is cloudy".

Of course, a suitable interpretation of the quantifier must be supplied in order to determine a meaningful ranking. Because we are working with discrete, digitized images throughout, it is first necessary to translate the original criterion into a corresponding condition on discrete pixels in the digitized image, viz

"As many pixels as possible that belong to Southern Germany
are classified as cloudy".

The ranking is then computed by determining the document score ϱ_d of each image d in the database, and by presenting the images in decreasing order of their associated scores. Based on the model $\mathcal{M}$, the score ϱ_d achieved by a given image d can readily be computed as

$$\varrho_d = \mathcal{M}(\textbf{\textsf{as many as possible}})(\textbf{\textsf{Southern Germany}}, \textbf{\textsf{cloudy}}_d)\,,$$

where **as many as possible** is the quantifier defined by (13), which denotes the relative share, while $\textbf{\textsf{cloudy}}_d$ is the fuzzy region of cloudiness grades depicted in image d.

In order to illustrate this approach to query processing with fuzzy quantifiers, the example query was run on twelve images, a random sample of the images in the document base. The resulting ranking of cloudiness situations under the criterion "As much as possible of Southern Germany is cloudy" is presented in Fig. 12. The computed ordering appears to be in good conformance with our expectations, i.e. the resulting image sequence indeed shows a steady decrease of cloudiness in Southern Germany. This experimental evidence in favour of the chosen interpretation of fuzzy quantifiers is not surprising, of course, because the essential aspects of linguistic adequacy have been encoded in the axiom system for fuzzy quantification, which I devised in the first part of the report.

Apart from its illustrative value, the example also provides some first experimental data on typical processing times. Due to the extent of the domain of quantification,

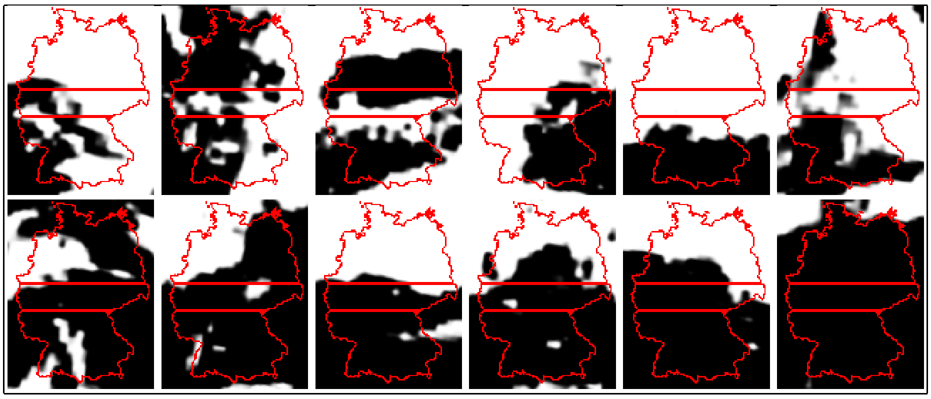

Figure 12: Ranking computed for query "As much as possible of Southern Germany is cloudy". Most relevant results in top left corner, quality of match decreases from left to right and from top to bottom.

which comprises the full set of pixel positions in the images, and also due to the modest demands on accuracy, I decided to apply the algorithm based on integer arithmetics using 8 bit accuracy (see table 15). In order to compute the ranking, a quantification result had to be calculated for each image, based on the domain of 219063 (411x533) pixel coordinates. This took approx. 0.03 seconds processing time per image on an AMD Athlon 600 MHz PC running Linux, assuming that the image is already loaded into the working memory.

Up to this point, I have emphasized the utility of fuzzy quantifiers for weighted aggregation and importance qualification. However, an implementation of fuzzy quantifiers offers further benefits to applications. Today, it is often not the lack of data, but rather their abundance that hinders decisions. It therefore becomes vitally important to develop methods which extract the interesting features or regularities from a large stock of data, thus elucidating the global 'shape' of the given collection. Due to the massive amounts of raw data, those techniques come into focus that automatically generate summaries of the given data sets, in order to make them intelligible and reveal the hidden characteristics. It is natural and often favourable to express these summaries linguistically (rather than using descriptive statistics). In particular, this is the case if the database comprises fuzzy attributes; if the data items can vary with respect to their importance to the summarisation; and finally if the observable regularities, too, are only a matter of tendency. Following Yager [163], the linguistic summary is usually constructed from a linguistic quantifier, which represents the *quantity in agreement*, and from a natural language predicate, which functions as the *summarizer*. For purposes of illustration, consider the following linguistic summary cited from Kacprzyk and Strykowski [73, p. 31], who detail a system for generating summaries of sales data at a computer retailer:

"Much sales of components is with a high commission".

In this case, the summary can be split into the domain ‘sales of components’, the quantity in agreement ‘much’ and finally the summarizer, ‘[sales] with a high commission’. We can therefore represent the summary in terms of the following quantifying expression,

$$\widetilde{\textbf{much}}(\textbf{sales with a high commission})\,,$$

where $\widetilde{\textbf{much}}$ must be a suitable fuzzy quantifier defined on the universe of component sales. Of course, it is also possible to use two-place quantifiers, which let us relativize summaries to fuzzily defined special circumstances, and which also make it possible to incorporate importances. For example, imagine a summary

“Much recent sales of components is with a high commission”.

In this case, we have an additional summarizer ‘recent [sales of components]’ which expresses a soft constraint on those sales of components covered by the summary. Such relative summaries can be modelled by a quantifying expression built from a two-place quantifier, viz

$$\widetilde{\textbf{much}}(\textbf{recent sales of components}, \textbf{sales with a high commission})\,.$$

Some experimental systems for linguistic data summarization which fit into this general framework are described in [72, 73, 126, 193].

All of these systems rely on structured, relational representations, and are therefore taylored to classical database applications.

Data summarization with fuzzy quantifiers is not restricted to relational data, though. In particular, the same techniques can also improve our understanding of images and image sequences. In order to illustrate this, I will again resort to the familiar example domain.

We start from a large collection of intensity images (each also annotated by a ‘time stamp’), which represent cloudiness situations at certain points of time. For simplicity, I will assume that these images are periodically sampled at equidistant time intervals. Obviously, users of the weather information system do not care about the specific points of time at which the images were shot, which is considered a purely technical matter. By contrast, these users are interested in the weather in certain ‘natural’ geographic areas and the distribution and flow of weather in certain ‘natural’ time intervals. By ‘natural’ I mean that these regions of interest have an associated cognitive representation, and can therefore be *expressed linguistically*.

For purposes of illustration, consider some user interested in characteristics of the recent weather situation. Let us further assume that the user expresses the time of interest in a linguistic way, by specifying the temporal summarizer “in the last days”. The chosen term “in the last days” then imposes a soft constraint on those images in the database which must be considered in the summary. For purposes of demonstration, we shall limit ourselves to a small sequence of cloudiness situations, which is depicted in the upper row of Fig. 13; the relevance of these images with respect to the criterion "in the last days" is also shown. Obviously, the sequence of interest will grow much larger in a real-world application, and the system is therefore supposed to condense the individual pieces of information by generating image summaries. This summarization will effectively eliminate the temporal dimension, i.e. the summarizing image is no longer anchored at a certain point of time. Fuzzy two-place quantifiers are suitable operators for performing this abstraction, because they are capable of summarizing the data and simultaneously incorporating the importances expressed by "in the last days". The resulting criteria for generating the summarizing images then assume the following form,

"Q-times cloudy in the last days",

where Q is the chosen quantifier. In order to describe the data from several perspectives, the system is supposed to apply several distinct quantifiers Q, each of which highlights a certain characteristic of the images.

Let me now detail the mechanism which will be used for generating the summary images. The domain of quantification E consists of the given set of images, which represent cloudiness situations. It is understood that all of these images are defined on the same set of pixel coordinates, i.e. all images share the same dimensions. The images are also annotated with a time stamp. The time stamp can be used to define a fuzzy set $X_1 \in \widetilde{\mathcal{P}}(E)$, which judges membership of the images in the fuzzy time interval "in the last days". In order to generate a summarizing image from the given data, we first choose some $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ which represents the quantifier of interest. The target image is defined on the same set of pixel coordinates as the original images. It is generated by pixel-wise application of the following procedure. We have already fixed the quantifier Q and the restriction $X_1 \in \widetilde{\mathcal{P}}(E)$ which encodes the importances. In dependence on the given pixel p, we now define a second fuzzy subset $X_{2,p} \in \widetilde{\mathcal{P}}(E)$ of the set of images, where $\mu_{X_{2,p}}(e)$ is the degree of cloudiness observed in cloudiness situation e at the location of p, i.e. the degree to which p is classified as 'cloudy'. By applying the quantifier to the restriction X_1 and scope $X_{2,p}$, we finally compute the intensity of the considered pixel in the result image, which is the score of the criterion

"Q-times cloudy in pixel p in the last days".

In other words, the resulting image R has pixel intensities

$$R_p = \mathcal{F}(Q)(X_1, X_{2,p}) \,,$$

for all pixels p, where $\mathcal{F}$ is the chosen model of fuzzy quantification, in this case $\mathcal{F} = \mathcal{M}$. In the example, I have used $\mathcal{F} = \mathcal{M}$, which was the only known DFS at the time the experiments were carried out, but the other models $\mathcal{F}_{\mathrm{Ch}}$ and $\mathcal{M}_{CX}$ are also conceivable choices. As has already been remarked, it is advantageous to build more than one image summary. Noticing that every quantifier will accentuate a certain regularity observed in the images, it is advisable to use more than one quantifier, in order to achieve a richer description of the data. In practice, I decided to use the universal quantifier for modelling "always"; the existential quantifier for modelling “at least once”; and some ‘intermediate’ quantifiers for modelling “sometimes” (i.e., at least a few times), “often” (assumed to increase linearly with frequency) and “almost always”. The latter quantifiers were all derived from a trapezoidal proportional quantifier $\textbf{trp}_{a,b,c} : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ defined by

$$\textbf{trp}_{a,b,c}(Y_1, Y_2) = \begin{cases} t_{a,b}(|Y_1 \cap Y_2|/|Y_1|) & : \ Y_1 \neq \varnothing \\ c & : \ Y_1 = \varnothing \end{cases}$$

$$t_{a,b}(z) = \begin{cases} 0 & : \ z < a \\ \frac{z-a}{b-a} & : \ a \leq z \leq b \\ 1 & : \ z > b \end{cases}$$

where $Y_1, Y_2 \in \mathcal{P}(E)$, $a, b, c, z \in \mathbf{I}$, $a < b$. The precise definition of the derived quantifiers in terms of the parameters $a, b, c \in \mathbf{I}$ as well as the corresponding summaries are shown in Fig. 13. Now considering the summary images, one immediately recognizes those areas that were sunny all of the time (those depicted black in the leftmost summary), those areas that were overcast all of the time (those shown white in the rightmost image) and the exact graduation of the areas in between (center images). In the crisp case, it would be possible to display all of this information in one single image, which depicts frequency. However, this cannot be done in the fuzzy case, because the importances of the summarized images (specified by “in the last days”) and also the cloudiness grades are both a matter of tendency. It is therefore invaluable to have a more general technique at our disposal, which also accounts for the gradual nature of the data. This is particularly true if one moves to collections of realistic size, which can no longer be judged by looking at the individual data items.

When working with such large collections, processing times become a critical factor. Hence let us also consider some performance data. In the example shown in Fig. 13, a total number of 219063 quantification results involving a domain of 8 images had to be computed for each result image (i.e. one quantifying expression for each of the 411x533 pixels). Due to the emphasis on processing speed rather than accuracy, it was again decided to use the integer-based implementation of proportional quantifiers in $\mathcal{M}$ with 8 bit accuracy. The summarisation process then took a total of about 5 seconds per result image on the same computer platform as above. These numbers suggest that the proposed algorithms are indeed very efficient, and not likely to create a performance bottleneck in applications.

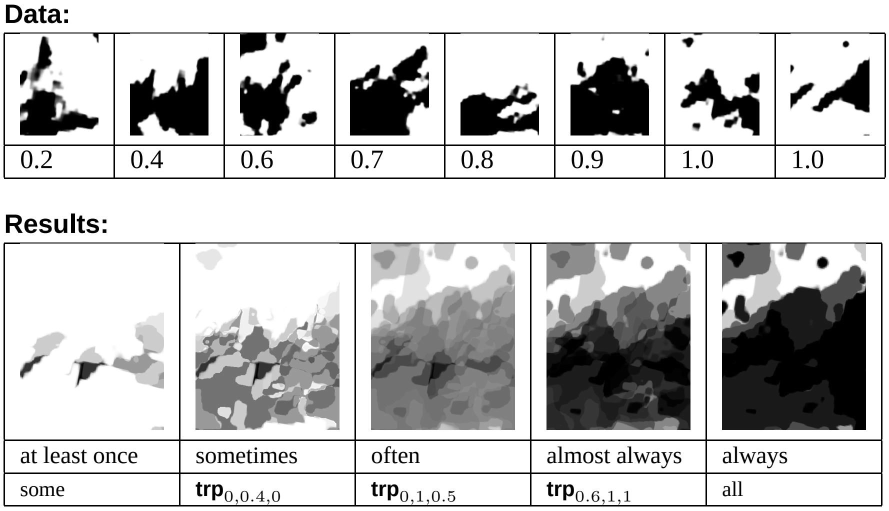

Figure 13: Image sequence and summarization results for various choices of the criterion “Q-times cloudy in the last days”. Regions that meet the criterion are depicted white.

12 Multiple variable binding and branching quantification

12.1 Motivation and chapter overview

The framework for fuzzy quantification which I proposed in Chap. 2 and then elaborated in the later chapters, was motivated by the failure of earlier work to cover the variety of individual quantifiers found in natural languages. Thus, Zadeh's traditional framework was judged much too limited in scope, while my improved framework was specifically designed to capture a substantial class of linguistic quantifiers (although not all of them). The main part of the report was then concerned with the development of a complete and useful theory of fuzzy quantification within the proposed skeleton: by demarcating the range of intended models, by devising constructions for such models, and finally by implementing quantifiers in these models. It is important to notice at this point that all of these efforts were directed at the modelling of individual quantifiers. However, it is not only the diversity of possible quantifiers in NL which poses difficulties to a systematic and comprehensive modelling of linguistic quantification. Further peculiarities become visible when the quantifiers are no longer treated in isolation. Even in simple cases like "most", the ways in which these quantifiers interact when combined in meaningful propositions can be complex and sometimes even puzzling. Consider the quantifying proposition "Most men and most women admire each other", for example, in which we find a reciprocal predicate, "admire each other". Barwise [5] argues that so-called branching quantification is needed to capture the meaning of propositions involving reciprocal predicates. Without branching quantifiers, the above example must be linearly phrased as either a. or b.,

a. $[\mathrm{most}\, x : \mathrm{men}(x)][\mathrm{most}\, y : \mathrm{women}(y)]\, \mathrm{adm}(x, y)$

b. $[\mathrm{most}\, y : \mathrm{women}(y)][\mathrm{most}\, x : \mathrm{men}(x)]\, \mathrm{adm}(x, y)\,.$

(Obviously, we need a logic which supports generalized quantifiers in order for these expressions to make sense. For a suitable system of first-order logic with fuzzy quantifiers see Barwise and Feferman [7, Ch. 2]). Neither interpretation captures the expected symmetry with respect to the men and women involved. Surely the Boolean conjunction of a. and b. will be symmetric, but it does not capture the expected meaning either. In fact, we need a construction like

$$\left.\begin{array}{l} [Q_1\, x : \quad \mathrm{men}(x)] \\ [Q_2\, y : \mathrm{women}(y)] \end{array}\right\rangle \mathrm{adm}(x, y)$$

where the quantifiers $Q_1 = Q_2 = \mathrm{most}$ operate in parallel and independently of each other. This branching use of quantifiers can be analysed in terms of Lindström quantifiers [97], i.e. multi-place quantifiers capable of binding several variables. In this case, we have three arguments, and the quantifier should bind x in $\mathrm{men}(x)$, y in $\mathrm{women}(y)$ and both variables in $\mathrm{adm}(x, y)$. Thus anticipating the use of Lindström quantifiers which will be formally defined in the next section, the above branching expression can be modelled by a Lindström quantifier Q of type $\langle 1, 1, 2\rangle$:

$$Q_{x,y,xy}(\mathrm{men}(x), \mathrm{women}(y), \mathrm{adm}(x, y))$$

(For details on the logical language and the definition of its semantics, see Lindström's original presentation [97]). Let us now consider the problem of assigning an interpretation to such quantifiers. Obviously, Q depends on the meaning of "most". Given a fixed choice of base set or 'universe' $E \neq \varnothing$ over which the quantification ranges, the quantifier "most" in its precise sense (majority of) can be expressed as

$$\textbf{most}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1 \cap Y_2| > \frac{1}{2}|Y_1| \\ 0 & : \quad \text{else} \end{cases}$$

for all crisp subsets Y_1, Y_2 of E (provided that E be finite), see (6). This lets us evaluate "most men are married" by calculating $\textbf{most}(\textbf{men}, \textbf{married})$, where $\textbf{men}, \textbf{married}$ are the sets of men and of married people, respectively. The definition can be extended to base sets of infinite cardinality if so desired. A possible interpretation of "most" in this case is

$$\textbf{most}(Y_1, Y_2) = \begin{cases} 1 & : \quad Y_1 \cap Y_2 \cap \widehat{\beta}(Y_1 \cap Y_2) \neq \varnothing \text{ for all bijections } \beta : Y_1 \longrightarrow Y_1 \\ 0 & : \quad \text{else} \end{cases}$$

where $Y_1, Y_2 \in \mathcal{P}(E)$.

Let us now attempt a similar analysis of the above quantifier Q. We first notice that the quantifier must accept three arguments, i.e. the sets $A, B \in \mathcal{P}(E)$ of men and women, and the binary relation $R \in \mathcal{P}(E^2)$ of people admiring each other, assumed to be crisp for simplicity. The quantifier then determines a two-valued quantification result $Q(A, B, R) \in \{0, 1\}$ from these data. Barwise [5, p. 63] showed how to define Q in a special case; here I adopt Westerståhl's reformulation for binary quantifiers [158, p. 274, (D1)]. Hence consider a choice of $Q_1, Q_2 : \mathcal{P}(E)^2 \longrightarrow \{0, 1\}$. Let us further assume that the Q_i's, like "most", are nondecreasing in their second argument, i.e. $Q_i(Y_1, Y_2) \leq Q_i(Y_1, Y_2')$ whenever $Y_2 \subseteq Y_2'$. In this case, the complex quantifier Q becomes:

$$Q(A, B, R) = \begin{cases} 1 & : \quad \exists U \times V \subseteq R : Q_1(A, U) = 1 \wedge Q_2(B, V) = 1 \\ 0 & : \quad \text{else} \end{cases} \tag{235}$$

Hence "Most men and most women admire each other" means that there exists a mutual admiration group $U \times V \subseteq \mathrm{adm}$ such that most men and most women belong to that group. In my view, this analysis is correct and expresses the intended meaning of the example. It should be remarked at this point that Westerståhl, unlike Barwise, is only concerned with conservative quantifiers, thus assuming that $Q_i(Y_1, Y_2) = Q_i(Y_1, Y_1 \cap Y_2)$. This permits him to restrict attention to $U \times V \subseteq R \cap (A \times B)$ rather than $U \times V \subseteq R$ without changing the interpretation. Westerståhl also extends this analysis to a generic construction which admits non-monotonic quantifiers (p. 281, Def. 3.1). But, how can we incorporate approximate quantifiers and fuzzy arguments, as in "Many young and most old people respect each other"?

The goal of the present chapter is devising a method which assigns meaningful interpretations to such cases. To this end, I first incorporate Lindström-like quantifiers capable of binding several variables into the QFM framework. Following this, the axiom system for ordinary models of fuzzy quantification is adapted to the new cases.

This extension is modelled closely after the DFS solution and thus permits to re-use the models presented so far. I will also initiate research into more specialized conditions which might further constrain the plausible models of fuzzy quantification in the more general sense. Specifically, I will connect Lindström-like quantifications and the analysis of branching constructions to the formation of quantifier nestings. Finally I explain how the above analysis of reciprocal constructions in terms of branching quantifiers and its generalization due to Westerståhl can be extended towards fuzzy branching quantification. The proposed analysis is of special importance to linguistic data summarization because the full meaning of reciprocal summarizers, which describe factors being “correlated” or “associated” with each other, can only be captured by branching quantification.

12.2 Lindström quantifiers

Let us start from the notion of crisp quantifiers which incorporate multiple-variable binding. A suitable semantical concept which achieves this is defined as follows.

Definition 168

A Lindström quantifier *is a class $\mathcal{Q}$ of (relational) structures of some common type $t = \langle t_1, \ldots, t_n \rangle$, such that $\mathcal{Q}$ is closed under isomorphism [97, p. 186].*

The cardinal $n \in \mathbb{N}$ specifies the number of arguments; the individual components $t_i \in \mathbb{N}$ specify the number of variables that the quantifier binds in its i-th argument position. For example, the existential quantifier, which accepts one argument and binds one variable, has type $t = \langle 1 \rangle$. The corresponding class $\mathcal{E}$ comprises all structures $\langle E, A \rangle$ where $E \neq \varnothing$ is a base set and A is a nonempty subset of E. In the introduction to this chapter, we have already met with a more complex quantifier Q of type $\langle 1, 1, 2 \rangle$ given by (235). In this case, $\mathcal{Q}$ is the class of all structures $\langle E, A, B, R \rangle$ with $Q(A, B, R) = 1$, where $E \neq \varnothing$ is the base set and $A, B \in \mathcal{P}(E)$, $R \in \mathcal{P}(E^2)$. As shown by Lindström, such quantifiers can be used to assign an interpretation to quantifying expressions like $Q_{x,y,xy}(\varphi(x), \psi(y), \chi(x, y))$ where Q binds x in $\varphi(x)$, y in $\psi(y)$, and both x, y in $\chi(x, y)$. Thus, Q is capable of binding multiple variables in its third argument position. Informally, the semantical interpretation of $\varphi(x)$ results in the set $A = \{x : \varphi(x)\}$ of all things which satisfy $\varphi(x)$; ψ will result in the set B of all things which satisfy $\psi(y)$; and $\chi(x, y)$ will result in the relation $R = \{(x, y) : \chi(x, y)\} \in \mathcal{P}(E^2)$. The resulting sets (or more precisely, t_i-ary relations) can then be used to determine the truth status of the quantifying sentence, by checking whether $(A, B, R) \in \mathcal{Q}$ or not. The formal definition of the logical system and its semantics are described in [97].

To sum up, Lindström quantifiers introduce a very powerful notion of quantifiers. However, as already pointed out in the introduction, the condition that $\mathcal{Q}$ be closed under isomorphism, which is useful for describing logical or mathematical quantifiers, is too restrictive when we turn to the modelling of natural language. In order to express non-quantitative examples of quantifiers, e.g. proper names “Ronald” or constructions involving proper names like “all except Lotfi” which depend on specific individuals, it is therefore necessary to drop invariance under isomorphisms. The following definition of a generalized Lindström quantifier accounts for these considerations.

Definition 169
A generalized Lindström quantifier *is a class $\mathcal{Q}$ of relational structures of a common type $t = \langle t_1, \ldots, t_n \rangle$.*

Again, the arity $n \in \mathbb{N}$ specifies the number of arguments, and the components $t_i \in \mathbb{N}$ specify the number of variables that the quantifier binds in its i-th argument position. In principle, we have now arrived at the two-valued concept which I would like to generalize to the fuzzy case, by incorporating approximate quantification and fuzzy arguments.

In order to avoid the introduction of a 'fuzzy class of relational structures', it is convenient to organize the information gathered in the total class $\mathcal{Q}$ into more localized representations. In order to accomplish this, I will follow the same basic strategy that I pursued in the case of ordinary two-valued, semi-fuzzy and fuzzy quantifiers, which were all defined relative to a given domain, E. Hence let us introduce a notion of two-valued L-quantifiers (where 'L' signifies Lindström) which are no longer concerned with all base sets at the same time, but rather relativized to a single choice of E.

Definition 170
A two-valued L-quantifier *of type $t = \langle t_1, \ldots, t_n \rangle$ on a base set $E \neq \varnothing$ is a mapping $Q : \mathop{\times}\limits_{i=1}^{n} \mathcal{P}(E^{t_i}) \longrightarrow \mathbf{2}$. Hence Q assigns a crisp quantification result $Q(Y_1, \ldots, Y_n) \in \{0, 1\}$ to each choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ \ldots, \ n\}$. A* full two-valued L-quantifier *Q of type t assigns a two-valued L-quantifier Q_E of type t on E to each base set $E \neq \varnothing$.*

Let me briefly explain how the full two-valued L-quantifiers Q so defined are related to generalized Lindström quantifiers $\mathcal{Q}$. Starting from a Lindström quantifier $\mathcal{Q}$, we can define a full two-valued L-quantifier by

$$Q_E(Y_1, \ldots, Y_n) = 1 \Leftrightarrow (E, Y_1, \ldots, Y_n) \in \mathcal{Q}$$

for all base sets $E \neq \varnothing$ and all crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i = 1, \ldots, n$. Starting from a full two-valued L-quantifier Q, on the other hand, we can define a corresponding generalized Lindström quantifier as the class $\mathcal{Q}$ of all relational structures $(E, Y_1, \ldots, Y_n)$ of type t which satisfy the condition $Q_E(Y_1, \ldots, Y_n) = 1$. These constructions are obvious inverses of each other, i.e. we can always switch from a generalized Lindström quantifier to the corresponding full L-quantifier and vice versa. Consequently, both concepts express basically the same thing, and are thus interchangeable. Here I will opt for the latter concept of two-valued L-quantifiers, which is better suited for my purposes. In addition, it is convenient to keep the base set E fixed. I will therefore focus on the relativized notion of two-valued L-quantifiers of type t on E (rather than 'full' ones). These provide the point of departure for my generalizations to be presented in the next section. It should be obvious how to develop unrelativized notions like the above 'full two-valued L-quantifiers' from these concepts, but such extensions are irrelevant for my current purposes.

12.3 (Semi-)fuzzy L-quantifiers and a suitable notion of QFMs

Now that the basic issues concerning two-valued Lindström quantifiers have been clarified, we can turn to the problem of introducing the intended support for fuzziness. The basic strategy that proved itself useful for developing the ordinary framework then guides us to introducing suitable definitions of semi-fuzzy L-quantifiers, fuzzy L-quantifiers and L-QFMs.

Definition 171
A semi-fuzzy L-quantifier *of type $t = \langle t_1, \dots, t_n\rangle$ on a base set $E \neq \varnothing$ is a mapping $Q : \mathop{\times}\limits_{i=1}^{n} \mathcal{P}(E^{t_i}) \longrightarrow \mathbf{I}$ which assigns a gradual quantification result $Q(Y_1, \dots, Y_n) \in [0,1]$ to each choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1,\ \dots, n\}$.*

Thus, Q accepts crisp arguments of the indicated types, but it can express approximate quantifications. 'Ordinary' semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ can be viewed as a special case of semi-fuzzy L-quantifiers; they roughly correspond to n-ary semi-fuzzy L-quantifiers of type $t = \langle 1, \dots, 1\rangle$ on the given base set, E. However, I must be particular about the point that semi-fuzzy L-quantifiers are not literally *identical* to this special case of semi-fuzzy L-quantifiers; we only have a one-to-one correspondence (see below).

Semi-fuzzy L-quantifiers are proposed as a uniform specification medium for arbitrary quantifiers suitable to express multiple variable binding. Again, we also need operational quantifiers, which are not restricted to crisp inputs:

Definition 172
A fuzzy L-quantifier of type *t on a base-set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \mathop{\times}\limits_{i=1}^{n} \widetilde{\mathcal{P}}(E^{t_i}) \longrightarrow \mathbf{I}$ which assigns a gradual quantification result $\widetilde{Q}(X_1, \dots, X_n) \in [0,1]$ to each choice of fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1,\ \dots, n\}$.*

Fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ again correspond to the special case of an n-ary fuzzy L-quantifier of type $\langle 1, \dots, 1\rangle$ on E, but they are not literally identical to these quantifiers, and only related by a one-to-one correspondence.

A suitable fuzzification mechanism must be used for associating specifications to target quantifiers. The notion of an L-QFM can be developed in total analogy to the earlier proposal of QFMs for ordinary quantifiers.

Definition 173
An L-quantifier fuzzification mechanism *(L-QFM) $\mathcal{F}$ assigns to each semi-fuzzy L-quantifier Q a corresponding fuzzy L-quantifier $\mathcal{F}(Q)$ of the same type $t = \langle t_1, \dots, t_n\rangle$ and on the same base set $E \neq \varnothing$.*

These definitions give birth to a new framework for fuzzy quantification which encompasses Lindström quantifiers and hence, multiple variable binding (provided a suitable definition of a logic on top of such quantifiers. Such a system of logic might be modelled closely after Lindström's original stipulations in [97]). The models for 'L-quantification', i.e. L-QFMs, are totally unconstrained, though, and again we need to make provisions in order to guarantee plausible interpretations and shrink down the totality of possible L-QFMs to the intended choices.

12.4 Constructions on L-quantifiers

It is possible to develop plausibility criteria for L-QFMs in total analogy to those for ordinary QFMs. However, some preparations are necessary.

The notion of an underlying semi-fuzzy L-quantifier of a given fuzzy L-quantifier is defined pretty much the same way as in the ordinary case:

Definition 174
Let $\widetilde{Q}$ *be a fuzzy L-quantifier of type* $t = \langle t_1, \ldots, t_n \rangle$ *on a base set* E*. The* underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q}) : \mathop{\times}\limits_{i=1}^{n} \mathcal{P}(E^{t_i}) \longrightarrow \mathbf{I}$ *is the semi-fuzzy quantifier of type* t *on* E *defined by*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n)\,,$$

for all $Y_1 \in \mathcal{P}(E^{t_1}), \ldots, Y_n \in \mathcal{P}(E^{t_n})$*.*

Hence again, $\mathcal{U}(\widetilde{Q})$ 'forgets' that $\widetilde{Q}$ can be applied to fuzzy arguments, and only considers its behaviour on crisp arguments. This notion will prove useful to express that an L-QFM properly generalizes the semi-fuzzy quantifiers it is applied to, in analogy to (Z-1).

Recalling the constructions defined for ordinary semi-fuzzy quantifiers in Chap. 3, I further needed projection quantifiers $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ and fuzzy projection quantifiers $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{2}$ in order to describe membership assessments as a special kind of quantification; the corresponding condition imposed on ordinary models was (Z-2). However, I will not introduce a special notation for two-valued projection L-quantifiers and fuzzy projection L-quantifiers. To see why, consider a two-valued projection quantifier Q_e of type $\langle 1 \rangle$ on some base set E, $e \in E$. Generalizing the definition of ordinary projection quantifiers, Q_e is the mapping $Q_e : \mathcal{P}(E^1) \longrightarrow \mathbf{2}$ defined by $Q_e(Y) = \chi_Y((e))$ for all $Y \in \mathcal{P}(E^1)$. It is then apparent that Q_e can be expressed as $Q_e = \pi_{(e)}$, where $\pi_{(e)} : \mathcal{P}(E^1) \longrightarrow \mathbf{I}$ is an ordinary projection quantifier defined by Def. 9. Hence two-valued projection L-quantifiers are special cases of ordinary projection quantifiers, and we need not introduce a new symbol to signify them. By similar reasoning, a fuzzy projection quantifier $\widetilde{Q}_e$ of type $\langle 1 \rangle$ on a base set $E \neq \varnothing$ is the mapping $\widetilde{Q}_e : \widetilde{\mathcal{P}}(E^1) \longrightarrow \mathbf{I}$ defined by $\widetilde{Q}_e(X) = \mu_X((e))$ for all $X \in \widetilde{\mathcal{P}}(E^1)$. Hence $\widetilde{Q}_e$ can be viewed as a special case of 'ordinary' fuzzy projection quantifier $\widetilde{Q}_e = \widetilde{\pi}_{(e)}$, where $\widetilde{\pi}_{(e)} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is defined by Def. 10. Again, there is no need to introduce a new notation because every fuzzy projection L-quantifier can be expressed as $\widetilde{\pi}_{(e)}$ for some $e \in E$.

As noted above, semi-fuzzy quantifiers and fuzzy quantifiers can be viewed as special cases of semi-fuzzy L-quantifiers and fuzzy L-quantifiers by means of a suitable embedding. This correspondence is important because the formal machinery introduced so far, in particular the construction of induced truth functions and the definition of the induced extension principle, is developed in terms of ordinary quantifiers. This suggests that we might utilize this correspondence in order to define the induced truth functions and induced extension principle of an L-QFM in terms of the definitions which are already available for ordinary QFMs.

Hence let us make this correspondence explicit. We first consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. As already remarked above, a matching semi-fuzzy L-quantifier Q' has arity n, type $t = \langle 1, \ldots, 1 \rangle$, and is also defined on E. Hence Q' is a mapping $Q' : \mathcal{P}(E^1)^n \longrightarrow \mathbf{I}$. This demonstrates why Q' and Q are in general not identical: it is true that the semi-fuzzy L-quantifier Q' is a also an ordinary n-ary semi-fuzzy quantifier, but it is defined on a different base set, i.e. E^1 rather than E. Here, E^1 is the set of all one-tuples $E^1 = \{(e) : e \in E\}$, i.e. the set of all mappings $f : \{1\} \longrightarrow E$, which is usually different from the set $E = \{e : e \in E\}$. A similar situation occurs when we turn attention to fuzzy quantifiers vs. fuzzy L-quantifiers. In this case, a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is also different from its possible counterparts $\widetilde{Q}' : \widetilde{\mathcal{P}}(E^1)^n \longrightarrow \mathbf{I}$, essentially for the same reason. Let me now describe the obvious one-to-one correspondence between the elements of these sets. Let $\beta : E \longrightarrow E^1$ denote the mapping which maps the elements $e \in E$ to one-tuples

$$\beta(e) = (e) \tag{236}$$

The mapping which maps the one-tuples $(e) \in E$ to their first component $(e)_1 = e$ will be denoted $\vartheta : E^1 \longrightarrow E$, i.e.

$$\vartheta((e)) = e \tag{237}$$

for all $(e) \in E^1$. Obviously, β and ϑ are isomorphisms, and ϑ is the inverse mapping of β. Based on β and ϑ, I am now able to express the correspondence between ordinary (semi-)fuzzy quantifiers and their generalized counterparts in formal terms. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier. The semi-fuzzy L-quantifier Q' of type $\langle 1, \ldots, 1 \rangle \in \mathbb{N}^n$ on E which corresponds to Q can be expressed as

$$Q' = Q \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta}, \tag{238}$$

i.e.

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{\vartheta}(Y_1), \ldots, \widehat{\vartheta}(Y_n)) \tag{239}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. The inverse of ϑ, i.e. β, can be used in a similar way to translate back from semi-fuzzy L-quantifiers of arity n and type $\langle 1, \ldots, 1 \rangle$ to corresponding ordinary semi-fuzzy quantifiers, i.e. assuming crisp arguments. In the fuzzy case, we can also proceed analogously. In order to translate a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ into an n-ary fuzzy L-quantifier of type $\langle 1, \ldots, 1 \rangle$ on E, we can proceed as in (238); but now, the arguments range over fuzzy sets, so we will replace the crisp powerset mapping $\widehat{\vartheta}$ with the fuzzy image mapping $\hat{\hat{\vartheta}}$ obtained from the standard extension principle. Again, we can also relate a fuzzy L-quantifier $\widetilde{Q}$ of arity n and type $\langle 1, \ldots, 1 \rangle$ on E to a corresponding ordinary fuzzy quantifier $\widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, which then becomes

$$\widetilde{Q}' = \widetilde{Q} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\beta}}$$

i.e.

$$\widetilde{Q}'(X_1, \ldots, X_n) = \widetilde{Q}(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_n))$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. In the following definition, I combine these constructions in order to accomplish an interpretation of ordinary quantifiers in L-QFMs.

Definition 175
Let $\mathcal{F}$ be an L-QFM. The corresponding ordinary QFM $\mathcal{F}_R$ is defined by

$$\mathcal{F}_R(Q) = \mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\beta}},$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

The construction of a corresponding 'ordinary' QFM $\mathcal{F}_R$ for a given L-QFM $\mathcal{F}$ permits us to define the induced truth functions and the induced extension principle of an L-DFS $\mathcal{F}$ as the fuzzy truth functions and the extension principle, respectively, which are induced by the ordinary DFS $\mathcal{F}_R$:

Definition 176
Let $\mathcal{F}$ be an L-QFM and $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ be a given mapping (i.e. a 'semi-fuzzy truth function') of arity $n \in \mathbb{N}$. The induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{\mathcal{F}}(f) = \widetilde{\mathcal{F}_R}(f),$$

where $\mathcal{F}_R$ is the ordinary QFM defined by Def. 175.

Notes

- whenever $\mathcal{F}$ is clear from context, I will again use abreviations $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$, $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ etc.
- The definition of induced truth functions is extended to the induced fuzzy set operations of complementation $\widetilde{\neg}$, intersection $\widetilde{\cap}$ and $\widetilde{\cup}$ in the obvious ways, i.e. again by elementwise application of the induced negation, conjunction and disjunction on membership grades.

Based on the induced fuzzy truth functions and fuzzy set operations of an L-QFM, I can now develop constructions on semi-fuzzy and fuzzy L-quantifiers which parallel my definitions of antonyms, negation and duals for ordinary quantifiers. Here, I will confine myself to exemplifying the definition of dualization, which I need for generalizing (Z-3) to L-QFMs; the other concepts can be developed in a similar way but will not be significant in the following.

Definition 177
Given a semi-fuzzy L-quantifier Q of type $t = \langle t_1, \ldots, t_n \rangle$, $n > 0$ on some base set $E \neq \varnothing$, the semi-fuzzy L-quantifier $Q\widetilde{\square}$ of type t on E is defined by

$$Q\widetilde{\square}(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)$$

for all $Y_1 \in \mathcal{P}(E^{t_1}), \ldots, Y_n \in \mathcal{P}(E^{t_n})$. $Q\widetilde{\square}$ is called the dual of Q. The dual of a fuzzy L-quantifier $\widetilde{Q}$ of type t on E is defined analogously.

In order to express a condition constraining the behaviour of $\mathcal{F}$ for unions of argument sets which parallels (Z-4), I must also generalize the construction of quantifiers from such unions. The definition of semi-fuzzy and fuzzy L-quantifiers resulting from unions should be apparent:

Definition 178
Given a semi-fuzzy L-quantifier Q of type $t = \langle t_1, \ldots, t_n \rangle$, $n > 0$ on some base set $E \neq \varnothing$, the semi-fuzzy L-quantifier $Q\cup$ of arity $n+1$ and type $\langle t_1, \ldots, t_n, t_n \rangle$ on E is defined by

$$Q\cup(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$$

for all $Y_1 \in \mathcal{P}(E^{t_1}), \ldots, Y_n \in \mathcal{P}(E^{t_n})$ and $Y_{n+1} \in \mathcal{P}(E^{t_n})$. For fuzzy L-quantifiers $\widetilde{Q}$ of type t on E, the $(n+1)$-ary fuzzy L-quantifier $\widetilde{Q}\widetilde{\cup}$ of type $\langle t_1, \ldots, t_n, t_n \rangle$ on E is defined analogously.

Monotonicity of semi-fuzzy and fuzzy L-quantifiers is defined as in the ordinary case. We only need to make sure that all arguments involved are chosen from their appropriate ranges $Y_i \in \mathcal{P}(E^{t_i})$ for semi-fuzzy L-quantifiers, and $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$ in the fuzzy case:

Definition 179
A semi-fuzzy L-quantifier Q of type $t = \langle t_1, \ldots, t_n \rangle$, $n > 0$ on some base set E is nondecreasing in its i-th argument, *$i \in \{1, \ldots, n\}$, if*

$$Q(Y_1, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

for all $Y_1 \in \mathcal{P}(E^{t_1}), \ldots, Y_n \in \mathcal{P}(E^{t_n})$ and $Y_i' \in \mathcal{P}(E^{t_i})$ with $Y_i \subseteq Y_i'$. Q is said to be nonincreasing in its i-th argument *if under the same conditions, it always holds that*

$$Q(Y_1, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)\,.$$

The corresponding definitions for fuzzy quantifiers $\widetilde{Q}$ of type t on E are analogous. In this case, the arguments range over $\widetilde{\mathcal{P}}(E^{t_i})$, and '$\subseteq$' is the fuzzy inclusion relation.

Based on semi-fuzzy L-quantifiers which are nonincreasing in their last argument, I will be able to express a condition on L-QFMs which parallels (Z-5).

Finally, I would also like to extend the condition that $\mathcal{F}$ be compatible with functional composition (Z-6) towards the new type of fuzzification mechanisms. I will resort to the ordinary QFM $\mathcal{F}_R$ associated with $\mathcal{F}$ in order to define the induced extension principle of an L-QFM.

Definition 180
The induced extension principle $\widehat{\mathcal{F}}$ *of an L-QFM* $\mathcal{F}$ *is defined by* $\widehat{\mathcal{F}} = \widehat{\mathcal{F}_R}$*, i.e.* $\widehat{\mathcal{F}}$ *is the induced extension principle of the QFM* $\mathcal{F}_R$ *defined by Def. 175.*

12.5 The class of plausible L-models

Based on the above constructions, I can now define my axiom system for a class of plausible L-QFMs, called ‘L-DFSes’ or ‘L-models’ of fuzzy quantification. It should be obvious from the earlier characterization of DFSes and the generalization of the relevant constructions how the criteria for L-DFSes which support Lindström-like quantifiers can be expressed:

Definition 181
An L-QFM $\mathcal{F}$ *is called an* L-DFS *if the following conditions are satisfied for all semi-fuzzy L-quantifiers* $Q : \mathop{\times}\limits_{i=1}^{n} \mathcal{P}(E^{t_i}) \longrightarrow \mathbf{I}$ *of arbitrary types* $t = \langle t_1, \ldots, t_n \rangle$ *and on arbitrary base sets* $E \neq \varnothing$*:*

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $t \in \{\langle\rangle, \langle 1 \rangle\}$	(L-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_{(e)}$ *if* $Q = \pi_{(e)}$ *for some* $e \in E$	(L-2)
Dualisation	$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square} \quad n > 0$	(L-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup} \quad n > 0$	(L-4)
Preservation of monotonicity	*If* Q *is nonincreasing in the* n*-th arg, then* $\mathcal{F}(Q)$ *is nonincreasing in the* n*-th arg,* $n > 0$	(L-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_i : {E'}^{t'_i} \longrightarrow E^{t_i}$*,* $t' \in \mathbb{N}^n$ *(same* n*),* $E' \neq \varnothing$*.*	(L-6)

The axiom system is constructed in total analogy to my basic system for ordinary QFMs. Thus it is not surprising that the generalized L-models are also suitable for carrying out ‘ordinary’ quantification, i.e. the model $\mathcal{F}_R$ associated with $\mathcal{F}$ is indeed a DFS:

Theorem 271 *Let* $\mathcal{F}$ *be an L-QFM and* $\mathcal{F}_R$ *the corresponding QFM defined by Def. 175. If* $\mathcal{F}$ *is an L-DFS, then* $\mathcal{F}_R$ *is a DFS.*
(Proof: D.32, p.531+)

I will not comment further on the axioms because they are so closely modelled after my conditions (Z-1)–(Z-6) imposed on plausible choices of QFMs. For these I gave ample motivation in Chap. 3, and there is no point repeating these arguments here. However, another issue needs clarification. As witnessed by the previous chapters, the comprehensive discussion of ordinary DFSes, the development of constructive principles for useful models, and finally the implementation of such models, required considerable effort. Obviously, there would be a lot to gain if we could reuse these results on ordinary models. Ideally, it would be possible to express all L-DFSes in terms of suitable ordinary DFSes. In this case, my results on properties of the models, constructive classes of plausible models and finally the techniques developed for implementation would directly translate into corresponding results on the new L-models.

In order to effect the desired reduction, we need a generic method which lets us construct a 'suitable' L-QFM $\mathcal{F}_L$ from a given $\mathcal{F}$. Here 'suitable' means that: (a) $\mathcal{F}_L$ properly extends $\mathcal{F}$, i.e. $\mathcal{F} = \mathcal{F}_{LR}$; and (b) $\mathcal{F}_L$ is an L-DFS whenever $\mathcal{F}$ is a DFS, i.e. the construction 'works'. In addition, the new construction should be compatible with the earlier construction of $\mathcal{F}_R$ from a given L-QFM. Hence we should also have (c) $\mathcal{F} = \mathcal{F}_{RL}$ for all L-DFSes $\mathcal{F}$, i.e. $\mathcal{F}$ can be recovered from $\mathcal{F}_R$. A generic construction of L-models which answers these requirements, will eliminate the problem of establishing classes of L-DFSes altogether, because every L-DFS $\mathcal{F}'$ can then be expressed as $\mathcal{F}' = \mathcal{F}_L$ for an ordinary DFS $\mathcal{F}$. I propose the following construction to accomplish this.

Let $\mathcal{F}$ be an ordinary QFM and let Q be a semi-fuzzy L-quantifier of type $t = \langle t_1, \ldots, t_n \rangle$ on some base set E. I will abbreviate

$$m = \max\{t_1, \ldots, t_n\}\,, \tag{240}$$

in particular $t_i \le m$ for all $i \in \{1, \ \ldots, n\}$. I can therefore introduce injections $\zeta_i : E^{t_i} \longrightarrow E^m$ which embed E^{t_i} into a common extended domain E^m. These are defined by

$$\zeta_i(e_1, \ldots, e_{t_i}) = (e_1, \ldots, e_{t_i}, \underbrace{e_{t_i}, \ldots, e_{t_i}}_{(m - t_i)\text{ times}}) \tag{241}$$

for all $(e_1, \ldots, e_{t_i}) \in E^{t_i}$, $i \in \{1, \ \ldots, n\}$. Hence ζ_i fills in the $(m - t_i)$ missing components by repeating e_{t_i}. It should be apparent that ζ_i is indeed an injection. We also need mappings in the reverse direction. These will be written $\kappa_i : E^m \longrightarrow E^{t_i}$, $i \in \{1, \ \ldots, n\}$. The definition is as follows,

$$\kappa(e_1, \ldots, e_m) = (e_1, \ldots, e_{t_i}) \tag{242}$$

for all $(e_1, \ldots, e_m) \in E^m$, $i \in \{1, \ \ldots, n\}$. Hence κ_i is the projection on the first t_i components of $(e_1, \ldots, e_m)$, i.e. it simply drops the final $(m - t_i)$ components. In particular, the κ_i's are onto (surjective). It should be apparent from (241) and (242) that subsequent application of ζ_i (which adds $m - t_i$ components) and κ_i (which drops these components) will take us back to the original t_i-tuple. Hence for all $i \in \{1, \ \ldots, n\}$,

$$\kappa_i \circ \zeta_i = \mathrm{id}_{E^{t_i}}\,. \tag{243}$$

In dependence on the given semi-fuzzy L-quantifier Q, I now define a mapping $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ by

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{\kappa}_1(Y_1 \cap \operatorname{Im} \zeta_1), \ldots, \widehat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)) \,, \tag{244}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E^m)$. It is instructive to observe that Q' is, at the same time, an n-ary semi-fuzzy L-quantifier of type $\langle m, \ldots m \rangle$ on E, and an ordinary semi-fuzzy quantifier $Q : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ on a different base set, E^m. In fact, this is the basic idea underlying the following construction of L-QFMs from ordinary QFMs.

Definition 182
The L-QFM $\mathcal{F}_L$ associated with a given QFM $\mathcal{F}$ is defined as follows. For all semi-fuzzy L-quantifiers Q of some type $t = \langle t_1, \ldots, t_n \rangle$, on a base set $E \neq \varnothing$,

$$\mathcal{F}_L(Q) = \mathcal{F}(Q') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}_i$$

where Q' is the semi-fuzzy quantifier defined by (244)*, and the ζ_i's given by* (241) *are extended to mappings $\hat{\hat{\zeta}}_i : \widetilde{\mathcal{P}}(E^{t_i}) \longrightarrow \widetilde{\mathcal{P}}(E^m)$ by applying the standard extension principle.*

Now let us consider the above criteria (a)–(c) for a successful reduction of L-DFSes to ordinary DFSes in turn.

Theorem 272 *If $\mathcal{F}$ is a DFS, then $\mathcal{F}_{LR} = \mathcal{F}$.*
(Proof: D.33, p.539+)

($\mathcal{F}_L$ properly generalizes the original model $\mathcal{F}$.)

Theorem 273 *If $\mathcal{F}$ is a DFS, then the corresponding L-QFM $\mathcal{F}_L$ is an L-DFS.*
(Proof: D.34, p.543+)

(The proposed construction results in plausible models.)

Theorem 274 *If $\mathcal{F}$ is an L-DFS, then $\mathcal{F}_{RL} = \mathcal{F}$.*
(Proof: D.35, p.553+)

The latter theorem is of particular relevance because it eliminates the problem of establishing classes of L-DFSes altogether. To sum up, my criteria (a)–(c) are all valid and by the above reasoning, we need not be concerned with developing classes of L-DFSes any longer. The ordinary models $\mathcal{F}$ that have already been studied in some depth are sufficient to represent every L-DFS $\mathcal{F}'$ as $\mathcal{F}' = \mathcal{F}_L$, given a suitable choice of $\mathcal{F}$. This is of invaluable help because the investigation of constructive principles for plausible models, like those presented in in Chap. 3, turned out to be a rather complex matter. The canonical construction of $\mathcal{F}_L$, then, permits the use of the proven models $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{\text{Ch}}$ to handle the new cases of fuzzy L-quantification. In the following, I will identify these models with their extensions for simplicity, thus writing $\mathcal{M}_{CX}$ rather than $(\mathcal{M}_{CX})_L$ etc.

12.6 Nesting of quantifiers

The axioms for L-DFSes are directly modelled after the corresponding axioms for ordinary DFSes; they do not refer to details of multiple variable binding (on the syntactic level), or handling t_i-ary (fuzzy) relations which occur as arguments of L-quantifiers (i.e. on the level of modelling devices). It is an interesting question if additional axioms specifically concerned with L-quantifiers might prove useful for identifying those L-DFSes best suited for modelling branching quantification.

A construction that comes to mind is *quantifier nesting*. Consider $Q'xQ''y\varphi(x,y)$, for example. It should be apparent how to develop a semantical interpretation of such formulas in the proposed framework. Based on semi-fuzzy L-quantifiers, we can either analyze this in terms of two quantifiers of type $t_1 = t_2 = \langle 1 \rangle$ applied in succession. Alternatively, we can look at the whole block $Q_{xy} = Q'xQ''y$, which corresponds to a single semi-fuzzy L-quantifier of type $t = \langle 2 \rangle$. Now let $R \in \widetilde{\mathcal{P}}(E^2)$ be the interpretation of $\varphi(x,y)$ as a fuzzy relation. The analysis in terms of two separate quantifiers will result in an interpretation $\mathcal{F}(Q')(Z)$, where $\mu_Z(e) = \mathcal{F}(Q'')(eR)$ and $\mu_{eR}(e') = \mu_R(e,e')$ for all $e, e' \in E$. This analysis thus corresponds to a fuzzy L-quantifier $\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')$ of type $\langle 2 \rangle$ defined by

$$\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')(R) = \mathcal{F}(Q')(Z)$$

for all fuzzy relations $R \in \mathcal{P}(E^2)$. The quantifier block Q_{xy}, on the other hand, corresponds to a semi-fuzzy L-quantifier $Q' \mathbin{\widetilde{@}} Q''$ of type $\langle 2 \rangle$ defined by

$$Q' \mathbin{\widetilde{@}} Q''(S) = \mathcal{F}(Q')(Z)\,,$$

$\mu_Z(e) = Q''(eS)$, $eS = \{e' : (e,e') \in S\}$ for all *crisp* relations $S \in \mathcal{P}(E^2)$, and it results in a second interpretation, $\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'')(R)$. (I am using the 'tilde'-notation $\widetilde{@}$ here to signify that $Q' \mathbin{\widetilde{@}} Q''$ depends on the chosen L-QFM). It is natural to require that the two interpretations coincide, i.e.

$$\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') = \mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')\,.$$

More generally, we can define a nesting operation for semi-fuzzy L-quantifiers and fuzzy L-quantifiers of arbitrary types (the quantifier Q' must offer an argument slot to nest in, though, and thus needs positive arity $n > 0$).

Definition 183

Let $E \neq \varnothing$ be a given base set and Q' a semi-fuzzy L-quantifier on E of type $t = \langle t_1, \ldots, t_n \rangle$, $n > 0$. Further let Q'' be a semi-fuzzy L-quantifier on E of arbitrary type $t' = \langle t'_1, \ldots, t'_{n'} \rangle$, $n' \in \mathbb{N}$. The semi-fuzzy L-quantifier $Q' \mathbin{\widetilde{@}} Q''$ on E of type $t^* = \langle t_1, \ldots, t_{n-1}, t_n + t'_1, \ldots, t_n + t'_{n'} \rangle$ which results from the nesting of Q'' into the last argument of Q' is defined by

$$Q' \mathbin{\widetilde{@}} Q''(Y_1, \ldots, Y_{n-1}, S_1, \ldots, S_{n'}) = \mathcal{F}(Q')(Y_1, \ldots, Y_{n-1}, Z)$$

for all $Y_i \in E^{t_i}$*,* $i \in \{1, \dots, n-1\}$ *and* $S_j \in E^{t_n + t'_j}$*,* $j \in \{1, \dots, n'\}$*, where* $Z \in \widetilde{\mathcal{P}}(E^{t_n})$ *is defined by*

$$\mu_Z(e_1, \dots, e_{t_n}) = \mathcal{F}(Q'')((e_1, \dots, e_{t_n})S_1, \dots, (e_1, \dots, e_{t_n})S_{n'}) \tag{245}$$

for all $e_1, \dots, e_{t_n} \in E$*, and where the* t'_j*-ary relation* $(e_1, \dots, e_{t_n})S_j \in \mathcal{P}(E^{t'_j})$ *is given by*

$$(e_1, \dots, e_{t_n})S_j = \{(e'_1, \dots, e'_{t'_j}) : (e_1, \dots, e_{t_n}, e'_1, \dots, e'_{t'_j}) \in S_j\} \tag{246}$$

for $j \in \{1, \dots, n'\}$*.*

A similar construction is possible for fuzzy L-quantifiers:

Definition 184
Let $E \neq \varnothing$ *be a given base set and* $\widetilde{Q}'$ *a fuzzy L-quantifier on* E *of type* $t = \langle t_1, \dots, t_n \rangle$*,* $n > 0$*. Further let* $\widetilde{Q}''$ *be a fuzzy L-quantifier on* E *of arbitrary type* $t' = \langle t'_1, \dots, t'_{n'} \rangle$*,* $n' \in \mathbb{N}$*. The fuzzy L-quantifier* $\widetilde{Q}' \widetilde{@} \widetilde{Q}''$ *of type* $t^* = \langle t_1, \dots, t_{n-1}, t_n + t'_1, \dots, t_n + t'_{n'} \rangle$ *on* E *is defined by*

$$\widetilde{Q}' \mathbin{@} \widetilde{Q}''(X_1, \dots, X_{n-1}, R_1, \dots, R_{n'}) = \widetilde{Q}'(X_1, \dots, X_{n-1}, Z)$$

for all $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$*,* $i \in \{1, \dots, n-1\}$ *and* $R_j \in \widetilde{\mathcal{P}}(E^{t_n + t'_j})$*,* $j \in \{1, \dots, n'\}$*, where* $Z \in \widetilde{\mathcal{P}}(E^{t_n})$ *is defined by*

$$\mu_Z(e_1, \dots, e_{t_n}) = \widetilde{Q}''((e_1, \dots, e_{t_n})R_1, \dots, (e_1, \dots, e_{t_n})R_{n'}) \tag{247}$$

for all $e_1, \dots, e_{t_n} \in E$*, and where* $(e_1, \dots, e_{t_n})R_j \in \widetilde{\mathcal{P}}(E^{t'_j})$*,* $j \in \{1, \dots, n'\}$*, is the fuzzy relation defined by*

$$\mu_{(e_1, \dots, e_{t_n})R_j}(e'_1, \dots, e'_{t'_j}) = \mu_{R_j}(e_1, \dots, e_{t_n}, e'_1, \dots, e'_{t'_j}) \tag{248}$$

for all $e'_1, \dots, e'_{t'_j} \in E$*.*

In this general case, too, we would expect that plausible choices of $\mathcal{F}$ comply with the nesting operation.

Definition 185
An L-QFM $\mathcal{F}$ *is said to be compatible with* quantifier nesting *(in the last argument) if the equality*

$$\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') = \mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'') \tag{QN}$$

is valid for all semi-fuzzy L-quantifiers Q' *of type* $t = \langle t_1, \dots, t_n \rangle$*,* $n > 0$ *and* Q'' *of arbitrary type* $t' = \langle t'_1, \dots, t'_{n'} \rangle$*,* $n' \in \mathbb{N}$ *defined on the same base set* $E \neq \varnothing$*.*

Judging from some first tests, it appears that quantifier nesting expresses a very restrictive condition and excludes many useful models. In fact, I have the following result on general quantifier nestings:

Theorem 275 *Suppose that an L-DFS $\mathcal{F}$ is compatible with quantifier nesting. Then $\mathcal{F}_R$ is compatible with fuzzy argument insertion.*
(Proof: D.36, p.554+)

In other words, the only standard model which potentially admits the nesting of quantifiers is $\mathcal{M}_{CX}$, as shown by Th-97 and Th-98. I have not yet established whether $\mathcal{M}_{CX}$ validates the criterion but I would rather guess this is not the case. It is perfectly possible – as in the case of conservativity Def. 70 or convexity Def. 67 – that the unrestricted or 'strong' requirement of compatibility with quantifier nestings conflicts with other reasonable criteria and must thus be weakened to a compatible 'core' requirement. Obviously, further research should be directed into these issues in order to determine the status of $\mathcal{M}_{CX}$ and develop suitable weakenings of the nesting criterion if necessary.

Let me further comment on the limitation of (QN) to nestings in the last argument of the quantifier Q'. Nestings in another argument position k can be reduced to this case by flipping argument positions k and n, performing the nesting in the last argument, and then reordering the arguments again into their intended target positions. But, every DFS is known to be compatible with such permutations of argument positions, a property which apparently generalizes to the L-models. Thus the compatibility of a DFS with nestings in the last argument position is sufficient to ensure that it also comply with nestings in other argument positions. By repeating this process, (QN) even ensures the compatibility of conforming L-DFSes with multiple nestings of arbitrary depths. Thus, the condition is sufficient to cover general nestings involving quantifiers of arbitrary types with an arbitrary number of embedded quantifiers.

It would be useful however, to relate the proposed general nestings in the last argument to the simple nesting of unary quantifiers considered earlier, or to other simplified nesting criteria. This might facilitate the check that a model of interest (not necessary a standard model like $\mathcal{M}_{CX}$) comply with the nesting condition, and also eliminate some redundancy from the axiom system. I expect that the general nestings described by (QN) can be reduced to a simpler criterion, but further research is necessary to clarify this matter. Should a substantial weakening of the original requirement be necessary, the investigation of simplified criteria might also guide us to a useful weakening which still covers the nestings of linguistic relevance.

12.7 Application to the modelling of branching quantification

Let me now explain how the motivating example, "Many young and most old people respect each other" can be interpreted in the proposed framework. In this case, we have semi-fuzzy quantifiers $Q_1 =$ **many**, defined by **many**$(Y_1, Y_2) = |Y_1 \cap Y_2|/|Y_1|$, say, and $Q_2 =$ **most**, defined as above. Both quantifiers are nondecreasing in their second argument, i.e. we can adopt the analysis proposed by Barwise. The modification of equality (235) towards gradual truth values will be accomplished in the usual way, i.e. by replacing existential quantifiers with $\sup$ and conjunctions with $\min$ (non-standard connectives are not possible here). The semi-fuzzy L-quantifier Q of type $\langle 1, 1, 2\rangle$

constructed from Q_1, Q_2 then becomes

$$Q(A, B, R) = \sup\{\min(Q_1(A, U), Q_2(B, V)) : U \times V \subseteq R\}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. Hence "Many men and many women are relatives of each other", for example, which rests on crisp arguments, is now taken to denote the maximum degree to which many men and many women belong to a group $U \times V \subseteq R$ of mutual relatives. In my experience, this is a conclusive analysis; but we still need to extend it to fuzzy arguments. To this end, it is sufficient to apply the chosen L-DFS $\mathcal{F}$. We then obtain the fuzzy L-quantifier $\mathcal{F}(Q)$ of type $\langle 1, 1, 2\rangle$ suited to handle this case. Returning to the original example involving young and old people, we have fuzzy subsets $\textbf{young}, \textbf{old} \in \widetilde{\mathcal{P}}(E)$ of young and old persons, respectively, and a fuzzy relation $\textbf{rsp} \in \widetilde{\mathcal{P}}(E^2)$ of persons who respect each other. Thus, a meaningful interpretation of "Many young and most old people respect each other" is now given by $\mathcal{F}(Q)(\textbf{young}, \textbf{old}, \textbf{rsp})$.

In the event that either Q_1 or Q_2 fail to be increasing in the second argument, we must adopt Westerståhl's generic method for interpreting branching quantifiers. Hence let me describe how the method can be applied in the fuzzy case. Suppose that Q_1, Q_2 are arbitrary semi-fuzzy quantifiers of arity $n = 2$ on some base set E. Following Westerståhl, I introduce nondecreasing and nonincreasing approximations of the Q_i's, defined by $Q_i^+(Y_1, Y_2) = \sup\{Q_i(Y_1, L) : L \subseteq Y_2\}$ and $Q_i^-(Y_1, Y_2) = \sup\{Q_i(Y_1, U) : U \supseteq Y_2\}$, respectively. With the usual replacement of existential quantification with $\sup$ and conjunction with $\min$, Westerståhls interpretation formula [158, p. 281, Def. 3.1] becomes:

$$\begin{aligned} Q(A, B, R) = \sup\{\min\{&Q_1^+(A, U_1), Q_2^+(B, V_1), Q_1^-(A, U_2), Q_2^-(B, V_2)\} : \\ &(U_1 \cap A) \times (V_1 \cap B) \subseteq R \cap (A \times B) \subseteq (U_2 \cap A) \times (V_2 \cap B)\} \end{aligned}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. Application of an L-DFS then determines the corresponding fuzzy L-quantifier $\mathcal{F}(Q)$ of type $\langle 1, 1, 2\rangle$ suitable for interpretation.

As shown by Westerståhl [158, p.284], his method results in meaningful interpretations provided that (a) the Q_i's are 'logical', i.e. $Q_1(Y_1, Y_2)$ and $Q_2(Y_1, Y_2)$ can be expressed as a function of $|Y_1|$ and $|Y_1 \cap Y_2|$, see van Benthem [8, p. 446] and [9, p. 458]; and (b), the Q_i's are *convex* in their second argument, or 'CONT' in Westerståhl's terminology, i.e. $Q_i(Y_1, Y_2) \geq \min(Q_i(Y_1, L), Q_i(Y_1, U))$ for all $L \subseteq Y_2 \subseteq U$ according to Def. 67. The latter condition ensures that Q_1 and Q_2 can be recovered from their nondecreasing approximations Q_i^+ and their nonincreasing approximations Q_i^-, i.e. $Q_i = \min(Q_i^+, Q_i^-)$. This is generally the case when Q_1 and Q_2 are either nondecreasing in their second argument ("many"), nonincreasing ("few"), or of unimodal shape ("about ten", "about one third"). An example of branching quantification with unimodal quantifiers, which demand the generic method, is "About fifty young and about sixty old persons respect each other".

12.8 Chapter summary

Recognizing the utility of branching quantifiers to linguistic modelling, I have proposed an extension of the DFS theory of fuzzy quantification which incorporates these cases. Specifically, I introduced fuzzy L-quantifiers (generalizations of Lindström quantifiers to approximate quantifiers and fuzzy arguments), semi-fuzzy L-quantifiers (uniform specifications of such quantifiers) and plausible models of fuzzy quantification involving these quantifiers, called L-DFSes. The criteria for plausible L-models of fuzzy quantification parallel my requirements on QFMs, and the rationale for these conditions is the same as in the case of ordinary QFMs. I have further shown that the issue of identifying useful L-DFSes can be reduced to the known analysis of classes of ordinary models. Thus

(a) every plausible model of 'ordinary' fuzzy quantification (DFS) can be extended to a unique L-DFS for quantifiers with multiple variable binding;

(b) no other L-DFSes exist beyond those obtained from (a).

Westerståhl's analysis of branching NL quantification in terms of Lindström quantifiers is easily generalized to semi-fuzzy L-quantifiers and thus, branching type III quantifications. By applying the chosen model of fuzzy quantification, one then obtains a meaningful interpretation for branching type IV quantifications, thus admitting both approximate quantifiers and fuzzy arguments. At this point, it should be remarked that Westerståhl also extends his method to other types of branching quantification in NL, which in general can span more than two quantifiers. However, the formulas he proposes to cope with these cases can be adapted to my analysis in terms of (semi-)fuzzy L-quantifiers in total analogy to the example chosen for demonstration. Thus, the methods must be extended to semi-fuzzy L-quantifiers in the apparent ways; applying the L-QFM $\mathcal{F}$ then lets us fetch the final interpretation.

The identification of those L-DFSes specifically suited for modelling branching quantification in NL is an advanced topic that should be tackled by future research. I have already hinted at a possible strengthening of the system based on quantifier nesting. The nesting construction is linked to branching quantification in the following way: in the motivating example of this chapter, the introduction of the complex quantifier Q of type $\langle 1, 1, 2 \rangle$ rests on the grouping of several (independent) quantifiers into a block of quantifiers, which is then treated as an integral, singular quantifier. Applying the same idea to a linear sequence of quantifiers takes us to the construction of nested quantifiers discussed in section 12.6. Naturally, a model suited for branching quantification, which rests on the grouping of quantifiers into blocks, should also be compatible with such nestings of linear quantifiers. However, the resulting criterion appears to be extremely restrictive and it cannot be adopted without sacrificing useful models like $\mathcal{M}$ and $\mathcal{F}_{\mathrm{Ch}}$. It is not even clear if the full compatibility with quantifier nestings is consistent with the basic axioms at all, and it might hence be necessary to weaken the criterion to typical classes of linguistic quantifiers, like the quantitative variety. Obviously a more thorough discussion of quantifier nestings will be necessary to clarify these issues.

The proposed analysis of reciprocal constructions in terms of fuzzy branching quantifiers is of particular relevance to linguistic data summarization [73, 173]. Many summarizers of interest express mutual (or symmetric) relationships and can therefore be verbalized by a reciprocal construction. (Asymmetrical relations R' can also be used in reciprocal constructions after symmetrization, i.e. they must be replaced with $R = R' \cap R'^{\mathrm{op}}$). An ordinary summary like "Q_1 X_1's are strongly correlated with Q_2 X_2's" does not capture the symmetrical nature of correlations, and it neglects the resulting groups of mutually correlated objects. The proposed modelling in terms of branching quantifiers, by contrast, permits me to support a novel type of linguistic summaries specifically suited for describing groups of interrelated objects. Branching quantification, in this sense, is a natural language technique for detecting such groups in the data. A possible summary involving a reciprocal predicate is "The intake of most vegetables and many health-related indicators are strongly associated with each other".

13 Discussion

13.1 The fundamentals of fuzzy quantification

Natural language is pervaded with fuzziness, but the fuzziness of language does not normally interfere with our daily communication. In typical situations there is simply no need for guessing in order to grasp the meaning of everyday language. Thus, the fuzziness observed in NL cannot be totally random and uncontrolled in nature. By contrast, there must be some hidden regularities underlying linguistic fuzziness which explain the apparent possibility of systematical interpretation. It is this methodological assumption on which I based my research into fuzzy quantifiers. These linguistic constructions, too, are frequently used in language and normally do not result in misunderstanding. This suggests that the hidden structures necessary for understanding these quantifiers might be uncovered and made available for the automatic interpretation of these expressions on computer systems.

To achieve this, it was first necessary to identify the typical characteristics of linguistic quantifiers. In addition, the total range of quantifying phenomena in language had to be ascertained. Knowing these peculiarities was necessary to judge whether the existing account of quantifiers in logic and fuzzy set theory already achieved the goal of a plausible and comprehensive analysis, an issue which was resolved to the negative. Specifically, my comparison of logical versus linguistic quantifiers revealed some clear structural differences, which demonstrate that linguistic quantifiers call for a rather different kind of modelling. As opposed to logical quantifiers, the quantifiers found in NL are typically restricted by a qualifying argument and thus accept more than one argument; they are often not logical symbols; they are often not quantitative (i.e. definable in terms of cardinalities), and they are often not definable in first-order predicate logic. This structural mismatch can already be demonstrated for precise linguistic quantifiers; it is not necessary to resort to approximate examples. The above criteria have something to say about the proper modelling of linguistic quantifiers in general. But, an adequate modelling should not only match the internal structure of natural language quantifiers; it is further necessary to account for the fuzziness of language. For that purpose I explained the analysis of vague predicates common to the theory of vagueness, which clearly demonstrates that natural languages are paradigmatically vague. It even appears that vagueness serves an important purpose in NL because it offers ways of expressing ourselves when there is only imprecise knowledge. In addition, the support for vagueness increases robustness and effects a complexity reduction. Hence the modelling of language on computer systems should not deny the phenomenon of vagueness but rather utilize it in a similar way that NL profits from vagueness. Having demonstrated the necessity and practical utility of modelling vagueness, I then turned to fuzzy set theory which I introduced as a mathematical model of vague NL expressions. In the terminology of fuzzy set theory, vague predicates are of course called ‘fuzzy’, and vagueness now becomes ‘fuzziness’. I then explained the two independent sources of fuzziness which affect quantification, i.e. fuzziness in the quantifiers themselves versus the fuzziness of linguistic terms that occur in their arguments. Noticing that these types of fuzziness can occur in combination, it is necessary to support type IV quantifications [99], i.e. quantifications which admit both approximate quantifiers and fuzzy arguments. The traditional framework for fuzzy quantification proposed by Zadeh [188, 190] permits an interpretation of type IV quantifications for the considered types of quantifiers – i.e. the absolute and proportional kinds only. Compared to the wealth of linguistic quantifier types routinely treated in linguistics [61, 82, 83], the traditional framework and the approaches to fuzzy quantification derived from it are clearly too narrow and fall below the standards that have long been set. The traditional framework not only shows a lack of coverage, though. It also adopts a representation of linguistic quantifiers which does not square up with the established linguistic analysis. To be sure, Zadeh himself is aware of this problem, pointing out that his own approach be 'different' from the linguistic analysis of quantification [188, p. 149]. However, it appears that few people have contemplated the consequences of this departure from the competent scientific discipline. Not surprisingly, then, there is negative evidence against existing approaches to fuzzy quantification as to their linguistic plausibility (see also discussion in sections 1.10 to 1.15, and the evaluation of existing approaches detailed in appendix A). To sum up, there are precursors to the present work both in linguistics and fuzzy set theory. However, the linguistic analysis is only concerned with crisp quantifications. And research in fuzzy set theory has focused on the treatment of linguistic vagueness in the first place, and it was not that successful yet in devising interpretations that are linguistically conclusive. What is missing, then, is a unifying perspective which incorporates both the linguistic considerations and the benefits of a fuzzy-sets modelling of vagueness.

It was the goal of this work to develop a novel theory of fuzzy quantification which appeals both to the linguist and the fuzzy set theorist. In the report, I presented the fundamentals of such a theory of fuzzy quantification. The theory comprises the following components, which correspond to the subordinate tasks into which the overall problem can be organized:

- a conceptual framework for analysing fuzzy quantification with the desired comprehensiveness and precision;
- a system of formal postulates which effectively characterize those models which are linguistically plausible;
- a thorough evaluation of the class of these models under various semantical criteria;
- the description of concrete models derived from a given constructive principle, and the identification of models of special significance within the total class of such models;
- the development of efficient algorithms for implementing the main types of quantifiers in the dedicated models.

Let us now discuss these parts of the proposed theory in turn. The new framework for fuzzy quantification, to begin with, was expected to embed both traditions of analysing linguistic quantification and thus had to establish a joint perspective on these different conceptions. In order to achieve the required coverage of linguistic phenomena, I decided to start from the linguistic analysis, i.e. from the Theory of Generalized Quantifiers [6, 8, 9]. The concept of a two-valued generalized quantifier known from TGQ was considered a suitable starting point for developing a similar notion for fuzzy quantifications, which can involve fuzziness both in quantifiers and arguments. The basic purpose of my framework is that of permitting the formal analysis of such fuzzy quantifications, thus making the expressive power controllable which results from the incorporation of fuzziness.

In order to cope with the notorious modelling problem of explaining the relationship between linguistic quantifiers and the available modelling devices, I embarked on the general strategy of separating the specification of a quantifier from its operational form, which can be applied to fuzzy arguments for the purpose of quantification. The specification medium will be chosen such as to best support the description of target NL quantifiers, which needs only fix the essential aspects of the quantifier. Given the specification of a linguistic quantifier, it is then the responsibility of fuzzy quantification theory to establish the matching operational quantifier. This relationship is analyzed in terms of an interpretation mechanism. Pursuing this general strategy, I first introduced the representational foundations of the framework. The novel concept of a semi-fuzzy quantifier, to begin with, serves as a compact description of the linguistic quantifiers of interest. It thus plays a similar part as membership functions μ_Q in the earlier approaches. My proposed notion of semi-fuzzy quantifiers rests on the observation that the modelling of approximate quantifiers like “almost all” can be isolated from the problem of dealing with fuzzy arguments. Thus semi-fuzzy quantifiers are capable of expressing approximate quantification, but they only accept crisp arguments. They can hence be considered simplified representations of NL quantifiers taylored to the type III quantifications of Liu and Kerre [99]. Semi-fuzzy quantifiers are sufficiently powerful to embed all crisp generalized quantifiers known to the linguistic theory. Specifically, the phenomena of linguistic significance that were previously studied in TGQ can all be discussed in the context of semi-fuzzy quantifiers as well. Hence it is possible to express all issues of linguistic interest on the proposed level of specifications. Due to the restriction to crisp arguments, it is usually easy to grasp the meaning of a given semi-fuzzy quantifier. In particular semi-fuzzy quantifiers can be conveniently defined in terms of the usual cardinality for crisp sets. In this way, the restriction to crisp arguments makes it easy to define the desired specification in most cases. For the same reason, however, semi-fuzzy quantifiers are only suited as a specification medium because fuzzy arguments cannot be handled. We therefore need a complementary operational medium which permits the interpretation of arbitrary quantifications. For that purpose I introduced a general notion of fuzzy quantifiers which adopts Zadeh’s view of fuzzy quantifiers as fuzzy second-order predicates. Unlike semi-fuzzy quantifiers these operations now account both for approximate quantifiers and fuzzy arguments, and are hence suited for modelling arbitrary type IV quantifications. Due to their ample expressiveness, which forbids a rendering in terms of the usual crisp cardinalities, a suitable choice of fuzzy quantifier for a given linguistic prototype is typically hard to establish, i.e. we really need separate specifications like those offered by semi-fuzzy quantifiers. The expressiveness of fuzzy quantifiers is necessary, on the other hand, to fully describe the linguistic target operations, which demand the processing of fuzzy arguments and thus call for a model of type IV quantifications.

By connecting the specification of a linguistic quantifier and its associated operational interpretation, the notion of a QFM completes the theoretical skeleton in which fuzzy quantification can be discussed with the desired generality and formal rigor. Not all QFMs will result in plausible and coherent interpretations, though. The proposed notion merely avails us with a class of 'raw' models, from which the plausible examples must now be distilled. In other words, QFMs render possible the formalization of semantical criteria which can then be used to identify the intended models.

The structuring into specifications and interpretations, which is fundamental to my proposed framework, also separates the modelling problem into application-dependent and independent parts. The domain-specific task consists of selecting the precise interpretation that best fits the meaning of an approximate quantifier in a given application context. The problems that arise on this level are encapsulated by the assumed base descriptions in terms of semi-fuzzy quantifiers, which must be fixed in the course of building an application. The core problem of fuzzy quantification, which is application-independent, then consists of explaining how the description of the target operator in terms of a semi-fuzzy quantifier should be extended to the general case of fuzzy arguments. In other words, we must establish a coherent selection of fuzzy quantifiers which extrapolate the given specifications in the intended way. It is the proposed QFMs which assume responsibility for the generic issues centered around the processing of fuzzy arguments. To sum up, I tackle the modelling problem by isolating a certain core competence. In this way, part of the complexity of the original problem is effectively delegated to the QFM. This splitting into generic and application specific factors allows me to analyse the core aspects of fuzzy quantification without getting stuck in discussions which particular interpretation should be assigned to a given linguistic quantifier. From the perspective of applications, this division of labour is also of obvious interest, because we can now confine ourselves to simplified specifications of the intended quantifiers and need no longer be concerned with the intricacies of processing fuzzy arguments.

