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Ingo Glöckner

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Technische Fakultät der Universität Bielefeld,
Abteilung Informationstechnik, Postfach 10 01 31, 33501 Bielefeld

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Ingo Glöckner

Universität Bielefeld, Technische Fakultät, AG Technische Informatik,
Postfach 10 01 31, 33501 Bielefeld,
mailto:ingo@techfak.uni-bielefeld.de

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Abstract

Quantifiers are at the heart of human language. A formalization of natural language (NL) quantification and its subsequent computer implementation promises to enhance a broad range of applications including NL interfaces, linguistic data summarisation, multi-criteria decision making, database querying and others. However, the software implementations available for NL quantifiers will remain insufficient and linguistically implausible unless the inherent fuzziness of natural language is explicitly modelled. In order to remedy this situation and to provide better support for applications that need fuzzy quantifiers, the report presents an in-depth discussion of the standard models of fuzzy quantification, which best comply with our linguistic expectations. After reviewing the known classes of models that have already been identified in previous work on the axiomatic theory of fuzzy quantification (DFS theory), it introduces a novel class of models which embeds all of the previous classes. Two independent constructions are developed and thoroughly investigated which both establish the target class of models, and hence provide a justification of the resulting class from two perspectives:

- as an extension of the known class of $\mathcal{F}_\xi$-DFSes, it represents the full class of models definable in terms of three-valued cuts. This style of presentation lends itself to the development of algorithms which implement quantifiers in the models;
- as an abstraction of the models $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ proposed in [7], which were inspired by the fuzzification mechanism proposed by Gaines, it captures the class of models definable in terms of the extension principle, and hence links the analysis of fuzzy quantification to the fundamental principle underlying fuzzy set theory.

The report also describes some typical examples of the new models. In addition, it presents the exact conditions required to check if a model of interest obeys the adequacy properties discovered by DFS theory. The report hence reaches an important milestone in the superordinate endeavour of providing a solid theoretical foundation for the use of fuzzy quantifiers in applications.

1 The axiomatics of fuzzy quantification

Natural language (NL) is pervaded by fuzzy concepts like *tall* or *rich* which lack clear boundaries. The fuzziness of language is not restricted to its concepts, though. Approximate quantifiers like *almost all* or *many* are very frequently used, and serve the important purpose of abstracting from details, and summarising a large number of observations into a total view of the given situation. In order to make this expressive power available to machines and enable computer programs to handle this important aspect of natural language, the problem must be solved of how to assign a reasonable interpretation to quantifying natural language expressions, which might involve fuzziness both in the quantifier and its arguments.

Following Zadeh's pioneering ideas [23], a number of approaches have been developed to model approximate quantification with fuzzily defined concepts in the framework of fuzzy set theory [24, 15, 19, 20]. However, it was soon recognized that the resulting interpretations can be counter-intuitive [1, 15, 21]. A systematic investigation of the traditional approaches according to linguistic criteria produced negative findings in all cases [8].

DFS theory [7, 9] is an attempt to solve the puzzles of fuzzy quantification by embarking on an axiomatic solution. The basic idea is that linguistically plausible results can only be guaranteed if we succeed (a) to formalize the relevant aspects of linguistic adequacy, and (b) to develop computational models of the resulting axiom system. The natural starting point for putting this venture into action was considered the *linguistic* theory of natural language quantification, viz the Theory of Generalized Quantifiers (TGQ), see e.g. [2, 3, 4]. DFS theory adopts the notion of a two-valued generalized quantifier developed by TGQ, which is then extended to the key concepts of semi-fuzzy quantifiers and fuzzy quantifiers. Fuzzy quantifiers form the class of operators for approximate quantification with fuzzy arguments. However, these operators are too complex to be defined directly. DFS theory hence proposes an intermediate layer of semi-fuzzy quantifiers, which provide a more compact description of a fuzzy quantifying operator. Semi-fuzzy quantifiers are able to represent approximate quantification but avoid the intricacies caused by fuzziness in a quantifier's arguments. This makes it possible to define semi-fuzzy quantifiers conveniently in terms of the familiar cardinality of crisp sets (which is not of direct use for defining fuzzy quantifiers that accept fuzzy argument sets). Introducing semi-fuzzy quantifiers therefore greatly facilitates the modelling of NL base quantifiers, i.e. of non-composite quantifiers like *many* and *a few*, which cannot be reduced to combinations of other known quantifiers. The mapping from simplified descriptions, i.e. semi-fuzzy quantifiers, to corresponding fuzzy quantifiers is established through a quantifier fuzzification mechanism (QFM). DFS theory approaches the problem of reasonable interpretation by imposing formal conditions on admissable choices of QFMs. These conditions ensure that the essential properties of quantifiers and their relationships are preserved when applying the fuzzification mechanism. They can be likened to the well-known concept of a homomorphism (a structure preserving mapping compatible with a number of given constructions).

In the following I sketch the core of DFS theory. The exposition is intended to give a rough overview and to introduce all concepts required for the theorems and proofs to follow. Most of the material has been compiled from [11]. A more thorough introduction and motivating examples of the constructions or axioms can be found in the primary sources on DFS theory: the original presentation is [7]. The current terminology and a simplified axiom system have been introduced in [9]. Further sources of interest are [11], which develops a broad class of standard models, and [10], which summarizes the current state of DFS theory.

In order to introduce DFS theory, we first define two-valued generalized quantifiers in accordance with TGQ:

Definition 1 *An* n-ary two-valued quantifier *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$*, where* $E \neq \varnothing$ *is a nonempy set (the base set or domain),* $\mathcal{P}(E)$ *is the powerset (set of subsets) of* E*,* $n \in \mathbb{N}$ *is the arity (number of arguments), and* $\mathbf{2} = \{0, 1\}$ *denotes the set of two-valued truth values.*

A two-valued quantifier hence assigns a crisp interpretation $Q(Y_1, \ldots, Y_n) \in \mathbf{2}$ to each choice of crisp arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We allow for the case of nullary quantifiers ($n = 0$), which can be identified with the constants 0 and 1. Some examples of two-place quantifiers are

$$\textbf{all}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \subset Y_2$$
$$\textbf{some}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 \neq \varnothing$$
$$\textbf{no}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 = \varnothing$$
$$\textbf{at least k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \geq k$$
$$\textbf{more than k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| > k$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$; $|\bullet|$ denotes cardinality. The subscript E is dropped when the base set E is understood. In order to cover the approximate variety of NL quantifiers (e.g. *about 10*) and to be able to apply these quantifiers to arguments like *tall* and *rich*, we need to enhance this concept of quantifiers and incorporate ideas from fuzzy set theory. A *fuzzy subset* X of a given set E assigns to each element $e \in E$ a membership grade $\mu_X(e) \in \mathbf{I}$, where $\mathbf{I} = [0, 1]$ is the unit interval. A fuzzy subset is uniquely characterised by its membership function $\mu_X : E \longrightarrow \mathbf{I}$. For example, a fuzzy subset **tall** of a set E of people can be defined by stipulating a membership grade $\mu_{\textbf{tall}}(e)$ for each person $e \in E$. We shall denote the fuzzy powerset, i.e. the collection of all fuzzy subsets of E, by $\widetilde{\mathcal{P}}(E)$. It is convenient to assume that $\widetilde{\mathcal{P}}(E)$ is an ordinary set. In particular, crisp subsets will be viewed as a special case of fuzzy subsets, and it is understood that $\mathcal{P}(E) \subseteq \widetilde{\mathcal{P}}(E)$.[1] We are now ready to introduce fuzzy quantifiers:

Definition 2 *An* n-ary fuzzy quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

A fuzzy quantifier hence assigns to each n-tuple of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ an interpretation $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$, which is allowed to be gradual. As

[1] Note that this subsumption relationship does not hold if one identifies fuzzy subsets and their membership functions, i.e. if one stipulates that $\widetilde{\mathcal{P}}(E) = \mathbf{I}^E$, where $\mathbf{I}^E$ denotes the set of mappings $f : E \longrightarrow \mathbf{I}$. I assume that the appropriate transformations (e.g. from a crisp subset $A \subseteq E$ to its characteristic function $\chi_A \in \mathbf{2}^E \subseteq \mathbf{I}^E$) are carried out and for the sake of readability, these will be omitted in the notation.

opposed to two-valued quantifiers, fuzzy quantifiers accept fuzzy input (we could e.g. have $X_1 = \textbf{tall}, X_2 = \textbf{rich} \in \widetilde{\mathcal{P}}(E)$). In addition, fuzzy quantifiers produce fuzzy (gradual) output, thus providing a more natural account of approximate quantifiers like **about ten**, **almost all**, **many** etc. However, fuzzy quantifiers pose a new problem. Consider the expression *more than 10 percent*, for example. Given a finite base set E, we can easily define a corresponding two-valued quantifier **more than 10 percent** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$, viz

$$\textbf{more than 10 percent}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1 \cap Y_2| > |Y_1|/10 \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, utilizing the cardinality $|\bullet|$ of crisp sets. However, it is not that easy to provide a straightforward definition of a corresponding fuzzy quantifier $\widetilde{\textbf{more than 10 percent}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$. This is because X_1, X_2 in

$$\widetilde{\textbf{more than 10 percent}}(X_1, X_2)$$

are fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. The familiar cardinality of crisp sets is not applicable to the fuzzy arguments, and it hence cannot be used to define the fuzzy quantifier. There is no generally accepted notion of cardinality of fuzzy sets which could serve as a substitute for $|\bullet|$ in the fuzzy case. In order to overcome this problem, DFS theory introduces the intermediary concept of semi-fuzzy quantifiers.

Definition 3 *An* n-ary semi-fuzzy quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Q hence assigns to each n-tuple of crisp subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ a gradual interpretation $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$. Semi-fuzzy quantifiers share the expressiveness of fuzzy quantifiers because they support fuzzy (gradual) quantification results. Like fuzzy quantifiers, they are hence suited to model approximate quantification. On the other hand, semi-fuzzy quantifiers are defined for crisp arguments only, thus alleviating the need to provide a definition for arbitrary fuzzy arguments, which made it so hard to define fuzzy quantifiers and to justify a particular choice of their definition. Every semi-fuzzy quantifier depends on crisp arguments only and can conveniently be defined in terms of the crisp cardinality of its arguments and their Boolean combinations. In particular, every two-valued quantifier (like the above choice of **more than 10 percent**) is a semi-fuzzy quantifier by definition.

Because of these benefits, semi-fuzzy quantifiers are considered a suitable base representation for NL quantifiers, sufficiently expressive to capture all quantifiers in the sense of TGQ as well as approximate quantifiers, and still sufficiently simple to allow for a straightforward definition. But of course, semi-fuzzy quantifiers cannot be applied to fuzzy arguments like **tall** or **rich**. I hence suggest the use of a mechanism which accepts a description of the target quantifier, stated as a semi-fuzzy quantifier, and returns a corresponding fuzzy quantifier which properly generalises the semi-fuzzy quantifier to the case of fuzzy arguments.

Definition 4 *A* quantifier fuzzification mechanism *(QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ of the same arity $n \in \mathbb{N}$ and on the same base set E.*

There is an underlying assumption here, which I presupposed when introducing semi-fuzzy quantifiers and quantifier fuzzification mechanisms. In fact, the QFM framework can only be successully applied to model natural language quantification if the considered NL quantifiers indeed permit a reduction to the simplified representation format provided by semi-fuzzy quantifiers. Anticipating the construction of underlying semi-fuzzy quantifiers $\mathcal{U}(\widetilde{Q})$ (defined below in Def. 5), which simply restricts the fuzzy quantifier $\widetilde{Q}$ to crisp arguments, we can then express the following *quantification framework assumption* that must be fulfilled:

> **Quantification framework assumption (QFA):**
> *If two base quantifiers of interest (i.e. NL quantifiers to be defined directly) have distinct interpretations $\widetilde{Q} \neq \widetilde{Q}'$ as fuzzy quantifiers, then they are already distinct on crisp arguments, i.e. $\mathcal{U}(\widetilde{Q}) \neq \mathcal{U}(\widetilde{Q}')$.*

This condition ensures the applicability of the QFM framework because we can then represent $\widetilde{Q}, \widetilde{Q}'$ by $Q = \mathcal{U}(\widetilde{Q})$ and $Q' = \mathcal{U}(\widetilde{Q}')$, without compromising the existence of a QFM $\mathcal{F}$ which takes Q to $\widetilde{Q} = \mathcal{F}(Q)$ and Q' to $\widetilde{Q}' = \mathcal{F}(Q')$. If the QFA is violated by $\widetilde{Q}$ and $\widetilde{Q}'$, however, then it is impossible for any QFM to separate the quantifiers, because $\mathcal{U}(\widetilde{Q}) = \mathcal{U}(\widetilde{Q}')$ entails that the same interpretation $\mathcal{F}(\mathcal{U}(\widetilde{Q})) = \mathcal{F}(\mathcal{U}(\widetilde{Q}'))$ is assigned to both quantifiers. This fundamental assumption underlying the quantification framework makes so elementary a requirement, that it is hard to conceive how it could be violated in human language. In the following, I will hence assume that the QFA holds, because it does not seem to exclude any phenomena of interest, and also because the QFA is justified from the current linguistic standpoint (the linguistic theory of quantification, TGQ, silently makes the same assumption by restriction attention to two-valued arguments only).

The above definition of 'raw', totally unrestricted QFMs must now be taylored to a class of 'reasonable' fuzzification mechanisms. We expect a fuzzification mechanism to be 'systematic' or 'well-behaved' and in conformance to linguistic considerations. In the following, I introduce the set of criteria adopted by DFS theory. For a more comprehensive treatment and motivation, see [9].
Perhaps the most elementary condition on a fuzzification mechanism is that it properly generalizes the original semi-fuzzy quantifier. We can express this succintly if we introduce the following notion of underlying semi-fuzzy quantifiers.

Definition 5 *Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ be a fuzzy quantifier. The* underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n)\,,$$

for all n-tuples of crisp *subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.*

It is natural to require that $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. $\mathcal{F}(Q)$ properly generalizes Q in the sense that $\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$ when all arguments are crisp.

Another adequacy constraint is based on the relationship of crisp and fuzzy membership assessments with quantification. We make this relationship explicit through the following definitions of projection quantifiers:

Definition 6 *Suppose E is a base set and $e \in E$. The projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by*

$$\pi_e(Y) = \chi_Y(e)\,,$$

where $\chi_Y : E \longrightarrow \mathbf{2}$ is the characteristic function of $Y \in \mathcal{P}(E)$, thus

$$\chi_Y(e) = \begin{cases} 1 & : \quad e \in Y \\ 0 & : \quad \textit{else} \end{cases} \tag{1}$$

For example, we can use the crisp projection quantifier π_{John} to evaluate crisp membership assessments like *Is John married?*, which can be evaluated by computing $\pi_{\mathrm{John}}(\textbf{married})$, where **married** $\in \mathcal{P}(E)$ is the crisp subset of married people in E. A corresponding definition of fuzzy projection quantifiers is straightforward.

Definition 7 *Let a base set E be given and $e \in E$. The* fuzzy projection quantifier *$\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is defined by*

$$\widetilde{\pi}_e(X) = \mu_X(e)$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

For example, we can evaluate $\widetilde{\pi}_{\mathrm{John}}(\textbf{tall})$ to assess the grade to which John is tall, and we can compute $\widetilde{\pi}_{\mathrm{John}}(\textbf{rich})$ to determine $\mu_{\textbf{rich}}(\mathrm{John})$, the degree to which John is rich. Because crisp and fuzzy projection quantifiers play the same role, viz. that of crisp/fuzzy membership assessment, we expect a reasonable choice of QFM $\mathcal{F}$ to recognize this relationship and map each crisp projection quantifier π_e to the corresponding fuzzy projection quantifier, i.e. $\widetilde{\pi}_e = \mathcal{F}(\pi_e)$.

