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Branching of Fuzzy Quantifiers and Multiple Variable Binding: An Extension of DFS Theory

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Abstract

Lindström [11] introduced a very powerful notion of quantifiers, which permits multi-place quantification and the simultaneous binding of several variables. A special case, 'branching' quantifiers, was found to be useful in linguistics, specifically for modelling reciprocal constructions like "Most men and most women admire each other". Westerståhl [13] showed how to compute the three-place Lindström quantifier for "Q_1 A's and Q_2 B's R each other" from the two-place quantifiers Q_1 and Q_2, assuming precise quantifiers and crisp arguments. In the report, I generalize Westerståhl's method to approximate quantifiers like "many" and fuzzy arguments like "young". A consistent interpretation is achieved by incorporating Lindström quantifiers into the DFS theory of fuzzy quantification [5, 8], which rests on a system of formal adequacy criteria. The proposed analysis is of special importance to linguistic data summarization because the full meaning of reciprocal summarizers (e.g. describing factors which are "correlated" or "associated" with each other), can only be captured by branching quantification.[1]

1. Introduction

The quantifiers found in natural language (NL) are by no means restricted to the absolute and proportional types usually considered in fuzzy set theory. The linguistic theory of quantification, i.e. the Theory of Generalized Quantifiers [2] (TGQ), recognizes more than thirty different types of quantifiers, including quantifiers of exception like "all except about ten", cardinal comparatives like "many more than" and many others [10]. These quantifiers can be unary (like proper names in "Ronald is X") or multi-place; quantitative (like "about ten") or non-quantitative (like "all except Lotfi"); and they can be simplex or constructed, like "most married X's are Y's or Z's". However, it is not only the diversity of possible quantifiers in NL which poses difficulties to a systematic and comprehensive modelling of NL quantification. Even in simple cases like "most", the ways in which these quantifiers interact when combined in meaningful propositions can be complex and sometimes even puzzling. Consider "Most men and most women admire each other", for example, in which we find a reciprocal predicate, "admire each other". Barwise [1] argues that so-called branching quantification is needed to capture the meaning of propositions involving reciprocal predicates. Without branching quantifiers, the above example must be linearly phrased as either

a. $[\mathrm{most}\, x : \mathrm{men}(x)][\mathrm{most}\, y : \mathrm{women}(y)]\, \mathrm{adm}(x, y)$

b. $[\mathrm{most}\, y : \mathrm{women}(y)][\mathrm{most}\, x : \mathrm{men}(x)]\, \mathrm{adm}(x, y)$.

Neither interpretation captures the expected symmetry with respect to the men and women involved. In fact, we need a construction like

$$\left.\begin{array}{l}[Q_1\, x : \quad \mathrm{men}(x)] \\ [Q_2\, y : \mathrm{women}(y)]\end{array}\right\rangle \mathrm{adm}(x, y)$$

where the quantifiers $Q_1 = Q_2 = \mathrm{most}$ operate in parallel and independently of each other. This branching use of quantifiers can be analysed in terms of Lindström quantifiers [11], i.e. multi-place quantifiers capable of binding several variables. In this case, we have three arguments, and the quantifier should bind x in $\mathrm{men}(x)$, y in $\mathrm{women}(y)$ and both variables in $\mathrm{adm}(x, y)$. Thus, the above branching expression can be modelled by a Lindström quantifier Q of type $\langle 1, 1, 2\rangle$:

$$Q_{x,y,xy}(\mathrm{men}(x), \mathrm{women}(y), \mathrm{adm}(x, y)) .$$

Let us now consider the problem of assigning an interpretation to such quantifiers. Obviously, Q depends on the meaning of "most". Assuming a fixed choice of base set or 'universe' $E \neq \varnothing$, over which the quantification ranges, the quantifier "most" in its precise sense (majority of) can be expressed as

$$\textbf{most}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1 \cap Y_2| > \frac{1}{2}|Y_1| \\ 0 & : \quad \text{else}\end{cases}$$

[1] The proofs of all theorems cited here are listed in [8], where the subject is discussed at some more length. The report is available from the author's homepage http://www.techfak.uni-bielefeld.de/techfak/ags/ti/personen/ingo/.

for all crisp subsets Y_1, Y_2 of E (assuming that E be finite). This lets us evaluate "most men are married" by calculating $\textbf{most}(\textbf{men}, \textbf{married})$, where **men**, **married** are the sets of men and of married people, respectively. The definition can be extended to base sets of infinite cardinality if so desired. A possible interpretation of "most" in this case is

$$\textbf{most}(Y_1, Y_2) = \begin{cases} 1 : & Y_1 \cap Y_2 \cap \beta(Y_1 \cap Y_2) \neq \varnothing \\ & \text{for all bijections } \beta : Y_1 \longrightarrow Y_1 \\ 0 : & \text{else} \end{cases}$$

where $Y_1, Y_2 \in \mathcal{P}(E)$. Now attempting a similar analysis of the above quantifier Q, we notice that the quantifier must accept three arguments, i.e. the sets $A, B \in \mathcal{P}(E)$ of men and women, and the binary relation $R \in \mathcal{P}(E^2)$ of people admiring each other, assumed to be crisp for simplicity. The quantifier then determines a two-valued quantification result $Q(A, B, R) \in \{0, 1\}$ from these data. Barwise [1, p. 63] showed how to define Q in a special case; here I adopt Westerståhl's reformulation for binary quantifiers [13, p. 274, (D1)]. Hence consider a choice of $Q_1, Q_2 : \mathcal{P}(E)^2 \longrightarrow \{0, 1\}$. Let us further assume that the Q_i's, like "most", are nondecreasing in their second argument, i.e. $Q_i(Y_1, Y_2) \le Q_i(Y_1, Y_2')$ whenever $Y_2 \subseteq Y_2'$. In this case, the complex quantifier Q becomes:

$$Q(A, B, R) \qquad (*)$$
$$= \begin{cases} 1 : & \exists U \times V \subseteq R : Q_1(A, U) = 1 \wedge Q_2(B, V) = 1 \\ 0 : & \text{else} \end{cases}$$

Hence "Most men and most women admire each other" means that there exists a mutual admiration group $U \times V \subseteq \mathrm{adm}$ such that most man and most women belong to that group. In my view, this analysis is correct and expresses the intended meaning of the example. It should be remarked at this point that Westerståhl, unlike Barwise, is only concerned with 'conservative' quantifiers, thus assuming that $Q_i(Y_1, Y_2) = Q_i(Y_1, Y_1 \cap Y_2)$. This permits him to restrict attention to $U \times V \subseteq R \cap (A \times B)$ rather than $U \times V \subseteq R$ without changing the interpretation. Westerståhl then extends this analysis to a generic construction which also admits non-monotonic quantifiers (p. 281, Def. 3.1).