The particular choice of a semi-fuzzy quantifier suited for modelling a given linguistic quantifier can apparently depend on the context. To be sure, this is not that surprising because the precise meaning of quantifiers like "many" or "almost all" is obviously context-dependent as well and determined by contextual factors like an assumed standard of comparison, which is not explicitly given. Thus, the problem of context dependence it is not an artifact of my proposed modelling devices, but simply reflects a characteristic of language itself. Specifically, the problem will not go away if we switch to the familiar μ_Q-based representations. The notion of semi-fuzzy quantifiers has been designed such that all contextual factors fall into the realm of specification. Hence when describing the quantifiers of interest, the users must resolve these factors and commit to a single, unambiguous interpretation. In this way, the context-dependent factors are effectively isolated from the core problems of fuzzy quantification, i.e. from those factors that can be subjected to a general solution. This organization of fuzzy quantification permits me to develop the theory of fuzzy quantification independently of the intricacies of context dependence. From a practical perspective, though, it makes sense to assist the user or application designer who must decide on the precise meaning of these context-dependent quantifiers. This problem of determining a suitable choice of quantifiers is actually a special case of knowledge acquisition, and can hence be solved by suitable methods for the construction of membership functions. In the future, it might be possible to integrate some experimental findings on the effects of external factors like size or shape on the denotation of quantifiers that depend on the context [114, 113]. However, these ideas have not yet reached practical significance.

In order to make the basic framework useful in practice, it was necessary to further constrain the admissible mechanisms for interpretation to those choices which let us build reliable applications based on fuzzy quantifiers. QFMs, however, merely serve as a placeholder which lets us talk about models of fuzzy quantifications on a formal level. They do not account for any considerations regarding the linguistic quality of interpretations. Having introduced the QFM framework, I hence focused on the pretty complex problem how plausible choices of QFMs can be ascertained, i.e. models of fuzzy quantification which determine coherent interpretations and which best answer the linguistic expectations. To achieve this, it was necessary to state explicitly which choices of QFMs should be regarded plausible. Consequently, I investigated the characteristic properties of the intended models. Some of the concepts necessary to express these regularities were already known from TGQ, e.g. the constructions of antonym and dual, which only needed to be generalized to the fuzzy case. Some other criteria take care of the coherence of interpretations and the proper treatment of fuzziness. By turning these criteria into postulates for admissible models of fuzzy quantification, we can ensure that the 'essential' properties of quantifiers and their relationships be preserved when applying the fuzzification mechanism. This can be likened to the familiar mathematical concept of a homomorphism, i.e. of a structure-preserving mapping compatible with a number of relevant constructions. By compiling a system of such desiderata, we can encircle the envisioned class of plausible models. These requirements then comprise a catalogue of formal criteria against which every model of interest can be tested. In order to avoid redundant effort in proofs, the catalogue should further condensed into a succinct description of the admissible models in terms of an independent (and hence, minimal) system of axioms. In Chap. 3, I presented a complete system of such criteria, which also exhibits the desired minimality.

The semantical postulates (Z-1) to (Z-6) which I imposed on the intended models account for a number of very elementary requirements on plausible and coherent interpretations. The first criterion, (Z-1), requires the models to correctly generalize the original specification for a special case of quantifiers. The second criterion (Z-2) asserts that the models be compatible with membership assessments. Based on a canonical construction of induced fuzzy connectives, (Z-3) requires the compatibility of the interpretations with dualisation. The criterion (Z-4) in turn, makes sure that the models comply with unions of arguments. The requirement (Z-5) accounts for the desired monotonicity of interpretations, which is enforced for a special case. Finally (Z-6) links the interpretations obtained for different universes of quantification, by requiring the compatibility of the model with its matching choice of extension principle. As mentioned above, these axioms are known to be minimal, i.e. they capture independent aspects of linguistic adequacy. Taken together these axioms establish a type of plausible model for which I coined the term 'determiner fuzzification scheme', or DFS for short. In pretty much the same way that t-norms have proven a useful abstraction from the possible fuzzy conjunctions, my postulates for admissible interpretations are intended to identify a useful class of models for fuzzy quantification, which shows just the right degree of permissiveness. Hence the axioms should be broad enough to admit an interesting variety of models, but also sufficiently restrictive to to ensure that the computed interpretations be linguistically conclusive.

In order to judge if my semantical postulates (Z-1)–(Z-6) reach the goal of identifying a useful class of plausible models, it was necessary to consider various additional desiderata that can legitimately be expected of interpretations of fuzzy quantifiers. Specifically, I first showed that the models correctly generalize arbitrary specifications for crisp arguments, and that all of these models induce a plausible set of fuzzy connectives. I then showed that the models are compatible with a considerable number of operations on the arguments, thus preserving the essential argument structure of a given quantifier. In particular every DFS admits permutations of arguments, cylindrical extensions (augmentation with vacuous arguments), complementation (formation of antonyms), but also external negation and the formation of duals. In addition it admits intersections of arguments, their symmetrical difference, and the insertion of crisp arguments. The models further exhibit desirable monotonicity properties including the preservation of 'local' monotonicity for restricted ranges of arguments. The induced extension principle as well as the induced fuzzy inverse images are also reasonable. The models will plausibly interpret both quantitative and non-quantitative cases of quantifiers, and they will also preserve the property of 'having extension', which is shared by most linguistic quantifiers. In addition, the models are 'contextual', i.e. insensitive to the behaviour of a quantifier outside the context given by the fuzzy arguments. Finally the standard quantifiers will be interpreted plausibly in all of my models, judging from Thiele's characterization of fuzzy universal and existential quantifiers in terms of T- and S-quantifiers, respectively. The validity of these semantical requirements provides evidence that my core axioms are indeed useful because they entail the essential conditions. An investigation of these semantical criteria is also interesting in its own right. In the long run, these conditions will capture the intuitive expectations on plausible interpretations which should be answered by arbitrary models of fuzzy quantification, i.e. not only by a DFS. Specifically, I showed that catalog is also useful for evaluating the existing methods for fuzzy quantification, see appendix A. In this case, the criteria served to explain the covert inconsistencies of the traditional approaches.

To sum up, there is a variety of semantical properties shared by arbitrary choices of plausible models. Apart from studying such general characteristics of the models, it is also instructive to research into the structure of natural subclasses, i.e. collections of models grouped some property that they have in common. Due to the relative homogeneity of members in the same subclass, this strategy allowed me to define interesting constructions on these models. First of all I grouped the models by their induced negation, and I developed a model translation scheme which demonstrates that without loss of generality, we can focus on those models which induce the standard negation. Now grouping these $\neg$-DFSes by their induced disjunction, I further refined the class of these models. The resulting $\widetilde{\vee}$-DFSes are sufficiently homogeneous to permit model aggregations. For example, these classes are closed under symmetric sums, convex combinations etc. Next I assessed the structure of these classes by investigating a a natural order defined on these models. Due to the symmetry of a DFS with respect to negation, there is nothing like a 'least' or 'greatest' $\widetilde{\vee}$-DFS. However, the models are partially ordered by specificity, i.e. under the usual fuzziness order introduced by Mukaidono [110]. I showed that for non-empty collections of $\widetilde{\vee}$-models, a greatest lower bound in terms of specificity will always exist, which can be expressed in terms of the generalized fuzzy median. I further introduced the criterion of specificity consistency and proved that a non-empty collection of $\widetilde{\vee}$-DFSes has a least upper bound in terms of specificity exactly if it is specificity consistent. Again it is possible to give an explicit description of the resulting model. Finally I considered the class of standard DFSes, which induce the usual set of fuzzy connectives and thus comprise the standard models of fuzzy quantification. I presented an axiomatic description of this class of models and further ascertained various characteristics of its members. For example, I proved that for two-valued quantifiers, all standard models coincide with the fuzzification mechanism proposed by Gaines [44].

Few is known about non-standard models which depend on a different choice of fuzzy connectives. In an effort to keep open the possibility of non-standard models, I expressed the criteria (Z-1)–(Z-5) for plausible models in terms of the canonical construction of induced connectives, rather than resorting to the standard choices. For similar reasons, equality (Z-6) avoids the use of the standard extension principle, in favour of the natural extension principle induced by the given model. However, the research into these general (non-standard) models is still in its infant stage, and it is not clear at this point if the axioms really admit plausible models which induce a disjunction different from the standard choice, '$\max$'.

Having considered the essential properties shared by all models, as well as some natural subclasses of the models as well as constructions on such homogeneous models, I then discussed complementary properties of linguistic or practical interest. First of all, I formalized two distinct aspects of continuity, which ensure a certain stability of quantifications against changes in the quantifier and in its arguments. It also seems natural to assume that plausible models will 'propagate fuzziness', i.e. less specific input (quantifier or arguments) should not result in more specific interpretations. A suitable notion of specificity for quantifiers and their arguments can be derived from Mukaidono's fuzziness order. I further investigated the theoretical limits on the total catalogue of semantical properties that can simultaneously be verified by a QFM. Conjunctions and disjunctions of quantifiers, as well as the identification of several variables, are examples of constructions to which a QFM cannot be fully compatible even under much weaker conditions than the DFS axioms. The existence of such critical cases is not surprising, though, due to the known impossibility of satisfying all axioms of Boolean algebra in the continuous-valued case. It would be interesting to ascertain whether the identification of variables will result in more specific results under the fuzziness order $\preceq_c$. I would expect this kind of behaviour if a model propagates fuzziness, but this conjecture has not been proven yet. Two other semantical criteria must also be formulated very carefully, notably the preservation of convexity (which applies to quantifiers of unimodal shape), and the preservation of conservativity. In both cases, the original ‘naive’ rendering of the criterion for fuzzy sets turns out too strong. However, it is possible to give weaker definitions consistent with the core axioms, which still cover most cases of linguistic interest. These ‘realistic’ formalizations are validated by all of my plausible models. Finally I discussed the construction of fuzzy argument insertion, which is necessary for a compositional treatment of quantifiers like “many young” in “Many young A’s are B’s”. As I proved later on, there is only one standard model, $\mathcal{M}_{CX}$, which is compatible with this linguistic construction.

Due to the postulates for admissible mechanisms for interpretation, we know how the plausible models behave. However, we do not know at this point how actual instances of such models look like. In order to populate the abstract space of interpretations so defined with concrete models, I turned to the issue of developing prototypical examples. Knowing such models is essential from a practical perspective because it is impossible to evaluate actual quantifications in an application without committing to a particular model of fuzzy quantification. In order to avail us with an interesting range of examples, I investigated general constructions capable of expressing a diversity of models. In other words I was concerned with developing explicit classes of models, which are based on a common constructive principle. Two examples of such principles that come to mind are α-cutting and the resolution principle. However, these constructions cannot express the intended models due to their lack of symmetry with respect to negation. I therefore propose the use of three-valued cuts, which exhibit the desired symmetry. The three-valued subsets obtained from these cuts are in turn expanded into a range of crisp sets. By considering all choices of arguments in these cut ranges, we obtain the set $\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$ of possible quantifications at the cut-level $\gamma \in \mathbf{I}$. Thus my approach determines a set of alternatives which must then be aggregated into the final quantification result. This basic approach I refined into three classes of constructive models. My proposal of $\mathcal{M}_{\mathcal{B}}$-DFSes is obtained from a simple aggregation method based on the generalized fuzzy median $\mathrm{m}_{\frac{1}{2}}$. The median-based aggregation rests on a principle of ‘cautiousness’ or ‘least specificity’, and thus generalizes the basic strategy of supervaluationism [80, p. 24]. This method generates a class of pretty regular models with appealing formal properties. For example, less specific input cannot result in more specific outputs in these models, i.e. every $\mathcal{M}_{\mathcal{B}}$-DFS is known to propagate fuzziness. I further identified a model $\mathcal{M}_{CX}$ distinguished by its semantical properties, e.g. compatibility with fuzzy argument insertion and fuzzy adjectival restriction, weak preservation of convexity etc. The model can be shown to consistently generalize the Sugeno integral and hence the basic FG-count approach. My second class of $\mathcal{F}_{\xi}$-models replaces the median-based aggregation scheme with a more general construction based on upper and lower bounds $\top = \sup \mathcal{S}_{Q,X_1,\ldots,X_n}$ and $\bot = \inf \mathcal{S}_{Q,X_1,\ldots,X_n}$ on the alternatives in $\mathcal{S}_{Q,X_1,\ldots,X_n}$. The resulting models, which include the $\mathcal{M}_{\mathcal{B}}$-type, are less regularly structured. For example, they need not propagate fuzziness, i.e. less specific input can result in more specific interpretations. The class contains a model $\mathcal{F}_{\mathrm{Ch}}$ of particular interest, which consistently generalizes the Choquet integral and hence the core OWA approach to fuzzy quantification. Finally I discussed an even broader class of $\mathcal{F}_{\Omega}$-models which admits arbitrary constructions that operate on $\mathcal{S}_{Q,X_1,\ldots,X_n}$. The class is of minor practical relevance because all of its ‘robust’ members already belong to the $\mathcal{F}_{\xi}$-type. However, these $\mathcal{F}_{\Omega}$-models coincide with the $\mathcal{F}_{\psi}$-DFSes, i.e. with the natural class of models defined in terms of the standard extension principle. Thus, the broadest class that I described in this work is mainly of theoretical interest. It is an issue of future research to relate the $\mathcal{F}_\Omega$/$\mathcal{F}_\psi$-models to the full class of standard DFSes, and to develop even more general constructions if necessary to catch hold of all standard models. Moreover, it would be worthwhile finding constructions which give us a grip on non-standard models. This might eventually decide the issue if such non-standard models exist at all, which is currently unresolved. For practical purposes it is also important to identify those examples of models which are likely most useful in applications. In the report, the models $\mathcal{M}_{CX}$ and $\mathcal{F}_{\mathrm{Ch}}$ as well as a third example $\mathcal{M}$, which is some kind of compromise between $\mathcal{M}_{CX}$ and $\mathcal{F}_{\mathrm{Ch}}$, were chosen as prototypes which serve as the candidates for practical implementation.

The level of abstraction achieved by my theoretical skeleton and the clear separation of responsibilities, specification versus interpretation and model versus implementation, permitted me to develop the theoretical basis of fuzzy quantification at a fast pace. It was even possible to identify specific classes of prototypical models without getting involved into the practical concern how these models can eventually be implemented. This strict separation of theoretical analysis and computational aspects allowed me to resolve the semantical issues into a more elegant solution compared to previous approaches. However, it remained to be shown that the new 'nice' models also measure up with the well-known and semantically imperfect ones when it comes to efficient implementation. I hence had to solve the problem how actual quantifications can be carried out in the candidate models. In particular, it had to be shown how the involved calculations can be organized into efficient computational procedures. Only this will make a practical advance from what appears nice judging from the formulas. In order to provide the theory with sufficient computational backing, I hence developed efficient algorithms for implementing the main types of quantifiers in the new models. Specifically, I proved that for finite E, the computation of quantification results in $\mathcal{F}_\xi$-models can always be based on coefficients $\top_j$, $\bot_j \in \mathbf{I}$ obtained from a finite sample of cut levels γ_j, i.e. there is no need to consider all choices of γ in the unit range. I presented an analysis of 'quantitative' quantifiers in terms of cardinality information sampled from the arguments, and I went on explaining how the computation of $\top_j$ and $\bot_j$ can be optimized based on this analysis. I also showed how the required cardinality information can be computed efficiently from histograms of the involved fuzzy arguments and their Boolean combinations. The practical utility of this methodology was illustrated by deriving algorithms for implementing the main types of quantifiers in the prototypical models mentioned above, i.e. $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{\mathrm{Ch}}$. The issue which of these models should be preferred in a given application like information retrieval and database querying, multi-criteria decisionmaking, data summarization etc., should now be decided by practical experiments and applications in prototypical systems. In particular, it will be instructive to compare $\mathcal{M}_{CX}$, which generalizes the Sugeno integral, and $\mathcal{F}_{\mathrm{Ch}}$, which abstracts from the Choquet integral, and to substantiate which type of model is perceived more natural by the users.

13.2 Comparison to earlier work on fuzzy quantification

Let me now compare my own results to existing work on fuzzy quantication, in order to highlight the main contributions. Roughly speaking, my proposed solution offers the following conceptual advantages. The novel theory of fuzzy quantification is

- a compatible extension of the Theory of Generalized Quantifiers [6, 8];
- not limited to absolute and proportional quantifiers;
- a genuine theory of fuzzy multi-place quantification;
- not limited to finite universes of discourse;
- not limited to quantitative (automorphism-invariant) examples of quantifiers;
- based on a rigid axiomatic foundation which enforces plausible and coherent interpretations;
- fully compatible to the formation of negation, antonyms, duals, and other constructions of linguistic relevance.

These achievements will now be explained at some more length. First of all, the novel approach improves upon existing work as to the coverage of linguistic phenomena related to fuzzy quantification. The traditional framework is essentially limited to absolute and proportional quantifiers. However, the quantifiers found in NL are by no means restricted to the absolute and proportional types. Quite the reverse, the comprehensive classification of natural language quantifiers by Keenan & Stavi [82, pp. 253-256] distinguishes 16 main classes of quantifiers (among other things, quantifiers of exception like "all except one", definites like "the ten", bounding quantifiers like "only", etc). The wealth of these heterogeneous quantifiers is covered by TGQ, assuming crisp quantifications. My own theory embeds these cases as well as their generalization to type IV quantifications. It is true that in the report, I confined myself to formally defining and implementing the four main types of quantifiers, i.e. absolute quantifiers, proportional quantifiers, quantifiers of exception and cardinal comparatives. These classes are significant from the linguistic perspective and their implementations which I detailed in Chap. 11, are of obvious utility to the envisioned applications. Further types of quantifiers, like proportional comparatives, are described in the literature on TGQ, e.g. Keenan and Moss [81], Keenan and Stavi [82], or Keenan and Westerståhl [83]. The classification of (semi-)fuzzy quantifiers into categories of linguistic and practical value can be developed in total analogy to the existing classification of TGQ for precise quantifications. An implementation of these types in the prototypical models is straightforward from the general analysis expounded in Chap. 11. To sum up, the novel framework easily incorporates arbitrary types of NL quantifiers, while existing approaches which are closely intertwined with the specifics of absolute and proportional quantifiers, lack the desired coverage and extensibility.

In particular, these approaches do not avail us with uniform specifications suited for all quantifiers: It is different kinds of membership functions $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ vs.

$\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ that they postulate for representing absolute and proportional quantifiers. An extension to additional types of quantifiers will necessitate the introduction of further representation formats and corresponding membership functions. The assumed first-order representations merely depict the intended behaviour of the quantifier in graphical form. In particular, they fail to make explicit the quantificational structure, i.e. the precise way in which the arguments of the quantifier enter into the quantification results. The responsibility to account for argument structure is therefore delegated to the model of quantification: for each type of quantifier, there must be some part of the model which takes care of the particular kind of quantificational structure and the specific way in which the arguments must be combined to calculate the final quantification results. It is this structuring of fuzzy quantification which forces the traditional approaches to introduce one specification format per quantifier type. The lack of a uniform representation results in the scattering of traditional methods into several submodels, i.e. one rule for two-place proportional quantifiers, another rule for unary absolute quantifiers etc. Internal coherence then becomes a critical issue, due to the separate definitions for each type of quantifiers. Consequently, existing approaches to fuzzy quantification do not offer a good starting point for developing general models of NL quantification. The delegation of quantificational structure to the model (rather than treating it as part of the specification) obstructs the formalization of universal models which are capable of interpreting arbitrary quantifiers, including future, unexpected examples. As witnessed by the new framework for fuzzy quantification that I proposed in the report, these intricacies of existing approaches can be easily avoided, if quantificational structure is alloted its proper place, and viewed as part of the specification of a quantifier. The proposed semi-fuzzy quantifiers which underly my own analysis account for these considerations. In this case irregular details like the specifics of argument structure will now be encoded in the basic specifications of quantifiers. Without any need to alter the chosen representation medium, it hence remains possible to incorporate novel cases of quantifiers at any time, even if these do not match the known patterns of argument structure. In this way, semi-fuzzy quantifiers establish a uniform representation for arbitrary quantifiers. These uniform representations no longer permit any details of quantifiers to be hard-coded in the model. This frees the model from any idiosyncratic aspects of NL quantifiers, and lets it focus on the problem of incorporating fuzzy arguments. By isolating the core issues of fuzzy quantification from idiosyncratic aspects, the proposed uniform specifications in terms of semi-fuzzy quantifiers rendered possible the development of universal models. Unlike the specialized methods for interpreting fuzzy quantifiers previously described in the literature, the generic models which I discussed in this report accept arbitrary types of quantifiers and also support arbitrary ways of combining the arguments into resulting interpretations. Specifically, the uniform representation facilitates the development of models which also show the desired coherence across quantifiers types.

Let us now return to the issue of possible quantifier interpretations. Among other things, it is surely worth noticing that certain types of quantifiers can be three-place, e.g. “more than” in the quantifying proposition “More men than women are smokers” ($n = 3$). Such quantifications cannot be interpreted in terms of the absolute and proportional quantifiers supported by the traditional models, simply because “more than” and similar cases are known to be irreducible to two-place quantifiers, see Hamm [61, pp. 23+]. Composite quantifiers like “Most Y_1's or Y_2's are Y_3's and Y_4's” can involve an even higher number of arguments, which is potentially unbounded. It is therefore important to know that all of these cases be covered by my proposed analysis, which avails us with a general theory of fuzzy multiplace quantification. In particular the new models of fuzzy quantification achieve an adequate treatment of restricted quantification or ‘importance qualification’. To existing approaches, however, the complexity of restricted quantification has been prohibitive, and the counterexamples listed in Chap. 1 clearly demonstrate the failure of the cardinality-based methods. An extension of these approaches to ternary quantifiers has been suggested by Zadeh, who mentions the “third kind” of quantifiers in [188, p. 149] and [190, p. 757]. However, it appears that nobody pursued this direction further due to the difficulties of existing approaches with proportional quantifiers which involve only two (rather than three or more) arguments. My novel analysis does away with Zadeh's idea of using some measure of ‘relative cardinality’ to evaluate proportional quantification, and it also avoids ad-hoc devices like the ‘degree of orness’ used in [156, 170, 174]. In my theory, the notorious problem of proportional quantification weighted by importances is now solved in passing, because it offers a general solution for arbitrary two-place, and even for arbitrary multi-place quantifiers, which includes importance qualification as a special case.

Existing approaches to fuzzy quantification usually assume that the base set be finite.[45] My own analysis, by contrast, permits quantifiers on infinite domains, and every interpretation mechanism (QFM) must be well-defined in this case as well. It is true that the universes of quantification are typically finite in practical applications. Nevertheless the mastery of quantifications over infinite collections is of great practical value, because it lays the foundation for a modelling of fuzzy mass quantification. The examples of quantifiers which I focussed on in this work usually involved count nouns (“men”, “patients”, “Swedes”, . . .) and were thus concerned with concrete objects. Mass nouns like “water” or “wine” however, call for a different kind of quantification which ranges over continuous, possibly atomless masses like “some water” or “much wine”. An adequate modelling of quantification over such masses will necessarily involve infinite sets, of which masses are special instances. The proposed notion of QFMs and the plausible models (DFSes) developed from it, which support infinite universes of quantification, hence provide a good starting point for approaching fuzzy mass quantification. In practice, it might be necessary to add some more structure in order to deal with mass quantification. For example, it might be convenient to assume that E be a measurement space with a measure P defined on it. We can then define a proportional quantifier Q, intended to model “more than 30 percent” on E, by

$$Q(Y_1, Y_2) = \begin{cases} 1 & : \quad P(Y_1 \cap Y_2) \geq 0.3\, P(Y_1) \\ 0 & : \quad \text{else} \end{cases}$$

assuming that $Y_1, Y_2 \subseteq E$ be measurable subsets. In the case that Y_1 or Y_2 are not measurable, it is most natural to declare $Q(Y_1, Y_2)$ undefined. This suggests that quantification over masses will involve an extension to partial (rather than total) semi-fuzzy quantifiers, and a suitable notion of QFMs that map these to fuzzy quantifiers. Hence

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[45] The Σ-Count approach can be generalised to measurable fuzzy subsets of sets of infinite cardinality by replacing summation with integration [184, p. 167]. However, as we have seen in Chap. 1, the Σ-count approach does not offer a suitable departure for developing plausible models of fuzzy NL quantification.

the support for quantifiers on infinite base sets, which is already available in my current proposal, is an important prerequisite of treating mass quantification but it needs to be enhanced with a model of partial quantifiers and basic concepts of measure theory.

Unlike the approaches to fuzzy quantification described in the literature, the proposed theory also achieves an adequate account of non-quantitative quantifiers. By Th-25, every model of the DFS axioms maps quantitative semi-fuzzy quantifiers to quantitative fuzzy quantifiers (e.g. "almost all"). But the theorem also states that $\mathcal{F}$ maps non-quantitative semi-fuzzy quantifiers to corresponding *non-quantitative* fuzzy quantifiers. Some examples are proper names like **john** $= \pi_{\mathrm{John}}$, and non-quantitative composite quantifiers like "All X's except the men are Y's" or "All male X's are Y's". Existing approaches to fuzzy quantification, however, are defined in terms of absolute or relative cardinality measures for fuzzy sets, which must be invariant under automorphisms in order to adequately capture the notion of 'cardinality'. Consequently, the traditional models *always* result in quantitative quantifiers and are thus unable to model any non-quantitative cases. This failure of existing approaches gains some weight as soon as infinite base sets are admitted (e.g. in mass quantification). In this case most important quantifiers including e.g., "most", become non-quantitative in the sense of automorphism invariance, see van Benthem [8, p.473+].

Taken together, these observations demonstrate that from the perspective of coverage, only my new proposal conforms to the linguistic data, because it embeds the established Theory of Generalized Quantifiers. The traditional approaches to fuzzy quantification, by contrast, are too narrow from a linguistic perspective and for structural reasons, they bear the risk of producing incoherent results. The coverage of my new theory with respect to quantificational phenomena compared to that of TGQ and of existing approaches to fuzzy quantification, is summarized in Fig. 14.

Having explained the difference in scope, I would now like to discuss the issue of linguistic plausibility. With the notable exception of H. Thiele's analysis of specialized classes of quantifiers [149, 150], existing approaches to fuzzy quantification were mainly concerned with the development of new rules for interpreting fuzzy quantifiers, the linguistic plausibility of which was 'proven' with a few positive examples. My novel theory of fuzzy quantification, however, is based on a rigid axiomatic foundation which ensures the intended behaviour of the resulting models. Due to the considerable effort which I spent on the discovery and formalization of the intuitive criteria for plausible interpretations of quantifiers, these criteria will indeed ensure that reasonable and coherent results be obtained. The desired quality of interpretations can also be guaranteed across quantifier types, because of my uniform representations which fit arbitrary quantifiers. Concerning specific constructions of linguistic relevance, it the proposed models are fully compatible with Aristotelian squares, i.e. they admit the formation of external negation, antonyms, and duals. This property is offered by *none* of the earlier approaches. Furthermore, every DFS is compatible to the Piaget group of transformations, the significance of which stems from empirical findings in developmental psychology. For a note on the importance of the Piaget group of transformations to fuzzy logic, see Dubois & Prade [36, p. 158+]. In Chap. 4, various properties have been discussed which clearly distinguish the novel approach from ad-hoc interpretation rules. For example, the DFS models are known to 'preserve extension' by Th-26 and

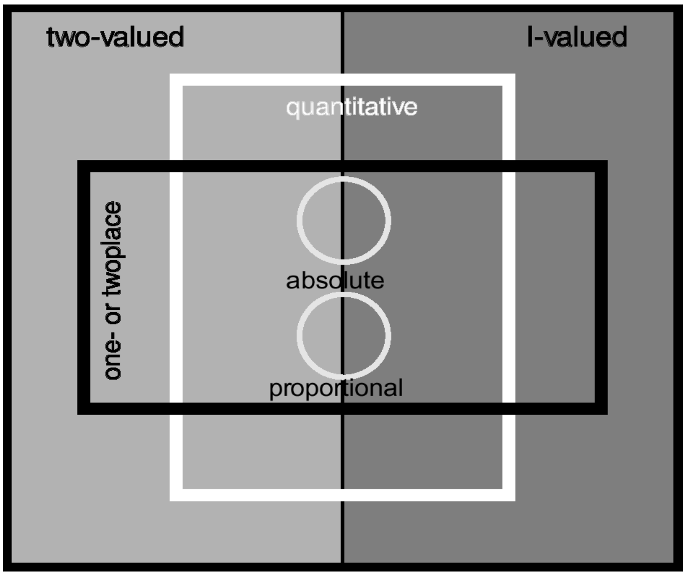

Figure 14: Coverage of linguistic quantifiers: DFS (outer rectange) vs. TGQ (left half) and existing approaches (circles)

hence reproduce the insensitivity of NL against the precise choice of domain. It should also be pointed out that by committing to a model of fuzzy quantification, a unique definition of all fuzzy connectives is induced. For example, every DFS is compatible with exactly one choice of conjunction, disjunction, implication, etc – there is no degree of freedom left in the definition of these connectives. Every DFS also induces a unique choice of extension principle and of fuzzy inverse images. Thus, the proposed models are not only capable of interpreting arbitrary quantifiers in the sense of TGQ and their generalizations to type IV quantifications. By fixing a coherent choice of fuzzy connectives and other basic constructs of fuzzy set theory, they also establish a consistent view of fuzzy set theory. The approaches described in the literature by contrast, are essentially limited compared to TGQ, and struggling with internal difficulties that stem from their lack of an axiomatic foundation.

13.3 Future perspectives

To sum up, the proposed theoretical framework is much more complete and linguistically faithful compared to earlier work. The axiomatic procedure in particular, i.e. the formalization of semantical postulates to constrain the admissible models, allowed me to investigate fuzzy quantification with unparalleled formal rigor, that has previously only been achieved for ‘special cases’ like the T- and S-quantifiers of Thiele [149] and his median quantifiers [150]. As to the coverage of possible quantifications, the proposed framework accounts for the majority of quantificational phenomena in natural language. But there remain a few cases not treated yet, that future research efforts should be directed into. I already mentioned the issue of quantification over masses (e.g. “much wine”), which necessitates an extension of my basic framework with measure-theoretic notions. Another topic that deserves more attention is fuzzy branching quantification. I have made some first steps towards the modelling of branching quantification in Chap. 12, where I introduced Lindström-like quantifiers into the fuzzy sets framework, which are capable of binding several variables. However it is not yet clear how branching quantifications can be implemented efficiently in the proposed models. In addition it is necessary to gain a better understanding of these powerful quantifications and develop additional criteria for plausibility to control them. The postulate of compatibility with nestings of quantifiers I consider a natural choice, but it appears to be extremely restrictive (possibly even conflicting with my core axioms). I therefore suggested that the criterion be restricted to cases of special linguistic relevance if necessary, thus assuming quantitativity, conservativity, monotonic or convex shape, etc. In order to achieve unrestricted compatibility with the nesting of quantifiers, it would also be possible to explore some alternatives to the core axioms. Specifically, the following criterion of *compatibility with inverse images* might be useful in this context, viz

$$\mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} f_i^{-1}) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}^{-1}(f_i)$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all mappings $f_i : E \longrightarrow E'$, $i \in \{1, \ldots, n\}$, see Def. 47. Like ‘functional application’ (Z-6), this criterion will ensure the systematicity of $\mathcal{F}$ across domains. Thus, it would be instructive investigating an alternative axiom system where (Z-6) is replaced with the novel postulate (and further criteria if necessary).

Another topic worth investigating is concerned with the epistemic difficulty of ascertaining the precise values of attributes like height, age, temperature etc. Rather than assuming exact knowledge of these attributes and thus infinite precision, a more realistic approach would admit imperfect knowledge of attribute values. The research into fuzzy quantification under these weaker assumptions has been initiated by H. Prade [118, 119], who approaches the problem in the setting of possibility theory. It would likely be useful to incorporate similar ideas into my core framework and consider its possible extensions to interval-valued quantifications.

Due to the open-endedness of language, which also shows up in its wealth of quantifiers, I was only able to develop the fundamentals of a theory of fuzzy NL quantification, and certain phenomena like intensional quantifiers had to be omitted to keep things manageable. Still the proposed analysis achieves a greater coverage than earlier work, and it also demonstrates a clear and rigorous treatment of its subject. In particular each of the criteria (Z-1) to (Z-6) for linguistic plausibility gives us a grip on an independent aspect of fuzzy quantification. Future research into such criteria will elucidate more and more dimensions of fuzzy quantification in natural language. If the basic analysis proposed in this work will be pursued further, we might eventually be able to fully reproduce the semantics of natural language quantifiers. And, we might also gain some insights into the linguistic regularities which elucidate the role of vagueness in human language and our daily problem solving. In this connection the understanding of fuzzy quantification marks a small but important step into the great wide open of human cognition.

Appendix

A An evaluation framework for approaches to fuzzy quantification

A.1 Motivation and chapter overview

Fuzzy quantifiers will not unfold their full potential for applications unless the models used in implementations are linguistically adequate and catch the intended meaning of NL quantifiers. For example, if an operator is labelled ‘most’, it is essential that it behave like the NL quantifier “most”, in order to avoid misunderstandings. In the introductory chapter to the report, I already sketched some examples of implausible interpretations for the ‘traditional’ models. Here, I will reconsider these examples and elaborate them in full detail. These observations on the existing approaches (as well as further evidence) will then be discussed from the perspective of linguistic plausibility. This is important because up to now, there are only sporadic results on this issue, e.g. in [3, 37, 124, 125, 175]. In particular, the traditional approaches have never been confronted with that analysis of NL quantifiers, which is now common in linguistics. An evaluation from a linguistic perspective is somewhat peculiar due to the different representations chosen by fuzzy set theory and linguistics (i.e., membership functions μ_Q vs. generalized quantifiers in the sense of TGQ). Accordingly, Zadeh [188] confines himself to mentioning TGQ, but does not attempt any cross-comparison; and there is no mutual influence between these research areas.

Based on the framework for fuzzy quantification presented in the main part of the report, it now becomes possible to conduct this evaluation of the traditional approaches to fuzzy quantification with respect to linguistic plausibility criteria. However, these approaches must first be fitted into the QFM format, i.e. some explicit means are needed to make the formal apparatus developed for QFMs applicable. To this end, I will present a canonical construction which associates with each ‘traditional’ approach a corresponding partial QFM. The construction is always possible if a natural condition is satisfied, which will be made precise in the so-called Evaluation Framework Assumption (EFA). The canonical construction will spare us any individual correspondence assertions: In dependence on the given traditional model, it will automatically determine a matching fuzzy quantifier for the semi-fuzzy quantifiers of interest. However, such manual assertions are also possible and can indeed be useful in some cases, e.g. if the EFA is violated, or if one wants to improve upon the standard construction. The case-by-case procedure, which then acts as a substitute for the canonical construction, still makes applicable the linguistic criteria developed in the report. It might therefore disclose some results concerning plausibility which would otherwise have remained unaccessible.

Having set up the evaluation framework, I will then discuss the main approaches to fuzzy quantification with the desired formal rigor. In particular, the conspicious example images shown in the introduction will now be analyzed in some more depth. Usually I will also add ‘minimal’ examples, which reproduce the critical behaviour for a handful of participating objects, and thus can be examined using paper and pencil.

All examples will be analyzed carefully, in order to unveil the precise causes of failure, which can usually be traced to a violation of the proposed adequacy criteria for QFMs. Because of these explanatory merits, I hope that my analysis will also be of interest to those who take sides with the traditional view of fuzzy quantification.

Note. The central results of this evaluation have been anticipated in Chap. 1, so there is some inevitable overlap of the introduction and the present chapter. For the sake of legibility, I have decided to accept an occasional repetition of concepts already mentioned in the introduction. In particular, the counter-examples will now be developed from scratch and presented in full detail, without any reference to their summaries in the first chapter.

A.2 The evaluation framework

In this section, I will introduce the formal framework suited for evaluating the existing approaches to fuzzy quantification. Specifically, the framework is taylored to those 'traditional' interpretation mechanisms which rest on Zadeh's representation of fuzzy linguistic quantifiers in terms of fuzzy subsets of the non-negative reals or of the unit interval. In order to make a uniform framework possible, it is convenient to fit these approaches in the following general structure.

The approach of interest, which will be symbolized by $\mathcal{Z}$, must be defined both for absolute quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ (Zadeh's quantifiers of the first kind) and relative/proportional quantifiers $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ (Zadeh's quantifiers of the second kind).[46] In each case, it is necessary to model both the unrestricted use of the quantifier, where the quantification involves only one explicit and possibly fuzzy argument ("Most things are tall"), and the two-place use, where the quantification can be fuzzily restricted, as in "Most young are poor". Labelling the absolute and proportional cases by *abs* and *prp*, respectively, and also marking the unrestriced and restricted uses by superscripts (1) and (2), a 'full' approach $\mathcal{Z}$ to fuzzy quantification in Zadeh's setting can then be specified by defining the following interpretation mechanisms for all choices of finite domains E:

- $\mathcal{Z}^{(1)}_{\mathrm{abs}}$, which maps $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ to $\mathcal{Z}^{(1)}_{\mathrm{abs}}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, for modelling unrestricted absolute quantification;
- $\mathcal{Z}^{(2)}_{\mathrm{abs}}$, which maps $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ to $\mathcal{Z}^{(2)}_{\mathrm{abs}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, for modelling restricted absolute quantification;
- $\mathcal{Z}^{(1)}_{\mathrm{prp}}$, which maps $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ to $\mathcal{Z}^{(1)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, for modelling unrestricted proportional quantification;
- $\mathcal{Z}^{(2)}_{\mathrm{prp}}$, which maps $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ to $\mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$, for modelling restricted proportional quantification.

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[46] See Zadeh [188, p. 149] for the distinction of quantifiers of the first kind (based on absolute counts) and those of the second kind (based on relative counts).

The approaches described in the literature do not always define the complete scheme, and I will therefore allow that certain cases be omitted. In addition, the definition of

$$\mathcal{Z}^{(2)}_{\mathrm{abs}}(\mu_Q)(X_1, X_2) = \mathcal{Z}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1 \cap X_2) \tag{249}$$

is usually assumed without mention, and the two-place use of absolute quantifiers only shows up in examples which demonstrate the application of these approaches. Among other things, the above equality will reduce "About fifteen children cry" to "About fifteen 'things' are children and cry", because

$$\mathcal{Z}^{(2)}_{\mathrm{abs}}(\mu_{\textbf{about 15}})(\textbf{children}, \textbf{cry}) = \mathcal{Z}^{(1)}_{\mathrm{abs}}(\mu_{\textbf{about 15}})(\textbf{children} \cap \textbf{cry})$$

Finally, $\mathcal{Z}^{(1)}_{\mathrm{prp}}$ can be expressed in terms of $\mathcal{Z}^{(2)}_{\mathrm{prp}}$ whenever the latter mechanism is available, and then becomes

$$\mathcal{Z}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) = \mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_Q)(E, X)\,.$$

For example, "Most things are expensive" can be interpreted thus,

$$\mathcal{Z}^{(1)}_{\mathrm{prp}}(\mu_{\textbf{most}})(\textbf{expensive}) = \mathcal{Z}^{(2)}_{\mathrm{prp}}(\mu_{\textbf{most}})(E, \textbf{expensive})$$

where $\mu_{\textbf{most}}(x) = 1$ for $x > 0.5$ and 0 otherwise, and $X \in \textbf{expensive} \in \widetilde{\mathcal{P}}(E)$.

Now that we have a uniform notation for the approaches of interest, we can turn to the issue of developing the evaluation framework. The basic problem to be solved is that the concepts developed in the main part of the report are not directly applicable to existing approaches. The point is that approaches based on fuzzy linguistic quantifiers are not defined in terms of a quantifier fuzzification mechanism, and hence fail to establish a direct interpretation of semi-fuzzy quantifiers. We have to bridge the gap between semi-fuzzy quantifiers and fuzzy linguistic quantifiers in a systematic way. To this end, I recall the notion of an *underlying semi-fuzzy quantifier* $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, which is nothing but the restriction of the given fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ to crisp arguments, see Def. 7. Now let us consider one of the approaches $\mathcal{Z}$ based on fuzzy linguistic quantifiers. We cannot expect that $\mathcal{Z}$ give rise to a 'full' (totally defined) quantifier fuzzification mechanism. Compared to the scope of a QFM, which aims at NL quantification in general, the quantificational phenomena addressed by existing approaches are simply too limited, and cover only a fraction of the possible cases. In other words, $\mathcal{Z}$ is not sufficient to determine a total QFM defined for arbitrary quantifiers. However, it is often possible to reconstruct a *partially defined* quantifier fuzzification mechanism $\mathcal{F}$ based on $\mathcal{Z}$ as follows. Given the membership function μ_Q of a fuzzy linguistic quantifier, we first obtain the corresponding semi-fuzzy quantifier relative to $\mathcal{Z}$ as $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$, and use this to define $\mathcal{F}(Q) = \mathcal{Z}(\mu_Q)$. Obviously, the construction of $\mathcal{F}$ succeeds only if $\mathcal{U}(\mathcal{Z}(\mu_Q)) \mapsto \mathcal{Z}(\mu_Q)$ is functional, but this is a plausible requirement anyway. It will be called the *evaluation framework assumption* (EFA).

- The EFA is very closely related to the QFA, which underlies the quantification framework described in the main part of the report, see section 2.7. To be specific, the QFA states that every base quantifier $\widetilde{Q}$ of interest is uniquely determined by its behaviour on crisp arguments. By contrast, the EFA requires that those quantifiers $\widetilde{Q} = \mathcal{Z}(\mu_Q)$ which result from the considered interpretation mechanism $\mathcal{Z}$ are fully specified by their quantification results for crisp arguments. If we assume that the mechanism $\mathcal{Z}$ is linguisticially plausible, i.e. all $\widetilde{Q} = \mathcal{Z}(\mu_Q)$ are potential denotations of NL quantifiers, then the EFA demands that the QFA be valid for all quantifiers in the responsibility of $\mathcal{Z}$, and hence comes out as the QFA in disguise.

- In case the EFA holds unconditionally for $\mathcal{Z}$, we can use the constructed partial fuzzification mechanism $\mathcal{F}$ to establish or reject the preservation and homomorphism properties of interest. We only need to take care of the fact that we now have a partial QFM, rather than a totally defined QFM, and adapt the adequacy conditions accordingly, which were originally developed for total QFMs. For example, a partial QFM $\mathcal{F}$ is said to preserve negation if $\mathcal{F}(Q)$ is defined exactly if $\mathcal{F}(\neg Q)$ is defined; and in case both are defined, we have $\mathcal{F}(\neg Q) = \neg\mathcal{F}(Q)$. As we shall see below, the required modifications to these criteria are only too obvious, and will not cause any kind of difficulty.
  When evaluating the existing approaches, it will be beneficial not to insist on the induced connectives, which were needed to set up the generic quantification framework. in the case of the generic quantification framework. With the traditional approaches, I consider the standard connectives more appropriate, because the literature on these approaches generally assumes this particular choice of connectives. In addition, it is not always unambiguously clear how to define the induced connectives from these approaches, and the decision to refer to the standard choices helped me to keep things simple, and consistent with common practice. Consequently, I will generally assume the standard truth functions (negation $1 - x$, conjunction $\min$, disjunction $\max$ etc.) in this chapter, and it is understood that all constructions on (semi-) fuzzy quantifiers like dualisation etc. be defined in terms of these connectives.

- If the EFA is violated by $\mathcal{Z}$, the concepts developed for QFMs can still be useful to assess certain properties of $\mathcal{Z}$. In this case, the EFA can always be enforced by restricting the set of considered membership functions μ_Q, thus eliminating the conflicting cases. To see this, suppose $E \neq \varnothing$ is a base set, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, and let us abbreviate

  $$\mathrm{Ch}(Q) = \{\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I} \,:\, \mathcal{U}(\widetilde{Q}) = Q \wedge \text{ exists } \mu_Q \text{ s.th. } \widetilde{Q} = \mathcal{Z}(\mu_Q)\}\,.$$

  If the EFA holds, then $\mathrm{Ch}(Q)$ is either empty, in which case $\mathcal{F}(Q)$ is undefined, or it is a singleton set $\mathrm{Ch}(Q) = \{\widetilde{Q}\}$, in which case $\mathcal{F}(Q)$ is defined and $\mathcal{F}(Q) = \widetilde{Q}$. If the EFA does not hold, then $\mathrm{Ch}(Q)$ has more than one element in some cases. By reducing the set of considered μ_Q, we can always obtain a reduced system in which all $\mathrm{Ch}(Q)$ – now determined from the restricted set of μ_Q – are singleton or empty. We can hence view $\mathrm{Ch}(Q)$ as providing a number of choices for $\mathcal{F}(Q) \in \mathrm{Ch}(Q)$.
  It therefore makes sense to say that $\mathcal{Z}$ can *represent* a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ if there exists some μ_Q such that $Q = \mathcal{U}(\mathcal{Z}(\mu_Q))$. We can then refute a property of interest by proving that $\mathcal{Z}$ cannot represent Q without violating the property.

Now that the basic framework has been installed, it will be applied in order to assess the plausibility of the main approaches to fuzzy quantification, viz the Σ-count, FG-count, OWA- and FE-count models. When presenting counter-examples in the evaluation to follow, I will generally prefer precise quantifiers like "all" or $[\textbf{rate} \geq r]$, because I assume that there are strong intuitions about the intended behaviour of these quantifiers, and my claims that certain results are counter-intuitive should be uncontroversial. Of course, this does not mean that I am only interested in two-valued quantifiers.

A.3 The Sigma-count approach

Let us first consider the Σ-count approach already mentioned in the introduction. The model rests on the notion of a *Σ-count*[47] [31], which is is defined as the sum of the membership values of a given fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ (E finite), thus

$$\Sigma\text{-Count}(X) = \sum_{e \in E} \mu_X(e)\,.$$

The Σ-count is claimed to provide a (coarse) summary of the cardinality of the fuzzy set X, expressed as a non-negative real number. A corresponding scalar definition of fuzzy proportion, the *relative Σ-count* or *relative cardinality* [180], is defined by

$$\Sigma\text{-Count}(X_2/X_1) = \Sigma\text{-Count}(X_1 \cap X_2) / \, \Sigma\text{-Count}(X_1)\,.$$

The Σ-count approach to fuzzy quantification, introduced by Zadeh [184, 188], uses Σ-Count(X) and Σ-Count(X_2/X_1) to model the absolute and proportional kinds of fuzzy linguistic quantifiers. The two types of quantifiers are hence treated differently. As explained above, the unrestricted and restricted interpretations of both types of quantifiers must be discerned, i.e. quantification relative to the domain as a whole, or relative to an explicitly given, and possibly fuzzy, restriction. Referring to my notation for the mechanism $\mathcal{Z} = \text{SC}$ ('Sigma Count'), the definition of the Σ-count approach can now be stated as:

$$\begin{aligned}
&\text{SC}^{(1)}_{\text{abs}}(\mu_Q)(X) = \mu_Q\left(\Sigma\text{-Count}(X)\right)\\
&\text{SC}^{(2)}_{\text{abs}}(\mu_Q)(X_1, X_2) = \text{SC}^{(1)}_{\text{abs}}(\mu_Q)(X_1 \cap X_2)\\
&\text{SC}^{(1)}_{\text{prp}}(\mu_Q)(X) = \text{SC}^{(2)}_{\text{prp}}(\mu_Q)(E, X)\\
&\text{SC}^{(2)}_{\text{prp}}(\mu_Q)(X_1, X_2) = \mu_Q\left(\Sigma\text{-Count}(X_2/X_1)\right)
\end{aligned}$$

Note. The split definition in terms of four separate formulas simply instantiates the generic scheme for a given mechanism $\mathcal{Z}$ that was presented above. In particular, the 'abs'-versions apply to the absolute kind where $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$, and the 'prp'-versions to the proportional kind $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$. Furthermore, the superscripts (1) and (2) denote unrestricted and restricted quantification, respectively.

Now that the definition of the Σ-count approach has been presented in the assumed format, we can set out to investigate the linguistic plausibility of the approach. To

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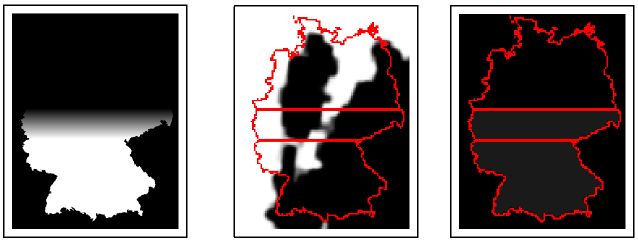
[47] also known as the 'power' of a fuzzy set

begin with, the Σ-count approach is somewhat peculiar because (unlike the other approaches), it does not comply with the EFA. This is most apparent with absolute fuzzy linguistic quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$: The assumption is then violated by any pair of membership functions $\mu_Q, \mu_{Q'} : \mathbb{R}^+ \longrightarrow \mathbf{I}$ with $\mu_Q|_\mathbb{N} = \mu_{Q'}|_\mathbb{N}$ but $\mu_Q \neq \mu_{Q'}$. The trouble is that with absolute quantifiers, the Σ-count approach requires the specification of quantification result $\mu_Q(x)$ in the case that the computed Σ-count x is not a cardinal number. The decision on which quantification results to assign for such x affects the results obtained, but intuitions are scarce in this unfamiliar case.

Next I will recast Yager's example [175, p.257] on counter-intuitive behaviour of the Σ-count approach in my setting. Hence suppose that $E = \{\mathrm{Hans}, \mathrm{Maria}, \mathrm{Anton}\}$ is a set of persons, and using Zadeh's notation, that the fuzzy subset **blond** $\in \widetilde{\mathcal{P}}(E)$ is defined by **blond** $= \frac{1}{3}/\mathrm{Hans} + \frac{1}{3}/\mathrm{Maria} + \frac{1}{3}/\mathrm{Anton}$, i.e. all three are blond to a degree of $\frac{1}{3}$. Now consider the quantifying statement "There is exactly one blond person". Its interpretation in the Σ-count approach is $\mathrm{SC}^{(1)}_{\mathrm{abs}}(\mu_Q)($**blond**$)$, where Q is a quantifier suited to model (the unrestricted use of) "exactly one". In order for the condition of *correct generalisation* to be respected, we are forced to have $\mu_Q(1) = 1$. It follows that the above statement evaluates to 1 (fully true), although there is clearly *not* exactly one blond person in the base set (which one should that be?) but rather *a total amount of blondness* of one, as one might say, i.e. nothing of linguistic significance.[48]

A similar example from the image domain is presented in Fig. 15. In this case, the Σ-count approach is used to evaluate the condition that "About 10 percent of Southern Germany are cloudy"; and it is assumed that $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ is chosen such that, say, $\mu_Q(x) = 1$ in the range $x \in [0.08, 0.12]$, and that μ_Q decays to zero outside this range (the precise shape of μ_Q in latter case is inessential to the example). As shown in Fig. 15, both cloudiness situations (a) and (b) are considered cloudy to a degree of one (fully true) if the Σ-count approach is used to evaluate "About 10 percent of Southern Germany are cloudy". While a result of 1 is plausible in case (a), this is not the case in situation (b), in which *all* of Southern Germany is cloudy to a very low degree (viz. one tenth), which certainly does not mean that one tenth of Southern Germany is cloudy.



Southern Germany (a) SC: 1 (OK) (b) SC: 1 (implausible)

Figure 15: About 10 percent of Southern Germany are cloudy (Sigma-Count)

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[48] Zadeh [188] proposes to apply some threshold on X in these cases, in order to prevent accumulative effects of small membership grades, but I consider this an ad-hoc device because there is no obvious way in which any particular choice of threshold can be justified.

'Trivial' or 'degenerate' cases often require particular attention. One such case is that of a quantifier supplied with an argument tuple of empty sets. As I will now show, the Σ-count approach does not treat this case consistently. To see this, let us choose $0 < r_2 < r_1 < 1$ and consider the two-valued proportional quantifiers $Q_1 = [\textbf{rate} \geq r_1]$, $Q_2 = [\textbf{rate} > r_2] : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ (see Def. 3), which have $Q_1(\varnothing, \varnothing) = 1$, but $Q_2(\varnothing, \varnothing) = 0$. The problem is that Zadeh does not specify the denotation of Σ-Count$(\varnothing/\varnothing)$. So let us assume that Σ-Count$(\varnothing/\varnothing) = c \in \mathbf{I}$; further suppose that μ_{Q_1}, $\mu_{Q_2} : \mathbf{I} \longrightarrow \mathbf{I}$ are used to model Q_1 and Q_2, respectively. By not satisfying the EFA, the Σ-count approach leaves us with some choices which μ_{Q_1}, μ_{Q_2} to use as a model of Q_1 and Q_2. However, every reasonable choice of μ_{Q_1} should have $\mu_{Q_1}(x) = 1$ only if $x \geq r_1$, and $\mu_{Q_2}(x) = 0$ only if $x \leq r_2$.[49] *Correct generalization* then demands that $\mu_{Q_1}(c) = 1$, i.e. $c \geq r_1$, but also that $\mu_{Q_2}(c) = 0$, i.e. $c \leq r_2$, which contradicts the assumption that $r_2 < r_1$.

Let me further remark that the Σ-count approach yields potentially satisfying results only if μ_Q is genuinely fuzzy, because a two-valued quantifier (with corresponding two-valued membership function μ_Q) is mapped to a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{2}$, the results of which are *always crisp*.[50] One might object that, although a two-valued quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is to be modelled, an adequate choice of μ_Q should be continuous-valued. For example, if $E = \{a, b\}$, can't we model the universal quantifier $\forall$ by

$$\mu_Q(x) = \begin{cases} 0 & : \quad x \leq \frac{1}{2} \\ 2x - 1 & : \quad x > \frac{1}{2} \end{cases}$$

rather than using the two-valued membership function $\mu_\forall : \mathbf{I} \longrightarrow \mathbf{I}$,

$$\mu_\forall(x) = \begin{cases} 0 & : \quad x < 1 \\ 1 & : \quad x = 1 \end{cases}$$

which results in a crisp operator when applying $\mathrm{SC}^{(1)}_{\mathrm{prp}}$? To see that this objection is invalid, let us assume that the two-valued quantifier Q to be 'fuzzified' is of the proportional type; I will utilize the fact that such quantifiers have extension.[51] Now suppose that $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ is a proper choice for interpreting Q, and $q \in \mathbb{Q} \cap \mathbf{I}$ is some rational number in $\mathbf{I}$. Firstly, we can choose $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ such that Σ-Count$(X_2/X_1) = q$. Because q is rational and nonnegative, there exist $z, m \in \mathbb{N}$ s.th. $q = z/m$; we may also require that $m \geq |E|$. We extend E by arbitrary elements to some superset $E' \supseteq E$ with $|E'| = m$, and choose an arbitrary *crisp* subset $Z \in \mathcal{P}(E')$ with $|Z| = z$. *Correct generalisation* then yields

$$\mu_{Q_{E'}}(q) = \mu_{Q_{E'}}(\Sigma\text{-Count}(Z/E)) = Q_{E'}(E, Z) \in \mathbf{2}\,.$$

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[49] it will be shown below that in order to have the Σ-count preserve extension, these conditions on $\mu_{Q_1}(x)$ and $\mu_{Q_2}(x)$ must hold for all $x \in \mathbf{I} \cap \mathbb{Q}$. If we also require that monotonicity properties be preserved, this result extends to the full range $x \in \mathbf{I}$.

[50] This problem has been obscured by Zadeh's use of the quantifier **most**, which he views as being genuinely fuzzy.

[51] i.e. f, v_0 in Def. 166 can be chosen independently of E. This is apparent from equality (23).

In order to avoid a violation of the semantic requirement of *extensionality* (see Def. 40), which would disqualify the Σ-count approach anyway, we are then justified to assume that

$$\mu_{Q_E}(\Sigma\text{-Count}(X_2/X_1)) = \mu_{Q_{E'}}(\Sigma\text{-Count}(X_2/X_1)) = \mu_{Q_{E'}}(q) \in \mathbf{2}\,.$$

The membership functions suited for modelling two-valued proportional quantifiers are therefore restricted to $\{0, 1\}$ on $\mathbf{I}\cap\mathbb{Q}$. The option of selecting intermediate membership grades in the open interval $(0, 1)$ for the remaining 'definition gaps' on $\mathbf{I} \setminus \mathbb{Q}$ has few practical relevance. In particular, the Σ-count approach produces two-valued results, and thus cannot determine useful *gradual* evaluations, in the case of frequently used quantifiers like "all" and $[\textbf{rate} \geq r]$. This peculiarity of the Σ-count approach is undesirable in at least two ways.

Firstly, there is a well-known relationship between conjunction and universal quantification, which I already mentioned in section 4.16: a conjunction $c_1 \wedge \cdots \wedge c_m$ of m two-valued criteria $c_1, \ldots, c_m \in \mathbf{2}$ corresponds to the quantified statement "All criteria $c_1, \ldots, c_m$ are true", i.e.

$$c_1 \wedge \cdots \wedge c_m = \forall_E(C)$$

where $E = \{1, \ldots, m\}$ and $C = \{j : c_j = 1\}$. We should expect this relationship between conjunction and universal quantification to be preserved in the fuzzy case. (As it is in all models of DFS theory – see theorem Th-30). The point is that $\mu_\forall : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\mu_\forall(1) = 1$, $\mu_\forall(x) = 0$ otherwise, results in

$$\mathrm{SC}^{(1)}_{\mathrm{prp}}(\mu_\forall)(X) = \begin{cases} 1 & : \quad X = E \\ 0 & : \quad \text{else} \end{cases}$$

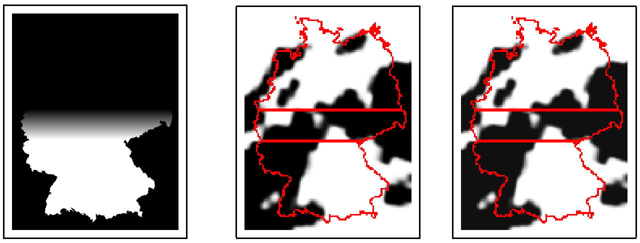
In particular, if $E = \{1, 2\}$, we obtain the induced conjunction operator $c : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ defined by $c(x_1, x_2) = \mathrm{SC}^{(1)}_{\mathrm{prp}}(\mu_\forall)(X)$, where $X = x_1/1 + x_2/2$, i.e.

$$c(x_1, x_2) = \begin{cases} 1 & : \quad c_1 = c_2 = 1 \\ 0 & : \quad \text{else} \end{cases}$$

This is certainly not a reasonable fuzzy conjunction operator (in particular, it is not a t-norm).

Apart from this observation that the Σ-count approach assigns an implausible interpretation to the universal quantifier, there is another undesirable consequence of the fact that two-valued μ_Q are mapped to fuzzy quantifiers $\mathrm{SC}(\mu_Q)$ which always return crisp results: such quantifiers are discontinuous, i.e. very slight changes in the membership grades of their arguments can drastically change the quantification result. More generally, we observe that whenever μ_Q is discontinuous, then the corresponding fuzzy quantifier under the Σ-count approach is also a discontinuous function of the membership grades of its argument sets. In practical applications, there is almost always some amount of noise (e.g. due to the finite precision of floating point operations), which can have drastic effects when using the Σ-count approach for modelling this kind of quantifiers.

An example which demonstrates this sensitivity to noise is presented in Fig. 16. In this case, image (b) is a slightly modified version of image (a) – all pixels with a cloudiness grade of 0 have been set to a slightly higher grade. Although the difference is small and hardly perceptible, the Σ-count approach 'jumps' from 0 to 1 when we move from the cloudiness situation (a) to the slightly modified situation depicted in (b).



Southern Germany (a) Result: 0 (b) Result: 1

Figure 16: At least 60 percent of Southern Germany are cloudy (Sigma-Count)

Finally, the Σ-count is not compatible with antonyms, as pointed out by Zadeh [188, p. 167]. Noticing that the Σ-count method admits external negation, we conclude that the approach is also not compatible with dualisation (which can be decomposed into the antonym of the negation).

A.4 The OWA approach

In this section, we shall review Yager's [170, 175] approach to fuzzy quantification, which is named after its use of ordered weighted averaging (OWA) operators. Only the proportional type $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ of fuzzy linguistic quantifiers is considered. In addition, μ_Q is assumed to be *regular nondecreasing*, i.e. $\mu_Q(0) = 0$, $\mu_Q(1) = 1$, and $\mu_Q(x) \leq \mu_Q(y)$ for all $x, y \in \mathbf{I}$ such that $x \leq y$. In order to define the OWA approach, some more notation must be introduced. Given a finite base set $E \neq \varnothing$, $m = |E|$ and $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, let us define $\mu_{Q,E} : \{0, \dots, m\} \longrightarrow \mathbf{I}$ by

$$\mu_{Q,E}(j) = \mu_Q(\tfrac{j}{m})$$

for $j = 0, \dots, m$. Let us also recall the coefficient $\mu_{[j]}(X) \in \mathbf{I}$ introduced in Def. 99, which denotes the j-th largest membership value of X (including duplicates). We can then define the OWA approach for unrestricted proportional quantification by

$$\mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) = \sum_{j=1}^{m} (\mu_{Q,E}(j) - \mu_{Q,E}(j-1)) \cdot \mu_{[j]}(X)\,,$$

where $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ is regular nondecreasing, and $X \in \widetilde{\mathcal{P}}(E)$. In order to model two-place quantifiers, Yager introduces a weighting formula parametrised by the so-called *degree of orness* of the considered operator, which is defined by

$$\mathrm{orness}(\mu_{Q,E}) = \frac{1}{m-1}\sum_{j=1}^{m}(m-j)(\mu_{Q,E}(j) - \mu_{Q,E}(j-1)) \qquad (250)$$

(adapted to my notation). In other words,

$$\mathrm{orness}(\mu_{Q,E}) = \frac{1}{m-1}\sum_{j=1}^{m-1}\mu_{Q,E}(j)\,.$$

In addition, a weight function $g_\alpha : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is needed, which depends on the degree of orness $\alpha = \mathrm{orness}(\mu_{Q,E})$ of the given μ_Q. This weight function is defined by

$$g_\alpha(x_1, x_2) = (x_1 \vee (1-\alpha)) \cdot {x_2}^{x_1 \vee \alpha}\,,$$

for all $x_1, x_2 \in \mathbf{I}$.[52] The weighting function must now be extended to fuzzy arguments: for fuzzy sets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, we define $g_\alpha(X_1, X_2) \in \widetilde{\mathcal{P}}(E)$ pointwise by

$$\mu_{g_\alpha(X_1,X_2)}(e) = g_\alpha(\mu_{X_1}(e), \mu_{X_2}(e))\,,$$

for all $e \in E$. Based on these concepts, we can now succinctly state the formula for two-place quantification in the OWA framework proposed in [170]. The fuzzy quantifier $\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ determined by the OWA approach can then be expressed as

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q) = \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(g_\alpha(X_1, X_2))\,,$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where μ_Q is regular nondecreasing and $\alpha = \mathrm{orness}(\mu_{Q,E})$.

Let me note in passing that the degree of orness is *undefined* when the cardinality of the base set is $|E| = 1$. Consequently the OWA approach is unable to provide an interpretation of two-place quantifiers in the case of singleton base sets.

On the positive side, it is easily observed that the OWA approach fulfills the EFA. But there is negative evidence as regards its adequacy. Firstly, $\mathrm{orness}(\mu_{Q,E})$ typically depends on the cardinality of E. It follows that most two-place fuzzy quantifiers $\widetilde{Q} = \mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)$ do not have extension.[53] To provide an example, let us consider $\mu_{[\mathrm{rate}>0.5]} : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\mu_{[\mathrm{rate}>0.5]}(x) = 1$ if $x > 0.5$, and 0 otherwise. Further suppose $E = \{\mathrm{Joan}, \mathrm{Clarissa}\}$ is a set of persons, **men** $= \varnothing$ is the set of those persons in E which are men (this happens to be empty), and **rich** $= E$ is the fuzzy set of persons in E which are rich (this happens to be crisp and coincide with E). Then, because $\mu_{[\mathrm{rate}>0.5],E} = \mu_{\forall,E}$,

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.5]})(\textbf{men}, \textbf{rich}) = \mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_\forall)(\varnothing, E) = 1\,.$$

Let us now extend E to the larger set $E' = \{\mathrm{Joan}, \mathrm{Clarissa}, \mathrm{Mary}\}$, and let us suppose that the extensions of 'men' and 'rich' remain unchanged in E', i.e. $\mathrm{Mary} \notin$ **men** and

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[52] We shall assume that $0^0 = 1$, i.e. $g_0(0, 0) = 1$.

[53] The effect is of practical relevance only if $|E|$ is small.

$\mathrm{Mary} \notin$ **rich**. We should then expect, because the quantifier "more than half" has extension, that the quantification result does not change when we replace E with the larger base set E'. However, application of OWA in the extended domain E' yields

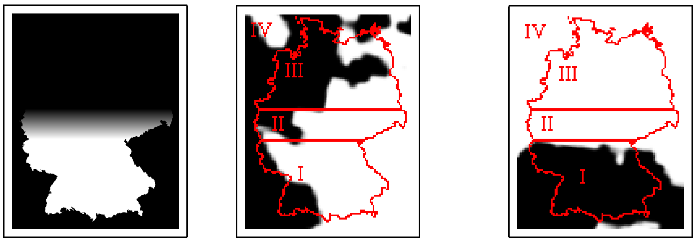
$$\begin{aligned}&\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.5]})(\textbf{men}, \textbf{rich})\\&= \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.5]})(0.5/\mathrm{Joan} + 0.5/\mathrm{Clarissa} + 0/\mathrm{Mary})\\&= 0.5\,.\end{aligned}$$

The observed change in $\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.5]})$ is clearly implausible. It is caused by the change in the degree of orness of $\mu_{[\mathrm{rate}>0.5]}$ from $\alpha = 0$ (in E) to $\alpha = 0.5$ (in E'). Similar effects are observed with most other $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, which also result in a failure of OWA with respect to the property of having extension.

This is not the only flaw of this approach, though. Another defect of $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$, which I consider even more serious, can be demonstrated for all quantifiers with 'intermediate' degrees of orness, i.e. $\alpha = \mathrm{orness}(\mu_{Q,E}) \in (0, 1)$.[54] In this case

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)(\varnothing, \varnothing) = 0 \neq (1 - \alpha) = \mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)(\varnothing, E)\,, \qquad (251)$$

which shows that, except "all" and "some", no semi-fuzzy quantifier $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ which can be represented by OWA is conservative. In particular, *the OWA approach cannot represent any proportional semi-fuzzy quantifiers except for "all" and "some"*.[55,56] But this is exactly the type of quantifiers OWA is intended to model.



Southern Germany (a) desired: 1, OWA: 0.1 (b) desired: 0, OWA: 0.6

Figure 17: At least 60 percent of Southern Germany are cloudy

If we still try to use OWA for interpreting proportional quantifiers, implausible results must be expected. The canonical construction of $\mathcal{F}$, which has been presented in section A.2 above, is then replaced with direct assertions that a given semi-fuzzy quantifier

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[54] In other words, the quantifier must be distinct from $\forall$ and $\exists$.

[55] all proportional quantifiers are conservative by Def. 166.

[56] It is rather questionable anyway that "all" and "some" are considered proportional in Yager's setting. For example, "all" and "some" are easily defined for infinite domains, which is in clear opposition to the situation with proportional quantifiers. In my view, "some" should rather be considered an absolute quantifier, and "all" should be considered a special case of quantifier of exception (i.e., "all except 0", which permits no exceptions at all).

Q should be mapped to a certain fuzzy quantifier $\mathcal{F}(Q)$ under the OWA approach. The property of correct generalisation, which was enforced by the canonical construction, is not necessarily guaranteed by such ad hoc correspondences. Nevertheless, part of the evaluation framework remains applicable, because we can still use the semantical postulates for plausible approaches in order to judge the adequacy of the resulting model. For example, let us consider the fuzzy linguistic quantifier $\mu_{[\mathrm{rate}\geq 0.6]}$, defined by

$$\mu_{[\mathrm{rate}\geq 0.6]}(x) = \begin{cases} 1 & : \quad x \geq 0.6 \\ 0 & : \quad \text{else} \end{cases}$$

and make the explicit assertion that $[\textbf{rate} \geq r_1]$ be interpreted by the fuzzy quantifier $\mathcal{F}([\textbf{rate} \geq r_1]) = \mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}\geq 0.6]})$. In fact, this is the common choice for “at least 60 percent” when using the OWA approach. Now let us consider the suitability of the resulting fuzzy quantifier for interpreting “At least 60 percent of Southern Germany are cloudy”, given the cloudiness situation displayed in Fig. 17. In case (a), we expect the result of 1, because so many pixels which fully belong to Southern Germany (I) are classified as fully cloudy that, regardless of whether we view the intermediate cases (II) as belonging to Southern Germany or not, its cloud coverage is always larger than 60 percent. Likewise in (b), we expect a result of 0 because regardless of whether the pixels in (II) are viewed as belonging to Southern Germany, its cloud coverage is always smaller than 60 percent. OWA, however, ranks image (b) much higher than image (a). This counter-intuitive result is explained by OWA’s lack of conservativity: the cloudiness grades of pixels in areas (III) and (IV), which do not belong to Southern Germany at all, still have a strong (and undesirable) impact on the computed results.

To see that this failure of OWA is caused by the lack of conservativity (in the weak sense of Def. 71) of the quantifier $\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}\geq 0.6]})$ let us consider the synthetic examples depicted in Fig. 18. In images (a) and (b), Southern Germany is fully covered by clouds in the sense that each pixel in the support of Southern Germany[57] has a cloudiness grade of 1. We hence expect that in both (a) and (b), the condition “More than 60 percent of Southern Germany are cloudy” is fully satisfied. In particular, image (b) is the intersection of image (a) and the support of Southern Germany. Because “more than 60 percent” is conservative, we should expect that the results for image (a) and (b) coincide.

In images (c) and (d), there is no cloudiness at all in Southern Germany in the sense that each pixel in the support of Southern Germany has a cloudiness grade of 0. We should therefore expect that in both cases, the condition “More than 60 percent of Southern Germany are cloudy” fails completely (i.e. the result should be 0). In particular, the results of (c) and (d) should coincide because (d) is the intersection of (c) with the support of Southern Germany, and “more than 60 percent” is conservative.
As shown in Fig. 18, the OWA approach neither computes the same results for (a) and (b), nor for (c) and (d). Even worse, its lack of weak conservativity causes OWA not to produce the intended order of results, which is (a) and (b) best, (c) and (d) worst. By contrast, image (c), which shows no cloudiness at all in Southern Germany, and

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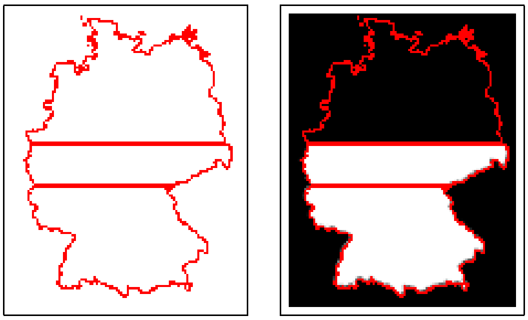
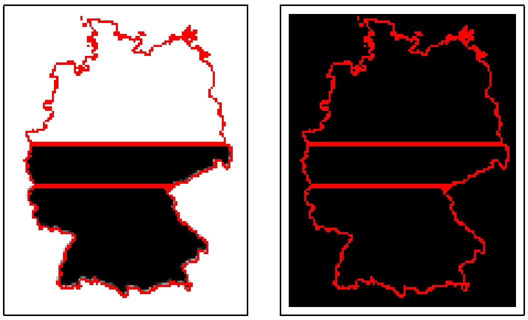
[57] i.e. each pixel which belongs to Southern Germany to a degree larger than 0



(a) desired: 1, OWA: 0.6 (b) desired: 1, OWA: 0



(c) desired: 0, OWA: 0.6 (d) desired: 0, OWA: 0

Figure 18: More than 60 percent of southern germany are cloudy. Synthetic examples: desired results vs. results computed by $\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}\geq 0.6]})$

image (a), in which Southern Germany is totally covered by clouds, are both assigned the same (rather high) score of 0.6; while image (b), in which Southern Germany is totally covered by clouds, and image (d), showing now clouds at all, both receive the lowest possible score of 0.

Apart from these negative findings concerning OWA's ability to represent conservative proportional quantifiers in the case of restricted (two-place) quantification, there are some further pecularities of the OWA approach that also deserve attention. It has already been mentioned that the 'core' OWA approach is limited to the regular nondecreasing type of $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$. Yager [165] proposes a quantifier synthesis method to handle the remaining quantifiers of the proportional kind, i.e. those choices of μ_Q that are not regular nondecreasing. This construction can be rephrased as follows. For each considered μ_Q, we define $\mu_{\mathrm{ant}\,Q} : \mathbf{I} \longrightarrow \mathbf{I}$ by $\mu_{\mathrm{ant}\,Q}(x) = \mu_Q(1-x)$. $\mu_{\mathrm{ant}\,Q}$ is intended to model the antonym of μ_Q [188]. Let us call $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ *regular nonincreasing* if $\mu_{\mathrm{ant}\,Q}$ is regular nondecreasing. Yager [175] suggests to interpret such quantifiers in terms of their antonyms, i.e. to define

$$\mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) = \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{\mathrm{ant}\,Q})(\neg X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, provided that μ_Q is regular nonincreasing. Of course, we could also have used the negation, defined by $\mu_{\neg Q}(x) = 1 - \mu_Q(x)$, to state an alternative definition

$$\mathrm{OWA}^{(1)}_{\mathrm{prp}}{}'(\mu_Q)(X) = 1 - \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{\neg Q})(X)\,.$$

However, both alternatives coincide because $\mathrm{OWA}^{(1)}_{\mathrm{prp}}$ preserves the duality between the quantifiers which correspond to $\mu_{\mathrm{ant}\,Q}$ and $\mu_{\neg Q}$.

In the general case where $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ is neither regular nondecreasing nor nonincreasing, Yager suggests to decompose μ_Q into a conjunction (or possibly other Boolean combination) of regular nondecreasing and nonincreasing quantifiers. For example, if $\mu_Q = \mu_{Q_1} \wedge \mu_{Q_2}$, where μ_{Q_1} is regular nondecreasing and μ_{Q_2} is regular nonincreasing, the quantifier synthesis method yields

$$\mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) = \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{Q_1})(X) \wedge \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{\mathrm{ant}\,Q_2})(\neg X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$. Unfortunately, the obtained results depend on the decomposition of μ_Q chosen, as I will now show. For example, consider

$$\mu_Q(x) = \begin{cases} 0 & : \; x \in [0, 0.25) \\ 0.3 & : \; x \in [0.25, 0.75] \\ 0 & : \; x \in (0.75, 1] \end{cases} \tag{252}$$

which can be decomposed into a conjunction of

$$\mu_{Q_1}(x) = \begin{cases} 0 & : \; x \in [0, 0.25) \\ 0.3 & : \; x \in [0.25, 0.75] \\ 1 & : \; x \in (0.75, 1] \end{cases} \qquad \mu_{Q_2}(x) = \begin{cases} 1 & : \; x \in [0, 0.25) \\ 0.3 & : \; x \in [0.25, 0.75] \\ 0 & : \; x \in (0.75, 1] \end{cases}$$

but also into a conjunction of

$$\mu_{Q_1'}(x) = \begin{cases} 0 & : \; x \in [0, 0.25) \\ 0.3 & : \; x \in [0.25, 0.5) \\ 1 & : \; x \in [0.5, 1] \end{cases} \qquad \mu_{Q_2'}(x) = \begin{cases} 1 & : \; x \in [0, 0.5) \\ 0.3 & : \; x \in [0.5, 0.75] \\ 0 & : \; x \in (0.75, 1] \end{cases}$$

If we interpret the regular nondecreasing μ_{Q_2} and $\mu_{Q_2'}$ by means of $\mu_{\mathrm{ant}\ Q_2}$ and $\mu_{\mathrm{ant}\ Q_2'}$, as suggested by Yager, and choose $E = \{a, b, c, d\}$, and $X \in \widetilde{\mathcal{P}}(E)$ with $X = 0/a + 0.5/b + 0.5/c + 1/d$, then

$$\begin{aligned} \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{Q_1})(X) \wedge \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{\mathrm{ant}\,Q_2})(\neg X) &= 0.3 \\ \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{Q_1'})(X) \wedge \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_{\mathrm{ant}\,Q_2'})(\neg X) &= 0.65 \end{aligned}$$

i.e. the quantifier synthesis method depends on the decomposition chosen, and is hence ill-defined. These problems carry over to two-place quantification based on $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$. We then have

$$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_Q)(E, X) = \mathrm{OWA}^{(1)}_{\mathrm{prp}}(\mu_Q)(X)\,.$$

This demonstrates that $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$ embeds all counter-examples for one-place quantification.