We can also evaluate a QFM from the perspective of propositional fuzzy logic. By a canonical construction, every QFM also gives rise to induced fuzzy truth functions, i.e. to a unique choice of fuzzy conjunction, disjunction etc. In order to establish this link between logical connectives and quantifiers, we first observe that $\mathbf{2}^n \cong \mathcal{P}(\{1, \ldots, n\})$, using the bijection $\eta : \mathbf{2}^n \longrightarrow \mathcal{P}(\{1, \ldots, n\})$ defined by

$$\eta(x_1, \ldots, x_n) = \{k \in \{1, \ldots, n\} : x_k = 1\}\,,$$

for all $x_1, \ldots, x_n \in \mathbf{2}$. We can use an analogous construction in the fuzzy case. We then have $\mathbf{I}^n \cong \widetilde{\mathcal{P}}(\{1, \ldots, n\})$, based on the bijection $\widetilde{\eta} : \mathbf{I}^n \longrightarrow \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ defined by

$$\mu_{\widetilde{\eta}(x_1, \ldots, x_n)}(k) = x_k\,,$$

for all $x_1, \ldots, x_n \in \mathbf{I}$ and $k \in \{1, \ldots, n\}$. These bijections can be utilized for a translation between semi-fuzzy truth functions $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ and corresponding semi-fuzzy quantifiers $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$, and similarly the translation from fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ into fuzzy truth functions $\widetilde{f} : \mathbf{I}^n \longrightarrow \mathbf{I}$.

Definition 8 *Let a QFM $\mathcal{F}$ and a mapping ('semi-fuzzy truth function') $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ of arity $n > 0$ be given. The semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ is defined by*

$$Q_f(Y) = f(\eta^{-1}(Y))$$

for all $Y \in \mathcal{P}(\{1, \ldots, n\})$. The induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ *is defined by*

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n)),$$

for all $x_1, \ldots, x_n \in \mathbf{I}$. If $f : \mathbf{2}^0 \longrightarrow \mathbf{I}$ is a nullary semi-fuzzy truth function (i.e., a constant), we shall define $\widetilde{\mathcal{F}}(f) : \mathbf{I}^0 \longrightarrow \mathbf{I}$ by $\widetilde{\mathcal{F}}(f)(\varnothing) = \mathcal{F}(c)(\varnothing)$, where $c : \mathcal{P}(\{\varnothing\})^0 \longrightarrow \mathbf{I}$ is the constant $c(\varnothing) = f(\varnothing)$.[2,3]

We shall not impose restrictions on the induced connectives directly; these will be entailed by the remaining axioms.
Induced operations on fuzzy sets like fuzzy complement $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$, fuzzy intersection $\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ and fuzzy union $\widetilde{\cup} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$, can be defined element-wise in terms of the induced negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$, conjunction $\widetilde{\wedge} : \mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$ or disjunction $\widetilde{\vee} : \mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$, respectively. For example, the induced complement $\widetilde{\neg} X \in \widetilde{\mathcal{P}}(E)$ of $X \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_{\widetilde{\neg} X}(e) = \widetilde{\neg}\, \mu_X(e),$$

for all $X \in \widetilde{\mathcal{P}}(E)$ and $e \in E$.
Based on the induced fuzzy negation and complement, we can express important constructions on quantifiers like negation, formation of antonyms, and dualisation.

Definition 9 *The* external negation $\widetilde{\neg} Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$(\widetilde{\neg} Q)(Y_1, \ldots, Y_n) = \widetilde{\neg}(Q(Y_1, \ldots, Y_n)),$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. The definition of $\widetilde{\neg}\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ in the case of fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is analogous.

[2]The special treatment of nullary truth functions is necessary to avoid the use of $Q_f : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which is not a semi-fuzzy quantifier because the base set is empty. More information on the construction of induced fuzzy truth functions may be found in [9].

[3]To facilitate understanding: $\mathcal{P}(\{\varnothing\})^0 = \{f | f : \varnothing \longrightarrow \mathcal{P}(\{\varnothing\})\} = \{\varnothing\}$, i.e. '$\varnothing$' in $c(\varnothing)$ and $f(\varnothing)$ denotes the empty tuple.

For example, **no** is the negation of **some**.[4]

Definition 10 *Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. The* antonym $Q\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of* Q *is defined by*

$$Q\neg(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *The antonym* $\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined analogously, based on the given fuzzy complement* $\widetilde{\neg}$.

For example, **no** is the antonym of **all**. The dual $Q\widetilde{\Box}$ of a quantifier is the negation of the antonym, or equivalently, the antonym of the negation:

Definition 11 *The* dual $Q\widetilde{\Box} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $n > 0$ *is defined by*

$$Q\widetilde{\Box}(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *The dual* $\widetilde{Q}\widetilde{\Box} = \widetilde{\neg}\,\widetilde{Q}\widetilde{\neg}$ *of a fuzzy quantifier* $\widetilde{Q}$ *is defined analogously.*

For example, **some** is the dual of **all**. We expect that a given QFM $\mathcal{F}$ be compatible with these constructions on quantifiers. Hence $\mathcal{F}$(**no**) should be the negation of $\mathcal{F}$(**some**), $\mathcal{F}$(**no**) should be the antonym of $\mathcal{F}$(**all**) and $\mathcal{F}$(**some**) should be the dual of $\mathcal{F}$(**all**).

Apart from negation/complementation, we can also form intersections and unions of argument sets to construct new quantifiers from given ones.

Definition 12 *Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. We define quantifiers* $Q\cup, Q\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ *by*

$$Q\cup(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$$
$$Q\cap(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1})$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E)$. *In the case of fuzzy quantifiers,* $\widetilde{Q}\widetilde{\cup}$ *and* $\widetilde{Q}\widetilde{\cap}$ *are defined analogously, based on the given fuzzy set operations* $\widetilde{\cup}$ *and* $\widetilde{\cap}$, *resp.*

In some proofs, I will also need another construction, that of permuting arguments of a quantifier. Here I restrict to a special type of argument transpositions. It is apparent that every permutation of the argument positions can be decomposed into a sequence of the following simple transpositions:

Definition 13 *For every semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *and all* $i \in \{1, \ \ldots, n\}$, *the semi-fuzzy quantifier* $Q\tau_i : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$Q\tau_i(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{i-1}, Y_n, Y_{i+1}, \ldots, Y_{n-1}, Y_i)$$

[4] assuming that $\widetilde{\neg}\, 0 = 1$ and $\widetilde{\neg}\, 1 = 0$, which will be assured by the axioms.

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. (In other words, the i-th argument is exchanged with the last argument). The definition of $\widetilde{Q}\tau_i$ for fuzzy quantifiers is analogous.

As we shall later see, every intended model is compatible with argument permutations. This ensures e.g. that symmetries of a quantifier are preserved in the fuzzy case.
Another important characteristic of quantifiers expresses through their monotonicity properties.

Definition 14 *A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is said to be* nonincreasing in its i-th argument, *$i \in \{1, \ldots, n\}$, if*

$$Q(Y_1, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

whenever $Y_1, \ldots, Y_n, Y_i' \in \mathcal{P}(E)$ such that $Y_i \subseteq Y_i'$. Q is said to be nondecreasing in the i-th argument *if the reverse inequation holds. The definitions for fuzzy quantifiers are analogous.*

For example, **all** is nonincreasing in the first argument and nondecreasing in the second argument. We expect each reasonable choice of QFM $\mathcal{F}$ to preserve such monotonicity properties. Hence $\mathcal{F}$(**all**) should be nonincreasing in the first and nondecreasing in the second argument.

We can also utilize a QFM to construct fuzzy powerset mappings. Let us first recall the concept of a powerset mapping in the crisp case.

Definition 15 *To each mapping $f : E \longrightarrow E'$, we can associate a mapping $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ (the powerset mapping of f) which is defined by*

$$\widehat{f}(Y) = \{f(e) : e \in Y\}, \quad (2)$$

for all $Y \in \mathcal{P}(E)$.[5]

In order to generalise this concept to the fuzzy case, we need a mechanism which associates fuzzy powerset mappings $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ to given mappings $f : E \longrightarrow E'$. Such a mechanism is called an *extension principle*.[6] The standard extension principle, proposed by Zadeh [22], is defined by

$$\mu_{\hat{f}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}, \quad (3)$$

for all $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $e' \in E'$. With each QFM, we can associate a corresponding extension principle through a canonical construction.

Definition 16 *Every QFM $\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E \longrightarrow E'$ (where $E, E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ defined by*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X),$$

[5] Often the same symbol is used to denote both the original mapping and the powerset mapping.
[6] For our purposes, it will be convenient to assume that $E, E' \neq \varnothing$.

for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$.

We require that every 'reasonable' choice of $\mathcal{F}$ be compatible with its induced extension principle in the following sense. Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $f_1, \dots, f_n : E' \longrightarrow E$ are given mappings, $E' \neq \varnothing$. We can construct the semi-fuzzy quantifier $Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by composing Q with the powerset mappings $\widehat{f}_i, \dots, \widehat{f_n}$, i.e.

$$(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i)(Y_1, \dots, Y_n) = Q(\widehat{f}_1(Y_1), \dots, \widehat{f_n}(Y_n))\,, \tag{4}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E')$. By utilizing the induced extension principle $\widehat{\mathcal{F}}$ of a QFM, we can perform a similar construction on fuzzy quantifiers, thus composing $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ with $\widehat{\mathcal{F}}(f_1), \dots, \widehat{\mathcal{F}}(f_n)$ to form the fuzzy quantifier $\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$ defined by

$$(\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i))(X_1, \dots, X_n) = \widetilde{Q}(\widehat{\mathcal{F}}(f_1)(X_1), \dots, \widehat{\mathcal{F}}(f_n)(X_n))\,,$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E')$. We require that a QFM $\mathcal{F}$ be compatible with this construction, i.e.

$$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)\,.$$

This condition is of particular importance because it is the only criterion which relates the results of $\mathcal{F}$ for different base sets E, E'. It hence grants that $\mathcal{F}$ behave consistently across domains. We can combine the above conditions in order to capture our expectations on plausible models of fuzzy quantification in a condensed set of axioms.

Definition 17 *A QFM $\mathcal{F}$ is called a* determiner fuzzification scheme *(DFS) if the following conditions are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(Z-2)
Dualisation	$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}$ $n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}$ $n > 0$	(Z-4)
Preservation of monotonicity	*If Q is nonincreasing in n-th arg, then $\mathcal{F}(Q)$ is nonincreasing in n-th arg,* $n > 0$	(Z-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \dots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(Z-6)

The original definition of DFSes in [7] was based on nine axioms. These were subsequently condensed into the equivalent axiom system presented above, and the independence of the new axioms (Z-1) to (Z-6) has been proven [9].

The conditions (Z-1)–(Z-6) are intended to cover those adequacy criteria that are essential from the perspective of linguistics and fuzzy logic, and to provide a formalisation of these criteria in terms of a system of independent axioms. Due to the goal of obtaining an independent system, it was not possible to include all of these adequacy criteria directly into the axiom set, thus compromising its independence. However, it has been shown in [9] that DFSes comply with a large number of linguistic and logical adequacy criteria. The following excerpt is not intended to review these results on adequacy properties of DFSes, which can be found in full detail in [9]. By contrast, only those definitions and theorems are highlighted, that are necessary to understand and prove the new theorems. Unless otherwise stated, the proofs of all theorems cited can be found in [7, 9].

First we review some results on the fuzzy truth functions induced by a DFS. Let us recall the definition of a strong negation (i.e. 'reasonable' fuzzy negation operator):

Definition 18 $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *iff it satisfies*

a. $\widetilde{\neg}\, 0 = 1$ *(boundary condition)*

b. $\widetilde{\neg}\, x_1 \geq \widetilde{\neg}\, x_2$ *for all* $x_1, x_2 \in \mathbf{I}$ *such that* $x_1 < x_2$ *(i.e.* $\widetilde{\neg}$ *is monotonically decreasing)*

c. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e.* $\widetilde{\neg}$ *is involutive).*

Note. Whenever the standard negation $\neg x = 1 - x$ is being assumed, we shall drop the 'tilde'-notation. Hence the standard fuzzy complement is denoted $\neg X$, where $\mu_{\neg X}(e) = 1 - \mu_X(e)$. Similarly, the external negation of a (semi-) fuzzy quantifier with respect to the standard negation is written $\neg Q$, and the antonym of a fuzzy quantifier with respect to the standard fuzzy complement is written as $\widetilde{Q}\neg$.

We also need the concepts of a t-norm (i.e. 'reasonable' fuzzy conjunction) and s-norm ('reasonable' fuzzy disjunction), see [16]. The fuzzy truth functions induced by a DFS are guaranteed to belong to the class of such reasonable operators:

Theorem 1 *In every DFS* $\mathcal{F}$*,*

a. $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$ *is the identity truth function;*

b. $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator;*

c. $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ *is a* t*-norm;*

d. $x_1 \,\widetilde{\vee}\, x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \,\widetilde{\wedge}\, \widetilde{\neg}\, x_2)$*, i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$*.*

In the proofs to follow we also need the following theorem, which is a consequence of Th-1, (Z-4) and (Z-3).

Theorem 2 *Every DFS $\mathcal{F}$ is compatible with the external negation of quantifiers, i.e. for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{F}(\widetilde{\neg}\, Q) = \widetilde{\neg}\, \mathcal{F}(Q)$.*

We further need a result on the monotonicity of DFSes. For semi-fuzzy $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, we say that $Q \leq Q'$ iff $Q(Y_1, \ldots, Y_n) \leq Q'(Y_1, \ldots, Y_n)$ for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. For fuzzy quantifiers $\widetilde{Q}, \widetilde{Q}'$, we use an analogous definition, i.e. $\widetilde{Q} \leq \widetilde{Q}'$ iff $\widetilde{Q}(X_1, \ldots, X_n) \leq \widetilde{Q}'(X_1, \ldots, X_n)$ for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Theorem 3 *Every DFS $\mathcal{F}$ is* monotonic, *i.e. if $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are given semi-fuzzy quantifiers and $Q \leq Q'$, then $\mathcal{F}(Q) \leq \mathcal{F}(Q')$.*

Finally I cite a result concerning the preservation of symmetries in a quantifier's arguments.

Theorem 4 *Every DFS $\mathcal{F}$ is compatible with argument transpositions, i.e. $\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i$ for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and all $i \in \{1, \ldots, n\}$.*

This theorem establishes e.g. that the meanings of 'some rich people are lucky' and 'some lucky people are rich' coincide.
Next we turn to special subclasses of DFSes, in order to single out a class of standard models for fuzzy quantification.

Definition 19 *Suppose $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ is strong negation operator. A DFS $\mathcal{F}$ is called a $\widetilde{\neg}$-DFS if its induced negation coincides with $\widetilde{\neg}$, i.e. $\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}$. In particular, we will call $\mathcal{F}$ a $\neg$-DFS if it induces the standard negation $\neg x = 1 - x$.*

As has been shown in [7, Th-28, p. 44], no models of interest are lost if we restrict attention to $\neg$-DFSes only (i.e. to DFSes which induce the standard negation). This is because all other DFSes can be transformed into $\neg$-DFSes and vice versa.
It is convenient to group the models by their induced disjunctions.