But, how can we incorporate approximate quantifiers and fuzzy arguments, like in "Many young and most old people respect each other"?

In the following, I present a method which assigns meaningful interpretations to such cases. I first describe the formal framework in which I will carry out this endeavour. In this connection I also present some practical models for 'ordinary' fuzzy quantification; as we shall see, these will also prove useful for fuzzy branching quantification. Following this, I show how Lindström-like quantifiers capable of binding several variables can be incorporated into the chosen framework. Finally I explain how the above analysis of branching quantifiers and its generalization by Westerståhl can be utilized for interpreting fuzzy branching quantification.

2. The Linguistic Theory of Fuzzy Quantification

The Theory of Generalized Quantifiers [2, 10] rests on a simple but expressive model of two-valued quantifiers, which offers a uniform representation for the diversity of NL quantifiers mentioned above. However, TGQ was not developed with fuzzy sets in mind, and its semantical analysis is essentially two-valued. In order to reconcile fuzzy quantifiers and linguistic consensus, the author developed a linguistic theory of fuzzy quantification, known as 'DFS theory' [8, 5, 6], which generalises the basic concepts of TGQ to approximate quantifiers and fuzzy arguments, thus achieving a comprehensive treatment fuzzy NL quantifiers, and coherent interpretations which follow the linguistic expectations. To accomplish this, the theory introduces the following basic notions.

Definition 1 *An n-ary* fuzzy quantifier $\widetilde{Q}$ *on a base set $E \neq \varnothing$ assigns a gradual quantification result $\widetilde{Q}(X_1, \dots, X_n) \in [0, 1]$ to each choice of fuzzy subsets $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.*

($\widetilde{\mathcal{P}}(E)$ denotes the fuzzy powerset). Fuzzy quantifiers constitute an expressive class of operators. However, they are often hard to define because the familiar concept of cardinality of crisp sets is not applicable to the fuzzy sets that form the arguments of a fuzzy quantifier. It is therefore necessary to permit a simplified specification, however powerful enough to embed all quantifiers in the sense of TGQ.

Definition 2 *An n-ary* semi-fuzzy quantifier *on a base set $E \neq \varnothing$ assigns a gradual result $Q(Y_1, \dots, Y_n) \in [0, 1]$ to all* crisp *subsets $Y_1, \dots, Y_n \in \mathcal{P}(E)$.*

Compared to fuzzy quantifiers, semi-fuzzy quantifiers are clearly better a specification medium: they are much easier to define because only crisp inputs must be considered. In particular, the usual crisp cardinality is applicable to their arguments, which is essential to describing the meaning of quantifiers. An interpretation mechanism is used to associate these specifications with their matching fuzzy quantifiers.

Definition 3 *A quantifier fuzzification mechanism (QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier Q a fuzzy quantifier $\mathcal{F}(Q)$ of the same arity and on the same base set.*

The resulting fuzzy quantifiers $\mathcal{F}(Q)$ can then be applied to fuzzy arguments. Assuming a 'good' choice of $\mathcal{F}$, i.e. a *plausible model* of fuzzy quantification, this method will result in meaningful interpretations of NL quantification involving approximate quantifiers and fuzzy arguments. In order to ensure coherent and plausible results, the QFM should conform to all requirements of linguistic relevance. This preservation of certain properties, or compatibility of $\mathcal{F}$ with constructions on (semi-)fuzzy quantifiers, can be likened to the mathematical concept of a homomorphism (structure-preserving mapping). The formalization of a large number of such properties and the subsequent extraction of a compact description converged into the following system of six basic criteria.

(Z-1) **Correct generalisation.** For all crisp arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, we require that

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n) .$$

(Combined with the other axioms, this condition can be restricted to $n \leq 1$).
Rationale: a semi-fuzzy quantifier Q is defined only for crisp arguments, while $\mathcal{F}(Q)$ is defined for arbitrary fuzzy arguments. If all arguments are crisp, Q and $\mathcal{F}(Q)$ must match.

(Z-2) **Membership assessment.** The two-valued quantifier defined by $\pi_e(Y) = 1$ if $e \in Y$ and $\pi_e(Y) = 0$ otherwise for crisp Y, has the obvious fuzzy counterpart $\widetilde{\pi}_e(X) = \mu_X(e)$ for fuzzy subsets of E. I require that $\mathcal{F}(\pi_e) = \widetilde{\pi}_e$.
Rationale: Membership assessment (crisp or fuzzy) can be modelled through quantifiers. For an element e of the base set, we can define a two-valued quantifier π_e which checks if e is present in its argument. Similarly, we can define a fuzzy quantifier $\widetilde{\pi}_e$ which returns the degree to which e is contained in its argument. It is natural to require that the crisp quantifier π_e be mapped to $\widetilde{\pi}_e$, which serves the same purpose in the fuzzy case.

The constructions considered next will depend on the complement and union of fuzzy sets. Thiele [12] has shown that various plausible choices of fuzzy existential quantifiers exist, which are closely tied to a corresponding fuzzy disjunction (and hence, union). Consequently, we must allow some variability here. The fuzzy connectives which best match the behaviour of a QFM on quantifiers, called its induced connectives, are obtained from a canonical construction.

Definition 4 *The* induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : [0,1]^n \longrightarrow [0,1]$ *of a 'semi-fuzzy' truth function $f : \{0,1\}^n \longrightarrow [0,1]$ is defined by $\widetilde{\mathcal{F}}(f) = \mathcal{F}(f \circ \eta^{-1}) \circ \widetilde{\eta}$, where $\eta(y_1, \ldots, y_n) = \{i : y_i = 1\}$ for all $y_1, \ldots, y_n \in \{0,1\}$ and $\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(i) = x_i$ for all $x_i \in [0,1]$, $i \in \{1, \ldots, n\}$.*

Whenever $\mathcal{F}$ is understood, I abbreviate $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$, $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ etc. The induced fuzzy connectives are extended to fuzzy set operations in the usual ways, e.g. $\mu_{\widetilde{\neg} X}(e) = \widetilde{\neg}\, \mu_X(e)$ for complementation and $\mu_{X_1 \widetilde{\cup} X_2}(e) = \mu_{X_1}(e) \,\widetilde{\vee}\, \mu_{X_2}(e)$ for unions of fuzzy sets. The desired criteria involving these operations can now be expressed as follows.