In the two-place case, however, there are some additional problems because $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$, unlike $\mathrm{OWA}^{(1)}_{\mathrm{prp}}$, does not preserve duals. Suppose we wish to evaluate the criterion "Less than 60 percent of Southern Germany are cloudy". Noticing that the quantifier is not of the regular nondecreasing type, we must resort to one of the following equivalent statements:

i. "It is not the case that at least 60 percent of Southern Germany are cloudy", i.e. use negation and compute $\neg\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}\geq 0.6]})(X_1, X_2)$;

ii. "More than 40 percent of Southern Germany are not cloudy", i.e. use the antonym and compute $\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.4]})(X_1, \neg X_2)$.

Unlike in NL, however, these statements are not equivalent under the OWA approach. When applied to the images in Fig. 17, we obtain different results as shown in Table 23. In this case, both i. and ii. compute the same (wrong) ranking[58], which demonstrates that neither alternative is correct; in other cases, their rankings can differ. The problem is that OWA, which cannot model "less than 60 percent" directly, forces us to choose one of i., ii.; but due to their expected equivalence, there is no preference for either choice. In an appropriate model, the computed results coincide in both cases and the chosen alternative for computing the result becomes inessential. The OWA approach, however, fails to preserve the duality of "at least 60 percent" vs. "more than 40 percent".

Quantifier	**Fig. 17(a)**	**Fig. 17(b)**
$\neg\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}\geq 0.6]})$	0.9	0.4
$\mathrm{OWA}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate}>0.4]})\neg$	0.4	0
(desired result)	0	1

Table 23: Less than 60 percent of Southern Germany are cloudy

A.5 The FG-count approach

Apart from the Σ-count approach, Zadeh [188] also introduces a second approach to fuzzy quantification. The FG-count approach rests on the idea that the cardinality of a fuzzy set cannot be fully described by a single scalar number (as in the Σ-count approach). By contrast, the cardinality of a fuzzy set should rather be be viewed as a fuzzy subset of the non-negative integers. If E is a finite base set, and $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E, then the FG-count of X [188, p.156,p.157], denoted $\text{FG-count}(X) \in \widetilde{\mathcal{P}}(\mathbb{N})$, is defined by

$$\begin{aligned}\mu_{\text{FG-count}(X)}(j) &= \sup\{\alpha \in \mathbf{I} : \, |X_{\geq\alpha}| \geq j\} \\ &= \mu_{[j]}(X)\,,\end{aligned}$$

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[58] i.e. result order with respect to $\geq$

for all $j \in \mathbb{N}$, recalling in particular that $\mu_{[0]}(X) = 1$ and $\mu_{[j]}(X) = 0$ for all $j > |E|$, see Def. 99. Intuitively, the FG-count expresses the degree to which $X \in \widetilde{\mathcal{P}}(E)$ has at least j elements. The FG-count can then be used to interpret absolute quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ as follows [165]:

$$\mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q) = \max_{j=0,\dots,m} \mu_Q(j) \wedge \mu_{[j]}(X)$$

Following the canonical strategy (249) for defining the restricted use of absolute quantifiers, $\mathrm{FG}^{(2)}_{\mathrm{abs}}$ can be modelled by

$$\mathrm{FG}^{(2)}_{\mathrm{abs}}(\mu_Q)(X_1, X_2) = \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)(X_1 \cap X_2)\,.$$

Yager [165] proposes a method for interpreting proportional quantifiers $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, which is obviously related to Zadeh's FG-count approach. $\mathrm{FG}^{(1)}_{\mathrm{prp}}$ then becomes

$$\mathrm{FG}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) = \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{Q,E})(X)\,,$$

(adapted to my notation), where $\mu_{Q,E}(j) = \mu_Q(j/m)$, $m = |E|$, as in the case of the OWA approach. As concerns the restricted use of proportional quantifiers, Zadeh proposes the *relative* FG-count for pairs of fuzzy sets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, defined by

$$\text{FG-Count}(X_2/X_1) = \sum_{\alpha \in [0,1]} \alpha / \tfrac{|X_{1_{\geq\alpha}} \cap X_{2_{\geq\alpha}}|}{|X_{1_{\geq\alpha}}|}$$

and emphasises *"that the right-hand member" [of the defining equation] "should be treated as a fuzzy multiset, which implies that terms of the form α_1/u and α_2/u should not be combined into a single term $(\alpha_1 \vee \alpha_2)/u$, as they would be in the case of a fuzzy set"* [188, p.157]. However, if the right-hand member of the equation is a fuzzy multiset, then obviously FG-Count(X_2/X_1) is a fuzzy multiset as well. It is therefore *not* possible to formulate $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ in analogy to the definition of $\mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_Q)$, i.e. the following proposal is *not* licensed by Zadeh's definition:

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) = \sup\{\mu_Q(r) \wedge \mu_{\text{FG-Count}(X_2/X_1)}(r) : r \in \mathbf{I}\}$$

(This definition would be ill-formed because it identifies the cases which Zadeh explicitly wants to be kept separate). To the best of my knowledge, no attempt has been made to provide a definition of $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ in terms of the above FG-Count(X_2/X_1). In particular, Zadeh's examples on the application of the FG-count approach in [188] only demonstrate its application in the unrestricted case, and can thus be handled by $\mathrm{FG}^{(1)}_{\mathrm{prp}}$.

As in the OWA case, it is easily shown that the FG-count approach complies with the EFA. However, the range of fuzzy quantifiers which can be handled by the FG-count approach is essentially limited, because the two-place use of proportional quantifiers cannot be modelled, which is of obvious relevance to most applications.

Another limitation of the model becomes visible once we investigate the monotonicity properties of the fuzzy quantifiers $\widetilde{Q} = \mathrm{FG}(\mu_Q)$ obtained from the FG-count approach. It can be observed that, regardless of μ_Q, the resulting fuzzy quantifiers are *always monotonically nondecreasing* in their arguments. Consequently the FG-count approach is unable to represent any quantifiers except for those which are nondecreasing.[59] In analogy to the solution proposed for OWA, we might reduce monotonically nonincreasing quantifiers to their antonym or their negation, and consider more complex quantifiers as Boolean combinations of such monotonic quantifiers. However, the very same example already used to discuss the OWA approach reveals that the quantifier synthesis method fails in the FG-count case, too. Using the same abbreviations as in (252), we have

$$\begin{aligned}\mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{Q_1})(X) \wedge \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{\mathrm{ant}\,Q_2})(\neg X) &= 0.3\\ \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{Q_1'})(X) \wedge \mathrm{FG}^{(1)}_{\mathrm{abs}}(\mu_{\mathrm{ant}\,Q_2'})(\neg X) &= 0.5\,,\end{aligned}$$

i.e. we obtain different results for different decompositions of μ_Q.

Yager [171, p.72] proposes a weighting formula, which can be conceived of as providing a definition of $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$, however not based on Zadeh's definition of relative FG-count. It is defined by

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) = \max\left\{\min\left(\mu_Q\left(\frac{\sum_{e\in S}\mu_{X_1}(e)}{\sum_{e\in E}\mu_{X_1}(e)}\right), H_S\right) : S \in \mathcal{P}(E)\right\}$$

where $H_S = \min\{\max(1-\mu_{X_1}(e), \mu_{X_2}(e)) : e \in S\}$.
To see that this formula is related to the FG-count approach, suffice it to observe that

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(E, X) = \mathrm{FG}^{(1)}_{\mathrm{prp}}(\mu_Q)(X) \tag{253}$$

for all $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(E)$, provided that $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(E, X)$ is defined in terms of the above formula.

Before turning to the issue of plausibility, let us observe that

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(\varnothing, X_2)$$

is undefined, regardless of $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$, E, and $X_2 \in \widetilde{\mathcal{P}}(E)$. Strictly speaking, $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ is not a fuzzy quantifier according to my definition, it is only a partial mapping from $\widetilde{\mathcal{P}}(E)^2$ to $\mathbf{I}$ which is defined whenever $X_1 \neq \varnothing$. As we have seen in the case of the Σ-Count approach, extending the approach in such a way as to provide reasonable results in this case can be rather difficult (perhaps impossible). However, for the sake of making the framework applicable to $\mathrm{FG}^{(2)}_{\mathrm{prp}}$, we shall assume that each $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ is completed to a total mapping $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ in some (arbitrary) way, and we shall not consider the results obtained in the case $X_1 = \varnothing$ further.

As regards the plausibility of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$, we should first notice that the EFA is violated. This is apparent when we consider a pair of nondecreasing $\mu_Q, \mu_{Q'} : \mathbf{I} \longrightarrow \mathbf{I}$ and a

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[59] Yager [165] therefore explicitly restricts application of the FG-count approach to monotonically nondecreasing μ_Q. His approach is criticized by Ralescu [124], however based on an invalid example $\mu_{\textbf{a few}}$ of a quantifier which fails to be nondecreasing.

base set E, $m = |E| \geq 2$ such that

$$\mu_Q(\tfrac{p}{q}) = \mu_Q(\tfrac{p}{q}) \qquad (254)$$

for all $q \in \{1, \ldots, m\}$, $p \in \{0, \ldots, q\}$, but $\mu_Q(z) \neq \mu_{Q'}(z)$ for some other choice of $z \in \mathbf{I}$. In particular, we then know from (254) that $z \neq 1$ and hence $1 - z \neq 0$, which will be needed below. For purpose of demonstration, we shall further assume that $z \leq \frac{1}{2}$. As concerns the results of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ in this case, we first observe that $\mathcal{U}(\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)) = \mathcal{U}(\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{Q'}))$, because for all crisp $Y_1, Y_2 \in \mathcal{P}(E)$,

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(Y_1, Y_2) = \mu_Q(\tfrac{p}{q}) = \mu_{Q'}(\tfrac{p}{q}) = \mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{Q'})(Y_1, Y_2)$$

by (254), where $p = |Y_1 \cap Y_2|$ and $q = |Y_1|$. Now pick an arbitrary but fixed pair of elements $e_0, e_1 \in E$, $e_0 \neq e_1$, and abbreviate $\lambda = 1 - \min(\mu_Q(z), \mu_{Q'}(z))$. Defining $X_1 \in \widetilde{\mathcal{P}}(E)$ by

$$\mu_{X_1}(e) = \begin{cases} \frac{\lambda z}{1-z} & : \quad e = e_0 \\ \lambda & : \quad e = e_1 \\ 0 & : \quad \text{else} \end{cases}$$

and letting $X_2 = \{e_0\}$ then results in

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) = \mu_Q(z) \neq \mu_{Q'}(z) = \mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{Q'})(X_1, X_2)\,.$$

Hence $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ and $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)$ agree for all crisp arguments, but they can disagree for certain choices of fuzzy arguments. This demonstrates that the EFA is indeed violated.

Beyond this failure of the EFA, there are some further undesirable properties. Let us consider the quantifier $[\textsf{\textbf{rate}} > \frac{p}{q}]$, where $q \in \mathbb{N} \setminus \{0\}$, $p \in \{1, \ldots, q-1\}$. Due to the fact that $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ does not fulfill the EFA, we have some choices which $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ to use as a model of $[\textsf{\textbf{rate}} > \frac{p}{q}]$. $\mu_{[\mathrm{rate} > \frac{p}{q}]}$ is a proper choice for interpreting $[\textsf{\textbf{rate}} > \frac{p}{q}]$ using $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ because

$$\mathcal{U}(\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate} > \frac{p}{q}]})) = [\textsf{\textbf{rate}} > \tfrac{p}{q}]$$

(assuming a proper completion in the case $X_1 = \varnothing$), but other μ_Q can also fulfill this property. As in the case of the Σ-count approach, one can show that in order to prevent $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ from violating the extensionality criterion (in the sense of Def. 40), every reasonable choice of μ_Q must satisfy $\mu_Q(x) = \mu_{[\mathrm{rate} > \frac{p}{q}]}(x)$ for all $x \in \mathbf{I} \cap \mathbb{Q}$. In the following, I will use $\mu_{[\mathrm{rate} > \frac{p}{q}]}$, but the effects demonstrated will also occur for other possible choices of μ_Q.

Now suppose that E is a base set such that $m = |E| > q$. Then E can be written as $E = \{e_1, \ldots, e_m\}$, where the e_i are pairwise distinct elements of E. Let us define $X_1^{(\lambda)} \in \widetilde{\mathcal{P}}(E)$, $X_2 \in \mathcal{P}(E)$ by

$$\begin{aligned} X_1^{(\lambda)} &= 1/e_1 + \cdots + 1/e_q + \lambda/e_{q+1} \\ X_2 &= \{e_1, \ldots, e_p\} \end{aligned}$$

for all $\lambda \in \mathbf{I}$. We should expect that $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate} > \frac{p}{q}]})(X_1^{(\lambda)}, X_2) = 0$, because $[\textsf{\textbf{rate}} > \frac{p}{q}]$ is constantly zero in the range (U, V), where

$$U = (\{e_1, \dots, e_q\},\, X_2) = (X_1^{(0)}, X_2)\,,$$
$$V = (\{e_1, \dots, e_{q+1}\}, X_2) = (X_1^{(1)},\, X_2)\,.$$

However, instead of the desired result stated above, what we really get is the following,

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate} > \frac{p}{q}]})(X_1^{(\lambda)}, X_2) = \begin{cases} 0 & : \quad \lambda = 0 \\ 1 - \lambda & : \quad \lambda > 0 \end{cases} \tag{255}$$

In fact, the example reveals that $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ does not preserve local monotonicity properties, because the resulting fuzzy quantifier fails to be locally nondecreasing in (U, V). Moreover, it is not contextual (see Def. 42 for the definition of contextuality): its result $1 - \lambda$ in the considered range is nonzero, although the original quantifier, i.e. the specification for crisp sets, is constantly zero in (U, V).

Apart from the failure of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ to preserve local monotonicity, equation (255) also points at another weakness of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$: if μ_Q is discontinuous (in particular if it is two-valued), then $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2)$ can be discontinuous in the membership grades of X_1 as well. In the example, a very slight change of $X_1^{(\lambda)}$ (from $\lambda = 0$ to $\lambda' = \varepsilon > 0$) can result in a drastic change of $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{[\mathrm{rate} > \frac{p}{q}]})(X_1^{(\lambda)}, X_2)$. This is not acceptable in practical applications due to the inevitable presence of noise: a very slight change in X_1, X_2 might cause totally different results of $\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2)$.

In order to demonstrate these effects in the image ranking task, let us consider the quantifier $[\textsf{\textbf{rate}} \geq 0.05]$, i.e. **at least five percent**. Some results of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ for the chosen quantifier are shown in Fig. 19. Let us first consider the case that X_1 is the standard choice for interpreting Southern Germany depicted in 19(a), and where X_2 is the cloudiness situation depicted in (c). We expect a result of zero in this case because there are no clouds in the support of X_1. The FG-count approach, however, returns a


(a) Southern Germany#1 (b) Southern Germany#2 (c) cloudy

Result for Southern Germany#1: 0.55
Result for Southern Germany#2: 0.95
Desired result: 0

Figure 19: At least 5 percent of Southern Germany are cloudy (FG-Count)

score of 0.55. One might subject that, as in the case of OWA, this implausible result could be caused by a lack of conservativity – after all, there are some clouds in the upper part of image (c). However, it is easily verified that all fuzzy quantifiers obtained from $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ are conservative. It is the failure of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ to preserve local monotonicity properties which causes the implausible behaviour.

The sensitivity of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ to slight changes in X_1 becomes apparent when the standard definition of the fuzzy region 'Southern Germany' (a) is replaced with the slightly modified Southern Germany#2 depicted in 19(b). The image has been generated from (a) by replacing black pixels ($\mu_X(e) = 0$) in the lower two thirds of (a) with a slightly lighter black, $\mu_X(e) = 0.05$. Because the upper third of the image is unchanged, i.e. the clouds in the northern part of (c) are still outside the support of Southern Germany#2, we expect a result of 0 in this case, too. The FG-count approach, however, results in a score of 0.95, again because local monotonicity is not preserved. The large change in the result, although we have modified the representation of Southern Germany in terms of X_1 only very slightly, demonstrates that the operators obtained from the FG-count approach can indeed be very brittle.

We have already seen that $\mathrm{FG}^{(1)}_{\mathrm{prp}}$ is essentially limited to nondecreasing quantifiers. Owing to (253), then, the proposed extension to restricted quantification is subject to the same limitations, i.e. $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ will behave reasonably only if μ_Q is nondecreasing. In particular, the failure of the 'quantifier synthesis' method to extend the coverage of $\mathrm{FG}^{(1)}_{\mathrm{abs}}$ and $\mathrm{FG}^{(1)}_{\mathrm{prp}}$ directly carries over to $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ (in this case, E must be used for the first argument, while X becomes the second argument. Here I will present an additional example, which shows that in the case of two-place quantification, the decomposition can also fail because $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ is not compatible with dualisation. Hence let us consider

$$\mu_Q(x) = \begin{cases} 1 & : \quad x \geq \frac{1}{6} \\ 0 & : \quad x < \frac{1}{6} \end{cases} \qquad\qquad \mu_{Q'}(x) = \begin{cases} 1 & : \quad x > \frac{5}{6} \\ 0 & : \quad x \leq \frac{5}{6} \end{cases}$$

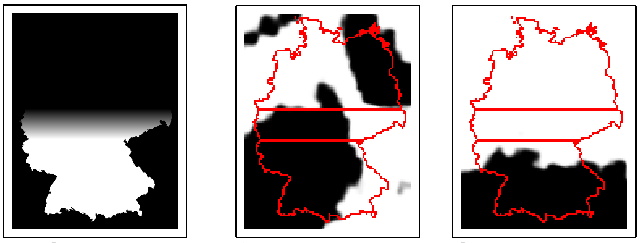
representing $[\textbf{rate} \geq \frac{1}{6}]$ and its dual $[\textbf{rate} > \frac{5}{6}]$, respectively, and choose $E = \{e_1, e_2\}$, $X_1 = X_2 = \frac{1}{3}/e_1 + \frac{1}{3}/e_2$. We then have

$$\mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_Q)(X_1, X_2) = \tfrac{2}{3} \neq \tfrac{1}{3} = 1 - \tfrac{2}{3} = 1 - \mathrm{FG}^{(2)}_{\mathrm{prp}}(\mu_{Q'})(X_1, \neg X_2)\,.$$

As in the case of $\mathrm{OWA}^{(2)}_{\mathrm{prp}}$, the failure of $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ to preserve duals is of particular importance because $\mathrm{FG}^{(2)}_{\mathrm{prp}}$ cannot provide a reasonable interpretation of quantifiers like "less than 30 percent" directly. In order to evaluate this quantifier, we either have to use the negation "at least 30 percent", or the antonym "more than 70 percent", which are duals of each other. However, because duality is not preserved by $\mathrm{FG}^{(2)}_{\mathrm{prp}}$, both choices produce different results as depicted in Fig. 20.

A.6 The FE-count approach

Apart from introducing the FG-count and suggesting its use for the modelling of fuzzy quantification, Zadeh [188] proposes yet another definition of fuzzy cardinality. The



Southern Germany (a) (b)

Quantifier	**Image (a)**	**Image (b)**
not at least 30 percent	0.72	0
more than 70 percent not	1	0.29

Figure 20: Less than thirty percent of Southern Germany are cloudy (FG-count approach)

FE-count of $X \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_{\text{FE-count}(X)}(j) = \mu_{[j]}(X) \wedge \neg\mu_{[j+1]}(X)\,.$$

Unlike FG-count(X), which expresses the degree to which $X \in \widetilde{\mathcal{P}}(E)$ has at least j elements, the FE-count is intended to express the degree to which X has exactly j elements. The adequacy of the FE-count as a measure of fuzzy cardinality has been questioned by Dubois & Prade [37].

Ralescu [124] suggests that the interpretation of fuzzy quantifiers be based on the computation of FE-counts. He only considers the unrestricted use of absolute quantifiers $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$. In this case, the FE-count model is defined thus,

$$\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_Q)(X) = \max_{j=0,\ldots,m} \mu_Q(j) \wedge \mu_{[j]}(X) \wedge \neg\mu_{[j+1]}(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$. The model complies with the EFA, but the phenomenon treated (only one-place absolute quantifiers) is of course very limited. Nevertheless, the FE-count approach explicitly targets at an improvement upon the FG-count model, so we should be curious about its score on the plausibility scale.

Given a domain E with $|E| = m$ and $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$, we can conveniently restrict attention to $\mu_{Q,E} : \{0, \ldots, m\} \longrightarrow \mathbf{I}$ (as we have already done in the case of OWA), which for absolute quantifiers becomes $\mu_{Q,E}(j) = \mu_Q(j)$ for all $j = 0, \ldots, m$. We can then view $\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_Q)$ as a function of $\mu_{Q,E}$. In the case of a two-element domain, say $E = \{a, b\}$, the existential and universal quantifiers are modelled thus,

$$\mu_{\exists,E}(j) = \begin{cases} 0 & : \quad j = 0 \\ 1 & : \quad \text{else} \end{cases} \qquad \mu_{\forall,E}(j) = \begin{cases} 0 & : \quad j < 2 \\ 1 & : \quad j = 2 \end{cases}$$

for $j \in \{0, 1, 2\}$. Letting $X = 1/a + 0/b$ and $Y = 1/a + 0.5/b$, we observe that $\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_{\exists,E})(X) = 1$, but $\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_{\exists,E})(Y) = 0.5$, although $X \subseteq Y$ (in the sense of inclusion of fuzzy sets). It follows that the FE-count approach does not preserve monotonicity properties of quantifiers. In addition, the example shows that the FE-count does not generate a reasonable interpretation of the existential quantifier, which should be expressable in terms of an s-norm.[60] The example further demonstrates that the FE-count approach does not preserve duality, because $\mu_{\forall,E}$ is mapped to the intended

$$\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_{\forall,E})(Z) = \min(\mu_Z(a), \mu_Z(b))\,,$$

rather than the dual of $\mathrm{FE}^{(1)}_{\mathrm{abs}}(\mu_{\exists,E})$. Finally, the FE-count approach also does not preserve negation.

The failure of the FE-count approach to preserve monotonicity can also be demonstrated in my example scenario of fuzzy image regions. Hence let us consider the statement that a fuzzy image region X is nonempty.[61] The condition corresponds to existential quantification, and the FE-count formula is hence applied to $\mu_\exists : \mathbb{R}^+ \longrightarrow \mathbf{I}$ defined by $\mu_\exists(0) = 0$, $\mu_\exists(j) = 1$ else. Now consider the results shown in Fig. 21. The result in case (a) is satisfactory because X is a crisp nonempty subset of the set of pixel coordinates E. The fuzzy image region X' depicted in (b) is much larger and contains X. Due to the monotonicity of the existential quantifier, we should expect that the fuzzy image region X' depicted in (b) is nonempty, too. However, due to the presence of fuzziness, the FE-count approach results in a score of 0.5, which is clearly implausible.


(a) desired: 1, FE: 1 (b) desired: 1, FE: 0.5

Figure 21: X is not empty (FE-count approach)

To sum up, the FE-count is limited in applicability and even in the (simplest) case of unrestricted absolute quantifiers, it generates inacceptable interpretations.

A.7 Chapter summary

The traditional approaches to fuzzy quantification rest on the very simple representation offered by membership functions $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$ and $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$. Conceptually

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[60] see section 4.16 on the intended interpretation of fuzzy existential and universal quantification, which reviews Thiele's analysis of T- and S-quantifiers.

[61] Note that I cannot use the examples of two-place proportional quantifiers here because the FE-count approach is unable to model these.

these approaches are clearly less developed compared to the theory expounded in this report. However, these approaches are particularly easy to implement, which makes them appealing to applications at first sight. This made it important to investigate if these approaches perform well in those cases they intend to model, well knowing that they have too few coverage for a general treatment of fuzzy NL quantification (see section 1.15 above).

In order to assess the plausibility of the approaches described in the literature, I have introduced a canonical construction which fits these approaches into the framework pursued in this report, and hence assigns to each of these approaches an (incompletely determined) QFM $\mathcal{F}$.

The framework is directly applicable if its underlying assumption is satisfied, which I made precise in terms of the 'EFA' (evaluation framework assumption). Basically, the EFA requires that all fuzzy quantifiers obtained from the considered approach are already discernable from their behaviour on crisp arguments. This simplification is justified by the fundamental assumption of the quantification framework, i.e. the QFA described in Chap. 2, which poses the related requirement that all base quantifiers of interest can also be discerned on the level of crisp arguments. This suggests that beyond its part in the construction of the canonical embedding, the EFA is also a plausibility criterion which is interesting in its own right. Those approaches that violate the EFA can still be discussed in the analysis framework; instead of a unique fuzzy quantifier assigned by the construction, we then have a number of available choices. In this case, we can still check if certain adequacy criteria can be fulfilled, by making an appropriate selection from the available interpretations. If no verifying choice exists, the considered approach to fuzzy quantification must be considered incompatible with the criterion of interest.

The proposed method permits a rigorous evaluation of existing approaches under their compliance with the known semantical postulates, which were originally introduced for testing the plausibility of QFMs. Furthermore, the criteria formalized in Chap. 3-6 can also guide the systematic search for critical examples which result in implausible interpretations. The discovery of such counter-examples then raises the new question of the causes of the unexpected behaviour. Obviously, similar effects can only be avoided in improved approaches if we understand the defects of existing approaches. It is the main merit of the proposed evaluation framework that it can *explain* the failure of existing approaches, which now becomes a violation of elementary adequacy requirements.

In the chapter, I have instantiated the framework for the main approaches to fuzzy quantification, i.e. the Σ-count, FG-count, OWA and the FE-count models. Based on the known plausibility criteria, these approaches were then checked against the intended semantics of fuzzy NL quantification. This evaluation has substantiated my doubts on the adequacy of the 'traditional' account to fuzzy quantification, which reduces everything to calculations on simple membership functions $\mu_Q : \mathbb{R}^+ \longrightarrow \mathbf{I}$ or $\mu_Q : \mathbf{I} \longrightarrow \mathbf{I}$. For each of the approaches in this tradition, I have presented several counter-examples in which its results are inacceptable. It is true that the OWA- and FG-count approaches have a reasonable 'core' for nondecreasing unary quantifiers.

However, their extensions to non-monotonic quantifiers and importance qualification are not conclusive. Despite the algorithmic simplicity of existing approaches, these findings discourage the use of these models in practice, especially in those applications of fuzzy quantifiers which depend on their natural language meaning. In fact, it was this massive evidence against the traditional view of fuzzy quantification which forced me to abandon the common assumptions of these approaches, and develop a theory fundamentally different, focussing on the notions of semi-fuzzy quantifiers and QFMs. In conforming models like $\mathcal{M}_{CX}$ or $\mathcal{F}_{\mathrm{Ch}}$, then, faults comparable to those of existing approaches become impossible on formal grounds.

B A note on the fuzzification pattern

B.1 Motivation and chapter overview

The analysis of fuzzy quantification pursued in this report depends heavily on the notion of a fuzzification mechanism. As I have argued in Chap. 2, where the QFM framework for fuzzy quantification was introduced, the use of fuzzification mechanisms makes a basic commitment, which I called the QFA (quantification framework assumption). In order to shed some light on the abstract structure underlying the proposed framework and further elucidate the precise contribution of the QFA, I will now discuss general fuzzification mechanisms in some depth. Specifically, I will use the term *fuzzification pattern* to denote that proven strategy for solving problems in the fuzzy sets framework, which rests on the use of fuzzification mechanisms. In order to gain a better understanding of these mechanisms, we shall consider the necessary components of the pattern, and make explicit the preconditions for its successful application. The resulting analysis will be particularly helpful because it explains the quantification framework assumption in a more general setting.

B.2 The basic problem

Abstracting from details, we will now be concerned with a rather typical situation in the fuzzy sets framework, which can be described as follows. Suppose we would like to solve a certain class of problems $\mathbb{P}$, which involve one or more variables that range over fuzzy sets. Each problem $p \in \mathbb{P}$ is known to find its solution $\widetilde{s} = \widetilde{\mathcal{A}}(p)$ in a collection of potential solutions $\widetilde{\mathbb{S}}$. In formal terms, we can describe the association between problems and corresponding full solutions by a mapping $\widetilde{\mathcal{A}} : \mathbb{P} \longrightarrow \widetilde{\mathbb{S}}$. Due to the presence of fuzzy variables, though, the particular solution $\widetilde{\mathcal{A}}(p)$ of a given problem $p \in \mathbb{P}$ will typically be hard to ascertain, and we might find ourselves unable to develop an explicit description of $\widetilde{\mathcal{A}}$. The problem is that precise outputs (e.g. numerical membership grades) must be computed from imprecise, fuzzy inputs. Because in many cases, the presence of fuzziness might blur our intuitions concerning a 'good' solution, we must consult auxiliary techniques in order to ascertain consistent outputs $\widetilde{\mathcal{A}}(p)$.

B.3 The reduction equality

One possible approach to solve the problem is a reduction to the crisp case: for the moment, we simply 'forget' that some of the variables range over fuzzy sets, and confine ourselves to crisp instantiations only. In doing this, each full solution $\widetilde{s} \in \widetilde{\mathcal{A}}$ is pruned to an incomplete solution $s = \mathcal{U}(\widetilde{s})$, which sets aside certain fuzzy cases. Of course, one will take pains to cut off those cases which comprise the major source of difficulty. In formal terms, excluding part of the fuzzy cases means a passage from complete solutions $\widetilde{s} \in \widetilde{\mathbb{S}}$ to simplified counterparts $\mathcal{U}(\widetilde{s})$ in an intermediate domain, which will be denoted $\mathbb{S}$. The passage can then be modelled by a mapping $\mathcal{U} : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$, which 'forgets' part of a full solution $\widetilde{s}$, and cuts it down to the underlying incomplete solution

$\mathcal{U}(\widetilde{s})$. By composing $\widetilde{\mathcal{A}}$ and $\mathcal{U}$, we associate with each problem p the 'incomplete' or 'simplified' solution $\mathcal{U}(\widetilde{\mathcal{A}}(p))$, which is only concerned with that part of the problem where everything looks neat and crisp. The resulting simplified answer mapping will be denoted $\mathcal{A} : \mathbb{P} \longrightarrow \mathbb{S}$. It is defined by the following *reduction equality*,

$$\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}} . \tag{256}$$

The dependencies between the full answer mapping $\widetilde{\mathcal{A}}$, the incomplete answer mapping $\mathcal{A}$, and the mapping to underlying simplified solutions $\mathcal{U}$, are summarized in the following diagram:

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\widetilde{\mathcal{A}}"] \arrow[dr, "\mathcal{A}"'] & \widetilde{\mathbb{S}} \arrow[d, "\mathcal{U}"] \\
 & \mathbb{S}
\end{tikzcd}$$

B.4 Fuzzification mechanisms and the fuzzification equality

Provided that the simplification $\mathcal{U}$ be chosen carefully, describing the expected outcome of $\mathcal{A}(p)$ will be much easier than ascertaining $\widetilde{\mathcal{A}}(p)$, because the intriguing fuzzy cases are now left aside. By specifying the desired choice of $\mathcal{A}(p)$, we take one important step towards the solution of the full problem – but of course, there still remains a gap between the simplified solution $\mathcal{A}(p)$ and the corresponding full solution $\widetilde{\mathcal{A}}(p)$. This is precisely where fuzzification mechanisms enter the scene. In order to bridge the gap, we need a suitable mapping $\mathcal{F} : \mathbb{S} \longrightarrow \widetilde{\mathbb{S}}$, which takes each incomplete solution $s \in \mathbb{S}$ to the associated complete solution $\widetilde{s} = \mathcal{F}(s)$, and thus surmounts the restriction to 'easy' cases. It is this mapping $\mathcal{F}$, which I will call a *fuzzification mechanism*.

When delegating part of the problem to a fuzzification mechanism, special care must be taken to ensure the accuracy of the resulting model, i.e. for all $p \in \mathbb{P}$, the final solution $\mathcal{F}(\mathcal{A}(p))$ calculated by the fuzzification mechanism must coincide with the true solution $\widetilde{\mathcal{A}}(p)$. Consequently, the admissible choices of $\mathcal{F}$ are constrained by the following *fuzzification equality*:

$$\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A} . \tag{257}$$

The current state of this analysis can be summarized by the following diagram, which shows the dependencies between the full answer mapping $\widetilde{\mathcal{A}}$, the fuzzification mechanism $\mathcal{F}$, and finally the mapping $\mathcal{A}$ to provisional answers:

$$\begin{tikzcd}
\mathbb{P} \arrow[r, "\widetilde{\mathcal{A}}"] \arrow[dr, "\mathcal{A}"'] & \widetilde{\mathbb{S}} \\
& \mathbb{S} \arrow[u, "\mathcal{F}"']
\end{tikzcd}$$

B.5 The separation criterion

The proposed analysis of $\widetilde{\mathcal{A}}$ depends heavily on the fuzzification mechanism $\mathcal{F}$. However, it is not a matter of course that a conforming choice of $\mathcal{F}$ will exist. In order to develop this analysis further, we must therefore identify the precise conditions under which the modelling of $\widetilde{\mathcal{A}}$ in terms of a fuzzification mechanism will succeed, i.e. *when exactly* will the existence of a fuzzification mechanism be granted, which validates the fuzzification equality (257)? To see this, we start from the apparent observation that

$$\forall p,\ p' \in \mathbb{P}, \quad \mathcal{A}(p) = \mathcal{A}(p') \implies \mathcal{F}(\mathcal{A}(p)) = \mathcal{F}(\mathcal{A}(p'))\,.$$

We can then rewrite $\mathcal{F}(\mathcal{A}(p)) = \widetilde{\mathcal{A}}(p)$ and $\mathcal{F}(\mathcal{A}(p')) = \widetilde{\mathcal{A}}(p')$ in accordance with the fuzzification equality $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$, thus

$$\forall p,\ p' \in \mathbb{P}, \quad \mathcal{A}(p) = \mathcal{A}(p') \implies \widetilde{\mathcal{A}}(p) = \widetilde{\mathcal{A}}(p')\,. \tag{258}$$

Now utilizing the reduction $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$, we finally obtain the following necessary condition for the analysis of $\widetilde{\mathcal{A}}$ in terms of a fuzzification mechanism, which I will call the *separation criterion*:

$$\forall p,\ p' \in \mathbb{P}, \quad \mathcal{U}(\widetilde{\mathcal{A}}(p)) = \mathcal{U}(\widetilde{\mathcal{A}}(p')) \implies \widetilde{\mathcal{A}}(p) = \widetilde{\mathcal{A}}(p')\,. \tag{259}$$

The criterion asserts that those problems which are partially solved by a simplified solution s, must all admit the same full solution $\widetilde{s}$. The chosen name becomes clear once we reformulate the separation criterion in terms of its contraposition,

$$\forall p,\ p' \in \mathbb{P}, \quad \widetilde{\mathcal{A}}(p) \neq \widetilde{\mathcal{A}}(p') \implies \mathcal{U}(\widetilde{\mathcal{A}}(p)) \neq \mathcal{U}(\widetilde{\mathcal{A}}(p'))\,. \tag{260}$$

In other words, all problems which need different full solutions must be separated by $\mathcal{A}$, i.e. the difference must already show up in the provisional solutions.

As I will now prove, the separation criterion is not only necessary, but also sufficient for achieving a decomposition of $\widetilde{\mathcal{A}}$ in terms of a fuzzification mechanism. Let us say that a potential solution $s \in \mathbb{P}$ is *visible* or *indispensible* if it shows up as a solution $s = \mathcal{A}(p)$ of some problem $p \in \mathbb{P}$. Hence $s \in \mathbb{P}$ is visible if and only if $\mathcal{A}^{-1}(s) \neq \varnothing$ or equivalently, $s \in \operatorname{Im}\mathcal{A}$. A potential solution $s \in \mathbb{S}$ which is not visible is called *dispensible*; these solutions are characterized by the condition $\mathcal{A}^{-1}(s) = \varnothing$ or equivalently, $s \notin \operatorname{Im}\mathcal{A}$. Due to the fact that the dispensible solutions are not visible to the problems in $\mathbb{P}$, the behaviour of the fuzzification mechanism $\mathcal{F}$ for such choices of

s is also not visible judging from $\mathbb{P}$. Hence if $\mathcal{F}, \mathcal{F}' : \mathbb{S} \longrightarrow \widetilde{\mathbb{S}}$ are two fuzzification mechanisms with $\mathcal{F}|_{\mathrm{Im}\,\mathcal{A}} = \mathcal{F}'|_{\mathrm{Im}\,\mathcal{A}}$, i.e. $\mathcal{F}$ and $\mathcal{F}'$ are identical for visible solutions, then

$$\mathcal{F} \circ \mathcal{A} = \mathcal{F}' \circ \mathcal{A}\,, \tag{261}$$

i.e. the mechanisms cannot be discerned under the perspective of interest. This observation demonstrates that the precise choice of $\mathcal{F}(s)$ for dispensible solutions is inessential and does not affect the validity of the fuzzification equality. We can therefore define a fuzzification mechanism $\mathcal{F}$ based on $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ in the following way:

$$\mathcal{F}(s) = \begin{cases} \widetilde{\mathcal{A}}(p) & : \quad \mathcal{A}^{-1}(s) \neq \varnothing \\ \widetilde{s} & : \quad \mathcal{A}^{-1}(s) = \varnothing \end{cases}$$

where p is an arbitrary element $p \in \mathcal{A}^{-1}(s) \neq \varnothing$, and $\widetilde{s}$ an arbitrary choice of $\widetilde{s} \in \widetilde{\mathbb{S}}$. Assuming that the separation criterion applies, it is now apparent from (261) that the fuzzification mechanism so defined will indeed comply with the fuzzification equality, and thus achieve the desired decomposition $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$ of the target mapping $\widetilde{\mathcal{A}}$. Consequently, the separation criterion is both necessary and sufficient for a successful modelling of $\widetilde{\mathcal{A}}$ in terms of a fuzzification mechanism.

B.6 Turning the analysis into a practical pattern

Building on the concepts developed so far, we can now discuss how the fuzzification pattern will be applied in practice. In particular, we must account for the apparent change in the configuration of dependent and independent variables (input and output parameters). In the previous sections, it was convenient to assume that $\widetilde{\mathcal{A}}$ and $\mathcal{U}$ be given, and the mapping $\mathcal{A}$ to incomplete solutions was then defined in terms of these input parameters. Hence $\widetilde{\mathcal{A}}$ and $\mathcal{U}$ were treated as independent variables, and $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$ was a dependent variable defined in terms of the reduction equality. When using the pattern to solve real problems, though, there is a shift of responsibilities, and the participants of the pattern then assume different roles.

In this case, the mapping $\widetilde{\mathcal{A}} : \mathbb{P} \longrightarrow \widetilde{\mathbb{S}}$ is unknown in advance, and we would like to avoid the complexity of directly specifying the correspondence between the given problems $p \in \mathbb{P}$ and their associated full solutions $\widetilde{\mathcal{A}}(p)$ in the collection $\widetilde{\mathbb{S}}$. We will hence try and set up an instance of the above decomposition in order to establish the correspondence. The pattern then serves to identify a reasonable choice of the target mapping $\widetilde{\mathcal{A}}$ from problems to complete solutions. This can be accomplished by deriving $\widetilde{\mathcal{A}}$ from the fuzzification equality, i.e. the mapping changes into a dependent variable defined by $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$. In order to achieve this rendering of $\widetilde{\mathcal{A}}$, it is mandatory that the separation criterion described above be satisfied, which was shown to be necessary for a successful decomposition in terms of the fuzzification equality. Thus $\widetilde{\mathcal{A}}$ can only be expressed in the desired form if admissible choices are made for the relevant participants of the pattern: otherwise no proper instance of the fuzzification pattern will result, but only some approximation of the ‘true’ $\widetilde{\mathcal{A}}$. As shown by expression (259) above, it is $\widetilde{\mathcal{A}}$ and the input parameter $\mathcal{U}$ which constitute the separation criterion. Before specifying the remaining parameters, we should therefore make a suitable choice of the structural [input] parameter $\mathcal{U}$, which admits a definition of the target mapping in terms of a fuzzification mechanism defined on $\mathbb{S}$. The question then arises how to determine a good system of partial solutions $\mathbb{S}$ and the 'forgetful' mapping $\mathcal{U} : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$ which connects the full solutions $\widetilde{s} \in \widetilde{\mathbb{S}}$ with their reductions $s = \mathcal{U}(\widetilde{s})$.

B.7 The instantiation condition

The choice of $\mathcal{U}$ and its codomain $\mathbb{S}$ is critical to the successful application of the fuzzification pattern: it decides upon the structure of the decomposition, and hence controls the reduction of complexity at the level of incomplete solutions. In any event, $\mathcal{U}$ will represent a trade-off between two contrary forces. First of all, $\mathcal{U}$ must suppress enough detail in order to facilitate the specification of $\mathcal{A}$. Obviously, the more cases are eliminated, the easier we will arrive at the simplified solutions. However, $\mathcal{U}$ must discern sufficient structure for the separation criterion to remain valid and should hence not become too coarse-grained. Consequently, great care must be taken to make a good choice of the forgetful mapping $\mathcal{U}$ and the associated range of partial solutions $\mathbb{S}$, in order to create a firm basis for specifying the remaining components of the pattern. A poor choice of $\mathcal{U}$, which violates the separation criterion (259), will spoil the application of the pattern from the very beginning.

Unfortunately, it is not visible from the point of view of the pattern whether $\mathcal{U}$ really complies with the separation criterion (259), simply because the criterion also depends on the unknown target mapping $\widetilde{\mathcal{A}}$. In order to decide upon this issue, we therefore need the extrinsic perspective of the human expert who applies the pattern. The expert must have sufficient insight into the general characteristics of $\widetilde{\mathcal{A}}$, to justify the trust that the separation criterion is valid (of course, this does not mean that the target mapping must be known in detail). This is a fundamental commitment on part of the expert applying the pattern. Acknowledging that the use of fuzzification mechanisms always presupposes the separation criterion to hold, one must subscribe to the validity of the criterion whenever applying the pattern. In this sense, the separation criterion now represents the basic *instantiation condition* of the fuzzification pattern, which is absolutely mandatory for its successful application.

Provided that the pattern is applied correctly, i.e. the instantiation condition is verified by the chosen $\mathcal{U}$, it is hence guaranteed that a specification of incomplete solutions $\mathcal{A} : \mathbb{P} \longrightarrow \mathbb{S}$ and a fuzzification mechanism $\mathcal{F} : \mathbb{S} \longrightarrow \widetilde{\mathbb{S}}$ exist which solve the problem. Given such a collection of solution fragments $\mathbb{S}$ and their semantics ascribed by $\mathcal{U}$, it is then up to the expert to specify suitable choices of provisional solutions $\mathcal{A}$ and correspondence assertions $\mathcal{F}$. In principle, these input parameters must be fully defined when applying the pattern. However, I envision that establishing an instance $\mathcal{A}(p)$ of incomplete solution might still require considerable effort. Consequently, it will greatly improve the practicality of the pattern, if we do not assume that $\mathcal{A}$ be fully known in advance. In order to make the pattern more generic and maximise its range of applications, we should rather try and permit the user to specify only those instances of $\mathcal{A}(p)$, which currently attract most interest. Avoiding the reference to the 'full' mapping $\mathcal{A}$ will be rewarding because it makes the fuzzification approach applicable even when our knowledge of $\mathcal{A}$ is incomplete, and consists only of a limited number of assertions $s = \mathcal{A}(p)$.

B.8 Relativized generalization

When unfolding the basic analysis in section B.4, I also took some steps to constrain the admissible choices of $\mathcal{F}$. Specifically, it was the fuzzification equality $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$ which I introduced for this purpose. When applying the pattern in practice, though, the fuzzification equality loses its constraining function. Hence suppose that a choice of partial solutions $s = \mathcal{A}(p)$ and corresponding results $\mathcal{F}(s)$ of the fuzzification mechanism are fixed. In this case, the fuzzification equality is used to fetch the full solutions by applying the fuzzification mechanism to $\mathcal{A}(p)$. In other words, the fuzzification equality (257) has now become the defining equation of the dependent variable $\widetilde{\mathcal{A}}$. We must therefore develop other measures against poor examples, in order to ensure a 'good' choice of $\mathcal{F}$ and set up a valid instance of the underlying pattern. The basic idea is that of imposing formal constraints on the admissible choices of $\mathcal{F}$, i.e. the desired quality standards will be achieved by enforcing logical and application-specific requirements. In general terms, $\mathcal{F}$ should be chosen such that plausible choices of s be mapped to plausible choices of full solutions $\widetilde{s} = \mathcal{F}(s)$. The precise choice of such requirements is application-specific. In fact, systems which reach for high quality standards can become rather specialized, as witnessed by the DFS axioms described in Chap. 3, which further constrain the admissible models of fuzzy quantification. There is one minimal requirement, however, which fits into this discussion of the generic pattern, because it is structural in nature, and mandatory for a successful application of fuzzification mechanisms in general. To wit, it is the former reduction equality $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$ which gives us a grip on the legal choices of $\mathcal{F}$. Because $\mathcal{A}$ has now become an input parameter, the reduction equality no longer serves a defining purpose, but rather constrains the admissible fuzzification mechanisms. To see this, recall that $\widetilde{\mathcal{A}}$ has now become a dependent variable, defined by $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$. By substituting $\mathcal{F} \circ \mathcal{A}$ for $\widetilde{\mathcal{A}}$, the reduction equality $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$ expands as follows,

$$\mathcal{A} = \mathcal{U} \circ \mathcal{F} \circ \mathcal{A}\,. \tag{262}$$

The former criterion then changes into the condition of *relativized generalisation*, which constrains $\mathcal{F}$ to those conforming choices that result in a valid reduction $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$. The chosen name becomes clear if we consider the following equivalent rendering of relativized generalization,

$$\forall s \in \operatorname{Im} \mathcal{A}, \quad \mathcal{U}(\mathcal{F}(s)) = s\,. \tag{263}$$

Hence relativized generalization expresses the requirement that $\mathcal{F}(s)$ correctly generalizes s to the fuzzy case. When judging from the 'easy' cases only, and hence reducing a full solution to $\mathcal{U}(\mathcal{F}(s))$ again, it will always coincide with the original base solution s. The condition is called 'relativized' because it only applies to visible solutions $s = \mathcal{A}(p) \in \operatorname{Im} \mathcal{A}$, which actually solve some problem $p \in \mathbb{P}$.

B.9 Correct generalization

At this point, we have managed to enforce the reduction equality. However, things are not settled yet, because the proposed condition of relativized generalization still assumes complete knowledge of $\mathrm{Im}\,\mathcal{A}$, i.e. we must be able to tell dispensible from indispensible solutions. This dependency will now be eliminated in favour of the flexible use of the pattern. In practice, the full definition of $\mathcal{A}$ and the precise extent of $\mathrm{Im}\,\mathcal{A}$ will often be unknown in advance, because the user will usually elaborate only those instances of $\mathcal{A}(p)$, which relate to a problem of actual interest. In order to preserve the validity of the reduction equality and simultaneously avoid any reference to $\mathrm{Im}\,\mathcal{A}$, we will simply to drop the relativization to $\mathrm{Im}\,\mathcal{A}$, and strengthen the original condition (263) to the unrestricted requirement that

$$\forall s \in \mathbb{S}, \quad \mathcal{U}(\mathcal{F}(s)) = s\,. \tag{264}$$

As a matter of caution, the resulting condition of *correct generalization*, now catches hold of arbitrary elements in the surrounding collection $\mathbb{S} \supseteq \mathrm{Im}\,\mathcal{A}$. Abstracting from individual elements, it can be rewritten more succinctly,

$$\mathcal{U} \circ \mathcal{F} = \mathrm{id}_{\mathbb{S}}\,, \tag{265}$$

where $\mathrm{id}_{\mathbb{S}}$ is the identity $\mathrm{id}_{\mathbb{S}}(s) = s$ for all $s \in \mathbb{S}$. The latter equality no longer depends on $\mathcal{A}$ and hence makes a practical filtering criterion, which only admits those choices of $\mathcal{F}$ that comply with the reduction equality.

The novel requirement also means a strengthening of the original system, though, which might result in a loss of intended models. It is therefore important that we analyse the precise assumptions made by introducing this criterion. We shall start with an apparent observation: due to the fact that the identity $\mathrm{id}_{\mathbb{S}}$ is a bijection, the composition $\mathcal{U} \circ \mathcal{F} = \mathrm{id}_{\mathbb{S}}$ reveals that the condition of correct generalization can only be fulfilled if $\mathcal{U}$ is a surjection (onto). Hence let us check if the requirement that $\mathcal{U}$ be onto will limit the scope of the fuzzification pattern. Owing to the reduction equality, we can expand $\mathcal{A}$ into $\mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}}$. In particular, $\mathrm{Im}\,\mathcal{A} = \mathrm{Im}\mathcal{U} \circ \widetilde{\mathcal{A}} \subseteq \mathrm{Im}\mathcal{U}$, which shows that all visible solutions are also within reach of $\mathcal{U}$. In other words, the remaining solutions outside $\mathrm{Im}\mathcal{U}$ are all dispensible. Therefore no solutions of relevance (which actually solve one of the problems in $\mathbb{P}$) will be lost, if we replace the surrounding collection $\mathbb{S}$ of potential solutions with the smaller collection $\mathbb{S}' = \mathrm{Im}\mathcal{U}$. This will make $\mathcal{U} : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}'$ a surjection. Because $\mathrm{Im}\,A \subseteq \mathbb{S}'$ and $\mathrm{Im}\mathcal{U} \subseteq \mathbb{S}'$, $\mathcal{A}$ and $\mathcal{U}$ can remain unchanged, i.e. the reduction equality will not be touched. The shift from $\mathbb{S}$ to $\mathbb{S}' \subseteq \mathbb{S}$ makes it necessary to restrict the fuzzification mechanism $\mathcal{F}$ to $\mathcal{F}' = \mathcal{F}|_{\mathbb{S}'}$. This will not affect the fuzzification equality $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$, though, because $\mathrm{Im}\,\mathcal{A} \subseteq \mathbb{S}'$. Consequently, the requirement that $\mathcal{U}$ be onto can always be fulfilled by restricting $\mathbb{S}$ to $\mathrm{Im}\mathcal{U}$. This modification will not result in a loss of generality, because the interesting properties of the system are left intact. In the following, I will therefore assume that $\mathcal{U}$ be a surjection, in order to meet the liabilities of correct generalization.

Based on this finding, we can now show that correct generalization does not mean a real strengthening compared to relativized generalization, because it still spans the range of possible models. Hence suppose that $\mathcal{U}$ be onto. Starting from the given $\mathcal{F}$, we can then define a modified fuzzification mechanism $\mathcal{F}^*$ by

$$\mathcal{F}^*(s) = \begin{cases} \mathcal{F}(s) & : \quad s \in \operatorname{Im} A \\ \widetilde{s} & : \quad s \notin \operatorname{Im} A \end{cases}$$

for all $s \in \mathbb{S}$, where $\widetilde{s}$ is an arbitrary choice of $\widetilde{s} \in \mathcal{U}^{-1}(s)$, which is known to exist because $\mathcal{U}$ is surjective. Clearly $\mathcal{F}^* \circ A = \mathcal{F} \circ A$, i.e. both mechanisms result in the same target mapping $\widetilde{\mathcal{A}}$. In addition, the modified fuzzification mechanism $\mathcal{F}^*$ now satisfies correct generalization, provided that $\mathcal{F}$ complies with the relativized criterion. For proving this, we consider the following two cases. If s is indispensible, i.e. $s \in \operatorname{Im} A$, then $\mathcal{F}^*(s) = \mathcal{F}(s)$ and consequently $\mathcal{U}(\mathcal{F}^*(s)) = \mathcal{U}(\mathcal{F}(s)) = s$ by (263). In the remaining case that s is dispensible, we know that $\mathcal{F}^*(s) = \widetilde{s} \in \mathcal{U}^{-1}(s) \neq \varnothing$, i.e. $\mathcal{U}(\mathcal{F}^*(s)) = \mathcal{U}(\widetilde{s}) = s$. This demonstrates that $\mathcal{F}^*$ indeed complies with correct generalization. The remaining claim that $\mathcal{F}^* \circ \mathcal{A} = \mathcal{F} \circ \mathcal{A}$ is apparent from the fact that $\mathcal{F}$ and $\mathcal{F}^*$ coincide on $\operatorname{Im} A$ by (261).

To sum up, the shift to correct generalization only means a superficial strengthening of the original system, and does not cause any loss of relevant models. In particular, we can always use the practical criterion of correct generalization in lieu of relativized generalization, in order to avoid the requirement that $\operatorname{Im} \mathcal{A}$ be known in advance. Following common practice [91, 103], I will generally assume that the unrestricted condition of correct generalization be used, and hence made an integral part of the fuzzification pattern[62] Knowing that the target mapping $\widetilde{\mathcal{A}}$ is obtained from $\mathcal{A}$ by applying the fuzzification mechanism, i.e. $\widetilde{\mathcal{A}} = \mathcal{F} \circ \mathcal{A}$, correct generalization will then guarantee the desired reduction

$$\mathcal{A} = \mathrm{id}_{\mathbb{S}} \circ \mathcal{A} = \mathcal{U} \circ \mathcal{F} \circ \mathcal{A} = \mathcal{U} \circ \widetilde{\mathcal{A}} .$$

Consequently, the constraint will ensure that all admissible choices of $\mathcal{F}$ result in a valid instantiation of the original system, which confirms the analysis in terms of the fuzzification and reduction equality. Just like its relativized cousin, correct generalization should be viewed as a minimal requirement on plausible models, because it filters out only those mechanisms which must be rejected for purely formal reasons. In practice, the criterion should be backed with additional problem-specific requirements which further specify the optimal choices of $\mathcal{F}$ for the given application. It is then understood that only the best mechanisms which comply with all requirements, will be regarded valid instances of the fuzzification pattern.

B.10 Chapter summary

In this chapter, I have shed some light on the methodological origins of the proposed framework, by analysing the generic pattern which underlies the use of fuzzification mechanisms. The results of this investigation can now be summarized as follows. The fuzzification pattern is a proven strategy for solving problems in the fuzzy sets framework. The pattern is useful in situations where the presence of fuzziness makes it

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[62] The criterion is called ‘Normalbedingung’ (normality condition) by Kreiser et al. [91]; Menhardt [103, p. 4-11] uses the term ‘Minimalbedingung’ (minimal condition).

prohibitively difficult to establish the solutions $\widetilde{A}(p) \in \widetilde{\mathbb{S}}$ of a given class of problems $p \in \mathbb{P}$. Under these circumstances, it makes sense to try a reduction to the crisp case. We hence introduce a collection $\mathbb{S}$ of solution fragments $s \in \mathbb{S}$, which drop some of the fuzzy cases. The relationship between the two types of solutions is established by a mapping $\mathcal{U} : \widetilde{\mathbb{S}} \longrightarrow \mathbb{S}$, which assigns to each full solution $\widetilde{s} \in \widetilde{\mathbb{S}}$ its simplified fraction $s = \mathcal{U}(\widetilde{s}) \in \mathbb{S}$. Without loss of generality, we shall further assume that $\mathcal{U}$ be surjective (onto). The choice of $\mathcal{U}$ and $\mathbb{S}$ is critical to the successful application of the pattern. The expert who sets up the pattern must ensure that $\mathcal{U}$ validate the instantiation condition explained in section B.7. Under these circumstances, the existence of a fuzzification mechanism $\mathcal{F}$ with the desired properties will be granted. $\mathcal{F}$ should be chosen in accordance with the the condition of correct generalization. Apart from this minimal requirement, application-specific criteria will usually also be considered. Once $\mathcal{F}$ is fixed, we are no longer forced to solve a given problem $p \in \mathbb{P}$ directly, and elaborate each new solution $\widetilde{\mathcal{A}}(p)$ from scratch. By contrast, we can now content ourselves with establishing the incomplete solution $\mathcal{A}(p)$, and delegate the responsibility of completing this proposal to the fuzzification mechanism. Starting from a plausible choice of $s = \mathcal{A}(p)$, we simply apply $\mathcal{F}$ in order to fetch the final interpretation $\widetilde{s} = \mathcal{F}(s) = \widetilde{\mathcal{A}}(p)$, which now covers the full problem including all fuzzy cases.

It should be evident how the proposed framework can be mapped to this structure and viewed as an instance of the fuzzification pattern. In this case, $\mathbb{P}$ comprises problems p of the type

> *How can we model the NL quantifier Q?*

A *complete solution* $\widetilde{Q} = \widetilde{\mathcal{A}}(p)$ to the problem must specify the quantifier in all detail, including its behaviour for fuzzy arguments. In order to give a full solution, the problem must hence be rephrased as follows,

> *What is the proper interpretation of the given NL quantifier in terms of a fuzzy quantifier?*

Obviously, this is the correspondence problem described in section 2.4, which conjures up the dilemma of fuzzy quantifiers. In order to eliminate part of this complexity and better support the specification of fuzzy quantifiers in practice, I suggested to temporarily focus on crisp arguments only. We hence content ourselves with providing incomplete solutions, which no longer solve p in full, and only account for the following subproblem:

> *What is the proper specification of the given NL quantifier in terms of a semi-fuzzy quantifier*?

This simplification now permits us to specify the target quantifier by supplying a matching choice of semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, thus stipulating the correspondence $\mathcal{A}(p) = Q$. The interpretation will be completed by applying a QFM. This will take Q to the corresponding fuzzy quantifier $\widetilde{Q} = \mathcal{F}(Q)$, which then represents the solution of the original problem, i.e. $\widetilde{Q} = \widetilde{\mathcal{A}}(p)$.

We know from the above general analysis that the chosen QFM $\mathcal{F}$ must at least satisfy the condition of correct generalization. In addition, I have proposed the use of application-specific criteria which further constrain the admissible choices. Of course, this is precisely the strategy pursued in Chap. 3, where I investigated the intuitive expectations on plausible interpretations, which where then compiled into the axiomatic description of intended models. A look back at the framework assumption now confirms that the QFA is really just a rendering of the separation criterion.[63] The QFA therefore makes sure that a successful interpretation in terms of a fuzzification mechanism $\mathcal{F}$ will indeed be possible. In particular, the full solution (fuzzy quantifier) $\widetilde{Q}$ of the modelling problem can always be computed by applying $\mathcal{F}$ to the semi-fuzzy quantifiers (incomplete solutions), which I proposed as the base representations for fuzzy NL quantification.

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[63] To be precise, the condition of correct generalization has not been included into the definition of QFMs. However, the proposed DFS axioms ensure that all conforming models comply with this criterion, see Th-2.

C Theorem reference chart

Theorems cited in Chap. 3:
Th-1: see [48], Th-69/72/73, p. 52

Theorems cited in Chap. 4:
Th-2: see [46], Th-1, p. 27
Th-3: see [46], Th-2, p. 27
Th-4: see [46], Th-3, p. 28
Th-5: see [46], Th-6, p. 30
Th-6: see [46], Th-7a, p. 31
Th-7: see [46], Th-7b, p. 31
Th-8: see [48], Th-36, p. 36
Th-9: see [48], Th-38, p. 38
Th-10: see [48], Th-82, p. 58
Th-11: see [48], Th-38, p. 38
Th-12: see [48], Th-38, p. 38
Th-13: see [46], Th-9, p. 35
Th-14: see [48], Th-38, p. 38
Th-15: see [48], Th-38, p. 38
Th-16: see [46], Th-4, p. 28
Th-17: see [46], Th-12, p. 36
Th-18: see [46], Th-13, p. 36
Th-19: see [46], Th-14, p. 37
Th-20: see [46], Th-16, p. 38
Th-21: see [46], Th-17, p. 38
Th-22: see [46], Th-19, p. 39
Th-23: see [46], Th-21, p. 40
Th-24: see [46], Th-22, p. 40
Th-25: see [46], Th-18, p. 39
Th-26: see [49], Th-16, p. 18
Th-27: see [48], Th-75, p. 55
Th-28: see [49], Th-9, p. 16
Th-29: see [147], Th-8.1, p. 42
Th-30: see [48], Th-24, p. 24
Th-31: see [147], Th-8.2, p. 48
Th-32: see [48], Th-25, p. 25
Th-33: see [48], Th-26, p. 25
Th-34: see [46], Th-23, p. 41
Th-35: see [48], Th-33, p. 34
Th-36: see [48], Th-34, p. 36
Th-37: see [48], Th-35, p. 36

Theorems cited in Chap. 5:
Th-38: see [46], Th-27, p. 43
Th-39: see [46], Th-28, p. 44

Th-40: see [46], Th-29, p. 44
Th-41: see [48], Th-30, p. 27
Th-42: see [49], Th-27, p. 22
Th-43: see [46], Th-32, p. 48
Th-44: see [46], Th-33, p. 48
Th-45: see [46], Th-34, p. 49
Th-46: see [48], Th-125, p. 74
Th-47: see here on p. 168

Theorems cited in Chap. 6:

Th-48: see [46], note, p. 50
Th-49: see [46], note, p. 52
Th-50: see [48], Th-76, p. 56
Th-51: see [48], Th-77, p. 56
Th-52: see [48], Th-78, p. 56
Th-53: see [48], Th-79, p. 57
Th-54: see [48], Th-81, p. 57
Th-55: see [48], Th-84, p. 59
Th-56: see [48], Th-85, p. 59
Th-57: see [49], Th-34, p. 27
Th-58: see [49], Th-35, p. 27

Theorems cited in Chap. 7:

Th-59: see [46], Th-40, p. 57
Th-60: see [46], 41, p. 57
Th-61: see here on p. 199
Th-62: see [46], Th-42, p. 61
Th-63: see [46], Th-44, p. 63
Th-64: see [48], Th-40, p. 42
Th-65: see [48], Th-41, p. 43
Th-66: see [48], Th-62, p. 49
Th-67: see [48], Th-62, p. 49
Th-68: see [48], Th-58, p. 49
Th-69: see [48], Th-65 to Th-70, p. 51/52
Th-70: see [48], note, p. 47
Th-71: see [48], Th-64, p. 50
Th-72: see [48], Th-64, p. 50
Th-73: see [48], Th-71, p. 52
Th-74: see [48], note, p. 48
Th-75: see [48], Th-87, p. 61
Th-76: see [48], Th-91, p. 62
Th-77: see [48], Th-93, p. 62
Th-78: see [48], Th-86, p. 61
Th-79: see [50], Th-18, p. 23
Th-80: see [48], Th-88, p. 61
Th-81: see [48], Th-89, p. 61

Th-82: see [48], Th-90, p. 62
Th-83: see [48], Th-92, p. 62
Th-84: see [48], Th-103, p. 65
Th-85: see [48], Th-104, p. 65
Th-86: see [48], Th-105, p. 65
Th-87: see [48], Th-106, p. 66
Th-88: see [48], Th-107, p. 66
Th-89: see [48], Th-108, p. 66
Th-90: see [48], Th-109, p. 66
Th-91: see [48], Th-110, p. 67
Th-92: see [48], Th-111, p. 67
Th-93: see [48], Th-112, p. 67
Th-94: see [48], Th-113, p. 67
Th-95: see [48], Th-44, p. 46
Th-96: see [48], Th-46, p. 46
Th-97: see [48], Th-102, p. 65
Th-98: see [48], Th-124, p. 74
Th-99: see here on p. 216
Th-100: see [48], Th-100, p. 65
Th-101: see [48], Th-101, p. 65
Th-102: see [48], Th-123, p. 73
Th-103: see [48], Th-126, p. 74
Th-104: see [48], Th-132, p. 76
Th-105: see [48], Th-127, p. 74
Th-106: see [48], Th-128, p. 75

Theorems cited in Chap. 8:

Th-107: see [50], Th-20, p. 25
Th-108: see [50], Th-21, p. 25
Th-109: see [50], Th-22, p. 26
Th-110: see [50], Th-23, p. 26
Th-111: see [50], Th-24, p. 27
Th-112: see [50], Th-25, p. 27
Th-113: see [50], Th-26, p. 27
Th-114: see [50], Th-27, p. 28
Th-115: see [50], Th-28, p. 28
Th-116: see [50], Th-29, p. 28
Th-117: see [50], Th-30, p. 29
Th-118: see [50], Th-31, p. 29
Th-119: see [50], Th-32, p. 29
Th-120: see [50], Th-33, p. 30
Th-121: see [50], Th-34, p. 30
Th-122: see [50], Th-35, p. 30
Th-123: see [50], Th-36, p. 30
Th-124: see [50], Th-37, p. 30
Th-125: see [50], Th-38, p. 31

Th-126: see [50], Th-39, p. 31
Th-127: see [50], Th-40, p. 31
Th-128: see [50], Th-41, p. 31
Th-129: see [50], Th-42, p. 31
Th-130: see [50], Th-43, p. 31
Th-131: see [50], Th-44, p. 32
Th-132: see [50], Th-45, p. 32
Th-133: see [50], Th-46, p. 32
Th-134: see [50], Th-47, p. 32
Th-135: see [50], Th-48, p. 33
Th-136: see [50], Th-49, p. 33
Th-137: see [50], Th-50, p. 33
Th-138: see [50], Th-51, p. 34
Th-139: see [50], Th-52, p. 34
Th-140: see [50], Th-53, p. 34
Th-141: see [50], Th-54, p. 34
Th-142: see [50], Th-55, p. 34
Th-143: see [50], Th-56, p. 34
Th-144: see [50], Th-57, p. 35
Th-145: see [50], Th-58, p. 35
Th-146: see [50], Th-59, p. 35
Th-147: see [50], Th-60, p. 35
Th-148: see [50], Th-61, p. 35
Th-149: see [50], Th-62, p. 35
Th-150: see [50], Th-63, p. 36
Th-151: see [50], Th-64, p. 36
Th-152: see [50], Th-65, p. 36

Theorems cited in Chap. 9:

Th-153: see [51], Th-32, p. 34
Th-154: see [51], Th-33, p. 35
Th-155: see [51], Th-34, p. 36
Th-156: see [51], Th-35, p. 37
Th-157: see [51], Th-36, p. 37
Th-158: see [51], Th-37, p. 37
Th-159: see [51], Th-38, p. 38
Th-160: see [51], Th-39, p. 38
Th-161: see [51], Th-40, p. 38
Th-162: see [51], Th-41, p. 39
Th-163: see [51], Th-42, p. 39
Th-164: see [51], Th-43, p. 40
Th-165: see [51], Th-44, p. 40
Th-166: see [51], Th-45, p. 40
Th-167: see [51], Th-46, p. 41
Th-168: see [51], Th-47, p. 41
Th-169: see [51], Th-49, p. 42

Th-170: see [51], Th-50, p. 42
Th-171: see [51], Th-51, p. 43
Th-172: see [51], Th-52, p. 43
Th-173: see [51], Th-53, p. 43
Th-174: see [51], Th-54, p. 44
Th-175: see [51], Th-55, p. 44
Th-176: see [51], Th-56, p. 44
Th-177: see [51], Th-57, p. 44
Th-178: see [51], Th-58, p. 45
Th-179: see [51], Th-59, p. 45
Th-180: see [51], Th-60, p. 46
Th-181: see [51], Th-61, p. 46
Th-182: see [51], Th-62, p. 46
Th-183: see [51], Th-63, p. 46
Th-184: see [51], Th-64, p. 47
Th-185: see [51], Th-65, p. 47
Th-186: see [51], Th-66, p. 47
Th-187: see [51], Th-67, p. 47
Th-188: see [51], Th-68, p. 47
Th-189: see [51], Th-69, p. 47
Th-190: see [51], Th-70, p. 47
Th-191: see [51], Th-71, p. 48
Th-192: see [51], Th-72, p. 48
Th-193: see [51], Th-73, p. 48
Th-194: see [51], Th-74, p. 48
Th-195: see [51], Th-75, p. 48
Th-196: see [51], Th-76, p. 48
Th-197: see [51], Th-77, p. 48
Th-198: see [51], Th-78, p. 49
Th-199: see [51], Th-79, p. 49
Th-200: see [51], Th-80, p. 49
Th-201: see [51], Th-81, p. 49
Th-202: see [51], Th-82, p. 49
Th-203: see [51], Th-83, p. 49
Th-204: see [51], Th-84, p. 50
Th-205: see [51], Th-85, p. 50
Th-206: see [51], Th-86, p. 50
Th-207: see [51], Th-87, p. 50
Th-208: see [51], Th-88, p. 50
Th-209: see [51], Th-89, p. 50
Th-210: see [51], Th-90, p. 50
Th-211: see [51], Th-91, p. 51

Theorems cited in Chap. 10:

Th-212: see [51], Th-92, p. 56
Th-213: see [51], Th-93, p. 57

Th-214: see [51], Th-94, p. 57
Th-215: see [51], Th-95, p. 57
Th-216: see [51], Th-99, p. 61
Th-217: see [51], Th-107, p. 62
Th-218: see [51], Th-116, p. 63
Th-219: see [51], Th-121, p. 64
Th-220: see [51], Th-97, p. 61
Th-221: see [51], Th-98, p. 61
Th-222: see [51], Th-117, p. 63
Th-223: see [51], Th-118, p. 64
Th-224: see [51], Th-119, p. 64
Th-225: see [51], Th-120, p. 64
Th-226: see [51], Th-122, p. 66
Th-227: see [51], Th-123, p. 66
Th-228: see [51], Th-124, p. 67
Th-229: see [51], Th-125, p. 67
Th-230: see [51], Th-126, p. 68
Th-231: see [51], Th-127, p. 68
Th-232: see [51], Th-128, p. 68
Th-233: see [51], Th-129, p. 68
Th-234: see [51], Th-130, p. 69

Theorems cited in Chap. 11:

Th-235: see [46], Th-35, p. 49
Th-236: see [48], Th-119, p. 72
Th-237: see [48], Th-120, p. 72
Th-238: see here on p. 291
Th-239: see here on p. 291
Th-240: see [48], 95, p. 63
Th-241: see here on p. 292
Th-242: see [48], 96, p. 63
Th-243: see [48], Th-130, p. 75
Th-244: see [48], 98, p. 64
Th-245: see [48], 129, p. 75
Th-246: see here on p. 296
Th-247: see here on p. 298
Th-248: see here on p. 299
Th-249: see here on p. 299
Th-250: see here on p. 300
Th-251: see here on p. 300
Th-252: see here on p. 301
Th-253: see here on p. 301
Th-254: see here on p. 302
Th-255: see here on p. 305
Th-256: see here on p. 306
Th-257: see here on p. 306

Th-258: see here on p. 307
Th-259: see here on p. 311
Th-260: see here on p. 312
Th-261: see here on p. 314
Th-262: see here on p. 315
Th-263: see here on p. 316
Th-264: see here on p. 320
Th-265: see here on p. 331
Th-266: see here on p. 338
Th-267: see here on p. 341
Th-268: see here on p. 341
Th-269: see here on p. 342
Th-270: see here on p. 347

Theorems cited in Chap. 12:
Th-271: see here on p. 370
Th-272: see here on p. 372
Th-273: see here on p. 372
Th-274: see here on p. 372
Th-275: see here on p. 375

Note. Some of the cited theorems are based on a different formalization of the DFS axioms that was later replaced with the current system (Z-1)–(Z-6). The equivalence of these formalizations has been proven in [48, Th-38, p. 38]. It is therefore guaranteed that those theorems cited above, which stem from the old axioms, remain valid in the new system. The original formalization of the models also referred to a different construction of induced fuzzy truth functions. However, the fuzzy truth functions induced by the old construction have been shown to coincide with those obtained from the new construction assumed here, see [48, Th-36, p. 36]. This ensures that all ‘old’ results concerning induced connectives remain valid for the new construction.

D Proofs of the new theorems

Any proposition which occurs in the main text is called a *theorem*, and any proposition which only occurs in the proofs a *lemma*. Theorems are referred to as Th-n, where n is the number of the theorem, while lemmata are referred to as L-n, where n is the number of the lemma. Equations which are embedded in proofs are referred to as (n), where n is the number of the equation.

D.1 Proof of Theorem 47

Lemma 1
Let a QFM $\mathcal{F}$ be given, and suppose that $\mathcal{F}$ satisfies (S-2) *and* (S-6). *Then $\mathcal{F}$ induces the standard fuzzy disjunction, i.e. $\widetilde{\mathcal{F}}(\vee) = \vee$, where $x_1 \vee x_2 = \max(x_1, x_2)$.*

Proof To see this, let $x_1, x_2 \in \mathbf{I}$ be given. Recalling the construction of induced connectives, $\widetilde{\mathcal{F}}(\vee)(x_1, x_2)$ becomes

$$\widetilde{\mathcal{F}}(\vee)(x_1, x_2) = \mathcal{F}(Q_\vee)(\widetilde{\eta}(x_1, x_2)) \,,$$

where $Q_\vee : \mathcal{P}(\{1,2\}) \longrightarrow \mathbf{2}$ is the following quantifier,

$$Q_\vee(Y) = \begin{cases} 1 & : \quad Y \neq \varnothing \\ 0 & : \quad Y = \varnothing \end{cases} \tag{266}$$

for all $Y \in \mathcal{P}(\{1,2\})$, see Def. 11. In other words, $Q_\vee = \exists_{\{1,2\}}$. Let us now notice that $\exists_{\{1,2\}}$ can be expressed as

$$\exists_{\{1,2\}} = \pi_\varnothing \circ \widehat{!} \,,$$

where $\pi_\varnothing$ is the projection quantifier $\pi_\varnothing : \mathcal{P}(\{)\varnothing\} \longrightarrow \mathbf{2}$, and $! : \{1,2\} \longrightarrow \{\varnothing\}$ is the mapping defined by $!(e) = \varnothing$ for all $e \in \{1,2\}$. (This rephrasing of the existential quantifier has also been utilized in the earlier theorem Th-24). Combining these results, we now obtain that

$$Q_\vee = \pi_\varnothing \circ \widehat{!} \,. \tag{267}$$

Let us now consider $\hat{\hat{!}} : \widetilde{\mathcal{P}}(\{1,2\}) \longrightarrow \widetilde{\mathcal{P}}(\varnothing)$. By Def. 21, the fuzzy powerset mapping becomes

$$\begin{aligned} \mu_{\hat{\hat{!}}(X)}(()\varnothing) &= \sup\{\mu_X(e) : e \in !^{-1}(\varnothing)\} \\ &= \sup\{\mu_X(e) : e \in \{1,2\}\} \\ &= \max(\mu_X(1), \mu_X(2)) \,, \end{aligned}$$

for all $X \in \widetilde{\mathcal{P}}(\{1,2\})$. In particular

$$\mu_{\hat{\hat{!}}(\widetilde{\eta}(x_1,x_2))}(()\varnothing) = \max(x_1, x_2) \tag{268}$$

by the definition of $\widetilde{\eta}(x_1, x_2)$, see equation (16). We can hence proceed as follows.

$$\begin{aligned}
\widetilde{\mathcal{F}}(\vee)(x_1, x_2) &= \mathcal{F}(Q_\vee)(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 11} \\
&= \mathcal{F}(\pi_\varnothing \circ \widehat{!})(\widetilde{\eta}(x_1, x_2)) && \text{by (267)} \\
&= \widetilde{\pi}_\varnothing(\hat{\hat{!}}(\widetilde{\eta}(x_1, x_2))) && \text{by (S-2) and (S-6)} \\
&= \mu_{\hat{\hat{!}}(\widetilde{\eta}(x_1,x_2))}(\varnothing) && \text{by Def. 10} \\
&= \max(x_1, x_2)\,, && \text{by (268)}
\end{aligned}$$

as desired.

Lemma 2
Suppose a QFM $\mathcal{F}$ satisfies (S-2) *and* (S-6). *Then $\mathcal{F}$ induces the standard extension principle, i.e. $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$.*

Proof Consider some mapping $f : E \longrightarrow E'$ where $E, E' \neq \varnothing$. Further let $X \in \widetilde{\mathcal{P}}(E)$ and $e' \in E'$ be given. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(e') &= \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X) && \text{by Def. 22} \\
&= \mathcal{F}(\pi_{e'} \circ \widehat{f})(X) && \text{apparent from Def. 8 and Def. 9} \\
&= \widetilde{\pi}_{e'}(\hat{\hat{f}}(X)) && \text{by (S-2), (S-6)} \\
&= \mu_{\hat{\hat{f}}(X)}(()e')\,. && \text{by Def. 10}
\end{aligned}$$

Recalling that $e' \in E'$ was arbitrarily chosen, this proves that $\widehat{\mathcal{F}}(f)(X) = \hat{\hat{f}}(X)$. Now observing that the choice of f and X was arbitrary as well, we obtain the final result that indeed $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$.

Lemma 3
Suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 1$ and $A \in \mathcal{P}(E)$. Further let a choice of $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ be given, which is assumed to be involutive. Then

$$Q \triangleleft A = \langle Q \rangle \circ \widehat{h}\widetilde{\Box} \circ \widehat{k}\widetilde{\Box}\cup^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\imath}_i^{\,n-1,E} \tag{269}$$

where

$$E'' = E_{n-1} \cup \{(e, n) : e \in A\}\,, \tag{270}$$

$k : E_{n-1} \longrightarrow E''$ is the inclusion

$$k(e, i) = (e, i) \tag{271}$$

for all $(e, i) \in E_{n-1}$, and $h : E'' \longrightarrow E_n$ is the inclusion

$$h(e, i) = (e, i) \tag{272}$$

for all $(e, i) \in E''$.