Definition 20 *A $\neg$-DFS $\mathcal{F}$ which induces a fuzzy disjunction $\widetilde{\vee}$ is called a $\widetilde{\vee}$-DFS.*

Definition 21 *A DFS $\mathcal{F}$ is called a* standard DFS *if and only if $\mathcal{F}$ is a $\max$-DFS, i.e. a DFS which induces the standard negation $\neg x = 1 - x$ and the standard disjunction $x \vee y = \max(x, y)$.*

Note. It is then apparent from earlier work [9, Th-17.a, p. 20 and Th-25, p. 25] that standard DFSes are exactly those $\neg$-DFSes which induce the standard extension principle $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$ and the standard connectives of fuzzy logic. It is hence suggested that standard DFSes be considered the standard models of fuzzy quantification.
The following result that has been proven for standard DFSes will be needed in later proofs.

Theorem 5 *All standard DFSes coincide on two-valued quantifiers. Hence if $\mathcal{F}, \mathcal{F}'$ are standard DFSes and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued quantifier, then $\mathcal{F}(Q) = \mathcal{F}'(Q)$.*

The $\neg$-DFSes can be partially ordered by 'specificity' or 'fuzziness', in the sense of closeness to $\frac{1}{2}$. We define a partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ by

$$x \preceq_c y \Leftrightarrow y \leq x \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} \leq x \leq y\,, \tag{5}$$

for all $x, y \in \mathbf{I}$. $\preceq_c$ is Mukaidono's ambiguity relation, see [14]. We extend this basic definition of $\preceq_c$ for scalars to the case of DFSes in the obvious way:

Definition 22 *Suppose $\mathcal{F}$, $\mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is* consistently less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, iff for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}'(Q)(X_1, \ldots, X_n)\,.$$

We now wish to establish the existence of consistently least specific $\widetilde{\vee}$-DFSes. As it turns out, the greatest lower specificity bound of a collection of $\widetilde{\vee}$-DFSes can be expressed using the fuzzy median [17, 5].

Definition 23 *The* fuzzy median $\mathrm{med}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{med}_{\frac{1}{2}}(u_1,\, u_2) = \begin{cases} \min(u_1,\, u_2) & : \ \min(u_1,\, u_2) > \frac{1}{2} \\ \max(u_1,\, u_2) & : \ \max(u_1,\, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

The basic connective can be generalised to an operator $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.

Definition 24 *The* generalised fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{med}_{\frac{1}{2}}(\inf X,\ \sup X)\,,$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Now we can state the desired theorem.

Theorem 6 *Suppose that $\widetilde{\vee}$ is an s-norm $\mathbb{F}$ a non-empty collection of $\widetilde{\vee}$-DFSes $\mathcal{F} \in \mathbb{F}$. Then there exists a greatest lower specificity bound on $\mathbb{F}$, i.e. a $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{glb}}$ such that $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}$ for all $\mathcal{F} \in \mathbb{F}$ (i.e. $\mathcal{F}_{\mathrm{glb}}$ is a lower specificity bound), and for all other lower specificity bounds $\mathcal{F}'$, $\mathcal{F}' \preceq_c \mathcal{F}_{\mathrm{glb}}$.*
$\mathcal{F}_{\mathrm{glb}}$ is defined by

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}\,,$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

In particular, the theorem asserts the existence of least specific $\widetilde{\vee}$-DFSes, i.e. whenever $\widetilde{\vee}$ is an s-norm such that $\widetilde{\vee}$-DFSes exist, then there exists a least specific $\widetilde{\vee}$-DFS (just apply the above theorem to the collection of all $\widetilde{\vee}$-DFSes).
As concerns the converse issue of most specific DFSes, i.e. least upper bounds with respect to $\preceq_c$, the following definition of 'specificity consistence' turns out to provide the key concept:

Definition 25 *Suppose $\widetilde{\vee}$ is an s-norm and $\mathbb{F}$ is a non-empty collection of $\widetilde{\vee}$-DFSes $\mathcal{F} \in \mathbb{F}$. $\mathbb{F}$ is called* specificity consistent *iff for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, either $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ or $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$, where*

$$R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}\,.$$

We can now express the exact conditions under which a collection of $\widetilde{\vee}$-DFSes has a least upper specificity bound.

Theorem 7 *Suppose $\widetilde{\vee}$ is an s-norm and $\mathbb{F}$ is a non-empty collection of $\widetilde{\vee}$-DFSes $\mathcal{F} \in \mathbb{F}$.*

a. $\mathbb{F}$ has upper specificity bounds exactly if $\mathbb{F}$ is specificity consistent.

b. If $\mathbb{F}$ is specificity consistent, then its least upper specificity bound is the $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{lub}}$ defined by

$$\mathcal{F}_{\mathrm{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}(Q)(X_1, \ldots, X_n) : \mathcal{F} \in \mathbb{F}\}$.

Let us now consider some additional adequacy criteria for approaches to fuzzy quantification, which are not necessarily required for arbitrary DFSes. The first two criteria are concerned with the 'propagation of fuzziness', i.e. the way in which the amount of imprecision in the model's inputs affects changes of the model's outputs. To this end, let us recall the partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ defined by equation (5). We can extend $\preceq_c$ to fuzzy sets $X \in \widetilde{\mathcal{P}}(E)$, semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ as follows:

$$\begin{aligned} X \preceq_c X' &\iff \mu_X(e) \preceq_c \mu_{X'}(e) && \text{for all } e \in E; \\ Q \preceq_c Q' &\iff Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n) && \text{for all } Y_1, \ldots, Y_n \in \mathcal{P}(E); \\ \widetilde{Q} \preceq_c \widetilde{Q}' &\iff \widetilde{Q}(X_1, \ldots, X_n) \preceq_c \widetilde{Q}'(X_1, \ldots, X_n) && \text{for all } X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\,. \end{aligned}$$

Intuitively, we expect that the quantification results become less specific whenever the quantifier or the argument sets become less specific: the fuzzier the input, the fuzzier the output.

Definition 26 *We say that a QFM $\mathcal{F}$* propagates fuzziness in arguments *if and only if the following property is valid for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n, X_1', \ldots, X_n'$:*

If $X_i \preceq_c X'_i$ for all $i = 1, \dots, n$, then $\mathcal{F}(Q)(X_1, \dots, X_n) \preceq_c \mathcal{F}(Q)(X'_1, \dots, X'_n)$.
We say that $\mathcal{F}$ propagates fuzziness in quantifiers *if and only if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.*

Both conditions are certainly natural to require, and I consider them as desirable but optional. A more thorough discussion of propagation of fuzziness and its tradeoffs can be found in [11].
Finally, I introduce two adequacy criteria concerned with distinct aspects of the 'smoothness' or 'continuity' of a DFS. These conditions are essential for DFSes to be *practical* because it is extremely important for applications that the results of a DFS be stable under slight changes in the inputs. These 'changes' can either occur in the fuzzy argument sets (e.g. due to noise), or they can affect the semi-fuzzy quantifier. For example, if a person A has a slightly different interpretation of quantifier Q compared to person B, then we still want them to understand each others, and the quantification results obtained from the two models of the target quantifier should be very similar in such cases.
In order to express the robustness criterion with respect to slight changes in the fuzzy arguments, a metric on fuzzy subsets is needed, which serves as a numerical quantity of the similarity of the arguments. For all base sets $E \neq \varnothing$ and all $n \in \mathbb{N}$, we define the metric $d : \widetilde{\mathcal{P}}(E)^n \times \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$d((X_1, \dots, X_n), (X'_1, \dots, X'_n)) = \max_{i=1}^{n} \sup\{|\mu_{X_i}(e) - \mu_{X'_i}(e)| : e \in E\}\,, \qquad (6)$$

for all $X_1, \dots, X_n, X'_1, \dots, X'_n \in \widetilde{\mathcal{P}}(E)$. Based on this metric, we can now express the desired criterion for continuity *in arguments*.

Definition 27 *We say that a QFM $\mathcal{F}$ is* arg-continuous *if and only if $\mathcal{F}$ maps all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to continuous fuzzy quantifiers $\mathcal{F}(Q)$, i.e. for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\mathcal{F}(Q)(X_1, \dots, X_n), \mathcal{F}(Q)(X'_1, \dots, X'_n)) < \varepsilon$ for all $X'_1, \dots, X'_n \in \widetilde{\mathcal{P}}(E)$ with $d((X_1, \dots, X_n), (X'_1, \dots, X'_n)) < \delta$.*

A second robustness criterion is intended to capture the idea that slight changes in a semi-fuzzy quantifier should not cause the quantification results to change drastically. To introduce this criterion, we must first define suitable distance measures for semi-fuzzy quantifiers and for fuzzy quantifiers. Hence for all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$d(Q, Q') = \sup\{|Q(Y_1, \dots, Y_n) - Q'(Y_1, \dots, Y_n)| : Y_1, \dots, Y_n \in \mathcal{P}(E)\}\,, \qquad (7)$$

and similarly for all fuzzy quantifiers $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$,

$$d(\widetilde{Q}, \widetilde{Q}') = \sup\{|\widetilde{Q}(X_1, \dots, X_n) - \widetilde{Q}'(X_1, \dots, X_n)| : X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)\}\,. \qquad (8)$$

Definition 28 *We say that a QFM $\mathcal{F}$ is* Q-continuous *if and only if for each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mathcal{F}(Q), \mathcal{F}(Q')) < \varepsilon$ whenever $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ satisfies $d(Q, Q') < \delta$.*

Both condition are crucial to the utility of a DFS and should be possessed by every model employed in practical applications. They are not part of the DFS axioms because I wanted to have DFSes for general t-norms (including the discontinuous variety).

This completes the first chapter, which was intended to give a brief introduction into DFS theory. After motivating the need for a formal analysis and computer modelling of approximate quantification, I first reviewed existing approaches which try and embed approximate quantification in the framework of fuzzy set theory, based on operators called fuzzy linguistic quantifiers. In addition, some pointers to the literature were given, which reveal that these approaches are not plausible from a linguistic perspective, and can produce unexpected results in important situations.
The DFS theory of fuzzy quantification, by contrast, starts from the linguistic theory of quantification, TGQ, which is then extended to handle the inherent fuzziness, which is observed both in a quantifier and in its arguments. As opposed to existing approaches which only consider absolute quantifiers (like *about ten*) and proportional quantifiers (like *most*), its anchoring into the linguistic framework permits DFS theory to cover the complete range of NL quantification, which is known from linguistics. The concept of a fuzzy quantifier was proposed, which embeds all quantifiers in the sense of TGQ and constitutes the class of target operators for fuzzy NL quantification. These operators pose a problem, though, because they live on fuzzy arguments and do not permit a direct definition in terms of a cardinality measure. This makes it very hard to justify that a particular choice of fuzzy quantifier be the proper model of a given NL quantifier. In order to solve this problem of defining appropriate models of given NL quantifiers, the novel concept of semi-fuzzy quantifiers was then introduced, which are (a) capable of expressing approximate quantification, and (b) restricted to two-valued arguments, i.e. definable in terms of the usual cardinality measure (whenever appropriate). Due to their conceptual simplicity, semi-fuzzy quantifiers are good base representations of NL quantifiers. However, they do not solve the problem of handling fuzzy arguments (like in *most rich people are bald*). To provide a full account of fuzzy quantification and support fuzziness both in quantifiers and their arguments, it is hence necessary to translate semi-fuzzy quantifiers into corresponding fuzzy quantifiers, which can then be applied to fuzzy arguments. This translation is accomplished by quantifier fuzzification mechanisms (or QFMs for short). QFMs are one of the central concepts of DFS theory, and span the class of 'raw', totally unrestricted fuzzification mechanisms.
In turn, a number of adequacy criteria were presented which shrink down the unrestricted class of QFMs to its subclass of plausible models, and thus ensure a systematic transfer from semi-fuzzy quantifiers Q to corresponding target operators, i.e. fuzzy quantifiers $\mathcal{F}(Q)$, which indeed extrapolate the meaning of the base quantifier. Most of these criteria have either been adopted from TGQ or reflect logical considerations. Taken together, they constitute the set of 'DFS axioms' (Z-1)–(Z-6), which provide a characterisation of the intended models, dubbed DFSes (determiner fuzzification schemes), in terms of an independent axiom system.
Finally I presented a small number of additional properties, which are either fulfilled by arbitrary DFSes (like monotonicity), or limited to special cases of DFSes (like propagation of fuzziness). The most important property of practical models has also been formalized, which is certainly that of *stability*, in order to absorb slight variations e.g. due to noise, quantization errors etc., which are typical of real-world applications. In for-

malizing this property, I recognized that it actually has two distinct faces, and consequently developed separate criteria that capture (a) the robustness of the quantification results under slight changes in the arguments (arg-continuity), and (b) the robustness of the quantification results under slight changes in the quantifier (Q-continuity).
The chapter also introduces the class of standard DFSes, which is formed by those models which conform to the standard operations of fuzzy set theory ($\min$, $\max$ etc.). Due to the comprehensive adequacy properties observed with these models, and due to the natural embedding of the established core of fuzzy set theory (standard connectives, standard extension principle etc.), the assumption is made in the report that it is these models, i.e. the standard DFSes, which constitute the standard models of fuzzy quantification.

2 The class of $\mathcal{M}_\mathcal{B}$-DFSes

In [7], the first three models of the DFS axioms have been presented. An investigation of the common principle underlying these DFSes has led to the introduction of $\mathcal{M}_\mathcal{B}$-DFSes in [9], the class of DFSes defined in terms of three-valued cuts of arguments and subsequent aggregation based on the fuzzy median. Here I recall the definition of $\mathcal{M}_\mathcal{B}$-QFMs and the characterisation of $\mathcal{M}_\mathcal{B}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping $\mathcal{B}$. Important models are also presented and some interesting properties of $\mathcal{M}_\mathcal{B}$-DFSes are highlighted. Most of the material is compiled from [11] and its exposition is mainly intended to introduce the basic concepts that will be generalized lateron. Unless otherwise stated, the proofs of all theorems cited in this chapter can be found in [9], which provides a comprehensive discussion of $\mathcal{M}_\mathcal{B}$-DFSes.
I first define the unrestricted class of $\mathcal{M}_\mathcal{B}$-QFMs, which will then be shrunk to the reasonable cases of $\mathcal{M}_\mathcal{B}$-DFSes by imposing conditions on the aggregation mapping. To this end, we need some notation. We recall the concept of α-cuts and strict α-cuts of fuzzy subsets:

Definition 29 *Let E be a given set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\alpha \in \mathbf{I}$. By $X_{\geq\alpha} \in \mathcal{P}(E)$ we denote the α-cut*

$$X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\}\,.$$

Definition 30 *Let $X \in \widetilde{\mathcal{P}}(E)$ be given and $\alpha \in \mathbf{I}$. By $X_{>\alpha} \in \mathcal{P}(E)$ we denote the strict α-cut*

$$X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\}\,.$$

In terms of these α-cuts, we define the cut range $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$, which represents a three-valued cut at the 'cautiousness level' $\gamma \in \mathbf{I}$ by a set of alternatives $\{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\}$. The reason for introducing three-valued cuts is that we need a cutting mechanism compatible with complementation. α-cuts, however, have $(\neg X)_{\geq\alpha} \neq \neg(X_{\geq\alpha})$. The desired symmetry is easily obtained with three-valued cuts, defined as follows:

Definition 31 *Suppose E is some set, $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. $X_\gamma^{\min}, X_\gamma^{\max} \in \mathcal{P}(E)$ and $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ are defined by*

$$X_\gamma^{\min} = \begin{cases} X_{>\frac{1}{2}} & : \ \gamma = 0 \\ X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} & : \ \gamma > 0 \end{cases}$$

$$X_\gamma^{\max} = \begin{cases} X_{\geq\frac{1}{2}} & : \ \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \ \gamma > 0 \end{cases}$$

$$\mathcal{T}_\gamma(X) = \{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\}\,.$$

Note. The relationship of cut ranges $\mathcal{T}_\gamma(X)$ and three-valued sets is discussed in [7, p. 58+] and [9, p. 39+].