(Z-3) **Dualisation.** $\mathcal{F}$ must preserve dualisation of quantifiers, i.e. $\mathcal{F}(Q')(X_1, \ldots, X_n) = \widetilde{\neg}\, \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, \widetilde{\neg}\, X_n)$ for all fuzzy arguments $X_1, \ldots, X_n$ whenever $Q'(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)$ for all crisp arguments $Y_1, \ldots, Y_n$.
Rationale: Obviously, a phrase like "all X's are Y's" should have the same result as "it is not the case that some X's are not Y's".

(Z-4) **Union.** $\mathcal{F}$ must preserve unions of arguments, i.e. we expect that $\mathcal{F}(Q')(X_1, \ldots, X_{n+1}) = \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, X_n \,\widetilde{\cup}\, X_{n+1})$ whenever $Q'(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$.
Rationale: It should not matter whether "many X's are Y's or Z's" is computed by evaluating $\mathcal{F}(\textbf{many})(X, Y \,\widetilde{\cup}\, Z)$ or by computing $\mathcal{F}(Q)(X, Y, Z)$ with $Q(X, Y, Z) = \textbf{many}(X, Y \cup Z)$.

(Z-5) **Monotonicity in arguments.** $\mathcal{F}$ must preserve monotonicity in arguments, i.e. if Q is nondecreasing/nonincreasing in the i-th argument, then $\mathcal{F}(Q)$ has the same property. (When combined with the other axioms, the condition can be restricted to the case that Q is nonincreasing in its n-th argument.)
Rationale: There must be a systematically different interpretation of statements like "all men are tall" and "all young men are tall" where the former statement expresses the stricter condition.

The last criterion which I will state requires the extension of mappings $f : E \longrightarrow E'$ to fuzzy powerset mappings $f' : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$. The usual way of doing this is by applying the standard extension principle. In this case, the extension $f' = \hat{\hat{f}}$ becomes $\mu_{\hat{\hat{f}}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}$ for all $e' \in E'$. However, due to the close relationship between the extension principle and existential quantification, I must admit other choices of extension principles, which match the existential quantifiers $\mathcal{F}(\exists)$ of $\mathcal{F}$.

Definition 5 *The* induced extension principle *of $\mathcal{F}$, denoted $\widehat{\mathcal{F}}$, maps f to the extension $f' = \widehat{\mathcal{F}}(f)$ defined by $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\pi_{e'} \circ \widehat{f})$, where $\widehat{f}$ is the crisp image mapping $\widehat{f}(Y) = \{f(e) : e \in Y\}$ for all $Y \in \mathcal{P}(E)$.*

Based on $\widehat{\mathcal{F}}$, I can now define the sixth criterion:

(Z-6) **Functional application.** $\mathcal{F}$ must be compatible with the requirement of 'functional application', i.e. we expect that $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n))$, where the semi-fuzzy quantifier Q' is defined by $Q'(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n))$. Rationale: This abstract axiom ensures that $\mathcal{F}$ behave consistently over different domains E.

Definition 6 *A QFM $\mathcal{F}$ which satisfies (Z-1) to (Z-6) is called a* determiner fuzzification scheme (DFS).

(In linguistics, "most", "almost all" etc. are called 'determiners'). If $\mathcal{F}$ induces the standard negation $\neg x = 1 - x$ and the standard extension principle, then it is called a *standard DFS*. These DFSes constitute the natural class of standard models of fuzzy quantification.

A large number of properties of linguistic or logical relevance result from the above axioms: If $\mathcal{F}$ is a DFS, then

- $\mathcal{F}$ induces a reasonable set of fuzzy propositional connectives, i.e. $\widetilde{\neg}$ is a strong negation $\widetilde{\wedge}$ is a t-norm, $\widetilde{\vee}$ is an s-norm etc.
- $\mathcal{F}(\forall)$ is a T-quantifier and $\mathcal{F}(\exists)$ is an S-quantifier in the sense of Thiele [12]. This means that the universal quantifier $\forall$ and the existential quantifier $\exists$ are interpreted plausibly in every DFS.
- $\mathcal{F}$ is compatible with the negation of quantifiers. Hence the equality $\mathcal{F}(Q')(X_1, \ldots, X_n) = \widetilde{\neg}\,\mathcal{F}(Q)(X_1, \ldots, X_n)$ is valid provided that $Q'(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_n)$ be valid. For example, the meanings of "at least one tall men is lucky" and "it is not the case that no tall man is lucky" coincide in every DFS;
- $\mathcal{F}$ is compatible with the formation of antonyms. Therefore $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, \widetilde{\neg}\, X_n)$ is valid whenever $Q'(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)$. For example, the meanings of "every tall men is bald" and "no tall men is not bald" coincide in every DFS.
- $\mathcal{F}$ is compatible with intersections. This means that equality $\mathcal{F}(Q')(X_1, \ldots, X_{n+1}) = \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, X_n \,\widetilde{\cap}\, X_{n+1})$ holds, provided that the equality $Q'(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1})$ holds. For example, the meanings of "at least two X's are Y's' and 'the set of X's that are Y's contains at least two elements" coincide in every DFS.
- $\mathcal{F}$ is compatible with argument permutations. In other words $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(X_{\beta(1)}, \ldots, X_{\beta(n)})$ is valid whenever $Q'(Y_1, \ldots, Y_n) = Q(Y_{\beta(1)}, \ldots, Y_{\beta(n)})$, where β is a permutation of $\{1, \ldots, n\}$. In particular, symmetry properties of a quantifier are preserved by applying $\mathcal{F}$. Hence the meaning of "about 50 X's are Y's" and "about 50 Y's are X's" coincide in every DFS.
- Finally, $\mathcal{F}$ is compatible with argument insertion. This means that $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(X_1, \ldots, X_n, A)$ is valid whenever $Q'(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n, A)$, for a fixed crisp argument $A \in \mathcal{P}(E)$. For example, the meanings of "many (married X)'s are Y's" and "(many married) X's are Y's" coincide in $\mathcal{F}$.