Proof See [48, L-16, p. 96].

Lemma 4

Consider a QFM $\mathcal{F}$, and let us assume that $\mathcal{F}$ satisfies (S-2), (S-3), (S-4) *and* (S-6). *Then $\mathcal{F}$ induces the standard fuzzy negation $\neg x = 1 - x$, i.e. $\widetilde{\mathcal{F}}(\neg) = \neg$.*

Proof To see this, let $x \in \mathbf{I}$ be given. We further abbreviate

$$X = \widetilde{\eta}(x)\,, \tag{273}$$

i.e. $X \in \widetilde{\mathcal{P}}(\{1\})$ is defined by

$$\mu_X(1) = x\,. \tag{274}$$

We shall further assume that $Q_\neg : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$ is defined in accordance to Def. 11, and hence validates

$$\widetilde{\mathcal{F}}(\neg)(x) = \mathcal{F}(Q_\neg)(X)\,. \tag{275}$$

In order to prove the lemma, it is necessary to introduce another quantifier $Q' : \mathcal{P}(\{1\})^2 \longrightarrow \mathbf{2}$, which is constructed from the projection quantifier $\pi_1 : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$, viz

$$Q' = \pi_1 \cup \Box\,. \tag{276}$$

In other words,

$$Q'(Y_1, Y_2) = \begin{cases} 1 & : \quad Y_1 = \varnothing \wedge Y_2 = \{1\} \\ 0 & : \quad \text{else} \end{cases} \tag{277}$$

for all $Y_1, Y_2 \in \mathcal{P}(\{1\})$, which is apparent from Def. 9, Def. 14 and Def. 15. Let us further observe that $Q_\neg$ can be expressed in terms of Q' by utilizing the concept of argument insertion. $Q_\neg$ then becomes

$$Q_\neg = Q' \triangleleft \{1\}\,. \tag{278}$$

This is immediate from Def. 34 and (277).

It is here that we can profit from the previous lemma L-3. To this end, let us instantiate the lemma by the base set $E = \{1\}$, the quantifier Q' of arity $n = 2$ and the crisp argument $A = \{1\}$. The lemma then permits us to rewrite the quantifier $Q' \triangleleft \{1\}$ as follows,

$$Q' \triangleleft \{1\} = \langle Q' \rangle \circ \widehat{h} \Box \circ \widehat{k} \Box \cup^1 \circ \mathop{\times}_{i=1}^{1} \widehat{\imath}_i^{\,1,E} \tag{279}$$

where $k : E_1 \longrightarrow E''$, $h : E'' \longrightarrow E_2$ and E'' are defined by (271), (272) and (270), respectively. Next we should simplify the above expression. Firstly, it is apparent from the fact that the full domain $A = \{1\} = E$ has been inserted into the quantifier, that E'' collapse into the original domain $E'' = E_2 = \{(1,1), (1,2)\}$. We also need

E_1, which becomes $E_1 = \{(1,1)\}$. It is then obvious that k is in fact the inclusion $k : \{(1,1)\} \longrightarrow \{(1,1),(1,2)\}$, and hence defined by

$$k(1,1) = (1,1)\,, \tag{280}$$

see (271). In addition, the result on E'' discloses that h is in fact the inclusion $h : \{(1,1),(1,2)\} \longrightarrow \{(1,1),(1,2)\}$, i.e. defined by $h(1,1) = (1,1)$, $h(1,2) = (1,2)$, see (272). In other words, h is the identity $h = \mathrm{id}_{\{(1,1),(1,2)\}}$. Consequently, the fuzzy powerset mapping $\widehat{h} = \mathrm{id}_{\mathcal{P}(\{(1,1),(1,2)\})}$ is an identity as well, and can hence be cancelled in the right-hand member of (279). By further eliminating trivial cartesian products, the above equation can then be simplified into

$$Q' \triangleleft \{1\} = \langle Q' \rangle \Box \circ \widehat{k} \Box \circ \widehat{\imath}_1^{\,1,E}\,. \tag{281}$$

In order to prove the claim of the lemma, we need a final preparation. Hence let us show that

$$\langle \mathcal{F}(Q') \rangle \Box \circ \hat{\hat{k}} \Box \circ \hat{\hat{\imath}}_1^{\,1,E}(X) = \neg x\,. \tag{282}$$

The proof of this equation is rather tedious but elementary.
Given a base set E', a fuzzy set $Z \in \widetilde{\mathcal{P}}(E')$, and a crisp set $A \in \mathcal{P}(E')$, let us define $A \setminus Z$ by

$$\mu_{A \setminus Z}(e) = \begin{cases} \neg \mu_Z(e) & : \quad e \in A \\ 0 & : \quad e \notin A \end{cases} \tag{283}$$

for all $e \in E'$. In particular, we can express the fuzzy complement $\neg Z$ by $\neg Z = E' \setminus Z$. In the following, I will prefer this notation of the fuzzy complement, because it makes explicit the assumed choice of base set.
By expanding the operator-based notation for duals and functional application according to Def. 14 and (20), the left-hand member of (282) becomes

$$\langle \mathcal{F}(Q') \rangle \Box \circ \hat{\hat{k}} \Box \circ \hat{\hat{\imath}}_1^{\,1,E}(X) = \neg\neg \langle \mathcal{F}(Q') \rangle (\{(1,1),(1,2)\} \setminus \hat{\hat{k}}(\{(1,1)\} \setminus \hat{\hat{\imath}}_1^{\,1,E}(X)))\,. \tag{284}$$

Let us now notice from Def. 21, (16) and (273) that $U = \widehat{\imath}_1^{\,1,E}(X) \in \widetilde{\mathcal{P}}(\{(1,1)\})$ is the fuzzy subset defined by $\mu_U(1,1) = x$. By (283), then,

$$V = \{(1,1)\} \setminus U = \{(1,1)\} \setminus \hat{\hat{\imath}}_1^{\,1,E}(X) \in \widetilde{\mathcal{P}}(\{(1,1)\})$$

is the fuzzy subset defined by $\mu_V(1,1) = \neg x$. Let us now consider the fuzzy subset $W \in \widetilde{\mathcal{P}}(\{(1,1),(1,2)\})$ defined by

$$W = \hat{\hat{k}}(V) = \hat{\hat{k}}(\{(1,1)\} \setminus \hat{\hat{\imath}}_1^{\,1,E}(X))\,.$$

Recalling (280) and Def. 21, it is then immediate that W is defined by $\mu_W(1,1) = \neg x$, $\mu_W(1,2) = 0$. The complement

$$Z = \{(1,1),(1,2)\} \setminus W = \{(1,1),(1,2)\} \setminus \hat{\hat{k}}(\{(1,1)\} \setminus \hat{\hat{\imath}}_1^{\,1,E}(X)) \tag{285}$$

is hence the fuzzy subset $Z \in \widetilde{\mathcal{P}}(\{(1,1),(1,2)\})$ defined by

$$\mu_Z(1,1) = x, \qquad \mu_Z(1,2) = 1\,. \tag{286}$$

Let us now turn attention to $\langle \mathcal{F}(Q') \rangle$. We first deduce from (276) that $\mathcal{F}(Q') = \widetilde{\pi}_1 \widetilde{\cup} \Box$, which is justified by (S-2), (S-3) and (S-4). By expanding the operator-based notation according to Def. 14, Def. 15, and Def. 10, it now comes out that $\mathcal{F}(Q') : \widetilde{\mathcal{P}}(\{1\})^2 \longrightarrow \mathbf{I}$ is the fuzzy quantifier defined by

$$\mathcal{F}(Q')(X_1, X_2) = \neg(\mu_{X_1}(1) \vee \neg\mu_{X_2}(1))\,,$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(\{1\})$. In turn, we obtain from Def. 50 that the fuzzy quantifier $\langle \mathcal{F}(Q') \rangle : \widetilde{\mathcal{P}}(\{(1,1),(1,2)\}) \longrightarrow \mathbf{I}$ is defined by

$$\langle \mathcal{F}(Q') \rangle (Z') = \neg(\mu_{Z'}((1,1)) \vee \neg\mu_{Z'}((1,2)))\,, \tag{287}$$

for all $Z' \in \widetilde{\mathcal{P}}(\{(1,1),(1,2)\})$. In particular, for the given choice of Z,

$$\begin{aligned}
&\langle \mathcal{F}(Q') \rangle (\{(1,1),(1,2)\} \setminus \hat{\hat{k}}(\{(1,1)\} \setminus \hat{\hat{\imath}}_1^{1,E}(X))) \\
&= \langle \mathcal{F}(Q') \rangle (Z) && \text{by (285)} \\
&= \neg(\mu_Z(1,1) \vee \neg\mu_Z(1,2)) && \text{by (287)} \\
&= \neg(x \vee \neg 1) && \text{by (286)} \\
&= \neg(x \vee 0) \\
&= \neg x\,.
\end{aligned}$$

This completes the proof of equation (282).
Building on these preparations, the proof of the lemma has now become straightforward:

$$\begin{aligned}
\widetilde{\mathcal{F}}(\neg)(x) &= \mathcal{F}(Q_\neg)(X) && \text{by (273), Def. 11} \\
&= \mathcal{F}(Q' \triangleleft \{1\})(X) && \text{by (278)} \\
&= \mathcal{F}(\langle Q' \rangle \Box \circ \widehat{k} \Box \circ \widehat{\imath}_1^{1,E})(X) && \text{by (281)} \\
&= \mathcal{F}(\langle Q' \rangle) \Box \circ \hat{\hat{k}} \Box \circ \hat{\hat{\imath}}_1^{1,E}(X) && \text{by (S-3), (S-6)} \\
&= \langle \mathcal{F}(Q') \rangle \Box \circ \hat{\hat{k}} \Box \circ \hat{\hat{\imath}}_1^{1,E}(X) && \text{by Th-37} \\
&= \neg x\,. && \text{by (282)}
\end{aligned}$$

Noticing that $x \in \mathbf{I}$ was arbitrarily chosen, this completes the proof that $\mathcal{F}$ induces the standard negation $\widetilde{\mathcal{F}}(\neg) = \neg$.

Proof of Theorem 47

I will first prove that the conditions (S-1)–(S-6) are necessary for $\mathcal{F}$ to be a standard DFS. Hence consider a standard model $\mathcal{F}$. By Def. 61 and Def. 59, then, $\mathcal{F}$ is known to induce the standard negation $\widetilde{\mathcal{F}}(\neg) = \neg$, $\neg x = 1 - x$, and the standard disjunction

$\widetilde{\mathcal{F}}(\vee) = \vee$, where $x_1 \vee x_2 = \max(x_1, x_2)$. We can now profit from Th-33 and deduce that $\mathcal{F}$ also induces the standard extension principle $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$. Hence the conditions (S-3), (S-4) and (S-6) reduce to the original conditions (Z-3), (Z-4) and (Z-6). The remaining conditions (S-1) and (S-5) coincide with the original choices (Z-1) and (Z-5) anyway. To sum up, the special conditions (S-1)–(S-6) collapse into the original axiom system (Z-1)–(Z-6). The latter system is validated by every model of the theory, and in particular by the standard models. We hence conclude that the model $\mathcal{F}$ also satisfies the new system (S-1)–(S-6). Because no special assumptions have been made on the choice of the standard DFS $\mathcal{F}$, this proves that every standard model fulfills the novel requirements (S-1)–(S-6). In particular, these conditions are necessary in order to make $\mathcal{M}$ a standard DFS.

In order to show that the conditions are also sufficient for $\mathcal{F}$ to be a standard model, suppose that the given QFM $\mathcal{F}$ satisfies (S-1)–(S-6). We can then apply L-1, L-2 and L-4, which substantiate that $\mathcal{F}$ induces the standard disjunction, the standard extension principle, and the standard negation, respectively. In turn, the induced fuzzy union of $\mathcal{F}$ is the standard fuzzy union. The DFS axioms (Z-1)–(Z-6), which are built from the induced connectives and induced extension principle, hence collapse into the new axioms (S-1)–(S-6), which are expressed in terms of the standard connectives and the standard extension principle. Because the latter system of conditions is already known to hold for $\mathcal{F}$, and noticing that the given $\mathcal{F}$ makes the original axioms (Z-1)–(Z-6) coincide with this system, this completes the proof that (Z-1)–(Z-6) are satisfied by the considered model $\mathcal{F}$. Hence $\mathcal{F}$ is a DFS. Recalling that $\mathcal{F}$ induces the standard negation and disjunction, we finally conclude from Def. 59 and Def. 61 that $\mathcal{F}$ is a standard DFS, as desired.

D.2 Proof of Theorem 61

Let $E \neq \varnothing$ be a given base set and $X, X' \in \widetilde{\mathcal{P}}(E)$.

a.: In order to prove the claim on complementation, let us first observe that for $\gamma = 0$,

$$(\neg X)^{\min}_0 = (\neg X)_{> \frac{1}{2}} = \neg(X_{\geq \frac{1}{2}}) = \neg(X^{\max}_0)$$

and

$$(\neg X)^{\max}_0 = (\neg X)_{\geq \frac{1}{2}} = \neg(X_{> \frac{1}{2}}) = \neg(X^{\min}_0)\,,$$

which is apparent from Def. 82 and the known property of α-cuts that $\neg X_{\geq \alpha} = \neg X_{> 1-\alpha}$ and $\neg X_{> \alpha} = \neg X_{\geq 1-\alpha}$ for all $\alpha \in \mathbf{I}$.
For $\gamma > 0$, we obtain that

$$(\neg X)^{\min}_\gamma = (\neg X)_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = \neg(X_{> \frac{1}{2} - \frac{1}{2}\gamma}) = \neg(X^{\max}_\gamma)$$
$$(\neg X)^{\max}_\gamma = (\neg X)_{> \frac{1}{2} - \frac{1}{2}\gamma} = \neg(X_{\geq \frac{1}{2} + \frac{1}{2}\gamma}) = \neg(X^{\min}_\gamma)\,,$$

again due to Def. 82 and the known behaviour of α-cuts with respect to complementation.

b.: The claim that $X \cap {X'}^{\min}_{\gamma} = X^{\min}_{\gamma} \cap {X'}^{\min}_{\gamma}$ and $X \cap {X'}^{\max}_{\gamma} = X^{\max}_{\gamma} \cap {X'}^{\max}_{\gamma}$ is apparent from Def. 82, which reduces $(\bullet)^{\min}_{\gamma}$ and $(\bullet)^{\max}_{\gamma}$ to α-cuts or strict α-cuts, and the known compatibility of α-cuts and strict α-cuts with respect to intersections.

c.: For similar reasons, it is clear that $X \cap {X'}^{\min}_{\gamma} = X^{\min}_{\gamma} \cap {X'}^{\min}_{\gamma}$ and $X \cap {X'}^{\max}_{\gamma} = X^{\max}_{\gamma} \cap {X'}^{\max}_{\gamma}$, because Def. 82 reduces $(\bullet)^{\min}_{\gamma}$ and $(\bullet)^{\max}_{\gamma}$ to α-cuts or strict α-cuts, and because α-cuts and strict α-cuts are known to be compatible with unions.

D.3 Proof of Theorem 99

In order to prove the theorem, it is sufficient to notice that (M-1)–(M-6) coincide with the known conditions (S-1)–(S-6), respectively. Recalling theorem Th-47, we can then conclude that the models of (M-1)–(M-6) are precisely the standard DFSes. Because the remaining condition (M-7) merely requires the distinguishing property of $\mathcal{M}_{CX}$ among the standard models which has been established in theorem Th-98, it is then evident that the proposed axiom system indeed achieves a unique characterisation of $\mathcal{M}_{CX}$, as claimed by the theorem.

D.4 Proof of Theorem 238

Lemma 5

Let $E \neq \varnothing$ be a given base set and suppose that $V = (V_1, \dots, V_n)$, $V' = (V'_1, \dots, V'_n)$, $W = (W_1, \dots, W_n)$, $W' = (W'_1, \dots, W'_n) \in \mathcal{P}(E)^n$ satisfy

$$V'_i \subseteq V_i \subseteq W_i \subseteq W'_i \tag{288}$$

for all $i \in \{1, \dots, n\}$. Then for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$,

$$\Xi_{V,W}(X_1, \dots, X_n) \leq \Xi_{V',W'}(X_1, \dots, X_n)\,.$$

Proof Apparent. We first notice that

$$\begin{aligned}
&\inf\{\mu_{X_i}(e) : e \in V_i\} \\
&= \min(\inf\{\mu_{X_i}(e) : e \in V'_i\}, \\
&\qquad \inf\{\mu_{X_i}(e) : e \in V_i \setminus V'_i\}) && \text{because } V'_i \subseteq V_i \\
&\leq \inf\{\mu_{X_i}(e) : e \in V'_i\}\,,
\end{aligned}$$

i.e.

$$\inf\{\mu_{X_i}(e) : e \in V_i\} \leq \inf\{\mu_{X_i}(e) : e \in V'_i\} \tag{289}$$

for all $i \in \{1, \dots, n\}$. By similar reasoning,

$$\begin{aligned}
&\inf\{1 - \mu_{X_i}(e) : e \notin W_i\} \\
&= \min(\inf\{1 - \mu_{X_i}(e) : e \notin W'_i\}, \\
&\qquad \inf\{1 - \mu_{X_i}(e) : e \in W'_i \setminus W_i\}) && \text{because } W_i \subseteq W'_i \\
&\leq \inf\{1 - \mu_{X_i}(e) : e \notin W'_i\}\,,
\end{aligned}$$

i.e.

$$\inf\{1 - \mu_{X_i}(e) : e \notin W_i\} \leq \inf\{1 - \mu_{X_i}(e) : e \notin W_i'\} \tag{290}$$

for all $i \in \{1, \ \ldots, \ n\}$. Therefore

$$\begin{aligned}
&\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n) \\
&= \min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) \quad \text{by (55)} \\
&\leq \min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i'\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i'\}) \quad \text{by (289), (290)} \\
&= \widetilde{\widetilde{\Xi}}_{V',W'}(X_1, \ldots, X_n)\,.
\end{aligned}$$

Lemma 6

Let $E \neq \varnothing$ be a finite base set, $X \in \widetilde{\mathcal{P}}(E)$ and $k \in \{0, \ldots, |E|\}$. Then

a. $\max\{\min\{\mu_X(e) : e \in V\} : |V| = k\} = \mu_{[k]}(X)$*;*

b. $\max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\} = 1 - \mu_{[k+1]}(X)$*;*

c. $\max\{\min(\min\{\mu_X(e) : e \in V\}, \min\{1 - \mu_X(e) : e \notin V\}) : |V| = k\} = \min(\mu_{[k]}(X), 1 - \mu_{[k+1]}(X)$.

Proof Due to the fact that E is finite, we can rewrite it as $E = \{e_1, \ldots, e_{|E|}\}$, where the elements are ordered such that $\mu_X(e_1) \geq \mu_X(e_2) \geq \cdots \geq \mu_X(e_{|E|})$, i.e. $e_1, \ldots, e_{|E|}$ are distinct elements with

$$\mu_X(e_j) = \mu_{[j]}(X)\,. \tag{291}$$

for all $j \in \{0, \ldots, |E|\}$. Let us now consider the two claims made by the lemma.

a. For a given choice of $k \in \{0, \ldots, |E|\}$, letting $V' = \{e_1, \ldots, e_k\}$ apparently results in the set of cardinality $|V'| = k$ which maximizes $\min\{\mu_X(e) : e \in V\}$, $|V| = k$, because V' contains those k elements which achieve the highest scores of $\mu_X(e)$. Therefore

$$\max\{\min\{\mu_X(e) : e \in V\} : |V| = k\} = \min\{\mu_X(e) : e \in V'\} \tag{292}$$

and in turn,

$$\begin{aligned}
&\max\{\min\{\mu_X(e) : e \in V\} : |V| = k\} \\
&= \min\{\mu_X(e) : e \in V'\} && \text{by (292)} \\
&= \min\{\mu_X(e_1), \ldots, \mu_X(e_k)\} && \text{by definition of } V' \\
&= \min\{\mu_{[1]}(X), \ldots, \mu_{[k]}(X)\}\,. && \text{by (291)}
\end{aligned}$$

Recalling from Def. 99 that $\mu_{[1]}(X) \geq \mu_{[2]}(X) \geq \cdots \geq \mu_{[|E|]}(X)$, we can therefore conclude that

$$\max\{\min\{\mu_X(e) : e \in V\} : |V| = k\} = \mu_{[j]}(X)\,,$$

as desired.

b. Next we consider $\max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\}$. It is apparent from the chosen order of $e_1, \ldots, e_{|E|}$ that the same choice of $V' = \{e_1, \ldots, e_k\}$ as in the previous case maximizes $\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\}$ as well, because V' contains those k elements which result in the lowest scores of $1 - \mu_X(e)$. Consequently, the complement of V' contains those $|E| - k$ elements which result in the maximal scores of $1 - \mu_X(e)$. We can hence assert that

$$\max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\} = \min\{1 - \mu_X(e) : e \notin V'\}\,. \tag{293}$$

In particular

$$\begin{aligned}
&\max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\} \\
&= \min\{1 - \mu_X(e) : e \notin V'\} && \text{by (293)} \\
&= \min\{1 - \mu_X(e_{k+1}), \ldots, 1 - \mu_X(e_{|E|})\} && \text{because } V' = \{e_1, \ldots, e_k\} \\
&= \min\{1 - \mu_{[k+1]}(X), \ldots, 1 - \mu_{[|E|]}(X)\} && \text{by (291)} \\
&= 1 - \max\{\mu_{[k+1]}(X), \ldots, \mu_{[|E|]}(X)\}\,. && \text{by De Morgan's law}
\end{aligned}$$

Again utilizing that $\mu_{[1]}(X) \geq \mu_{[2]}(X) \geq \cdots \geq \mu_{[|E|]}(X)$ by Def. 99, we then obtain that

$$\max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\} = 1 - \mu_{[k+1]}(X)\,,$$

which completes the proof of the second part of the lemma.

c. In order to prove the equality stated in part **c.**, we again abbreviate $V' = \{e_1, \ldots, e_k\}$. Then

$$\begin{aligned}
&\min(\min\{\mu_X(e) : e \in V'\}, \min\{1 - \mu_X(e) : e \notin V'\}) \\
&\leq \max\{\min(\min\{\mu_X(e) : e \in V\}, \\
&\qquad \min\{1 - \mu_X(e) : e \notin V\}) : |V| = k\} && \text{because } |V'| = k \\
&\leq \min(\max\{\min\{\mu_X(e) : e \in V\} : |V| = k\}, \\
&\qquad \max\{\min\{1 - \mu_X(e) : e \notin V\} : |V| = k\}) \\
&= \min(\min\{\mu_X(e) : e \in V'\}, \min\{1 - \mu_X(e) : e \notin V'\})\,. && \text{by (292), (293)}
\end{aligned}$$

Hence indeed

$$\begin{aligned}
&\max\{\min(\min\{\mu_X(e) : e \in V\}, \min\{1 - \mu_X(e) : e \notin V\}) : |V| = k\} \\
&= \min(\min\{\mu_X(e) : e \in V'\}, \min\{1 - \mu_X(e) : e \notin V'\})\,.
\end{aligned} \tag{294}$$

Let us now recall that $V' = \{e_1, \dots, e_k\}$ where $\mu_X(e_j) = \mu_{[j]}(X)$. Therefore

$$\begin{aligned}
\min\{\mu_X(e) : e \in V\} &= \min\{\mu_X(e_1), \dots, \mu_X(e_k)\} && \text{by definition of } V' \\
&= \min\{\mu_{[1]}(X), \dots, \mu_{[k]}(X)\} && \text{by chosen order of } e_1, \dots, e_{|E|} \\
&= \mu_{[k]}(X)\,, && \text{by Def. 99}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\min\{1 - \mu_X(e) : e \notin V\} \\
&= 1 - \max\{\mu_X(e) : e \notin V\} && \text{by De Morgan's law} \\
&= 1 - \max\{\mu_X(e_{k+1}), \dots, \mu_X(e_{|E|})\} && \text{because } V' = \{e_1, \dots, e_k\} \\
&= 1 - \max\{\mu_{[k+1]}(X), \dots, \mu_{[|E|]}(X)\} && \text{by chosen order of } e_1, \dots, e_{|E|} \\
&= 1 - \mu_{[k+1]}(X)\,. && \text{by Def. 99}
\end{aligned}$$

By utilizing these equalities, the former result (294) can be further simplified into the desired

$$\begin{aligned}
&\max\{\min(\min\{\mu_X(e) : e \in V\}, \min\{1 - \mu_X(e) : e \notin V\}) : |V| = k\} \\
&= \min(\mu_{[k]}(X), 1 - \mu_{[k+1]}(X))\,.
\end{aligned}$$

Lemma 7
Let $E \neq \varnothing$ be some finite base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Then

$$\mathcal{M}_{CX}([\textbf{card} \geq])(X_1, X_2) = \max\{\min(\mu_{[k]}(X_1), 1 - \mu_{[k+1]}(X_2)) : 0 \leq k \leq |E|\}\,.$$

Proof To see this, consider a finite base set $E \neq \varnothing$ and a choice of fuzzy arguments $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Let us now consider the coefficients $L([\textbf{card} \geq], V, W)$ defined by (57). It is obvious from (139) that these coefficients become

$$L([\textbf{card} \geq], (V_1, V_2), (W_1, W_2)) = \begin{cases} 1 & : \quad |V_1| \geq |W_2| \\ 0 & : \quad \text{else}\,. \end{cases}$$

In turn, we obtain from (51) that

$$\widetilde{[\textbf{card} \geq]}^{L}_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) = \begin{cases} \widetilde{\Xi}_{(V_1,V_2),(W_1,W_2)}(X_1, X_2) & : \quad |V_1| \geq |W_2| \\ 0 & : \quad \text{else} \end{cases} \tag{295}$$

for all $V_1, V_2, W_1, W_2 \in \mathcal{P}(E)$ with $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. Hence

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\geq])(X_1, X_2)\\
&= \sup\{\widetilde{[\textbf{card}\geq]}^{L}_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) : \\
&\qquad V_1, V_2, W_1, W_2 \in \mathcal{P}(E), V_1 \subseteq W_1, V_2 \subseteq W_2\} && \text{by Th-102}\\
&= \sup\{\widetilde{\Xi}_{(V_1,V_2),(W_1,W_2)}(X_1, X_2) : V_1 \subseteq W_1, V_2 \subseteq W_2, |V_1| \geq |W_2|\} && \text{by (295)}\\
&= \sup\{\widetilde{\Xi}_{(V_1,\varnothing),(E,W_2)}(X_1, X_2) : |V_1| \geq |W_2|\} && \text{by L-5}\\
&= \sup\{\widetilde{\Xi}_{(V,\varnothing),(E,W)}(X_1, X_2) : |V| = |W|\} && \text{by L-5}\\
&= \sup\{\min(\inf\{\mu_{X_1}(e) : e \in V\},\\
&\qquad \inf\{1 - \mu_{X_2}(e) : e \notin W\}) : |V| = |W|\}\,, && \text{by (55)}
\end{aligned}$$

i.e.

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\geq])(X_1, X_2)\\
&\ = \max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) : |V| = |W|\}
\end{aligned}$$

because E is finite. From this we proceed in the following way,

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\geq])(X_1, X_2)\\
&= \max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) :\\
&\qquad |V| = k, |W| = k, 0 \leq k \leq |E|\}\\
&= \max\{\max\{\min(\min\{\mu_{X_1}(e) : e \in V\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin W\}) : |V| = k\} :\\
&\qquad |W| = k, 0 \leq k \leq |E|\}\\
&= \max\{\min(\max\{\min\{\mu_{X_1}(e) : e \in V\} : |V| = k\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin W\}) :\\
&\qquad |W| = k, 0 \leq k \leq |E|\}\\
&= \max\{\max\{\min(\max\{\min\{\mu_{X_1}(e) : e \in V\} : |V| = k\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin W\}) :\\
&\qquad |W| = k\} : 0 \leq k \leq |E|\}\\
&= \max\{\min(\max\{\min\{\mu_{X_1}(e) : e \in V\} : |V| = k\},\\
&\qquad \max\{\min\{1 - \mu_{X_2}(e) : e \notin W\} : |W| = k\}) :\\
&\qquad 0 \leq k \leq |E|\}\\
&= \max\{\min(\mu_{[k]}(X_1), 1 - \mu_{[k+1]}(X_2)) : 0 \leq k \leq |E|\}\,, && \text{by L-6}
\end{aligned}$$

as desired.

Lemma 8
Suppose that $E \neq \varnothing$ is some finite base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Then

$$\mathcal{M}_{CX}([\textbf{card}>])(X_1, X_2) = \max\{\min(\mu_k(X_1), 1 - \mu_k(X_2)) : 1 \leq k \leq |E|\}\,.$$

Proof Hence let $E \neq \varnothing$ be a given finite base set and suppose that $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. We first consider the coefficients $L([\textbf{card} >], (V_1, V_2), (W_1, W_2))$ defined by (57). It is apparent from the defining equality (140) of $[\textbf{card} >]$ that the coefficient $L([\textbf{card} >], (V_1, V_2), (W_1, W_2))$ becomes

$$L([\textbf{card} >], (V_1, V_2), (W_1, W_2)) = \begin{cases} 1 & : \quad |V_1| > |W_2| \\ 0 & : \quad \text{else} \end{cases}$$

for all $V_1, V_2, W_1, W_2 \in \mathcal{P}(E)$ with $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$. In particular, we obtain that

$$\widetilde{(V1, V_2)}^{L}_{(W_1,W_2),(}X_1, X_2) = \begin{cases} \widetilde{\Xi}_{(V_1,V_2),(W_1,W_2)}(X_1, X_2) & : \quad |V_1| > |W_2| \\ 0 & : \quad \text{else} \end{cases} \tag{296}$$

which is immediate from the former result and (51).
Now turning to $\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2)$, we proceed as follows.

$$\begin{aligned} &\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2) \\ &= \sup\{\widetilde{Q}^{L}_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) : \\ &\qquad V_1, V_2, W_1, W_2 \in \mathcal{P}(E), V_1 \subseteq W_1, V_2 \subseteq W_2\} && \text{by Th-102} \\ &= \sup\{\widetilde{\Xi}_{(V_1,V_2),(W_1,W_2)}(X_1, X_2) : \\ &\qquad V_1, V_2, W_1, W_2 \in \mathcal{P}(E), V_1 \subseteq W_1, V_2 \subseteq W_2, |V_1| > |W_2|\} && \text{by (296)} \\ &= \sup\{\widetilde{\Xi}_{(V,\varnothing),(E,W)}(X_1, X_2) : V, W \in \mathcal{P}(E), |V| > |W|\} && \text{by L-5} \\ &= \sup\{\widetilde{\Xi}_{(V,\varnothing),(E,W)}(X_1, X_2) : V, W \in \mathcal{P}(E), |W| = |V| - 1\} && \text{by L-5} \\ &= \sup\{\min(\inf\{\mu_{X_1}(e) : e \in V\}, \inf\{1 - \mu_{X_2}(e) : e \notin W\}) : \\ &\qquad V, W \in \mathcal{P}(E), |W| = |V| - 1\}\,, && \text{by (55)} \end{aligned}$$

i.e.

$$\begin{aligned} &\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2) \\ &= \max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) : \\ &\qquad V, W \in \mathcal{P}(E), |W| = |V| - 1\} \end{aligned}$$

because E is finite. This rendering of $\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2)$ can be further simplified based on the following considerations.

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2) \\
&= \max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) : \\
&\qquad V, W \in \mathcal{P}(E), |W| = |V| - 1\} \\
&= \max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) : \\
&\qquad V, W \in \mathcal{P}(E), |W| = k - 1, |V| = k, 1 \leq k \leq |E|\} \\
&= \max\{\max\{\max\{\min(\min\{\mu_{X_1}(e) : e \in V\}, \min\{1 - \mu_{X_2}(e) : e \notin W\}) : \\
&\qquad |V| = k\} : |W| = k - 1\} : 1 \leq k \leq |E|\} \\
&= \max\{\max\{\min(\max\{\min\{\mu_{X_1}(e) : e \in V\} : |V| = k\}, \\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin W\}) : \\
&\qquad |W| = k - 1\} : 1 \leq k \leq |E|\} \\
&= \max\{\min(\max\{\min\{\mu_{X_1}(e) : e \in V\} : |V| = k\}, \\
&\qquad \max\{\min\{1 - \mu_{X_2}(e) : e \in W\} : |W| = k - 1\}) : 1 \leq k \leq |E|\} \,,
\end{aligned}$$

which proves the desired

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card} >])(X_1, X_2) \\
&= \max\{\min(\mu_{[k]}(X_1), 1 - \mu_{[k]}(X_2)) : 1 \leq k \leq |E|\} \,,
\end{aligned}$$

noticing that L-6 is now applicable.

Lemma 9
Consider a finite base set $E \neq \varnothing$ and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card} =])(X_1, X_2) \\
&= \max\{\min\{\mu_{[k]}(X_1), 1 - \mu_{[k+1]}(X_1), \mu_{[k]}(X_2), 1 - \mu_{[k+1]}(X_2)\} : 0 \leq k \leq |E|\} \,.
\end{aligned}$$

Proof In this case, the coefficients $L([\textbf{card} =], (V_1, V_2), (W_1, W_2))$ defined by (57) become

$$L([\textbf{card} =], (V_1, V_2), (W_1, W_2)) = \begin{cases} 1 & : \quad V_1 = W_1, V_2 = W_2, |V_1| = |V_2| \\ 0 & : \quad \text{else} \end{cases}$$

for all $V_1, V_2, W_1, W_2 \in \mathcal{P}(E)$, $V_1 \subseteq W_1$, $V_2 \subseteq W_2$, which is immediate from (141). In particular, $\widetilde{[\textbf{card} =]}^{L}_{(V_1, V_2), (W_1, W_2)}(X_1, X_2)$ becomes

$$\begin{aligned}
&\widetilde{[\textbf{card} =]}^{L}_{(V_1, V_2), (W_1, W_2)}(X_1, X_2) \\
&= \begin{cases} \widetilde{\Xi}_{(V_1, V_2), (V_1, V_2)}(X_1, X_2) & : \quad V_1 = W_1, V_2 = W_2, |V_1| = |V_2| \\ 0 & : \quad \text{else} \end{cases}
\end{aligned} \tag{297}$$

see (51). Building on these preparations, I can now proceed as follows. Firstly

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\,{=}])(X_1, X_2)\\
&= \sup\{\widetilde{[\textbf{card}\,{=}]}^{L}_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) :\\
&\qquad V_1, V_2, W_1, W_2 \in \mathcal{P}(E), V_1 \subseteq W_1, V_2 \subseteq W_2\} && \text{by Th-102}\\
&= \sup\{\widetilde{\Xi}_{(A,B),(A,B)}(X_1, X_2) :\\
&\qquad A, B \in \mathcal{P}(E), |A| = |B|\} && \text{by (297)}\\
&= \sup\{\min\{\inf\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \inf\{1 - \mu_{X_1}(e) : e \notin A\},\\
&\qquad \inf\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \inf\{1 - \mu_{X_2}(e) : e \notin B\}\} : |A| = |B|\}\,, && \text{by (55)}
\end{aligned}$$

i.e.

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\,{=}])(X_1, X_2)\\
&= \max\{\min\{\min\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\},\\
&\qquad \min\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\}\} : |A| = |B|\}
\end{aligned}$$

because E is finite. This can be further simplified as follows.

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}\,{=}])(X_1, X_2)\\
&= \max\{\min\{\min\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\},\\
&\qquad \min\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\}\} : |A| = k, |B| = k, 0 \le k \le |E|\}\\
&= \max\{\max\{\min(\min(\min\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\}),\\
&\qquad \min(\min\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\})) : |A| = k\} : |B| = k, 0 \le k \le |E|\}\\
&= \max\{\min(\max\{\min(\min\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\}) : |A| = k\},\\
&\qquad \min(\min\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\})) : |B| = k, 0 \le k \le |E|\}\\
&= \max\{\max\{\min(\max\{\min(\min\{\mu_{X_1}(e) : e \in A\},\\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\}) : |A| = k\},\\
&\qquad \min(\min\{\mu_{X_2}(e) : e \in B\},\\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\})) : |B| = k\} : 0 \le k \le |E|\}
\end{aligned}$$

and consequently

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}=])(X_1, X_2) \\
&= \max\{\min(\max\{\min(\min\{\mu_{X_1}(e) : e \in A\}, \\
&\qquad \min\{1 - \mu_{X_1}(e) : e \notin A\}) : |A| = k\}, \\
&\qquad \max\{\min(\min\{\mu_{X_2}(e) : e \in B\}, \\
&\qquad \min\{1 - \mu_{X_2}(e) : e \notin B\}) : |B| = k\}) : 0 \le k \le |E|\}\,.
\end{aligned}$$

The right-hand member of the above equality can now be further simplified by utilizing part **c.** of L-6. We then obtain the desired rendering of $\mathcal{M}_{CX}([\textbf{card}=])$, viz

$$\begin{aligned}
&\mathcal{M}_{CX}([\textbf{card}=])(X_1, X_2) \\
&= \max\{\min(\min(\mu_k(X_1), 1 - \mu_{k+1}(X_1)), \\
&\qquad \min(\mu_k(X_2), 1 - \mu_{k+1}(X_2))) : 0 \le k \le |E|\} \\
&= \max\{\min\{\mu_k(X_1), 1 - \mu_{k+1}(X_1), \mu_k(X_2), 1 - \mu_{k+1}(X_2)\} : 0 \le k \le |E|\}\,.
\end{aligned}$$

Proof of Theorem 238

Let $E \neq \varnothing$ be some finite base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Further suppose that $\mathcal{F}$ is a given standard DFS. Now consider a choice of fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. We first notice that

$$\begin{aligned}
&\mathcal{F}([\textbf{card}\geq])(X_1, X_2) \\
&= \mathcal{M}_{CX}([\textbf{card}\geq])(X_1, X_2) && \text{by Th-46, Th-77} \\
&= \max\{\min(\mu_{[k]}(X_1), 1 - \mu_{[k+1]}(X_2)) : 0 \le k \le |E|\}\,. && \text{by L-7}
\end{aligned}$$

By similar reasoning, we obtain that

$$\begin{aligned}
&\mathcal{F}([\textbf{card}>])(X_1, X_2) \\
&= \mathcal{M}_{CX}([\textbf{card}>])(X_1, X_2) && \text{by Th-46, Th-77} \\
&= \max\{\min(\mu_k(X_1), 1 - \mu_k(X_2)) : 1 \le k \le |E|\}\,. && \text{by L-8}
\end{aligned}$$

Finally, the quantifier [**card** =] can be handled in an analogous way,

$$\begin{aligned}
&\mathcal{F}([\textbf{card}=])(X_1, X_2) \\
&= \mathcal{M}_{CX}([\textbf{card}=])(X_1, X_2) && \text{by Th-46, Th-77} \\
&= \max\{\min\{\mu_{[k]}(X_1), 1 - \mu_{[k+1]}(X_1), \\
&\qquad \mu_{[k]}(X_2), 1 - \mu_{[k+1]}(X_2)\} : 0 \le k \le |E|\}\,. && \text{by L-9}
\end{aligned}$$

D.5 Proof of Theorem 239

Lemma 10
Let $E \neq \varnothing$ be some base set. Further let $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ be fuzzy subsets of E. Then

$$\begin{aligned}
&\mathcal{M}_{CX}(\textbf{eq})(X_1, X_2)\\
&= \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) :\\
&\qquad\qquad \min(\mu_{X_1}(e), \mu_{X_2}(e)) \geq 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e))\},\\
&\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) :\\
&\qquad\qquad 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) > \min(\mu_{X_1}(e), \mu_{X_2}(e))\})\,.
\end{aligned}$$

Proof Suppose that $E \neq \varnothing$ is some base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. It is apparent from (142) and (57) that

$$L(\textbf{eq}, (V_1, V_2), (W_1, W_2)) = \begin{cases} 1 & : \quad V_1 = V_2 = W_1 = W_2 \\ 0 & : \quad \text{else} \end{cases}$$

for all $V_1, V_2, W_1, W_2 \in \mathcal{P}(E)$ with $V_1 \subseteq V_2, W_1 \subseteq W_2$. In turn, we obtain from (51) that

$$\widetilde{\textbf{eq}}^L_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) = \begin{cases} \widetilde{\Xi}_{(V_1,V_1),(V_1,V_1)}(X_1, X_2) & : \quad V_1 = V_2 = W_1 = W_2 \\ 0 & : \quad \text{else}\,, \end{cases}$$

for all $V_1, V_2, W_1, W_2 \in \mathcal{P}(E)$ with $V_1 \subseteq W_1$, $V_2 \subseteq W_2$, i.e.

$$\widetilde{\textbf{eq}}^L_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) \tag{298}$$

$$= \begin{cases} \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in V_1\}, & \\ \qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin V_1\}) & : \quad V_1 = V_2 = W_1 = W_2 \\ 0 & : \quad \text{else}\,, \end{cases} \tag{299}$$

which is straightforward from (55). Consequently,

$$\begin{aligned}
&\mathcal{M}_{CX}(\textbf{eq})(X_1, X_2)\\
&= \sup\{\widetilde{\textbf{eq}}^L_{(V_1,V_2),\,(W_1,W_2)}(X_1, X_2) : V_1, V_2, W_1, W_2 \in \mathcal{P}(E),\\
&\qquad\qquad V_1 \subseteq W_1, V_2 \subseteq W_2\} && \text{by Th-102}\\
&= \sup\{\min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y\},\\
&\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y\}) : Y \in \mathcal{P}(E)\}\,, && \text{by (299)}
\end{aligned}$$

i.e.

$$\begin{aligned}
&\mathcal{M}_{CX}(\textbf{eq})(X_1, X_2)\\
&= \sup\{\min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y\},\\
&\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y\}) : Y \in \mathcal{P}(E)\}\,.
\end{aligned} \tag{300}$$

Now consider the following choice of $Y' \in \mathcal{P}(E)$,

$$Y' = \{e \in E : \min(\mu_{X_1}(e), \mu_{X_2}(e)) \geq 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e))\}\,. \tag{301}$$

It is immediate from the definition of Y' that for all $Y \in \mathcal{P}(E)$,

$$\begin{aligned}&\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \setminus Y'\}\\ &\leq \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \setminus Y'\}\end{aligned} \tag{302}$$

and

$$\begin{aligned}&\inf\{1 - \max(\mu_{X_1}(e), 1 - \max(\mu_{X_2}(e)) : e \in Y' \setminus Y\}\\ &\leq \inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y' \setminus Y\}\,,\end{aligned} \tag{303}$$

because (301) ensures that these inequalities are valid for all considered elements $e \in Y \setminus Y'$ and $e \in Y' \setminus Y$, respectively. Therefore

$$\begin{aligned}&\min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y\})\\ &= \min\{\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \cap Y'\},\\ &\qquad \inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \setminus Y'\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y' \setminus Y\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y \cup Y'\}\}\\ &\leq \min\{\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \cap Y'\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y \setminus Y'\},\\ &\qquad \inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y' \setminus Y\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y \cup Y'\}\} \qquad \text{by (302), (303)}\\ &= \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y',\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y'\})\,,\end{aligned}$$

i.e.

$$\begin{aligned}&\min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y\}, \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y\})\\ &\leq \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y', \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y'\})\,.\end{aligned}$$

This demonstrates that the supremum in (300) is attained for $Y = Y'$. We can therefore conclude from (300) and (301) that indeed

$$\begin{aligned}&\mathcal{M}_{CX}(\textbf{eq})(X_1, X_2)\\ &= \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in Y', \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : e \notin Y'\})\\ &= \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) :\\ &\qquad\qquad \min(\mu_{X_1}(e), \mu_{X_2}(e)) \geq 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e))\},\\ &\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) :\\ &\qquad\qquad 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) > \min(\mu_{X_1}(e), \mu_{X_2}(e))\})\,,\end{aligned}$$

which completes the proof of the theorem.

Proof of Theorem 239

Let $E \neq \varnothing$ be some base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Further suppose that $\mathcal{F}$ is a standard DFS. The equality quantifier **eq** then becomes

$$\begin{aligned}
&\mathcal{F}(\textbf{eq})(X_1, X_2) \\
&= \mathcal{M}_{CX}(\textbf{eq})(X_1, X_2) && \text{by Th-46, Th-77} \\
&= \min(\inf\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : \\
&\qquad\qquad \min(\mu_{X_1}(e), \mu_{X_2}(e)) \geq 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e))\}, \\
&\qquad \inf\{1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) : \\
&\qquad\qquad 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) > \min(\mu_{X_1}(e), \mu_{X_2}(e))\}) \,. && \text{by L-10}
\end{aligned}$$

D.6 Proof of Theorem 241

Lemma 11
Let $E \neq \varnothing$ be a set of finite cardinality $|E| = m$ and let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy subset of E. Then for all automorphisms $\beta : E \longrightarrow E$ and all $j \in \{1, \dots, m\}$,

$$\mu_{[j]}(\hat{\hat{\beta}}(X)) = \mu_{[j]}(X) \,,$$

i.e. $\mu_{[j]}(\bullet)$ is automorphism-invariant.

Proof Let $E \neq \varnothing$ be the given base set of cardinality $|E| = m$ and $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E. Further let an automorphism $\beta : E \longrightarrow E$ be given. By Def. 99, E can then be written $E = \{e_1, \dots, e_m\}$ for pairwise distinct elements $e_1, \dots, e_m \in E$ such that

$$\mu_X(e_1) \geq \mu_X(e_2) \geq \dots \geq \mu_X(e_m) \tag{304}$$

and

$$\mu_{[j]}(X) = \mu_X(e_j) \tag{305}$$

for all $j \in \{1, \dots, m\}$. Now define $e'_1, \dots, e'_m \in E$ by

$$e'_j = \beta(e_j) \tag{306}$$

for $j = 1, \dots, m$. Recalling that β is a bijection, the e'_j's are pairwise distinct because the e_j's are. Therefore $E = \{e'_1, \dots, e'_m\}$. In addition

$$\begin{aligned}
\mu_{\hat{\hat{\beta}}(X)}(e'_j) &= \mu_X(\beta^{-1}(e'_j)) && \text{by Def. 21} \\
&= \mu_X(\beta^{-1}(\beta(e_j))) && \text{by (306)} \\
&= \mu_X(e_j) \,,
\end{aligned}$$

i.e.

$$\mu_{\hat{\hat{\beta}}(X)}(e'_j) = \mu_X(e_j) \tag{307}$$

for all $j \in \{1, \dots, m\}$. Hence by (304),

$$\mu_{\hat{\beta}(X)}(e'_1) \geq \mu_{\hat{\beta}(X)}(e'_2) \geq \dots \geq \mu_{\hat{\beta}(X)}(e'_m)$$

as well. By Def. 99, then, we conclude that

$$\mu_j(\hat{\hat{\beta}}(X)) = \mu_{\hat{\beta}(X)}(e'_j) \tag{308}$$

for $j = 1, \dots, m$. Therefore

$$\begin{aligned} \mu_{[j]}(\hat{\hat{\beta}}(X)) &= \mu_{\hat{\beta}(X)}(e'_j) && \text{by (308)} \\ &= \mu_X(e_j) && \text{by (307)} \\ &= \mu_{[j]}(X)\,, && \text{by (305)} \end{aligned}$$

which completes the proof of the lemma.

Proof of Theorem 241

Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ be a unary fuzzy quantifier on a base set of finite cardinality $|E| = m$.

To show that **a.** entails **b.**, suppose that $\widetilde{Q}$ is quantitative. Noticing that $|E| = m$, we can choose pairwise distinct elements $e_1, \dots, e_m \in E$ such that $E = \{e_1, \dots, e_m\}$. For $z_1, \dots, z_m \in \mathbf{I}$, let $Z \in \widetilde{\mathcal{P}}(E)$ be the fuzzy subset defined by

$$\mu_Z(e_j) = z_j \tag{309}$$

for all $e_j \in E$. In terms of Z, let us now define the mapping $g : \mathbf{I}^m \longrightarrow \mathbf{I}$ by

$$g(z_1, \dots, z_m) = Q(Z) \tag{310}$$

for all $z_1, \dots, z_n \in \mathbf{I}$. Now let $X \in \widetilde{\mathcal{P}}(E)$ be an arbitrary fuzzy subset of E. Then the elements $e_1, \dots, e_m$ of E can be reordered such that $E = \{e_{\rho(1)}, \dots, e_{\rho(m)}\}$ (i.e. the $e_{\rho(j)}$ are pairwise distinct) and

$$\mu_{[j]}(X) = \mu_X(e_{\rho(j)}) \tag{311}$$

for all $j \in \{1, \dots, m\}$. Now define an automorphism $\beta : E \longrightarrow E$ by

$$\beta(e_{\rho(j)}) = \beta(e_j) \tag{312}$$

for all $e_{\rho(j)} \in E$. Abbreviating

$$Z = \hat{\hat{\beta}}(X)\,, \tag{313}$$

we then obtain that

$$\begin{aligned} \mu_Z(e_j) &= \mu_X(e_{\rho(j)}) && \text{by Def. 21, (312), (313)} \\ &= \mu_{[j]}(X) && \text{by (311)} \\ &= \mu_{[j]}(Z) && \text{by L-11, (313)} \end{aligned}$$

Therefore

$$\begin{aligned}
\widetilde{Q}(X) &= \widetilde{Q}(\hat{\hat{\beta}}(X)) && \text{by Def. 39} \\
&= \widetilde{Q}(Z) && \text{by (313)} \\
&= g(\mu_{[1]}(Z), \ldots, \mu_{[m]}(Z)) && \text{by (310)} \\
&= g(\mu_{[1]}(X), \ldots, \mu_{[m]}(X))\,, && \text{by L-11, (313)}
\end{aligned}$$

as desired.

In order to prove that **b.** entails **a.**, suppose that $\widetilde{Q}$ can be expressed in terms of (144), based on a mapping $g : \mathbf{I}^m \longrightarrow \mathbf{I}$. Now let $\beta : E \longrightarrow E$ be some automorphism and $X \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\widetilde{Q}(\hat{\hat{\beta}}(X)) &= g(\mu_{[1]}(\hat{\hat{\beta}}(X)), \ldots, \mu_{[m]}(\hat{\hat{\beta}}(X))) && \text{by (144)} \\
&= g(\mu_{[1]}(X), \ldots, \mu_{[m]}(X)) && \text{by L-11} \\
&= \widetilde{Q}(X)\,, && \text{by (144)}
\end{aligned}$$

i.e. $\widetilde{Q}$ is indeed quantitative according to Def. 39.

D.7 Proof of Theorem 246

Lemma 12
Let $X \in \widetilde{\mathcal{P}}(E)$ be some fuzzy subset and suppose that $\ell', \ell, u, u' \in \mathbb{N}$ are given integers such that $\ell' \leq \ell \leq u \leq u'$. Then $\mu_{\|X\|_{\mathrm{iv}}}(\ell', u') \geq \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)$.

Proof To see this, we recall from Def. 99 that

$$\mu_{[j]}(X) \geq \mu_{[k]}(X) \tag{314}$$

whenever $j \leq k$, i.e. the $\mu_{[j]}(X)$ form a nonincreasing sequence. We may hence conclude from $\ell' \leq \ell$ that

$$\mu_{[\ell']}(X) \geq \mu_{[\ell]}(X)\,. \tag{315}$$

Similarly, we obtain from (314) and $u \leq u'$ that $\mu_{[u+1]}(X) \geq \mu_{[u'+1]}(X)$. By means of negation, the latter inequality becomes

$$1 - \mu_{[u'+1]}(X) \geq 1 - \mu_{[u+1]}(X)\,. \tag{316}$$

Therefore

$$\begin{aligned}
\mu_{\|X\|_{\mathrm{iv}}}(\ell', u') &= \min(\mu_{[\ell']}(X), 1 - \mu_{[u'+1]}(X)) && \text{by Def. 162} \\
&\geq \min(\mu_{[\ell]}(X), 1 - \mu_{[u+1]}(X)) && \text{by (315) and (316)} \\
&= \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)\,, && \text{by Def. 162}
\end{aligned}$$

as desired.

Lemma 13
Let $E \neq \varnothing$ be some finite base set and $r, s \in \mathbb{N}$, $r \leq s$. Then $\mathcal{M}_{CX}([\geq r : \leq s]) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is the fuzzy quantifier defined by

$$\mathcal{M}_{CX}([\geq r : \leq s])(X) = \mu_{\|X\|_{\mathrm{iv}}}(r, s) = \min(\mu_{[r]}(X), 1 - \mu_{[s+1]}(X)),$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Proof It is obvious from Def. 163 that $[\geq r : \leq s](Y)$ can be expressed in terms of the cardinality of Y, i.e.

$$[\geq r : \leq s](Y) = q(|Y|)$$

for all $Y \in \mathcal{P}(E)$, where the mapping $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{2}$ becomes

$$q(j) = \begin{cases} 1 & : \quad r \leq j \leq s \\ 0 & : \quad \text{else} \end{cases} \tag{317}$$

for all $j \in \{0, \ldots, |E|\}$. In particular, we can apply Th-240 and conclude that $[\geq r : \leq s]$ is quantitative. In turn, knowing this renders Th-245 applicable. For a given choice of $X \in \widetilde{\mathcal{P}}(E)$, this lets us deduce that $\mathcal{M}_{CX}([\geq r : \leq s])(X)$ can be expressed as

$$\mathcal{M}_{CX}([\geq r : \leq s])(X) = \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell, u), q^{\min}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\} \tag{318}$$

for all $X \in \widetilde{\mathcal{P}}(E)$, where the coefficient $q^{\min}(\ell, u)$ is sampled from q according to (145). Now taking a closer look at this coefficient, it quite obvious from (317) that

$$q^{\min}(\ell, u) = \begin{cases} 1 & : \quad r \leq \ell \leq u \leq s \\ 0 & : \quad \text{else} \end{cases} \tag{319}$$

for all $\ell, u \in \{0, \ldots, |E|\}$, $\ell \leq u$. For the given $X \in \widetilde{\mathcal{P}}(E)$, we can hence proceed as follows.

$$\begin{aligned} &\mathcal{M}_{CX}([\geq r : \leq s])(X) \\ &= \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell, u), q^{\min}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\} && \text{by (318)} \\ &= \max\{\mu_{\|X\|_{\mathrm{iv}}}(\ell, u) : q^{\min}(\ell, u) = 1\} && \text{because } q^{\min} \text{ two-valued} \\ &= \max\{\mu_{\|X\|_{\mathrm{iv}}}(\ell, u) : r \leq \ell \leq u \leq s\} && \text{by (319)} \\ &= \mu_{\|X\|_{\mathrm{iv}}}(r, s)\,. && \text{by L-12} \end{aligned}$$

Now recalling Def. 162, we finally obtain that $\mathcal{M}_{CX}([\geq r : \leq s])(X) = \mu_{\|X\|_{\mathrm{iv}}}(r, s) = \min(\mu_{[r]}(X), 1 - \mu_{[s+1]}(X))$, which completes the proof of the lemma.

Proof of Theorem 246

In order to prove the main theorem, consider a finite base set $E \neq \varnothing$ and a choice of $r, s \in \mathbb{N}$ with $r \leq s$. Now let $\mathcal{F}$ be an arbitrary standard DFS. I prove that

$\mathcal{F}([\geq r : \leq s])$ takes the desired form by considering a choice of fuzzy argument $X \in \widetilde{\mathcal{P}}(E)$. Then $\mathcal{F}([\geq r : \leq s])(X) = \mathcal{M}_{CX}([\geq r : \leq s])(X)$ by Th-46, because the base quantifier $[\geq r : \leq s]$ is two-valued. The above lemma L-13 lets us deduce the desired $\mathcal{F}([\geq r : \leq s])(X) = \mu_{\|X\|_{\mathrm{iv}}}(r, s) = \min(\mu_{[r]}(X), 1 - \mu_{[s+1]}(X))$.

D.8 Proof of Theorem 247

Let a base set $E \neq \varnothing$ be given.

a.: In order to prove the first part of the theorem, we consider a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$. Clearly

$$\begin{aligned} A(\neg X) &= \operatorname{Im} \mu_{\neg X} && \text{by (151)} \\ &= \{\mu_{\neg X}(e) : e \in E\} \\ &= \{1 - \mu_X(e) : e \in E\} \\ &= \{1 - \alpha : \alpha \in \{\mu_X(e) : e \in E\}\} \\ &= \{1 - \alpha : \alpha \in \operatorname{Im} \mu_X\} \\ &= \{1 - \alpha : \alpha \in A(X)\}\,, && \text{by (151)} \end{aligned}$$

i.e.

$$A(\neg X) = \{1 - \alpha : \alpha \in A(X)\}\,. \tag{320}$$

Consequently

$$\begin{aligned} A(\neg X) \cap [\tfrac{1}{2}, 1] &= \{\alpha \in A(\neg X) : \alpha \in [\tfrac{1}{2}, 1]\} \\ &= \{1 - \alpha : \alpha \in A(X), 1 - \alpha \in [\tfrac{1}{2}, 1]\} && \text{by (320)} \\ &= \{1 - \alpha : \alpha \in A(X), \alpha \in [0, \tfrac{1}{2}]\}\,, \end{aligned}$$

i.e.

$$A(\neg X) \cap [\tfrac{1}{2}, 1] = \{1 - \alpha : \alpha \in A(X) \cap [0, \tfrac{1}{2}]\}\,. \tag{321}$$

By similar reasoning,

$$\begin{aligned} A(\neg X) \cap [0, \tfrac{1}{2}] &= \{\alpha \in A(\neg X) : \alpha \in [0, \tfrac{1}{2}]\} \\ &= \{1 - \alpha : \alpha \in A(X), 1 - \alpha \in [0, \tfrac{1}{2}]\} && \text{by (320)} \\ &= \{1 - \alpha : \alpha \in A(X), \alpha \in [\tfrac{1}{2}, 1]\} \end{aligned}$$

and hence

$$A(\neg X) \cap [0, \tfrac{1}{2}] = \{1 - \alpha : \alpha \in A(X) \cap [\tfrac{1}{2}, 1]\}\,. \tag{322}$$

Therefore

$$\begin{aligned}\Gamma(\neg X) &= \{2\alpha - 1 : \alpha \in A(\neg X) \cap [\tfrac{1}{2}, 1]\} \\ &\quad \cup \{1 - 2\alpha : \alpha \in A(\neg X) \cap [0, \tfrac{1}{2}]\} \\ &\quad \cup \{0, 1\} && \text{by (152)} \\ &= \{2(1-\alpha) - 1 : \alpha \in A(X) \cap [0, \tfrac{1}{2}]\} \\ &\quad \cup \{1 - 2(1-\alpha) : \alpha \in A(X) \cap [\tfrac{1}{2}, 1]\} \\ &\quad \cup \{0, 1\} && \text{by (321), (322)} \\ &= \{1 - 2\alpha : \alpha \in A(X) \cap [0, \tfrac{1}{2}]\} \\ &\quad \cup \{2\alpha - 1 : \alpha \in A(X) \cap [\tfrac{1}{2}, 1]\} \\ &\quad \cup \{0, 1\} \\ &= \Gamma(X)\,, && \text{by (152)}\end{aligned}$$

as desired.

b.: In order to see that the second part of the theorem is valid, consider $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. We notice that for all $e \in E$,

$$\begin{aligned}\mu_{X_1 \cap X_2}(e) &= \min(\mu_{X_1}(e), \mu_{X_2}(e)) \\ &\in \{\mu_{X_1}(e), \mu_{X_2}(e)\} \\ &\subseteq \operatorname{Im} \mu_{X_1} \cup \operatorname{Im} \mu_{X_2} \\ &= A(X_1, X_2)\,, && \text{by (151)}\end{aligned}$$

i.e.

$$\mu_{X_1 \cap X_2}(e) \in A(X_1, X_2)\,. \tag{323}$$

Consequently

$$\begin{aligned}A(X_1 \cap X_2) &= \operatorname{Im} \mu_{X_1 \cap X_2} && \text{by (151)} \\ &= \{\mu_{X_1 \cap X_2}(e) : e \in E\} \\ &\subseteq A(X_1, X_2)\,, && \text{by (323)}\end{aligned}$$

i.e.

$$A(X_1 \cap X_2) \subseteq A(X_1, X_2)\,. \tag{324}$$

In turn,

$$\begin{aligned}\Gamma(X_1 \cap X_2) &= \{2\alpha - 1 : \alpha \in A(X_1 \cap X_2) \cap [\tfrac{1}{2}, 1]\} \\ &\quad \cup \{1 - 2\alpha : \alpha \in A(X_1 \cap X_2) \cap [0, \tfrac{1}{2}]\} \\ &\quad \cup \{0, 1\} \\ &\subseteq \{2\alpha - 1 : \alpha \in A(X_1, X_2) \cap [\tfrac{1}{2}, 1]\} \\ &\quad \cup \{1 - 2\alpha : \alpha \in A(X_1, X_2) \cap [0, \tfrac{1}{2}]\} \\ &\quad \cup \{0, 1\} && \text{by (324)} \\ &= \Gamma(X_1, X_2)\,. && \text{by (152)}\end{aligned}$$

This completes the proof for the fuzzy intersection $X_1 \cap X_2$. The remaining claim on the standard union of X_1 and X_2 is then apparent from De Morgan's law, i.e.

$$\begin{aligned}
\Gamma(X_1 \cup X_2) &= \Gamma(\neg(\neg X_1 \cap \neg X_2)) && \text{by De Morgan's law}\\
&= \Gamma(\neg X_1 \cap \neg X_2) && \text{by part } \textbf{a.} \text{ of the theorem}\\
&\subseteq \Gamma(\neg X_1, \neg X_2) && \text{by part } \textbf{b.} \text{ on intersections}\\
&= \Gamma(\neg X_1) \cup \Gamma(\neg X_2) && \text{by (153)}\\
&= \Gamma(X_1) \cup \Gamma(X_2) && \text{by part } \textbf{a.} \text{ of the theorem}\\
&= \Gamma(X_1, X_2)\,, && \text{by (153)}
\end{aligned}$$

as desired.

D.9 Proof of Theorem 248

Lemma 14
Let $X \in \widetilde{\mathcal{P}}(E)$ be some fuzzy subset such that $A(X)$ is finite, and further suppose that $A = \{\alpha_0, \dots, \alpha_m\} \supseteq A(X)$ is chosen such that $\alpha_0 < \cdots < \alpha_m$. Then for all $j \in \{0, \dots, m-1\}$ and $\alpha \in (\alpha_j, \alpha_{j+1})$,

$$\begin{aligned}
X_{\geq \alpha} &= X_{\geq \overline{\alpha}_j}\\
X_{> \alpha} &= X_{> \overline{\alpha}_j}
\end{aligned}$$

where $\overline{\alpha} = (\alpha_j + \alpha_{j+1})/2$.

Proof Let $j \in \{0, \dots, m-1\}$ be given and consider some $\alpha \in (\alpha_j, \alpha_{j+1})$. Due to the fact that $A(X) = \operatorname{Im} \mu_X \subseteq A = \{\alpha_0, \dots, \alpha_m\}$ and further noticing that the α_k form a strictly increasing sequence, it is apparent that $A \cap (\alpha_j, \alpha_{j+1}) = \varnothing$ and hence $A(X) \cap (\alpha_j, \alpha + j + 1) = \varnothing$ as well. Because $A(X) = \{\mu_X(e) : e \in E\}$, this proves that for all $e \in E$, either $\mu_X(e) \leq \alpha_j$ or $\mu_X(e) \geq \alpha_{j+1}$. Noticing that $\alpha_j < \alpha < \alpha_{j+1}$ and $\alpha_j < \overline{\alpha}_j < \alpha_{j+1}$, the following equivalences are now straightforward. For all $e \in E$,

$$\mu_X(e) \geq \alpha \iff \mu_X(e) \geq \alpha_{j+1} \iff \mu_X(e) \geq \overline{\alpha}_j \tag{325}$$

$$\mu_X(e) > \alpha \iff \mu_X(e) > \alpha_j \iff \mu_X(e) > \overline{\alpha}_j\,. \tag{326}$$

Therefore

$$\begin{aligned}
X_{\geq \alpha} &= \{e \in E : \mu_X(e) \geq \alpha\} && \text{by Def. 75}\\
&= \{e \in E : \mu_X(e) \geq \overline{\alpha}_j\} && \text{by (325)}\\
&= X_{\geq \overline{\alpha}_j}\,, && \text{by Def. 75}
\end{aligned}$$

and

$$\begin{aligned}
X_{> \alpha} &= \{e \in E : \mu_X(e) > \alpha\} && \text{by Def. 76}\\
&= \{e \in E : \mu_X(e) > \overline{\alpha}_j\} && \text{by (326)}\\
&= X_{> \overline{\alpha}_j}\,, && \text{by Def. 76}
\end{aligned}$$

as desired.

Lemma 15

Let $E \neq \varnothing$ be some base set and suppose that $X \in \widetilde{\mathcal{P}}(E)$ is a choice of fuzzy subset which makes $\Gamma(X)$ a finite set. Further let $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X)$ be chosen such that $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m$. Then for all $j \in \{0, \ldots, m-1\}$ and all $\gamma \in (\gamma_j, \gamma_{j+1})$,

$$\mathcal{T}_\gamma(X) = \mathcal{T}_{\overline{\gamma}_j}(X)\,.$$

Proof To see this, let $j \in \{0, \ldots, m-1\}$ and consider a choice of $\gamma \in (\gamma_j, \gamma_{j+1})$. Then in particular $\gamma > \gamma_j \geq \gamma_0 = 0$, i.e. $\gamma > 0$. We hence obtain that

$$\begin{aligned} X^{\min}_\gamma &= X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} && \text{by Def. 82 because } \gamma > 0 \\ &= X_{\geq \frac{1}{2} + \frac{1}{2}\overline{\gamma}_j} && \text{by L-14} \\ &= X^{\min}_{\overline{\gamma}_j}\,, && \text{by Def. 82 because } \overline{\gamma}_j > 0 \end{aligned}$$

and

$$\begin{aligned} X^{\max}_\gamma &= X_{> \frac{1}{2} - \frac{1}{2}\gamma} && \text{by Def. 82 because } \gamma > 0 \\ &= X_{> \frac{1}{2} - \frac{1}{2}\overline{\gamma}_j} && \text{by L-14} \\ &= X^{\max}_{\overline{\gamma}_j}\,. && \text{by Def. 82 because } \overline{\gamma}_j > 0 \end{aligned}$$

Hence $X^{\min}_\gamma = X^{\min}_{\overline{\gamma}_j}$ and $X^{\max}_\gamma = X^{\max}_{\overline{\gamma}_j}$ and in turn,

$$\begin{aligned} \mathcal{T}_\gamma(X) &= \{Y : X^{\min}_\gamma \subseteq Y \subseteq X^{\max}_\gamma\} \\ &= \{Y : X^{\min}_{\overline{\gamma}_j} \subseteq Y \subseteq X^{\max}_{\overline{\gamma}_j}\} \\ &= \mathcal{T}_{\overline{\gamma}_j}(X)\,. \end{aligned}$$

Proof of Theorem 248

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. We shall further suppose that $\Gamma(X_1, \ldots, X_n)$ be finite. In addition, a choice of $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X_1, \ldots, X_n)$ will be assumed with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. Now let $j \in \{0, \ldots, m-1\}$ and $\gamma \in (\gamma_j, \gamma_{j+1})$. Then

$$\begin{aligned} &\top_{Q,X_1,\ldots,X_n}(\gamma) \\ &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_i), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 100} \\ &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_{\overline{\gamma}_j}(X_i), \ldots, Y_n \in \mathcal{T}_{\overline{\gamma}_j}(X_n)\} && \text{by L-15} \\ &= \top_{Q,(X_1,\ldots,X_n)}(\overline{\gamma}_j) && \text{by Def. 100} \\ &= \top_j\,, && \text{by (155)} \end{aligned}$$

and similarly

$$\begin{aligned}
&\bot_{Q,X_1,\ldots,X_n}(\gamma) && \\
&= \inf\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_i),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 100} \\
&= \inf\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_{\overline{\gamma}_j}(X_i),\ldots,Y_n \in \mathcal{T}_{\overline{\gamma}_j}(X_n)\} && \text{by L-15} \\
&= \bot_{Q,(X_1,\ldots,X_n)}(\overline{\gamma}_j) && \text{by Def. 100} \\
&= \bot_j\,, && \text{by (155)}
\end{aligned}$$

as desired. From this , we conclude that

$$Q_\gamma(X_1,\ldots,X_n) = \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\ldots,X_n}(\gamma),\bot_{Q,X_1,\ldots,X_n}(\gamma)) = \mathrm{med}_{\frac{1}{2}}(\top_j,\bot_j) = C_j\,,$$

recalling Th-111 and (157).

D.10 Proof of Theorem 249

Lemma 16
Suppose that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2) *and* (X-4). *Then*

$$\xi(\top^\sharp,\bot^\sharp) = \xi(\top^\flat,\bot^\flat)$$

for all $(\top,\bot) \in \mathbb{T}$.

Proof Trivial.

$$\begin{aligned}
\xi(\top^\sharp,\bot^\sharp) &= \xi(\top^\flat,\bot^\sharp) && \text{by (X-4)} \\
&= 1 - \xi(1-\bot^\sharp, 1-\top^\flat) && \text{by (X-2)} \\
&= 1 - \xi((1-\bot)^\sharp, (1-\top)^\flat) && \text{apparent from Def. 87} \\
&= 1 - \xi((1-\bot)^\flat, (1-\top)^\flat) && \text{by (X-4)} \\
&= 1 - \xi(1-\bot^\flat, 1-\top^\flat) && \text{apparent from Def. 87} \\
&= \xi(\top^\flat,\bot^\flat)\,. && \text{by (X-2)}
\end{aligned}$$

Lemma 17
Suppose that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2), (X-4) *and* (X-5). *Then*

$$\xi(\top,\bot) = \xi(\top^\flat,\bot) = \xi(\top^\sharp,\bot) = \xi(\top,\bot^\sharp) = \xi(\top,\bot^\flat)$$

for all $(\top,\bot) \in \mathbb{T}$.

Proof It has already been shown by Glöckner [50, L-20, p. 58] that

$$\xi(\top,\bot) = \xi(\top^\flat,\bot) = \xi(\top^\sharp,\bot)\,. \tag{327}$$

The remaining equalities are then apparent from

$$\begin{aligned}
\xi(\top, \bot) &= 1 - \xi(1 - \bot, 1 - \top) && \text{by (X-2)} \\
&= 1 - \xi((1 - \bot)^\sharp, 1 - \top) && \text{by (327)} \\
&= 1 - \xi(1 - \bot^\sharp, 1 - \top) && \text{apparent from Def. 87} \\
&= \xi(\top, \bot^\sharp) && \text{by (X-2)}\,,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\xi(\top, \bot) &= 1 - \xi(1 - \bot, 1 - \top) && \text{by (X-2)} \\
&= 1 - \xi((1 - \bot)^\flat, 1 - \top) && \text{by (327)} \\
&= 1 - \xi(1 - \bot^\flat, 1 - \top) && \text{apparent from Def. 87} \\
&= \xi(\top, \bot^\flat) && \text{by (X-2)}\,.
\end{aligned}$$

Proof of Theorem 249

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and suppose that $(\gamma_j)_{j \in \{0,\dots,m\}}$ is an $\mathbf{I}$-valued sequence such that

$$0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1\,.$$

Now consider a choice of $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$\begin{aligned}
\top(\gamma) &= \top'(\gamma') \\
\bot(\gamma) &= \bot'(\gamma')
\end{aligned}$$

for all $j \in \{0, \dots, m-1\}$ and $\gamma, \gamma' \in (\gamma_j, \gamma_{j+1})$. Further suppose that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2), (X-4) and (X-5). In order to establish the desired equality $\xi(\top, \bot) = \xi(\top', \bot')$, we simply notice that

$$(\top^{\flat\sharp}, \bot^{\flat\sharp}) = (\top'^{\flat\sharp}, \bot'^{\flat\sharp})\,. \tag{328}$$

Therefore

$$\begin{aligned}
\xi(\top, \bot) &= \xi(\top^{\flat\sharp}, \bot^{\flat\sharp}) && \text{by L-17} \\
&= \xi(\top'^{\flat\sharp}, \bot'^{\flat\sharp}) && \text{by (328)} \\
&= \xi(\top', \bot')\,, && \text{by L-17}
\end{aligned}$$

as desired.

D.11 Proof of Theorem 250

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \in \mathcal{P}(\mathbf{I})$ such that $\Gamma \supseteq \Gamma(X_1, \ldots, X_n)$ and $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. Then

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Ch}}(Q)(X_1, \ldots, X_n) \\
&= \tfrac{1}{2} \int_0^1 \top_{Q,(X_1,\ldots,X_n)}(\gamma)\, d\gamma + \tfrac{1}{2} \int_0^1 \bot_{Q,(X_1,\ldots,X_n)}(\gamma)\, d\gamma && \text{by Def. 104, Def. 102} \\
&= \tfrac{1}{2} \int_0^1 \top_{Q,(X_1,\ldots,X_n)}(\gamma) + \bot_{Q,(X_1,\ldots,X_n)}(\gamma)\, d\gamma \\
&= \tfrac{1}{2} \sum_{j=0}^{m-1} (\gamma_{j+1} - \gamma_j)(\top_j + \bot_j)\,, && \text{by Th-248 and (155), (156)}
\end{aligned}$$

where the last equality results from the usual definition of the Riemann integral for step functions with a finite number of support points. The integrand $\top_{Q,X_1,\ldots,X_n}(\gamma) + \bot_{Q,X_1,\ldots,X_n}(\gamma)$ is known to belong to this class of mappings from Th-248.

D.12 Proof of Theorem 251

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given such that $\Gamma(X_1, \ldots, X_n)$ is finite. Further let some $\Gamma = \{\gamma_0, \ldots, \gamma_m\} \supseteq \Gamma(X_1, \ldots, X_n)$ be given with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. We are interested in the case that $C_0 = \frac{1}{2}$, i.e. $Q_{\overline{\gamma}_0}(X_1, \ldots, X_n) = \frac{1}{2}$ by (157). By Th-248, then, we know that $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma \in (\gamma_0, \gamma_1] = (0, \gamma_1]$. Recalling Th-62, we can now conclude that in fact, $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma \in (0, 1]$. We then obtain from Def. 87 that

$$(Q_\gamma(X_1, \ldots, X_n)_{\gamma \in \mathbf{I}})^\sharp = \mathrm{c}_{\frac{1}{2}}\,. \tag{329}$$

In addition, it is apparent from the above analysis of $Q_\gamma(X_1, \ldots, X_n)$ that

$$(Q_\gamma(X_1, \ldots, X_n)_{\gamma \in \mathbf{I}})^\flat = (Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}\,. \tag{330}$$

Now consider a choice of $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ such that $\mathcal{M}_\mathcal{B}$ becomes a DFS. By Th-67, then, we know that $\mathcal{B}$ satisfies (B-4) and (B-3). Therefore

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) &= \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 86} \\
&= \mathcal{B}((Q_\gamma(X_1, \ldots, X_n)_{\gamma \in \mathbf{I}})^\flat) && \text{by (330)} \\
&= \mathcal{B}((Q_\gamma(X_1, \ldots, X_n)_{\gamma \in \mathbf{I}})^\sharp) && \text{by (B-4)} \\
&= \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) && \text{by (329)} \\
&= \tfrac{1}{2}\,. && \text{by (B-3)}
\end{aligned}$$

D.13 Proof of Theorem 252

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and suppose that $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy arguments such that $\Gamma(X_1, \ldots, X_n)$ is a finite subset of $\mathbf{I}$. Further assume a choice of

$\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X_1, \dots, X_n)$ with $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$.

a.: $\bot_0 > \frac{1}{2}$.
In order to prove this case, we first observe that for all $x \in [\frac{1}{2}, 1]$ and $y \in [0, x]$,

$$\mathrm{med}_{\frac{1}{2}}(x, y) = \max(\tfrac{1}{2}, y)\,, \tag{331}$$

which is apparent from Def. 56. Let us further notice that $\top_j \geq \bot_j$ by (155), (156) and Th-107. In particular, we can conclude from $\bot_0 > \frac{1}{2}$ and $\top_0 \geq \bot_0$ that $\top_0 \geq \frac{1}{2}$ as well. Now utilizing that $\top_j \geq \top_0$ for all $j \in \{0, \dots, m-1\}$ again by (155) and Th-107, this proves that both $\top_j \in [\frac{1}{2}, 1]$ and $\bot_j \leq \top_j$ for all $j \in \{0, \dots, m-1\}$. Recalling from (157) that $C_j = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j)$, we can now apply (331) and conclude that $C_j = \mathrm{med}_{\frac{1}{2}}(\top_j, \bot_j) = \max(\bot_j, \frac{1}{2})$, i.e.

$$C_j = \begin{cases} \bot_j & : \quad \bot_j > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot_j \leq \frac{1}{2}\,. \end{cases} \tag{332}$$

Let us further observe that $C_j = \frac{1}{2}$ exactly if $\bot_j \leq \frac{1}{2}$, which is apparent from (332). Hence J^* defined by (159) becomes

$$J^* = \{j \in \{0, \dots, m-1\} : C_j = \tfrac{1}{2}\} = \{j \in \{0, \dots, m-1\} : \bot_j \leq \tfrac{1}{2}\}\,. \tag{333}$$

Next we utilize that $\bot_j$ is nonincreasing by (156) and Th-107. Hence $\bot_{j'} \leq \frac{1}{2}$ for some $j' \in \{0, \dots, m-1\}$ entails that $\bot_j \leq \frac{1}{2}$ for all $j \geq j'$ as well. In the case that J^* is non-empty, we can substitute $j^* = \min J^*$ for j', and conclude that $\bot_j > \frac{1}{2}$ for all $j < j^*$, $\bot_j \leq \frac{1}{2}$ for all $j \geq j^*$. The above equation (332) then becomes

$$C_j = \begin{cases} \bot_j & : \quad j < j^* \\ \frac{1}{2} & : \quad j \geq j^*\,, \end{cases} \tag{334}$$

for all $j \in \{0, \dots, m-1\}$, as desired. In the remaining case that $J^* = \varnothing$, we know from (333) that $C_j \neq \frac{1}{2}$ and $\bot_j > \frac{1}{2}$ for all $j \in \{0, \dots, m-1\}$, i.e. $C_j = \bot_j$ for all $j \in \{0, \dots, m-1\}$ by (332). Letting $j^* = m$, as proposed in (160), hence makes (334) valid in the case $J^* = \varnothing$ as well, because the latter case '$j \geq j^*$' then becomes vacuous, while the former case '$j < j^*$' in (334) ensures that we always get the desired result of $C_j = \bot_j$. Hence the claimed equality is valid for all $j \in \{0, \dots, m-1\}$ and regardless of J^*, which completes the proof of part **a.** of the theorem.

b.: $\top_0 < \frac{1}{2}$.
The proof of case **b.** is completely analogous to that of the former case. The above equation (331) must then be replaced with

$$\mathrm{med}_{\frac{1}{2}}(x, y) = \min(x, \tfrac{1}{2})\,,$$

for all $y \in [0, \frac{1}{2}]$ and $x \in [y, 1]$, which is again straightforward from Def. 56.