How can we use these cut ranges to evaluate fuzzy quantifiers? The basic idea is that we can view the crisp range $\mathcal{T}_\gamma(X)$ as providing a set of alternatives to be checked. For example, in order to evaluate a quantifier Q at a certain cut level γ, we have to consider all choices of $Q(Y_1, \ldots, Y_n)$, where $Y_i \in \mathcal{T}_\gamma(X_i)$. The set of results obtained in this way must then be aggregated to a single result in the unit interval, which we denote as $Q_\gamma(X_1, \ldots, X_n) \in \mathbf{I}$. The generalised fuzzy median (see Def. 24) is well-suited to carry out this aggregation. The use of the fuzzy median for this purpose was originally motivated by the observation that the resulting fuzzification mechanisms embed Kleene's three-valued logic. This is useful because the targeted class of models (viz, standard DFSes) are known to embed Kleene's logic, too.
Let us hence use the crisp ranges $\mathcal{T}_\gamma(X_i)$ of the argument sets to define a family of QFMs $(\bullet)_\gamma$, indexed by the cautiousness parameter $\gamma \in \mathbf{I}$:

Definition 32 *For every $\gamma \in \mathbf{I}$, we denote by $(\bullet)_\gamma$ the QFM defined by*

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

None of the QFMs $(\bullet)_\gamma$ is a DFS, because the required information is spread over various cut levels. Hence in order to define DFSes based on these QFMs, we must simultaneously consider the results obtained at all levels of cautiousness γ, i.e. the γ-index family $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$. We can then apply various aggregation operators on these γ-indexed results to obtain new QFMs, which have a chance of being DFSes. We now define the domain on which these aggregation operators can act.

Definition 33 *$\mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^-$ and $\mathbb{B} \subseteq \mathbf{I}^\mathbf{I}$ are defined by*

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^\mathbf{I} : f(0) > \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [\tfrac{1}{2}, 1] \text{ and } f \text{ nonincreasing }\}\\
\mathbb{B}^{\frac{1}{2}} &= \{\mathrm{c}_{\frac{1}{2}}\}\\
\mathbb{B}^- &= \{f \in \mathbf{I}^\mathbf{I} : f(0) < \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [0, \tfrac{1}{2}] \text{ and } f \text{ nondecreasing }\}\\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^-\,.
\end{aligned}$$

Note. In the definition of $\mathbb{B}^{\frac{1}{2}}$, $\mathrm{c}_{\frac{1}{2}} : \mathbf{I} \longrightarrow \mathbf{I}$ is the constant $\mathrm{c}_{\frac{1}{2}}(x) = \frac{1}{2}$ for all $x \in \mathbf{I}$. More generally, we stipulate for all $a \in \mathbf{I}$ that $\mathrm{c}_a : \mathbf{I} \longrightarrow \mathbf{I}$ be the constant mapping

$$\mathrm{c}_a(x) = a\,, \tag{9}$$

for all $x \in \mathbf{I}$.

Theorem 8

a. *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *are given. Then*

$$(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} \in \begin{cases} \mathbb{B}^+ & : \quad Q_0(X_1, \ldots, X_n) > \frac{1}{2} \\ \mathbb{B}^{\frac{1}{2}} & : \quad Q_0(X_1, \ldots, X_n) = \frac{1}{2} \\ \mathbb{B}^- & : \quad Q_0(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

b. *For each* $f \in \mathbb{B}$ *there exists* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *such that* $f = (Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$.

Given an aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$, we define the corresponding QFM $\mathcal{M}_\mathcal{B}$ as follows.

Definition 34 *Suppose* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given. The QFM* $\mathcal{M}_\mathcal{B}$ *is defined by*

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) \,, \tag{10}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1 \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{M}_\mathcal{B}$-QFMs we mean the class of all QFMs $\mathcal{M}_\mathcal{B}$ defined in this way. It is apparent that if we do not impose restrictions on admissible choices of $\mathcal{B}$, the resulting QFMs will often fail to be DFSes. Hence let us state the necessary and sufficient conditions that $\mathcal{B}$ must satisfy in order to make $\mathcal{M}_\mathcal{B}$ a DFS. To express these conditions, we first need some constructions on $\mathbb{B}$.

Definition 35 *Suppose* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e., nondecreasing or nonincreasing). The mappings* $f^\flat$, $f^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$ *are defined by:*

$$f^\sharp = \begin{cases} \lim\limits_{y \to x^+} f(y) & : \quad x < 1 \\ f(1) & : \quad x = 1 \end{cases}$$

$$f^\flat = \begin{cases} \lim\limits_{y \to x^-} f(y) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases} \qquad \text{for all } f \in \mathbb{B},\ x \in \mathbf{I}.$$

It is apparent that if $f \in \mathbb{B}$, then $f^\sharp \in \mathbb{B}$ and $f^\flat \in \mathbb{B}$. $f^\sharp$ and $f^\flat$ are obviously very 'similar' to each others (and to f) and every reasonable $\mathcal{B}$ should map $f^\flat$ and $f^\sharp$ to the same aggregation result. This turns out to be essential for $\mathcal{M}_\mathcal{B}$ to satisfy (Z-6), because $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$ is not compatible with (Z-6) in a precise sense, but only modulo $\sharp/\flat$.
We shall further introduce several coefficients which describe certain aspects of a mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Definition 36 *For every monotonic mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$ (i.e., either nondecreasing or nonincreasing), we define*

$$f_0^* = \lim_{\gamma \to 0^+} f(\gamma) \tag{11}$$

$$f_*^0 = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{12}$$

$$f_*^{\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \tag{13}$$

$$f_1^* = \lim_{\gamma \to 1^-} f(\gamma) \tag{14}$$

$$f_*^{1\uparrow} = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} \tag{15}$$

$$f_*^{0\uparrow} = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{16}$$

$$f_*^{1\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 1\}\,. \tag{17}$$

We only need $f_*^{\frac{1}{2}}$ to define the desired conditions on $\mathcal{B}$; it turns out to be essential for ensuring a proper behaviour of $\mathcal{M}_{\mathcal{B}}$ in the case of three-valued quantifiers, and in particular to ensure the desired results for the two-valued projection quantifiers of (Z-2). We will use the remaining coefficients later to define examples of $\mathcal{M}_{\mathcal{B}}$-DFSes.

Definition 37 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathcal{B}$, we define the following conditions on $\mathcal{B}$:*

$$\mathcal{B}(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{B-1}$$

$$\mathcal{B}(1-f) = 1 - \mathcal{B}(f) \tag{B-2}$$

If $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$, *then* (B-3)

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases}$$

$$\mathcal{B}(f^\sharp) = \mathcal{B}(f^\flat) \tag{B-4}$$

If $f \leq g$, *then* $\mathcal{B}(f) \leq \mathcal{B}(g)$ (B-5)

As witnessed by the next theorem, these conditions capture precisely the requirement on $\mathcal{B}$ for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.

Theorem 9

a. *The conditions* (B-1) *to* (B-5) *are sufficient for $\mathcal{M}_{\mathcal{B}}$ to be a standard DFS.*

b. *The conditions* (B-1) *to* (B-5) *are necessary for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.*

c. *The conditions* (B-1) *to* (B-5) *are independent.*

In particular, $\mathcal{B}(f) = 1 - \mathcal{B}(1-f)$ for all $f \in \mathbb{B}$, and $\mathcal{B}(f) \geq \frac{1}{2}$ whenever $f \in \mathbb{B}^+$. We can hence give a more concise description of $\mathcal{M}_\mathcal{B}$-DFSes, because it is sufficient to consider their behaviour on $\mathbb{B}^+$ only:

Definition 38 *By $\mathbb{H} \subseteq \mathbf{I}^\mathbf{I}$ we denote the set of nonincreasing $f : \mathbf{I} \longrightarrow \mathbf{I}$, $f \neq 0$,*

$$\mathbb{H} = \{ f \in \mathbf{I}^\mathbf{I} : \ f \text{ nonincreasing and } f(0) > 0 \} .$$

We can associate with each $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ a $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ as follows:

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{18}$$

Theorem 10 *If $\mathcal{M}_\mathcal{B}$ is a DFS, then $\mathcal{B}$ can be defined in terms of a mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to equation* (18). *$\mathcal{B}'$ is defined by*

$$\mathcal{B}'(f) = 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 \, . \tag{19}$$

We can hence focus on mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ without loosing any desired models.

Definition 39 *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathbb{H}$, we define the following conditions on $\mathcal{B}'$:*

$$\mathcal{B}'(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{C-1}$$

$$\text{If } \widehat{f}(\mathbf{I}) \subseteq \{0,1\}, \text{ then } \mathcal{B}'(f) = f_*^0, \tag{C-2}$$

$$\mathcal{B}'(f^\sharp) = \mathcal{B}'(f^\flat) \quad \text{if } \widehat{f}((0,1]) \neq \{0\} \tag{C-3}$$

$$\text{If } f \leq g, \text{ then } \mathcal{B}'(f) \leq \mathcal{B}'(g) \tag{C-4}$$

A theorem analogous to Th-9 can be proven for (C-1) to (C-4):

Theorem 11

a. *The conditions* (C-1) *to* (C-4) *are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*

b. *The conditions* (C-1) *to* (C-4) *are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

c. *The conditions* (C-1) *to* (C-4) *are independent.*

Our introducing of $\mathcal{B}'$ is only a matter of convenience, because the definition of $\mathcal{B}'$ is usually shorter than the definition of the corresponding $\mathcal{B}$. We now present some examples of $\mathcal{M}_\mathcal{B}$-QFMs.

Definition 40 *By $\mathcal{M}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_\int(f) = \int_0^1 f(x)\,dx \, , \qquad \text{for all } f \in \mathbb{H}.$$

Theorem 12 *$\mathcal{M}$ is a standard DFS.*

$\mathcal{M}$ is Q-continuous and arg-continuous and hence a good choice for applications.

Definition 41 *By $\mathcal{M}_U$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_U(f) = \max(f_*^{1\uparrow}, f_1^*) \qquad \textit{for all } f \in \mathbb{H}\textit{, see } (14) \textit{ and } (15).$$

Theorem 13 *Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^{1\uparrow} \oplus f_1^*\,,$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equations (10) *and* (18)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_U$ is a standard DFS. It is neither Q-continuous nor arg-continuous and hence not practical. However, $\mathcal{M}_U$ is of theoretical interest because it represents an extreme case of $\mathcal{M}_\mathcal{B}$-DFS in terms of specificity:

Theorem 14 *$\mathcal{M}_U$ is the least specific $\mathcal{M}_\mathcal{B}$-DFS.*

Let us now consider the issue of most specific $\mathcal{M}_\mathcal{B}$-DFSes.

Definition 42 *By $\mathcal{M}_S$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_S(f) = \min(f_*^0, f_0^*) \qquad \textit{for all } f \in \mathbb{H}\textit{; see } (11) \textit{ and } (12).$$

Theorem 15 *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the QFM $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (10) *and* (18)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_S$ is a standard DFS. $\mathcal{M}_S$ fails on both continuity conditions, but:

Theorem 16 *$\mathcal{M}_S$ is the most specific $\mathcal{M}_\mathcal{B}$-DFS.*

Definition 43 *By $\mathcal{M}_{CX}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_{CX}(f) = \sup\{\min(x, f(x)) : x \in \mathbf{I}\} \qquad \textit{for all } f \in \mathbb{H}.$$

Theorem 17 *Suppose $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a continuous t-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\}$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (10) *and* (18)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

Therefore $\mathcal{M}_{CX}$ is a standard DFS. It is Q-continuous and arg-continuous and hence a good choice for applications.
As has been shown in [9], $\mathcal{M}_{CX}$ exhibits unique properties. In fact, it is the only standard DFS which is compatible with a construction called 'fuzzy argument insertion', which ensures a compositional interpretation of adjectival restriction with fuzzy adjectives. $\mathcal{M}_{CX}$ can be shown to generalize the well-known Sugeno integral to the case of multiplace and non-monotonic quantifiers. Hence $\mathcal{M}_{CX}$ consistently generalises the basic FG-count approach of [19, 24], which is restricted to quantitative and non-decreasing one-place quantifiers. In addition, $\mathcal{M}_{CX}$ can be shown to implement the so-called 'substitution approach' to fuzzy quantification [18], i.e. the fuzzy quantifier is modelled by constructing an equivalent logical formula (involving fuzzy connectives). The reader interested in details is invited to consult [9].
Returning to $\mathcal{M}_{\mathcal{B}}$-DFSes in general, we can state that:

Theorem 18

- *All $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued arguments, i.e. if $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ satisfy $\mu_{X_i}(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$;*
- *all $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$.*

This is different from general standard DFSes, which are guaranteed to coincide only for two-valued quantifiers. An issue first addressed in [11] is whether $\preceq_c$ is a genuine partial order:

Theorem 19 *$\preceq_c$ is not a total order on $\mathcal{M}_{\mathcal{B}}$-DFSes.*

In particular, the standard DFSes are only partially ordered by $\preceq_c$.
One of the characteristic properties of $\mathcal{M}_{\mathcal{B}}$-DFSes is that they propagate fuzziness.

Theorem 20

- *Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in quantifiers.*
- *Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in arguments.*

This important theorem completes the review of $\mathcal{M}_{\mathcal{B}}$-DFSes.

Summarizing, I have presented the required definitions of three-valued cuts and of the median-based aggregation mechanism, and subsequently introduced the corresponding class of $\mathcal{M}_{\mathcal{B}}$-QFMs, which are built from these base constructions. In addition, a sketch of those results on $\mathcal{M}_{\mathcal{B}}$-DFSes was given, that are relevant for the purposes of this report. In particular, I have presented an analysis of the precise conditions which make $\mathcal{M}_{\mathcal{B}}$ a DFS. I have also included prominent examples of $\mathcal{M}_{\mathcal{B}}$-DFSes. Some of these models play a special role even to the broader classes of models which will be introduced in the subsequent chapters. In addition, characteristic properties of

$\mathcal{M}_\mathcal{B}$-DFSes have been discussed. Here I want to capitalize on the last theorem that every $\mathcal{M}_\mathcal{B}$-DFS propagates fuzziness in quantifiers as well as in arguments. I consider this an important adequacy property because it appears implausible that the results should become more specific when the input (quantifier or argument) gets fuzzier. Nevertheless, there seems to be a price one has to pay for the propagation of fuzziness: as the input becomes less specific, the result of an $\mathcal{M}_\mathcal{B}$-DFS is likely to attain the least specific value of $\frac{1}{2}$, see [11, Th-34/Th-40]. In some applications, it might be preferable to sacrifice the propagation of fuzziness, in order to obtain specific results (e.g. a fine-grained result ranking) even in those cases where the input is overly fuzzy. A suitable class of models which embeds the $\mathcal{M}_\mathcal{B}$-DFSes will be presented in the next chapter.