A number of further important properties are possessed by every DFS. Every DFS maps quantitative (automorphism-invariant) quantifiers like "almost all" or "a few" to quantitative fuzzy quantifiers; and it maps non-quantitative cases like "John" or "most married" to non-quantitative fuzzy quantifiers. In addition, every DFS is *contextual*, i.e. $\mathcal{F}(Q)(X_1, \ldots, X_n)$ only depends on the behaviour of Q inside the ambiguity ranges $\mathrm{core}(X_i) \subseteq Y_i \subseteq \mathrm{support}(X_i)$, where $\mathrm{core}(X_i)$ denotes the elements with unity membership and $\mathrm{support}(X_i)$ denotes elements with non-zero membership. Every DFS is also known to *preserve extension*, i.e. insensitive to the exact choice of the domain as a whole. For example, we should expect that the interpretation of "Most tall people are bald" does not depend on the precise choice of the universe, as long as it is large enough to contain the fuzzy subsets of interest. For a comprehensive discussion of adequacy properties of DFSes, see [8].

3. Examples of practical models

In order to be useful in practice, the abstract framework so defined must be populated with concrete models. Hence let me introduce a constructive principle for such models and discuss some prototypical examples. In order to define these models, we need the cut range $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ of a fuzzy subset X at the cutting level $\gamma \in [0,1]$, which corresponds to a symmetric, three-valued cut of X at γ:

$$\mathcal{T}_\gamma(X) = \{Y \subseteq E : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\}$$

where

$$X_\gamma^{\min} = \begin{cases} X_{\geq \frac{1}{2}+\frac{1}{2}\gamma} & : \gamma \in (0,\, 1] \\ X_{>\frac{1}{2}} & : \gamma = 0 \end{cases}$$
$$X_\gamma^{\max} = \begin{cases} X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma \in (0,\, 1] \\ X_{\geq\frac{1}{2}} & : \gamma = 0 \end{cases}$$

Here $X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\}$ denotes α-cut, and $X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\}$ the strict α-cut. (γ can be thought of as a parameter of 'cautiousness'.) I now introduce a pair of mappings which specify upper and lower bounds of the quantification results obtained for all choices of $Y_1, \ldots, Y_n$ in the cut ranges (thus, my approach is basically supervaluationist):

$$\top_{Q,X_1,\ldots,X_n}(\gamma) = \sup\{Q(Y_1,\ \ldots,\ Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$$
$$\bot_{Q,X_1,\ldots,X_n}(\gamma) = \inf\{Q(Y_1,\ \ldots,\ Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}\,.$$

In order to define DFSes based on $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$, the results of these mappings for all levels of cautiousness must be aggregated. In [7], the full class of models definable in this way, i.e. by $\mathcal{F}_\xi(Q)(X_1,\ \ldots,\ X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n})$, has been investigated and the necessary and sufficient conditions on ξ have been presented which ensure that $\mathcal{F}_\xi$ be a DFS. Here I will confine myself to presenting three examples.

The model $\mathcal{M}$ uses the fuzzy median,

$$\mathrm{med}_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & : \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & : \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \text{else} \end{cases}$$

for all $u_1, u_2 \in [0,1]$, to combine $\top_Q$ and $\bot_Q$. It then eliminates the cutting parameter by an integration. Thus,

$$\mathcal{M}(Q)(X_1,\ \ldots,\ X_n) = \int_0^1 \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\ldots,X_n}(\gamma), \bot_{Q,X_1,\ldots,X_n}(\gamma))\, d\gamma\,.$$

It can be shown that $\mathcal{M}$ is a standard DFS. It is continuous in arguments and quantifiers, i.e. robust against slight variations in $X_1, \ldots, X_n$ and in the quantifier Q. Moreover $\mathcal{M}$ is known to propagate fuzziness in arguments and quantifiers, i.e. less specific input cannot result in more specific outputs.

The integral, which I used in the definition of $\mathcal{M}$, is not the only possible way of abstracting from γ. In the course of my investigation into the possible models, there emerged a model $\mathcal{M}_{\mathrm{CX}}$ which is the provably best choice from a linguistic perspective, even among the full class of standard models. For example, it is the *only standard model* which permits the compositional interpretation of adjectival restriction by a fuzzy adjective, like in "almost all young A's are B's". It is hence guaranteed that $\mathcal{M}_{\mathrm{CX}}(\textbf{almost all young})(A, B) = \mathcal{M}_{\mathrm{CX}}(\textbf{almost all})(\textbf{young} \cap A, B)$. Like $\mathcal{M}$, the model is also continuous in arguments and in quantifiers, and it also propagates fuzziness. Because of its unique properties, $\mathcal{M}_{\mathrm{CX}}$ is the preferred choice for all applications that need to capture NL semantics (additional properties are discussed in [8]).

The model can be defined in terms of $\top_Q, \bot_Q$ and $\mathrm{med}_{1/2}$-based aggregation, but also in the following more compact form.

$$\mathcal{M}_{\mathrm{CX}}(Q)(X_1,\ \ldots,\ X_n) = \sup\{Q^L_{V,\,W}(X_1,\ \ldots,\ X_n) : V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}$$

where

$$Q^L_{V,\,W}(X_1,\ \ldots,\ X_n) = \min(\Xi_{V,W}(X_1,\ \ldots,\ X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i\}$$
$$\Xi_{V,W}(X_1,\ \ldots,\ X_n) = \min_{i=1}^n \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\})\,.$$

Another interesting aspect of $\mathcal{M}_{\mathrm{CX}}$ is that it consistently generalises the Sugeno integral and hence the 'basic' FG-count approach to arbitrary n-place quantifiers, and to quantifiers that do not fulfill any special monotonicity requirements.

My last example $\mathcal{F}_{\mathrm{owa}}$ is defined by

$$\mathcal{F}_{\mathrm{owa}}(Q)(X_1,\ \ldots,\ X_n) = \tfrac{1}{2}\int_0^1 \top_{Q,X_1,\ldots,X_n}(\gamma)d\gamma + \tfrac{1}{2}\int_0^1 \bot_{Q,X_1,\ldots,X_n}(\gamma)d\gamma\,.$$

$\mathcal{F}_{\mathrm{owa}}$ is a standard DFS. The model is of particular interest because it consistently generalises the Choquet integral and hence the 'basic' OWA approach, to the 'hard' cases of general multiplace and non-monotonic quantifiers. $\mathcal{F}_{\mathrm{owa}}$ is a practical model because it is continuous both in arguments and in quantifiers, which ensures a certain stability. However, $\mathcal{F}_{\mathrm{owa}}$ does not propagate fuzziness in arguments nor in quantifiers. It is therefore inferior to the other models from an adequacy perspective because less specific input can result in more specific output. Nevertheless, $\mathcal{F}_{\mathrm{owa}}$ can be useful when the inputs are overly fuzzy and one still needs a fine-grained result ranking, because it discerns cases in which models that propagate fuzziness are no longer informative.