D.14 Proof of Theorem 253

Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier, $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments such that $\Gamma(X_1, \dots, X_n)$ is finite, and and $\Gamma = \{\gamma_0, \dots, \gamma_m\}$ a finite subset of $\mathbf{I}$ with $\Gamma \supseteq \Gamma(X_1, \dots, X_n)$ and $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$. By expanding the definition of the Riemann integral in the case of step functions, we then obtain that

$$
\begin{aligned}
&\mathcal{M}(Q)(X_1, \dots, X_n) \\
&= \int_0^1 Q_\gamma(X_1, \dots, X_n)\, d\gamma && \text{by Def. 84} \\
&= \sum_{j=0}^{m-1} (\gamma_{j+1} - \gamma_j) C_j && \text{by Th-248} \\
&= \sum_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) C_j + \sum_{j=j^*}^{m-1} (\gamma_{j+1} - \gamma_j) C_j && \text{by splitting the sum} \\
&= \sum_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) C_j + \sum +j = j^{*m-1} (\gamma_{j+1} - \gamma_j) \tfrac{1}{2}\,, && \text{by (161)}
\end{aligned}
$$

i.e.

$$
\mathcal{M}(Q)(X_1, \dots, X_n) = \left(\sum_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) C_j \right) + \tfrac{1}{2}(1 - \gamma_{j^*})\,, \qquad (335)
$$

where the last step is apparent recalling that $\gamma_m = 1$. This can be further simplified if we notice that $\mathcal{M}$ is an $\mathcal{M}_\mathcal{B}$-DFS by Th-63, and hence satisfies

$$
\mathcal{M}(Q)(X_1, \dots, X_n) = \tfrac{1}{2} \qquad (336)
$$

in the case that $\bot_0 \leq \frac{1}{2} \leq \top_0$ or equivalently $C_0 = \frac{1}{2}$, see Th-251. In the case that $\bot_0 > \frac{1}{2}$, we can profit from the rendering of C_j that was achieved in Th-252.a, and hence rewrite (335) as

$$
\mathcal{M}(Q)(X_1, \dots, X_n) = \left(\sum_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) \bot_j \right) + \tfrac{1}{2}(1 - \gamma_{j^*})\,. \qquad (337)
$$

An analogous simplification is possible in the remaining case that $\top_0 < \frac{1}{2}$. We then obtain the desired

$$
\mathcal{M}(Q)(X_1, \dots, X_n) = \left(\sum_{j=0}^{j^*-1} (\gamma_{j+1} - \gamma_j) \top_j \right) + \tfrac{1}{2}(1 - \gamma_{j^*}) \qquad (338)
$$

by replacing C_j in (335) with $\top_j$, which is justified by part b. of Th-252. Having considered all possible cases, we can finally synthesize (336)–(338) into the desired result, viz

$$\mathcal{M}(Q)(X_1,\ldots,X_n) = \begin{cases} \left(\sum_{j=0}^{j^*-1}(\gamma_{j+1}-\gamma_j)\bot_j\right) + \frac{1}{2}(1-\gamma_{j^*}) & : \quad C_0 > \frac{1}{2} \\ \frac{1}{2} & : \quad C_0 = \frac{1}{2} \\ \left(\sum_{j=0}^{j^*-1}(\gamma_{j+1}-\gamma_j)\top_j\right) + \frac{1}{2}(1-\gamma_{j^*}) & : \quad C_0 < \frac{1}{2} \end{cases} .$$

D.15 Proof of Theorem 254

Lemma 18
For all $f \in \mathbb{H}$,

$$\mathcal{B}'_{CX}(f) = \text{the unique } x \text{ s.th. } f(y) > y \text{ for all } y < x \text{ and } f(y) < y \text{ for all } y > x\,.$$

Proof See Glöckner [48, Th-94,p. 63].

Proof of Theorem 254

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$ be given such that $\Gamma(X_1,\ldots,X_n)$ is finite. Further suppose that $\Gamma = \{\gamma_0,\ldots,\gamma_m\} \supseteq \Gamma(X_1,\ldots,X_n)$ is a subset of $\mathbf{I}$ with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$. In the following, I will use the abbreviations introduced in the body of the theorem. It is convenient to discern three cases which parallel the case-wise rendering of $\mathcal{M}_{CX}(Q)(X_1,\ldots,X_n)$ in the theorem.

a.: $\bot_0 > \frac{1}{2}$. In this case, we first utilize (45) and L-18, which lets us express the quantification result $\mathcal{M}_{CX}(Q)(X_1,\ldots,X_n)$ as

$$\mathcal{M}_{CX}(Q)(X_1,\ldots,X_n) = \tfrac{1}{2} + \tfrac{1}{2}\widehat{\gamma}\,, \tag{339}$$

where

$$\begin{aligned} \widehat{\gamma} = \text{the unique } \gamma \text{ s.th. } & Q_{\gamma'}(X_1,\ldots,X_n) > \tfrac{1}{2} + \tfrac{1}{2}\gamma' \text{ for all } \gamma' < \gamma \quad \text{ and} \\ & Q_{\gamma'}(X_1,\ldots,X_n) < \tfrac{1}{2} + \tfrac{1}{2}\gamma' \text{ for all } \gamma' > \gamma. \end{aligned} \tag{340}$$

It remains to be shown that

$$\widehat{\gamma} = \max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})\,. \tag{341}$$

To this end, I first prove that

$$\widehat{\gamma} \geq \gamma_{\widehat{\jmath}}\,. \tag{342}$$

Let us notice in advance that for $j \in \{0,\ldots,\widehat{\jmath}-1\}$, $2\bot_j - 1 = B_j > \gamma_{j+1}$ by (162) (164) and (165). Therefore

$$\bot_j > \tfrac{1}{2} + \tfrac{1}{2}\gamma_{j+1} \tag{343}$$

for all $j \in \{0, \ldots, \widehat{\jmath} - 1\}$.
In order to prove the desired inequality (342), consider some $\gamma < \gamma_{\widehat{\jmath}}$. Then there exists $\gamma' \geq \gamma$ with $\gamma' \in (\gamma_{\widehat{\jmath}-1}, \gamma_j)$. We notice that $Q_{\overline{\gamma}_0}(X_1, \ldots, X_n) = C_0 > \frac{1}{2}$ entails that $Q_0(X_1, \ldots, X_n) > 0$. In particular, $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$ is known to be nonincreasing by Th-62. Therefore

$$\begin{aligned}
Q_\gamma(X_1, \ldots, X_n) &\geq Q_{\gamma'}(X_1, \ldots, X_n) && \text{because } \gamma' \geq \gamma \\
&= C_{\widehat{\jmath}-1} && \text{by Th-248} \\
&\geq \bot_{\widehat{\jmath}-1} && \text{by Th-252} \\
&> \tfrac{1}{2} + \tfrac{1}{2}\gamma_{\widehat{\jmath}} && \text{by (343)} \\
&> \tfrac{1}{2} + \tfrac{1}{2}\gamma\,,
\end{aligned}$$

because $\gamma < \gamma_{\widehat{\jmath}}$ by assumption. Hence $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2} + \frac{1}{2}\gamma$, and by (340), $\widehat{\gamma} > \gamma$. Noticing that $\gamma < \gamma_{\widehat{\jmath}}$ was arbitrarily chosen, this proves that $\widehat{\gamma} \geq \gamma_{\widehat{\jmath}}$, i.e. (342) is valid.

Now suppose that $B_{\widehat{\jmath}} < \gamma_{\widehat{\jmath}}$ and let $\gamma > \gamma_{\widehat{\jmath}}$. We can then choose some $\gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1})$ with $\gamma' < \gamma$. As concerns γ, we first observe that

$$\begin{aligned}
\tfrac{1}{2} + \tfrac{1}{2}\gamma &> \tfrac{1}{2} + \tfrac{1}{2}\gamma_{\widehat{\jmath}} && \text{because } \gamma > \gamma_{\widehat{\jmath}} \\
&> \tfrac{1}{2} + \tfrac{1}{2}B_{\widehat{\jmath}} && \text{by assumption} \\
&= \tfrac{1}{2} + \tfrac{1}{2}(2\bot_{\widehat{\jmath}} - 1) && \text{by (162)} \\
&= \bot_{\widehat{\jmath}}\,,
\end{aligned}$$

i.e.

$$\tfrac{1}{2} + \tfrac{1}{2}\gamma > \bot_{\widehat{\jmath}} \tag{344}$$

We further conclude from $\gamma > \gamma_{\widehat{\jmath}}$ that $\gamma > 0$ and hence

$$\tfrac{1}{2} + \tfrac{1}{2}\gamma > \tfrac{1}{2}\,. \tag{345}$$

Therefore

$$\begin{aligned}
&\tfrac{1}{2} + \tfrac{1}{2}\gamma \\
&> \max(\tfrac{1}{2}, \bot_{\widehat{\jmath}}) && \text{by (344), (345)} \\
&= C_{\widehat{\jmath}} && \text{by Th-252} \\
&= Q_{\gamma'}(X_1, \ldots, X_n) && \text{by Th-248, } \gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}) \\
&\geq Q_\gamma(X_1, \ldots, X_n) && \text{because } \gamma \geq \gamma' \text{ and } Q_\gamma(X_1, \ldots, X_n) \text{ nonincreasing}\,,
\end{aligned}$$

i.e. $Q_\gamma(X_1, \ldots, X_n) < \frac{1}{2} + \frac{1}{2}\gamma$. We can then conclude from (340) that $\widehat{\gamma} < \gamma$. Noticing that $\gamma > \gamma_{\widehat{\jmath}}$ was arbitrarily chosen, this substantiates that $\widehat{\gamma} \leq \gamma_{\widehat{\jmath}}$. However, we already know from (342) that $\widehat{\gamma} \geq \gamma_{\widehat{\jmath}}$ as well. Under our previous assumption that $B_{\widehat{\jmath}} < \gamma_{\widehat{\jmath}}$, we therefore obtain the desired

$$\widehat{\gamma} = \gamma_{\widehat{\jmath}} = \max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})\,. \tag{346}$$

It remains to be shown that this equality is also valid in the case that $B_{\widehat{\jmath}} \geq \gamma_{\widehat{\jmath}}$. Hence suppose that indeed $B_{\widehat{\jmath}} \geq \gamma_{\widehat{\jmath}}$. Recalling that $\widehat{\jmath} \in \widehat{J}$ by (165), we then obtain from (164) that $B_{\widehat{\jmath}} \leq \gamma_{\widehat{\jmath}+1}$ in particular. Hence in fact,

$$B_{\widehat{\jmath}} \in [\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}] \, .$$

In order to establish a lower bound on $\widehat{\gamma}$, we consider a choice of $\gamma \in [\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})$. Then there exists $\gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1})$ with $\gamma' \geq \gamma$. We now proceed as follows.

$$\begin{array}{ll}
\frac{1}{2} + \frac{1}{2}\gamma & \\
< \frac{1}{2} + \frac{1}{2}B_{\widehat{\jmath}} & \text{because } \gamma < B_{\widehat{\jmath}} \\
= \frac{1}{2} + \frac{1}{2}(2\bot_{\widehat{\jmath}} - 1) & \text{by (162)} \\
= \bot_{\widehat{\jmath}} & \\
\leq C_{\widehat{\jmath}} & \text{by Th-252} \\
= Q_{\gamma'}(X_1, \ldots, X_n) & \text{by Th-248, } \gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}) \\
\leq Q_{\gamma}(X_1, \ldots, X_n) \, , &
\end{array}$$

where the last step is valid because $Q_\gamma(X_1, \ldots, X_n)$ is known to be nonincreasing and $\gamma' \geq \gamma$. Therefore $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2} + \frac{1}{2}\gamma$ and in turn, $\widehat{\gamma} > \gamma$ by (340). Because $\gamma \in [\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})$ was arbitrarily chosen, we conclude that

$$\widehat{\gamma} \geq B_{\widehat{\jmath}} \, . \tag{347}$$

Finally consider some $\gamma \in (B_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}]$. It is then possible to choose some $\gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1})$ with $\gamma' \leq \gamma$. We now observe that

$$\begin{array}{rll}
\frac{1}{2} + \frac{1}{2}\gamma & > \frac{1}{2} + \frac{1}{2}B_{\widehat{\jmath}} & \text{because } \gamma > B_{\widehat{\jmath}} \\
& = \frac{1}{2} + \frac{1}{2}(2\bot_{\widehat{\jmath}} - 1) & \text{by (162)} \\
& = \bot_{\widehat{\jmath}} \, , &
\end{array}$$

i.e.

$$\frac{1}{2} + \frac{1}{2}\gamma > \bot_{\widehat{\jmath}} \, . \tag{348}$$

In addition, $\gamma > B_{\widehat{\jmath}} \geq \gamma_{\widehat{\jmath}}$ entails that $\gamma > 0$ and hence

$$\frac{1}{2} + \frac{1}{2}\gamma > \frac{1}{2} \, . \tag{349}$$

Combining both inequalities, we finally obtain that

$$\begin{array}{rll}
\frac{1}{2} + \frac{1}{2}\gamma & > \max(\frac{1}{2}, \bot_{\widehat{\jmath}}) & \text{by (348), (349)} \\
& = C_{\widehat{\jmath}} & \text{by Th-252} \\
& = Q_{\gamma'}(X_1, \ldots, X_n) & \text{by Th-248, } \gamma' \in (\gamma_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}) \\
& \geq Q_{\gamma}(X_1, \ldots, X_n) \, . & \text{because } Q_\gamma(X_1, \ldots, X_n) \text{ nonincreasing and } \gamma \geq \gamma'
\end{array}$$

Hence $Q_\gamma(X_1, \ldots, X_n) < \frac{1}{2} + \frac{1}{2}\gamma$ and $\widehat{\gamma} < \gamma$ by (340). Noticing that $\gamma \in (B_{\widehat{\jmath}}, \gamma_{\widehat{\jmath}+1}]$ was arbitrarily chosen, this proves that $\widehat{\gamma} \leq B_{\widehat{\jmath}}$. Combining this with the converse inequality (347), we conclude that

$$\widehat{\gamma} = B_{\widehat{\jmath}} = \max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}}) \tag{350}$$

under the assumption that $B_{\widehat{\jmath}} \geq \gamma_{\widehat{\jmath}}$. Recalling the earlier result (346) in the alternative case that $B_{\widehat{\jmath}} < \gamma_{\widehat{\jmath}}$, it is now apparent that $\widehat{\gamma} = \max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})$ holds unconditionally provided that $C_0 > \frac{1}{2}$. Substituting $\max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})$ into (339) then yields the desired $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) = \frac{1}{2} + \frac{1}{2}\max(\gamma_{\widehat{\jmath}}, B_{\widehat{\jmath}})$.

b.: $\bot_0 \leq \frac{1}{2} \leq \top_0$**.** Then $C_0 = \mathrm{med}_{\frac{1}{2}}(\bot_0, \top_0) = \frac{1}{2}$ and hence $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) = \frac{1}{2}$ by Th-77 and Th-251.

c.: $\top_0 < \frac{1}{2}$**.** Due to the symmetry of $\mathcal{M}_{CX}$ with respect to negation, the proof of this case is entirely analogous to that of part **a.** of the theorem.

D.16 Proof of Theorem 255

Lemma 19
Let $\beta : E \longrightarrow E'$ be some bijection. Then for all $Z, Z' \in \mathcal{P}(E)$,

a. $\widehat{\beta}(\neg Z) = \neg\widehat{\beta}(Z)$;

b. $\widehat{\beta}(Z \cup Z') = \widehat{\beta}(Z) \cup \widehat{\beta}(Z')$;

c. $\widehat{\beta}(Z \cap Z') = \widehat{\beta}(Z) \cap \widehat{\beta}(Z')$.

Proof Let us first notice that $\neg\widehat{\beta}(Z) \subseteq \widehat{\beta}(\neg Z)$ because β is surjective (onto), and conversely $\widehat{\beta}(\neg Z) \subseteq \neg\widehat{\beta}(Z)$ because β is injective (mono). Therefore

$$\widehat{\beta}(\neg Z) = \neg\widehat{\beta}(Z)\,,$$

which proves part **a.** of the lemma.
As to **b.**, the claimed

$$\widehat{\beta}(Z \cup Z') = \widehat{\beta}(Z) \cup \widehat{\beta}(Z')$$

is a well-known properties of arbitrary powerset mappings $\widehat{\beta}$, see Def. 19, and does not actually require β to be a bijection.
By De Morgan's law, we finally obtain from parts **a.** and **b.** of the lemma that indeed

$$\widehat{\beta}(Z \cap Z') = \widehat{\beta}(Z) \cap \widehat{\beta}(Z')$$

in the case of intersections, i.e. part **c.** is valid as well.

Lemma 20
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *be a semi-fuzzy quantifier on a finite base set* $E \neq \varnothing$*. If* Q *can be computed from the cardinalities of its arguments and their Boolean combinations then* Q *is quantitative.*

Proof Hence suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is such a quantifier. Recalling (169), Q can then be expressed as

$$Q(Y_1, \dots, Y_n) = q(|\Phi_1(Y_1, \dots, Y_n)|, \dots, |\Phi_K(Y_1, \dots, Y_n)|) \tag{351}$$

for all $Y_1, \dots, Y_n \in \widetilde{\mathcal{P}}(E)$, where the $\Phi_1, \dots, \Phi_K : \mathcal{P}(E)^n \longrightarrow \mathcal{P}(E)$ are Boolean combinations of the arguments, and q is a mapping $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$. In order to prove that Q is quantitative according to Def. 38, we consider an automorphism (bijection) $\beta : E \longrightarrow E$. Let us now recall from the previous lemma L-19 that $\widehat{\beta}$ commutes with intersections, unions and negations. Because $\Phi_1(Y_1, \dots, Y_n), \dots, \Phi_K(Y_1, \dots, Y_n)$ are constructed from the arguments of the quantifiers by applying these basic operations, we conclude that

$$\Phi_\ell(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n)) = \widehat{\beta}(\Phi_\ell(Y_1, \dots, Y_n))$$

and in turn, Therefore

$$|\Phi_\ell(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n))| = |\widehat{\beta}(\Phi_\ell(Y_1, \dots, Y_n))| = |\Phi_\ell(Y_1, \dots, Y_n)| \tag{352}$$

for all $\ell \in \{1, \dots, K\}$, where the reduction $|\widehat{\beta}(Z)| = |Z|$ is possible because β is a bijection. We conclude that

$$\begin{aligned}
&Q(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n)) \\
&= q(|\Phi_1(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n))|, \dots, |\Phi_K(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n))|) && \text{by (351)} \\
&= q(|\Phi_1(Y_1, \dots, Y_n)|, \dots, |\Phi_K(Y_1, \dots, Y_n)|) && \text{by (352)} \\
&= Q(Y_1, \dots, Y_n)\,. && \text{by (351)}
\end{aligned}$$

Due to the fact that the automorphism β was arbitrarily chosen, this proves that Q is automorphism-invariant. Hence Q is indeed quantitative, as desired.

In order to prove the converse claim made by the theorem, that every quantitative quantifier on a finite base set can be expressed in terms of the cardinalities of its arguments and their Boolean combinations, we need a series of preparations.

Hence consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$. Let us recall the notation introduced in (167), i.e. the Boolean combination $\Phi_{\ell_1, \dots, \ell_n}(Y_1, \dots, Y_n)$ of $Y_1, \dots, Y_n \in \mathcal{P}(E)$ for given $\ell_1, \dots, \ell_n \in \{0, 1\}$. For a choice of arguments $Y_1, \dots, Y_n \in \mathcal{P}(E)$, it is then apparent that the $\Phi_{\ell_1, \dots, \ell_n}(Y_1, \dots, Y_n)$ form a partition of E, viz

$$E = \dot{\cup}\{\Phi_{\ell_1, \dots, \ell_n}(Y_1, \dots, Y_n) : \ell_1, \dots, \ell_n \in \{0, 1\}\}\,, \tag{353}$$

where $\dot\cup$ denotes the disjoint union. In addition, the individual arguments $Y_1, \ldots, Y_n$ can be recovered from the $\Phi_{\ell_1,\ldots,\ell_n}$'s in the obvious way, i.e.

$$Y_i = \dot\cup\{\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n) : \ell_i = 1\} \tag{354}$$

for all $i \in \{1, \ldots, n\}$. In the following, it is useful to introduce a binary relation $\sim\, \subseteq \mathcal{P}(E)^n \times \mathcal{P}(E)^n$, defined by

$$(Y_1, \ldots, Y_n) \sim (Y_1', \ldots, Y_n') \iff |\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)| = |\Phi_{\ell_1,\ldots,\ell_n}(Y_1', \ldots, Y_n')| \\ \text{for all } \ell_1, \ldots, \ell_n \in \{0, 1\}\,, \tag{355}$$

where $(Y_1, \ldots, Y_n), (Y_1', \ldots, Y_n') \in \mathcal{P}(E)^n$. It is apparent from (355) that $\sim$ is an equivalence relation on $\mathcal{P}(E)^n$.

We further notice that given $(Y_1, \ldots, Y_n), (Y_1', \ldots, Y_n') \in \mathcal{P}(E)^n$ with $(Y_1, \ldots, Y_n) \sim (Y_1', \ldots, Y_n')$, there always exists a bijection $\beta : E \longrightarrow E$ which translates these argument tuples into each others. To see this, we first consider some choice of $\ell_1, \ldots, \ell_n \in \{0, 1\}$. We can then conclude from (355) and $(Y_1, \ldots, Y_n) \sim (Y_1', \ldots, Y_n')$ that $|\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)| = |\Phi_{\ell_1,\ldots,\ell_n}(Y_1', \ldots, Y_n')|$. In particular, there exists a bijection

$$\beta_{\ell_1,\ldots,\ell_n} : \Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n) \longrightarrow \Phi_{\ell_1,\ldots,\ell_n}(Y_1', \ldots, Y_n')\,.$$

Noticing that the domain of a given $\beta_{\ell_1,\ldots,\ell_n}$ is

$$\mathrm{Dom}\,\beta_{\ell_1,\ldots,\ell_n} = \Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)\,, \tag{356}$$

we then obtain from (353) that

$$E = \dot\cup\{\mathrm{Dom}\,\beta_{\ell_1,\ldots,\ell_n} : \ell_1, \ldots, \ell_n \in \{0, 1\}\}\,.$$

In other words, the base set E is the disjoint union of the domains of the local bijections $\beta_{\ell_1,\ldots,\ell_n}$. It is therefore possible to combine the $\beta_{\ell_1,\ldots,\ell_n}$'s into a piecewise definition of a mapping $\beta : E \longrightarrow E$, by stating

$$\beta(e) = \beta_{\ell_1,\ldots,\ell_n}(e)\,, \tag{357}$$

where $\ell_1, \ldots, \ell_n \in \{0, 1\}$ is the unique choice of coefficients with $e \in \mathrm{Dom}\,\beta_{\ell_1,\ldots,\ell_n} = \Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)$. We now notice that

$$\mathrm{Im}\,\beta_{\ell_1,\ldots,\ell_n} = \Phi_{\ell_1,\ldots,\ell_n}(Y_1', \ldots, Y_n')\,.$$

Recalling from (353) that E is the disjoint union of all $\Phi_{\ell_1,\ldots,\ell_n}$, it is now obvious that E is the disjoint union of the images of all $\beta_{\ell_1,\ldots,\ell_n}$. In particular, the mapping β defined by (357) is an injection (mono), because all $\beta_{\ell_1,\ldots,\ell_n}$ are injections and their images are known not to overlap. However, the union of all $\mathrm{Im}\,\beta_{\ell_1,\ldots,\ell_n}$ also exhausts E, which shows that β is surjective. In sum, $\beta : E \longrightarrow E$ is known to be a bijection. Let us now assert that β also achieves the desired transformation between the argument tuples $(Y_1, \ldots, Y_n)$ and $(Y_1', \ldots, Y_n')$.

Lemma 21

Let $E \neq \varnothing$ be some finite base set, consider a choice of $Y_1, \dots, Y_n, Y'_1, \dots, Y'_n \in \mathcal{P}(E)$, $n \in \mathbb{N}$, such that $Y_1, \dots, Y_n \sim Y'_1, \dots, Y'_n$. Then

$$(Y'_1, \dots, Y'_n) = (\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n))$$

where $\beta : E \longrightarrow E$ is the bijection defined by (357).

Proof To see this, consider some $i \in \{1, \dots, n\}$. We first observe that for all $\ell_1, \dots, \ell_n \in \{0, 1\}$,

$$\begin{aligned} \widehat{\beta}(\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)) &= \widehat{\beta}_{\ell_1,\dots,\ell_n}(\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)) && \text{by (357)} \\ &= \widehat{\beta}_{\ell_1,\dots,\ell_n}(\operatorname{Dom} \beta_{\ell_1,\dots,\ell_n}) && \text{by (356)} \\ &= \operatorname{Im} \beta_{\ell_1,\dots,\ell_n} , \end{aligned}$$

i.e.

$$\widehat{\beta}(\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)) = \Phi_{\ell_1,\dots,\ell_n}(Y'_1, \dots, Y'_n) . \tag{358}$$

Therefore

$$\begin{aligned} & Y'_i \\ &= \dot{\cup}\{\Phi_{\ell_1,\dots,\ell_n}(Y'_1, \dots, Y'_n) : \ell_i = 1\} && \text{by (354)} \\ &= \dot{\cup}\{\widehat{\beta}(\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)) : \ell_i = 1\} && \text{by (358)} \\ &= \widehat{\beta}(\dot{\cup}\{\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)) : \ell_i = 1\} && \text{by L-19} \\ &= \widehat{\beta}(Y_i) , && \text{by (354)} \end{aligned}$$

as desired.

Based on the previous lemma, we can now assert that every quantitative quantifier is $\sim$-invariant, i.e. insensitive against the concrete choice of arguments in a given equivalence class under $\sim$.

Lemma 22

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a quantitative semi-fuzzy quantifier on a finite base set $E \neq \varnothing$. Then Q is $\sim$-invariant, i.e.

$$Q(Y_1, \dots, Y_n) = Q(Y'_1, \dots, Y'_n)$$

for all $(Y_1, \dots, Y_n), (Y'_1, \dots, Y'_n) \in \mathcal{P}(E)^n$ with $(Y_1, \dots, Y_n) \sim (Y'_1, \dots, Y'_n)$.

Proof Let $E \neq \varnothing$ be a finite base set and suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is quantitative. Further consider a choice of $Y_1, \dots, Y_n, Y'_1, \dots, Y'_n \in \mathcal{P}(E)$ with $(Y_1, \dots, Y_n) \sim (Y'_1, \dots, Y'_n)$. Now choosing the automorphism $\beta : E \longrightarrow E$ according to (357) immediately yields

$$\begin{aligned} Q(Y_1, \dots, Y_n) &= Q(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n)) && \text{because } Q \text{ quantitative, see Def. 38} \\ &= Q(Y'_1, \dots, Y'_n) , && \text{by L-21} \end{aligned}$$

which proves that Q is indeed $\sim$-invariant.

Lemma 23
Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ be given. If Q is quantitative, then Q can be computed from the cardinalities of its arguments and their Boolean combinations in the sense of (169).

Proof Consider a quantitative semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite base set. Abbreviating $K = 2^n$, we define $D \subseteq \{0, \ldots, |E|\}^K$ by

$$D = \{d = (d_1, \ldots, d_K) \in \{0, \ldots, |E|\}^K : \sum_{j=1}^{K} d_j = |E|\} .$$

In order to simplify the rendering of the proof, it is convenient to depart from the indexing of the d_j by integers j and use their representation as binary numbers. In addition, it will be convenient to reverse the bit sequence (starting with the least significant bit). The d's will hence be indexed by n binary digits and listed in the following order:

$$d_{0,\ldots,0}, d_{1,0,\ldots,0}, d_{0,1,0,\ldots,0}, d_{1,1,0,\ldots,0}, \ldots, d_{1,\ldots,1,0}, d_{1,\ldots,1} .$$

Incorporating these notational conventions, the above definition of D then becomes

$$D = \{d = (d_{0,\ldots,0}, \ldots, d_{1,\ldots,1}) \in \{0, \ldots, |E|\}^K : \sum_{\ell_1,\ldots,\ell_n \in \{0,1\}} d_{\ell_1,\ldots,\ell_n} = |E|\} . \tag{359}$$

Now consider a choice of $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Then

$$\begin{aligned} &\sum_{\ell_1,\ldots,\ell_n \in \{0,1\}} |\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n)| \\ &= |\dot{\bigcup} \{\Phi_{\ell_1,\ldots,\ell_n}(Y_1, \ldots, Y_n) : \ell_1, \ldots, \ell_n \in \{0,1\}\}| && \text{because of disjoint union} \\ &= |E| . && \text{by (353)} \end{aligned}$$

It is therefore obvious from (359) that

$$d = (|\Phi_{0,\ldots,0}(Y_1, \ldots, Y_n)|, \ldots, |\Phi_{1,\ldots,1}(Y_1, \ldots, Y_n)|) \in D . \tag{360}$$

Conversely, it is obvious that for all $d = (d_{0,\ldots,0}, \ldots, d_{1,\ldots,1}) \in \{0, \ldots, |E|\}^K$, there exists a choice of $Y_1^d, \ldots, Y_n^d \in \mathcal{P}(E)$ with

$$d_{\ell_1,\ldots,\ell_n} = |\Phi_{\ell_1,\ldots,\ell_n}(Y_1^d, \ldots, Y_n^d)| \tag{361}$$

for all $\ell_1, \ldots, \ell_n \in \{0,1\}$. To this end, we simply split E into a partition

$$E = \dot{\bigcup}\{Z_{\ell_1,\ldots,\ell_n} : \ell_1, \ldots, \ell_n \in \{0,1\}\} ,$$

where the $Z_{\ell_1,\ldots,\ell_n} \in \mathcal{P}(E)$ are subsets with

$$|Z_{\ell_1,\ldots,\ell_n}| = d_{\ell_1,\ldots,\ell_n} .$$

Letting

$$Y_j^d = \dot{\cup}\{Z_{\ell_1,\dots,\ell_n} : \ell_j = 1\}$$

then results in a choice of $Y_1^d, \dots, Y_n^d \in \mathcal{P}(E)$ which satisfies

$$\Phi_{\ell_1,\dots,\ell_n}(Y_1^d, \dots, Y_n^d) = Z_{\ell_1,\dots,\ell_n} ,$$

and consequently elicits the desired property that

$$|\Phi_{\ell_1,\dots,\ell_n}(Y_1^d, \dots, Y_n^d)| = |Z_{\ell_1,\dots,\ell_n}| = d_{\ell_1,\dots,\ell_n}$$

for all $\ell_1, \dots, \ell_n \in \{0, 1\}$.
In the following, we can hence assume an arbitrary but fixed choice of $(Y_1^d, \dots, Y_n^d) \in \mathcal{P}(E)$ for all $d \in D$, which satisfy condition (361).
Now consider a choice of $Y_1, \dots, Y_n \in \widetilde{\mathcal{P}}(E)$. From (360), we know that $d = (|\Phi_{0,\dots,0}(Y_1, \dots, Y_n)|, \dots, |\Phi_{1,\dots,1}(Y_1, \dots, Y_n)|) \in D$. In turn, we conclude from (361) that there exists a representative $(Y_1^d, \dots, Y_n^d) \in \mathcal{P}(E)^n$ with

$$|\Phi_{\ell_1,\dots,\ell_n}(Y_1^d, \dots, Y_n^d)| = d_{\ell_1,\dots,\ell_n} = |\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)|$$

for all $\ell_1, \dots, \ell_n \in \{0, 1\}$. Recalling the defining criterion (355) of $\sim$, it becomes obvious that indeed

$$(Y_1, \dots, Y_n) \sim (Y_1^d, \dots, Y_n^d) , \tag{362}$$

where $d \in D$ is obtained from (360). In other words, the $(Y_1^d, \dots, Y_n^d) \in \mathcal{P}(E)^n$, $d \in D$ form a system of representatives for the equivalence classes of $\sim$.

Based on the given choice of representatives $(Y_1^d, \dots, Y_n^d) \in \mathcal{P}(E)^n$, $d \in D$, we now define a mapping $q : \{0, \dots, |E|\}^K \longrightarrow \mathbf{I}$ by

$$q(d) = \begin{cases} Q(Y_1^d, \dots, Y_n^d) & : \quad d \in D \\ 0 & : \quad \text{else} \end{cases} \tag{363}$$

for all $d = (d_{0,\dots,0}, \dots, d_{1,\dots,1}) \in \{0, \dots, |E|\}^K$, again assuming the above conventions on the indexing of d.

Let us now complete the proof and show that the original quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ can be expressed in terms of the mapping q, which is then applied to the cardinalities of the Boolean combinations $\Phi_{\ell_1,\dots,\ell_n}$. Hence consider a choice of arguments $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Let us further suppose that $d \in D$ is derived from the given arguments according to (360). Then

$$\begin{aligned} & Q(Y_1, \dots, Y_n) \\ & = Q(Y_1^d, \dots, Y_n^d) && \text{by (362) and L-22} \\ & = q(d) && \text{by (363) and (361)} \\ & = q(|\Phi_{0,\dots,0}|(Y_1, \dots, Y_n), \dots, |\Phi_{1,\dots,1}(Y_1, \dots, Y_n)|) . && \text{by (360)} \end{aligned}$$

Noticing that the arguments $Y_1, \dots, Y_n \in \mathcal{P}(E)$ were chosen arbitrarily, this proves that the quantifier Q can indeed be expressed as a function q of the cardinalities of Boolean combinations $\Phi_{\ell_1,\dots,\ell_n}(Y_1, \dots, Y_n)$ of its arguments, as desired.

The proof of the superordinate theorem now becomes a simple corollary of the above lemmata L-20 and L-23:

Proof of Theorem 255

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier on a finite base set $E \neq \varnothing$.

a. If Q can be expressed in terms of the cardinalities of Boolean combinations of its arguments, then it is known to be quantitative from lemma L-20.

b. Conversely if Q is quantitative, then we know from lemma L-23 that Q can be expressed in terms of the cardinalities of Boolean combinations of its arguments.

This completes the proof that the quantitative semi-fuzzy quantifiers on finite base sets are exactly those quantifiers which solely depend on the cardinality of Boolean combinations of their arguments.

D.17 Proof of Theorem 256

This is a simple corollary to Th-255 for $n = 2$. However, the direct proof is also very simple. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given two-place quantifier on a finite base set, and further suppose that Q is quantitative. We then know from Th-255 that $Q(Y_1, Y_2)$ can be expressed in terms of the cardinalities of Boolean combinations of Y_1 and Y_2. Now suppose that $\Phi(Y_1, Y_2)$ is such a Boolean combination of Y_1 and Y_2. Then $\Phi(Y_1, Y_2)$ can be reformulated into disjunctive normal form, and hence expressed as a disjoint union of the min-terms $\Phi_1(Y_1, Y_2) = Y_1 \cap Y_2$, $\Phi_2(Y_1, Y_2) = Y_1 \cap \neg Y_2 = Y_1 \backslash Y_2$, $\Phi_3(Y_1, Y_2) = \neg Y_1 \cap Y_2 = Y_2 \backslash Y_1$ and $\Phi_4(Y_1, Y_2) = \neg Y_1 \cap \neg Y_2 = E \backslash (Y_1 \cup Y_2)$, i.e.

$$\Phi(Y_1, Y_2) = \dot{\cup}\{\Phi_\ell : \ell \in L\}$$

for some choice of $L \subseteq \{1, 2, 3, 4\}$. Due to disjoint union, the corresponding cardinalities of the involved subsets sum up to the total cardinality. Therefore

$$|\Phi(Y_1, Y_2)| = \sum_{\ell \in L} |\Phi_\ell(Y_1, Y_2)| \,, \tag{364}$$

where $a = |\Phi_1(Y_1, Y_2)|$, $b = |\Phi_2(Y_1, Y_2)|$, $c = |\Phi_3(Y_1, Y_2)|$, $d = |\Phi_4(Y_1, Y_2)|$. This proves that $Q(Y_1, Y_2)$ can be computed from a, b, c and d, because $Q(Y_1, Y_2)$ can be expressed in terms of the cardinalities of certain Boolean combinations of Y_1 and Y_2, and because these cardinalities can be computed from a, b, c and d according to (364).

D.18 Proof of Theorem 257

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a conservative and quantitative two-place quantifier on a finite base set. We already know from Th-256 that there exists $q : \{0, \ldots, |E|\}^4 \longrightarrow \mathbf{I}$ such that

$$Q(Y_1, Y_2) = q(a, b, c, d) \tag{365}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where $a = |Y_1 \setminus Y_2|$, $b = |Y_2 \setminus Y_1|$, $c = |Y_1 \cap Y_2|$ and $d = |E \setminus (Y_1 \cup Y_2)|$.
Now consider such a choice of arguments $Y_1, Y_2 \in \mathcal{P}(E)$. We abbreviate $Y_2' = Y_1 \cap Y_2$. In terms of the modified second argument, the cardinality coefficients now become

$$a' = |Y_1 \setminus Y_2'| = |Y_1 \setminus (Y_1 \cap Y_2)| = |Y_1 \setminus Y_2| = a \tag{366}$$

$$b' = |Y_2' \setminus Y_1| = |(Y_1 \cap Y_2) \setminus Y_1| = |\varnothing| = 0 \tag{367}$$

$$c' = |Y_1 \cap (Y_1 \cap Y_2)| = |Y_1 \cap Y_2| = c \tag{368}$$

and

$$\begin{aligned} d' &= |E \setminus (Y_1 \cup Y_2')| = |E \setminus (Y_1 \cup (Y_1 \cap Y_2))| = |E \setminus Y_1| \\ &= |E| - |Y_1| = |E| - |(Y_1 \setminus Y_2) \,\dot\cup\, (Y_1 \cap Y_2)| \\ &= |E| - |Y_1 \setminus Y_2| - |Y_1 \cap Y_2| = |E| - a - c\,. \end{aligned} \tag{369}$$

Let us hence define $q' : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$ by

$$q'(x, y) = \begin{cases} q(x, 0, y, |E| - x - y) & : \quad x + y \le |E| \\ 0 & : \quad x + y > |E| \end{cases} \tag{370}$$

for all $x, y \in \{0, \dots, |E|\}$. Then

$$\begin{aligned} Q(Y_1, Y_2) &= Q(Y_1, Y_2') && \text{by Def. 70} \\ &= q(a', b', c', d') && \text{by (365)} \\ &= q(a, 0, b, |E| - a - c) && \text{by (366)–(369)} \\ &= q'(a, c)\,. && \text{by (370)} \end{aligned}$$

Hence $Q(Y_1, Y_2)$ can be expressed in terms of $a = |Y_1 \setminus Y_2|$ and $c = |Y_1 \cap Y_2|$, as desired.

D.19 Proof of Theorem 258

Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ on a finite base set, and suppose that Q is both quantitative and conservative. By Th-257, there exists a mapping $q : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$ such that

$$Q(Y_1, Y_2) = q(a, c) \tag{371}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where $a = |Y_1 \setminus Y_2|$ and $c = |Y_1 \cap Y_2|$. We define $q' : \{0, \dots, |E|\}^2 \longrightarrow \mathbf{I}$ by

$$q'(x, y) = \begin{cases} q(x - y, y) & : \quad x \ge y \\ 0 & : \quad x < y \end{cases} \tag{372}$$

for all $x, y \in \{0, \dots, |E|\}$. Now let $Y_1, Y_2 \in \mathcal{P}(E)$ be given. We abbreviate $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$. Then apparently

$$a = c_1 - c_2 \tag{373}$$

$$c = c_2\,. \tag{374}$$

Therefore

$$\begin{aligned} Q(Y_1, Y_2) &= q(a, c) && \text{by (371)} \\ &= q(c_1 - c_2, c_2) && \text{by (373), (374)} \\ &= q'(c_1, c_2)\,, && \text{by (372)} \end{aligned}$$

i.e. $Q(Y_1, Y_2) = q'(c_1, c_2)$ for $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$, as desired.

D.20 Proof of Theorem 259

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Let us denote by $R = R_\gamma^{\Phi_{0,\ldots,0}, \ldots, \Phi_{1,\ldots,1}}(X_1, \ldots, X_n)$ the relation $R \subseteq \{0, \ldots, |E|\}^{(2^n)}$ defined by (172). We shall further denote by $R' \subseteq \{0, \ldots, |E|\}^{(2^n)}$ the relation defined by

$$R' = \{c : \{0,1\}^n \longrightarrow \{0, \ldots, |E|\} \mid \text{there exists } (\lambda_p)_{p \in \{0,*,1\}^n} \in \Lambda \text{ such that } c_d = \sum_{p \in \{0,*,1\}^n} \lambda_p(d)\}\,. \tag{375}$$

The claimed equality (186) can therefore be proven by showing that $R = R'$. I accomplish this by by considering both inclusions $R \subseteq R'$ and $R' \subseteq R$.

a.: Proof of the inclusion $R \subseteq R'$.
Hence suppose that $c \in R = R_\gamma^{\Phi_{0,\ldots,0}, \ldots, \Phi_{1,\ldots,1}}(X_1, \ldots, X_n)$ for some $c : \{0,1\}^n \longrightarrow \{0, \ldots, |E|\}$. By (172), then, there exist $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$ such that

$$c_d = |\Phi_d(Y_1, \ldots, Y_n)| = |Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| \tag{376}$$

for all $d = (d_1, \ldots, d_n) \in \{0,1\}^n$. Now consider the $\{0,*,1\}^n$-indexed family $(\lambda_p)_{p \in \{0,*,1\}^n}$ of mappings $\lambda_p : \{0,1\}^n \longrightarrow \{0, \ldots, |E|\}$ defined by

$$\lambda_p(d) = |S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| \tag{377}$$

for all $p = (p_1, \ldots, p_n) \in \{0,*,1\}^n$ and $d \in \{0,1\}^n$. In the following, I will prove that $(\lambda_p)_{p \in \{0,*,1\}^n} \in \Lambda$. According to (185), it must hence be shown that for all $p \in \{0,*,1\}^n$,

i. $\lambda_p(d) = 0$ for all $d \in \{0,1\}^n \setminus D(p)$; and

ii. $c(p) = \sum_{d \in \{0,1\}^n} \lambda_p(d)$.

ad i.: Suppose that $d \notin D(p)$. We then know from (182) that there exists $j \in \{1, \ \ldots, n\}$ such that

$$(p_j, d_j) \in \{(1,0), (0,1)\}\,.$$

In the first case that $p_j = 1$ and $d_j = 0$, we recall that $Y_j \in \mathcal{T}_\gamma(X_j) = \{Y : (X_j)_\gamma^{\min} \subseteq Y \subseteq (X_j)_\gamma^{\max}\}$. In particular, $(X_j)_\gamma^{\min} \subseteq Y_j$ and hence

$$(X_j)_\gamma^{\min} \cap \neg Y_j = \varnothing\,. \tag{378}$$

Consequently

$$\begin{aligned} X_{i_j}^{[p_j]} \cap Y_{i_j}^{(d_j)} &= X_j{}^{[1]} \cap Y_j{}^{(0)} && \text{because } p_j = 1,\ d_j = 0 \\ &= (X_j)_\gamma^{\min} \cap \neg Y_j && \text{by (181), (168)} \\ &= \varnothing\,. && \text{by (378)} \end{aligned}$$

In the remaining case that $p_j = 0$ and $d_j = 1$, we observe that $Y_j \in \mathcal{T}_\gamma(X_j)$ entails $Y_j \subseteq (X_j)_\gamma^{\max}$; in particular

$$\neg(X_j)_\gamma^{\max} \cap Y_j = \varnothing\,. \tag{379}$$

Therefore

$$\begin{aligned} X_{i_j}^{[p_j]} \cap Y_{i_j}^{(d_j)} &= X_j{}^{[0]} \cap Y_j{}^{(1)} && \text{because } p_j = 0,\ d_j = 1 \\ &= \neg(X_j)_\gamma^{\max} \cap Y_j && \text{by (181), (168)} \\ &= \varnothing\,. && \text{by (379)} \end{aligned}$$

This substantiates that in both cases,

$$X_{i_j}^{[p_j]} \cap Y_{i_j}^{(d_j)} = \varnothing\,. \tag{380}$$

Consequently

$$\begin{aligned} \lambda_p(d) &= |S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{by (377)} \\ &= |X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]} \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{by (183)} \\ &\leq |X_{i_j}^{[p_j]} \cap Y_{i_j}^{(d_j)}| \\ &= |\varnothing| && \text{by (380)} \\ &= 0\,, \end{aligned}$$

i.e. $\lambda_p(d) = 0$, as desired.

ad ii.: In order to prove that the second precondition is valid, we first notice that the min-terms $Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$, $d \in \{0,1\}^n$, are mutually disjoint, and cover the total domain E. Hence

$$E = \dot{\cup}\{Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)} : d = (d_1, \ldots, d_n) \in \{0,1\}^n\}\,. \tag{381}$$

Therefore

$$S(p) = \dot{\cup}\{S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)} : d = (d_1, \ldots, d_n) \in \{0,1\}^n\}\,. \tag{382}$$

The disjoint union permits us to compute the cardinality of $S(p)$ by summing up the cardinalities of all sets which participate in the union. Consequently

$$\begin{aligned} c(p) &= |S(p)| && \text{by (184)} \\ &= \sum_{d \in \{0,1\}^n} |S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{by (382)} \\ &= \sum_{d \in \{0,1\}^n} \lambda_p(d)\,. && \text{by (377)} \end{aligned}$$

This demonstrates that both conditions i. and ii. are satisfied, i.e. $(\lambda_p)_{p\in\{0,*,1\}^n} \in \Lambda$. It remains to be shown that $c_d = \sum_{p\in\{0,*,1\}^n} \lambda_p(d)$. To this end, let us observe from Def. 82 that for all $Z \in \mathcal{P}(E)$, $Z_\gamma^{\min} \subseteq Z_\gamma^{\max}$. Therefore

$$E = (\neg Z_\gamma^{\max}) \mathbin{\dot\cup} (Z_\gamma^{\max} \setminus Z_\gamma^{\min}) \mathbin{\dot\cup} Z_\gamma^{\min} \,. \tag{383}$$

Recalling from (181) that $Z^{[0]} = \neg Z_\gamma^{\max}$, $Z^{[*]} = Z_\gamma^{\max} \setminus Z_\gamma^{\min}$ and $Z^{[1]} = Z_\gamma^{\min}$, this substantiates that

$$E = Z^{[0]} \mathbin{\dot\cup} Z^{[*]} \mathbin{\dot\cup} Z^{[1]} \,.$$

Consequently, the full domain E can be partitioned as follows,

$$E = \dot\cup\{X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_n}^{[p_n]} : p \in \{0,*,1\}^n\} = \dot\cup\{S(p) : p \in \{0,*,1\}^n\} \,. \tag{384}$$

Based on this decomposition of E, it is now that for all $d \in \{0,1\}^n$,

$$\begin{aligned} c_d &= |Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{by def. of cardinality coefficients} \\ &= |\dot\bigcup_{p\in\{0,*,1\}^n} S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{by (384)} \\ &= \sum_{p\in\{0,*,1\}^n} |S(p) \cap Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| && \text{due to disjoint union} \\ &= \sum_{p\in\{0,*,1\}^n} \lambda_p(d) \,. && \text{by (377)} \end{aligned}$$

This completes the proof that indeed $c \in R'$. Noticing that $c \in R$ was arbitrarily chosen, we conclude that $R \subseteq R'$.

b.: Proof of the inclusion $R' \subseteq R$.
Hence consider a choice of $c \in R'$. We then know from (375) that there exists a choice of $(\lambda_p)_{p\in\{0,*,1\}^n} \in \Lambda$ such that

$$c_d = \sum_{p\in\{0,*,1\}^n} \lambda_p(d) \tag{385}$$

for all $d \in \{0,1\}^n$. Now consider some $p \in \{0,*,1\}^n$. We then know from (184) and (185) that

$$|S(p)| = c(p) = \sum_{d\in\{0,1\}^n} \lambda_p(d) \,.$$

In particular, there exists a choice of subsets $S_{p,d} \subseteq S(p)$, $d \in \{0,1\}^n$, such that

$$|S_{p,d}| = \lambda_p(d) \tag{386}$$

and

$$S(p) = \dot\cup\{S_{p,d} : d \in \{0,1\}^n\} \tag{387}$$

for all $d \in \{0,1\}^n$. We hence obtain that

$$\begin{aligned} E &= \dot\cup\{S(p) : p \in \{0,*,1\}^n\} && \text{by (384)} \\ &= \dot\cup\{\dot\cup\{S_{p,d} : d \in \{0,1\}^n\} : p \in \{0,*,1\}\}\,, && \text{by (387)} \end{aligned}$$

i.e.

$$E = \dot\cup\{S_{p,d} : p \in \{0,*,1\}^n, d \in \{0,1\}^n\}\,. \tag{388}$$

Let us now define $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ by

$$Y_j = \dot\cup\{S_{p,d'} : d'_j = 1\} \tag{389}$$

for all $j = 1, \ldots, n$. It is then apparent from (388) and (389) that the complement $\neg Y_j$ becomes

$$\neg Y_j = \dot\cup\{S_{p,d'} : d'_j = 0\}\,. \tag{390}$$

Recalling (168), we can also express (389) and (390) as ${Y_j}^{(1)} = \dot\cup\{S_{p,d'} : d'_j = 1\}$ and ${Y_j}^{(0)} = \dot\cup\{S_{p,d'} : d'_j = 0\}$, i.e.

$$Y_{i_j}^{(d_j)} = \dot\cup\{S_{p,d'} : d'_j = d_j\} \tag{391}$$

where $d_j \in \{0,1\}$. Therefore

$$\begin{aligned} & Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)} \\ &= \dot\cup\{S_{p,d'} : d'_1 = d_1, \ldots, d'_n = d_n\} && \text{by (391)} \\ &= \dot\cup\{S_{p,d} : p \in \{0,*,1\}^n\}\,. \end{aligned}$$

Due to the fact that $Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$ is decomposed into a disjoint union, we can compute its cardinality as the sum of component cardinalities. Hence for all $d \in \{0,1\}^n$,

$$\begin{aligned} & |Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}| \\ &= \sum_{p \in \{0,*,1\}^n} |S_{p,d}| && \text{due to disjoint sum} \\ &= \sum_{p \in \{0,*,1\}^n} \lambda_p(d) && \text{by (386)} \\ &= c_d\,. && \text{by (385)} \end{aligned}$$

We conclude from (172) that $c \in R = R_\gamma^{\Phi_{0,\ldots,0},\ldots,\Phi_{1,\ldots,1}}(X_1, \ldots, X_n)$. Noticing that $c \in R'$ was arbitrarily chosen, this shows that $R' \subseteq R$. When combined with the previous result that $R \subseteq R'$, this completes the proof that $R = R'$, i.e. the claimed equality (186) is indeed valid.

D.21 Proof of Theorem 260

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Further suppose that $\Phi'_1(Y_1, \ldots, Y_n), \ldots, \Phi'_K(Y_1, \ldots, Y_n)$ are Boolean combinations of the crisp variables $Y_1, \ldots, Y_n$, and let $R' = R_\gamma^{\Phi'_1, \ldots, \Phi'_K}(X_1, \ldots, X_n) \subseteq \{0, \ldots, |E|\}^K$ be the associated relation as defined by (172). I will further denote by $\Phi_d(Y_1, \ldots, Y_n)$, $d \in \{0,1\}^n$, the min-term $Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$ built from the variables $Y_1, \ldots, Y_n$. In order to prove the claimed equality (189), we first recall that each Boolean combination $\Phi'_j(Y_1, \ldots, Y_n)$, $j \in \{1, \ldots, K\}$, can be expressed in disjunctive normal form (DNF), i.e. as a union of min-terms $Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)}$. In fact, it is quite obvious that

$$\begin{aligned}\Phi'_j(Y_1, \ldots, Y_n) &= \dot\cup\{Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)} : d = (d_1, \ldots, d_n) \in \{0,1\}^n, \\ &\qquad \Phi'_j(Y_1, \ldots, Y_n) \cap \Phi_d(Y_1, \ldots, Y_n) \neq \varnothing\} \\ &= \dot\cup\{Y_{i_1}^{(d_1)} \cap \cdots \cap Y_{i_n}^{(d_n)} : d \in D_j\}\,, && \text{by (190)}\end{aligned}$$

and consequently

$$\Phi'_j(Y_1, \ldots, Y_n) = \dot\cup\{\Phi_d(Y_1, \ldots, Y_n) : d \in D_j\} \tag{392}$$

for all $j \in \{1, \ldots, K\}$. The disjoint union in (392) permits us to compute the cardinality of $\Phi'_j(Y_1, \ldots, Y_n)$ by adding up the cardinalities of all sets that participate in the disjoint union. Therefore

$$|\Phi'_j(Y_1, \ldots, Y_n)| = \sum_{d \in D_j} |\Phi_d(Y_1, \ldots, Y_n)|\,.$$

Now referring to the abbreviations $c_{d_1,\ldots,d_n} = |\Phi_{d_1,\ldots,d_n}(Y_1, \ldots, Y_n)|$ for all $d = (d_1, \ldots, d_n) \in \{0,1\}^n$, and $c'_j = |\Phi'_j(Y_1, \ldots, Y_n)|$ for all $j \in \{1, \ldots, K\}$, the above result becomes

$$c'_j = \sum_{d \in D_j} c_d\,, \tag{393}$$

for all $j \in \{1, \ldots, K\}$. Consequently

$$\begin{aligned}R' &= \{(c'_1, \ldots, c'_K) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n), \\ &\qquad c'_j = |\Phi'_j(Y_1, \ldots, Y_n)| \text{ for all } j \in \{1, \ldots, K\}\} && \text{by (172), (188)} \\ &= \{(\sum_{d \in D_1} c_d, \ldots, \sum_{d \in D_K} c_d) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n), \\ &\qquad c_d = |\Phi_d(Y_1, \ldots, Y_n)| \text{ for all } d \in \{0,1\}^n\} && \text{by (393)} \\ &= \{(\sum_{d \in D_1} c_d, \ldots, \sum_{d \in D_K} c_d) : c = (c_{0,\ldots,0}, \ldots, c_{1,\ldots,1}) \in R\}\,. && \text{by (172), (187)}\end{aligned}$$

In particular

$$R' = \{(c'_1, \ldots, c'_K) : (c_{0,\ldots,0}, \ldots, c_{1,\ldots,1}) \in R, c'_j = \sum_{d \in D_j} c_d,\ j = 1, \ldots, K\}\,,$$

i.e. (189) is indeed valid.

D.22 Proof of Theorem 261

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Further suppose that $P_* \in \mathbb{P}^*$ is given. The proof is by induction on the coefficient $i_* = i_*(P)$ defined by (196).

a.: $i_* = 0$**.**
In this case, we know from (197) that

$$\sigma_{P_*}(P_*) = 1 \tag{394}$$

and

$$\sigma_{P_*}(P) = 0 \tag{395}$$

for $P \in \mathbb{P}$, $P \neq P_*$. In addition, we conclude from (196) $i_* = 0$ that $\max\{i \in \operatorname{Dom} P_* : P_*(i) = *\} = 0$. Knowing that $\operatorname{Dom} P_* \subseteq \{1, \ldots, n\}$, this entails that $\{i \in \operatorname{Dom} P_* : P_*(i) = *\} = \varnothing$, i.e. $P_*(i) \neq *$ for all $i \in \operatorname{Dom} P_*$. Hence indeed

$$P \in \mathbb{P}\,, \tag{396}$$

see (194). Consequently

$$\begin{aligned}
c(P_*) &= 1 \cdot c(P_*) + \sum_{P \in \mathbb{P}\setminus\{P_*\}} 0 \cdot c(P) \\
&= \sigma_{P_*}(P_*) + \sum_{P \in \mathbb{P}\setminus\{P_*\}} \sigma_{P_*}(P) \cdot c(P) && \text{by (394), (395)} \\
&= \sum_{P \in \mathbb{P}} \sigma_{P_*}(P) \cdot c(P)\,. && \text{by (396)}
\end{aligned}$$

b.: $i_* > 0$**.**
In this case, we shall assume by induction on i_* that (195) has been shown for all $i'_* < i_*$. Due to the fact that $i_* = i_*(P_*) > 0$, we know from (198) that $\sigma_{P_*}(P) = \sigma_{P_*}(P') - \sigma_{P_*}(P'')$, where $P', P'' \in \mathbb{P}^*$ are defined by (199) and (200), respectively. Let us abbreviate $i'_* = i_*(P')$ and $i''_* = i_*(P'')$. In order for the induction to be well-founded, it must first be shown that $i'_* < i_*$ and $i''_* < i_*$. Hence let

$$D^- = \{i \in \operatorname{Dom} P_* : i < i_*\} \tag{397}$$

$$D^+ = \{i \in \operatorname{Dom} P_* : i > i_*\}\,. \tag{398}$$

It is apparent from (199) and (200) that

$$\operatorname{Dom} P_* = \operatorname{Dom} P' = \operatorname{Dom} P'' \tag{399}$$

and

$$P_*(i) = P'(i) = P''(i) \tag{400}$$

whenever $i \in D^+ \cup D^- = \operatorname{Dom} P_* \setminus \{i_*\}$. Now consider $i_* = \max\{i \in \operatorname{Dom} P_* : P_*(i) = *\}$. We first observe from (398) that

$$P_*(i) \neq * \tag{401}$$

for all $i \in D^+$. Combining this with (400), we know that $P'(i) \neq *$ and $P''(i) \neq *$ for all $i \in D^+$ as well. In addition, we know from (199) and (200) that $P'(i_*) = + \neq *$ and $P''(i_*) = 1 \neq *$. Noticing that $D^+ \cup \{i_*\} = \operatorname{Dom} P_* \setminus D^-$ and recalling (399), this proves that

$$\{i \in \operatorname{Dom} P' : P'(i) = *\} = \{i \in D^- : P'(i) = *\} \subseteq D^-$$

and

$$\{i \in \operatorname{Dom} P'' : P''(i) = *\} = \{i \in D^- : P''(i) = *\} \subseteq D^- .$$

Therefore $i'_* = \max\{i \in \operatorname{Dom} P' : P'(i) = *\} \leq \max D^- < i_*$ and $i''_* = \max\{i \in \operatorname{Dom} P'' : P''(i) = *\} \leq \max D^- < i_*$, which is apparent from (196) and (397). This substantiates that indeed $i'_* < i_*$ and $i''_* < i_*$, i.e. the induction is well-founded. We can then conclude from the induction hypothesis, i.e. (195) being valid for all $i < i_*$, that in fact

$$c(P') = \sum_{P \in \mathbb{P}} \sigma_{P'}(P) \cdot c(P) \tag{402}$$

$$c(P'') = \sum_{P \in \mathbb{P}} \sigma_{P''}(P) \cdot c(P) . \tag{403}$$

Referring to the graph representation $P = \{(i_1, p_1), \ldots, (i_m, p_m)\}$ where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $p_j = P(i_j)$ for all $j \in \{1, \ldots, m\}$, let j_* denote the unique choice of $j_* \in \{1, \dots, m\}$ with $i_* = i_{j_*}$. Then

$$\begin{aligned}
S(P') = {} & X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap X_{i_{j_*}}{}^{[+]} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]} && \text{by (192), (199)} \\
= {} & X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap X_{i_{j_*}}{}^{\max}_{\gamma} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]} && \text{by (181)} \\
= {} & (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})_{\gamma}^{\max} \cap (X_{i_{j_*}})_{\gamma}^{\min} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) \\
& \dot{\cup}\, (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})_{\gamma}^{\max} \cap \neg (X_{i_{j_*}})_{\gamma}^{\min} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) \\
= {} & (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})_{\gamma}^{\min} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) \\
& \dot{\cup}\, (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})_{\gamma}^{\max} \cap \neg (X_{i_{j_*}})_{\gamma}^{\min} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) && \text{because } (X_{i_{j_*}})_{\gamma}^{\min} \subseteq (X_{i_{j_*}})_{\gamma}^{\max} \\
= {} & (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})^{[1]} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) \\
& \dot{\cup}\, (X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_{j_*-1}}^{[p_{j_*-1}]} \\
& \cap (X_{i_{j_*}})^{[*]} \\
& \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \cdots \cap X_{i_m}^{[p_m]}) && \text{by (181)} \\
= {} & S(P'') \,\dot{\cup}\, S(P)\,. && \text{by (192), (199)}
\end{aligned}$$

Therefore $c(P') = |S(P')| = |S(P'')| + |S(P)| = c(P'') + c(P)$, i.e.

$$c(P) = c(P') - c(P'')\,. \tag{404}$$

In particular

$$\begin{aligned}
c(P) &= c(P') - c(P'') && \text{by (404)}\\
&= \sum_{P\in\mathbb{P}} \sigma_{P'}(P)\cdot c(P) - \sum_{P\in\mathbb{P}} \sigma_{P''}(P)\cdot c(P) && \text{by (402), (403)}\\
&= \sum_{P\in\mathbb{P}} (\sigma_{P'}(P) - \sigma_{P''}(P))\cdot c(P)\\
&= \sum_{P\in\mathbb{P}} \sigma_{P_*}(P)\cdot c(P)\,. && \text{by (198)}
\end{aligned}$$

This proves that (195) is valid for P_*, as desired.

D.23 Proof of Theorem 262

Lemma 24
Let $U, W \in \mathcal{P}(E)$, $X \in \widetilde{\mathcal{P}}(E)$, $\gamma \in \mathbf{I}$ and $p_* \in \{0, 1, +, -\}$. Further suppose that p' is defined in terms of p_* according to (211). *Then*

$$|U \cap X^{[p_*]} \cap V| = |U \cap V| - |U \cap X^{[p']} \cap V|\,.$$

Proof Let us first observe from (181) that

$$X^{[1]} = X_\gamma^{\min} = \neg(\neg X_\gamma^{\min}) = \neg X^{[-]} \tag{405}$$

and

$$X^{[0]} = \neg X_\gamma^{\max} = \neg X^{[+]}\,. \tag{406}$$

We further know from (211) that $(p_*, p') \in \{(0, +), (1, -), (+, 0), (-, 1)\}$. This permits us to deduce from (405) and (406) that

$$X^{[p']} = \neg X^{[p_*]}\,, \tag{407}$$

for all $p_* \in \{0, 1, +, -\}$ and the corresponding choice of p'. Abbreviating

$$Y = X^{[p_*]}\,, \tag{408}$$

we now proceed as follows. Firstly

$$U \cap V = (U \cap Y \cap V)\,\dot\cup\,(U \cap \neg Y \cap V)\,.$$

Due to the disjoint union, the corresponding cardinalities simply add up, i.e.

$$|U \cap V| = |U \cap Y \cap V| + |U \cap \neg Y \cap V|\,,$$

which can be rewritten as

$$|U \cap Y \cap V| = |U \cap V| - |U \cap \neg Y \cap V|\,.$$

Now expanding Y according to (408), we conclude that

$$|U \cap X^{[p_*]} \cap V| = |U \cap V| - |U \cap \neg X^{[p_*]} \cap V| \,.$$

We then obtain from (407) that indeed

$$|U \cap X^{[p_*]} \cap V| = |U \cap V| - |U \cap X^{[p']} \cap V| \,,$$

as desired.

The lemma will be useful in the following proof of the main theorem.

Proof of Theorem 262

Let $E \neq \varnothing$ be a finite base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. I will prove (204) by induction on the coefficient $i_* = i_*(P)$, $P \in \mathbb{P}$, as defined by (205).

a.: $i_* = 0$.
Let $P = \{(i_1, p_1), \ldots, (i_m, p_m)\}$ be the graph representation of P, where $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $p_j = P(i_j)$ for all $j \in \{1, \ldots, m\}$. Due to the fact that $\operatorname{Dom} P = \{i_1, \ldots, i_m\} \subseteq \{1, \ \ldots, n\}$, we can conclude from $i_* = \max\{i \in \operatorname{Dom} P : \operatorname{type}(P(i)) \neq \operatorname{type}(p_m)\} = 0$ by (205), that in fact

$$\{i \in \operatorname{Dom} P : \operatorname{type}(P(i)) \neq \operatorname{type}(p_m)\} = \varnothing \,.$$

In particular

$$\operatorname{type}(p_j) = y \tag{409}$$

for all $j \in \{1, \ldots, m\}$, where $y = \operatorname{type}(p_m)$. Therefore

$$\begin{aligned}
& c(P) \\
&= |X_{i_1}^{[p_1]} \cap \cdots \cap X_{i_m}^{[p_m]}| && \text{by (193)} \\
&= |(X_{i_1}^{\langle \operatorname{pol}(p_1) \rangle})_\gamma^{\operatorname{type}(p_1)} \cap \cdots \cap (X_{i_1}^{\langle \operatorname{pol}(p_m) \rangle})_\gamma^{\operatorname{type}(p_m)}| && \text{by (203)} \\
&= |(X_{i_1}^{\langle \operatorname{pol}(p_1) \rangle})_\gamma^{y} \cap \cdots \cap (X_{i_1}^{\langle \operatorname{pol}(p_m) \rangle})_\gamma^{y}| && \text{by (409)} \\
&= |(X_{i_1}^{\langle \operatorname{pol}(p_1) \rangle} \cap \cdots \cap X_{i_m}^{\langle \operatorname{pol}(p_m) \rangle})_\gamma^{y}| \,. && \text{by Th-61}
\end{aligned}$$

Abbreviating

$$V = \{(i_1, \operatorname{pol}(p_1)), \ldots, (i_m, \operatorname{pol}(p_m))\} \in \mathbb{V} \,, \tag{410}$$

this proves that

$$\begin{aligned}
c(P) &= |(X_{i_1}^{\langle v_1\rangle} \cap \cdots \cap X_{i_m}^{\langle v_m\rangle})_\gamma^y| && \text{by (176), (410)} \\
&= |(Z_V)_\gamma^y| && \text{by (178)} \\
&= \begin{cases} |(Z_V)_\gamma^{\min}| & : \quad y = \min \\ |(Z_V)_\gamma^{\max}| & : \quad y = \max \end{cases} \\
&= \begin{cases} \ell_V & : \quad y = \min \\ u_V & : \quad y = \max \end{cases} && \text{by (179) and (180)} \\
&= \sum_{V \in \mathbb{V}} \zeta_P(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_P(V, \max) \cdot u_V \,. && \text{by (206)}
\end{aligned}$$

This substantiates that (204) is valid in the case that $i_* = i_*(P) = 0$.

b.: $i_* > 0$**.**
In this case, we shall assume by induction on i_* that (204) has been shown for all $i'_* < i_*$. Due to the fact that $i_* = i_*(P) > 0$, we know from (207) that $\zeta_P(V, y) = \zeta_{P'}(V, y) - \zeta_{P''}(V, y)$ for all $V \in \mathbb{V}$ and $y \in \{\min, \max\}$, where $P', P'' \in \mathbb{P}$ are defined by (208) and (209), respectively. In the following, let $i'_* = i_*(P')$ and $i''_* = i_*(P'')$. In order for the induction to be well-founded, I will first show that $i'_* < i_*$ and $i''_* < i_*$. To this end, it is convenient to refer to the representation of P by in terms of its graph, i.e.

$$P = \{(i_1, p_1), \ldots, (i_m, p_m)\}\,, \tag{411}$$

where

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n\,, \tag{412}$$

and $p_j = P(i_j)$ for all $j \in \{1, \ldots, m\}$. Let us denote by j_* the unique choice of $j_* \in \{1, \ldots, m\}$ with

$$i_* = i_{j_*}\,. \tag{413}$$

According to (208) and (209), the graph representations of P' and P'' then become

$$P' = \{(i_1, p_1), \ldots, (i_{j_*-1}, p_{j_*-1}), (i_{j_*+1}, p_{j_*+1}), \ldots, (i_m, p_m)\} \tag{414}$$

$$P'' = \{(i_1, p_1), \ldots, (i_{j_*-1}, p_{j_*-1}), (i_*, p'), (i_{j_*+1}, p_{j_*+1}), \ldots, (i_m, p_m)\}\,, \tag{415}$$

where p' is given by (211). Now let us reconsider $i_* = i_*(P) > 0$. In terms of the graph representation, (205) becomes $i_* = i_*(P) = \max\{i_j : \mathrm{type}(p_j) \neq \mathrm{type}(p_m), j \in \{1, \ldots, m\}\}$. We can therefore conclude from (411) and (412) that

$$\mathrm{type}(p_j) = \mathrm{type}(p_m) \tag{416}$$

for all $j \in \{j_* + 1, \dots, m\}$. Therefore

$$\begin{aligned}
& i'_* \\
& = i_*(P') \\
& = \max\{i_j : \text{type}(p_j) \neq \text{type}(p_m), \\
& \qquad\qquad j \in \{1, \dots, j_* - 1, j_* + 1, \dots, m\}\} && \text{by (205), (414)} \\
& = \max\{i_j : \text{type}(p_j) \neq \text{type}(p_m), j \in \{1, \dots, j_* - 1\}\} && \text{by (416)} \\
& \leq i_{j_*-1} && \text{by (412)} \\
& < i_{j_*} && \text{by (412)} \\
& = i_*\,, && \text{by (413)}
\end{aligned}$$

i.e. $i'_* < i_*$, as desired. As concerns $i_* = i_*(P'')$, let us first notice that

$$\text{type}(P''(i_*)) = \text{type}(p') = \text{type}(p_m) \tag{417}$$

This is apparent because $\text{type}(p_*) \neq \text{type}(p_m)$ by (205) and (210). In addition, it is easily verified from (202) and (211) that $\text{type}(p') \neq \text{type}(*)$. Due to the fact that there are only two possible values of $\text{type}(p_m), \text{type}(p_*)$ and $\text{type}(p') \in \{\min, \max\}$, $\text{type}(p_*) \neq \text{type}(p_m)$ and $\text{type}(p_*) \neq \text{type}(p')$ implies that $\text{type}(p') = \text{type}(p_m)$, i.e. (417) is indeed valid. Based on these preparations, we now obtain in the case of $i''_* = i_*(P'')$ that

$$\begin{aligned}
& i''_* \\
& = i_*(P'') \\
& = \max\{i \in \text{Dom}\, P'' : \text{type}(P''(i)) \neq \text{type}(p_m)\} && \text{by (205)} \\
& = \max\{i \in \text{Dom}\, P'' \setminus \{i_*\} : \text{type}(P''(i)) \neq \text{type}(p_m)\} && \text{by (417)} \\
& = \max\{i_j : \text{type}(p_j) \neq \text{type}(p_m), \\
& \qquad\qquad j \in \{1, \dots, j_* - 1, j_* + 1, \dots, m\}\} && \text{by (412), (413)} \\
& = \max\{i_j : \text{type}(p_j) \neq \text{type}(p_m), j \in \{1, \dots, j_* - 1\}\} && \text{by (416)} \\
& \leq i_{j_*-1} && \text{by (412)} \\
& < i_{j_*} && \text{by (412)} \\
& = i_*\,, && \text{by (413)}
\end{aligned}$$

which proves that $i''_* < i_*$ as well. Knowing that $i_*(P') = i'_* < i_* = i_*(P)$ and $i_*(P'') = i''_* < i_* = i_*(P)$, the induction on i_* is indeed well-founded. In particular, we can now conclude from the induction hypothesis, which asserts that (204) be valid for all $i'_* < i_*$, that in fact

$$c(P') = \sum_{V \in \mathbb{V}} \zeta_{P'}(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_{P'}(V, \max) \cdot u_V \tag{418}$$

$$c(P'') = \sum_{V \in \mathbb{V}} \zeta_{P''}(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_{P''}(V, \max) \cdot u_V \tag{419}$$

for all $V \in \mathbb{V}$ and $y \in \{\min, \max\}$. Therefore

$$
\begin{aligned}
&c(P) \\
&|X_{i_1}^{[p_1]} \cap \dots X_{i_{j_*-1}}^{[p_{j_*-1}]} \cap X_{i_{j_*}}{}^{[p_*]} \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \dots \cap X_{i_m}^{[p_m]}| && \text{by (210), (413)} \\
&= |X_{i_1}^{[p_1]} \cap \dots X_{i_{j_*-1}}^{[p_{j_*-1}]} \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \dots \cap X_{i_m}^{[p_m]}| \\
&- |X_{i_1}^{[p_1]} \cap \dots X_{i_{j_*-1}}^{[p_{j_*-1}]} \cap X_{i_{j_*}}{}^{[p']} \cap X_{i_{j_*+1}}^{[p_{j_*+1}]} \cap \dots \cap X_{i_m}^{[p_m]}| && \text{by L-24} \\
&= c(P') - c(P'') && \text{by (193), (208), (209)} \\
&= (\sum_{V \in \mathbb{V}} \zeta_{P'}(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_{P'}(V, \max) \cdot u_V) \\
&- (\sum_{V \in \mathbb{V}} \zeta_{P''}(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_{P''}(V, \max) \cdot u_V) && \text{by (418) and (419)} \\
&= \sum_{V \in \mathbb{V}} (\zeta_{P'}(V, \min) - \zeta_{P''}(V, \min)) \cdot \ell_V \\
&+ \sum_{V \in \mathbb{V}} (\zeta_{P'}(V, \max) - \zeta_{P''}(V, \max)) \cdot u_V \\
&= \sum_{V \in \mathbb{V}} \zeta_P(V, \min) \cdot \ell_V + \sum_{V \in \mathbb{V}} \zeta_P(V, \max) \cdot u_V\,. && \text{by (207)}
\end{aligned}
$$

This completes the proof that (204) is valid in the case $i_* > 0$ as well.

D.24 Proof of Theorem 263

Let $E \neq \varnothing$ be a finite base set, $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. In order to make the proof more readable, I will usually drop the subscript γ, which is fixed during the proof. I will therefore abbreviate $X^{\min} = X_\gamma^{\min}$ and $X^{\max} = X_\gamma^{\max}$. It is further assumed throughout this proof that $\neg X^{\min} = \neg(X^{\min})$ and $\neg X^{\max} = \neg(X^{\max})$, omitting brackets. In addition, I will abbreviate $|X|_\gamma^{\min} = |X_\gamma^{\min}|$ and $|X|_\gamma^{\max} = |X_\gamma^{\max}|$. Again dropping the subscript, this becomes $|X|^{\min} = |X^{\min}|$ and $|X|^{\max} = |X^{\max}|$. Exposing the 'max' and 'min' helps me to save brackets, for example I can then write $|X_1 \cap X_2|^{\min}$ rather than $|(X_1 \cap X_2)^{\min}|$. Finally, I will refer to the coefficients introduced in the theorem, which now become $\ell_1 = |X_1|^{\min}$, $\ell_2 = |X_1 \cap X_2|^{\min}$ $\ell_3 = |X_1 \cap \neg X_2|^{\min}$, $u_1 = |X_1|^{\max}$, $u_2 = |X_1 \cap X_2|^{\max}$, and $u_3 = |X_1 \cap \neg X_2|^{\max}$.