3 The class of $\mathcal{F}_\xi$-DFSes

In order to show that standard DFSes exist which fail to propagate fuzziness in quantifiers and/or arguments, the median-based aggregation mechanism used to define $\mathcal{M}_\mathcal{B}$-DFSes was later replaced with a more general construction. This new construction, which results in a broader class of models, the $\mathcal{F}_\xi$-DFSes, provides the starting point of a further generalization which will be made in this report.
The material presented in this chapter is mainly compiled from [11], which also contains the proofs of all theorems cited, and a more detailled discussion of the structure and properties of the new models.
I now introduce the constructions necessary to define the broader class of models. We get an idea of how to abstract from $\mathcal{M}_\mathcal{B}$-QFMs if we simply expand the definition of the generalized fuzzy median and rewrite $(\bullet)_\gamma$ as

$$\begin{aligned} Q_\gamma(X_1, \ldots, X_n) \;&=\; \mathrm{med}_{\frac{1}{2}}(\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}, \\ &\qquad \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\})\,. \end{aligned} \tag{20}$$

This is apparent from Def. 24 and Def. 32. The fuzzy median can then be replaced with other connectives, e.g. the arithmetic mean $(x+y)/2$. If we view $\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ and $\inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ as mappings that depend on γ, then we can even eliminate the pointwise application of the connective and define more 'holistic' mechanisms.
Based on the definition of the crisp range $\mathcal{T}_\gamma(X)$ of a three-valued cut, which provides a set of alternative choices for crisp arguments, we define the upper and lower bounds of the quantification results given these alternatives as follows:

Definition 44 *Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n$ be given. We define the upper bound mapping $\top_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ and the lower bound mapping $\bot_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ by*

$$\begin{aligned} \top_{Q,X_1,\ldots,X_n}(\gamma) &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} \\ \bot_{Q,X_1,\ldots,X_n}(\gamma) &= \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\,. \end{aligned}$$

The domain $\mathbb{T}$ of the aggregation operators $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which combine the results of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ can be defined as follows.

Definition 45 *$\mathbb{T} \subseteq \mathbf{I}^\mathbf{I} \times \mathbf{I}^\mathbf{I}$ is defined by*

$$\mathbb{T} = \{(\top, \bot) : \ \top : \mathbf{I} \longrightarrow \mathbf{I} \text{ nondecreasing}, \bot : \mathbf{I} \longrightarrow \mathbf{I} \text{ nonincreasing}, \bot \leq \top\}\,.$$

Theorem 21

a. Suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given. Then $(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) \in \mathbb{T}$.

b. For all $(\top, \bot) \in \mathbb{T}$, there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ such that $(\top, \bot) = (\top_{Q,X}, \bot_{Q,X})$.

Based on the aggregation operator $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, we define a corresponding QFM $\mathcal{F}_\xi$ in the obvious way.

Definition 46 *For every mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, the QFM $\mathcal{F}_\xi$ is defined by*

$$\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}), \tag{21}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all fuzzy subsets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

The class of QFMs defined in this way will be called the class of $\mathcal{F}_\xi$-QFMs. Obviously it embeds the class of $\mathcal{M}_\mathcal{B}$-QFMs:

Theorem 22 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is a given aggregation mapping. Then $\mathcal{M}_\mathcal{B} = \mathcal{F}_\xi$, where $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is defined by*

$$\xi(\top, \bot) = \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) \tag{22}$$

for all $(\top, \bot) \in \mathbb{T}$, and $\mathrm{med}_{\frac{1}{2}}(\top, \bot)$ abbreviates

$$\mathrm{med}_{\frac{1}{2}}(\top, \bot)(\gamma) = \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)),$$

for all $\gamma \in \mathbf{I}$.

Hence all $\mathcal{M}_\mathcal{B}$-QFMs are $\mathcal{F}_\xi$-QFMs, and all $\mathcal{M}_\mathcal{B}$-DFSes are $\mathcal{F}_\xi$-DFSes. The full class of $\mathcal{F}_\xi$-QFMs contains a number of QFMs that do not fulfill the DFS axioms. We hence impose five elementary conditions on the aggregation mapping ξ, in order to characterize the well-behaved models, i.e. the class of $\mathcal{F}_\xi$-DFSes.

Definition 47 *For all $(\top, \bot) \in \mathbb{T}$, we impose the following conditions on aggregation mappings $\xi : \mathbb{T} \longrightarrow \mathbf{I}$.*

If $\top = \bot$, then $\xi(\top, \bot) = \top(0)$ (X-1)

$\xi(1 - \bot, 1 - \top) = 1 - \xi(\top, \bot)$ (X-2)

If $\top = \mathrm{c}_1$ and $\bot(\mathbf{I}) \subseteq \{0, 1\}$, then $\xi(\top, \bot) = \frac{1}{2} + \frac{1}{2}\bot_*^0$ (X-3)

$\xi(\top^\flat, \bot) = \xi(\top^\sharp, \bot)$ (X-4)

If $(\top', \bot') \in \mathbb{T}$ such that $\top \leq \top'$ and $\bot \leq \bot'$, then $\xi(\top, \bot) \leq \xi(\top', \bot')$ (X-5)

As stated in the following theorems, the conditions imposed on ξ capture exactly the requirements that make $\mathcal{F}_\xi$ a DFS.

Theorem 23

a. The conditions (X-1) *to* (X-5) *are sufficient for $\mathcal{F}_\xi$ to be a standard DFS.*

b. The conditions (X-1) *to* (X-5) *are necessary for* $\mathcal{F}_\xi$ *to be a DFS.*

c. The conditions (X-1) *to* (X-5) *are independent.*

Sometimes we should be aware of the relationship between the 'B-conditions' and the 'X-conditions' in the case of $\mathcal{M}_\mathcal{B}$-QFMs:

Theorem 24 *Suppose* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given and* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is defined by equation* (22)*. Then*

1. (B-1) *is equivalent to* (X-1)*;*
2. (B-2) *is equivalent to* (X-2)*;*
3. *(a)* (B-3) *entails* (X-3)*;*
   (b) the conjunction of (X-2) *and* (X-3) *entails* (B-3)*;*
4. *(a)* (B-4) *entails* (X-4)*;*
   (b) the conjunction of (X-2) *and* (X-4) *entails* (B-4)*;*
5. (B-5) *is equivalent to* (X-5)*.*

The theorem is useful, e.g. to show that the $\mathcal{M}_\mathcal{B}$-DFSes are exactly those $\mathcal{F}_\xi$-DFSes that propagate fuzziness in both quantifiers and arguments.

Theorem 25 *Suppose an* $\mathcal{F}_\xi$*-DFS propagates fuzziness in both quantifiers and arguments. Then* $\mathcal{F}_\xi$ *is an* $\mathcal{M}_\mathcal{B}$*-DFS.*

(The converse implication is already known from Th-20).
Let us now give examples of 'genuine' $\mathcal{F}_\xi$-DFSes (i.e. models that go beyond the special case of $\mathcal{M}_\mathcal{B}$-DFSes).

Definition 48 *The QFM* $\mathcal{F}_{\mathrm{Ch}} = \mathcal{F}_{\xi_{\mathrm{Ch}}}$ *is defined in terms of* $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ *by*

$$\xi_{\mathrm{Ch}}(\top, \bot) = \tfrac{1}{2} \int_0^1 \top(\gamma)\, d\gamma + \tfrac{1}{2} \int_0^1 \bot(\gamma)\, d\gamma\,,$$

for all $(\top, \bot) \in \mathbb{T}$*.*

Note. Both integrals are known to exist because $\top$ and $\bot$ are monotonic mappings. Hence ξ_{Ch} is well-defined.

Theorem 26 $\mathcal{F}_{\mathrm{Ch}}$ *is a standard DFS.*

The model $\mathcal{F}_{\mathrm{Ch}}$ is of special interest because it consistently generalizes the well-known Choquet integral and hence the 'basic' OWA approach to general multi-place quantifiers without any assumptions on monotonicity, see [11]. It is a practical model because

it is Q-continuous and arg-continuous, which grants the desired robustness against noise. Unlike the other models presented so far, $\mathcal{F}_{\mathrm{Ch}}$ does not propagate fuzziness, neither in quantifiers nor in arguments. Hence $\mathcal{F}_{\mathrm{Ch}}$ is a 'genuine' $\mathcal{F}_\xi$-DFS (i.e. not an $\mathcal{M}_\mathcal{B}$-DFS) by Th-20. In particular, this proves that $\mathcal{F}_\xi$-DFSes are indeed more general than $\mathcal{M}_\mathcal{B}$-DFSes.
Let us now discuss some aspects related to the specificity order on $\mathcal{F}_\xi$-DFSes.

Theorem 27 *$\mathcal{M}_U$ is the least specific $\mathcal{F}_\xi$-DFS.*

As concerns upper specificity bounds, it has been shown in [11, Th-60, p. 35] that the 'full' class of $\mathcal{F}_\xi$-QFMs is not specificity consistent. We hence know from Th-7 that a 'most specific $\mathcal{F}_\xi$-DFS' does not exist. However, we obtain a positive result if we restrict attention $\mathcal{F}_\xi$-DFSes which propagate fuzziness. Both the class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers, and those that propagate fuzziness in arguments, are specificity consistent and hence possess upper specificity bounds.

We shall now consider some more examples of models defined in terms of the new construction, and locate them within the full class by specificity.

Definition 49 *The QFM $\mathcal{F}_S$ is defined in terms of $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ by*

$$\xi_S(\top, \bot) = \begin{cases} \min(\top_1^*, \frac{1}{2} + \frac{1}{2}\bot_*^{\leq\frac{1}{2}}) & : \quad \bot(0) > \frac{1}{2} \\ \max(\bot_1^*, \frac{1}{2} - \frac{1}{2}\top_*^{\geq\frac{1}{2}}) & : \quad \top(0) < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$, where the coefficients $f_*^{\leq\frac{1}{2}}, f_*^{\geq\frac{1}{2}} \in \mathbf{I}$ are defined by

$$f_*^{\leq\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \leq \tfrac{1}{2}\} \tag{23}$$

$$f_*^{\geq\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \geq \tfrac{1}{2}\}\,, \tag{24}$$

for all $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 28 *$\mathcal{F}_S$ is a standard DFS.*

The model propagates fuzziness in quantifiers, but not in arguments. Hence $\mathcal{F}_S$ is a 'genuine' $\mathcal{F}_\xi$-DFS as well, see Th-20. Its relevance stems from the following theorem:

Theorem 29 *$\mathcal{F}_S$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in quantifiers.*

Hence $\mathcal{F}_S$ is of theoretical interest because it represents a boundary case of $\mathcal{F}_\xi$-DFSes. However, the model is not suited for applications because it fails on both continuity conditions.
Finally we consider the following QFM $\mathcal{F}_A$:

Definition 50 *The QFM $\mathcal{F}_A$ is defined in terms of $\xi_A : \mathbb{T} \longrightarrow \mathbf{I}$ by*

$$\xi_A(\top, \bot) = \begin{cases} \min(\bot_0^*, \frac{1}{2} + \frac{1}{2}\bot_*^0) & : \bot_0^* > \frac{1}{2} \\ \max(\top_0^*, \frac{1}{2} - \frac{1}{2}\top_*^{1\downarrow}) & : \top_0^* < \frac{1}{2} \\ \frac{1}{2} & : \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$.

Theorem 30 *$\mathcal{F}_A$ is a standard DFS.*

The model fails to propagate fuzziness in quantifiers, but it does propagate fuzziness in arguments. Hence $\mathcal{F}_A$ is a genuine $\mathcal{F}_\xi$-DFS as well, see Th-20. Recalling the symmetric situation with $\mathcal{F}_S$, the lack of both conditions with $\mathcal{F}_{\mathrm{Ch}}$ and the presence of both conditions in the case of $\mathcal{M}_\mathcal{B}$-DFSes, it is hence apparent that the conditions of propagating fuzziness in quantifiers and arguments are independent for $\mathcal{F}_\xi$-DFSes. The relevance of the model $\mathcal{F}_A$ stems from the following observation.

Theorem 31 *$\mathcal{F}_A$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments.*

Therefore $\mathcal{F}_A$ represents a boundary case of $\mathcal{F}_\xi$-QFMs. It is not suited for applications, though, because it violates both continuity conditions.

To sum up, the chapter sketched an effort to extend the class of known models, and to show that models exist which do not propagate fuzziness. The construction from which $\mathcal{M}_\mathcal{B}$-QFMs were built, namely that of $Q_\gamma(X_1, \ldots, X_n)$, was therefore dropped and replaced with a pair of mappings which represent the lower bound $\bot_{Q,X_1,\ldots,X_n}$ and upper bound $\top_{Q,X_1,\ldots,X_n}$ on the results obtained for the three-valued cuts. These mappings capture some important aspects of the quantifier and its intended behaviour for the considered fuzzy arguments. However, the important information is scattered across the cut levels and hence a subsequent aggregation step is needed. We accomplish this by applying the mapping ξ, which computes the final quantification result $\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n})$. In order to identify the subclass of well-behaved models within the unrestricted class of resulting $\mathcal{F}_\xi$-QFMs, an independent set of criteria was developed which capture the necessary and sufficient conditions on ξ that make $\mathcal{F}_\xi$ a DFS. In addition, some examples of 'genuine' $\mathcal{F}_\xi$-DFSes were given: firstly $\mathcal{F}_{\mathrm{Ch}}$, which is important because of its affinity to the Choquet integral/'basic' OWA approach; secondly $\mathcal{F}_S$, the most specific $\mathcal{F}_\xi$-DFS which propagates fuzziness in quantifiers; and finally $\mathcal{F}_A$, the most specific $\mathcal{F}_\xi$-DFS which propagates fuzziness in arguments. In addition, the $\mathcal{M}_\mathcal{B}$-DFSes were located within their apparent superclass of $\mathcal{F}_\xi$-DFSes, and characterised as precisely those $\mathcal{F}_\xi$-DFSes that propagate fuzziness both in quantifiers and arguments. Although it is not clear at this stage whether the new models form a 'natural' class with certain distinguished properties, the introduction of $\mathcal{F}_\xi$-DFSes clearly led to the discovery of some relevant models, like $\mathcal{F}_{\mathrm{Ch}}$, which can be expressed in terms of the new construction. In addition, it turned out that the upper and lower bound mappings $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ are easy to compute for common quantifiers. This is witnessed, for example, by a successful implementation of absolute and proportional quantifiers based on the model

$\mathcal{F}_{\mathrm{Ch}}$, which is described in [10, 12]. Apart from its theoretical merits, I hence consider the class of $\mathcal{F}_\xi$-DFSes a fruitful source of practical models, which can prove useful in future applications.

This brief discussion of $\mathcal{F}_\xi$-DFSes completes the review of known classes of standard DFSes, which included all definitions and theorems required to develop the novel material. The remaining part of the report is devoted to the search of more general types of models. In order to ensure that the new models subsume the known $\mathcal{F}_\xi$-DFSes, which form the broadest class of standard models developed in previous work on DFS theory, it was considered best to start from the underlying mechanism that was used to define ξ, and to pursue an apparent generalization. As we shall see, this generalization will result in a new class of models genuinely broader than $\mathcal{F}_\xi$-DFSes, the *full* class of models definable in terms of three-valued cuts. After introducing this class in the next chapter, and discussing its models and their properties to some depth, the subsequent chapter then departs from the three-valued cut mechanism. It succeeds in defining DFSes from a very different construction, which is theoretically appealing because these models utilize the extension principle for the transfer from semi-fuzzy to fuzzy quantifiers. Interestingly, both constructions span the same class of standard models.