To sum up, there is a basic stock of prototypical models, which can be used in applications. These models are computational. An efficient histogram-based method for implementing quantifiers in $\mathcal{M}$, $\mathcal{M}_{\mathrm{CX}}$ and $\mathcal{F}_{\mathrm{owa}}$ is described in [8]. The method, which is applicable to arbitrary quantifiers defined in terms of cardinalities, has been further optimized for absolute and proportional quantifiers, quantifiers of exception and cardinal comparatives.

4. Extension towards multiple variable binding

Let us start from the notion of crisp quantifiers which incorporate multiple-variable binding. A *Lindström quantifier* is a class $\mathcal{Q}$ of (relational) structures of some type $t = \langle t_1, \dots, t_n \rangle$, such that $\mathcal{Q}$ is closed under isomorphism [11, p. 186]. The cardinal $n \in \mathbb{N}$ specifies the number of arguments; the individual components $t_i \in \mathbb{N}$ specify the number of variables that the quantifier binds in its i-th argument position. For example, the existential quantifier, which accepts one argument and binds one variable, has type $t = \langle 1 \rangle$. The corresponding class $\mathcal{E}$ comprises all structures $\langle E, A \rangle$ where $E \neq \varnothing$ is a base set and A is a nonempty subset of E. In the introduction, we have already met with a more complex quantifier Q of type $\langle 1, 1, 2 \rangle$. In this case, $\mathcal{Q}$ is the class of all structures $\langle E, A, B, R \rangle$ with $Q(A, B, R) = 1$, where $E \neq \varnothing$ is the base set and $A, B \in \mathcal{P}(E)$, $R \in \mathcal{P}(E^2)$.

In order to model NL quantifiers like "all except Lotfi", which depend on specific individuals, we must drop the requirement of isomorphism closure. Hence, in principle, a *generalized Lindström quantifier* is a class $\mathcal{Q}$ of relational structures of a common type $t = \langle t_1, \dots, t_n \rangle$. For my purposes, it is convenient to break down the information gathered in the total class $\mathcal{Q}$ into more localized representations. This is accomplished by the following definition:

Definition 7 *A* two-valued L-quantifier *of type $t = \langle t_1, \dots, t_n \rangle$ on a base set $E \neq \varnothing$ assigns a crisp quantification result $Q(Y_1, \dots, Y_n) \in \{0, 1\}$ to each choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \dots, n\}$. A* full two-valued L-quantifier *Q of type t assigns a two-valued L-quantifier Q_E of type t on E to each base set $E \neq \varnothing$.*

It should be apparent that 'full' L-quantifiers are in one-to-one correspondence with generalized Lindström quantifiers. For my purposes, however, it is preferable to generalize the relativized notion. In order to develop a theory of fuzzy quantification which incorporates Lindström quantifiers, and hence multiple variable binding, I will pursue the same basic strategy that proved itself useful for analysing ordinary quantifiers, i.e. I will introduce (semi-)fuzzy L-quantifiers and L-QFMs.

Definition 8 *A* semi-fuzzy L-quantifier *of type $t = \langle t_1, \dots, t_n \rangle$ on $E \neq \varnothing$ assigns a gradual quantification result $Q(Y_1, \dots, Y_n) \in [0, 1]$ to each choice of crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \dots, n\}$.*

Thus, Q accepts crisp arguments of the indicated types, but it can express approximate quantifications. Semi-fuzzy L-quantifiers are proposed as a uniform specification medium for arbitrary quantifiers with multiple variable binding. Again, we also need operational quantifiers, which are not restricted to crisp inputs:

Definition 9 *A* fuzzy L-quantifier *of type t on $E \neq \varnothing$ assigns a gradual quantification result $\widetilde{Q}(X_1, \dots, X_n) \in [0, 1]$ to each choice of fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \dots, n\}$.*

A suitable fuzzification mechanism will be used for associating specifications to target quantifiers.

Definition 10 *An* L-QFM *$\mathcal{F}$ assigns to each semi-fuzzy quantifier Q of some type t on $E \neq \varnothing$ a corresponding fuzzy L-quantifier $\mathcal{F}(Q)$ of the same type t and on the same base set E.*

It is possible to develop plausibility criteria for L-QFMs in total analogy to those for ordinary QFMs. However, some preparations are necessary. In order to identify a matching choice of fuzzy connectives and of extension principle, I associate with every L-QFM $\mathcal{F}$ a corresponding 'ordinary' QFM $\mathcal{F}_R$ defined as follows. For every n-ary semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow [0, 1]$, on E, let Q' denote the n-ary quantifier on E^1 defined by $Q'(Y_1, \dots, Y_n) = Q'(\widehat{\vartheta}(Y_1), \dots, \widehat{\vartheta}(Y_n))$ for all $Y_1, \dots, Y_n \in \mathcal{P}(E^1)$, where $\vartheta : E^1 \longrightarrow E$ is the mapping $\vartheta((e)) = e$ for all $(e) \in E^1$. We then stipulate:

Definition 11 *The QFM $\mathcal{F}_R$ is defined by*

$$\mathcal{F}_R(Q)(X_1, \dots, X_n) = \mathcal{F}(Q')(\hat{\hat{\beta}}(X_1), \dots, \hat{\hat{\beta}}(X_n))$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, where $\hat{\hat{\beta}}$ is obtained from the mapping $\beta : E \longrightarrow E^1$ with $\beta(e) = (e)$ by applying the standard extension principle.

The induced fuzzy connectives and the induced extension principle of $\mathcal{F}$ are identified with the connectives and the extension principle induced by the ordinary QFM $\mathcal{F}_R$.

Based on these preparations, I can now develop criteria for plausible L-models of fuzzy quantification which parallel my requirements on QFMs. (The 'rationale' for these conditions is the same as above in each case).

(L-1) **Correct generalisation.** It is required that

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$$

for all crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ldots, n\}$; combined with the other axioms, this condition can be restricted to quantifiers of types $t = \langle\rangle$ or $t = \langle 1 \rangle$.

(L-2) **Membership assessment.** Quantifiers for membership assessment of the special form $\pi_{(e)} : \mathcal{P}(E^1) \longrightarrow \{0, 1\}$ for some $e \in E$ also qualify as two-valued L-quantifiers of type $\langle 1 \rangle$ on E. These quantifiers should be mapped to their fuzzy counterparts $\widetilde{\pi}_{(e)}$ of type $\langle 1 \rangle$ on E, i.e. we must have $\mathcal{F}(\pi_{(e)}) = \widetilde{\pi}_{(e)}$.