Let us now consider some choice of $Y_1 \in \mathcal{T}_\gamma(X_1)$ and $Y_2 \in \mathcal{T}_\gamma(X_2)$. We can then define the following cardinality coefficients:

$$\alpha = |Y_1 \cap \neg X_1^{\min} \cap \neg X_2^{\max}| \tag{420}$$

$$\beta = |Y_1 \cap \neg X_1^{\min} \cap X_2^{\min}| \tag{421}$$

$$\gamma = |Y_2 \cap X_1^{\min} \cap \neg X_2^{\min}| \tag{422}$$

$$\delta = |Y_1 \cap Y_2 \cap \neg X_1^{\min} \cap \neg X_2^{\min}| \tag{423}$$

$$\varepsilon = |Y_1 \cap \neg Y_2 \cap \neg X_1^{\min} \cap X_2^{\max}|\,. \tag{424}$$

Particularly, the sum of the last coefficients $\delta + \varepsilon$ becomes

$$\delta + \varepsilon = |Y_1 \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| \,. \tag{425}$$

This is straightforward from (423) and (424) once we notice that $Y_1 \cap X_1^{\max} = Y_1$, $Y_2 \cap X_2^{\max} = Y_2$ and $\neg Y_2 \cap \neg X_2^{\min} = \neg Y_2$.

As to the possible values of the coefficients $\alpha, \beta, \gamma, \delta, \varepsilon$, it is obvious that by choosing $Y_1 \in \mathcal{T}_\gamma(X_1)$ and $Y_2 \in \mathcal{T}_\gamma(X_2)$ appropriately, all combinations of values in the following ranges can be assumed (which also exhaust all possible options):

$$\alpha \in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}|\} \tag{426}$$

$$\beta \in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}|\} \tag{427}$$

$$\gamma \in \{0, \ldots, |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|\} \tag{428}$$

$$\delta \in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|\} \tag{429}$$

$$\varepsilon \in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| - \delta\} \,. \tag{430}$$

In the following, I will express $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$ in terms of these cardinality coefficients. To this end, let us notice that $X_1^{\max}$ can be decomposed into a disjoint union of the following components,

$$\begin{aligned} X_1^{\max} = \quad & (X_1^{\min} \cap \neg X_2^{\max}) \\ & \dot\cup (X_1^{\min} \cap X_2^{\min}) \\ & \dot\cup (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\ & \dot\cup (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}) \\ & \dot\cup (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}) \\ & \dot\cup (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \,, \end{aligned} \tag{431}$$

which is possible because $X_1^{\min} \subseteq X_1^{\max}$ and $X_2^{\min} \subseteq X_2^{\max}$. We now consider the given choice of $Y_1 \in \mathcal{T}_\gamma(X_1)$. We then know from Def. 82 that $Y_1 \subseteq X_1^{\max}$. In particular $Y_1 = Y_1 \cap X_1^{\max}$. By expanding $X_1^{\max}$ in $Y_1 \cap X_1^{\max}$ according to (431) and then utilizing the law of distributivity in order to move Y_1 into the members of the disjoint union, we now obtain that

$$\begin{aligned} Y_1 = & (Y_1 \cap X_1^{\min} \cap \neg X_2^{\max}) \\ & \dot\cup\, (Y_1 \cap X_1^{\min} \cap X_2^{\min}) \\ & \dot\cup\, (Y_1 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\ & \dot\cup\, (Y_1 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}) \\ & \dot\cup\, (Y_1 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}) \\ & \dot\cup\, (Y_1 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \,. \end{aligned}$$

Recalling that $X_1^{\min} \subseteq Y_1 \subseteq X_1^{\max}$, some simplifications to this result become possible, because $Y_1 \cap X_1^{\min} = X_1^{\min}$ and $Y_1 \cap X_1^{\max} = Y_1$. The above result then reduces to

$$\begin{aligned}Y_1 =&(X_1^{\min} \cap \neg X_2^{\max})\\ &\dot\cup\,(X_1^{\min} \cap X_2^{\min})\\ &\dot\cup\,(X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min})\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap \neg X_2^{\max})\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap X_2^{\min})\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min})\,,\end{aligned}$$

i.e.

$$\begin{aligned}Y_1 =&X_1^{\min}\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap \neg X_2^{\max})\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap X_2^{\min})\\ &\dot\cup\,(Y_1 \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min})\,.\end{aligned}$$

Due to the disjoint unions, the cardinalities of the components directly sum up to the total cardinality of Y_1, i.e.

$$\begin{aligned}|Y_1| =&|X_1|^{\min}\\ &+|Y_1 \cap \neg X_1^{\min} \cap \neg X_2^{\max}|\\ &+|Y_1 \cap \neg X_1^{\min} \cap X_2^{\min}|\\ &+|Y_1 \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|\,.\end{aligned}$$

Now utilizing the coefficients $\alpha, \beta, \delta, \epsilon$ introduced in (420), (421), (423) and (424), and further utilizing (425), we therefore obtain that

$$c_1 = |Y_1| = \ell_1 + \alpha + \beta + \delta + \varepsilon\,. \tag{432}$$

Next we consider $c_2 = |Y_1 \cap Y_2|$. Again, we notice that $Y_1 \cap Y_2 \subseteq Y_1 \subseteq X_1^{\max}$. Hence $Y_1 \cap Y_2 = Y_1 \cap Y_2 \cap X_1^{\max}$. Therefore the same procedure as above can be applied. By expanding $X_1^{\max}$ according to (431) and then utilizing distributivity in order to move $Y_1 \cap Y_2$ into the members of the disjoint union, we first obtain that

$$\begin{aligned}Y_1 \cap Y_2 =&(Y_1 \cap Y_2 \cap X_1^{\min} \cap \neg X_2^{\max})\\ &\dot\cup\,(Y_1 \cap Y_2 \cap X_1^{\min} \cap X_2^{\min})\\ &\dot\cup\,(Y_1 \cap Y_2 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min})\\ &\dot\cup\,(Y_1 \cap Y_2 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max})\\ &\dot\cup\,(Y_1 \cap Y_2 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min})\\ &\dot\cup\,(Y_1 \cap Y_2 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min})\,.\end{aligned}$$

Again, some simplifications are possible because $X_1^{\min} \subseteq Y_1 \subseteq X_1^{\max}$ and $X_2^{\min} \subseteq Y_2 \subseteq X_2^{\max}$, i.e. $Y_1 \cap X_1^{\min} = X_1^{\min}$, $Y_1 \cap X_1^{\max} = Y_1$, $Y_2 \cap X_2^{\min} = X_2^{\min}$,

$Y_2 \cap X_2^{\max} = Y_2$ and $Y_2 \cap \neg X_2^{\max} = \varnothing$. The above rendering of $Y_1 \cap Y_2$ can therefore be simplified into

$$\begin{aligned} Y_1 \cap Y_2 = & (X_1^{\min} \cap X_2^{\min}) \\ & \dot{\cup}\, (Y_2 \cap X_1^{\min} \cap \neg X_2^{\min}) \\ & \dot{\cup}\, (Y_1 \cap \neg X_1^{\min} \cap X_2^{\min}) \\ & \dot{\cup}\, (Y_1 \cap Y_2 \cap \neg X_1^{\min} \cap \neg X_2^{\min}) \,. \end{aligned}$$

The representation in terms of a disjoint union again permits us to sum up constituent cardinalities. Therefore

$$\begin{aligned} |Y_1 \cap Y_2| = & |X_1 \cap X_2|^{\min} \\ & + |Y_2 \cap X_1^{\min} \cap \neg X_2^{\min}| \\ & + |Y_1 \cap \neg X_1^{\min} \cap X_2^{\min}| \\ & + |Y_1 \cap Y_2 \cap \neg X_1^{\min} \cap \neg X_2^{\min}| \,. \end{aligned}$$

In terms of the cardinality coefficients, this can now be expressed in the following more succint form,

$$c_2 = |Y_1 \cap Y_2| = \ell_2 + \beta + \gamma + \delta \,. \tag{433}$$

Having expressed c_1 and c_2 in terms of the coefficients $\alpha, \beta, \gamma, \delta$ and ε, it is now useful to reconsider the ranges of these coefficients, which have been stated in (426)–(430). In particular, I would like to show that the upper bounds on these ranges can be expressed in terms of the given coefficients $\ell_1, \ell_2, \ell_3, u_1, u_2$ and u_3 only.

Let us first consider the maximal choice $\alpha_{\max}$ α, i.e. $\alpha_{\max} = |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}|$. To this end, we notice that $X_1^{\max}$ can be split into $X_1^{\max} = (X_1^{\max} \cap X_2^{\max}) \dot{\cup} (X_1^{\max} \cap \neg X_2^{\max})$. The constituent cardinalities then add up to the total cardinality of $X_1^{\max}$, thus

$$|X_1|^{\max} = |X_1 \cap X_2|^{\max} + |X_1^{\max} \cap \neg X_2^{\max}|$$

by Th-61. In terms of the cardinality coefficients introduced above, we can also express this as $u_1 = u_2 + |X_1^{\max} \cap \neg X_2^{\max}|$, or equivalently,

$$|X_1^{\max} \cap \neg X_2^{\max}| = u_1 - u_2 \,. \tag{434}$$

Turning to $\alpha_{\max}$, we decompose $X_1^{\max} \cap \neg X_2^{\max}$ into a disjoint union

$$X_1^{\max} \cap \neg X_2^{\max} = (X_1^{\max} \cap X_1^{\min} \cap \neg X_2^{\max}) \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}) \,,$$

i.e. $X_1^{\max} \cap \neg X_2^{\max} = (X_1^{\min} \cap \neg X_2^{\max}) \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max})$. Recalling Th-61, we can further refine this into $X_1^{\max} \cap \neg X_2^{\max} = X_1 \cap X_2{}^{\min} \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max})$. The corresponding cardinalities then become

$$|X_1^{\max} \cap \neg X_2^{\max}| = |X_1 \cap \neg X_2|^{\min} + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}| \,.$$

Building on (434), we can recast this in terms of the cardinality coefficients as follows,

$$u_1 - u_2 = \ell_3 + \alpha_{\max} \, .$$

Therefore $\alpha_{\max}$ has the simple rendering,

$$\alpha_{\max} = |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max}| = u_1 - u_2 - \ell_3 \, . \tag{435}$$

Next we consider the maximum choice of β, i.e. $\beta_{\max} = |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}|$ by (427). We first decompose $X_1^{\max}$ into $X_1^{\max} = (X_1^{\max} \cap X_2^{\min}) \,\dot{\cup}\, (X_1^{\max} \cap \neg X_2^{\min})$ or equivalently, $X_1^{\max} = (X_1^{\max} \cap X_2^{\min}) \dot{\cup} (X_1 \cap \neg X_2)^{\max}$, see Th-61. The corresponding cardinalities then become $|X_1|^{\max} = |X_1^{\max} \cap X_2^{\min}| + |X_1 \cap \neg X_2|^{\max}$, which can be abbreviated as $u_1 = |X_1^{\max} \cap X_2^{\min}| + u_3$. This proves that

$$|X_1^{\max} \cap X_2^{\min}| = u_1 - u_3 \, . \tag{436}$$

In order to obtain the desired representation of $\beta_{\max}$, we now rewrite $X_1^{\max} \cap X_2^{\min} = (X_1^{\max} \cap X_1^{\min} \cap X_2^{\min}) \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min})$, i.e. $X_1^{\max} \cap X_2^{\min} = (X_1^{\min} \cap X_2^{\min}) \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min})$ because $X_1^{\min} \subseteq X_1^{\max}$, and in turn $X_1^{\max} \cap X_2^{\min} = X_1 \cap {X_2}^{\min} \,\dot{\cup}\, (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min})$ by Th-61. We therefore obtain for the corresponding cardinalities that

$$|X_1^{\max} \cap X_2^{\min}| = |X_1 \cap X_2|^{\min} + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}| \, .$$

In terms of the cardinality coefficients, this can be abbreviated as $u_1 - u_3 = \ell_2 + \beta_{\max}$, recalling equation (436). In other words,

$$\beta_{\max} = |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min}| = u_1 - \ell_2 - u_3 \, . \tag{437}$$

Now we discuss the maximum choice of the coefficient γ, i.e. $\gamma_{\max} = |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|$ by (428). In order to prepare this investigation, I first split $X_1^{\min}$ into a disjoint union $X_1^{\min} = (X_1^{\min} \cap X_2^{\max}) \,\dot{\cup}\, (X_1^{\min} \cap \neg X_2^{\max})$. By utilizing Th-61, this can be slighly modified into $X_1^{\min} = (X_1^{\min} \cap X_2^{\max}) \,\dot{\cup}\, (X_1 \cap \neg X_2)^{\min}$. Again, the constituent cardinalities add up to the total cardinality $|X_1|^{\min}$, i.e.

$$|X_1|^{\min} = |X_1^{\min} \cap X_2^{\max}| + |X_1 \cap \neg X_2|^{\min} \, .$$

In terms of the cardinality coefficients, we therefore have $\ell_1 = |X_1^{\min} \cap X_2^{\max}| + \ell_3$, or equivalently,

$$|X_1^{\min} \cap X_2^{\max}| = \ell_1 - \ell_3 \, . \tag{438}$$

Based on this preparations, we can now consider $\gamma_{\max}$. To this end, we rewrite $X_1^{\min} \cap X_2^{\max}$ as follows,

$$\begin{aligned}
& X_1^{\min} \cap X_2^{\max} \\
& = (X_1^{\min} \cap X_2^{\max} \cap X_2^{\min}) \,\dot{\cup}\, (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\
& = (X_1^{\min} \cap X_2^{\min}) \,\dot{\cup}\, (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) && \text{because } X_2^{\min} \subseteq X_2^{\max} \\
& = X_1 \cap {X_2}^{\min} \,\dot{\cup}\, (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \, . && \text{by Th-61}
\end{aligned}$$

In particular, this proves that $|X_1^{\min} \cap X_2^{\max}| = |X_1 \cap X_2|^{\min} + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|$. Based on (438), we therefore obtain that $\ell_1 - \ell_3 = \ell_2 + \gamma_{\max}$, i.e.

$$\gamma_{\max} = |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| = \ell_1 - \ell_2 - \ell_3 \,. \tag{439}$$

Finally we consider $|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}|$, which constitutes the maximal choice of δ and ε. In order to prepare this, I first decompose $(X_1 \cap X_2)^{\max}$, viz

$$\begin{aligned}
&(X_1 \cap X_2)^{\max} \\
&= X_1^{\max} \cap X_2^{\max} && \text{by Th-61} \\
&= (X_1^{\max} \cap X_2^{\max} \cap X_2^{\min}) \mathbin{\dot\cup} (X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}) \\
&= (X_1^{\max} \cap X_2^{\min}) \mathbin{\dot\cup} (X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}) && \text{because } X_2^{\min} \subseteq X_2^{\max}.
\end{aligned}$$

The corresponding cardinalities then become

$$\begin{aligned}
u_2 &= |X_1 \cap X_2|^{\max} \\
&= |X_1^{\max} \cap X_2^{\min}| + |X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}| \\
&= u_1 - u_3 + |X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}| \,, && \text{by (436)}
\end{aligned}$$

i.e.

$$|X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}| = -u_1 + u_2 + u_3 \,. \tag{440}$$

Based on this preparation, we can now proceed as follows. Firstly

$$\begin{aligned}
&X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \\
&= (X_1^{\max} \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\
&= (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}) && \text{because } X_1^{\min} \subseteq X_1^{\max}.
\end{aligned}$$

In particular, this proves that

$$\begin{aligned}
&|X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min}| \\
&= |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| \,.
\end{aligned}$$

Now recalling (439) and (440), we can rewrite this as

$$-u_1 + u_2 + u_3 = \ell_1 - \ell_2 - \ell_3 + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| \,,$$

i.e.

$$|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min}| = -u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3 \,. \tag{441}$$

It is then apparent from (429) that the maximum choice of δ is

$$\delta_{\max} = -u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3 \,. \tag{442}$$

In this case, (430) then ties ε to the minimal choice $\varepsilon = 0$. Conversely, ε can assume the maximum value of

$$\varepsilon_{\max} = -u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3 \,, \tag{443}$$

but only if δ is tied to $\delta = 0$, see (429) and (430). From a more general perspective, (441) establishes an upper bound on the *sum* of δ and ε, i.e.

$$(\delta + \varepsilon)_{\max} = -u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3 \,. \tag{444}$$

This is apparent when combining (441) and (425).

To sum up, I have shown how the possible values of the cardinality coefficients $\alpha, \beta, \gamma, \delta$ and ε can be expressed in terms of $\ell_1, \ell_2, \ell_3, u_1, u_2$ and u_3.

Based on these renderings of the cardinality coefficients, it is now a straightforward task to identify the possible combinations of $c_1 = |Y_1|$ and $c_2 = |Y_1 \cap Y_2|$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$ and $Y_2 \in \mathcal{T}_\gamma(X_2)$, which are gathered in the target relation $R(X_1, X_2)$.

As to c_1, we know from (432) that $c_1 = |Y_1| = \ell_1 + \alpha + \beta + \delta + \varepsilon$. Hence the minimal choice $c_1^{\min}$ is observed when all involved cardinality coefficients assume their minimum $\alpha = \beta = \delta = \varepsilon = 0$. We then obtain

$$c_1^{\min} = \ell_1 \,.$$

By similar reasoning, the maximum $c_1^{\max}$ is observed when the involved coefficients assume their maxima, which I have made explicit in (435), (437) and (444). By instantiating the coefficients with their maximum values, (432) becomes

$$c_1^{\max} = \ell_1 + u_1 - u_2 - \ell_3 + u_1 - \ell_2 - u_3 - u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3 = u_1 \,.$$

It is apparent from the fact that $\mathcal{T}_\gamma(X_i)$, $i \in \{1, 2\}$, is a *closed range* of crisp sets $\mathcal{T}_\gamma(X_i) = \{Y : X_i^{\min} \subseteq Y \subseteq X_i^{\max}\}$, that the cardinality coefficients also assume all intermediate values between their minima and maxima. Therefore $c_1 = \ell_1 + \alpha + \beta + \delta + \varepsilon$ also assumes all intermediate values between $c_1^{\min} = \ell_1$ and $c_2^{\max} = u_1$, i.e.

$$\{c_1 = |Y_1| : Y_1 \in \mathcal{T}_\gamma(X_1)\} = \{c_1 : \ell_1 \leq c_1 \leq u_1\} \,.$$

In the following, I will assume a choice of $c_1 = |Y_1|$ within its range of legal values. In order to make explicit the target relation, it is now necessary to identify the possible values of $c_2 = |Y_1 \cap Y_2|$, given c_1. To this end, we first consider $\ell_1 + \alpha_{\max} + \varepsilon_{\max}$. By (435) and (443), this becomes $\ell_1 + \alpha_{\max} + \varepsilon_{\max} = \ell_1 + u_1 - u_2 - \ell_3 - u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3$, i.e.

$$\ell_1 + \alpha_{\max} + \varepsilon_{\max} = \ell_2 + u_3 \,. \tag{445}$$

It is convenient to discern the following two cases.

1. $c_1 \leq \ell_2 + u_3$.
   It is then apparent from (432), (445) that c_1 can be expressed as $c_1 = \ell_1 + \alpha + \varepsilon$

for a choice of $\alpha \leq \alpha_{\max}$ and $\varepsilon \leq \varepsilon_{\max}$ and $\beta = \gamma = \delta = 0$. We then obtain from (433) that $c_2 = \ell_2$, which is apparently the minimal choice (because the cardinality coefficients are always non-negative). Finally we notice that $c_1 - u_3 \leq \ell_2 + u_3 - u_3 = \ell_2$ by the assumption of case 1. Therefore $c_2^{\min} = \ell_2$ can also be expressed as $c_2^{\min} = \max(c_1 - u_3, \ell_2)$.

2. $c_1 > \ell_2 + u_3$.
   In this case, we know from (432) and (445) that c_1 exceeds $\ell_1 + \alpha_{\max} + \varepsilon_{\max}$. Therefore $c_2 = \ell_2 + \beta + \gamma + \delta$ is minimized if we express c_1 as $c_1 = \ell_1 + \alpha_{\max} + \beta + \varepsilon_{\max}$, where

   $$\beta = c_1 - \ell_1 - \alpha_{\max} - \varepsilon_{\max} = c_1 - \ell_2 - u_3 \, , \tag{446}$$

   and further choose $\delta = \gamma = 0$. By (433) and (446), c_2 then assumes its minimum

   $$c_2 = \ell_2 + \beta = \ell_2 + c_1 - \ell_2 - u_3 = c_1 - u_3 \, .$$

   Due to the assumption of case 2. that $c_1 > \ell_2 + u_3$, this entails that $c_1 - u_3 > \ell_2 + u_3 - u_3 = \ell_2$. Hence the minimal choice of c_2 can also be expressed as $c_2^{\min} = \max(c_1 - u_3, \ell_2)$.

To sum up, I have shown that in both cases,

$$c_2^{\min} = \max(c_1 - u_3, \ell_2) \, . \tag{447}$$

Finally we consider the maximum choice of $c_2 = |Y_1 \cap Y_2|$, given c_1. It is now useful to consider $\ell_1 + \beta_{\max} + \delta_{\max}$. By (437) and (442), this becomes $\ell_1 + \beta_{\max} + \delta_{\max} = \ell_1 + u_1 - \ell_2 - u_3 - u_1 + u_2 + u_3 - \ell_1 + \ell_2 + \ell_3$, i.e.

$$\ell_1 + \beta_{\max} + \delta_{\max} = u_2 + \ell_3 \, . \tag{448}$$

Again, it is convenient to separate two cases.

a. $c_1 \leq u_2 + \ell_3$.
   We then conclude from (432) and (448) that c_1 can be expressed as

   $$c_1 = \ell_1 + \beta + \delta \, , \tag{449}$$

   where $\beta \leq \beta_{\max}$, $\delta \leq \delta_{\max}$, and $\alpha = \varepsilon = 0$. Further choosing $\gamma = \gamma_{\max}$ then maximises c_2, which becomes

   $$\begin{aligned} c_2 &= \ell_2 + \beta + \gamma_{\max} + \delta && \text{by (433)} \\ &= \ell_2 + \beta + \delta + \ell_1 - \ell_2 - \ell_3 && \text{by (439)} \\ &= \ell_2 + c_1 - \ell_1 + \ell_1 - \ell_2 - \ell_3 && \text{by (449)} \\ &= c_1 - \ell_3 \, . \end{aligned}$$

   Due to the assumption of case a. that $c_1 \leq u_2 + \ell_3$, we know that $c_1 - \ell_3 \leq u_2 + \ell_3 - \ell_3 = u_2$. Therefore $c_2^{\max} = c_1 - \ell_3$ can also be expressed as $c_2^{\max} = \min(c_1 - \ell_3, u_2)$.

b. $c_1 > u_2 + \ell_3$.
In this case, we know from (432) and (448) that c_1 exceeds $\ell_1 + \beta_{\max} + \delta_{\max}$. Consequently, in order to maximize c_2, we choose

$$\alpha = c_1 - \ell_1 - \beta_{\max} - \delta_{\max} = c_1 - u_2 - \ell_3 \,,$$

and further let $\beta = \beta_{\max}$, $\delta = \delta_{\max}$ and hence $\varepsilon = 0$ by (425), and finally $\gamma = \gamma_{\max}$, which only affects c_2. The maximum value of c_2 then becomes

$$\begin{aligned} c_2^{\max} &= \ell_2 + \beta_{\max} + \gamma_{\max} + \delta_{\max} && \text{by (433)} \\ &= \ell_2 + \ell_1 - \ell_2 - \ell_3 + \beta_{\max} + \delta_{\max} && \text{by (439)} \\ &= \ell_2 + \ell_1 - \ell_2 - \ell_3 + u_2 + \ell_3 - \ell_1 && \text{by (448)} \\ &= u_2 \,. \end{aligned}$$

The assumption of this case that $c_1 > u_2 + \ell_3$ then entails that $c_1 - \ell_3 > u_2 + \ell_3 - \ell_3 = u_2$. Therefore the maximal choice of c_2 can again be given the rendering $c_2^{\max} = \min(c_1 - \ell_3, u_2)$.

We therefore know that indeed $c_2^{\max} = \min(c_1 - \ell_3, u_2)$, and it has also been shown that $c_2^{\min} = \max(c_1 - u_3, \ell_2)$. Due to the fact that all $c_2 \in \{c_2^{\min}, \dots, c_2^{\max}\}$ can be expressed as a sum of the cardinality coefficients $\alpha, \beta, \gamma, \delta$ and ε, and because these coefficients can assume all integers in their associated ranges (426)–(430), we conclude that for each given c_1,

$$\begin{aligned} R_{c_1} &= \{(c_1, |Y_1 \cap Y_2|) : Y_1 \in \mathcal{T}_\gamma(X_1), Y_2 \in \mathcal{T}_\gamma(X_2), c_1 = |Y_1|\} \\ &= \{(c_1, \ell_2 + \beta + \gamma + \delta) : c_1 = \ell_1 + \alpha + \beta + \delta + \varepsilon, \text{all coefficients within ranges}\} \\ &= \{(c_1, c_2) : c_2^{\min} \leq c_2 \leq c_2^{\max}\} \\ &= \{(c_1, c_2) : \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} \,. \end{aligned}$$

This proves the claim of the theorem because the target relation $R = R(X_1, X_2)$ can be expressed as the union of all R_{c_1}, for $\ell_1 = c_1^{\min} \leq c_1 \leq c_1^{\max} = u_1$.

D.25 Proof of Theorem 264

Lemma 25
Let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy subset such that $\Gamma(X)$ is finite, and let $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$ be given such that $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X)$. Then for all $j \in \{0, \dots, m-1\}$, $A(X) \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}) = \varnothing$ and $A(X) \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$.

Proof By assumption, the $\gamma_j \in [0,1]$, $j \in \{0, \dots, m\}$ form a strictly increasing sequence. Therefore

$$\Gamma \cap (\gamma_j, \gamma_{j+1}) = \varnothing \tag{450}$$

for all $j \in \{0, \dots, m+1\}$. In order to prove that $A(X) \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2_{\gamma_j+1}}) = \varnothing$, we consider the linear mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(\gamma) = \frac{1}{2} + \frac{1}{2}\gamma$. Noticing that f is a strictly increasing bijection, (450) translates into

$$\begin{aligned}
&\{\tfrac{1}{2} + \tfrac{1}{2}\gamma_k : k \in \{0, \dots, m\}\} \cap (\tfrac{1}{2} + \tfrac{1}{2}\gamma_j, \tfrac{1}{2} + \tfrac{1}{2}\gamma_{j+1}) \\
&= \{f(\gamma_k) : \gamma_k \in \Gamma\} \cap (f(\gamma_j), f(\gamma_{j+1})) \\
&= \widehat{f}(\Gamma) \cap \widehat{f}((\gamma_j, \gamma_{j+1})) \\
&= \widehat{f}(\Gamma \cap (\gamma_j, \gamma_{j+1})) \\
&= \widehat{f}(\varnothing) \\
&= \varnothing \,.
\end{aligned}$$

Now recalling from (218) that $A_\Gamma \cap [\frac{1}{2}, 1] = \{\frac{1}{2} + \frac{1}{2}\gamma_k : k \in \{0, \dots, m\}\}$, this proves that

$$(A_\Gamma \cap [\tfrac{1}{2}, 1]) \cap (\tfrac{1}{2} + \tfrac{1}{2}\gamma_j, \tfrac{1}{2} + \tfrac{1}{2}\gamma_{j+1}) = \varnothing \,.$$

In addition, we apparently have $(A_\Gamma \cap [0, \frac{1}{2})) \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}) = \varnothing$. Utilizing the apparent decomposition of A_Γ into $A_\Gamma = (A_\Gamma \cap [0, \frac{1}{2})) \cup (A_\Gamma \cap [\frac{1}{2}, 1])$, we hence obtain that

$$\begin{aligned}
&A_\Gamma \cap (\tfrac{1}{2} + \tfrac{1}{2}\gamma_j, \tfrac{1}{2} + \tfrac{1}{2}\gamma_{j+1}) \\
&= ((A_\Gamma \cap [0, \tfrac{1}{2})) \cup (A_\Gamma \cap [\tfrac{1}{2}, 1])) \cap (\tfrac{1}{2} + \tfrac{1}{2}\gamma_j, \tfrac{1}{2} + \tfrac{1}{2}\gamma_{j+1}) \\
&= \varnothing \cup \varnothing \\
&= \varnothing \,.
\end{aligned}$$

Finally we recall that by (219), $A(X) \subseteq A_\Gamma$ provided that $\Gamma \supseteq \Gamma(X)$. Therefore $A_\Gamma \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}) = \varnothing$ entails that $A(X) \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}) = \varnothing$ as well.

The second claim of the lemma can be proven in an analogous way. In this case, we utilize the linear mapping $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $g(\gamma) = \frac{1}{2} - \frac{1}{2}\gamma$. Noticing that g is a strictly decreasing bijection, (450) then translates into

$$\{\tfrac{1}{2} - \tfrac{1}{2}\gamma_k : k \in \{0, \dots, m\}\} \cap (\tfrac{1}{2} - \tfrac{1}{2}\gamma_{j+1}, \tfrac{1}{2} - \tfrac{1}{2}\gamma_j) = \varnothing \,.$$

Recalling (218), this proves that $(A_\Gamma \cap [0, \frac{1}{2}]) \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$. Combining this with the apparent $(A_\gamma \cap (\frac{1}{2}, 1]) \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$, we then obtain that $A_\Gamma \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$. But $A(X) \subseteq A_\Gamma$, hence $A(X) \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$, as desired.

Lemma 26

Consider a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ *such that* $\Gamma(X)$ *is finite and further suppose that* $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X)$ *is chosen such that* $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$. *Then for all* $j \in \{0, \dots, m-1\}$ *and all* $\gamma \in (\gamma_j, \gamma_{j+1})$,

$$\begin{aligned}
X_\gamma^{\min} &= X_{> \frac{1}{2} + \frac{1}{2}\gamma_j} = X_{\geq \frac{1}{2} + \frac{1}{2}\gamma_{j+1}} \\
X_\gamma^{\max} &= X_{\geq \frac{1}{2} - \frac{1}{2}\gamma_j} = X_{> \frac{1}{2} - \frac{1}{2}\gamma_{j+1}} \,.
\end{aligned}$$

Proof Consider $j \in \{0, \dots, m-1\}$ and $\gamma \in (\gamma_j, \gamma_{j+1})$. Then in particular $\gamma > 0$ and by Def. 82,

$$X^{\min}_{\gamma} = X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} \,.$$

We now recall from L-25 that $A(X) \cap (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}) = \varnothing$, i.e. $\mu_X(e) \notin (\frac{1}{2} + \frac{1}{2}\gamma_j, \frac{1}{2} + \frac{1}{2}\gamma_{j+1}$ for all $e \in E$. It is therefore immediate from $\gamma_j < \gamma_{j+1}$ and the definition of α-cuts and strict α-cuts that indeed $X^{\min}_{\gamma} = X_{> \frac{1}{2} + \frac{1}{2}\gamma_j} = X_{\geq \frac{1}{2} + \frac{1}{2}\gamma_{j+1}}$, see Def. 75 and Def. 76.

Let us now consider $X^{\max}_{\gamma}$. Because $\gamma > 0$, we obtain from Def. 82 that $X^{\max}_{\gamma} = X_{> \frac{1}{2} - \frac{1}{2}\gamma}$. In addition, we know from L-25 that $A(X) \cap (\frac{1}{2} - \frac{1}{2}\gamma_{j+1}, \frac{1}{2} - \frac{1}{2}\gamma_j) = \varnothing$. Again, the definition of α-cuts and strict α-cuts lets us conclude that $X^{\max}_{\gamma} = X_{\geq \frac{1}{2} - \frac{1}{2}\gamma_j} = X_{> \frac{1}{2} - \frac{1}{2}\gamma_{j+1}}$, knowing that $\gamma_j < \gamma_{j+1}$.

Proof of Theorem 264

Consider a finite base set $E \neq \varnothing$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$. Further let a choice of $\Gamma = \{\gamma_0, \dots, \gamma_m\} \supseteq \Gamma(X)$ be given such that $0 = \gamma_0 < \gamma_1 < \dots < \gamma_{m-1} < \gamma_m = 1$. For $j = 0$, we first observe that

$$\begin{aligned} X^{\min}_{\overline{\gamma}_0} &= X_{> \frac{1}{2}} && \text{by L-26} \\ &= \{e \in E : \mu_X(e) > \tfrac{1}{2}\} && \text{by Def. 76} \\ &= \{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_k \text{ for some } k \in \{1, \dots, m\}\} \end{aligned}$$

i.e.

$$X^{\min}_{\overline{\gamma}_0} = \dot{\cup}\{\mu_X^{-1}(\tfrac{1}{2} + \tfrac{1}{2}\gamma_k) : k \in \{1, \dots, m\}\} \,, \tag{451}$$

where '$\dot{\cup}$' denotes the disjoint union; the last two steps are valid because $A(X) \subseteq A_\Gamma$ by (219) and hence $A(X) \cap (\frac{1}{2}, 1] \subseteq A_\Gamma \cap (\frac{1}{2}, 1] = \{\frac{1}{2} + \frac{1}{2}\gamma_k : k \in \{1, \dots, m\}\}$ by (218). Due to the fact that $X^{\min}_{\overline{\gamma}_0}$ resolves into the disjoint union of its subsets $\mu_X^{-1}(\frac{1}{2} + \frac{1}{2}\gamma_k)$, equation (451) permits us to compute the cardinality of $X^{\min}_{\overline{\gamma}_0}$ by simply adding up the cardinalities of all subsets which participate in the disjoint union. Hence

$$\ell(0) = |X^{\min}_{\overline{\gamma}_0}| = \sum_{k=1}^{m} |\mu_X^{-1}(\tfrac{1}{2} + \tfrac{1}{2}\gamma_k)| = \sum_{k=1}^{m} H^+(k) \tag{452}$$

by (214) and (220). Now turning to $X^{\max}_{\overline{\gamma}_0}$, it is easily shown that

$$\begin{aligned} X^{\max}_{\overline{\gamma}_0} &= X_{\geq \frac{1}{2}} && \text{by L-26} \\ &= X_{> \frac{1}{2}} \,\dot{\cup}\, \{e \in E : \mu_X(e) = \tfrac{1}{2}\} \end{aligned}$$

where the last step is immediate from Def. 75 and Def. 76 because $A(X) \cap (\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\gamma_1) \subseteq A_\Gamma \cap (\frac{1}{2}, \frac{1}{2} + \frac{1}{2}\gamma_1) = \varnothing$, see L-25. Gaining from L-26, we can further simplify this into

$$X^{\max}_{\overline{\gamma}_0} = X^{\min}_{\overline{\gamma}_0} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2}\} .$$

Again, the disjoint union renders it possible to compute the total number of elements in $X^{\max}_{\overline{\gamma}_0}$ by summing up the constituent cardinalities. Utilizing (214) and (220), we therefore obtain that

$$u(0) = |X^{\max}_{\overline{\gamma}_0}| = \ell(0) + H^+(0) . \tag{453}$$

Next we consider the case that $j > 0$. It is then apparent that

$$\begin{aligned} X^{\max}_{\overline{\gamma}_j} &= X_{\geq \frac{1}{2} - \frac{1}{2}\gamma_j} && \text{by L-26} \\ &= X_{> \frac{1}{2} - \frac{1}{2}\gamma_j} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} - \tfrac{1}{2}\gamma_j\} && \text{by Def. 75, Def. 76} \\ &= X^{\max}_{\overline{\gamma}_{j-1}} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} - \tfrac{1}{2}\gamma_j\} , && \text{by L-26} \end{aligned}$$

i.e.

$$X^{\max}_{\overline{\gamma}_j} = X^{\max}_{\overline{\gamma}_{j-1}} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} - \tfrac{1}{2}\gamma_j\} .$$

Due to the disjoint union, we then obtain for the corresponding cardinalities that

$$|X^{\max}_{\overline{\gamma}_j}| = |X^{\max}_{\overline{\gamma}_{j-1}}| + |\{e \in E : \mu_X(e) = \tfrac{1}{2} - \tfrac{1}{2}\gamma_j\}| .$$

Recalling the abbreviations introduced in (215) and (221), this proves the desired

$$u(j) = u(j-1) + H^-(j) .$$

By similar reasoning, we conclude that

$$\begin{aligned} X^{\min}_{\overline{\gamma}_{j-1}} &= X_{\geq \frac{1}{2} + \frac{1}{2}\gamma_j} && \text{by L-26} \\ &= X_{> \frac{1}{2} + \frac{1}{2}\gamma_j} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\} && \text{by Def. 75, Def. 76} \\ &= X^{\min}_{\gamma_j} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\} , && \text{by L-26} \end{aligned}$$

i.e.

$$X^{\min}_{\overline{\gamma}_{j-1}} = X^{\min}_{\gamma_j} \mathbin{\dot\cup} \{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\} .$$

Turning to cardinalities, this shows that

$$|X^{\min}_{\overline{\gamma}_{j-1}}| = |X^{\min}_{\gamma_j}| + |\{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\}|$$

or equivalently,

$$|X^{\min}_{\gamma_j}| = |X^{\min}_{\overline{\gamma}_{j-1}}| - |\{e \in E : \mu_X(e) = \tfrac{1}{2} + \tfrac{1}{2}\gamma_j\}| .$$

Recalling abbreviations (214) and (220), this finally becomes

$$\ell(j) = \ell(j-1) - H^+(j) ,$$

as desired.

D.26 Proof of Theorem 265

Lemma 27

Let $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3 \in \mathbb{N}$ *be given integers with* $\ell_1 \leq u_1$, $\ell_2 \leq u_2$, $\ell_3 \leq u_3$, $\ell_1 \leq u_2 + \ell_3 \leq u_1$ *and* $\ell_1 \leq \ell_2 + u_3 \leq u_1$. *Further let*

$$R = \{(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} \,.$$

a. If $u_2 + \ell_3 > 0$, *then*

$$\max\{c_2/c_1 : (c_1, c_2) \in R, c_1 > 0\} = \tfrac{u_2}{u_2+\ell_3} \,.$$

b. If $\ell_2 + u_3 > 0$, *then*

$$\min\{c_2/c_1 : (c_1, c_2) \in R, c_1 > 0\} = \tfrac{\ell_2}{\ell_2+u_3} \,.$$

Proof Let us suppose that $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$ satisfy the requirements of the lemma. We first consider part **a.** of the lemma which is concerned with the maximal proportion. Hence let us assume that $u_2 + \ell_3 > 0$. For a given choice of $c_1 \in \{\ell_1, \ldots, u_1\}$ with $c_1 > 0$, we first notice that

$$\max\{c_2/c_1 : \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} = \min(c_1 - \ell_3, u_2)/c_1 \,,$$

which is apparent from the definition of R. Because c_1 was arbitrary, this means that

$$\begin{aligned}
&\max\{c_2/c_1 : (c_1, c_2) \in R, c_1 > 0\} \\
&= \max\{\max\{c_2/c_1 : \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} : \\
&\qquad\qquad \max(\ell_1, 1) \leq c_1 \leq u_1\}
\end{aligned}$$

can be further simplified into

$$\begin{aligned}
&\max\{c_2/c_1 : (c_1, c_2) \in R, c_1 > 0\} \\
&= \max\{\min(c_1 - \ell_3, u_2)/c_1 : \max(\ell_1, 1) \leq c_1 \leq u_1\} \,,
\end{aligned} \tag{454}$$

Let us now consider a choice of $c_1 \leq u_2+\ell_3$, $c_1 > 0$. Then $c_1-\ell_3 \leq u_2+\ell_3-\ell_3 = u_2$, i.e. $\min(c_1 - \ell_3, u_2) = c_1 - \ell_3$. Therefore

$$\begin{aligned}
&\tfrac{\min(c_1-\ell_3, u_2)}{c_1} \\
&= \tfrac{c_1-\ell_3}{c_1} \\
&\leq \tfrac{c_1-\ell_3+(u_2+\ell_3-c_1)}{c_1+(u_2+\ell_3-c_1)} && \text{because } u_2 + \ell_3 - c_1 \geq 0 \\
&= \tfrac{u_2}{u_2+\ell_3} \,,
\end{aligned}$$

i.e.

$$\tfrac{\min(c_1-\ell_3, u_2)}{c_1} \leq \tfrac{\min((u_2+\ell_3)-\ell_3, u_2)}{u_2+\ell_3} = \tfrac{u_2}{u_2+\ell_3} \,. \tag{455}$$

For $c_1 \geq u_2 + \ell_3$, we apparently have $c_1 - \ell_3 \geq u_2$ and in turn,

$$\begin{aligned} &\tfrac{\min(c_1-\ell_3,u_2)}{c_1} \\ &= \tfrac{u_2}{c_1} \\ &\leq \tfrac{u_2}{u_2+\ell_3}\,, && \text{because } c_1 \leq u_2+\ell_3 \end{aligned}$$

i.e.

$$\tfrac{\min(c_1-\ell_3,u_2)}{c_1} \leq \tfrac{\min((u_2+\ell_3)-\ell_3,u_2)}{u_2+\ell_3} = \tfrac{u_2}{u_2+\ell_3}\,.$$

Combining this with the above (455), we obtain that

$$\max\{\tfrac{\min(c_1-\ell_3,u_2)}{c_1} : \min(\ell_1,1) \leq c_1 \leq u_1\} = \tfrac{u_2}{u_2+\ell_3}\,,$$

provided that $u_2 + \ell_3 > 0$. Recalling (454), this completes the proof of part **a.**

Next we consider part **b.** of the lemma. Hence suppose that $\ell_2 + u_3 > 0$. We first rewrite

$$\begin{aligned} &\min\{c_2/c_1 : (c_1,c_2) \in R, c_1 > 0\} \\ &= \min\{\min\{c_2/c_1 : \max(c_1-u_3,\ell_2) \leq c_2 \leq \min(c_1-\ell_3,u_2)\} : \\ &\qquad \max(\ell_1,1) \leq c_1 \leq u_1\}\,. \end{aligned}$$

Now we identify the embedded minima. These are obviously attained for the minimal choice of c_2 given c_1, i.e. for $c_2 = \max(c_1-u_3,\ell_2)$. The above equation then becomes

$$\min\{c_2/c_1 : (c_1,c_2) \in R, c_1 > 0\} = \min\{\max(c_1-u_3,\ell_2)/c_1, \min(\ell_1,1) \leq c_1 \leq u_1\}\,. \tag{456}$$

Let us now consider $c_1 \leq \ell_2 + u_3$. Then $c_1 - u_3 \leq \ell_2 + u_3 - u_3 = \ell_2$, i.e. $\max(c_1 - u_3, \ell_2) = \ell_2$. Therefore

$$\begin{aligned} \tfrac{\max(c_1-u_3,\ell_2)}{c_1} &= \tfrac{\ell_2}{c_1} \\ &\geq \tfrac{\ell_2}{\ell_2+u_3}\,, && \text{because } c_1 \leq \ell_2+u_3 \end{aligned}$$

i.e.

$$\tfrac{\max(c_1-u_3,\ell_2)}{c_1} \geq \tfrac{\max((\ell_2+u_3)-u_3,\ell_2)}{\ell_2+u_3} = \tfrac{\ell_2}{\ell_2+u_3}\,. \tag{457}$$

In the remaining case that $c_1 \geq \ell_2 + u_3$, we know that $c_1 - u_3 \geq \ell_2 + u_3 - u_3 = \ell_2$, i.e. $\max(c_1 - u_3, \ell_2) = c_1 - u_3$. Therefore

$$\begin{aligned} &\tfrac{\max(c_1-u_3,\ell_2)}{c_1} \\ &= \tfrac{c_1-u_3}{c_1} \\ &\geq \tfrac{c_1-u_3+(\ell_2+u_3-c_1)}{c_1+(\ell_2+u_3-c_1)} && \text{because } \ell_2+u_3-c_1 < 0 \text{ and } \ell_2+u_3 > 0 \\ &= \tfrac{\ell_2}{\ell_2+u_3} \\ &= \tfrac{\max((\ell_2+u_3)-u_3,\ell_2)}{\ell_2+u_3}\,. \end{aligned}$$

Combining this with (457) proves that

$$\min\{\max(c_1 - u_3, \ell_2)/c_1 : \max(\ell_1, 1) \le c_1 \le u_1\} = \ell_2/(\ell_2 + u_3)\,.$$

The claim of part **b.** is therefore apparent from (456).

Lemma 28
Let $f : \mathbf{I} \longrightarrow \mathbf{I}$ *be a nondecreasing mapping and let* $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3 \in \mathbb{N}$ *be given integers with* $\ell_1 \le u_1$*,* $\ell_2 \le u_2$*,* $\ell_3 \le u_3$*,* $\ell_1 \le u_2 + \ell_3 \le u_1$ *and* $\ell_1 \le \ell_2 + u_3 \le u_1$*. Further let*

$$R = \{(c_1, c_2) : \ell_1 \le c_1 \le u_1, \max(c_1 - u_3, \ell_2) \le c_2 \le \min(c_1 - \ell_3, u_2)\}\,.$$

a. If $u_2 + \ell_3 > 0$*, then*

$$\max\{f(c_2/c_1) : (c_1, c_2) \in R, c_1 > 0\} = f\left(\tfrac{u_2}{u_2+\ell_3}\right)\,.$$

b. If $\ell_2 + u_3 > 0$*, then*

$$\min\{f(c_2/c_1) : (c_1, c_2) \in R, c_1 > 0\} = f\left(\tfrac{\ell_2}{\ell_2+u_3}\right)\,.$$

Proof Both claims of the lemma are immediate from L-27, noting that f is nondecreasing, and hence preserves minima and maxima.

Proof of Theorem 265

Let $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ be a given proportional quantifier on a finite base set. We then know from Def. 166 that Q can be expressed as

$$Q(Y_1, Y_2) = q(c_1, c_2) = \begin{cases} f(c_2/c_1) & : \quad c_1 > 0 \\ v_0 & : \quad c_1 = 0 \end{cases} \tag{458}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where $c_1 = |Y_1|$, $c_2 = |Y_1 \cap Y_2|$, $f : \mathbf{I} \longrightarrow \mathbf{I}$ and $v_0 \in \mathbf{I}$. As already stated in the theorem, we shall also assume that the mapping f is nondecreasing. In the following, I will again utilize the notational conventions introduced in the proof of Th-263, i.e. the subscript γ will usually be dropped, and I will often write $|X|^{\min}$ and $|X|^{\max}$ rather than $|X^{\min}|$ and $|X^{\max}|$, respectively.

Now consider a choice of fuzzy arguments $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ and a choice of the cutting parameter $\gamma \in \mathbf{I}$. We then know from Th-263 and Def. 100 that

$$\top_{Q,X_1,X_2}(\gamma) = \max\{q(c_1, c_2) : (c_1, c_2) \in R\}$$

and

$$\bot_{Q,X_1,X_2}(\gamma) = \max\{q(c_1, c_2) : (c_1, c_2) \in R\}$$

where

$$R = \{(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\}\,,$$

and $\ell_1, \ell_2, \ell_3, u_1, u_2, u_3$ are defined as in Th-263. We notice that $c_2 = |Y_1 \cap Y_2| \leq |Y_1| = c_1$. Therefore $(0, c_2) \in R$ is possible only if $c_2 = 0$ as well. We can therefore decompose R into $R' = R \setminus \{(0,0)\}$, which is known to contain pairs $(c_1, c_2) \in R'$ with $c_1 > 0$ only, and in $R'' = R \cap \{(0,0)\}$. It is then apparent from (458) that

$$\{q(c_1, c_2) : (c_1, c_2) \in R'\} = \{f(c_2/c_1) : (c_1, c_2) \in R'\}$$

and

$$\{q(c_1, c_2) : (c_1, c_2) \in R''\} = \{v_0 : (c_1, c_2) \in R''\} = \begin{cases} \{v_0\} : R'' \neq \varnothing & : \\ \varnothing & : \quad R'' = \varnothing\,. \end{cases}$$

Noticing that $R = R' \mathbin{\dot\cup} R''$, the above renderings of $\top_{Q,X_1,X_2}(\gamma)$ and $\bot_{Q,X_1,X_2}(\gamma)$ can then be split into

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max(\max\{f(c_2/c_1) : (c_1, c_2) \in R'\}, \max\{v_0 : (c_1, c_2) \in R''\}) \\ &= \begin{cases} \max(\max\{f(c_2/c_1) : (c_1, c_2) \in R'\}, v_0) & : \quad R'' \neq \varnothing \\ \max\{f(c_2/c_1) : (c_1, c_2) \in R'\} & : \quad R'' = \varnothing \end{cases} \end{aligned} \tag{459}$$

$$\begin{aligned} \bot_{Q,X_1,X_2}(\gamma) &= \min(\min\{f(c_2/c_1) : (c_1, c_2) \in R'\}, \min\{v_0 : (c_{,}c_2) \in R''\}) \\ &= \begin{cases} \min(\min\{f(c_2/c_1) : (c_1, c_2) \in R'\}, v_0) & : \quad R'' \neq \varnothing \\ \min\{f(c_2/c_1) : (c_1, c_2) \in R'\} & : \quad R'' = \varnothing\,. \end{cases} \end{aligned} \tag{460}$$

Let us also make sure in advance that the preconditions of lemma L-28 are satisfied. We clearly have $\ell_j = X_j^{\min} \leq X_j^{\max} = u_j$ for $j \in \{1, 2, 3\}$ because ${X_j}_\gamma^{\min} \subseteq {X_j}_\gamma^{\max}$. In addition,

$$\begin{aligned} \ell_1 &= |X_1|^{\min} \\ &= |(X_1^{\min} \cap X_2^{\max}) \mathbin{\dot\cup} (X_1^{\min} \cap \neg X_2^{\max})| \\ &= |X_1^{\min} \cap X_2^{\max}| + |X_1^{\min} \cap \neg X_2^{\max}| \\ &= |X_1^{\min} \cap X_2^{\max}| + |X_1 \cap \neg X_2|^{\min} && \text{by Th-61} \\ &\leq |X_1^{\max} \cap X_2^{\max}| + |X_1 \cap \neg X_2|^{\min} && \text{because } X_1^{\min} \subseteq X_1^{\max} \\ &= |X_1 \cap X_2|^{\max} + |X_1 \cap \neg X_2|^{\min} && \text{by Th-61} \\ &= u_2 + \ell_3\,. \end{aligned}$$

Similarly,

$$\begin{aligned}
u_2 + \ell_3 &= |X_1 \cap X_2|^{\max} + |X_1 \cap \neg X_2|^{\min} \\
&= |X_1^{\max} \cap X_2^{\max}| + |X_1^{\min} \cap \neg X_2^{\max}| && \text{by Th-61} \\
&\leq |X_1^{\max} \cap X_2^{\max}| + |X_1^{\max} \cap \neg X_2^{\max}| && \text{because } X_1^{\min} \subseteq X_1^{\max} \\
&= |(X_1^{\max} \cap X_2^{\max}) \mathbin{\dot\cup} (X_1^{\max} \mathbin{\dot\cup} \neg X_2^{\max})| \\
&= |X_1|^{\max} \\
&= u_1 \, .
\end{aligned}$$

Hence

$$\ell_1 \leq u_2 + \ell_3 \leq u_1 \, , \tag{461}$$

as desired. We further observe that

$$\begin{aligned}
\ell_1 &= |X_1|^{\min} \\
&= |(X_1^{\min} \cap X_2^{\max}) \mathbin{\dot\cup} (X_1^{\min} \cap \neg X_2^{\max})| \\
&= |X_1^{\min} \cap X_2^{\max}| + |X_1^{\min} \cap \neg X_2^{\max}| \\
&\leq |X_1^{\max} \cap X_2^{\max}| + |X_1^{\min} \cap \neg X_2^{\max}| && \text{because } X_1^{\min} \leq X_1^{\max} \\
&= |X_1 \cap X_2|^{\max} + |X_1 \cap \neg X_2|^{\min} && \text{by Th-61} \\
&= u_2 + \ell_3
\end{aligned}$$

and

$$\begin{aligned}
u_2 + \ell_3 &= |X_1 \cap X_2|^{\max} + |X_1 \cap \neg X_2|^{\min} \\
&= |X_1^{\max} \cap X_2^{\max}| + |X_1^{\min} \cap \neg X_2^{\max}| && \text{by Th-61} \\
&\leq |X_1^{\max} \cap X_2^{\max}| + |X_1^{\max} \cap \neg X_2^{\max}| && \text{because } X_1^{\min} \subseteq X_1^{\max} \\
&= |(X_1^{\max} \cap X_2^{\max}) \mathbin{\dot\cup} (X_1^{\max} \cap \neg X_2^{\max})| \\
&= |X_1|^{\max} \\
&= u_1 \, ,
\end{aligned}$$

i.e. it also holds that

$$\ell_1 \leq \ell_2 + u_3 \leq u_1 \, , \tag{462}$$

as required for applying L-28.
Based on these preparations, we can now consider the proposed formula for $q^{\min}$. Following the piecewise definition of $q^{\min}$ given in the theorem, the following cases must be discerned.

1. $\ell_1 > 0$.
   We then know from Th-263 that $c_1 \geq \ell_1 > 0$ for all $(c_1, c_2) \in R$, i.e. $R' = R$ and $R'' = \varnothing$. In addition, we know from (462) that $\ell_2 + u_3 \geq \ell_1 > 0$. Therefore

$$\begin{aligned}
\perp_{Q,X_1,X_2}(\gamma) &= \min\{f(c_2/c_1) : (c_1, c_2) \in R'\} && \text{by (460)} \\
&= f(\ell_2/(\ell_2 + u_3)) && \text{by L-28} \\
&= q^{\min}(\ell_1, \ell_2, u_1, u_3) \, ,
\end{aligned}$$

as desired.

2. $\ell_1 = 0$.
   Then $(0,0) \in R$ and $R'' = \{(0,0)\} \neq \varnothing$.
   a. In the case that $\ell_2 + u_3 > 0$, L-28 is applicable. We then obtain

$$\begin{aligned}\bot_{Q,X_1,X_2}(\gamma) &= \min(\min\{f(c_2/c_1) : (c_1,c_2) \in R'\}, v_0) && \text{by (460)}\\ &= \min(f(\ell_2/(\ell_2+u_3)), v_0) && \text{by L-28}\\ &= q^{\min}(\ell_1,\ell_2,u_1,u_3)\,.\end{aligned}$$

   b. If $\ell_2 + u_3 = 0$, is is useful to discern two further subcases.
      i. $u_1 > 0$. In this case, we first notice that $\ell_2 + u_3 = 0$ entails that

$$u_3 = |X_1 \cap \neg X_2|^{\max} = 0 \qquad (463)$$

and

$$\ell_2 = |X_1 \cap X_2|^{\min} = 0\,. \qquad (464)$$

Therefore

$$\begin{aligned}|X_1^{\max} \cap \neg X_2^{\max}| &\le |X_1^{\max} \cap \neg X_2^{\min}| && \text{because } X_2^{\min} \subseteq X_2^{\max}\\ &= |X_1 \cap \neg X_2|^{\max} && \text{by Th-61}\\ &= 0\,, && \text{by (463)}\end{aligned}$$

i.e.

$$|X_1^{\max} \cap \neg X_2^{\max}| = 0\,. \qquad (465)$$

In turn

$$\begin{aligned}u_1 &= |X_1^{\max}|\\ &= |(X_1^{\max} \cap X_2^{\max}) \,\dot\cup\, (X_1^{\max} \cap \neg X_2^{\max})|\\ &= |X_1^{\max} \cap X_2^{\max}| + |X_1^{\max} \cap \neg X_2^{\max}|\\ &= |X_1^{\max} \cap X_2^{\max}| && \text{by (465)}\\ &= |X_1 \cap X_2|^{\max} && \text{by Th-61}\\ &= u_2\,,\end{aligned}$$

i.e.

$$u_2 = u_1\,. \qquad (466)$$

We can further conclude from (463) and $\ell_3 \le u_3$ that

$$\ell_3 = 0 \qquad (467)$$

as well. Consequently, the relation R becomes

$$\begin{aligned} R &= \{(c_1,c_2) : \ell_1 \le c_1 \le u_1, \\ &\qquad \max(c_1 - u_3, \ell_2) \le c_2 \le \min(c_1 - \ell_3, u_2)\} \\ &= \{(c_1,c_2) : \ell_1 \le c_1 \le u_1, \\ &\qquad \max(c_1 - 0, 0) \le c_2 \le \min(c_1 - 0, u_1) && \text{by (463), (464),} \\ & && \text{(467) and (466)} \\ &= \{(c_1,c_2) : \ell_1 \le c_1 \le u_1, c_1 \le c_2 \le c_1\}\,, && \text{because } c_1 \le u_1 \end{aligned}$$

i.e.

$$R = \{(c_1,c_1) : \ell_1 \le c_1 \le u_1\}\,. \tag{468}$$

Due to the fact that $u_1 > 0$, we know that $(u_1,u_1) \in R'$, i.e. $R' \neq \varnothing$. Therefore

$$\begin{aligned} &\min\{f(c_2/c_1) : (c_1,c_2) \in R'\} \\ &= \min\{f(c_1/c_1) : \max(\ell_1, 1) \le c_1 \le u_1\} && \text{by (468)} \\ &= \min\{f(1)\}\,, \end{aligned}$$

and hence

$$\min\{f(c_2/c_1) : (c_1,c_2) \in R'\} = f(1)\,. \tag{469}$$

Let us also recall that $\ell_1 = 0$ entails that $R'' \neq \varnothing$. Therefore

$$\begin{aligned} \bot_{Q,X_1,X_2}(\gamma) &= \min(\min\{f(c_1/c_1) : (c_1,c_2) \in R'\}, v_0) && \text{by (460)} \\ &= \min(f(1), v_0) && \text{by (469)} \\ &= q^{\min}(\ell_1,\ell_2,u_1,u_3)\,. \end{aligned}$$

ii. Finally if $u_1 = 0$, then $R = \{(0,0)\}$ and in turn,

$$\begin{aligned} \bot_{Q,X_1,X_2}(\gamma) &= \min\{q(c_1,c_2) : (c_1,c_2) \in R\} && \text{by Th-263} \\ &= \min\{q(0,0)\} && \text{because } R = \{(0,0)\} \\ &= q(0,0) \\ &= v_0 && \text{by (458)} \\ &= q^{\min}(\ell_1,\ell_2,u_1,u_3)\,. \end{aligned}$$

This completes the proof that indeed $\bot_{Q,X_1,X_2}(\gamma) = q^{\min}(\ell_1,\ell_2,u_1,u_3)$, and we shall address the additional claim of the theorem that $\top_{Q,X_1,X_2}(\gamma) = q^{\max}(\ell_1,\ell_3,u_1,u_2)$. Again, it is convenient to follow the piecewise definition of $q^{\max}$ presented in the theorem, and consequently discern the following cases.

1. $\ell_1 > 0$. Then $c_1 \ge \ell_1 > 0$ for all $(c_1,c_2) \in R$, i.e. $R' = R$ and $R'' = \varnothing$. Therefore

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max\{f(c_2/c_1) : (c_1,c_2) \in R'\} && \text{by (459)} \\ &= f(u_2/(u_2 + \ell_3)) && \text{by L-28} \\ &= q^{\max}(\ell_1,\ell_3,u_1,u_2)\,. \end{aligned}$$

2. $\ell_1 = 0$. Then in particular $\ell_2 = |X_1 \cap X_2|^{\min} \leq |X_1|^{\min} = \ell_1 = 0$, i.e. $\ell_2 = 0$ as well. This shows that $(0,0) \in R$ and therefore $R'' = \{(0,0)\} \neq \varnothing$.

   a. $u_2 + \ell_3 > 0$. In this case, we can profit from L-28, viz

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max(\max\{f(c_2/c_1) : (c_1,c_2) \in R', v_0) && \text{by (459)} \\ &= \max(f(u_2/(u_2+\ell_3)), v_0) && \text{by L-28} \\ &= q^{\max}(\ell_1, \ell_3, u_1, u_2)\,, \end{aligned}$$

   as desired.

   b. In the remaining case that $u_2 + \ell_3 = 0$, we know that

$$u_2 = |X_1 \cap X_2|^{\max} = 0\,, \tag{470}$$

   in particular

$$\ell_2 = 0\,, \tag{471}$$

   because $\ell_2 = |Y_1 \cap Y_2|^{\min} \leq |Y_1 \cap Y_2|^{\max} = u_2 = 0$; and

$$\ell_3 = 0\,. \tag{472}$$

   We shall discern two more subcases.

      i. $u_1 > 0$. In this case we observe that $|X_1^{\max} \cap X_2^{\min}| \leq |X_1^{\max} \cap X_2^{\max}| = |X_1 \cap X_2|^{\max} = u_2 = 0$ by Th-61 and (470), in particular

$$|X_1^{\max} \cap X_2^{\min}| = 0\,. \tag{473}$$

      In turn

$$\begin{aligned} u_1 &= |X_1^{\max}| \\ &= |(X_1^{\max} \cap X_2^{\min}) \,\dot\cup\, (X_1^{\max} \cap \neg X_2^{\min})| \\ &= |X_1^{\max} \cap X_2^{\min}| + |X_1^{\max} \cap \neg X_2^{\min}| \\ &= |X_1^{\max} \cap \neg X_2^{\min}| && \text{by (473)} \\ &= |X_1 \cap \neg X_2|^{\max} && \text{by Th-61} \\ &= u_3\,, \end{aligned}$$

      i.e.

$$u_1 = u_3 \tag{474}$$

      The relation R therefore becomes

$$\begin{aligned} &R \\ &= \{(c_1,c_2) : \ell_1 \leq c_1 \leq u_1, \\ &\quad \max(c_1 - u_3, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3, u_2)\} && \text{by Th-263} \\ &= \{(c_1,c_2) : \ell_1 \leq c_1 \leq u_1, \\ &\quad \max(c_1 - u_1, 0) \leq c_2 \leq \min(c_1 - 0, 0)\} && \text{by (474), (471), (472) and (470)} \\ &= \{(c_1,c_2) : \ell_1 \leq c_1 \leq u_1, 0 \leq c_2 \leq 0\} && \text{because } c_1 - u_1 \leq u_1 - u_1 = 0 \\ &= \{(c_1, 0) : \ell_1 \leq c_1 \leq u_1\}\,. \end{aligned}$$

Therefore

$$\max\{f(c_2/c_1) : (c_1, c_2) \in R'\} = \max\{f(0)\} = f(0)\,, \qquad (475)$$

noticing that $R' \neq \varnothing$ because $(u_1, 0) \in R'$. From this we obtain the desired

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max(\max\{q(c_2/c_1) : (c_1, c_2) \in R', v_0) && \text{by (459)} \\ &= \max(f(0), v_0) && \text{by (475)} \\ &= q^{\max}(\ell_1, \ell_3, u_1, u_2)\,. \end{aligned}$$

ii. $u_1 = 0$. Then $R = \{(0, 0)\}$, i.e.

$$\begin{aligned} \top_{Q,X_1,X_2}(\gamma) &= \max\{q(c_1, c_2) : (c_1, c_2) \in R\} && \text{by Th-263} \\ &= \max\{q(0, 0)\} && \text{because } R = \{(0, 0)\} \\ &= \max\{v_0\} && \text{by (458)} \\ &= v_0 \\ & q^{\max}(\ell_1, \ell_3, u_1, u_2)\,. \end{aligned}$$

This completes proof of the second claim of the theorem, i.e. the equality

$$\top_{Q,X_1,X_2}(\gamma) = q^{\max}(\ell_1, \ell_3, u_1, u_2)$$

is indeed valid.

D.27 Proof of Theorem 266

Let $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ be the given proportional quantifier based on $f : \mathbf{I} \longrightarrow \mathbf{I}$ and $v_0 \in \mathbf{I}$. Further let $V \in \mathcal{P}(E)$, $V \neq \varnothing$ be a crisp subset of E, and $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy subset of E. In addition, let $X' \in \widetilde{\mathcal{P}}(V)$ be defined as stated in the theorem, i.e.

$$\mu_{X'}(e) = \mu_X(e) \qquad (476)$$

for all $e \in V$. Furthermore, we shall suppose that $q' : \{0, \dots, |V|\} \longrightarrow \mathbf{I}$ is defined by $q'(c) = f(c/|V|)$ for all $c \in \{0, \dots, |V|\}$. Based on q', we can then define $Q' : \mathcal{P}(V) \longrightarrow \mathbf{I}$ by $Q'(Y) = q'(|Y|)$ for all $Y \in \mathcal{P}(V)$. In order to prove the claim of the theorem, it is convenient to introduce an intermediate quantifier $Q^* : \mathcal{P}(E) \longrightarrow \mathbf{I}$, which I define as follows,

$$Q^*(Y) = Q(V, Y) \qquad (477)$$

for all $Y \in \mathcal{P}(E)$. In terms of the constructions introduced in Chap. 4, we can recast this

$$Q^* = Q\tau_1 \triangleleft V\,. \qquad (478)$$

It is also instructive to notice that

$$Q^* = Q^* \cap \triangleleft V\,, \qquad (479)$$

because

$$\begin{aligned}
Q^*\cap\triangleleft V(Y) &= Q^*(Y \cap V) && \text{by Def. 33, Def. 34}\\
&= Q(V, Y \cap V) && \text{by (477)}\\
&= Q(V, Y) && \text{because } Q \text{ conservative, see Def. 166}\\
&= Q^*(Y)\,, && \text{by (477)}
\end{aligned}$$

for all $Y \in \mathcal{P}(E)$. In the following, let i denote the inclusion $i : V \longrightarrow E$, i.e. $i(e) = e$ for all $e \in V$. It is then apparent from (476) and Def. 21 that in fact,

$$\hat{\hat{\imath}}(X') = X \cap V\,. \tag{480}$$

In addition, Q' can be defined in terms of Q^* according to the obvious rule $Q'(Y) = Q(V, Y) = Q^*(Y) = Q^*(\widehat{\imath}(Y))$ for all $Y \in \mathcal{P}(V)$, i.e.

$$Q' = Q^* \circ \widehat{\imath}. \tag{481}$$

Hence in every DFS $\mathcal{F}$,

$$\begin{aligned}
\mathcal{F}(Q)(V, X) &= \mathcal{F}(Q)\tau_1\triangleleft V(X) && \text{by Def. 29, Def. 34}\\
&= \mathcal{F}(Q\tau_1\triangleleft V)(X) && \text{by Th-9, Th-15}\\
&= \mathcal{F}(Q^*)(X) && \text{by (478)}\\
&= \mathcal{F}(Q^*\cap\triangleleft V)(X) && \text{by (479)}\\
&= \mathcal{F}(Q^*)\widetilde{\cap}\triangleleft V(X) && \text{by Th-14, Th-15}\\
&= \mathcal{F}(Q^*)(X \cap V) && \text{by Def. 33, Def. 34, } V \text{ crisp}\\
&= \mathcal{F}(Q^*)(\hat{\hat{\imath}}(X')) && \text{by (480)}\\
&= \mathcal{F}(Q^* \circ \widehat{\imath})(X') && \text{by (Z-6), Th-21}\\
&= \mathcal{F}(Q')(X')\,, && \text{by (481)}
\end{aligned}$$

as desired.

D.28 Proof of Theorem 267

Lemma 29
Let $E \neq \varnothing$ be a finite base set, $X \in \widetilde{\mathcal{P}}(E)$. Then

$$\{|Y| : Y \in \mathcal{T}_\gamma(X)\} = \{k : |X_\gamma^{\min}| \leq k \leq |X_\gamma^{\max}|\}\,,$$

for all $\gamma \in \mathbf{I}$.

Proof See [48, L-82, p. 209]

Lemma 30
Let $E \neq \varnothing$ be a finite base set, $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Further let $U, V \in \mathcal{P}(E)$. Then

a. $|U \cap X_\gamma^{\min} \cap V| = |U \cap V| - |U \cap \neg X_\gamma^{\min} \cap V|$;

b. $|U \cap X_\gamma^{\max} \cap V| = |U \cap V| - |U \cap \neg X_\gamma^{\max} \cap V|$;

c. $|U \cap X_\gamma^{\max} \cap \neg X_\gamma^{\min} \cap V| = |U \cap X_\gamma^{\max} \cap V| - |U \cap X_\gamma^{\min} \cap V|$.

Proof Let E be the given finite base set. In order to simplify notation, I will drop the subscript γ and abbreviate $X^{\min} = X_\gamma^{\min}$, $X^{\max} = X_\gamma^{\max}$ etc. Part **a.** and **b.** are trivial: let $Y \in \{X^{\min}, X^{\max}\}$. Then

$$U \cap V = (U \cap Y \cap V) \,\dot{\cup}\, (U \cap \neg Y \cap V)\,.$$

Due to the disjoint union, the total cardinality can be computed by a simple summation, i.e.

$$|U \cap V| = |U \cap Y \cap V| + |U \cap \neg Y \cap V|\,.$$

In particular $|U \cap Y \cap V| = |U \cap V| - |U \cap \neg Y \cap V|$ for $V \in \{X^{\min}, X^{\max}\}$, i.e. part **a.** and **b.** are indeed valid.

Part **c.** of the lemma is straightforward from $X^{\min} \subseteq X^{\max}$. Therefore $X^{\min} = X^{\max} \cap X^{\min}$ and in turn,

$$\begin{aligned} U \cap X^{\max} \cap V &= (U \cap X^{\max} \cap X^{\min} \cap V) \,\dot{\cup}\, (U \cap X^{\max} \cap \neg X^{\min} \cap V) \\ &= (U \cap X^{\min} \cap V) \,\dot{\cup}\, (U \cap X^{\max} \cap \neg X^{\min} \cap V)\,. \end{aligned}$$

The corresponding cardinalities then become

$$|U \cap X^{\max} \cap V| = |U \cap X^{\min} \cap V| + |U \cap X^{\max} \cap \neg X^{\min} \cap V|\,.$$

In particular $|U \cap X^{\max} \cap \neg X^{\min} \cap V| = |U \cap X^{\max} \cap V| - |U \cap X^{\min} \cap V|$, as desired.

Proof of Theorem 267

Let $E \neq \varnothing$ be a finite base set, $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Due to the fact that γ is fixed throughout the proof, I will seize the conventions made in the proof of Th-263 and drop the subscript γ. In particular, I abbreviate $X^{\min} = X_\gamma^{\min}$ and $X^{\max} = X_\gamma^{\max}$. In order to improve readability of the proof, I will further abbreviate $|X|^{\min} = |X^{\min}|$ and $|X|^{\max} = |X^{\max}|$. The coefficients referenced in (234) then become:

$$\begin{aligned} \ell_1 &= |X_1 \cap X_3|^{\min} \\ \ell_2 &= |X_2 \cap X_3|^{\min} \\ \ell_3 &= |X_1 \cap \neg X_2 \cap X_3|^{\min} \\ \ell_4 &= |\neg X_1 \cap X_2 \cap X_3|^{\min} \\ u_1 &= |X_1 \cap X_3|^{\max} \\ u_2 &= |X_2 \cap X_3|^{\max} \\ u_3 &= |X_1 \cap \neg X_2 \cap X_3|^{\max} \\ u_4 &= |\neg X_1 \cap X_2 \cap X_3|^{\max}\,. \end{aligned}$$

Let us now consider some choice of $(Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3)$. We can then define the following cardinality coefficients:

$$a = |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{482}$$
$$b = |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \tag{483}$$
$$c = |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{484}$$
$$d = |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| = |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| \tag{485}$$
$$e = |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{486}$$
$$f = |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \tag{487}$$
$$g = |Y_1 \cap Y_3 \cap X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{488}$$
$$h = |Y_1 \cap Y_3 \cap X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| = |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \tag{489}$$
$$i_1 = |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{490}$$
$$i_2 = |Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{491}$$
$$j_1 = |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \tag{492}$$
$$j_2 = |Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \,. \tag{493}$$

We further let

$$\begin{aligned} k_1 \quad &= |Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \end{aligned} \tag{494}$$

$$\begin{aligned} k_2 \quad &= |\neg Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |\neg Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \end{aligned} \tag{495}$$

and

$$l = |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \,. \tag{496}$$

In addition, we abbreviate

$$\begin{aligned} m_1 \quad &= |Y_1 \cap \neg Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |\neg Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \end{aligned} \tag{497}$$

$$\begin{aligned} m_2 \quad &= |Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \end{aligned} \tag{498}$$

and

$$n = |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \,. \tag{499}$$

Finally, we abbreviate

$$\begin{aligned} o \quad &= |Y_1 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \,, \end{aligned} \tag{500}$$

$$\begin{aligned} p \quad &= |Y_1 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ &= |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ &= |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \,, \end{aligned} \tag{501}$$

for all $(Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3)$, where the equality in (485) is valid because ${X_2}^{\min} \subseteq Y_2$ and ${X_3}^{\min} \subseteq Y_3$. The equality in (489) is valid because ${X_1}^{\min} \subseteq Y_1$ and ${X_3}^{\min} \subseteq Y_3$. The equalities in (494) and (495) are valid because $X_2^{\min} \subseteq Y_2$. The equalities in (497) and (498) are valid because $X_1^{\min} \subseteq Y_1$. The equality (500) is valid because $X_1^{\min} \subseteq Y_1$ and $X_2^{\min} \subseteq Y_2$. Finally, the equalities in (501) are valid because $X_1^{\min} \subseteq Y_1$, $X_2^{\min} \subseteq Y_2$ and $X_3^{\min} \subseteq Y_3$.