4 The class of models defined in terms of three-valued cuts

In this chapter, a further step will be taken to extend the class of known DFSes. By abstracting from the mechanism used to define $\mathcal{F}_\xi$-QFMs, I first introduce the full class of QFMs definable in terms of three-valued cuts: the class of $\mathcal{F}_\Omega$-QFMs. Unlike $\mathcal{F}_\xi$-QFMs, the definition of which is based on the upper and lower bounds on the results obtained for the three-valued cuts and a subsequent aggregation step, the new models are defined directly in terms of the 'raw' result set obtained for the cuts, to which an aggregation mapping Ω is then applied. Hence the new approach captures all models definable in terms of three-valued cuts, and promises to span a general class of models worthwhile investigating. After introducing the surrounding class of $\mathcal{F}_\Omega$-QFMs, the structure of its well-behaved members is then analysed, by making explicit the necessary and sufficient conditions on the aggregation mapping Ω that make $\mathcal{F}_\Omega$ a DFS. In addition, the required theory will be developed that permits us to check interesting properties of $\mathcal{F}_\Omega$-DFSes, e.g. whether a given $\mathcal{F}_\Omega$ propagates fuzziness, and how given $\mathcal{F}_\Omega$-QFMs are related by specificity. It is shown that the new class of DFSes is genuinely broader than $\mathcal{F}_\xi$-DFSes. However, it does not introduce any new 'practical' models because those $\mathcal{F}_\Omega$-DFSes which are Q-continuous, and hence potentially suited for applications, are in fact $\mathcal{F}_\xi$-DFSes. These findings hence provide a justification for $\mathcal{F}_\xi$-QFMs. It is also shown that the full class of standard models which propagate fuzziness both in quantifiers and arguments, is genuinely broader than the class of $\mathcal{M}_\mathcal{B}$-DFSes. But again, all models outside the known range of models fail to be Q-continuous. Apart from investigating these properties, a subclass of $\mathcal{F}_\Omega$-QFMs will also be introduced, the class of $\mathcal{F}_\omega$-QFMs. These QFMs can be expressed in terms of a simpler construction which excludes some of the 'raw' $\mathcal{F}_\Omega$-QFMs. I show how this subclass is related to the full class of $\mathcal{F}_\Omega$-QFMs. Among others, this investigation reveals that the considered subclass still contains all well-behaved models, and hence the $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFSes coincide. The relevance of $\mathcal{F}_\omega$-QFMs stems from the fact that they can easily be linked to the alternative classes of models introduced lateron. In other words, $\mathcal{F}_\omega$-QFMs are needed to establish the link between the models defined in terms of three-valued cuts and those defined in terms of the extension principle. An investigation of $\mathcal{F}_\omega$-QFMs is hence essential to the proof that these classes coincide, which is one of the main contributions to DFS theory made in this report.

To begin with, I will now extend the class of $\mathcal{F}_\xi$-QFMs to the full class of QFMs definable in terms of three-valued cuts of the argument sets. Hence let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. In order to spot a starting point for the desired generalization, we re-consider the definition of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$. Apparently, the upper and lower bound mappings can be decomposed into (a) the three-valued cut mechanism, and (b) a subsequent $\inf/\sup$-based aggregation:

$$\begin{aligned} &\top_{Q,X_1,\ldots,X_n}(\gamma) \\ &= \sup\{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}_\gamma(X_1,\ldots,X_n)\} \\ &= \sup S_{Q,X_1,\ldots,X_n}(\gamma) \end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\perp_{Q,X_1,\dots,X_n}(\gamma) \\
&= \inf\{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)\} \\
&= \inf S_{Q,X_1,\dots,X_n}(\gamma)
\end{aligned} \tag{26}$$

for all $\gamma \in \mathbf{I}$, provided we define $S_{Q,X_1,\dots,X_n}(\gamma)$ as follows.

Definition 51 *For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$, the mapping $S_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$S_{Q,X_1,\dots,X_n}(\gamma) = \{Q(Y_1,\dots,Y_n) : (Y_1,\dots,Y_n) \in \mathcal{T}_\gamma(X_1,\dots,X_n)\}\,,$$

for all $\gamma \in \mathbf{I}$.

Some basic properties of $S_{Q,X_1,\dots,X_n}$ are stated in this theorem.

Theorem 32 *Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and choice of fuzzy subsets $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Then*

a. $S_{Q,X_1,\dots,X_n}(0) \neq \varnothing$*;*

b. $S_{Q,X_1,\dots,X_n}(\gamma) \subseteq S_{Q,X_1,\dots,X_n}(\gamma')$ *whenever* $\gamma,\gamma' \in \mathbf{I}$ *with* $\gamma \leq \gamma'$.

(Proof: A.1, p.77+)

It is hence apparent that all possible choices of $S_{Q,X_1,\dots,X_n}$ are contained in the following set $\mathbb{K}$.

Definition 52 $\mathbb{K} \subseteq \mathcal{P}(\mathbf{I})^{\mathbf{I}}$ *is defined by*

$$\mathbb{K} = \{S \in \mathcal{P}(\mathbf{I})^{\mathbf{I}} : S(0) \neq \varnothing \text{ and } S(\gamma) \subseteq S(\gamma') \text{ whenever } \gamma \leq \gamma'\}\,.$$

As we shall now prove, $\mathbb{K}$ is the minimal set which contains all possible choices for $S_{Q,X_1,\dots,X_n}$. To this end, we first have to introduce coefficients $s(z) \in \mathbf{I}$ associated with $S \in \mathbb{K}$, which will play an essential role throughout the report.

Definition 53 *Consider $S \in \mathbb{K}$. We associate with S a mapping $s : \mathbf{I} \longrightarrow \mathbf{I}$ defined by*

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\}\,,$$

for all $z \in \mathbf{I}$.

It is convenient to define a notation for the $s(z)$'s obtained from a given quantifier and arguments.

Definition 54 *For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, we denote the mapping s obtained from $\mathcal{S}_{Q,X_1,\dots,X_n}$ by applying Def. 53 by $s_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$. The resulting mapping is hence defined by*

$$s_{Q,X_1,\dots,X_n}(z) = \inf\{\gamma \in \mathbf{I} : z \in \mathcal{S}_{Q,X_1,\dots,X_n}(\gamma)\}\,,$$

for all $z \in \mathbf{I}$.

As we shall see later, all $\mathcal{F}_\Omega$-DFSes can be defined in terms of $s_{Q,X_1,\dots,X_n}$.

Theorem 33 *Let $\mathcal{S} \in \mathbb{K}$ be given and define $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by*

$$Q(Y) = Q_{\inf Y'}(Y'') \tag{27}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where

$$Y' = \{y \in \mathbf{I} : (0, y) \in Y\} \tag{28}$$

$$Y'' = \{y \in \mathbf{I} : (1, y) \in Y\} \tag{29}$$

and the $Q_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $z \in \mathbf{I}$ are defined by

$$Q_z(Y'') = \begin{cases} z & : \sup Y'' > s(z) \\ z_0 & : \textit{else} \end{cases} \tag{30}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$ if $z \notin \mathcal{S}(s(z))$, and

$$Q_z(Y'') = \begin{cases} z & : \sup Y'' \geq s(z) \\ z_0 & : \textit{else} \end{cases} \tag{31}$$

in the case that $z \in \mathcal{S}(s(z))$. z_0 is an arbitrary element

$$z_0 \in \mathcal{S}(0)\,, \tag{32}$$

which exists by Th-32. Further suppose that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ is defined by

$$\mu_X(a, y) = \begin{cases} \frac{1}{2} & : a = 0 \\ \frac{1}{2} - \frac{1}{2}y & : a = 1 \end{cases} \tag{33}$$

for all $a \in \mathbf{2}$, $y \in \mathbf{I}$. Then $\mathcal{S}_{Q,X} = \mathcal{S}$.
(Proof: A.2, p.77+)

Hence $\mathbb{K}$ is exactly the set of all $\mathcal{S} = \mathcal{S}_{Q,X_1,\dots,X_n}$ obtained for arbitrary choices of quantifiers and arguments. In order to obtain a quantification result from $\mathcal{S}_{Q,X_1,\dots,X_n}$, we apply an aggregation operator $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ in the obvious way.

Definition 55 *Consider an aggregation operator $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$. The corresponding QFM $\mathcal{F}_\Omega$ is defined by*

$$\mathcal{F}_\Omega(Q)(X_1, \dots, X_n) = \Omega(\mathcal{S}_{Q,X_1,\dots,X_n})\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{F}_\Omega$-QFMs, we mean the collection of all QFMs defined in this way. As usual, we must impose conditions to shrink the full class of $\mathcal{F}_\Omega$ to its subclass of $\mathcal{F}_\Omega$-DFSes.

Definition 56 *For all $S \in \mathbb{K}$, we define $S^\sharp, S^\flat \in \mathbb{K}$ as follows.*

$$S^\sharp = \begin{cases} \bigcap\limits_{\gamma' > \gamma} S(\gamma') & : \quad \gamma < 1 \\ \mathbf{I} & : \quad \gamma = 1 \end{cases} \qquad S^\flat = \begin{cases} S(0) & : \quad \gamma = 0 \\ \bigcup\limits_{\gamma' < \gamma} S(\gamma') & : \quad \gamma > 0 \end{cases}$$

for all $\gamma \in \mathbf{I}$.

Note. The definition is slightly asymmetric; I have departed from the usual scheme of defining $S^\sharp(1) = S(1)$ in this case because the present definition of $S^\sharp(1) = \mathbf{I}$ allows for more compact conditions on Ω, and eventually for shorter proofs.
I further stipulate a definition of $S \sqsubseteq S'$ which will serve to express a monotonicity condition on Ω.

Definition 57 *For all $S, S' \in \mathbb{K}$, $S \sqsubseteq S'$ if and only if the following two conditions hold for all $\gamma \in \mathbf{I}$:*

1. *for all $z \in S(\gamma)$, there exists $z' \in S'(\gamma)$ with $z' \geq z$;*
2. *for all $z' \in S'(\gamma)$, there exists $z \in S(\gamma)$ with $z \leq z'$.*

It is apparent from this definition that $\sqsubseteq$ is reflexive and transitive, but not necessarily antisymmetric (i.e. $S \sqsubseteq S'$ and $S' \sqsubseteq S$ does not imply that $S = S'$). Hence $\sqsubseteq$ is a preorder.
We are now ready to state the conditions on reasonable choices of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, in analogy to the conditions (B-1)–(B-5) for $\mathcal{M}_\mathcal{B}$-DFSes and to the conditions (X-1)–(X-5) for $\mathcal{F}_\xi$-DFSes:

Definition 58 *Consider $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$. We impose the following conditions on Ω. For all $S \in \mathbb{K}$,*

If there exists $a \in \mathbf{I}$ with $S(\gamma) = \{a\}$ for all $\gamma \in \mathbf{I}$, then $\Omega(S) = a$. (Ω-1)

If $S'(\gamma) = \{1 - z : z \in S(\gamma)\}$ for all $\gamma \in \mathbf{I}$, then $\Omega(S') = 1 - \Omega(S)$. (Ω-2)

If $1 \in S(0)$ and $S(\gamma) \subseteq \{0, 1\}$ for all $\gamma \in \mathbf{I}$, then $\Omega(S) = \frac{1}{2} + \frac{1}{2}s(0)$. (Ω-3)

$\Omega(S) = \Omega(S^\sharp)$ (Ω-4)

If $S' \in \mathbb{K}$ satisfies $S \sqsubseteq S'$, then $\Omega(S) \leq \Omega(S')$. (Ω-5)

Note. The only condition which is slighly different from the usual scheme is (Ω-4). The departure from requiring $\Omega(S^\sharp) = \Omega(S^\flat)$ turned out to shorten the proofs. The latter equation is entailed by the above conditions, however.

Theorem 34 *The conditions (Ω-1)–(Ω-5) on $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ are sufficient for $\mathcal{F}_\Omega$ to be a standard DFS.*
(Proof: A.3, p.78+)

In the following, I introduce another construction which elucidates the exact properties of $S \in \mathbb{K}$ that a conforming choice of Ω can rely on.

Definition 59 *For all $S \in \mathbb{K}$, we define $S^\ddagger \in \mathbb{K}$ by*

$$S^\ddagger(\gamma) = \{z \in \mathbf{I} : \textit{there exist } z', z'' \in S(\gamma) \textit{ with } z' \le z \le z''\}$$

for all $\gamma \in \mathbf{I}$.

Note. It is apparent that indeed $S^\ddagger \in \mathbb{K}$. The effect of applying ‡ to S is that of "filling the gaps" in the interior of S. The resulting $S^\ddagger$ will be a closed, half-open, or open interval.

The importance of this construction with respect to $\mathcal{F}_\Omega$-QFMs stems from the invariance of well-behaved $\mathcal{F}_\Omega$-QFMs with respect to the gap-filling operation:

Theorem 35 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is a given mapping such that $\mathcal{F}_\Omega$ satisfies* (Z-5)*. Then*

$$\Omega(S) = \Omega(S^\ddagger)\,,$$

for all $S \in \mathbb{K}$.
(Proof: A.4, p.90+)

This means that a well-behaved choice of Ω may only depend on $\sup S(\gamma)$, $\inf S(\gamma)$, and the knowledge whether $\sup S(\gamma) \in S(\gamma)$ and $\inf S(\gamma) \in S(\gamma)$. Apart from this, the 'interior structure' of $S(\gamma)$ is irrelevant to the determination of $\Omega(S)$.
I have anticipated the discussion of the gap-filling operation because it facilitates the proof that (Ω-5) is necessary for $\mathcal{F}_\Omega$ to satisfy (Z-5). The other 'Ω-conditions' are easily shown to be necessary for $\mathcal{F}_\Omega$ to be a DFS, and require only minor adjustments of the corresponding proofs for $\mathcal{F}_\xi$-QFMs that were presented in [11].

Theorem 36 *The conditions* (Ω-1)–(Ω-5) *on $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ are necessary for $\mathcal{F}_\Omega$ to be a DFS.*
(Proof: A.5, p.94+)

Hence the 'Ω-conditions' are necessary and sufficient for $\mathcal{F}_\Omega$ to be a DFS, and all $\mathcal{F}_\Omega$-DFSes are indeed standard DFSes. In order to prove that the criteria are independent, we relate $\mathcal{F}_\xi$-QFMs to their apparent superclass of $\mathcal{F}_\Omega$-QFMs.

Theorem 37 *Consider an aggregation mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. Then $\mathcal{F}_\xi = \mathcal{F}_\Omega$, where $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined by*

$$\Omega(S) = \xi(\top_S, \bot_S)\,, \tag{34}$$

for all $S \in \mathbb{K}$, and $(\top_S, \bot_S) \in \mathbb{T}$ is defined by

$$\top_S(\gamma) = \sup S(\gamma) \tag{35}$$
$$\bot_S(\gamma) = \inf S(\gamma) \tag{36}$$

for all $\gamma \in \mathbf{I}$.
(Proof: A.6, p.109+)

This is apparent. Hence all $\mathcal{F}_\xi$-QFMs are $\mathcal{F}_\Omega$-QFMs and all $\mathcal{F}_\xi$-DFSes are $\mathcal{F}_\Omega$-DFSes. The next theorem permits to reduce the independence proof of the conditions on Ω to the independence proof of the conditions imposed on ξ.