(L-3) **Dualisation.** $\mathcal{F}$ must preserve dualisation of quantifiers, i.e. $\mathcal{F}(Q')(X_1, \ldots, X_n) = \widetilde{\neg}\, \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, \widetilde{\neg}\, X_n)$ for all fuzzy arguments $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$ if $Q'(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)$ for all crisp arguments $Y_i \in \mathcal{P}(E^{t_i})$, $i \in \{1, \ldots, n\}$.

(L-4) **Union.** $\mathcal{F}$ must preserve unions of arguments, i.e. we must have $\mathcal{F}(Q')(X_1, \ldots, X_{n+1}) = \mathcal{F}(Q)(X_1, \ldots, X_{n-1}, X_n \,\widetilde{\cup}\, X_{n+1})$ whenever $Q'(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$.

(L-5) **Monotonicity in arguments.** We require that $\mathcal{F}$ preserve monotonicity in arguments, i.e. if Q is nondecreasing/nonincreasing in the i-th argument, then $\mathcal{F}(Q)$ has the same property. (The condition can again be restricted to the case that Q is nonincreasing in its n-th argument).

(L-6) **Functional application.** Given a semi-fuzzy L-quantifier Q of type $t = \langle t_1, \ldots, t_n \rangle$ on E, another type $t' = \langle t'_1, \ldots, t'_n \rangle$ (same n), a set $E' \neq \varnothing$, and mappings $f_i : {E'}^{t'_i} \longrightarrow E^{t_i}$ for $i \in \{1, \ldots, n\}$, we can define a quantifier Q' of type t' on E' according to $Q'(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n))$ for all $Y_i \in \mathcal{P}({E'}^{t'_i})$, $i \in \{1, \ldots, n\}$.
It is required that $\mathcal{F}(Q')(X_1, \ldots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f)_1(X_1), \ldots, \widehat{\mathcal{F}}(f)_n(X_n))$ for all fuzzy arguments $X_i \in \widetilde{\mathcal{P}}({E'}^{t'_i})$, $i \in \{1, \ldots, n\}$.

Definition 12 *An L-QFM which satisfies (L-1) to (L-6) is called an* L-DFS.

Let us now consider some results on these models.

Theorem 1 *For every L-DFS $\mathcal{F}$, the corresponding QFM $\mathcal{F}_R$ is a DFS.*

(The generalized models are also suitable for carrying out 'ordinary' quantification.)

As I will now show, the converse claim is also true, i.e quantification in L-DFSes can be reduced to quantification in the simple models. The construction which acccomplishes this reduction is defined as follows. Let $\mathcal{F}$ be an ordinary QFM and let Q be a semi-fuzzy L-quantifier of type $t = \langle t_1, \ldots, t_n \rangle$ on $E \neq \varnothing$. Let $m = \max\{t_1, \ldots, t_n\}$ and define embeddings $\zeta_i : E^{t_i} \longrightarrow E^m$ and projections $\kappa_i : E^m \longrightarrow E^{t_i}$ by

$$\zeta_i(e_1, \ldots, e_{t_i}) = (e_1, \ldots, e_{t_i - 1}, e_{t_i}, e_{t_i}, \ldots, e_{t_i})$$
$$\kappa_i(e_1, \ldots, e_m) = (e_1, \ldots, e_{t_i})$$

for $i \in \{1, \ldots, n\}$. I now introduce an n-ary semi-fuzzy quantifier Q' on E^m defined by

$$Q'(Y_1, \ldots, Y_n) = Q(\widehat{\kappa}_1(Y_1 \cap \zeta_1(E^{t_1})), \ldots, \widehat{\kappa}_n(Y_n \cap \zeta_n(E^{t_n}))),$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E^m)$.

Definition 13 *For every QFM $\mathcal{F}$, the L-QFM $\mathcal{F}_L$ is defined by*

$$\mathcal{F}_L(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(\hat{\hat{\zeta}}_1(X_1), \ldots, \hat{\hat{\zeta}}_n(X_n))$$

for all $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \ldots, n\}$, where the $\hat{\hat{\zeta}}_i$'s are obtained from ζ_i by applying the standard extension principle.

I have the following results on this canonical construction of an L-QFM $\mathcal{F}_L$ from a given QFM $\mathcal{F}$.

Theorem 2 *If $\mathcal{F}$ is a DFS, then $\mathcal{F}_{LR} = \mathcal{F}$.*

($\mathcal{F}_L$ properly generalizes the original model $\mathcal{F}$.)

Theorem 3 *If $\mathcal{F}$ is a DFS, then $\mathcal{F}_L$ is an L-DFS.*

(The proposed construction results in plausible models.)

Theorem 4 *If $\mathcal{F}$ is an L-DFS, then $\mathcal{F}_{RL} = \mathcal{F}$.*

The latter theorem is of particular relevance because it eliminates the problem of establishing classes of L-DFSes altogether. Every L-DFS $\mathcal{F}'$ can now be expressed as $\mathcal{F}' = \mathcal{F}_L$, where $\mathcal{F}$ is one of the ordinary models which have already been studied to some depth. This is of invaluable help because the investigation of constructive principles for plausible models, like those presented in section 2, turned out to be a rather complex matter which requires considerable effort. The canonical construction of $\mathcal{F}_L$, then, permits the use of the proven models $\mathcal{M}$, $\mathcal{M}_{\mathrm{CX}}$ and $\mathcal{F}_{\mathrm{owa}}$ to handle the new cases of fuzzy L-quantification. In the following, I will identify these models with their extensions for simplicity, thus writing $\mathcal{M}_{\mathrm{CX}}$ rather than $(\mathcal{M}_{\mathrm{CX}})_L$ etc.

5. Quantifier nestings

The axioms for L-DFSes are directly modelled after the corresponding axioms for ordinary DFSes; they do not refer to details of multiple variable binding. It is an interesting question if additional axioms specifically concerned with L-quantifiers can be useful to identify those L-DFSes best suited for modelling branching quantification.