As to the possible values of the coefficients a, b, c, d, e, f, g, h, i_1, i_2, j_1, j_2, k_1, k_2, l, m_1, m_2, n, o and p, it is obvious that by choosing $(Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3)$ appropriately, all combinations of values in the following ranges can be assumed (which also exhaust all possible options):

$$
\begin{aligned}
a &\in \{0, \ldots, |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (502)\\
b &\in \{0, \ldots, |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\} && (503)\\
c &\in \{0, \ldots, |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (504)\\
e &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (505)\\
f &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}|\} && (506)\\
g &\in \{0, \ldots, |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (507)\\
i_1 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (508)\\
i_2 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (509)\\
j_1 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\} && (510)\\
j_2 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\} && (511)\\
k_1 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (512)\\
k_2 &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| - k_1\} && (513)\\
l &\in \{0, \ldots, |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\} && (514)\\
m_1 &\in \{0, \ldots, |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\} && (515)\\
m_2 &\in \{0, \ldots, |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| - m_1\} && (516)\\
n &\in \{0, \ldots, |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\} && (517)\\
o &\in \{0, \ldots, |X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\}\,. && (518)
\end{aligned}
$$

Next I will express $c_1 = |Y_1 \cap Y_3|$ and $c_2 = |Y_2 \cap Y_3|$ in terms of these cardinality coefficients. For that purpose, let us first conclude from $X_i^{\min} \subseteq X_i^{\max}$ that $X_i^{\max} = X_i^{\min} \,\dot{\cup}\, (X_i^{\max} \cap \neg X_i^{\min})$, $i \in \{1, 2, 3\}$. Further utilizing the apparent $X_1^{\max} \cap X_3^{\max} \cap \neg X_1^{\max} = \varnothing$ and $X_1^{\max} \cap X_3^{\max} \cap \neg X_3^{\max} = \varnothing$, I now resolve $(X_1 \cap X_3)^{\max} =$

$X_1^{\max} \cap X_2^{\max}$ into a disjoint union of the following components,

$$\begin{aligned}
&X_1^{\max} \cap X_3^{\max} \\
&= (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}) \\
&\ \dot\cup\ (\neg Y_2 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_2 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (X_1^{\min} \cap X_3^{\min})\,.
\end{aligned}$$

Next we observe that $Y_1 \cap Y_3 = (Y_1 \cap Y_3) \cap (X_1^{\max} \cap X_2^{\max})$ because $Y_1 \subseteq X_1^{\max}$ and $Y_3 \subseteq X_3^{\max}$. By expanding $X_1^{\max} \cap X_3^{\max}$ into the above disjoint union, and by utilizing the law of distributivity to move $Y_1 \cap Y_3$ into the disjoint union, we now obtain

$$\begin{aligned}
Y_1 \cap Y_3 &= (Y_1 \cap Y_3) \cap (X_1^{\max} \cap X_3^{\max}) \\
&= (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap \neg Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\ \dot\cup\ (Y_1 \cap Y_3 \cap X_1^{\min} \cap X_3^{\min})\,.
\end{aligned}$$

Due to the fact that $Y_1 \cap Y_3$ is now resolved into a disjoint union of components, we can compute the cardinality of $Y_1 \cap Y_3$ by a summation of the cardinalities of all components, i.e.

$$\begin{aligned}
c_1 &= |Y_1 \cap Y_3| \\
&= |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |Y_1 \cap \neg Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |Y_1 \cap Y_3 \cap X_1^{\min} \cap X_3^{\min}| \,.
\end{aligned}$$

This can be further simplified if we notice that $X_1^{\min} \subseteq Y_1$ and $X_3^{\min} \subseteq Y_3$, i.e. $Y_1 \cap Y_3 \cap X_1^{\min} \cap X_3^{\min} = X_1^{\min} \cap X_3^{\min} = (X_1 \cap X_3)^{\min}$, see Th-61. Now utilizing the coefficients $e, f, g, i_1, j_1, k_1, l, m_1, m_2$ and o defined by (486), (487), (488), (490), (492), (494), (496), (497), (498) and (500), respectively, and referring to the coefficient $\ell_1 = |X_1 \cap X_3|^{\min}$, the above equality can now be rewritten as

$$c_1 = e + f + g + i_1 + j_1 + k_1 + l + m_1 + m_2 + o + \ell_1 \,. \tag{519}$$

In order to express $c_2 = |Y_2 \cap Y_3|$ in terms of the coefficients as well, we can proceed in an analogous way. First we resolve $X_2^{\max} \cap X_3^{\max}$ into a disjoint union

$$\begin{aligned}
& X_2^{\max} \cap X_3^{\max} \\
&= (\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
&\mathbin{\dot\cup} (\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
&\mathbin{\dot\cup} (Y_1 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (\neg Y_1 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
&\mathbin{\dot\cup} (X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
&\mathbin{\dot\cup} (X_2^{\min} \cap X_3^{\min}) \,.
\end{aligned}$$

In the next step, we notice that $Y_2 \cap Y_3 = (Y_2 \cap Y_3) \cap (X_2^{\max} \cap X_3^{\max})$ because $Y_2 \subseteq X_2^{\max}$ and $Y_3 \subseteq X_3^{\max}$. Utilizing distributivity, we hence obtain from the above representation of $X_2^{\max} \cap X_3^{\max}$ that in fact

$$\begin{aligned}
Y_2 \cap Y_3 = {} & (Y_2 \cap Y_3) \cap (X_2^{\max} \cap X_3^{\max}) \\
& = (Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
& \,\dot{\cup}\, (Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (\neg Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}) \\
& \,\dot{\cup}\, (Y_2 \cap Y_3 \cap X_2^{\min} \cap X_3^{\min}) \,.
\end{aligned}$$

Due to the disjoint union, the cardinalities of all involved sets sum up to the total cardinality of $Y_2 \cap Y_3$, i.e.

$$\begin{aligned}
c_2 = {} & |Y_2 \cap Y_3| \\
& = |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap \neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
& + |Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |\neg Y_1 \cap Y_2 \cap Y_3 \cap X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
& + |Y_2 \cap Y_3 \cap X_2^{\min} \cap X_3^{\min}| \,.
\end{aligned}$$

Due to the fact that $X_2^{\min} \subseteq Y_2$ and $X_3^{\min} \subseteq Y_3$, the last summand can be simplified into $|X_2^{\min} \cap X_3^{\min}| = |X_2 \cap X_3|^{\min} = \ell_2$, see Th-61. Now utilizing the coefficients $a, b, c, i_2, j_2, k_1, k_2, m_2, n, o$ defined by (482), (483), (484), (491), (493), (494), (495), (498), (499) and (500), respectively, the above equality can be presented more succinctly, viz

$$\begin{aligned}
c_2 &= |Y_2 \cap Y_3| && (520) \\
&= a + b + c + i_2 + j_2 + k_1 + k_2 + m_2 + n + o + \ell_2 \,. && (521)
\end{aligned}$$

In order to avoid dealing with the individual coefficients, which would be too awkward and unnecessarily bloat the proof, I will now group these coefficients into certain blocks, according to the following considerations. Comparing (519) and (521), we observe that the coefficients e, f, g, i_1, j_1, m_1 and l only affect c_1; that the coefficients a, b, c, i_2, j_2, k_2 and n only affect c_2; and that the coefficients k_1, m_2 and o affect both c_1 and c_2. It is further worth noticing that according to (502)–(518), all coefficients except for k_2 and m_2 can be chosen independently. The range of k_2 and m_2, by contrast, depends on the choice of k_1 and m_1, respectively. In order to discern those coefficients that belong to c_1; to c_2; and to both c_1 and c_2, and also to simplify computations involving the dependent coefficients, it is convenient to introduce these blocks of coefficients:

$$A = e + f + g + i_1 + j_1 + l \tag{522}$$

$$B = k_1 + m_2 + o \tag{523}$$

$$C = a + b + c + i_2 + j_2 + n\,. \tag{524}$$

In terms of the block coefficients, the former representation of c_1 achieved in (519) can now be written as

$$c_1 = \ell_1 + A + m_1 + B\,, \tag{525}$$

and the representation of c_2 stated in (521) reduces to

$$c_2 = \ell_2 + C + k_2 + B\,. \tag{526}$$

Having expressed c_1 and c_2 in terms of the (block) coefficients, I will now identify the possible choices of (c_1, c_2) and express these in terms of the ℓ_r and u_r, $r \in \{1, 2, 3, 4\}$.

Let us first consider the range of c_1. I will treat c_1 as the independent coefficient and express the possible choices of c_2 in dependence on c_2. Due to the fact that c_1 can be chosen unconditionally, the above lemma L-29 can be applied, which asserts that $|(X_1 \cap X_3)_\gamma^{\min}| \leq c_1 \leq |(X_1 \cap X_3)_\gamma^{\max}|$, or more succintly,

$$\ell_1 \leq c_1 \leq u_1\,. \tag{527}$$

In the following, I will assume some choice of c_1 in the above range; I am now interested in identifying the range of c_2 in dependence on the chosen c_1. Let us first consider the minimal choice $c_2^{\min}$ of c_2, given c_1. It is apparent from (524) and the ranges of its component coefficients (502), (503), (504), (509) and (511) and and (517) that we can choose $C = 0$. In addition, it is clear from (513) that we can further assume $k_2 = 0$. Consequently, (526) reduces to

$$c_2^{\min} = \ell_2 + B^{\min}\,, \tag{528}$$

where $B^{\min}$ is the minimal choice of B, given c_1. Recalling (525), it is apparent that

$$B^{\min} = \max(c_1 - \ell_1 - A^{\max} - m_1^{\max}, 0)\,, \tag{529}$$

where $A^{\max}$ is the maximal choice of A, and $m_1^{\max}$ is the maximal choice of m_1 determined by (515). Let us now express

$$\Gamma = \ell_1 + A^{\max} + m_1^{\max} \tag{530}$$

in terms of ℓ_r and u_r. By expanding $\ell_1 = |X_1 \cap X_3|^{\min}$ into

$$\begin{aligned} \ell_1 = |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ + |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,, \end{aligned}$$

by further expanding $A^{\max}$ according to (522) and (505), (506), (507), (508), (510) and (514) into the following sum,

$$\begin{aligned} A^{\max} = {} & |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\ & + |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,, \end{aligned}$$

and finally by expanding $m_1^{\max}$ according to (515) into $m_1^{\max} = |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|$, Γ now becomes

$$\begin{aligned} \Gamma = {} & |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\ & + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ & + |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\ & + |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ & + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ & + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\,. \end{aligned}$$

Let us now notice that

$$\begin{aligned} & X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min} \\ & = X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min} \mathbin{\dot{\cup}} X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min} \end{aligned}$$

and hence

$$\begin{aligned} & |X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & -|X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ & = 0\,. \end{aligned} \tag{531}$$

By substituting (531) into the former representation of Γ and by re-ordering the summands, we now obtain

$$\begin{aligned}\Gamma &= |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ &\quad + |X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ &\quad + |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad + |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\ &\quad + |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\ &\quad + |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &\quad - |X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \,.\end{aligned}$$

Recalling that $X^{\max} = \min X \mathbin{\dot\cup} (X^{\max} \cap \neg X^{\min})$, and $E = (\neg X^{\max}) \mathbin{\dot\cup} (X^{\max} \cap \neg X^{\min}) \mathbin{\dot\cup} X^{\min}$ for all $X \in \widetilde{\mathcal{P}}(E)$, this can be simplified as follows,

$$\Gamma = |X_1 \cap X_3|^{\max} - |X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \,. \tag{532}$$

In order to achieve a further simplification, let us consider the expression $|X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|$, which is part of the right-hand member of (532):

$$\begin{aligned}&|X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &= |X_1^{\max} \cap X_2^{\min} \cap X_3^{\max}| - |X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| && \text{by L-30.c} \\ &= |X_1^{\max} \cap X_3^{\max}| - |X_1^{\max} \cap \neg X_2^{\min} \cap X_3^{\max}| \\ &- (|X_2^{\min} \cap X_3^{\min}| - |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|) && \text{by L-30.a+b} \\ &= |X_1 \cap X_3|^{\max} - |X_1 \cap \neg X_2 \cap X_3|^{\max} \\ &- |X_2 \cap X_3|^{\min} + |\neg X_1 \cap X_2 \cap X_3|^{\min} \,. && \text{by Th-61}\end{aligned}$$

By substituting this into (532), Γ now becomes

$$\begin{aligned}\Gamma &= |X_1 \cap X_3|^{\max} - |X_1 \cap X_3|^{\max} \\ &\quad + |X_1 \cap \neg X_2 \cap X_3|^{\max} + |X_2 \cap X_3|^{\min} - |\neg X_1 \cap X_2 \cap X_3|^{\min} \\ &= |X_2 \cap X_3|^{\min} + |X_1 \cap \neg X_2 \cap X_3|^{\max} - |\neg X_1 \cap X_2 \cap X_3|^{\min} \,,\end{aligned}$$

or more succintly,

$$\Gamma = \ell_2 + u_3 - \ell_4 \,. \tag{533}$$

Therefore

$$\begin{aligned}
c_2^{\min} &= \ell_2 + B^{\min} && \text{by (528)}\\
&= \ell_2 + \max(c_1 - \ell_1 - A^{\max} - m_1^{\max}, 0) && \text{by (529)}\\
&= \max(c_1 + \ell_2 - \ell_1 - A^{\max} - m_1^{\max}, \ell_2) &&\\
&= \max(c_1 + \ell_2 - \Gamma, \ell_2) && \text{by (530)}\\
&= \max(c_1 + \ell_2 - \ell_2 - u_3 + \ell_4, \ell_2)\,, && \text{by (533)}
\end{aligned}$$

which proves the desired

$$c_2^{\min} = \max(c_1 - u_3 + \ell_4, \ell_2)\,. \tag{534}$$

Having expressed $c_2^{\min}$ in terms of the cardinality coefficients ℓ_r and u_r, I will now identify the maximal choice $c_2^{\max}$ of c_2, given $c_1 \in \{\ell_1, \ldots, u_1\}$. For this purpose, it is convenient to split the B block of coefficients again. Hence c_1 and c_2 become

$$c_1 = \ell_1 + A + k_1 + m_1 + m_2 + o \tag{535}$$

$$c_2 = \ell_2 + C + k_1 + k_2 + m_2 + o\,, \tag{536}$$

which is apparent from (523), (525) and (526). In order to maximize c_2, we let $C = C^{\max}$, where $C^{\max}$ is the maximal choice of C, which is determined by (524), (502), (503), (504), (509), (511) and (517). In addition, we observe from (512) and (513) that choosing $k_2 = |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| - k_1$ maximizes $k_1 + k_2$, which then becomes

$$k^{\max} = k_1 + k_2 = |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \tag{537}$$

It is further apparent from (515), (516) and (535) that without loss of generality, we can assume that

$$m_1 = 0\,.$$

This ensures that m_2 can be maximized in (536). These considerations can be summarized as follows,

$$c_1 = \ell_1 + A + k_1 + m_2 + o \tag{538}$$

$$c_2^{\max} = \ell_2 + C^{\max} + k^{\max} + m_2 + o\,. \tag{539}$$

It is apparent from this representation that $m_2 + o$ must be maximized in order for c_2 to achieve its maximum. Recalling (522), and looking up the ranges of the involved coefficients, we can always have $A = 0$. Furthermore, (512) lets us choose $k_1 = 0$ if so desired, i.e. $c_1 = \ell_1 + m_2 + o$ or $m_2 + o = c_1 - \ell_1$. On the other hand, we know that $m_2 + o \leq m_2^{\max} + o^{\max}$, where $m_2^{\max}$ and $o^{\max}$ are the maximal choices of m_2 and o according to (516) and (518), given $m_1 = 0$. This demonstrates that the maximum choice of $m_2 + o$, given c_1, is

$$(m_2 + o)^{\max} = \min(c_1 - \ell_1, m_2^{\max} + o^{\max})$$

Substituting this into (539), we obtain

$$c_2^{\max} = \ell_2 + C^{\max} + k^{\max} + \min(c_1 - \ell_1, m_2^{\max} + o^{\max})\,,$$

or equivalently,

$$c_2^{\max} = \min(c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max}, \ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max})\,. \tag{540}$$

This can be further simplified and expressed in terms of the cardinality coefficients ℓ_r and u_r only. Let us first consider $c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max}$. By utilizing the apparent $E = (\neg X_2^{\max}) \mathbin{\dot\cup} (X_2^{\max} \cap \neg X_2^{\min}) \mathbin{\dot\cup} X_2^{\min}$, I first expand $\ell_1 = |X_1^{\min} \cap X_3^{\min}|$ into

$$\begin{aligned} \ell_1 = |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| + |X_1^{\min} \cap (X_2^{\max} \cap \neg X_2^{\min}) \cap X_3^{\min}| \\ +|X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,. \end{aligned} \tag{541}$$

In a similar way, I then use $E = (\neg X_1^{\max}) \mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min}) \mathbin{\dot\cup} X_1^{\min}$ to expand $\ell_2 = |X_2^{\min} \cap X_3^{\min}|$ into

$$\begin{aligned} \ell_2 = |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| + |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\ +|X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,. \end{aligned} \tag{542}$$

By further expanding $C^{\max}$ according to (524), and assuming the maximal choices of a, b, c, i_2, j_2, and n asserted in (502), (503), (504), (509), (511) and (517), we now obtain that

$$\begin{aligned} C^{\max} \quad &= |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &+|\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ &+|\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\ &+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\ &+|X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\,. \end{aligned} \tag{543}$$

Therefore

$$\begin{aligned}
&c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max} \\
&= c_1 - |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\
&- |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&- |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|
\end{aligned}$$

by (541), (542), (543) and (537). By eliminating summands that cancel out, and by re-ordering the remaining summands, this becomes

$$\begin{aligned}
&c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max} \\
&= c_1 - |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}| \\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\,.
\end{aligned}$$

Noticing that $\neg X_1^{\min} = \neg X_1^{\max} \dot\cup (X_1^{\max} \cap \neg X_1^{\min})$, $X_2^{\max} = X_2^{\min} \dot\cup (X_2^{\max} \cap \neg X_2^{\min})$ and $X_3^{\max} = X_3^{\min} \dot\cup (X_3^{\max} \cap \neg X_3^{\min})$ the above expressions can be further simplified into

$$\begin{aligned}
&c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max} \\
&= c_1 - |X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}| + |\neg X_1^{\min} \cap X_2^{\max} \cap X_3^{\max}| \\
&= c_1 - |X_1 \cap \neg X_2 \cap X_3|^{\min} + |\neg X_1 \cap X_2 \cap X_3|^{\max}\,, \qquad \text{by Th-61}
\end{aligned}$$

or equivalently

$$c_1 - \ell_1 + \ell_2 + C^{\max} + k^{\max} = c_1 - \ell_3 + u_4\,. \tag{544}$$

In order to express $c_2^{\max}$ in terms of the cardinality coefficients ℓ_r and u_r, it remains to be shown how the expression $\ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max}$ in (540) can be reduced to these coefficients. To this end, I first expand ℓ_2, $C^{\max}$, $k^{\max}$, $m_2^{\max}$ and $o^{\max}$ according to (542), (543), (537); (516) given $m_1 = 0$; and (518), respectively. We then obtain

$$\begin{aligned}
&\ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max}\\
&= |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\,.
\end{aligned}$$

By re-ordering the summands, this becomes

$$\begin{aligned}
&\ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max}\\
&= |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |\neg X_1^{\max} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+ |X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\,.
\end{aligned}$$

Noticing that $E = (\neg X_1^{\max}) \mathbin{\dot\cup} (X_1^{\max} \cap \neg X_1^{\min}) \mathbin{\dot\cup} X_1^{\min}$, $X_2^{\max} = X_2^{\min} \mathbin{\dot\cup} (X_2^{\max} \cap \neg X_2^{\min})$ and $X_3^{\max} = X_3^{\min} \mathbin{\dot\cup} (X_3^{\max} \cap \neg X_3^{\min})$, it is therefore apparent that in fact,

$$\begin{aligned} \ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max} &= |X_2^{\max} \cap X_3^{\max}| \\ &= |X_2 \cap X_3|^{\max} \qquad \text{by Th-61} \end{aligned}$$

or equivalently,

$$\ell_2 + C^{\max} + k^{\max} + m_2^{\max} + o^{\max} = u_2\,. \tag{545}$$

By substituting (544) and (545) into (540), we then obtain the desired

$$c_2^{\max} = \min(c_1 - \ell_3 + u_4, u_2)\,. \tag{546}$$

Apparently, all choices of $c \in \{c_2^{\min}, \ldots, c_2^{\max}\}$ can be attained by combinations of the coefficients a to p in their respective ranges, and hence by corresponding choices of $Y_1 \in \mathcal{T}_\gamma(X_1)$, ... , $Y_n \in \mathcal{T}_\gamma(X_n)$ in the three-valued cut ranges at γ. Combining (527), (534) and (546), the possible choices of (c_1, c_2) therefore comprise the following combinations,

$$R = \{(c_1, c_2) : \ell_1 \le c_1 \le u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\,,$$

in conformance with (234). This substantiates that the equality claimed by the theorem is indeed valid.

D.29 Proof of Theorem 268

Let $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{I}$ be a cardinal comparative on a finite base set $E \ne \varnothing$ and let $q : \{0, \ldots, |E|\}^2 \longrightarrow \mathbf{I}$ be the mapping defined by (232). I will prove the equivalence claimed in the theorem by showing that the following two implications are valid.

a. implies b.:

Suppose that Q is nondecreasing in the first and nonincreasing in its second argument. I will first show that q is nondecreasing in its first argument. Hence let $c_1, c_1', c_2 \in \{0, \ldots, |E|\}$ be given with $c_1 \le c_1'$. I abbreviate $p = \max(c_1', c_2)$. Due to the fact that E is finite, there exist pairwise distinct elements $e_1, \ldots, e_m \in E$, $m = |E|$ such that $E = \{e_1, \ldots, e_m\}$. I now define $Y_1 = \{e_1, \ldots, e_{c_1}\}$, $Y_1' = \{e_1, \ldots, e_{c_1'}\}$, $Y_2 = \{e_1, \ldots, e_{c_2}\}$ and $Y_3 = \{e_1, \ldots, e_p\}$. Apparently $Y_1 \subseteq Y_3$, $Y_1' \subseteq Y_3$ and $Y_2 \subseteq Y_3$. Therefore

$$Y_1 \cap Y_3 = Y_1 \tag{547}$$

$$Y_1' \cap Y_3 = Y_1' \tag{548}$$

$$Y_2 \cap Y_3 = Y_2\,. \tag{549}$$

In addition, the sets Y_1, Y_1' and Y_2 obviously have the following cardinalities,

$$|Y_1| = c_1 \tag{550}$$

$$|Y_1'| = c_1' \tag{551}$$

$$|Y_2| = c_2\,. \tag{552}$$

Therefore

$$
\begin{aligned}
q(c_1', c_2) &= q(|Y_1'|, |Y_2|) && \text{by (551), (552)} \\
&= q(|Y_1' \cap Y_3|, |Y_2 \cap Y_3|) && \text{by (548), (549)} \\
&= Q(Y_1', Y_2, Y_3) && \text{by (232)} \\
&\geq Q(Y_1, Y_2, Y_3) && \text{because } Y_1 \subseteq Y_1' \text{ and} \\
& && Q \text{ nondec. in first arg} \\
&= q(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|) && \text{by (232)} \\
&= q(|Y_1|, |Y_2|) && \text{by (547), (549)} \\
&= q(c_1, c_2)\,. && \text{by (550), (552)}
\end{aligned}
$$

This substantiates that q is nondecreasing in its first argument. It remains to be shown that q is nonincreasing in its second argument. Hence let $c_1, c_2, c_2' \in \{0, \ldots, |E|\}$ with $c_2 \leq c_2'$. Based on the above choice of $e_1, \ldots, e_m \in E$ with $E = \{e_1, \ldots, e_m\}$, I now define Y_1, Y_2, Y_2', Y_3 by $Y_1 = \{e_1, \ldots, e_{c_1}\}$, $Y_2 = \{e_1, \ldots, e_{c_2}\}$, $Y_2' = \{e_1, \ldots, e_{c_2'}\}$ and $Y_3 = \{e_1, \ldots, e_p\}$, where $p = \max(c_1, c_2')$. Clearly $Y_1 \subseteq Y_3$, $Y_2 \subseteq Y_3$ and $Y_2' \subseteq Y_3$, i.e.

$$Y_1 \cap Y_3 = Y_1 \tag{553}$$

$$Y_2 \cap Y_3 = Y_2 \tag{554}$$

$$Y_2' \cap Y_3 = Y_2'\,. \tag{555}$$

The cardinalities of Y_1, Y_2 and Y_2' are also obvious,

$$|Y_1| = c_1 \tag{556}$$

$$|Y_2| = c_2 \tag{557}$$

$$|Y_2'| = c_2'\,. \tag{558}$$

Consequently

$$
\begin{aligned}
q(c_1, c_2') &= q(|Y_1|, |Y_2'|) && \text{by (556), (558)} \\
&= q(|Y_1 \cap Y_3|, |Y_2' \cap Y_3|) && \text{by (553), (555)} \\
&= Q(Y_1, Y_2', Y_3) && \text{by (232)} \\
&\leq Q(Y_1, Y_2, Y_3) && \text{because } Y_2 \subseteq Y_2' \text{ and} \\
& && Q \text{ noninc. in 2nd arg} \\
&= q(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|) && \text{by (232)} \\
&= q(|Y_1|, |Y_2|) && \text{by (553), (554)} \\
&= q(c_1, c_2)\,. && \text{by (556), (557)}
\end{aligned}
$$

Hence q is both nondecreasing in the first argument and nonincreasing in the second argument, i.e. condition **b.** is indeed valid.

b. implies a.:
To see this, suppose that q is nondecreasing in the first and nonincreasing in the second

argument. Further let $Y_1, Y_1', Y_2, Y_2', Y_3 \in \mathcal{P}(E)$ such that $Y_1 \subseteq Y_1'$ and $Y_2 \subseteq Y_2'$. Then

$$\begin{aligned} Q(Y_1', Y_2, Y_3) &= q(|Y_1' \cap Y_3|, |Y_2 \cap Y_3|) && \text{by (232)} \\ &\geq q(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|) && \text{because } |Y_1' \cap Y_3| \geq |Y_1 \cap Y_3| \\ &= Q(Y_1, Y_2, Y_3)\,, && \text{by (232)} \end{aligned}$$

i.e. Q is nondecreasing in its first argument. In addition,

$$\begin{aligned} Q(Y_1, Y_2', Y_3) &= q(|Y_1 \cap Y_3|, |Y_2' \cap Y_3|) && \text{by (232)} \\ &\leq q(|Y_1 \cap Y_3|, |Y_2 \cap Y_3|) && \text{because } |Y_2' \cap Y_3| \geq |Y_2 \cap Y_3| \\ &= Q(Y_1, Y_2, Y_3)\,. && \text{by (232)} \end{aligned}$$

Hence Q is both nondecreasing in its first argument and nonincreasing in its second, which proves that condition **a.** is indeed satisfied.

D.30 Proof of Theorem 269

Let $Q : \mathcal{P}(E)^3 \longrightarrow \mathbf{I}$ be a cardinal comparative on a finite base set and let q be the mapping $q : \{0, \ldots, m\}^2 \longrightarrow \mathbf{I}$ defined by (232), where $m = |E|$. Further suppose that Q is nondecreasing in the first and nonincreasing in the second argument. We then know from Th-268 that q is also nondecreasing in the first argument and nonincreasing in the second argument. Now let $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$ be given. Then

$$\begin{aligned} &\top_{Q,X_1,X_2,X_3}(\gamma) \\ &= \max\{Q(Y_1, Y_2, Y_3) : (Y_1, Y_2, Y_2) \in \mathcal{T}_\gamma(X_1, X_2, X_3)\} && \text{by Def. 100, } E \text{ finite} \\ &= \max\{q(Y_1 \cap Y_3, Y_2 \cap Y_3) : (Y_1, Y_2, Y_2) \in \mathcal{T}_\gamma(X_1, X_2, X_3)\} && \text{by (232)} \\ &= \max\{q(c_1, c_2) : \ell_1 \leq c_1 \leq u_1, \\ &\qquad \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\} && \text{by Th-267} \\ &= \max\{\max\{q(c_1, c_2) \\ &\qquad : \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\} \\ &\qquad : \ell_1 \leq c_1 \leq u_1\} \\ &= \max\{q(c_1, \max(c_1 - u_3 + \ell_4, \ell_2)) : \ell_1 \leq c_1 \leq u_1\}\,, \end{aligned}$$

where the last equality is valid because q is nonincreasing in its second argument, and hence achieves its maximum for the minimum choice of c_2. Now turning to

$\bot_{Q,X_1,X_2,X_3}(\gamma)$, we can proceed analogously:

$$\begin{aligned}
&\bot_{Q,X_1,X_2,X_3}(\gamma)\\
&= \min\{Q(Y_1,Y_2,Y_3) : (Y_1,Y_2,Y_2) \in \mathcal{T}_\gamma(X_1,X_2,X_3)\} && \text{by Def. 100, } E \text{ finite}\\
&= \min\{q(Y_1 \cap Y_3, Y_2 \cap Y_3) : (Y_1,Y_2,Y_2) \in \mathcal{T}_\gamma(X_1,X_2,X_3)\} && \text{by (232)}\\
&= \min\{q(c_1,c_2) : \ell_1 \le c_1 \le u_1,\\
&\qquad \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\} && \text{by Th-267}\\
&= \min\{\min\{q(c_1,c_2)\\
&\qquad : \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\\
&\qquad : \ell_1 \le c_1 \le u_1\}\\
&= \min\{q(c_1, \min(c_1 - \ell_3 + u_4, u_2)) : \ell_1 \le c_1 \le u_1\}\,.
\end{aligned}$$

In this case, the last equality is valid because q is nonincreasing in its second argument, and hence achieves its minimum for the maximal choice of c_2, i.e. $\min(c_1 - \ell_3 + u_4, u_2)$.

D.31 Proof of Theorem 270

Let $E \neq \varnothing$ be a finite base set, $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. It is then known from Th-267 that

$$\begin{aligned}
&\{(c_1,c_2) : (Y_1,Y_2,Y_3) \in \mathcal{T}_\gamma(X_1,X_2,X_3), c_1 = |Y_1 \cap Y_3|, c_2 = |Y_2 \cap Y_3|\}\\
&= \{(c_1,c_2) : \ell_1 \le c_1 \le u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\,.
\end{aligned} \tag{559}$$

Let us now consider some $c_1 \in \{\ell_1, \dots, u_1\}$. Noticing that the difference $c_1 - c_2$ is nonincreasing in c_2, we first observe that

$$\begin{aligned}
&\max\{c_1 - c_2 : \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\\
&= c_1 - \max(c_1 - u_3 + \ell_4, \ell_2)\,,
\end{aligned}$$

i.e.

$$\begin{aligned}
&\max\{c_1 - c_2 : \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\\
&= \min(u_3 - \ell_4, c_1 - \ell_2)\,.
\end{aligned} \tag{560}$$

Now abstracting from the choice of $c_1 \in \{\ell_1, \dots, u_1\}$, we obtain that

$$\begin{aligned}
&\max\{c_1 - c_2 : \ell_1 \le c_1 \le u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\\
&= \min(u_3 - \ell_4, u_1 - \ell_2)\,,
\end{aligned} \tag{561}$$

which is apparent from (560) because $\min(u_3 - \ell_4, c_1 - \ell_2)$ is nondecreasing in c_1, and because $c_1 \in \{\ell_1, \dots, u_1\}$. As I will now show, this can be further simplified into

$$\begin{aligned}
&\max\{c_1 - c_2 : \ell_1 \le c_1 \le u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \le c_2 \le \min(c_1 - \ell_3 + u_4, u_2)\}\\
&= u_3 - \ell_4\,.
\end{aligned} \tag{562}$$

To this end, it must be shown that $u_3 - \ell_4 \leq u_1 - \ell_2$ or equivalently,

$$u_1 - \ell_2 - (u_3 - \ell_4) = u_1 - \ell_2 - u_3 + \ell_4 \geq 0\,. \tag{563}$$

Hence let us expand u_1, ℓ_2, u_3, ℓ_4 according to their definition in Th-267, and further utilize Th-61 to rewrite these coefficients as $u_1 = |X_1 \cap X_3|^{\max} = |X_1^{\max} \cap X_3^{\max}|$, $\ell_2 = |X_2 \cap X_3|^{\min} = |X_2^{\min} \cap X_3^{\min}|$, $u_3 = |X_1 \cap \neg X_2 \cap X_3|^{\max} = |X_1^{\max} \cap \neg X_2^{\min} \cap X_3^{\max}|$ and $\ell_4 = |\neg X_1 \cap X_2 \cap X_3|^{\min} = |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|$. These expressions can be further decomposed into a sum of terms $|Z_1 \cap Z_2 \cap Z_3|$, $Z_r \in \{\neg X_r^{\max}, (X_r^{\max} \cap \neg X_r^{\min}), X_r^{\min}\}$ for $r \in \{1, 2, 3\}$, recalling that $E = \neg X_r^{\max} \dot\cup (X_r^{\max} \cap \neg X_r^{\min}) \dot\cup X_r^{\min}$, $X_r^{\max} = (X_r^{\max} \cap \neg X_r^{\min}) \dot\cup X_r^{\min}$ and $\neg X_r^{\min} = \neg X_r^{\max} \dot\cup (X_r^{\max} \cap \neg X_r^{\min})$. We then obtain for u_1,

$$\begin{aligned}
&u_1\\
&= |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,.
\end{aligned} \tag{564}$$

Similarly, ℓ_2 becomes

$$\begin{aligned}
&\ell_2\\
&= |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\min} \cap X_3^{\min}|\,.
\end{aligned} \tag{565}$$

In the case of u_3,

$$\begin{aligned}
&u_3\\
&= |X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\min} \cap \neg X_2^{\max} \cap X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}|\\
&+|X_1^{\min} \cap X_2^{\max} \cap \neg X_2^{\min} \cap X_3^{\min}|\,.
\end{aligned} \tag{566}$$

In the case of ℓ_4, it is sufficient to recall the earlier result,

$$\ell_4 = |\neg X_1^{\max} \cap X_2^{\min} \cap X_3^{\min}|\,. \tag{567}$$

Combining (564)–(567), I now obtain that

$$
\begin{aligned}
&u_1 - \ell_2 - u_3 + \ell_4 \\
&= |X_1^{\max} \cap \neg X_1^{\min} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| + |X_1^{\min} \cap X_3^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&= |X_1^{\max} \cap X_2^{\min} \cap X_3^{\max} \cap \neg X_3^{\min}| \\
&\geq 0\,.
\end{aligned}
$$

This proves that (563), and hence (562), is indeed valid. Recalling (559), I have therefore shown that the maximal difference is

$$
\begin{aligned}
&\max\{c_1 - c_2 : (Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_2), c_1 = |Y_1 \cap Y_3|, c_2 = |Y_2 \cap Y_3|\} \\
&= u_3 - \ell_4\,.
\end{aligned}
\tag{568}
$$

Let us now consider the minimal difference. Hence let $c_1 \in \{\ell_1, \ldots, u_1\}$. Due to the fact that $c_1 - c_2$ is nonincreasing in c_2, its minimum becomes

$$
\begin{aligned}
&\min\{c_1 - c_2 : \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\} \\
&= c_1 - \min(c_1 - \ell_3 + u_4, u_2) \\
&= \max(\ell_3 - u_4, c_1 - u_2)\,.
\end{aligned}
$$

Let us notice that the resulting expression $\max(\ell_3 - u_4, c_1 - u_2)$ is nondecreasing in c_1, and hence becomes minimal when c_1 assumes its minimum, i.e. for $c_1 = \ell_1$. Therefore

$$
\begin{aligned}
&\min\{c_1 - c_2 : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\} \\
&= \max(\ell_3 - u_4, \ell_1 - u_2)\,.
\end{aligned}
$$

As I will now show, this can be further simplified to

$$
\begin{aligned}
&\min\{c_1 - c_2 : \ell_1 \leq c_1 \leq u_1, \max(c_1 - u_3 + \ell_4, \ell_2) \leq c_2 \leq \min(c_1 - \ell_3 + u_4, u_2)\} \\
&= \ell_3 - u_4\,.
\end{aligned}
\tag{569}
$$

To this end, it is sufficient to show that $\ell_3 - u_4 \geq \ell_1 - u_2$ or equivalently, $\ell_3 - u_4 - (\ell_1 - u_2) = \ell_3 - u_4 - \ell_1 + u_2 \geq 0$. Hence let us substitute $X_1' = X_2$, $X_2' = X_1$ and $X_3' = X_3$. It is easily verified that the corresponding coefficients ℓ_r', u_r', $r \in \{1, 2, 3, 4\}$ then become $\ell_1' = \ell_2$, $\ell_2' = \ell_1$, $\ell_3' = \ell_4$, $\ell_4' = \ell_3$, $u_1' = u_2$, $u_2' = u_1$, $u_3' = u_4$ and $u_4' = u_3$. Therefore $\ell_3 - u_4 - \ell_1 + u_2 = u_1' - \ell_2' - u_3' + \ell_4' \geq 0$, see (563). This proves that (569) is valid. The minimal difference can hence be expressed as

$$
\begin{aligned}
&\min\{c_1 - c_2 : (Y_1, Y_2, Y_3) \in \mathcal{T}_\gamma(X_1, X_2, X_3), c_1 = |Y_1 \cap Y_3|, c_2 = |Y_2 \cap Y_3|\} \\
&= \ell_3 - u_4\,,
\end{aligned}
\tag{570}
$$

see (559). It is further apparent from (559) that the range of possible $c_1 - c_2$ comprises all integers d in between the minimal and the maximal difference, and hence between $\ell_3 - u_4$ and $u_3 - \ell_4$, see (568) and (570). This completes the proof that the set of possible $c_1 - c_2$ coincides with $\{\ell_3 - u_4, \ldots, u_3 - \ell_4\}$, as desired.

D.32 Proof of Theorem 271

Lemma 31
If an L-QFM $\mathcal{F}$ satisfies (L-1)*, then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-1).

Proof Suppose that $\mathcal{F}$ satisfies (L-1) and let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier of arity $n \in \{0, 1\}$ on some base set $E \neq \varnothing$. The two possible cases $n = 0$ and $n = 1$ will be considered in turn.

a.: $n = 0$**.** Then Q is a nullary quantifier $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$, and we are interested in the result of the construction described in Def. 175, i.e.

$$\mathcal{F}_R(Q) = \mathcal{F}(Q \circ \mathop{\times}_{i=1}^{0} \widehat{\vartheta}) \circ \mathop{\times}_{i=1}^{0} \hat{\hat{\beta}}\,. \tag{571}$$

Let us recall at this point that for a given set A, A^0 denotes the set of all mappings $f : \varnothing \longrightarrow A$, i.e. $A^0 = \{\varnothing\}$, where '$\varnothing$' is the empty mapping (or empty 'tuple'). Hence $\mathcal{P}(E)^0 = \mathcal{P}(E^1)^0 = \widetilde{\mathcal{P}}(E^1)^0 = \widetilde{\mathcal{P}}(E)^0 = \{\varnothing\}$, and the empty product mappings $\mathop{\times}_{i=1}^{0} \widehat{\vartheta}$ and $\mathop{\times}_{i=1}^{0} \hat{\hat{\beta}}$ which occur in (571) reduce to the identities

$$\mathop{\times}_{i=1}^{0} \widehat{\vartheta} = \mathrm{id}_{\{\varnothing\}} \tag{572}$$

$$\mathop{\times}_{i=1}^{0} \hat{\hat{\beta}} = \mathrm{id}_{\{\varnothing\}}\,, \tag{573}$$

simply because this is the only mapping available. Therefore

$$\begin{aligned}
\mathcal{F}_R(Q) &= \mathcal{F}(Q \circ \mathop{\times}_{i=1}^{0} \widehat{\vartheta}) \circ \mathop{\times}_{i=1}^{0} \hat{\hat{\beta}} && \text{by (571)}\\
&= \mathcal{F}(Q \circ \mathrm{id}_{\{\varnothing\}}) \circ \mathrm{id}_{\{\varnothing\}} && \text{by (572), (573)}\\
&= \mathcal{F}(Q)\\
&= Q\,, && \text{by (L-1)}
\end{aligned}$$

as desired.

b.: $n = 1$**.** In this case, Q is a unary quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$. Now consider a crisp subset $Y \in \mathcal{P}(E)$. Then

$$\begin{aligned}
\mathcal{F}_R(Q)(Y) &= \mathcal{F}(Q \circ \widehat{\vartheta})(\hat{\hat{\beta}}(Y)) && \text{by Def. 175} \\
&= \mathcal{F}(Q \circ \widehat{\vartheta})(\widehat{\beta}(Y)) && \text{by Def. 21 because } Y \text{ crisp} \\
&= (Q \circ \widehat{\vartheta})(\widehat{\beta}(Y)) && \text{by (L-1)} \\
&= Q(\widehat{\vartheta \circ \beta}(Y)) && \text{by compositionality of } \widehat{(\bullet)} \\
&= Q(\widehat{\mathrm{id}}_E(Y)) && \text{because } \vartheta = \beta^{-1} \\
&= Q(Y)\,,
\end{aligned}$$

where the last equality is apparent from the properties of crisp powerset mappings. Because $Y \in \mathcal{P}(E)$ was arbitrary, this proves that indeed $\mathcal{U}(\mathcal{F}_R(Q)) = Q$.

Lemma 32
If an L-QFM $\mathcal{F}$ satisfies (L-2)*, then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-2)*.*

Proof Let $E \neq \varnothing$ be some base set and $e \in E$. Then

$$\begin{aligned}
\mathcal{F}_R(\pi_e) &= \mathcal{F}(\pi_e \circ \widehat{\vartheta}) \circ \hat{\hat{\beta}} && \text{by Def. 175} \\
&= \mathcal{F}(\pi_{(e)}) \circ \hat{\hat{\beta}} && \text{by Def. 9 and } \vartheta \text{ bijection} \\
&= \widetilde{\pi}_{(e)} \circ \hat{\hat{\beta}} && \text{by (L-2)} \\
&= \widetilde{\pi}_e\,,
\end{aligned}$$

where the last equality is apparent from Def. 10, Def. 21 and the fact that β is a bijection.

Lemma 33
If an L-QFM $\mathcal{F}$ satisfies (L-3) *then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-3)*.*

Proof Hence let $\mathcal{F}$ be an L-QFM which satisfies (L-3) and let $\mathcal{F}_R$ be the corresponding ordinary QFM defined by Def. 175. Let me remark in advance that $\mathcal{F}$ and $\mathcal{F}_R$ induce the same fuzzy truth functions and fuzzy set operations; this is apparent from Def. 176. Now consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$. Firstly let us notice that for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\neg\widehat{\vartheta}(Y_n) = \widehat{\vartheta}(\neg Y_n) \tag{574}$$

because ϑ is a bijection, and therefore

$$\begin{aligned}
&Q\widetilde{\square}(\widehat{\vartheta}(Y_1),\dots,\widehat{\vartheta}(Y_n)) \\
&= \widetilde{\neg}\, Q(\widehat{\vartheta}(Y_1),\dots,\widehat{\vartheta}(Y_{n-1}),\neg\widehat{\vartheta}(Y_n)) && \text{by Def. 14} \\
&= \widetilde{\neg}\, Q(\widehat{\vartheta}(Y_1),\dots,\widehat{\vartheta}(Y_{n-1}),\widehat{\vartheta}(\neg Y_n)) && \text{by (574)} \\
&= \widetilde{\neg}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})(Y_1,\dots,Y_{n-1},\neg Y_n) \\
&= (Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})\widetilde{\square}(Y_1,\dots,Y_n)\,, && \text{by Def. 177}
\end{aligned}$$

i.e.

$$Q\widetilde{\square} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta} = Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}\widetilde{\square}\,. \tag{575}$$

Secondly, we notice that for $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
&\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})\widetilde{\square}(\hat{\hat{\beta}}(X_1),\dots,\hat{\hat{\beta}}(X_n)) \\
&= \widetilde{\neg}\,\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1),\dots,\hat{\hat{\beta}}(X_{n-1}),\widetilde{\neg}\,\hat{\hat{\beta}}(X_n)) && \text{by Def. 177} \\
&= \widetilde{\neg}\,\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1),\dots,\hat{\hat{\beta}}(X_{n-1}),\hat{\hat{\beta}}(\widetilde{\neg}\,X_n)) && \text{by Def. 21 and } \beta \text{ bijection} \\
&= \widetilde{\neg}(\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}})(X_1,\dots,X_{n-1},\widetilde{\neg}\,X_n) \\
&= (\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}})\widetilde{\square}(X_1,\dots,X_n)\,, && \text{by Def. 14}
\end{aligned}$$

i.e.

$$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})\widetilde{\square} \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} = \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}}\widetilde{\square}\,. \tag{576}$$

Then

$$\begin{aligned}
&\mathcal{F}_R(Q\widetilde{\square}) \\
&= \mathcal{F}(Q\widetilde{\square} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by Def. 175} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}\widetilde{\square}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by (575)} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})\widetilde{\square} \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by (L-3)} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}}\widetilde{\square} && \text{by (576)} \\
&= \mathcal{F}_R(Q)\widetilde{\square}\,, && \text{by Def. 175}
\end{aligned}$$

as desired.

Lemma 34

Let $f : E \longrightarrow E'$ be an injective mapping, $E, E' \neq \varnothing$. Further suppose that a mapping $g : \mathbf{I}^m \longrightarrow \mathbf{I}$ and fuzzy sets $X_1, \dots, X_m \in \widetilde{\mathcal{P}}(E)$ are given. Let us define a fuzzy set $Z \in \widetilde{\mathcal{P}}(E)$ by

$$\mu_Z(e) = g(\mu_{X_1}(e), \dots, \mu_{X_n}(e))$$

for $e \in E$. Then for all $e' \in E'$,

- *a. if $e' \in \operatorname{Im} f$, then $\mu_{\hat{\hat{f}}(Z)}(e') = g(\mu_{\hat{\hat{f}}(X_1)}(e'), \dots, \mu_{\hat{\hat{f}}(X_m)}(e'))$.*
- *b. if $e' \notin \operatorname{Im} f$, then $\mu_{\hat{\hat{f}}(Z)}(e') = 0$ and $g(\mu_{\hat{\hat{f}}(X_1)}(e'), \dots, \mu_{\hat{\hat{f}}(X_n)}(e')) = g(0, \dots, 0)$.*

Proof I will consider the two cases in turn. If $e' \in \operatorname{Im} f$, then there exists a unique choice of $e = f^{-1}(e')$ with $e' = f(e)$ because f is injective. From Def. 21, we know that

$$\mu_{\hat{\hat{f}}(X_i)}(e') = \mu_{X_i}(e)$$

and

$$\mu_{\hat{\hat{f}}(Z)}(e') = \mu_Z(e) = g(\mu_{X_1}(e), \dots, \mu_{X_n}(e))$$

Hence indeed

$$g(\mu_{\hat{\hat{f}}(X_1)}(e'), \dots, \mu_{\hat{\hat{f}}(X_n)}(e')) = g(\mu_{X_1}(e), \dots, \mu_{X_n}(e)) = \mu_Z(e) = \mu_{\hat{\hat{f}}(Z)}(e)\,.$$

This proves case a. In the second case, it is apparent from Def. 21 that $\mu_{\hat{\hat{f}}(X_i)}(e') = 0$ for all $i \in \{1, \dots, m\}$, i.e. $g(\mu_{\hat{\hat{f}}(X_1)}(e'), \dots, \mu_{\hat{\hat{f}}(X_n)}(e')) = g(0, \dots, 0)$. The remaining claim that $\mu_{\hat{\hat{f}}(Z)}(e') = 0$ is also apparent from $e' \notin \operatorname{Im} f$ and the definition of the standard extension principle.

Lemma 35

If an L-QFM $\mathcal{F}$ satisfies (L-4)*, then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-4)*.*

Proof Hence let $\mathcal{F}$ be an L-QFM which satisfies (L-4) and let $\mathcal{F}_R$ be the corresponding QFM defined by Def. 175. Again, we should notice that $\mathcal{F}$ and $\mathcal{F}_R$ induce the same fuzzy truth functions and fuzzy set operations, see Def. 176. Now let us consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$. First of all, we notice that

$$Q\cup \circ \overset{n+1}{\underset{i=1}{\times}} \widehat{\vartheta} = Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta}\cup\,. \tag{577}$$

To see this, consider $Y_1, \dots, Y_n, Y_{n+1} \in \mathcal{P}(E)$. Then

$$\widehat{\vartheta}(Y_n) \cup \widehat{\vartheta}(Y_{n+1}) = \widehat{\vartheta}(Y_n \cup Y_{n+1})\,, \tag{578}$$

which is apparent from the properties of powerset mappings, and in turn,

$$\begin{aligned}
&Q\cup(\widehat{\vartheta}(Y_1), \ldots, \widehat{\vartheta}(Y_{n+1}))\\
&= Q(\widehat{\vartheta}(Y_1), \ldots, \widehat{\vartheta}(Y_{n-1}), \widehat{\vartheta}(Y_n) \cup \widehat{\vartheta}(Y_{n+1})) && \text{by Def. 15}\\
&= Q(\widehat{\vartheta}(Y_1), \ldots, \widehat{\vartheta}(Y_{n-1}), \widehat{\vartheta}(Y_n \cup Y_{n+1})) && \text{by (578)}\\
&= (Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta})(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})\\
&= (Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta})\cup(Y_1, \ldots, Y_{n+1})\,. && \text{by Def. 178}
\end{aligned}$$

Now let $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$ be given. Noticing from (236) that β is a bijection, we first apply L-34.a and conclude that

$$\hat{\hat{\beta}}(X_n \mathbin{\widetilde{\cup}} X_{n+1}) = \hat{\hat{\beta}}(X_n) \mathbin{\widetilde{\cup}} \hat{\hat{\beta}}(X_{n+1})\,. \tag{579}$$

Therefore

$$\begin{aligned}
&\mathcal{F}_R(Q\cup)(X_1, \ldots, X_{n+1})\\
&= \mathcal{F}(Q\cup \circ \overset{n+1}{\underset{i=1}{\times}})(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_{n+1})) && \text{by Def. 175}\\
&= \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta}\cup)(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_{n+1})) && \text{by (577)}\\
&= \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_{n-1}), \hat{\hat{\beta}}(X_n) \mathbin{\widetilde{\cup}} \hat{\hat{\beta}}(X_{n+1})) && \text{by (L-4)}\\
&= \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_{n-1}), \hat{\hat{\beta}}(X_n \mathbin{\widetilde{\cup}} X_{n+1})) && \text{by (579)}\\
&= \mathcal{F}_R(Q)(X_1, \ldots, X_{n-1}, X_n \mathbin{\widetilde{\cup}} X_{n+1}) && \text{by Def. 175}\\
&= \mathcal{F}_R(Q)\widetilde{\cup}(X_1, \ldots, X_{n+1})\,, && \text{by Def. 15}
\end{aligned}$$

i.e. $\mathcal{F}_R$ indeed satisfies (Z-4).

Lemma 36

If an L-QFM $\mathcal{F}$ satisfies (L-5)*, then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-5)*.*

Proof To see this, consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which is nonincreasing in its n-th argument, $n > 0$. According to Def. 175, $\mathcal{F}_R(Q)$ is given by

$$\mathcal{F}_R(Q) = \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta}) \circ \overset{n}{\underset{i=1}{\times}} \widehat{\beta}\,. \tag{580}$$

In order to ascertain that $\mathcal{F}_R(Q)$ is nonincreasing in its last argument, we first notice that quantifier $Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta}$ is nonincreasing in its last argument. This is apparent if we consider $Y_1, \ldots, Y_n, Y_n' \in \mathcal{P}(E^1)$ with $Y_n \subseteq Y_n'$. By the monotonicity of powerset mappings, then, we obtain

$$\widehat{\vartheta}(Y_n) \subseteq \widehat{\vartheta}(Y_n')\,.$$

We can hence conclude from the fact that Q is nonincreasing in its last argument that

$$Q(\widehat{\vartheta}(Y_1), \dots, \widehat{\vartheta}(Y_n)) \geq Q(\widehat{\vartheta}(Y_1), \dots, \widehat{\vartheta}(Y_{n-1}), \widehat{\vartheta}(Y'_n)) \,,$$

i.e. $Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}$ is indeed nonincreasing in its last argument. Now consider a choice of fuzzy arguments $X_1, \dots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ with $X_n \subseteq X'_n$. By the monotonicity of the standard extension principle, we have

$$\hat{\hat{\beta}}(X_n) \subseteq \hat{\hat{\beta}}(X'_n) \,. \tag{581}$$

Therefore

$$\begin{aligned}
&\mathcal{F}_R(Q)(X_1, \dots, X_n) \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1), \dots, \hat{\hat{\beta}}(X_n)) && \text{by Def. 175} \\
&\geq \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\beta}}(X_1), \dots, \hat{\hat{\beta}}(X_{n-1}), \hat{\hat{\beta}}(X'_n)) && \text{by (L-5), (581)} \\
&= \mathcal{F}_R(Q)(X_1, \dots, X_{n-1}, X'_n) \,. && \text{by Def. 175}
\end{aligned}$$

Hence $\mathcal{F}_R(Q)$ is nonincreasing in its n-th argument. Because Q was an arbitrary quantifier nonincreasing in its last argument, this proves that $\mathcal{F}_R$ satisfies (Z-5).

Lemma 37

Let $\mathcal{F}$ be an L-QFM which satisfies (L-2)*, and $f : E' \longrightarrow E$ a bijection, where E, E' are nonempty sets. Then*

$$\widehat{\mathcal{F}}_R(f) = \hat{\hat{f}} \,.$$

Proof We already know from L-32 that $\mathcal{F}_R$ satisfies (Z-2). Now consider $X \in \widetilde{\mathcal{P}}(E')$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}_R(f)(X)}(e) &= \mathcal{F}_R(\pi_e \circ \widehat{f})(X) && \text{by Def. 22} \\
&= \mathcal{F}_R(\pi_{f^{-1}(e)})(X) && \text{apparent from Def. 9 and } f \text{ bijection} \\
&= \widetilde{\pi}_{f^{-1}(e)}(X) && \text{by (Z-2)} \\
&= \mu_X(f^{-1}(e)) && \text{by Def. 10} \\
&= \mu_{\hat{\hat{f}}(X)}(e)
\end{aligned}$$

where the last equality is apparent from Def. 21 noticing that f is a bijection.

Lemma 38

Let $f : E \longrightarrow E'$ and $g : E' \longrightarrow E''$ be mappings, $E, E', E'' \neq \varnothing$. Further let $\mathcal{F}$ be an L-QFM which satisfies (L-2) *and* (L-6)*. Then*

$$\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f) \,.$$

Proof Let $\mathcal{F}$ be an L-QFM which satisfies (L-2) and (L-6), and suppose that $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ are given mappings, where E, E', E'' are arbitrary nonempty sets. Let me first clarify some notation: I will write $\beta : E \longrightarrow E^1$, $\vartheta : E^1 \longrightarrow E$ for the mappings defined by (236) and (237) when referring to the base set E; an I will use symbols $\beta' : E' \longrightarrow {E'}^1$, $\vartheta' : {E'}^1 \longrightarrow E'$ when referring to the base set E'.
Now let $X \in \widetilde{\mathcal{P}}(E)$. I first prove an auxiliary equality

$$\widehat{\mathcal{F}}(f) = \hat{\hat{\vartheta}} \circ \widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta) \circ \hat{\hat{\beta}}\,. \tag{582}$$

Hence let $e' \in E'$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(e') &= \mu_{\widehat{\mathcal{F}}_R(f)(X)}(e') && \text{by Def. 180}\\
&= \mathcal{F}_R(\pi_{e'} \circ \widehat{f})(X) && \text{by Def. 22}\\
&= \mathcal{F}(\pi_{e'} \circ \widehat{f} \circ \widehat{\vartheta})(\hat{\hat{\beta}}(X)) && \text{by Def. 175}\\
&= \mathcal{F}(\pi_{e'} \circ \widehat{\vartheta' \circ \beta'} \circ \widehat{f} \circ \widehat{\vartheta})(\hat{\hat{\beta}}(X)) && \text{because } \vartheta' = {\beta'}^{-1}\\
&= \mathcal{F}(\pi_{e'} \circ \widehat{\vartheta}' \circ \widehat{\beta' \circ f \circ \vartheta})(\hat{\hat{\beta}}(X)) && \text{by compositionality of } \widehat{(\bullet)}\\
&= \mathcal{F}(\pi_{(e')} \circ \widehat{\beta' \circ f \circ \vartheta})(\hat{\hat{\beta}}(X)) && \text{see Def. 9, } \vartheta' \text{ bijection}\\
&= \mathcal{F}(\pi_{(e')})(\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))) && \text{by (L-6)}\\
&= \widetilde{\pi}_{(e')}(\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))) && \text{by (L-2)}\\
&= \mu_{\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))}((e')) && \text{by Def. 10}\\
&= \mu_{\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))}({\vartheta'}^{-1}(e')) && \text{by (237)}\\
&= \mu_{\hat{\hat{\vartheta}}'(\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X)))}(e') && \text{by Def. 21 and } \vartheta' \text{ bijection}\,,
\end{aligned}$$

i.e. (582) is indeed valid. Based on (582), we can now proceed as follows. Consider some $e'' \in E''$. Then

$$\begin{aligned}
&\mu_{\widehat{\mathcal{F}}(g\circ f)(X)}(e'') && \\
&= \mu_{\widehat{\mathcal{F}}_R(g\circ f)(X)}(e'') && \text{by Def. 180} \\
&= \mathcal{F}_R(\pi_{e''} \circ \widehat{g \circ f})(X) && \text{by Def. 22} \\
&= \mathcal{F}_R(\pi_{e''} \circ \widehat{g} \circ \widehat{f})(X) && \text{by compositionality of } \widehat{(\bullet)} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{f} \circ \widehat{\vartheta})(\hat{\hat{\beta}}(X)) && \text{by Def. 175} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{\vartheta}' \circ \widehat{\beta}' \circ \widehat{f} \circ \widehat{\vartheta})(\hat{\hat{\beta}}(X)) && \text{because } \vartheta' = {\beta'}^{-1} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{\vartheta}' \circ \beta\widehat{' \circ f \circ}\vartheta)(\hat{\hat{\beta}}(X)) && \text{by compositionality of } \widehat{(\bullet)} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{\vartheta}')(\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))) && \text{by (L-6)} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{\vartheta}')(\hat{\hat{\beta}}'(\hat{\hat{\vartheta}}'(\widehat{\mathcal{F}}(\beta' \circ f \circ \vartheta)(\hat{\hat{\beta}}(X))))) && \text{because } \beta' = {\vartheta'}^{-1} \\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{\vartheta}')(\hat{\hat{\beta}}'(\widehat{\mathcal{F}}(f)(X))) && \text{by (582)} \\
&= \mathcal{F}_R(\pi_{e''} \circ \widehat{g})(\widehat{\mathcal{F}}(f)(X)) && \text{by Def. 175} \\
&= \mu_{\widehat{\mathcal{F}}_R(g)(\widehat{\mathcal{F}}(f)(X))}(e'') && \text{by Def. 22} \\
&= \mu_{\widehat{\mathcal{F}}(g)(\widehat{\mathcal{F}}(f)(X))}(e'')\,. && \text{by Def. 180}
\end{aligned}$$

Because $e'' \in E''$ was arbitrary, this proves that $\widehat{\mathcal{F}}(g \circ f)(X) = \widehat{\mathcal{F}}(g)(\widehat{\mathcal{F}}(f)(X))$. Noticing that $X \in \widetilde{\mathcal{P}}(E)$ was also arbitrary, we hence obtain $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$, as desired.

Lemma 39
If an L-QFM $\mathcal{F}$ satisfies (L-2) *and* (L-6), *then the corresponding ordinary QFM $\mathcal{F}_R$ defined by Def. 175 satisfies* (Z-6).

Proof Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and let $f_1, \ldots, f_n : E' \longrightarrow E$ be given mappings, $E' \neq \varnothing$. I will use symbols $\beta : E \longrightarrow E^1$, $\vartheta : E^1 \longrightarrow E$, $\beta' : E' \longrightarrow {E'}^1$ and $\vartheta' : {E'}^1 \longrightarrow E$ in order to distinguish the mappings obtained from (236) and (237) for the two considered base sets, E and E'. We can then proceed as follows:

$$\begin{aligned}
&\mathcal{F}_R(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i) && \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by Def. 175} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \vartheta' \circ \widehat{\beta' \circ f_i} \circ \vartheta) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{because } \vartheta' \circ \beta' = \mathrm{id}_{E'} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}' \circ \beta' \widehat{\circ f_i \circ} \vartheta) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by compositionality of } \widehat{(\bullet)} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} (\widehat{\mathcal{F}}(\beta' \circ f_i \circ \vartheta) \circ \hat{\hat{\beta}}) && \text{by (L-6)} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} (\widehat{\mathcal{F}}(\beta' \circ f_i \circ \vartheta) \circ \widehat{\mathcal{F}}(\beta)) && \text{see L-37, } \beta \text{ bijection} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(\beta' \circ f_i \circ \vartheta \circ \beta) && \text{by L-38} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(\beta' \circ f_i) && \text{because } \vartheta = \beta^{-1} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(\beta') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) && \text{by L-38} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}}' \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) && \text{see L-37, } \beta' \text{ bijection} \\
&= \mathcal{F}_R(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) && \text{by Def. 175} \\
&= \mathcal{F}_R(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}_R(f_i)\,, && \text{by Def. 180}
\end{aligned}$$

i.e. $\mathcal{F}_R$ indeed satisfies (Z-6).

Proof of Theorem 271

Let $\mathcal{F}$ be an L-DFS and $\mathcal{F}_R$ the corresponding ordinary QFM defined by Def. 175. It has already been shown in the series of lemmata L-31, L-32, L-33, L-35, L-36 and L-39 that $\mathcal{F}_R$ satisfies (Z-1), (Z-2), (Z-3), (Z-4), (Z-5) and (Z-6), respectively. Hence $\mathcal{F}_R$ is a DFS by Def. 24.

D.33 Proof of Theorem 272

Lemma 40
Let $\mathcal{F}$ be a QFM which satisfies (Z-2) *and* (Z-6)*. Then for all mappings $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ where E, E', E'' are nonempty sets:*

a. $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$*;*

b. if f is an injection, then $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mu_{\hat{\hat{f}}(X)}(e') = \mu_X(f^{-1}(e'))$ *for all*

$e' \in \operatorname{Im} f$*;*

c. if f is a bijection, then $\widehat{\mathcal{F}}(f) = \hat{\hat{f}}$;

d. in particular $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$.

Proof Let us consider parts **a.–d.** in turn.

a.: Consider $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ where E, E', E'' are arbitrary nonempty sets. Then for all $e'' \in E''$ and $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(g \circ f)(X)}(e'') &= \mathcal{F}(\pi_{e''} \circ \widehat{g \circ f})(X) && \text{by Def. 22} && (583)\\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g} \circ \widehat{f})(X) && \text{by compositionality of } \widehat{(\bullet)} && (584)\\
&= \mathcal{F}(\pi_{e''} \circ \widehat{g})(\widehat{\mathcal{F}}(f)(X)) && \text{by (Z-6)} && (585)\\
&= \mu_{\widehat{\mathcal{F}}(g)(\widehat{\mathcal{F}}(f)(X))}(e'')\,. && \text{by Def. 22} && (586)
\end{aligned}$$

Because $e'' \in E''$ and $X \in \widetilde{\mathcal{P}}(E)$ were arbitrarily chosen, this demonstrates that $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$, as desired.

b.: Suppose that $f : E \longrightarrow E'$, $E, E' \neq \varnothing$, is an injection. Further let $e' \in \operatorname{Im} f$. Then there exists a unique choice of $f^{-1}(e') \in E$ such that $f(f^{-1}(e')) = e'$. In addition,

$$\begin{aligned}
&\pi_{e'}(\widehat{f}(Y))\\
&= \begin{cases} 1 & : \quad e' \in \widehat{f}(Y) \\ 0 & : \quad \text{else} \end{cases} && \text{by Def. 9}\\
&= \begin{cases} 1 & : \quad f^{-1}(e') \cap Y \neq \varnothing \\ 0 & : \quad \text{else} \end{cases} && \text{obvious from Def. 19; } f^{-1} \text{ refers to the inverse image mapping } f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)\\
&= \begin{cases} 1 & : \quad f^{-1}(e') \in Y \\ 0 & : \quad \text{else} \end{cases} && \text{because } f \text{ is injective; here } f^{-1}(e') \text{ denotes the unique element of } E \text{ with } f(f^{-1}(e')) = e'\\
&= \pi_{f^{-1}(e')}(Y) && \text{by Def. 9}
\end{aligned}$$

for all $Y \in \mathcal{P}(E)$, i.e.

$$\pi_{e'} \circ \widehat{f} = \pi_{f^{-1}(e')}\,. \tag{587}$$

Hence for all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(e') &= \mathcal{F}(\pi_{e'} \circ \widehat{f})(X) && \text{by Def. 22}\\
&= \mathcal{F}(\pi_{f^{-1}(e')})(X) && \text{by (587)}\\
&= \widetilde{\pi}_{f^{-1}(e')}(X) && \text{by (Z-2)}\\
&= \mu_X(f^{-1}(e')) && \text{by Def. 10}\\
&= \mu_{\hat{\hat{f}}(X)}(e')\,,
\end{aligned}$$

where the last equality ist apparent from the definition of the standard extension principle, Def. 21.

c.: This is only a special case of **b.**: if f is a bijection, then *all* $e' \in E'$ are members of $\mathrm{Im}\, f$; therefore case **b.** applies unconditionally and results in

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mu_{\hat{\hat{f}}(X)}(e')$$

for all $e' \in E'$, i.e. $\widehat{\mathcal{F}}(f) = \hat{\hat{f}}$, as desired.

d.: This is apparent from **c.** noticing that id_E is a bijection, which the standard extension principle maps to $\hat{\hat{\mathrm{id}}}_E = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$.

Lemma 41

Let Q be a semi-fuzzy L-quantifier of type $t \in \mathbb{N}^n$, $n \in \mathbb{N}$ on some base set $E \neq \varnothing$. If $t = (m, \ldots, m)$ for some $m \in \mathbb{N}$, then the mappings ζ_i and κ_i defined by (241) *and* (242) *reduce to identities*

$$\zeta_i = \kappa_i = \mathrm{id}_{E^m}\,.$$

In particular

$$\widehat{\zeta}_i = \widehat{\kappa}_i = \mathrm{id}_{\mathcal{P}(E^1)} \qquad \textit{and} \qquad \hat{\hat{\zeta}}_i = \hat{\hat{\kappa}}_i = \mathrm{id}_{\widetilde{\mathcal{P}}(E^1)}$$

for all $i \in \{1, \ \ldots, \ n\}$.
The quantifier Q' defined by (244) *reduces to*

$$Q' = Q\,.$$

Proof In this case, $E^{t_i} = E^m$; therefore

$$\zeta_i(e_1, \ldots, e_m) = (e_1, \ldots, e_m) = \mathrm{id}_{E^m}(e_1, \ldots, e_m)$$

for all $(e_1, \ldots, e_m) \in E^m$ by (241), i.e. $\zeta_i = \mathrm{id}_{E^1}$ for all $i \in \{1, \ \ldots, \ n\}$. Similarly, we obtain from (242) that

$$\kappa_i(e_1, \ldots, e_m) = (e_1, \ldots, e_m) = \mathrm{id}_{E^m}(e_1, \ldots, e_m)$$

for all $(e_1, \ldots, e_m) \in E^m$ in this case, i.e. $\kappa_i = \mathrm{id}_{E^m}$, $i \in \{1, \ \ldots, \ n\}$. It is then apparent from the known properties of crisp powerset mappings that indeed $\widehat{\zeta}_i = \widehat{\kappa}_i = \mathrm{id}_{\mathcal{P}(E^m)}$, see Def. 19. Similarly, it is obvious from properties of the standard extension principle introduced in Def. 21 that $\hat{\hat{\zeta}}_i = \hat{\hat{\kappa}}_i = \mathrm{id}_{\widetilde{\mathcal{P}}(E^m)}$. Now turning to Q', we simply observe that

$$\begin{aligned}
&Q'(Y_1, \ldots, Y_n) \\
&= Q(\widehat{\kappa}_1(Y_1 \cap \mathrm{Im}\,\zeta_1), \ldots, \widehat{\kappa}_n(Y_n \cap \mathrm{Im}\,\zeta_n)) && \text{by (244)} \\
&= Q(\widehat{\mathrm{id}}_{E^m}(Y_1 \cap E^m), \ldots, \widehat{\mathrm{id}}_{E^m}(Y_n \cap E^m)) && \text{(as shown above)} \\
&= Q(\mathrm{id}_{\mathcal{P}(E^m)}(Y_1), \ldots, \mathrm{id}_{\mathcal{P}(E^m)}(Y_n)) && \text{by Def. 19} \\
&= Q(Y_1, \ldots, Y_n)
\end{aligned}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E^m)$, i.e. $Q' = Q$, which completes the proof of the lemma.

Lemma 42
If a QFM $\mathcal{F}$ satisfies (Z-2) *and* (Z-6)*, then* $\mathcal{F}_{LR} = \mathcal{F}$.

Proof To see this, consider $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Then

$$\begin{aligned} \mathcal{F}_{LR}(Q) &= \mathcal{F}_L(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}) \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by Def. 175} \\ &= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\zeta}}_i \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}}\,, && \text{by Def. 182} \end{aligned}$$

i.e.

$$\mathcal{F}_{LR}(Q) = \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\zeta}}_i \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}}\,, \tag{588}$$

where $Q' : \mathcal{P}(E^1)^n \longrightarrow \mathbf{I}$ is defined in terms of $Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}$ according to (244). Noticing that $Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}$ is a semi-fuzzy L-quantifier on E of type $t = \langle 1, \ldots, 1 \rangle$, we can apply L-41 and conclude that $\zeta_i : E^1 \longrightarrow E^1$ and $\kappa_i : E^1 \longrightarrow E^1$ reduce to identities

$$\zeta_i = \kappa_i = \mathrm{id}_{E^1} \tag{589}$$

and Q' reduces to

$$Q' = Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta}\,. \tag{590}$$

Therefore

$$\begin{aligned} &\mathcal{F}_{LR}(Q) \\ &= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\zeta}}_i \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by (588)} \\ &= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \mathrm{id}_{\widetilde{\mathcal{P}}(E^1)} \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} && \text{by (589) and Def. 21} \\ &= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{\beta}} \\ &= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(\beta) && \text{by L-40.c and } \beta \text{ bijection} \\ &= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\beta}) && \text{by (590) and (Z-6)} \\ &= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\vartheta} \circ \widehat{\beta})\,, \end{aligned}$$

due to the compositionality of product mappings. It is then apparent from $\vartheta \circ \beta = \mathrm{id}_E$ and the compositionality of powerset mappings that $\widehat{\vartheta} \circ \widehat{\beta} = \widehat{\vartheta \circ \beta} = \widehat{\mathrm{id}}_E = \mathrm{id}_{\mathcal{P}(E)}$.

We then obtain

$$\begin{aligned}\mathcal{F}_{LR}(Q) &= \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{\vartheta} \circ \widehat{\beta}) \\ &= \mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \mathrm{id}_{\mathcal{P}(E)}) \\ &= \mathcal{F}(Q)\,.\end{aligned}$$

Because Q was arbitrary, this proves the desired $\mathcal{F} = \mathcal{F}_{LR}$.

Proof of Theorem 272

The theorem is a corollary to L-42, noticing that every DFS $\mathcal{F}$ satisfies (Z-2) and (Z-6) by definition, see Def. 24.

D.34 Proof of Theorem 273

Lemma 43
If a QFM $\mathcal{F}$ satisfies (Z-1), *then the corresponding L-QFM $\mathcal{F}_L$ satisfies* (L-1).

Proof Hence let $\mathcal{F}$ be a QFM which satisfies (Z-1) and let Q be a semi-fuzzy L-quantifier of type $t \in \{\langle\rangle, \langle 1\rangle\}$ on some base set $E \neq \varnothing$. I will treat the cases $t = \langle\rangle$ and $t = \langle 1\rangle$ separately.

Case a.: $t = \langle\rangle$, i.e. Q is a quantifier of arity $n = 0$. Hence Q is a mapping $Q : \overset{0}{\underset{i=1}{\times}} \mathcal{P}(E^{t_i}) \longrightarrow \mathbf{I}$, i.e. $Q : \{\varnothing\} \longrightarrow \mathbf{I}$, recalling that $\{\varnothing\}$ represents the empty product. But $\mathcal{P}(E)^0 = \{\varnothing\}$ as well; this demonstrates that Q also qualifies as a nullary semi-fuzzy quantifier $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ on E. Now turning to $\mathcal{F}_L$, we first notice that the quantifier Q' defined by (244) reduces to $Q' = Q$ in this case, because there are no arguments, and because $\mathcal{P}(E^m)^n = \mathcal{P}(E^0)^0 = \mathcal{P}(\{\varnothing\})^0 = \{\varnothing\}$, too. Hence by Def. 182,

$$\mathcal{F}_L(Q) = \mathcal{F}(Q') \circ \overset{0}{\underset{i=1}{\times}} \hat{\hat{\zeta}}_i = \mathcal{F}(Q') \circ \mathrm{id}_{\{\varnothing\}} = \mathcal{F}(Q') = \mathcal{F}(Q) \qquad (591)$$

noticing that the empty product map is the identity $\mathrm{id}_{\{\varnothing\}}$ which maps the empty mapping $\varnothing$ to itself. We conclude that

$$\begin{aligned}\mathcal{U}(\mathcal{F}_L(Q)) &= \mathcal{U}(\mathcal{F}(Q)) && \text{by (591)} \\ &= Q\,, && \text{by (Z-1)}\end{aligned}$$

as desired.

Case b.: $t = \langle 1 \rangle$**.** Here we proceed as follows. We first observe that

$$\begin{aligned} \mathcal{F}_L(Q) &= \mathcal{F}(Q') \circ \hat{\hat{\hat{\zeta}}} && \text{by (244)} \\ &= \mathcal{F}(Q) \circ \mathrm{id}_{\widetilde{\mathcal{P}}(E^1)} \,. && \text{by L-41} \end{aligned}$$

From this it is now apparent that $\mathcal{U}(\mathcal{F}_L(Q)) = \mathcal{U}(\mathcal{F}(Q)) = Q$, because $\mathcal{F}$ satisfies (Z-1) by assumption of the lemma.

Lemma 44
If a QFM $\mathcal{F}$ satisfies (Z-2)*, then the corresponding L-QFM $\mathcal{F}_L$ satisfies* (L-2)

Proof Let $E \neq \varnothing$ be some base set and $e \in E$. In order to verify that (L-2) holds, we must show that $\mathcal{F}_L(\pi_{(e)}) = \widetilde{\pi}_{(e)}$. To this end, we notice that $\pi_{(e)}$ has type $\langle 1 \rangle$, i.e. L-41 is applicable. Therefore

$$\begin{aligned} \mathcal{F}_L(\pi_{(e)}) &= \mathcal{F}(Q') \circ \hat{\hat{\hat{\zeta}}} && \text{by Def. 182} && (592) \\ &= \mathcal{F}(\pi_{(e)}) \circ \mathrm{id}_{\widetilde{\mathcal{P}}(E^1)} && \text{by L-41} && (593) \\ &= \widetilde{\pi}_{(e)} \,, && \text{by (Z-2)} && (594) \end{aligned}$$

i.e. $\mathcal{F}_L(\pi_{(e)}) = \widetilde{\pi}_{(e)}$, as desired.

Next we shall turn to the conditions which involve the induced fuzzy set operations of complementation $\widetilde{\neg}$ and formation of unions $\widetilde{\cup}$. It is therefore necessary to establish the conditions under which the induced fuzzy truth functions and fuzzy set operations of the L-QFM $\mathcal{F}_L$ coincide with those induced by the original QFM $\mathcal{F}$.

Lemma 45
Let $\mathcal{F}$ be a QFM and $\mathcal{F}_L$ the corresponding L-QFM defined by Def. 182. If $\mathcal{F}$ satisfies (Z-2) *and* (Z-6)*, then $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_L$, i.e. $\mathcal{F}$ and $\mathcal{F}_L$ induce the same fuzzy truth functions and the same choice of fuzzy set operations.*

Proof Trivial. Let $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ be a given mapping, $n \in \mathbb{N}$. Then

$$\begin{aligned} \widetilde{\mathcal{F}}_L(f) &= \widetilde{\mathcal{F}}_{LR}(f) && \text{by Def. 176} \\ &= \mathcal{F}_{LR}(Q_f) \circ \widetilde{\eta} && \text{by Def. 11} \\ &= \mathcal{F}(Q_f) \circ \widetilde{\eta} && \text{by L-42} \\ &= \widetilde{\mathcal{F}}(f) \,, && \text{by Def. 11} \end{aligned}$$

i.e. $\widetilde{\mathcal{F}} = \widetilde{\mathcal{F}}_L$, as desired. Consequently, $\mathcal{F}$ and $\mathcal{F}_L$ also induce the same fuzzy set operations, which are defined in terms of elementwise combinations of membership grades, and thus reduce to an application of the induced truth functions.

Lemma 46
If $\mathcal{F}$ is a DFS, then the corresponding L-QFM $\mathcal{F}_L$ satisfies (L-3).