Theorem 38 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is given and* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is defined by* (34). *Then*

a. (X-1) *is equivalent to* (Ω-1)*;*

b. (X-2) *is equivalent to* (Ω-2)*;*

c. (X-3) *is equivalent to* (Ω-3)*;*

d. 1. *the conjunction of* (X-2), (X-4) *and* (X-5) *implies* (Ω-4)*;*
 2. (Ω-4) *implies* (X-4)*;*

e. (X-5) *is equivalent to* (Ω-5).

(Proof: A.7, p.110+)

Based on this theorem and the known independence of the conditions (X-1)–(X-5), it is now easy to prove the desired result concerning independence.

Theorem 39 *The conditions* (Ω-1)–(Ω-5) *imposed on* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *are independent.*
(Proof: A.8, p.125+)

As has been remarked above, every $\mathcal{F}_\Omega$-DFS can be defined in terms of the mapping $s_{Q,(X_1,\ldots,X_n)} : \mathbf{I} \longrightarrow \mathbf{I}$ and this usually makes a simpler representation. It therefore makes sense to introduce the class of QFMs definable in terms of $s_{Q,(X_1,\ldots,X_n)} : \mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 40 *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *a choice of fuzzy arguments. Then* ${s_{Q,X_1,\ldots,X_n}}^{-1}(0) \neq \varnothing$*, i.e. there exists* $z_0 \in \mathbf{I}$ *with* $s_{Q,X_1,\ldots,X_n}(z_0) = 0$.
(Proof: A.9, p.126+)

Hence all possible choices of $s_{Q,X_1,\ldots,X_n}$ are contained of the following set $\mathbb{L}$.

Definition 60 $\mathbb{L} \subseteq \mathbf{I}^{\mathbf{I}}$ *is defined by*

$$\mathbb{L} = \{s \in \mathbf{I}^{\mathbf{I}} : s^{-1}(0) \neq \varnothing\} .$$

The following theorem states that $\mathbb{L}$ is the minimum subset of $\mathbf{I}^{\mathbf{I}}$ which contains all possible mappings $s_{Q,X_1,\ldots,X_n}$:

Theorem 41 *For all $s \in \mathbb{L}$, let us define $S : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by*

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{37}$$

for all $\gamma \in \mathbf{I}$. It is apparent that $S \in \mathbb{K}$. Let us further suppose that $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ is defined by (27) *and that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ is the fuzzy subset defined by* (33)*. Then $s_{Q,X} = s$.*
(Proof: A.10, p.126+)

In order to define quantification results based on $s_{Q,X_1,\ldots,X_n}$, we need an aggregation mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. The corresponding QFM $\mathcal{F}_\omega$ is defined in the usual way.

Definition 61 *Let a mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given. By $\mathcal{F}_\omega$ we denote the QFM defined by*

$$\mathcal{F}_\omega(Q)(X_1, \ldots, X_n) = \omega(s_{Q,X_1,\ldots,X_n}) \,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

It is obvious from the definition of $s_{Q,X_1,\ldots,X_n}$ in terms of $S_{Q,X_1,\ldots,X_n}$ that all $\mathcal{F}_\omega$-QFMs are $\mathcal{F}_\Omega$-QFMs, using the apparent choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$,

$$\Omega(S) = \omega(s) \tag{38}$$

where $s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\}$, see Def. 54. It is then clear from Def. 55 and Def. 61 that

$$\mathcal{F}_\Omega = \mathcal{F}_\omega \,. \tag{39}$$

The converse is not true, i.e. it is not the case that all $\mathcal{F}_\Omega$-QFMs are $\mathcal{F}_\omega$-QFMs. However, if an $\mathcal{F}_\Omega$-QFMs is sufficiently 'well-behaved', then it is also an $\mathcal{F}_\omega$-QFM. In particular, this is the case for $\mathcal{F}_\Omega$-DFSes.

Theorem 42

a. If $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-4)*, then $\mathcal{F}_\Omega = \mathcal{F}_\omega$, provided we define $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ by*

$$\omega(s) = \Omega(S) \tag{40}$$

for all $s \in \mathbb{L}$, where

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{41}$$

for all $\gamma \in \mathbf{I}$.

b. If $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ does not satisfy (Ω-4)*, then $\mathcal{F}_\Omega$ is not an $\mathcal{F}_\omega$-QFM.*

(Proof: A.11, p.127+)

Therefore an $\mathcal{F}_\Omega$-QFM is an $\mathcal{F}_\omega$-QFM if and only if it satisfies (Ω-4). Let us recall that by Th-36, (Ω-4) is necessary for $\mathcal{F}_\Omega$ to be a DFS. This means that we do not loose any models of interest if we restrict attention to the class of those $\mathcal{F}_\Omega$-QFMs which satisfy (Ω-4), and can hence be expressed as $\mathcal{F}_\omega$-QFMs.
It is then convenient to switch from (Ω-1)–(Ω-5) to corresponding conditions on $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. To accomplish this, we first define a preorder $\sqsubseteq \subseteq \mathbb{L} \times \mathbb{L}$, which is needed to express a monotonicity condition.

Definition 62 *For all $s, s' \in \mathbb{L}$, $s \sqsubseteq s'$ if and only if the following two conditions hold:*

a. for all $z \in \mathbf{I}$, $\inf\{s'(z') : z' \geq z\} \leq s(z)$;

b. for all $z' \in \mathbf{I}$, $\inf\{s(z) : z \leq z'\} \leq s'(z')$.

In the case of $\mathcal{F}_\omega$-QFMs, we can express the conditions on $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ even more succintly.

Definition 63 *We impose the following conditions on $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. For all $s \in \mathbb{L}$,*

If $s^{-1}([0,1)) = \{a\}$, then $\omega(s) = a$. (ω-1)

If $s'(z) = s(1-z)$ for all $z \in \mathbf{I}$, then $\omega(s') = 1 - \omega(s)$. (ω-2)

If $s(1) = 0$ and $s^{-1}([0,1)) \subseteq \{0,1\}$, then $\omega(s) = \frac{1}{2} + \frac{1}{2}s(0)$. (ω-3)

If $s' \in \mathbb{L}$ with $s \sqsubseteq s'$, then $\omega(s) \leq \omega(s')$. (ω-4)

Theorem 43 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω according to* (38)*. Then*

a. Ω satisfies (Ω-1) *if and only if ω satisfies* (ω-1)*;*

b. Ω satisfies (Ω-2) *if and only if ω satisfies* (ω-2)*;*

c. Ω satisfies (Ω-3) *if and only if ω satisfies* (ω-3)*;*

d. Ω satisfies (Ω-4)*;*

e. Ω satisfies (Ω-5) *if and only if ω satisfies* (ω-4)*.*

(Proof: A.12, p.130+)

Due to these relationships, the following theorems are obvious from the corresponding results for Ω.

Theorem 44 *The conditions* (ω-1)–(ω-4) *are sufficient for $\mathcal{F}_\omega$ to be a standard DFS.*
(Proof: A.13, p.135+)

Theorem 45 *The conditions* (ω-1)–(ω-4) *are necessary for $\mathcal{F}_\omega$ to be a DFS.*
(Proof: A.14, p.135+)

Theorem 46 *The conditions* (ω-1)–(ω-4) *are independent.*
(Proof: A.15, p.135+)

To sum up, $\mathcal{F}_\omega$-DFSes comprise all $\mathcal{F}_\Omega$-DFSes, they are usually easier to define, and simpler conditions (ω-1)–(ω-4) have to be checked. However, the monotonicity condition (ω-4) on ω is somewhat more complicated compared to the monotonicity condition (Ω-5) on Ω. In the following, I hence introduce a simpler preorder $\trianglelefteq$ for expressing monotonicity, which when combined with an additional condition can replace $\sqsubseteq$ and the corresponding monotonicity condition (ω-4). $\trianglelefteq$ is defined as follows.

Definition 64 *For all* $s, s' \in \mathbb{L}$, $s \trianglelefteq s'$ *if and only if the following two conditions hold:*

a. *for all* $z \in \mathbf{I}$, *there exists* $z' \geq z$ *with* $s'(z') \leq s(z)$;

b. *for all* $z' \in \mathbf{I}$, *there exists* $z \leq z$ *with* $s(z) \leq s'(z')$.

In order to state the additional condition, it is necessary to introduce a construction on $s \in \mathbb{L}$ which corresponds to the gap-filling operation $S^\ddagger$ defined on $S \in \mathbb{K}$.

Definition 65 *For all* $s \in \mathbb{L}$, $s^\ddagger : \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$s^\ddagger(z) = \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z'') : z'' \geq z\}) \,,$$

for all $z \in \mathbf{I}$.

Some basic properties of ‡ are the following.

Theorem 47 *Let* $s \in \mathbb{L}$ *be given. Then*

a. $s^\ddagger \leq s$;

b. $s^\ddagger \in \mathbb{L}$;

c. $s^\ddagger$ *is concave, i.e.*

$$s^\ddagger(z_2) \leq \max(s^\ddagger(z_1), s^\ddagger(z_3)) \,.$$

whenever $z_1 \leq z_2 \leq z_3$.

(Proof: A.16, p.136+)

We will need this 'concavication construction' for the proof that ω's satisfying (Ω-5) entails that Ω defined by (38) satisfies (Ω-5). The connection between ‡ and monotonic behaviour of ω becomes visible in the next theorem.

Theorem 48 *Suppose* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *satisfies* (ω-4)*. Then* $\omega(s^\ddagger) = \omega(s)$ *for all* $s \in \mathbb{L}$.
(Proof: A.17, p.137+)

The theorem proves useful for establishing the following result, which connects (ω-4) to the simplified conditions.

Theorem 49 *For all* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$*, the monotonicity condition* (ω-4) *is equivalent to the conjunction of the following two conditions:*

a. for all $s, s' \in \mathbb{L}$ *with* $s \trianglelefteq s'$*, it holds that* $\omega(s) \leq \omega(s')$*;*

b. for all $s \in \mathbb{L}$*,* $\omega(s) = \omega(s^{\ddagger})$*.*

(Proof: A.18, p.138+)

I will now present four examples of ‘genuine’ $\mathcal{F}_\omega$-DFSes, i.e. of $\mathcal{F}_\omega$-DFSes which do not belong to the class of $\mathcal{F}_\xi$-DFSes. To this end, it is necessary to introduce some coefficients defined in terms of a given $s \in \mathbb{L}$.

Definition 66 *For all* $s \in \mathbb{L}$*, the coefficients* $s_*^{\top,0}, s_*^{\bot,0}, ,_1^{\top,*}, ,_1^{\bot,*} s_*^{\leq\frac{1}{2}}, s_*^{\geq\frac{1}{2}} \in \mathbf{I}$ *are defined by*

$$s_*^{\top,0} = \sup {s^{\ddagger}}^{-1}(0) \tag{42}$$

$$s_*^{\bot,0} = \inf {s^{\ddagger}}^{-1}(0) \tag{43}$$

$$s_1^{\top,*} = \sup s^{-1}([0,1)) \tag{44}$$

$$s_1^{\bot,*} = \inf s^{-1}([0,1)) \tag{45}$$

$$s_*^{\leq\frac{1}{2}} = \inf\{s(z) : z \leq \tfrac{1}{2}\} \tag{46}$$

$$s_*^{\geq\frac{1}{2}} = \inf\{s(z) : z \geq \tfrac{1}{2}\}\,. \tag{47}$$

Based on these coefficients, I now define the examples of $\mathcal{F}_\omega$-DFSes.

Definition 67 *By* $\omega_M : \mathbb{L} \longrightarrow \mathbf{I}$ *we denote the following mapping,*

$$\omega_M(s) = \begin{cases} \min(s_*^{\bot,0}, \frac{1}{2} + \frac{1}{2} s_*^{\leq\frac{1}{2}}) & : \quad s_*^{\bot,0} > \frac{1}{2} \\ \max(s_*^{\top,0}, \frac{1}{2} - \frac{1}{2} s_*^{\geq\frac{1}{2}}) & : \quad s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$*. The QFM* $\mathcal{F}_M$ *is defined in terms of* ω_M *according to Def. 61, i.e.* $\mathcal{F}_M = \mathcal{F}_{\omega_M}$*.*

Let us first notice that the QFM $\mathcal{F}_M$ so defined is indeed a DFS.

Theorem 50 $\mathcal{F}_M$ *is a standard DFS.*
(Proof: A.19, p.142+)

Let us also remark that $\mathcal{F}_M$ is indeed a 'genuine' $\mathcal{F}_\omega$-DFS.

Theorem 51 *$\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_M = \mathcal{F}_\xi$.*
(Proof: A.20, p.152+)

In particular, this proves that the $\mathcal{F}_\omega$-DFSes are really more general than $\mathcal{F}_\xi$-DFSes, i.e. the $\mathcal{F}_\xi$-DFSes form a proper subclass of the $\mathcal{F}_\omega$-DFSes.

Definition 68 *By $\omega_P : \mathbb{L} \longrightarrow \mathbf{I}$ we denote the mapping defined by*

$$\omega_P(s) = \begin{cases} \min(s_1^{\top,*}, \frac{1}{2} + \frac{1}{2} s_*^{\leq \frac{1}{2}}) & : \quad s_*^{\perp,0} > \frac{1}{2} \\ \max(s_1^{\perp,*}, \frac{1}{2} - \frac{1}{2} s_*^{\geq \frac{1}{2}}) & : \quad s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. We define the QFM $\mathcal{F}_P$ in terms of ω_P according to Def. 61, i.e. $\mathcal{F}_P = \mathcal{F}_{\omega_P}$.

Theorem 52 *$\mathcal{F}_P$ is a standard DFS.*
(Proof: A.21, p.154+)

Let us also observe that $\mathcal{F}_P$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 53 *$\mathcal{F}_P$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ such that $\mathcal{F}_P = \mathcal{F}_\xi$.*
(Proof: A.22, p.159+)

It is possible to obtain an even more specific DFS by slightly changing the definition of $\mathcal{F}_P$.

Definition 69 *By $\omega_Z : \mathbb{L} \longrightarrow \mathbf{I}$ we denote the mapping defined by*

$$\omega_Z(s) = \begin{cases} \min(s_1^{\top,*}, \frac{1}{2} + \frac{1}{2} s_*^{\leq \frac{1}{2}}) & : \quad s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1] \\ \max(s_1^{\perp,*}, \frac{1}{2} - \frac{1}{2} s_*^{\geq \frac{1}{2}}) & : \quad s^{\ddagger^{-1}}(0) \subseteq [0, \frac{1}{2}] \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. We define the QFM $\mathcal{F}_Z$ in terms of ω_Z according to Def. 61, i.e. $\mathcal{F}_Z = \mathcal{F}_{\omega_Z}$.

Note. To see that ω_Z is well-defined, consider a choice of $s \in \mathbb{L}$ such that $s^{\ddagger^{-1}}(0) = \{\frac{1}{2}\}$. Then $s^{-1}(0) = \{\frac{1}{2}\}$ as well because $s^{-1}(0) \neq \varnothing$ and $s^{-1}(0) \subseteq s^{\ddagger^{-1}}(0)$ by Th-47. Hence

$$s_1^{\top,*} = \sup s^{-1}([0,1)) \geq \sup s^{-1}(0) = \sup\{\tfrac{1}{2}\} = \tfrac{1}{2}\,. \tag{48}$$

by (44). Similarly

$$s_1^{\perp,*} = \inf s^{-1}([0,1)) \leq \inf s^{-1}(0) = \inf\{\tfrac{1}{2}\} = \tfrac{1}{2}\,. \tag{49}$$

Noticing that

$$s_*^{\leq \frac{1}{2}} = 0 \tag{50}$$

$$s_*^{\geq \frac{1}{2}} = 0 \tag{51}$$

by (46) and (47) because $s(\frac{1}{2}) = 0$, we obtain the desired

$$\begin{aligned} \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) &= \min(\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2} \cdot 0) && \text{by (48), (50)} \\ &= \tfrac{1}{2} \\ &= \max(\tfrac{1}{2}, \tfrac{1}{2} - \tfrac{1}{2} \cdot 0) \\ &= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}})\,, && \text{by (49), (51)} \end{aligned}$$

i.e. ω_Z is indeed well-defined.