A construction that comes to mind is *quantifier nesting*. Consider $Q'xQ''y\varphi(x,y)$, for example. Based on semi-fuzzy L-quantifiers, we can either analyse this in terms of two quantifiers of type $t_1 = t_2 = \langle 1\rangle$ applied in succession. Alternatively, we can look at the whole block $Q_{xy} = Q'xQ''y$, which corresponds to a single semi-fuzzy L-quantifier of type $t = \langle 2\rangle$. Now let $R \in \widetilde{\mathcal{P}}(E^2)$ be the interpretation of $\varphi(x,y)$ as a fuzzy relation. The analysis in terms of two separate quantifier will result in an interpretation $\mathcal{F}(Q')(Z)$, where $\mu_Z(e) = \mathcal{F}(Q'')(eR)$ and $\mu_{eR}(e') = \mu_R(e,e')$ for all $e, e' \in E$. This analysis thus corresponds to a fuzzy L-quantifier $\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')$ of type $\langle 2\rangle$ defined by

$$\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')(R) = \mathcal{F}(Q')(Z)$$

for all fuzzy relations $R \in \mathcal{P}(E^2)$. The quantifier block Q_{xy}, on the other hand, corresponds to a semi-fuzzy L-quantifier $Q' \mathbin{\widetilde{@}} Q''$ of type $\langle 2\rangle$ defined by

$$Q' \mathbin{\widetilde{@}} Q''(S) = \mathcal{F}(Q')(Z)\,,$$

$\mu_Z(e) = Q''(eS)$, $eS = \{e' : (e,e') \in S\}$ for all *crisp* relations $S \in \mathcal{P}(E^2)$, and it results in a second interpretation, $\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'')(R)$. (I am using the 'tilde'-notation $\widetilde{@}$ here to signify that $Q' \mathbin{\widetilde{@}} Q''$ depends on the chosen L-QFM). It is natural to require that the two interpretations coincide, i.e.

$$\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') = \mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')\,.$$

More generally, we can define a nesting operation for semi-fuzzy L-quantifiers Q' of type $t = \langle t_1, \dots, t_n\rangle$, $n > 0$ and Q'' of arbitrary type $t' = \langle t'_1, \dots, t'_{n'}\rangle$, $n' \in \mathbb{N}$. In this case, the semi-fuzzy L-quantifier $Q' \mathbin{\widetilde{@}} Q''$ of type $t^* = \langle t_1, \dots, t_{n-1}, t_n + t'_1, \dots, t_n + t'_{n'}\rangle$ which results from the nesting of Q'' into the last argument of Q', becomes

$$\begin{aligned}&Q' \mathbin{\widetilde{@}} Q''(Y_1, \dots, Y_{n-1}, S_1, \dots, S_{n'})\\ &= \mathcal{F}(Q')(Y_1, \dots, Y_{n-1}, Z)\end{aligned}$$

for all $Y_i \in E^{t_i}$, $i \in \{1, \dots, n-1\}$ and $S_j \in E^{t_n + t'_j}$, $j \in \{1, \dots, n'\}$, where $Z \in \widetilde{\mathcal{P}}(E^{t_n})$ is defined by

$$\begin{aligned}&\mu_Z(e_1, \dots, e_{t_n})\\ &= \mathcal{F}(Q'')((e_1, \dots, e_{t_n})S_1, \dots, (e_1, \dots, e_{t_n})S_{n'})\end{aligned}$$

and

$$\begin{aligned}&(e_1, \dots, e_{t_n})S_j\\ &= \{(e'_1, \dots, e'_{t'_j}) : (e_1, \dots, e_{t_n}, e'_1, \dots, e'_{t'_j}) \in S_j\}\end{aligned}$$

for $j \in \{1, \dots, n'\}$. Similarly, the fuzzy L-quantifier $\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')$ of type t^* on E is defined by

$$\begin{aligned}&\mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')(X_1, \dots, X_{n-1}, R_1, \dots, R_{n'})\\ &= \mathcal{F}(Q')(X_1, \dots, X_{n-1}, Z)\end{aligned}$$

for all $X_i \in \widetilde{\mathcal{P}}(E^{t_i})$, $i \in \{1, \dots, n-1\}$ and $R_j \in \widetilde{\mathcal{P}}(E^{t_n + t'_j})$, $j \in \{1, \dots, n'\}$, where

$$\begin{aligned}&\mu_Z(e_1, \dots, e_{t_n})\\ &= \mathcal{F}(Q'')((e_1, \dots, e_{t_n})R_1, \dots, (e_1, \dots, e_{t_n})R_{n'})\end{aligned}$$

and

$$\begin{aligned}&\mu_{(e_1, \dots, e_{t_n})R_j}(e'_1, \dots, e'_{t'_j})\\ &= \mu_{R_j}(e_1, \dots, e_{t_n}, e'_1, \dots, e'_{t'_j})\end{aligned}$$

for $j \in \{1, \dots, n'\}$. In this general case, too, we would expect that

$$\mathcal{F}(Q' \mathbin{\widetilde{@}} Q'') = \mathcal{F}(Q') \mathbin{@} \mathcal{F}(Q'')\,. \qquad \text{(QN)}$$

A conforming L-QFM will be said to be *compatible with quantifier nesting*; combining the new requirement with the axioms for L-DFSes then takes us to a notion of *strong* L-DFSes.

Judging from some first tests, it appears that quantifier nesting expresses a very restrictive condition which excludes many useful models; I even conjecture that the criterion invalidates all standard models except $\mathcal{M}_{\mathrm{CX}}$. Thus, some further work is necessary to substantiate this claim and achieve a precise characterization of the conforming models.

Let me further comment on the limitation of (QN) to nestings in the last argument of the quantifier Q'. Nestings in another argument position k can be reduced to this case by flipping argument positions k and n, performing the nesting in the last argument, and then reordering the arguments again into their intended target positions. But, every DFS is known to be compatible with such permutations of argument positions, a property which apparently generalizes to L-DFSes. Consequently, the compatibility of a DFS with nestings in the last argument position is sufficient to ensure that it also comply with nestings in other argument positions.

By repeating this process, (QN) even ensures the compatibility of conforming L-DFSes with multiple nestings of arbitrary depths. Thus, the condition is sufficient to cover general nestings involving quantifiers of arbitrary types with an arbitrary number of embedded quantifiers.

It would be useful however, to relate the proposed general nestings in the last argument to the simple nesting of unary quantifiers considered earlier, or other simplified nesting criteria. This might facilitate the check that a model of interest comply with the nesting condition, and also eliminate some redundancy from the axiom system. I expect that the general nestings described by (QN) can be reduced to a simpler criterion, but further research is necessary to clarify this matter.