Proof Let $\mathcal{F}$ be a DFS and Q a semi-fuzzy L-quantifier of type $t = \langle t_1, \dots, t_n \rangle \in \mathbb{N}^n$ on some base set $E \neq \varnothing$ where $n > 0$. We first consider the induced fuzzy truth functions. Here we can apply L-45 and conclude that $\widetilde{\mathcal{F}}_L = \widetilde{\mathcal{F}}$, i.e. both $\mathcal{F}$ and $\mathcal{F}_L$ induce the same fuzzy connectives. Correspondingly, I will not discern these in my notation. In order to establish the desired (L-3), let us first consider the semi-fuzzy L-quantifier $Q\widetilde{\Box}$ of type t on E. Here, we obtain from Def. 182 that

$$\mathcal{F}_L(Q\widetilde{\Box}) = \mathcal{F}(Q') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}_i \tag{595}$$

referring to the mappings $\zeta_i : E^{t_i} \longrightarrow E^m$, $i \in \{1, \dots, n\}$, $m = \max\{t_1, \dots, t_n\}$ defined by (241), and to the semi-fuzzy quantifier $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ constructed from $Q\widetilde{\Box}$ and the mappings and $\kappa_i : E^m \longrightarrow E^{t_i}$ given by (242) according to (244), i.e. Q' becomes

$$\begin{aligned} & Q'(Y_1, \dots, Y_n) \\ & = (Q\widetilde{\Box})(\hat{\kappa}_1(Y_1 \cap \operatorname{Im} \zeta_1), \dots, \hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)) && \text{by (244)} \end{aligned}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E^m)$. By expanding the construction of dualisation according to Def. 14, we then obtain

$$\begin{aligned} & Q'(Y_1, \dots, Y_n) \\ & = \widetilde{\neg}\, Q(\hat{\kappa}_1(Y_1 \cap \operatorname{Im} \zeta_1), \dots, \hat{\kappa}_{n-1}(Y_{n-1} \cap \operatorname{Im} \zeta_{n-1}), \neg\hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)) \,. \end{aligned} \tag{596}$$

Let us now consider the expression $\neg\hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)$ which appears in the last argument of Q' in (596); I would like to show that

$$\neg\hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n) = \hat{\kappa}_n((\neg Y_n) \cap \operatorname{Im} \zeta_n) \,. \tag{597}$$

To this end, we observe that

$$\begin{aligned} & \hat{\kappa}_n((\neg Y_n) \cap \operatorname{Im} \zeta_n) \\ & = \{\kappa_n(a) : a \notin Y_n, a \in \operatorname{Im} \zeta_n\} && \text{by Def. 19} \\ & = \{\kappa_n(\zeta_n(b)) : \zeta_n(b) \notin Y_n\} && \text{substituting } a = \zeta_n(b) \text{ for } a \in \operatorname{Im} \zeta_n \\ & = \{b : \zeta_n(b) \notin Y_n\} && \text{by (243)} \\ & = \neg\{b : \zeta_n(b) \in Y_n\} && \text{by definition of complementation} \\ & = \neg\{\kappa_n(\zeta(b)) : \zeta_n(b) \in Y_n\} && \text{by (243)} \\ & = \neg\{\kappa_n(a) : a \in Y_n, a \in \operatorname{Im} \zeta_n\} && \text{substituting } a = \zeta_n(b) \text{ for } a \in \operatorname{Im} \zeta_n \\ & = \neg\hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n) \,. && \text{by Def. 19} \end{aligned}$$

This proves (597). Let us now return to the original quantifier Q. When applying $\mathcal{F}_L$ to Q, we obtain

$$\mathcal{F}_L(Q) = \mathcal{F}(Q'') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}_i \,, \tag{598}$$

see Def. 182. Here $Q'' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ is the semi-fuzzy quantifier constructed from Q according to (244), i.e.

$$Q''(Y_1, \dots, Y_n) = Q(\hat{\kappa}_1(Y_1 \cap \operatorname{Im} \zeta_1), \dots, \hat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)) \tag{599}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E^m)$. We now observe from (596), (597), (599) and Def. 14 that indeed

$$Q' = Q''\widetilde{\Box}\,. \tag{600}$$

Now consider a choice of fuzzy argument $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \ \ldots, \ n\}$. I first prove the equality

$$(\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n))\,\widetilde{\cap}\,\mathrm{Im}\,\zeta_n = \hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)\,\widetilde{\cap}\,\mathrm{Im}\,\zeta_n \tag{601}$$

which I will need below. Hence consider $a \in E^m$. If $a \notin \mathrm{Im}\,\zeta_n$, then

$$\begin{aligned}
\mu_{(\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n))\widetilde{\cap}\mathrm{Im}\,\zeta_n}(a) &= \mu_{\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n)}(a)\,\widetilde{\wedge}\,\chi_{\mathrm{Im}\,\zeta_n}(a) && \text{by pointwise definition of } \widetilde{\cap} \\
&= \mu_{\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n)}(a)\,\widetilde{\wedge}\,0 && \text{because } a \notin \mathrm{Im}\,\zeta_n \\
&= 0 && \text{by Th-5} \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)}(a)\,\widetilde{\wedge}\,0 && \text{by Th-5} \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)}(a)\,\widetilde{\wedge}\,\chi_{\mathrm{Im}\,\zeta_n}(a) && \text{because } a \notin \mathrm{Im}\,\zeta_n \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)\widetilde{\cap}\mathrm{Im}\,\zeta_n}(a) && \text{by pointwise definition of } \widetilde{\cap}\,.
\end{aligned}$$

In the remaining case that $a \in \mathrm{Im}\,\zeta_n$, we know that $a = \zeta_n(b)$ for a unique $b \in E^{t_n}$, because ζ_n is an injection. In this case

$$\begin{aligned}
\mu_{(\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n))\widetilde{\cap}\mathrm{Im}\,\zeta_n}(a) &= \mu_{\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n)}(a)\,\widetilde{\wedge}\,\chi_{\mathrm{Im}\,\zeta_n}(a) && \text{by pointwise definition of } \widetilde{\cap} \\
&= \mu_{\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n)}(a)\,\widetilde{\wedge}\,1 && \text{because } a \in \mathrm{Im}\,\zeta_n \\
&= \mu_{\widetilde{\neg}\,\hat{\hat{\zeta}}_n(X_n)}(a) && \text{by Th-5} \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)}(a) && \text{by L-34} \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)}(a)\,\widetilde{\wedge}\,\chi_{\mathrm{Im}\,\zeta_n}(a) && \text{because } a \in \mathrm{Im}\,\zeta_n \\
&= \mu_{\hat{\hat{\zeta}}_n(\widetilde{\neg}\,X_n)\widetilde{\cap}\mathrm{Im}\,\zeta_n}(a) && \text{by pointwise definition of } \widetilde{\cap}\,.
\end{aligned}$$

This completes the proof of equality (601). Based on these preparations, we can now proceed as follows.

$$\begin{aligned}
&\mathcal{F}_L(Q\widetilde{\Box})(X_1, \dots, X_n) \\
&= \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \dots, \hat{\hat{\zeta}}_n(X_n)) && \text{by (595)} \\
&= \mathcal{F}(Q''\widetilde{\Box})(\hat{\hat{\zeta}}_1(X_1), \dots, \hat{\hat{\zeta}}_n(X_n)) && \text{by (600)} \\
&= \widetilde{\neg}\, \mathcal{F}(Q'')(\hat{\hat{\zeta}}_1(X_1), \dots, \hat{\hat{\zeta}}_{n-1}(X_{n-1}), \widetilde{\neg}\, \hat{\hat{\zeta}}_n(X_n)) && \text{by (Z-3)} \\
&= \widetilde{\neg}\, \mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n})(\hat{\hat{\zeta}}_1(X_1) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_1, \dots, \\
&\qquad\qquad \hat{\hat{\zeta}}_{n-1}(X_{n-1}) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_{n-1}, \\
&\qquad\qquad (\widetilde{\neg}\, \hat{\hat{\zeta}}_n(X_n)) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_n) && \text{by (599), Th-14, Th-9, Th-15} \\
&= \widetilde{\neg}\, \mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n})(\hat{\hat{\zeta}}_1(X_1) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_1, \dots, \\
&\qquad\qquad \hat{\hat{\zeta}}_{n-1}(X_{n-1}) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_{n-1}, \\
&\qquad\qquad \hat{\hat{\zeta}}_n(\widetilde{\neg}\, X_n) \mathbin{\widetilde{\cap}} \operatorname{Im} \zeta_n) && \text{by (601)} \\
&= \widetilde{\neg}\, \mathcal{F}(Q'')(\hat{\hat{\zeta}}_1(X_1), \dots, \hat{\hat{\zeta}}_{n-1}(X_{n-1}), \hat{\hat{\zeta}}_n(\widetilde{\neg}\, X_n)) && \text{by (599), Th-14, Th-9, Th-15} \\
&= \widetilde{\neg}\, \mathcal{F}_L(Q)(X_1, \dots, X_{n-1}, \widetilde{\neg}\, X_n) && \text{by (598)} \\
&= \mathcal{F}_L(Q)\widetilde{\Box}(X_1, \dots, X_n) && \text{by Def. 177}
\end{aligned}$$

Hence indeed $\mathcal{F}_L(Q\widetilde{\Box}) = \mathcal{F}_L(Q)\widetilde{\Box}$. Because the choice of Q was arbitrary, this proves that $\mathcal{F}_L$ satisfies (L-3), as desired.

Lemma 47
Let $\mathcal{F}$ be a QFM which satisfies (Z-1). *Then for every semi-fuzzy truth function $f : \mathbf{2}^n \longrightarrow \mathbf{I}$,*

$$\widetilde{\mathcal{F}_L}(f)(y_1, \dots, y_n) = f(y_1, \dots, y_n)$$

for all $y_1, \dots, y_n \in \mathbf{2}$.

Proof Consider a QFM $\mathcal{F}$ which satisfies (Z-1) and let $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ be given. Further suppose that $y_1, \dots, y_n \in \mathbf{2}$ are two-valued arguments. Then

$$\begin{aligned}
&\widetilde{\mathcal{F}}_L(f)(y_1, \dots, y_n) \\
&= \widetilde{\mathcal{F}}_{LR}(f)(y_1, \dots, y_n) && \text{by Def. 176} \\
&= \mathcal{F}_{LR}(Q_f)(\widetilde{\eta}(y_1, \dots, y_n)) && \text{by Def. 11} \\
&= \mathcal{F}_{LR}(Q_f)(\eta(y_1, \dots, y_n)) && \text{by (15), (16), } y_i \text{ crisp} \\
&= \mathcal{F}(Q_f \circ \widehat{\vartheta})(\hat{\hat{\beta}}(\eta(y_1, \dots, y_n))) \\
&= \mathcal{F}(Q_f \circ \widehat{\vartheta})(\widehat{\beta}(\eta(y_1, \dots, y_n))) && \text{because } \eta(y_1, \dots, y_n) \text{ crisp} \\
&= Q_f(\widehat{\vartheta}(\widehat{\beta}(\eta(y_1, \dots, y_n)))) && \text{by (Z-1)} \\
&= Q_f(\eta(y_1, \dots, y_n)) && \text{because } \vartheta \circ \widehat{\beta} = \mathrm{id}_E \\
&= f(\eta^{-1}(\eta(y_1, \dots, y_n))) && \text{by Def. 11} \\
&= f(y_1, \dots, y_n)\,,
\end{aligned}$$

as desired.

Lemma 48
If $\mathcal{F}$ is a DFS, then the corresponding L-QFM $\mathcal{F}_L$ satisfies (L-4).

Proof Hence let $\mathcal{F}$ be a DFS and Q a semi-fuzzy L-quantifier of type $t \in \mathbb{N}^n$, $n > 0$ on some base set E. Let us notice in advance that $\mathcal{F}$ and $\mathcal{F}_L$ induce the same fuzzy truth functions and fuzzy set operations, so that I need not separate these in the notation. In order to show that $\mathcal{F}_L$ is compatible with the formation of unions in the arguments, let us first consider the fuzzy L-quantifier $\mathcal{F}_L(Q)\widetilde{\cup}$ of type $t' = \langle t_1, \dots, t_n, t_n \rangle \in \mathbb{N}^{n+1}$. In this case, we obtain from Def. 182 that

$$\mathcal{F}_L(Q)\widetilde{\cup} = \mathcal{F}(Q') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}_i \widetilde{\cup} \tag{602}$$

where Q' is defined by (244). I will have to apply this construction twice; therefore let me explain the exact notation I will use to describe Q'. In this case, I will write $\zeta_i : E^{t_i} \longrightarrow E^m$, $\kappa_i : E^m \longrightarrow E^{t_i}$, $i \in \{1, \dots, n\}$, for the mappings defined by (241) and (242), respectively. The quantifier $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ then becomes

$$Q'(Y_1, \dots, Y_n) = Q(\widehat{\kappa}_1(Y_1 \cap \operatorname{Im} \zeta_1), \dots, \widehat{\kappa}_n(Y_n \cap \operatorname{Im} \zeta_n)) \tag{603}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E^m)$. Next we consider the semi-fuzzy L-quantifier $Q\cup$ of type $t' = \langle t_1, \dots, t_n, t_n \rangle \in \mathbb{N}^{n+1}$ on E. In this case, Def. 182 yields

$$\mathcal{F}_L(Q\cup) = \mathcal{F}(Q'') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}{}'_i \tag{604}$$

and the construction now rests on mappings

$$\zeta'_i = \zeta_i \tag{605}$$

$$\kappa'_i = \kappa_i \tag{606}$$

for $i \in \{1, \ldots, n\}$, and an additional pair of mappings

$$\zeta'_{n+1} = \zeta_n \tag{607}$$

$$\kappa'_{n+1} = \kappa_n \tag{608}$$

for $i = n + 1$. The semi-fuzzy quantifier $Q'' : \mathcal{P}(E^m)^{n+1} \longrightarrow \mathbf{I}$ then becomes

$$\begin{aligned}
&Q''(Y_1, \ldots, Y_n) \\
&= Q\cup(\widehat{\kappa}'_1(Y_1 \cap \operatorname{Im}\zeta'_1), \ldots, \widehat{\kappa}'_{n+1}(Y_{n+1} \cap \operatorname{Im}\zeta'_{n+1})) && \text{by (244)} \\
&= Q\cup(\widehat{\kappa}_1(Y_1 \cap \operatorname{Im}\zeta_1), \ldots, \\
&\qquad \widehat{\kappa}_n(Y_n \cap \operatorname{Im}\zeta_n), \widehat{\kappa}_n(Y_{n+1} \cap \operatorname{Im}\zeta_n)) && \text{by (605), (606), (607), (608)} \\
&= Q(\widehat{\kappa}_1(Y_1 \cap \operatorname{Im}\zeta_1), \ldots, \widehat{\kappa}_{n-1}(Y_{n-1}\operatorname{Im}\zeta_{n-1}), \\
&\qquad \widehat{\kappa}_n(Y_n \cap \operatorname{Im}\zeta_n) \cup \widehat{\kappa}_n(Y_{n+1} \cap \operatorname{Im}\zeta_n))\,, && \text{by Def. 15} \\
&= Q(\widehat{\kappa}_1(Y_1 \cap \operatorname{Im}\zeta_1), \ldots, \widehat{\kappa}_{n-1}(Y_{n-1}\operatorname{Im}\zeta_{n-1}), \\
&\qquad \widehat{\kappa}_n((Y_n \cup Y_{n+1}) \cap \operatorname{Im}\zeta_n))\,,
\end{aligned}$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E^m)$, where the last step is justified by the known properties of powerset mappings and of the familiar operations on crisp sets. We conclude from (603) and Def. 15 that

$$Q'' = Q'\cup\,. \tag{609}$$

Hence for all $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
&\mathcal{F}_L(Q\cup)(X_1, \ldots, X_{n+1}) \\
&= \mathcal{F}(Q'')(\hat{\hat{\zeta}}{}'_1(X_1), \ldots, \hat{\hat{\zeta}}{}'_{n+1}(X_{n+1})) && \text{by (604)} \\
&= \mathcal{F}(Q'\cup)(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_n(X_n), \hat{\hat{\zeta}}_n(X_{n+1})) && \text{by (605), (607), (609)} \\
&= \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_{n-1}(X_{n-1}), \hat{\hat{\zeta}}_n(X_n) \mathbin{\widetilde{\cup}} \hat{\hat{\zeta}}_n(X_{n+1})) && \text{by (Z-4)} \\
&= \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_{n-1}(X_{n-1}), \hat{\hat{\zeta}}_n(X_n \mathbin{\widetilde{\cup}} X_{n+1})) && \text{by L-47, L-34} \\
&= \mathcal{F}_L(Q)(X_1, \ldots, X_{n-1}, X_n \mathbin{\widetilde{\cup}} X_{n+1})\,, && \text{by (602)}
\end{aligned}$$

i.e. $\mathcal{F}_L(Q\cup) = \mathcal{F}_L(Q)\widetilde{\cup}$, as desired.

Lemma 49

If a QFM $\mathcal{F}$ satisfies (Z-5), *then the corresponding L-QFM $\mathcal{F}_L$ satisfies* (L-5).

Proof Suppose that Q is a semi-fuzzy L-quantifier of type $t \in \mathbb{N}^n$ $n > 0$ on some base set $E \neq \varnothing$, which is nonincreasing in its n-th argument. As I will now show, the semi-fuzzy quantifier $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ defined by (244) is also nonincreasing in its last argument. To see this, let us consider $Y_1, \ldots, Y_n, Y'_n \in \mathcal{P}(E^m)$ with $Y_n \subseteq Y'_n$. Clearly $Y_n \cap \operatorname{Im}\zeta_n \subseteq Y'_n \cap \operatorname{Im}\zeta_n$ as well. And, due to the monotonicity of crisp powerset mappings, we also have

$$\widehat{\kappa}_n(Y_n \cap \operatorname{Im}\zeta_n) \subseteq \widehat{\kappa}_n(Y'_n \cap \operatorname{Im}\zeta_n)\,. \tag{610}$$

But, Q is nonincreasing in its last argument, hence

$$\begin{aligned}&Q(\kappa_1(Y_1 \cap \operatorname{Im}\zeta_1), \ldots, \kappa_n(Y_n \cap \operatorname{Im}\zeta_n))\\ &\geq Q(\kappa_1(Y_1 \cap \operatorname{Im}\zeta_1), \ldots, \kappa_{n-1}(Y_{n-1} \cap \operatorname{Im}\zeta_{n-1}), \kappa_n(Y'_n \cap \operatorname{Im}\zeta_n))\,.\end{aligned}$$

Recalling (244), then, we have

$$Q'(Y_1, \ldots, Y_n) \geq Q'(Y_1, \ldots, Y_{n-1}, Y'_n)\,,$$

i.e. Q' is indeed nonincreasing in its last argument. We can now proceed as follows. Consider a choice of fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \ldots, n\}$. Further let $X'_n \in \widetilde{\mathcal{P}}(E^{t_n})$ such that $X_n \subseteq X'_n$. Due to the monotonicity of the standard extension principle, we then know that

$$\hat{\hat{\zeta}}_n(X_n) \subseteq \hat{\hat{\zeta}}_n(X'_n)\,. \tag{611}$$

Therefore

$$\begin{aligned}&\mathcal{F}_L(Q)(X_1, \ldots, X_n)\\ &= \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_n(X_n)) && \text{by Def. 182}\\ &\geq \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_{n-1}(X_{n-1}), \zeta_n(X'_n)) && \text{by (Z-5) and (611)}\\ &= \mathcal{F}_L(Q)(X_1, \ldots, X_{n-1}, X'_n)\,,\end{aligned}$$

i.e. $\mathcal{F}_L(Q)$ is indeed nonincreasing in its n-th argument according to Def. 179.

Lemma 50
If $\mathcal{F}$ satisfies (Z-2) *and* (Z-6)*, then $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_L$, i.e. $\mathcal{F}$ and $\mathcal{F}_L$ induce the same extension principle.*

Proof Consider a mapping $f : E \longrightarrow E'$, $E, E' \neq \varnothing$ and let $e' \in E'$. Then

$$\begin{aligned}\mu_{\widehat{\mathcal{F}}_L(f)}(e') &= \mu_{\widehat{\mathcal{F}}_{LR}(f)}(e') && \text{by Def. 180}\\ &= \mu_{\widehat{\mathcal{F}}(f)}(e')\,, && \text{by L-42}\end{aligned}$$

i.e. $\widehat{\mathcal{F}}_L(f) = \widehat{\mathcal{F}}(f)$, which completes the proof of the lemma.

Lemma 51
If $\mathcal{F}$ is a DFS, then the corresponding L-QFM $\mathcal{F}_L$ satisfies (L-6)*.*

Proof Let $\mathcal{F}$ be a DFS and $\mathcal{F}_L$ the L-QFM defined by Def. 182. In order to prove the lemma, we consider a semi-fuzzy L-quantifier Q of type $t \in \mathbb{N}^n$, $n \in \mathbb{N}$, another type $t' \in \mathbb{N}^n$ with the same number of components, a base set $E' \neq \varnothing$ and some choice of mappings $f_i : {E'}^{t'_i} \longrightarrow E^{t_i}$, $i \in \{1, \ldots, n\}$. We need some preparations. Firstly we know from Def. 182 that

$$\mathcal{F}_L(Q) = \mathcal{F}(Q') \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\zeta}}_i\,, \tag{612}$$

where $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ is the semi-fuzzy quantifier constructed from Q according to by (244). Because we will need this construction twice here, let me state which symbols I will use to describe Q'. In this case, we have $m = \max\{t_1, \dots, t_n\}$; the mappings $\zeta_i : E^{t_i} \longrightarrow E^m$ and $\kappa_i : E^m \longrightarrow E^{t_i}$, $i \in \{1, \dots, n\}$ are defined by (241) and (242), respectively; and Q' hence becomes:

$$Q'(Y_1, \dots, Y_n) = Q(\widehat{\kappa}_1(Y_1 \cap \mathrm{Im}\,\zeta_1, \dots, \widehat{\kappa}_n(Y_n \cap \mathrm{Im}\,\zeta_n)) \qquad (613)$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E^m)$. In the following, it is also necessary to consider $\mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i)$. In this case, we obtain from Def. 182 that

$$\mathcal{F}_L(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i) = \mathcal{F}(Q'') \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{\zeta}}{}'_i\,, \qquad (614)$$

where Q'' is the quantifier constructed from $Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i$ according to (244). In this case, we have $m' = \max\{t'_1, \dots, t'_n\}$; the mappings $\zeta'_i : {E'}^{t'_i} \longrightarrow E^{m'}$ and $\kappa'_i : {E'}^{m'} \longrightarrow {E'}^{t'_i}$, $i \in \{1, \dots, n\}$ are defined by (241) and (242), respectively; and $Q'' : \mathcal{P}({E'}^{m'})^n \longrightarrow \mathbf{I}$ becomes

$$\begin{aligned} & Q''(Y_1, \dots, Y_n) \\ & = (Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i)(\widehat{\kappa}'_1(Y_1 \cap \mathrm{Im}\,\zeta'_1), \dots, \widehat{\kappa}'_n(Y_n \cap \mathrm{Im}\,\zeta'_n))\,, && \text{by (244)} \end{aligned}$$

i.e.

$$Q''(Y_1, \dots, Y_n) = Q(\widehat{f}_1(\widehat{\kappa}'_1(Y_1 \cap \mathrm{Im}\,\zeta'_1)), \dots, \widehat{f}_n(\widehat{\kappa}'_n(Y_n \cap \mathrm{Im}\,\zeta'_n))) \qquad (615)$$

for all $Y_1, \dots, Y_n \in \mathcal{P}({E'}^{m'})$. Let us now consider a choice of $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} & f_i \circ \kappa'_i \\ & = \kappa_i \circ \zeta_i \circ f_i \circ \kappa'_i \end{aligned}$$

because $\kappa_i \circ \zeta_i = \mathrm{id}_{E^{t_i}}$ by (243). We conclude from the known properties of crisp powerset mappings that

$$\widehat{f_i \circ \kappa} = \widehat{\kappa}_i \circ \zeta_i \widehat{\circ f_i \circ} \kappa'_i\,. \qquad (616)$$

Now consider some $Y_i \in \mathcal{P}({E'}^{m'})$. We then have

$$\zeta_i \widehat{\circ f_i \circ} \kappa'_i(Y_i) \subseteq \mathrm{Im}\,\zeta_i\,,$$

i.e.

$$\zeta_i \widehat{\circ f_i \circ} \kappa'_i(Y_i) \cap \mathrm{Im}\,\zeta_i = \zeta_i \widehat{\circ f_i \circ} \kappa'_i(Y_i)\,. \qquad (617)$$

Hence for all $Y_1, \ldots, Y_n \in \mathcal{P}(E'^{m'})$,

$$
\begin{aligned}
&Q''(Y_1, \ldots, Y_n) \\
&= Q(\widehat{f_1 \circ \kappa'_1}(Y_1 \cap \operatorname{Im} \zeta'_1)), \ldots, \widehat{f_n \circ \kappa'_n}(Y_n \cap \operatorname{Im} \zeta'_n))) && \text{by (615)} \\
&= Q(\widehat{\kappa_1}(\zeta_1 \widehat{\circ f_1 \circ} \kappa'_1(Y_1 \cap \operatorname{Im} \zeta'_1)), \ldots, \widehat{\kappa_n}(\zeta_n \widehat{\circ f_n \circ} \kappa'_n(Y_n \cap \operatorname{Im} \zeta'_n))) && \text{by (616)} \\
&= Q(\widehat{\kappa_1}(\zeta_1 \widehat{\circ f_1 \circ} \kappa'_1(Y_1 \cap \operatorname{Im} \zeta'_1) \cap \operatorname{Im} \zeta_1), \ldots, \\
&\qquad\qquad \widehat{\kappa_n}(\zeta_n \widehat{\circ f_n \circ} \kappa'_n(Y_n \cap \operatorname{Im} \zeta'_n) \cap \operatorname{Im} \zeta_n)) && \text{by (617)} \\
&= Q'(\zeta_1 \widehat{\circ f_1 \circ} \kappa'_1(Y_1 \cap \operatorname{Im} \zeta'_1), \ldots, \zeta_n \widehat{\circ f_n \circ} \kappa'_n(Y_n \cap \operatorname{Im} \zeta'_n))
\end{aligned}
$$

i.e.

$$
Q''(Y_1, \ldots, Y_n) = Q'(\zeta_1 \widehat{\circ f_1 \circ} \kappa'_1(Y_1 \cap \operatorname{Im} \zeta'_1), \ldots, \zeta_n \widehat{\circ f_n \circ} \kappa'_n(Y_n \cap \operatorname{Im} \zeta'_n)) \,. \tag{618}
$$

Now consider a choice of fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E'^{t'_i})$, $i \in \{1, \ \ldots, \ n\}$. Then

$$
\begin{aligned}
&\mathcal{F}_L(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})(X_1, \ldots, X_n) \\
&\quad= \mathcal{F}(Q'')(\hat{\hat{\zeta}}'_1(X_1), \ldots, \hat{\hat{\zeta}}'_n(X_n)) && \text{by (614)} \\
&\quad= \mathcal{F}(Q')(\widehat{\mathcal{F}}(\zeta_1 \circ f_1 \circ \kappa'_1)(\hat{\hat{\zeta}}'_1(X_1) \cap \operatorname{Im} \zeta'_1), \ldots, \\
&\qquad\qquad \widehat{\mathcal{F}}(\zeta_n \circ f_n \circ \kappa'_n)(\hat{\hat{\zeta}}'_n(X_n) \cap \operatorname{Im} \zeta'_n)) && \text{by (618), (Z-6), Th-14,} \\
& && \text{Th-9 and Th-15} \\
&\quad= \mathcal{F}(Q')(\widehat{\mathcal{F}}(\zeta_1 \circ f_1 \circ \kappa'_1)(\hat{\hat{\zeta}}'_1(X_1)), \ldots, \\
&\qquad\qquad \widehat{\mathcal{F}}(\zeta_n \circ f_n \circ \kappa'_n)(\hat{\hat{\zeta}}'_n(X_n))) && \text{apparent from Def. 21} \\
&\quad= \mathcal{F}(Q')(\widehat{\mathcal{F}}(\zeta_1 \circ f_1 \circ \kappa'_1 \circ \zeta'_1)(X_1), \ldots, \\
&\qquad\qquad \widehat{\mathcal{F}}(\zeta_n \circ f_n \circ \kappa'_n \circ \zeta'_n)(X_n)) && \text{by Th-21, Th-20} \\
&\quad= \mathcal{F}(Q')(\widehat{\mathcal{F}}(\zeta_1 \circ f_1)(X_1), \ldots, \widehat{\mathcal{F}}(\zeta_n \circ f_n)(X_n)) && \text{by (243)} \\
&\quad= \mathcal{F}(Q')(\widehat{\mathcal{F}}(\zeta_1)(\widehat{\mathcal{F}}(f_1)(X_1)), \ldots, \\
&\qquad\qquad \widehat{\mathcal{F}}(\zeta_n)(\widehat{\mathcal{F}}(f_n)(X_n))) && \text{by Th-20} \\
&\quad= \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(\widehat{\mathcal{F}}(f_1)(X_1)), \ldots, \hat{\hat{\zeta}}_n(\widehat{\mathcal{F}}(f_n)(X_n))) && \text{by Th-21} \\
&\quad= \mathcal{F}_L(Q)(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n)) && \text{by (612)} \\
&\quad= \mathcal{F}_L(Q)(\widehat{\mathcal{F}}_L(f_1)(X_1), \ldots, \widehat{\mathcal{F}}_L(f_n)(X_n)) && \text{by L-50} \,.
\end{aligned}
$$

Because the arguments were chosen arbitrarily, this demonstrates that indeed

$$
\mathcal{F}_L(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}) = \mathcal{F}_L(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}_L(f_i) \,,
$$

i.e. $\mathcal{F}_L$ satisfies (L-6), as desired.

Proof of Theorem 273

Let $\mathcal{F}$ be a DFS. The claim of the theorem that $\mathcal{F}_L$ is an L-DFS is now apparent from the series of lemmata L-43, L-44, L-46, L-48, L-49, L-51, which prove that $\mathcal{F}_L$ satisfies (L-1), (L-2), (L-3), (L-4), (L-5) and (L-6), respectively.

D.35 Proof of Theorem 274

Let $\mathcal{F}$ be a given L-DFS. In order to show that $\mathcal{F} = \mathcal{F}_{RL}$, we consider an arbitrary semi-fuzzy L-quantifier Q of some type $t \in \mathbb{N}^n$, $n \in \mathbb{N}$ on a base set $E \neq \varnothing$. We need some preparations. For $i \in \{1, \ldots, n\}$, let $\lambda_i : E^{t_i} \longrightarrow (E^m)^1$ denote the composition

$$\lambda_i = \beta \circ \zeta_i \tag{619}$$

i.e.

$$\lambda_i(e_1, \ldots, e_{t_i}) = ((e_1, \ldots, e_{t_i}, \underbrace{e_{t_i}, \ldots, e_{t_i}}_{(m - t_i) \text{ times}})) \tag{620}$$

for all $(e_1, \ldots, e_{t_i}) \in E^{t_i}$. Due to the compositionality of the standard extension principle, we then have

$$\hat{\hat{\lambda}}_i = \hat{\hat{\beta}} \circ \hat{\hat{\zeta}}_i\,. \tag{621}$$

Next we observe from (620) that the λ_i's are injections. And, the QFM $\mathcal{F}_R$ constructed from $\mathcal{F}$ according to Def. 175, is known to be a DFS from Th-271. Hence by Th-21, $\widehat{\mathcal{F}}_R(\lambda_i) = \hat{\hat{\lambda}}_i$. But $\widehat{\mathcal{F}} = \widehat{\mathcal{F}}_R$ by Def. 180; we conclude that

$$\widehat{\mathcal{F}}(\lambda_i) = \hat{\hat{\lambda}}_i \tag{622}$$

for all $i \in \{1, \ldots, n\}$. Based on these preparations, we can now proceed as follows. Consider a choice of fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
&\mathcal{F}_{RL}(Q)(X_1, \ldots, X_n) \\
&= \mathcal{F}_R(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_n(X_n)) && \text{by Def. 182, (244)} \\
&= \mathcal{F}(Q' \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\beta}}(\hat{\hat{\zeta}}_1(X_1)), \ldots, \hat{\hat{\beta}}(\hat{\hat{\zeta}}_n(X_n))) && \text{by Def. 175} \\
&= \mathcal{F}(Q' \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta})(\hat{\hat{\lambda}}_1(X_1), \ldots, \hat{\hat{\lambda}}_n(X_n)) && \text{by (621)} \\
&= \mathcal{F}(Q' \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta})(\widehat{\mathcal{F}}(\lambda_1)(X_1), \ldots, \widehat{\mathcal{F}}(\lambda_n)(X_n)) && \text{by (622)} \\
&= \mathcal{F}(Q' \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta} \circ \mathop{\times}_{i=1}^{n} \widehat{\lambda}_i)(X_1, \ldots, X_n) && \text{by (L-6)} \\
&= \mathcal{F}(Q' \circ \mathop{\times}_{i=1}^{n} \widehat{\vartheta} \circ \widehat{\beta} \circ \widehat{\zeta}_i)(X_1, \ldots, X_n)\,,
\end{aligned}$$

where the last equality is apparent from (619) and the compositionality of product mappings. In order to prove the desired $\mathcal{F}_{RL}(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q)(X_1, \ldots, X_n)$, it remains to be shown that

$$Q' \circ \bigtimes_{i=1}^{n} \widehat{\vartheta} \circ \widehat{\beta} \circ \widehat{\zeta}_i = Q\,. \tag{623}$$

To this end, we first observe that

$$\widehat{\vartheta} \circ \widehat{\beta} = \widehat{\vartheta \circ \beta} = \widehat{\mathrm{id}}_{E^m} = \mathrm{id}_{\mathcal{P}(E^m)}\,,$$

this is apparent from the fact that ϑ and β are inverses of each others, and from the known properties of crisp powerset mappings. Hence (623) reduces to

$$Q' \circ \bigtimes_{i=1}^{n} \widehat{\zeta}_i = Q\,. \tag{624}$$

In order to prove this, we consider a choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
&Q'(\widehat{\zeta}_1(Y_1), \ldots, \widehat{\zeta}_n(Y_n)) \\
&= Q(\widehat{\kappa}_1(\widehat{\zeta}_1(Y_1) \cap \mathrm{Im}\,\zeta_1), \ldots, \\
&\qquad \widehat{\kappa}_n(\widehat{\zeta}_n(Y_n) \cap \mathrm{Im}\,\zeta_n)) && \text{by (244)} \\
&= Q(\widehat{\kappa}_1(\widehat{\zeta}(Y_1)), \ldots, \widehat{\kappa}_n(\widehat{\zeta}_n(Y_n))) && \text{because } \widehat{\zeta}_i(Y_i) \subseteq \widehat{\zeta}_i(E^{t_i}) = \mathrm{Im}\,\zeta_i \\
&= Q(\widehat{\kappa_1 \circ \zeta_1}(Y_1), \ldots, \widehat{\kappa_n \circ \zeta_n}(Y_n)) && \text{(compositionality of powerset mappings)} \\
&= Q(Y_1, \ldots, Y_n)\,.
\end{aligned}$$

In the last step, I have utilized that $\kappa_i \circ \zeta_i = \mathrm{id}_{E^{t_i}}$ by (243); therefore $\widehat{\kappa_i \circ \zeta_i} = \widehat{\mathrm{id}}_{E^{t_i}} = \mathrm{id}_{\mathcal{P}(E^{t_i})}$ by the known properties of crisp powerset mappings. Noticing that the choice of arguments $Y_1, \ldots, Y_n$ was arbitrary, this demonstrates that (624) holds; in particular $\mathcal{F}(Q) = \mathcal{F}_{RL}(Q)$. Because the semi-fuzzy L-quantifier Q was arbitrarily chosen, this completes the proof that $\mathcal{F} = \mathcal{F}_{RL}$.

D.36 Proof of Theorem 275

Lemma 52
Let Q be an n-ary semi-fuzzy L-quantifier of type $t = \langle m, \ldots, m\rangle$ for some $m \in \mathbb{N}$. Then $\mathcal{F}_L(Q) = \mathcal{F}(Q')$ in every QFM $\mathcal{F}$, where $Q' : \mathcal{P}(E^m)^n \longrightarrow \mathbf{I}$ is the semi-fuzzy quantifier defined by (244).

Proof Straightforward.

$$\begin{aligned}
\mathcal{F}_L(Q) &= \mathcal{F}(Q') \circ \bigtimes_{i=1}^{n} \hat{\hat{\zeta}}_i && \text{by Def. 182} \\
&= \mathcal{F}(Q) \circ \bigtimes_{i=1}^{n} \hat{\hat{\mathrm{id}}}_{E^m} && \text{by L-41} \\
&= \mathcal{F}(Q) \circ \bigtimes_{i=1}^{n} \mathrm{id}_{\widetilde{\mathcal{P}}(E^m)} && \text{apparent from Def. 21} \\
&= \mathcal{F}(Q)\,.
\end{aligned}$$

Proof of Theorem 275

By Th-272, Th-273 and Th-274, it is sufficient to show that for every (ordinary) DFS $\mathcal{F}$, $\mathcal{F}_L$ is compatible with quantifier nesting only if $\mathcal{F}$ is compatible with fuzzy argument insertion according to Def. 74. The proof of this claim is by contraposition. Hence let $\mathcal{F}$ be a DFS which does not verify fuzzy argument insertion. Then there exist $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ such that

$$\mathcal{F}(Q)(X_1, \dots, X_n) \neq \mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} X_n)(X_1, \dots, X_{n-1}) \tag{625}$$

Let us define an n-ary semi-fuzzy L-quantifier $Q^*$of type $\langle 1, \dots, 1\rangle$ on $E' = E \times \mathbf{I}$ by

$$Q^*(Y_1, \dots, Y_n) = Q(\widehat{\beta}(Y_1), \dots, \widehat{\beta}(Y_n)) \tag{626}$$

for all $Y_1, \dots, Y_n \in \widetilde{\mathcal{P}}({E'}^1)$, where $\beta : {E'}^1 \longrightarrow E$ is the surjection

$$\beta((e, v)) = e \tag{627}$$

for all $((e, v)) \in {E'}^1$. Further define an injection $\theta : E \longrightarrow E'$ by

$$\theta(e) = ((e, \mu_{X_n}(e))) \tag{628}$$

for all $e \in E$. Clearly $\beta(\theta(e)) = \beta((e, \mu_{X_n}(e))) = e$ for all $e \in E$, i.e.

$$\beta \circ \theta = \mathrm{id}_E \, . \tag{629}$$

Therefore

$$\begin{aligned}
Q(Y_1, \dots, Y_n) &= Q(\widehat{\mathrm{id}}_E(Y_1), \dots, \widehat{\mathrm{id}}_E(Y_n)) && \text{apparent from Def. 19} \\
&= Q(\widehat{\beta}(\widehat{\theta}(Y_1)), \dots, \widehat{\beta}(\widehat{\theta}(Y_n))) && \text{by (629)} \\
&= Q^*(\widehat{\theta}(Y_1), \dots, \widehat{\theta}(Y_n)) && \text{by (626)}
\end{aligned}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$ and in turn,

$$\begin{aligned}
\mathcal{F}(Q)(X_1, \dots, X_n) &= \mathcal{F}(Q^* \circ \mathop{\times}_{i=1}^{n} \widehat{\theta})(X_1, \dots, X_n) \\
&= \mathcal{F}(Q^*)(\hat{\hat{\theta}}(X_1), \dots, \hat{\hat{\theta}}(X_n))
\end{aligned}$$

by (Z-6) and Th-21. Hence by L-52,

$$\mathcal{F}(Q)(X_1, \dots, X_n) = \mathcal{F}_L(Q^*)(X'_1, \dots, X'_{n-1}, \hat{\hat{\theta}}(X_n)) \, , \tag{630}$$

where $X'_1, \dots, X'_{n-1} \in \widetilde{\mathcal{P}}({E'}^1)$ abbreviate

$$X'_i = \hat{\hat{\theta}}(X_i) \, . \tag{631}$$

Now let us express $\hat{\hat{\theta}}(X_n)$ in terms of a nested quantifier. To this end, let Q' be the semi-fuzzy L-quantifier of type $\langle 1 \rangle$ on E' defined by

$$Q'(Y) = \sup\{v : ((e, v)) \in Y\} \tag{632}$$

for all $Y \in \mathcal{P}({E'}^1)$, and let $X'_n \in \widetilde{\mathcal{P}}({E'}^2)$ denote the collection

$$X'_n = \{((e, \mu_{X_n}(e)), (e, \mu_{X_n}(e))) : e \in E\} \,. \tag{633}$$

In particular, X'_n is crisp. Now let us consider the fuzzy set $Z \in \widetilde{\mathcal{P}}({E'}^1)$ determined by (247), i.e.

$$\mu_Z((e, v)) = \mathcal{F}_L(Q')(((e, v))X'_n) \tag{634}$$

for all $((e, v)) \in {E'}^1$. Noticing from (633) that X'_n is crisp, $((e, v))X'_n$ defined by (248) is also crisp and coincides with the relation defined by (246). By correct generalization (L-1), then, the fuzzy set Z can also be described thus,

$$\mu_Z((e, v)) = Q'(((e, v))X'_n) \tag{635}$$

for all $((e, v)) \in {E'}^1$, where

$$\begin{aligned} ((e, v))X'_n &= \{((e', v')) : ((e, v), (e', v')) \in X'_n\} && \text{by (246)} \\ &= \begin{cases} \{((e, v))\} & : \quad v = \mu_{X_n}(e) \\ \varnothing & : \quad \text{else} \end{cases} \end{aligned}$$

by (633). Hence

$$\mu_Z((e, v)) = \begin{cases} v & : \quad v = \mu_{X_n}(e) \\ 0 & : \quad \text{else} \end{cases} \tag{636}$$

by (632) and (635). On the other hand

$$\begin{aligned} \mu_{\hat{\hat{\theta}}(X_n)}((e, v)) &= \begin{cases} \mu_{\theta^{-1}((e,v))}(X_n) & : \quad ((e, v)) \in \operatorname{Im} \theta \\ 0 & : \quad \text{else} \end{cases} && \text{apparent from Def. 21} \\ &= \begin{cases} v & : \quad v = \mu_{X_n}(e) \\ 0 & : \quad \text{else} \end{cases} && \text{by (628)} \\ &= \mu_Z((e, v)) \,, && \text{by (636)} \end{aligned}$$

i.e. indeed

$$Z = \hat{\hat{\theta}}(X_n) \,. \tag{637}$$

Therefore

$$\begin{aligned} \mathcal{F}(Q)(X_1, \ldots, X_n) &= \mathcal{F}_L(Q^*)(X'_1, \ldots, X'_{n-1}, \hat{\hat{\theta}}(X_n)) && \text{by (630)} \\ &= \mathcal{F}_L(Q^*)(X'_1, \ldots, X'_{n-1}, Z) \,, && \text{by (637)} \end{aligned}$$

from which we obtain

$$\mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}_L(Q^*) \mathbin{@} \mathcal{F}_L(Q')(X'_1, \ldots, X'_n) \tag{638}$$

by (634) and Def. 184. In order to complete the proof, I need some more observations. Firstly

$$\begin{aligned}
\mathrm{id}_{\widetilde{\mathcal{P}}(E)} &= \widehat{\mathcal{F}}(\mathrm{id}_E) && \text{by Th-20.b} \\
&= \widehat{\mathcal{F}}(\beta \circ \theta) && \text{by (629)} \\
&= \widehat{\mathcal{F}}(\beta) \circ \widehat{\mathcal{F}}(\theta) && \text{by Th-20.a} \\
&= \widehat{\mathcal{F}}(\beta) \circ \hat{\hat{\theta}}\,, && \text{by Th-21}
\end{aligned}$$

i.e.

$$\mathrm{id}_{\widetilde{\mathcal{P}}(E)} = \widehat{\mathcal{F}}(\beta) \circ \hat{\hat{\theta}}\,. \tag{639}$$

Next we notice that for all $Y_1, \ldots, Y'_{n-1} \in \mathcal{P}({E'}^1)$,

$$\begin{aligned}
& Q \mathbin{\widetilde{\triangleleft}} X_n(\widehat{\beta}(Y'_1), \ldots, \widehat{\beta}(Y'_{n-1})) \\
&= \mathcal{F}(Q)(\widehat{\beta}(Y'_1), \ldots, \widehat{\beta}(Y'_{n-1}), X_n) && \text{by Def. 73} \\
&= \mathcal{F}(Q)(\widehat{\beta}(Y'_1), \ldots, \widehat{\beta}(Y'_{n-1}), \widehat{\mathcal{F}}(\beta)(\hat{\hat{\theta}}(X_n))) && \text{by (639)} \\
&= \mathcal{F}(Q)(\widehat{\mathcal{F}}(\beta)(Y'_1), \ldots, \widehat{\mathcal{F}}(\beta)(Y'_{n-1}), \widehat{\mathcal{F}}(\beta)(\hat{\hat{\theta}}(X_n))) && \text{because } Y_1, \ldots, Y_{n-1} \text{ crisp} \\
&= \mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\beta})(Y'_1, \ldots, Y'_{n-1}, \hat{\hat{\theta}}(X_n)) && \text{by (Z-6)} \\
&= \mathcal{F}(Q^*)(Y'_1, \ldots, Y'_{n-1}, Z) && \text{by (626), (637)} \\
&= \mathcal{F}_L(Q^*)(Y'_1, \ldots, Y'_{n-1}, Z) && \text{by L-52} \\
&= Q^* \mathbin{\widetilde{@}} Q'(Y_1, \ldots, Y'_{n-1}, X'_n)\,, && \text{by Def. 183, (245), (635)}
\end{aligned}$$

i.e.

$$Q \mathbin{\widetilde{\triangleleft}} X_n \circ \mathop{\times}\limits_{i=1}^{n-1} \widehat{\beta} = (Q^* \mathbin{\widetilde{@}} Q') \triangleleft X'_n \tag{640}$$

by Def. 34. Therefore

$$\begin{aligned}
& \mathcal{F}_L(Q^*) \mathbin{@} \mathcal{F}_L(Q')(X'_1, \ldots, X'_n) \\
&= \mathcal{F}(Q)(X_1, \ldots, X_n) && \text{by (638)} \\
&\neq \mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} X_n)(X_1, \ldots, X_{n-1}) && \text{by (625)} \\
&= \mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} X_n)(\widehat{\mathcal{F}}(\beta)(\hat{\hat{\theta}}(X_1)), \ldots, \widehat{\mathcal{F}}(\beta)(\hat{\hat{\theta}}(X_{n-1}))) && \text{by (639)} \\
&= \mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} X_n)(\widehat{\mathcal{F}}(\beta)(X'_1), \ldots, \widehat{\mathcal{F}}(\beta)(X'_{n-1})) && \text{by (631)} \\
&= \mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} X_n \circ \mathop{\times}\limits_{i=1}^{n-1} \widehat{\beta})(X'_1, \ldots, X'_{n-1}) && \text{by (Z-6)} \\
&= \mathcal{F}((Q^* \mathbin{\widetilde{@}} Q') \triangleleft X'_n)(X'_1, \ldots, X'_{n-1}) && \text{by (640)} \\
&= \mathcal{F}_L((Q^* \mathbin{\widetilde{@}} Q') \triangleleft X'_n)(X'_1, \ldots, X'_{n-1}) && \text{by L-52} \\
&= \mathcal{F}_L(Q^* \mathbin{\widetilde{@}} Q')(X'_1, \ldots, X'_n)\,.
\end{aligned}$$

The last step is apparent from Th-15 and Def. 182 because X'_n is crisp.

References

[1] P.B. Andrews. *An introduction to mathematical logic and type theory: to truth through proof*. Academic Press, 1986.

[2] T. Arnould and A. Ralescu. From “and” to “or”. In B. Bouchon-Meunier, L. Valverde, and R.R. Yager, editors, *IPMU ’92 - Advanced Methods in Artificial Intelligence*, volume 682 of *Lecture Notes in Computer Science*, pages 116–128. Springer, 1993.

[3] S. Barro, A. Bugarín, P. Cariñena, and F. Díaz-Hermida. A framework for fuzzy quantification models. GSI-99-01, Departamento de Electrónica e Computación, Universidade de Santiago di Compostela, E15782 Santiago de Compostela, Spain, 1999.

[4] J. Barwise. Monotone quantifiers and admissible sets. In *Generalized Recursion Theory II*, pages 1–38. North-Holland, Amsterdam, 1978.

[5] J. Barwise. On branching quantifiers in English. *Journal of Philosophical Logic*, 8:47–80, 1979.

[6] J. Barwise and R. Cooper. Generalized quantifiers and natural language. *Linguistics and Philosophy*, 4:159–219, 1981.

[7] J. Barwise and S. Feferman, editors. *Model-Theoretic Logics*. Springer, Berlin, 1985.

[8] J. van Benthem. Determiners and logic. *Linguistics and Philosophy*, 6, 1983.

[9] J. van Benthem. Questions about quantifiers. *Journal of Symbolic Logic*, 49, 1984.

[10] J.C. Bezdek. *Pattern Recognition with Fuzzy Objective Function Algorithms*. Plenum Press, New York, 1981.

[11] J.C. Bezdek, D. Dubois, and H. Prade, editors. *Fuzzy Sets in Approximate Reasoning and Information Systems*. Kluwer Academic Publishers, Dordrecht, NL, 1999.

[12] Z. Bien and K.C. Min, editors. *Fuzzy Logic and its Applications to Engineering, Information Sciences, and Intelligent Systems*. Information Science and Intelligent Systems. Kluwer Academic Publishers, Dordrecht, 1995.

[13] M. Black. Vagueness: an exercise in logical analysis. *Philosophy of Science*, 4:427–455, 1937. Reprinted with omissions in [80, p. 69-81].

[14] N. Blanchard. Cardinal and ordinal theories about fuzzy sets. In M.M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 149–157. North-Holland, Amsterdam, 1982.

[15] I. Bloch. Information combination operators for data fusion: a comparative review with classification. *IEEE Trans. on Systems, Man, and Cybernetics*, 26(1):52–67, 1996.

[16] I.M. Bocheński. *A history of formal logic*. Chelsea Publishing Company, New York, 2 edition, 1970.

[17] G. Bordogna, P. Carrara, and G. Pasi. Query term weights as constraints in fuzzy information retrieval. *Information Processing & Management*, 27(1):15–26, 1990.

[18] G. Bordogna and G. Pasi. A fuzzy information retrieval system handling users' preferences on document sections. In Dubois et al. [40].

[19] P. Bosc, B.B. Buckles, F.E. Petry, and O. Pivert. Fuzzy databases. In D. Dubois, H. Prade, and J. Bezdek, editors, *Fuzzy Sets in Approximate Reasoning and Expert Systems*, pages 403–468. Kluwer, Dordrecht, NL, 1999.

[20] P. Bosc and L. Lietard. Monotonic quantified statements and fuzzy integrals. In *Proc. of the NAFIPS/IFI/NASA '94 Joint Conference*, pages 8–12, San Antonio, Texas, 1994.

[21] P. Bosc and L. Lietard. On the comparison of the Sugeno and Choquet fuzzy integrals for the evaluation of quantified statements. In *Proceedings of EUFIT '95*, pages 709–716, 1995.

[22] P. Bosc and O. Pivert. Extending SQL retrieval features for the handling of flexible queries. In Dubois et al. [40], pages 233–251.

[23] N.R. Campbell. The sorites paradox. *Philosophical Studies*, 26:175–191, 1974.

[24] J. Cargile. The sorites paradox. *British Journal for the Philosophy of Science*, 1969.

[25] S.J. Chen and C.L. Hwang. *Fuzzy Multiple Attribute Decision Making: Methods and Applications*. Springer, Heidelberg, 1992.

[26] A. Church. A formulation of the simple theory of types. *Journal of Symbolic Logic*, 5:56–68, 1940.

[27] M. Delgado, F. Herrera, E. Herrera-Viedma, and L. Martinez. Combining numerical and linguistic information in group decision making. *Information Sciences*, 107:177–194, 1998.

[28] M. Delgado, D. Sánchez, and M.A. Vila. Un enfoque lógico para calcular el grado de cumplimiento de sentencias con cuantificadores linguísticos. In *Actas VII Congreso Español Sobre Tecnologías y Lógica Fuzzy (ESTYLF '97)*, pages 15–20, 1997.

[29] M. Delgado, D. Sánchez, and M.A. Vila. A survey of methods for evaluating quantified sentences. In *Proceedings of the First European Society for Fuzzy Logic and Technologies Conference (EUSFLAT '99)*, pages 279–282, 1999.

[30] M. Delgado, D. Sánchez, and M.A. Vila. Fuzzy cardinality based evaluation of quantified statements. *International Journal of Approximate Reasoning*, 23(1):23–66, 2000.

[31] A. De Luca and S. Termini. A definition of non-probabilistic entropy in the setting of fuzzy sets theory. *Information and Control*, 20(4):301–312, 1972.

[32] P. Dienes. On an implication function in many-valued systems of logic. *Journal of Symbolic Logic*, 14:95–97, 1949.

[33] J. van der Does. The dynamics of sophisticated laziness. In J. Groenendijk, editor, *Plurals and Anaphora*, DYANA-2 (Dynamic Interpretation of Natural Language) ESPRIT Basic Research Project 6852, Deliverable R.2.2.A, Part I, pages 1–52, 1993.

[34] D. Driankov, H. Hellendoorn, and M. Reinfrank. *An Introduction to Fuzzy Control*. Springer, 1996.

[35] D. Dubois, L. Godo, R. Lopez de Mantaras, and H. Prade. Qualitative reasoning with imprecise probabilities. *Journal of Intelligent Information Systems*, 2:319–363, 1993.

[36] D. Dubois and H. Prade. *Fuzzy Sets and Systems: Theory and Applications*. Academic Press, New York, 1980.

[37] D. Dubois and H. Prade. Fuzzy cardinality and the modelling of imprecise quantification. *Fuzzy Sets and Systems*, 16:199–230, 1985.

[38] D. Dubois and H. Prade. Fuzzy sets in approximate reasoning I, II. *Fuzzy Sets and Systems*, 40(1):143–202, 202–244, 1991.

[39] D. Dubois, H. Prade, and R. Yager. Merging fuzzy information. In D. Dubois, H. Prade, and J. Bezdek, editors, *Fuzzy Sets in Approximate Reasoning and Expert Systems*, pages 335–402. Kluwer, Dordrecht, NL, 1999.

[40] D. Dubois, H. Prade, and R.R. Yager, editors. *Fuzzy Information Engineering*. Wiley, New York, 1997.

[41] D. Edgington. Vagueness by degrees. In R. Keefe and P. Smith, editors, *Vagueness: A Reader*, pages 294–316. MIT Press/A Bradford Book, Cambridge, MA, 1997.

[42] K. Fine. Vagueness, truth and logic. *Synthese*, 1975.

[43] G. Frege. Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik*, 100:25–50, 1892.

[44] B.R. Gaines. Foundations of fuzzy reasoning. *Int. Journal of Man-Machine Studies*, 8:623–668, 1978.

[45] L.T.F. Gamut. *Intensional Logic and Logical Grammar*, volume II of *Logic, Language, and Meaning*. The University of Chicago Press, Chicago & London, 1984.

[46] I. Glöckner. DFS – an axiomatic approach to fuzzy quantification. TR97-06, Technical Faculty, University Bielefeld, 33501 Bielefeld, Germany, 1997.

[47] I. Glöckner. A framework for evaluating approaches to fuzzy quantification. TR99-03, Technical Faculty, University Bielefeld, 33501 Bielefeld, Germany, 1999.

[48] I. Glöckner. Advances in DFS theory. TR2000-01, Technical Faculty, University of Bielefeld, 33501 Bielefeld, Germany, 2000.

[49] I. Glöckner. An axiomatic theory of quantifiers in natural languages. TR2000-03, Technical Faculty, University of Bielefeld, 33501 Bielefeld, Germany, 2000.

[50] I. Glöckner. A broad class of standard DFSes. TR2000-02, Technical Faculty, University of Bielefeld, 33501 Bielefeld, Germany, 2000.

[51] I. Glöckner. Standard models of fuzzy quantification. TR2001-01, Technical Faculty, University of Bielefeld, 33501 Bielefeld, Germany, 2001.

[52] I. Glöckner and A. Knoll. Fuzzy quantifiers for processing natural language queries in content-based multimedia retrieval systems. TR97-05, Technical Faculty, University of Bielefeld, 33501 Bielefeld, Germany, 1997.

[53] I. Glöckner and A. Knoll. Application of fuzzy quantifiers in image processing: A case study. In *Proceedings of KES'99*, Adelaide, Australia, 1999.

[54] I. Glöckner and A. Knoll. Natural-language navigation in multimedia archives: An integrated approach. In *Proceedings of the Seventh ACM Multimedia Conference (MM '99)*, pages 313–322, Orlando, Florida, 1999.

[55] I. Glöckner and A. Knoll. Query evaluation and information fusion in a retrieval system for multimedia documents. In *Proceedings of Fusion'99*, pages 529–536, Sunnyvale, CA, 1999.

[56] I. Glöckner and A. Knoll. Architecture and retrieval methods of a search assistant for scientific libraries. In W. Gaul and R. Decker, editors, *Classification and Information Processing at the Turn of the Millennium*. Springer, Heidelberg, 2000.

[57] I. Glöckner and A. Knoll. Fuzzy quantification in granular computing and its role for data summarisation. In W. Pedrycz, editor, *Granular Computing: An Emerging Paradigm*, Studies in Fuzziness and Soft Computing, pages 215–256. Physica-Verlag, Heidelberg, 2001.

[58] I. Glöckner and A. Knoll. Architecture and retrieval methods of a search assistant for scientific libraries. In W. Gaul and R. Decker, editors, *Classification and Information Processing at the Turn of the Millennium*. Springer, Heidelberg, to appear.

[59] I. Glöckner, A. Knoll, and A. Wolfram. Data fusion based on fuzzy quantifiers. In *Proceedings of EuroFusion98, International Data Fusion Conference*, pages 39–46, 1998.

[60] J.A. Goguen. The logic of inexact concepts. *Synthese*, 19:325–373, 1969.

[61] F. Hamm. *Natürlich-sprachliche Quantoren. Modelltheoretische Untersuchungen zu universellen semantischen Beschränkungen*. Number 236 in Linguistische Arbeiten. Niemeyer, Tübingen, 1989.

[62] C.J. Harris, C.G. Moore, and M. Brown. *Intelligent Control: Aspects of Fuzzy Logic and Neural Nets*. World Scientific, Singapore, 1993.

[63] J. van Heijenoort. *From Frege to Gödel – A Source Book in Mathematical Logic*. Harvard University Press, Cambridge, MA, 1967.

[64] I. Heim. *The semantics of definite and indefinite noun phrases*. PhD thesis, University of Massachusetts, Amherst, 1982.

[65] H. Helbig. *Die semantische Struktur natürlicher Sprache. Wissensrepräsentation mit MultiNet*. Springer, Berlin, Heidelberg, 2000.

[66] L. Henkin. Some remarks on infinitely long formulas. In *Infinitistic methods*, pages 167–183. Pergamon Press, New York, and Polish Scientific Publishers, Warsaw, 1961.

[67] F. Herrera and J.L. Verdegay. On group decision making under linguistic preferences and fuzzy linguistic quantifiers. In B. Bouchon, R.R. Yager, and L.A. Zadeh, editors, *Fuzzy Logic and Soft Computing*, pages 173–180. World Scientific, 1995.

[68] J. Hintikka. Quantifiers vs. quantification theory. *Dialectica*, 27:329–358, 1973.

[69] P. Hájek and L. Kohout. Fuzzy implications and generalized quantifiers. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 4(3):225–233, 1996.

[70] F. de Jongh and H. Verkuyl. Generalized quantifiers: The properness of their strength. In J. van Benthem and A. ter Meulen, editors, *Generalized Quantifiers in Natural Language*. Foris, Dordrecht, 1984.

[71] J. Kacprzyk. *Multistage Decision-Making under Fuzziness*. Verlag TÜV Rheinland, Cologne, 1983.

[72] J. Kacprzyk. Intelligent data analysis via linguistic data summaries: A fuzzy logic approach. In R. Decker and W. Gaul, editors, *Classification and Information Processing at the Turn of the Millenium*, pages 153–161. Springer, 2000.

[73] J. Kacprzyk and P. Strykowski. Linguistic summaries of sales data at a computer retailer: A case study. In *Proc. of 8th International Fuzzy Systems Association World Congress (IFSA '99)*, pages 29–33, 1999.

[74] J. Kacprzyk and S. Zadrożny. Fuzzy queries in Microsoft Access V.2. In Dubois et al. [40], pages 223–232.

[75] J. Kacprzyk, S. Zadrożny, and M. Fedrizzi. An interactive GDSS for consensus reaching using fuzzy logic with linguistic quantifiers. In Dubois et al. [40], pages 567–574.

[76] J. Kacprzyk and A. Ziólkowsi. Database queries with fuzzy linguistic quantifiers. *IEEE Trans. on Systems, Man, and Cybernetics*, 16(3):474–479, 1986.

[77] H. Kamp. A theory of truth and semantic representation. In J. Groenendijk, T. Janssen, and M. Stokhof, editors, *Formal Methods in the Study of Language*, Amsterdam: Mathematical Centre, 1981.

[78] J. Kamp. Two theories about adjectives. In Edward Keenan, editor, *Formal Semantics of Natural Language*. Cambridge University Press, 1975.

[79] A. Kaufmann. *Introduction to the Theory of Fuzzy Subsets*, volume 1: Fundamental theoretical elements. Academic Press, New York, 1975.

[80] R. Keefe and P. Smith, editors. *Vagueness: A Reader*. A Bradford Book/MIT Press, Cambridge, MA, 1997.

[81] E. Keenan and L. Moss. Generalized quantifiers and expressive power of natural language. In J. van Benthem and A. ter Meulen, editors, *Generalized Quantifiers in Natural Language,* pages 73–124. Foris, Dordrecht, 1984.

[82] E. Keenan and J. Stavi. A semantic characterization of natural language determiners. *Linguistics and Philosophy*, 9:253–326, 1986.

[83] E. Keenan and D. Westerståhl. Generalized quantifiers in linguistics and logic. In J. van Benthem and A. Ter Meulen, editors, *Handbook of Logic and Language,* chapter 15, pages 837–893. Elsevier, Amsterdam, NL, 1997.

[84] H.J. Keisler. Logic with the quantifier 'There Exist Uncountably Many'. *Annals of Mathematical Logic*, 1:1–93, 1969.

[85] S.C. Kleene. *Introduction to Metamathematics*. North-Holland, Amsterdam, 1952.

[86] E.P. Klement, R. Mesiar, and E. Pap. Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. *Fuzzy Sets and Systems*, 104:3–13, 1999.

[87] G.J. Klir and T.A. Folger. *Fuzzy Sets, Uncertainty, and Information*. Prentice Hall, Englewood Cliffs, NJ, 1988.

[88] G.J. Klir, U.H. St. Clair, and B. Yuan. *Fuzzy Set Theory: Foundations and Applications*. Prentice Hall, Upper Saddle River, NJ, 1997.

[89] G.J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, Upper Saddle River, NJ, 1995.

[90] G.J. Klir and B. Yuan, editors. *Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems: Selected Papers by Lotfi A. Zadeh*. World Scientific, Singapore, 1996.

[91] L. Kreiser, S. Gottwald, and W. Stelzner, editors. *Nichtklassische Logik*. Akademie-Verlag, Berlin, Germany, 1988.

[92] S. A. Kripke. Semantical analysis of modal logic II. In J.W. Addison and A. Tarski, editors, *The theory of models*, pages 206–220. North-Holland, Amsterdam, 1965.

[93] S.A. Kripke. A completeness theorem in modal logic. *Journal for Symbolic Logic*, 24:1–14, 1959.

[94] R. Kruse, E. Schwecke, and J. Heinsohn. *Uncertainty and Vagueness in Knowledge Based Systems: Numerical Methods*. Springer, Heidelberg, 1991.

[95] Y. Lai and C. Hwang. *Fuzzy Mathematical Programming*. Springer, Heidelberg, 1992.

[96] G. Lakoff. Hedges: a study in meaning criteria and the logic of fuzzy concepts. *Journal of Philosophical Logic*, 2:458–508, 1973.

[97] P. Lindström. First order predicate logic with generalized quantifiers. *Theoria*, 32:186–195, 1966.

[98] P. Lindström. On extensions of elementary logic. *Theoria*, 35:1–11, 1969.

[99] Y. Liu and E.E. Kerre. An overview of fuzzy quantifiers. (I). interpretations. *Fuzzy Sets and Systems*, 95:1–21, 1998.

[100] Y. Liu and E.E. Kerre. An overview of fuzzy quantifiers. (II). reasoning and applications. *Fuzzy Sets and Systems*, 96:135–146, 1998.

[101] K.F. Machina. Truth, belief and vagueness. *Journal of Philosophical Logic*, 5:47–78, 1976. Reprinted in [80, p. 174-203].

[102] E.H. Mamdani and S. Assilian. An experiment in linguistic synthesis with a fuzzy logic controller. *Int. Journal of Man-Machine Studies*, 7:1–13, 1975.

[103] Wido Menhardt. *Unscharfe Mengen (Fuzzy Sets) zur Behandlung von Unsicherheit in der Bildanalyse*. PhD thesis, Fachbereich Informatik, Universität Hamburg, Germany, 1990.

[104] R. Mesiar and M. Navara. Diagonals of continuous triangular norms. *Fuzzy Sets and Systems*, 104:35–41, 1999.

[105] S. Miyamoto. *Fuzzy Sets in Information Retrieval and Cluster Analysis*. Kluwer Academic Publishers, Dordrecht, 1990.

[106] R. Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[107] R. Montague. The proper treatment of quantification in ordinary English. In J. Hintikka, J. Moravcsik, and P. Suppes, editors, *Approaches to Natural Language*, pages 221–242. Reidel, Dordrecht, 1973.

[108] A. Mostowski. On a generalization of quantifiers. *Fundamenta Mathematicae*, 44:12–36, 1957.

[109] M. Mucientes, R. Iglesias, C. Regueiro, A. Bugarín, P. Cariñena, and S. Barro. Fuzzy temporal rules for mobile robot guidance in dynamic environments. *IEEE Trans. on Systems, Man, and Cybernetics, Part C*, 66(1), 2002.

[110] M. Mukaidono. On some properties of fuzzy logic. *Syst.—Comput.—Control*, 6(2):36–43, 1975.

[111] H. Nakanishi, I.B. Turksen, and M. Sugeno. A review and comparison of six reasoning methods. *Fuzzy Sets and Systems*, 57(3):257–294, 1993.

[112] C.V. Negoita. *Expert Systems and Fuzzy Systems*. Benjamin/Cummings, menlo park, ca edition, 1985.

[113] S.E. Newstead and K. Coventry. The role of context and functionality in the interpretation of quantifiers. *European Journal of Cognitive Psychology*, 12:243–259, 2000.

[114] S.E. Newstead, P. Pollard, and D. Riezebos. The effect of set size on the interpretation of quantifiers used in rating scales. *Applied Ergonomics*, 18(3):178–182, 1987.

[115] H.T. Nguyen and E.A. Walker. *A First Course in Fuzzy Logic*. Chapman & Hall/CRC Press, Boca Raton, FL, 2nd edition, 2000.

[116] S.K. Pal and D.K. Majumder. *Fuzzy Mathematical Approach to Pattern Recognition*. John Wiley, New York, 1986.

[117] W. Pedrycz. *Fuzzy Control and Fuzzy Systems*. John Wiley, New York, 1989.

[118] H. Prade. Lipski's approach to incomplete information data bases restated and generalized in the setting of zadeh's possibility theory. *Information Sciences*, 9(1):27–42, 1984.

[119] H. Prade. A two-layer fuzzy pattern matching procedure for the evaluation of conditions involving vague quantifiers. *Journal of Intelligent and Robotic Systems*, 3:93–103, 1990.

[120] H. Prade and C. Testemale. Generalizing database relational algebra for the treatment of incomplete or uncertain information and vague queries. *Information Sciences*, 34(2):115–143, 1984.

[121] W.V. Quine. *Word and Object*. Cambridge University Press, Cambridge, Mass., 1960.

[122] W.V. Quine. *Ontological Relativity and other Essays*. Columbia University Press, New York, 1969.

[123] A. Ralescu, B. Bouchon, and D. Ralescu. Combining fuzzy quantifiers. In W. Chiang and J. Lee, editors, *Fuzzy Logic for the Applications to Complex Systems*, pages 246–252. World Scientific, Singapore, 1995.

[124] A.L. Ralescu. A note on rule representation in expert systems. *Information Sciences*, 38:193–203, 1986.

[125] D. Ralescu. Cardinality, quantifiers, and the aggregation of fuzzy criteria. *Fuzzy Sets and Systems*, 69:355–365, 1995.

[126] D. Rasmussen and R.R. Yager. A fuzzy SQL summary language for data discovery. In Dubois et al. [40], pages 253–264.

[127] N. Rescher. Quantifiers in many-valued logic. *Logique et Analyse*, 7:181–184, 1964.

[128] N. Rescher. *Many-valued logic*. Mc Graw-Hill, 1969.

[129] J.B. Rosser. The introduction of quantification into a Three-valued logic. reprinted for the members of the Fifth International Congress for the Unity of Science (Cambridge, 1939) from *The Journal of Unified Science (Erkenntnis)*, vol. 9 (never published). Abstracted in: *Journal of Symbolic Logic*, vol. 4 (1939), 170-171, 1939.

[130] J.B. Rosser and A.R. Turquette. Axiom schemes for m-valued functional calculi of first order. Part I. Definition of axiom schemes and proof of plausibility. *Journal of Symbolic Logic*, 13:177–192, 1948.

[131] J.B. Rosser and A.R. Turquette. Axiom schemes for m-valued functional calculi of first order. Part II. Deductive completeness. *Journal of Symbolic Logic*, 16:22–34, 1951.

[132] J.B. Rosser and A.R. Turquette. *Many-valued logics*. North-Holland, Amsterdam, 1952.

[133] R. Rovatti and C. Fantuzzi. s-norm aggregation of infinite collections. *Fuzzy Sets and Systems*, 84:255–269, 1996.

[134] B. Russell. Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30:222–262, 1908.

[135] B. Russell. *History of Western Philosophy*. Unwin University Books, 1946.

[136] I.A. Sag and E.F. Prince. Bibliography of works dealing with presupposition. In Oh and Dinneen, editors, *Syntax and Semantics*, volume 11: Presupposition, pages 389–403. Academic Press, New York, 1979.

[137] R. van der Sandt. *Context and Presupposition*. Croom Helm, New York, 1988.

[138] K. Schlechta. Defaults as generalized quantifiers. *Journal of Logic and Computation*, 5(4):473–494, 1995.

[139] B. Schweizer and A. Sklar. *Probabilistic metric spaces*. North-Holland, Amsterdam, 1983.

[140] J. Sgro. Completeness theorems for topological models. *Annals of Mathematical Logic*, pages 173–193, 1977.

[141] W. Silvert. Symmetric summation: A class of operations on fuzzy sets. *IEEE Trans. on Systems, Man, and Cybernetics*, 9:657–659, 1979.

[142] R.A. Sorensen. *Blindspots*. Clarendon Press, Oxford, 1988.

[143] M. Spies. *Syllogistic inference under uncertainty*. Psychologie-Verlags-Union, München, 1989.

[144] M. Stokhof and J. Groenendijk, editors. *Studies in Discourse Representation Theory and the Theory of Generalized Quantifiers*. De Gruyter, 1986.

[145] M. Sugeno. Fuzzy measures and fuzzy integrals: A survey. In M.M. Gupta, G.N. Saridis, and B.R. Gaines, editors, *Fuzzy Automata and Decision Processes*, pages 89–102. North-Holland, Amsterdam, 1977.

[146] P. Suppes. *Introduction to logic*. Van Nostrand, Princeton, New Jersey, 1957.

[147] H. Thiele. Zum Aufbau einer systematischen Theorie von Fuzzy-Quantoren. In *13. Workshop “Interdisziplinäre Methoden in der Informatik” – Arbeitstagung des Lehrstuhls Informatik I, Forschungsbericht Nr. 506*, pages 35–55. Fachbereich Informatik der Universität Dortmund, 1993.

[148] H. Thiele. On T-quantifiers and S-quantifiers. In *The Twenty-Fourth International Symposium on Multiple-Valued Logic*, pages 264–269, Boston, MA, 1994.

[149] H. Thiele. On fuzzy quantifiers. In Bien and Min [12], pages 343–352.

[150] H. Thiele. On median quantifiers. Technical report No. CI-27/98, University of Dortmund, Dept. of Computer Science/XI, 44221 Dortmund, Germany, 1998.

[151] E. Thijsse. On some proposed universals of natural language. In A. ter Meulen, editor, *Studies in Model-Theoretic Semantics*, pages 19–36. Foris, Dordrecht, 1983.

[152] H. Thöne. *Precise conclusions under uncertainty and incompleteness in deductive databases*. PhD thesis, University of Tübingen, Germany, 1994.

[153] H. Thöne, U. Güntzer, and W. Kiessling. Toward precision of probability bound propagation. In *Proc. of the 8th Conf. on Uncertainty in AI*, pages 315–322, San Francisco, 1992. Morgan and Kaufmann.

[154] R. Thomason, editor. *Formal Philosophy: Selected Papers of Richard Montague*. Yale University Press, New Haven, CT, 1974.

[155] M. Tye. Sorites paradoxes and the semantics of vagueness. In J.E. Tomberlin, editor, *Philosophical Perspectives 8: Logic and Language*, pages 189–206. Ridgeview, Atascadero, CA, 1994. Reprinted with ommisions in [80, p. 281-293].

[156] M.A. Vila, J.C. Cubero, J.M. Medina, and O. Pons. Using OWA operator in flexible query processing. In R.R. Yager and J. Kacprzyk, editors, *The Ordered Weighted Averaging Operators: Theory, Methodology and Applications*, pages 258–274. Kluwer Academic Publishers, Boston, 1997.

[157] W.J. Walkoe Jr. Finite partially-ordered quantification. *Journal of Symbolic Logic*, 35:535–555, 1970.

[158] D. Westerståhl. Branching generalized quantifiers and natural language. In Peter Gärdenfors, editor, *Generalized Quantifiers*, pages 269–298. Reidel, 1987.

[159] T. Williamson. Vagueness and ignorance. *Proceedings of the Aristotelian Society, Supplementary Volume*, 66:145–162, 1992. Reprinted in [80, p. 265-280].

[160] C. Wright. Language-mastery and the sorites paradox. In G. Evans and J. McDowell, editors, *Truth and Meaning: Essays in Semantics*, pages 223–247. Clarendon Press, Oxford, 1976.

[161] M. Wygralak. An axiomatic approach to scalar cardinalities of fuzzy sets. *Fuzzy Sets and Systems 110*, pages 175–179, 2000.

[162] R.R. Yager. A new approach to the summarization of data. *Information Sciences*, 28:69–86, 1982.

[163] R.R. Yager. A new approach to the summarization of data. *Information Sciences*, 28:69–86, 1982.

[164] R.R. Yager. Quantified propositions in a linguistic logic. *Int. Journal of Man-Machine Studies*, 19:195–227, 1983.

[165] R.R. Yager. Approximate reasoning as a basis for rule-based expert systems. *IEEE Trans. on Systems, Man, and Cybernetics*, 14(4):636–643, Jul./Aug. 1984.

[166] R.R. Yager. General multiple-objective decision functions and linguistically quantified statements. *Int. Journal of Man-Machine Studies*, 21:389–400, 1984.

[167] R.R. Yager. Reasoning with fuzzy quantified statements. Part I. *Kybernetes*, 14:233–240, 1985.

[168] R.R. Yager. Reasoning with fuzzy quantified statements. Part II. *Kybernetes*, 15:111–120, 1985.

[169] R.R. Yager. A characterization of the extension principle. *Fuzzy Sets and Systems*, 18(3):205–217, 1986.

[170] R.R. Yager. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Trans. on Systems, Man, and Cybernetics*, 18(1):183–190, 1988.

[171] R.R. Yager. Connectives and quantifiers in fuzzy sets. *Fuzzy Sets and Systems*, 40:39–75, 1991.

[172] R.R. Yager. Fuzzy quotient operators for fuzzy relational databases. In *Proceedings of IFES '91*, pages 289–296, 1991.

[173] R.R. Yager. On linguistic summaries of data. In W. Frawley and G. Piatetsky-Shapiro, editors, *Knowledge Discovery in Databases*, pages 347–363. AAAI/MIT Press, 1991.

[174] R.R. Yager. A general approach to rule aggregation in fuzzy logic control. *Journal of Applied Intelligence*, 2:333–351, 1992.

[175] R.R. Yager. Counting the number of classes in a fuzzy set. *IEEE Trans. on Systems, Man, and Cybernetics*, 23(1):257–264, 1993.

[176] R.R. Yager. Families of OWA operators. *Fuzzy Sets and Systems*, 59:125–148, 1993.

[177] R.R. Yager and J. Kacprzyk. Linguistic data summaries: A perspective. In *Proc. of 8th International Fuzzy Systems Association World Congress (IFSA '99)*, pages 44–48, 1999.

[178] R.R. Yager, S. Ovchinnikov, R.M. Tong, and H.T. Nguyen, editors. *Fuzzy Sets and Applications—Selected papers by L.A. Zadeh*. John Wiley, New York, 1987.

[179] L.A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[180] L.A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences*, 8,9:199–249,301–357, 1975.

[181] L.A. Zadeh. Fuzzy logic and approximate reasoning. *Synthese*, 30:407–428, 1975.

[182] L.A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.

[183] L.A. Zadeh. PRUF—a meaning representation language for natural languages. *Int. Journal of Man-Machine Studies*, 10:395–460, 1978.

[184] L.A. Zadeh. A theory of approximate reasoning. In J. Hayes, D. Michie, and L. Mikulich, editors, *Machine Intelligence*, volume 9, pages 149–194. Halstead, New York, 1979.

[185] L.A. Zadeh. Possibility theory and soft data analysis. In L. Cobb and R.M. Thrall, editors, *Mathematical Frontiers of the Social and Policy Sciences*, pages 69–129. Westview Press, Boulder, 1981.

[186] L.A. Zadeh. Test-score semantics for natural languages and meaning representation via pruf. In B.B. Rieger, editor, *Empirical Semantics, Vol. 1*, pages 281–349. Brockmeyer, Bochum, 1981.

[187] L.A. Zadeh. Fuzzy probabilities and their role in decision analysis. In *Proc. of the IFAC Symp. on Theory and Application of Digital Control*, pages 15–23, New Delhi, 1982.

[188] L.A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Computers and Mathematics with Applications*, 9:149–184, 1983.

[189] L.A. Zadeh. A theory of commonsense knowledge. In H.J. Skala et al., editor, *Aspects of Vagueness*, pages 257–295. Reidel, Dordrecht, 1984.

[190] L.A. Zadeh. Syllogistic reasoning in fuzzy logic and its application to usuality and reasoning with dispositions. *IEEE Trans. on Systems, Man, and Cybernetics*, 15(6):754–763, 1985.

[191] L.A. Zadeh. A computational theory of dispositions. *International Journal of Intelligent Systems*, 2(1):39–63, 1987.

[192] L.A. Zadeh. The role of fuzzy logic in the management of uncertainty in expert systems. In R.R. Yager et al., editor, *Fuzzy Sets and Applications: Selected Papers by L.A. Zadeh*, pages 413–441. Wiley, New York, 1987.

[193] S. Zadrożny and J. Kacprzyk. On database summarization using a fuzzy querying interface. In *Proc. of 8th International Fuzzy Systems Association World Congress (IFSA '99)*, pages 39–43, 1999.

[194] M. Zemankova and A. Kandel. Implementing imprecision in information systems. *Information Sciences*, 37(1-3):107–141, 1985.

[195] M. Zemankova-Leech and A. Kandel. *Fuzzy Relational Data Bases: A Key to Expert Systems*. Verlag TÜV Rheinland, Cologne, 1984.

[196] H.J. Zimmermann. *Fuzzy Sets, Decision Making, and Expert Systems*. Kluwer, Boston, MA, 1987.

[197] H.J. Zimmermann. *Fuzzy Set Theory—and its Applications*. Kluwer, Boston, MA, (2nd rev. ed.) edition, 1991.