6. Application to fuzzy branching quantification

Let me now explain how the motivating example, "Many young and most old people respect each other" can be interpreted in the proposed framework. In this case, we have semi-fuzzy quantifiers $Q_1 = \textbf{many}$, defined by $\textbf{many}(Y_1, Y_2) = |Y_1 \cap Y_2|/|Y_1|$, say, and $Q_2 = \textbf{most}$, defined as above. Both quantifiers are nondecreasing in their second argument, i.e. we can adopt eq. (*) proposed by Barwise. The modification towards gradual truth values will be accomplished in the usual way, i.e. by replacing existential quantifiers with sup and conjunctions with $\min$ (non-standard connectives are not possible here). The semi-fuzzy L-quantifier Q of type $\langle 1, 1, 2\rangle$ constructed from Q_1, Q_2 then becomes

$$Q(A, B, R) = \sup\{\min(Q_1(A, U), Q_2(B, V)) : U \times V \subseteq R\}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. Hence "Many men and many women are relatives of each other", for example, which rests on crisp arguments, is now taken to denote the maximum degree to which many men and many women belong to a group $U \times V \subseteq R$ of mutual relatives. In my experience, this is a conclusive analysis; but we still need to extend it to fuzzy arguments. To this end, it is sufficient to apply the chosen L-DFS $\mathcal{F}$. We then obtain the fuzzy L-quantifier $\mathcal{F}(Q)$ of type $\langle 1, 1, 2\rangle$ suited to handle this case. In the original example involving fuzzy sets of young and old people, we have fuzzy subsets $\textbf{young}, \textbf{old} \in \widetilde{\mathcal{P}}(E)$ of young and old people, respectively, and a fuzzy relation $\textbf{rsp} \in \widetilde{\mathcal{P}}(E^2)$ of people who respect each other. Thus, a meaningful interpretation of "Many young and most old people respect each other" is now given by $\mathcal{F}(Q)(\textbf{young}, \textbf{old}, \textbf{rsp})$.

Finally let me describe how Westerståhl's generic method for interpreting branching quantifiers can be applied in the fuzzy case. Hence suppose that Q_1, Q_2 are arbitrary semi-fuzzy quantifiers of arity $n = 2$ on some base set E. Following Westerståhl, I introduce nondecreasing and nonincreasing approximations of the Q_i's, defined by $Q_i^+(Y_1, Y_2) = \sup\{Q_i(Y_1, L) : L \subseteq Y_2\}$ and $Q_i^-(Y_1, Y_2) = \sup\{Q_i(Y_1, U) : U \supseteq Y_2\}$, respectively. With the usual replacement of existential quantification with sup and conjunction with $\min$, Westerståhls interpretation formula [13, p. 281, Def. 3.1] becomes:

$$\begin{aligned} Q(A, B, R) = \sup\{&\min\{Q_1^+(A, U_1), Q_2^+(B, V_1),\\ &Q_1^-(A, U_2), Q_2^-(B, V_2)\} :\\ &(U_1 \cap A) \times (V_1 \cap B)\\ &\subseteq R \cap (A \times B) \subseteq (U_2 \cap A) \times (V_2 \cap B)\} \end{aligned}$$

for all $A, B \in \mathcal{P}(E)$ and $R \in \mathcal{P}(E^2)$. Application of an L-DFS then determines the corresponding fuzzy L-quantifier $\mathcal{F}(Q)$ of type $\langle 1, 1, 2\rangle$ suitable for interpretation. As shown by Westerståhl [13, p.284], his method results in meaningful interpretations provided that (a) the Q_i's are 'logical', i.e. $Q_1(Y_1, Y_2)$ and $Q_2(Y_1, Y_2)$ can be expressed as a function of $|Y_1|$ and $|Y_1 \cap Y_2|$, see van Benthem [3, p. 446] and [4, p. 458]; and (b), the Q_i's are *convex* in their second argument [6, p. 55, Def. 85], or 'CONT' in Westerståhl's terminology, i.e. $Q_i(Y_1, Y_2) \geq \min(Q_i(Y_1, L), Q_i(Y_1, U))$ for all $L \subseteq Y_2 \subseteq U$. The latter condition ensures that Q_1 and Q_2 can be recovered from their nondecreasing approximations Q_i^+ and their nonincreasing approximations Q_i^-, i.e. $Q_i = \min(Q_i^+, Q_i^-)$. This is generally the case when Q_1 and Q_2 are either nondecreasing in their second argument ("many"), nonincreasing ("few"), or of unimodal shape ("about ten", "about one third"). An example of branching quantification with unimodal quantifiers, which demand the generic method, is "About fifty young and about sixty old persons respect each other".

7. Conclusion

Recognizing the utility of branching quantifiers to linguistic modelling, I have proposed an extension of the DFS theory of fuzzy quantification which incorporates these cases. Specifically, I introduced fuzzy L-quantifiers (generalizations of Lindström quantifiers to approximate quantifiers and fuzzy arguments), semi-fuzzy L-quantifiers (uniform specifications of such quantifiers) and plausible models of fuzzy quantification involving these quantifiers, called L-DFSes, showing that

(a) every plausible model of 'ordinary' fuzzy quantification (DFS) can be extended to a unique L-DFS for quantifiers with multiple variable binding;

(b) no other L-DFSes exist beyond those obtained from (a).

Westerståhl's analysis of branching NL quantification in terms of Lindström quantifiers is easily generalized to semi-fuzzy L-quantifiers. By applying the chosen model of fuzzy quantification, one then obtains a meaningful interpretation for branching quantification involving approximate quantifiers and fuzzy arguments.

The identification of those L-DFSes specifically suited for modelling branching constructions is an advanced topic that should be tackled by future research. I have already hinted at a possible strengthening of the system based on quantifier nesting, but the resulting criterion appears to be extremely restrictive and it is not clear at this point if it can be required without sacrificing useful models like $\mathcal{M}$ and $\mathcal{F}_{\mathrm{owa}}$.

The proposed analysis of reciprocal constructions in terms of fuzzy branching quantifiers is of particular relevance to linguistic data summarization [9, 14]. Many summarizers of interest express mutual (or symmetric) relationships and can therefore be verbalized by a reciprocal construction. (Asymmetrical relations R' can also be used in reciprocal constructions after symmetrization, i.e. they must be replaced with $R = R' \cap R'^{\mathrm{op}}$). An ordinary summary like "$Q_1\ X_1$'s are strongly correlated with $Q_2\ X_2$'s" does not capture the symmetrical nature of correlations, and it neglects the resulting groups of mutually correlated objects. The proposed modelling in terms of branching quantifiers, by contrast, permits me to support a novel type of linguistic summaries specifically suited for describing groups of interrelated objects. Branching quantification, in this view, is a natural language technique for detecting such groups in the data. A possible summary involving a reciprocal predicate is "The intake of most vegetables and many health-related indicators are strongly associated with each other".

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