Theorem 54 *$\mathcal{F}_Z$ is a standard DFS.*
(Proof: A.23, p.160+)

Again, it is easily shown that $\mathcal{F}_Z$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 55 *$\mathcal{F}_Z$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ such that $\mathcal{F}_Z = \mathcal{F}_\xi$.*
(Proof: A.24, p.166+)

Definition 70 *By $\omega_R : \mathbb{L} \longrightarrow \mathbf{I}$ we denote the mapping defined by*

$$\omega_R(s) = \begin{cases} \min(s_*^{\perp,0}, \frac{1}{2} + \frac{1}{2}s(0)) & : \; s_*^{\perp,0} > \frac{1}{2} \\ \max(s_*^{\top,0}, \frac{1}{2} - \frac{1}{2}s(1)) & : \; s_*^{\top,0} < \frac{1}{2} \\ \frac{1}{2} & : \; \textit{else} \end{cases}$$

for all $s \in \mathbb{L}$. We define the QFM $\mathcal{F}_R$ in terms of ω_R according to Def. 61, i.e. $\mathcal{F}_R = \mathcal{F}_{\omega_R}$.

Theorem 56 *$\mathcal{F}_R$ is a standard DFS.*
(Proof: A.25, p.167+)

Again, we can assert that $\mathcal{F}_R$ is a genuine $\mathcal{F}_\omega$-DFS.

Theorem 57 *$\mathcal{F}_R$ is not an $\mathcal{F}_\xi$-DFS, i.e. there exists no $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_R = \mathcal{F}_\xi$.*
(Proof: A.26, p.171+)

Now that the defining conditions of $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFS have been established and examples of the new classes of models have been given, we turn to additional properties like propagation of fuzziness. Usually I state the corresponding conditions both for the representation in terms of $\mathcal{F}_\Omega$ and in terms of $\mathcal{F}_\omega$. This provides maximum flexibility in later proofs whether a model at hand does or does not possess these properties.

Definition 71 *For all $S, S' \in \mathbb{K}$, we say that S is fuzzier (less crisp) than S', in symbols: $S \preceq_c S'$, if and only if the following conditions are satisfied for all $\gamma \in \mathbf{I}$.*

$$\text{for all } z' \in S'(\gamma)\text{, there exists } z \in S(\gamma) \text{ such that } z \preceq_c z'; \tag{52}$$

$$\text{for all } z \in S(\gamma)\text{, there exists } z' \in S'(\gamma) \text{ such that } z \preceq_c z'. \tag{53}$$

Definition 72 *Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given. We say that Ω* propagates fuzziness *if and only if*

$$\Omega(S) \preceq_c \Omega(S')$$

whenever $S, S' \in \mathbb{K}$ satisfy $S \preceq_c S'$.

Theorem 58 *For all $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers if and only if Ω propagates fuzziness.*
(Proof: A.27, p.173+)

The following condition permits a simplified check if a given Ω propagates fuzziness.

Theorem 59 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies* (Ω-1)–(Ω-5)*. Then Ω propagates fuzziness if and only if*

$$\Omega(S) = \Omega(S^\ddagger \cap [\tfrac{1}{2}, 1])\,,$$

for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$.
(Proof: A.28, p.178+)

Definition 73 *Let $S, S' \in \mathbb{K}$ be given. We say that S is less specific than S', in symbols $S \Subset S'$, if and only if*

$$S(\gamma) \supseteq S'(\gamma)$$

for all $\gamma \in \mathbf{I}$.

Definition 74 *Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given. We say that Ω* propagates unspecificity *if and only if*

$$\Omega(S) \preceq_c \Omega(S')$$

for every choice of $S, S' \in \mathbb{K}$ with $S \Subset S'$.

Theorem 60 *For all $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ propagates fuzziness in arguments if and only if Ω propagates unspecificity.*
(Proof: A.29, p.182+)

The above criterion for Ω propagating unspecificity can be simplified as follows.

Theorem 61 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies* (Ω-1), (Ω-2), (Ω-4) *and* (Ω-5). *Then the following conditions are equivalent:*

a. *Ω propagates unspecificity;*

b. *for all $s \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$, it holds that $\Omega(S) = \Omega(S')$, where $S' \in \mathbb{K}$ is defined by*

$$S'(\gamma) = \begin{cases} [z_*, 1] & : & z_* \in S(\gamma) \\ (z_*, 1] & : & z_* \notin S(\gamma) \end{cases} \tag{54}$$

for all $\gamma \in \mathbf{I}$, and where $z_* = z_*(\gamma)$ abbreviates

$$z_* = \inf S(\gamma)\,. \tag{55}$$

(Proof: A.30, p.185+)

Definition 75 *For all $s, s' \in \mathbb{L}$, we say that s is fuzzier (less crisp) than s', in symbols $s \preceq_c s'$, if and only if*

$$\text{for all } z \in \mathbf{I}, \text{ there exists } z' \in \mathbf{I} \text{ with } z \preceq_c z' \text{ and } s'(z') \leq s(z); \tag{56}$$

$$\text{for all } z' \in \mathbf{I}, \text{ there exists } z \in \mathbf{I} \text{ with } z \preceq_c z' \text{ and } s(z) \leq s'(z'). \tag{57}$$

Definition 76 *A mapping $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is said to* propagate fuzziness *if and only if $\omega(s) \preceq_c \omega(s')$ for all choices of $s, s' \in \mathbb{L}$ with $s \preceq_c s'$.*

Theorem 62 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is ‡-invariant, i.e. $\omega(s^\ddagger) = \omega(s)$ for all $s \in \mathbb{L}$. Then $\mathcal{F}_\omega$ propagates fuzziness in quantifiers if and only if ω propagates fuzziness.*
(Proof: A.31, p.188+)

If ω is well-behaved, then we can further simplify the condition that must be tested for establishing or rejecting that ω propagate fuzziness.

Theorem 63 *Suppose that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies* (ω-1)–(ω-4). *ω propagates fuzziness if and only if for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega(s) = \omega(s')$, where*

$$s'(z) = \begin{cases} s^\ddagger(z) & : & z \geq \frac{1}{2} \\ 1 & : & z < \frac{1}{2} \end{cases} \tag{58}$$

for all $z \in \mathbf{I}$.
(Proof: A.32, p.196+)

Definition 77 *A mapping* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is said to* propagate unspecificity *if and only if* $\omega(s) \preceq_c \omega(s')$ *whenever* $s, s' \in \mathbb{L}$ *satisfy* $s \leq s'$*.*

Theorem 64 *Let* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be a given mapping. Then* $\mathcal{F}_\omega$ *propagates fuzziness in arguments if and only if* ω *propagates unspecificity.*
(Proof: A.33, p.201+)

Again, it is possible to simplify the condition imposed on ω.

Theorem 65 *Suppose* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *satisfies* (ω-1), (ω-2) *and* (ω-4)*. Then the following conditions are equivalent.*

a. ω *propagates unspecificity;*

b. for all $s \in \mathbb{L}$ *with* $s^{-1} \cap [\frac{1}{2}, 1] \neq \varnothing$*, it holds that* $\omega(s) = \omega(s')$*, where* $s' \in \mathbb{L}$ *is defined by*

$$s'(z) = \inf\{s(z') : z' \leq z\} \tag{59}$$

for all $z \in \mathbf{I}$*.*

(Proof: A.34, p.203+)

Now let us apply these criteria to the examples of $\mathcal{F}_\omega$-DFSes.

Theorem 66 $\mathcal{F}_M$ *propagates fuzziness in quantifiers.*
(Proof: A.35, p.205+)

Theorem 67 $\mathcal{F}_M$ *propagates fuzziness in arguments.*
(Proof: A.36, p.209+)

Let us recall from Th-51 that $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, in particular not an $\mathcal{M}_\mathcal{B}$-DFS. Hence the class of $\mathcal{M}_\mathcal{B}$-DFSes, which propagate fuzziness in both arguments and quantifiers, does not include all standard DFSes with this property. $\mathcal{F}_M$ is a counterexample which demonstrates that the class of standard DFSes which propagate fuzziness both in quantifiers and arguments is genuinely broader than the class of $\mathcal{M}_\mathcal{B}$-DFSes.

Theorem 68 $\mathcal{F}_P$ *propagates fuzziness in quantifiers.*
(Proof: A.37, p.211+)

Theorem 69 $\mathcal{F}_P$ *does not propagate fuzziness in arguments.*
(Proof: A.38, p.212+)

Theorem 70 $\mathcal{F}_Z$ *propagates fuzziness in quantifiers.*
(Proof: A.39, p.213+)

Theorem 71 *$\mathcal{F}_Z$ does not propagate fuzziness in arguments.*
(Proof: A.40, p.214+)

As concerns $\mathcal{F}_R$, we have the following results.

Theorem 72 *$\mathcal{F}_R$ does not propagate fuzziness in quantifiers.*
(Proof: A.41, p.214+)

Theorem 73 *$\mathcal{F}_R$ propagates fuzziness in arguments.*
(Proof: A.42, p.215+)

Hence there are $\mathcal{F}_\omega$-DFSes beyond $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers, but not in arguments. In particular, the class of standard DFSes that propagate fuzziness in quantifiers but not in arguments is genuinely broader than the class of $\mathcal{F}_\xi$-DFSes with this property. We shall check later that the class of $\mathcal{F}_\omega$-DFSes with this property is still specificity consistent and investigate its least upper specificity bound.

Definition 78 *A collection $\mathfrak{Q}$ of mappings $\Omega \in \mathfrak{Q}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is called* specificity consistent *if and only if for all $S \in \mathbb{K}$, either $\{\Omega(S) : \Omega \in \mathbb{K}\} \subseteq [\frac{1}{2}, 1]$ or $\{\Omega(S) : \Omega \in \mathbb{K}\} \subseteq [0, \frac{1}{2}]$.*

Theorem 74 *Suppose $\mathfrak{Q}$ is a collection of mappings $\Omega \in \mathfrak{Q}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ and let $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathfrak{Q}\}$ be the corresponding collection of QFMs. Then $\mathbb{F}$ is specificity consistent if and only if $\mathfrak{Q}$ is specificity consistent.*
(Proof: A.43, p.216+)

Theorem 75 *Suppose $\mathfrak{Q}$ is a collection of mappings $\Omega \in \mathfrak{Q}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ which satisfy (Ω-5), and let $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathfrak{Q}\}$ be the corresponding collection of DFSes. Further suppose that every $\Omega \in \mathfrak{Q}$ has the additional property that $\Omega(S) = \frac{1}{2}$ for all $S \in \mathbb{K}$ with $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\mathbb{F}$ is specificity consistent.*
(Proof: A.44, p.217+)

Definition 79 *We say that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is fuzzier (less crisp) than $\Omega' : \mathbb{K} \longrightarrow \mathbf{I}$, in symbols: $\Omega \preceq_c \Omega'$, if and only if $\Omega(S) \preceq_c \Omega'(S)$ for all $S \in \mathbb{K}$.*

Theorem 76 *Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given mappings and let $\mathcal{F}_\Omega$, $\mathcal{F}_{\Omega'}$ be the corresponding QFMs defined by Def. 55. Then $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ if and only if $\Omega \preceq_c \Omega'$.*
(Proof: A.45, p.218+)

This criterion for comparing specificity can be further simplified in the frequent case that some basic assumptions can be made on Ω, Ω'.

Theorem 77 *Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given mappings which satisfy (Ω-2) and (Ω-5). Further suppose that $\Omega(S) = \frac{1}{2} = \Omega'(S)$ whenever $S \in \mathbb{K}$ has $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\Omega \preceq_c \Omega'$ if and only if $\Omega(S) \leq \Omega'(S)$ for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$.*
(Proof: A.46, p.218+)

Similar criteria can be established in the case of mappings $\omega : \mathbb{L} \longrightarrow \mathbf{I}$.

Definition 80 *A collection* ϖ *of mappings* $\omega \in \varpi$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is called specificity consistent if and only if for all* $s \in \mathbb{L}$, *either* $\{\omega(s) : \omega \in \mathbb{L}\} \subseteq [\frac{1}{2}, 1]$ *or* $\{\omega(s) : \omega \in \mathbb{L}\} \subseteq [0, \frac{1}{2}]$.

Theorem 78 *Suppose* ϖ *is a collection of mappings* $\omega \in \varpi$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$, *and let* $\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \varpi\}$ *be the corresponding collection of QFMs. Then* $\mathbb{F}$ *is specificity consistent if and only if* ϖ *is specificity consistent.*
(Proof: A.47, p.219+)

Theorem 79 *Suppose* ϖ *is a collection of mappings* $\omega \in \varpi$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *which satisfy* (ω-1)–(ω-4), *and let* $\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \varpi\}$ *be the corresponding collection of DFSes. Further suppose that every* $\omega \in \varpi$ *has the additional property that* $\omega(s) = \frac{1}{2}$ *for all* $s \in \mathbb{L}$ *with* $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. *Then* $\mathbb{F}$ *is specificity consistent.*
(Proof: A.48, p.220+)

The following theorems show that the above property is possessed both by $\mathcal{F}_\omega$-DFSes that propagate fuzziness in quantifiers and by those that propagate fuzziness in arguments:

Theorem 80 *Let* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be a given mapping which satisfies* (ω-1)–(ω-4) *and suppose that the corresponding DFS* $\mathcal{F}_\omega$ *propagates fuzziness in quantifiers. Then* $\omega(s) = \frac{1}{2}$ *for all* $s \in \mathbb{L}$ *with* $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.
(Proof: A.49, p.221+)

In particular,

Theorem 81 *The collection of* $\mathcal{F}_\omega$*-DFSes that propagate fuzziness in quantifiers is specificity consistent.*
(Proof: A.50, p.222+)

Theorem 82 *Let* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be a given mapping which satisfies* (ω-1)–(ω-4) *and suppose that the corresponding DFS* $\mathcal{F}_\omega$ *propagates fuzziness in arguments. Then* $\omega(s) = \frac{1}{2}$ *for all* $s \in \mathbb{L}$ *with* $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.
(Proof: A.51, p.222+)

Therefore

Theorem 83 *The collection of* $\mathcal{F}_\omega$*-DFSes that propagate fuzziness in arguments is specificity consistent.*
(Proof: A.52, p.222+)

Definition 81 *We say that* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *is fuzzier (less crisp) than* $\omega' : \mathbb{L} \longrightarrow \mathbf{I}$, *in symbols:* $\omega \preceq_c \omega'$, *if and only if* $\omega(s) \preceq_c \omega'(s)$ *for all* $s \in \mathbb{L}$.

Theorem 84 *Let* $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ *be given mappings and let* $\mathcal{F}_\omega$, $\mathcal{F}_{\omega'}$ *be the corresponding QFMs defined by Def. 61. Then* $\mathcal{F}_\omega \preceq_c \mathcal{F}_{\omega'}$ *if and only if* $\omega \preceq_c \omega'$.
(Proof: A.53, p.223+)

Again, it is possible to simplify the condition in typical situations.

Theorem 85 *Let* $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ *be given mappings which satisfy* (ω-2) *and* (ω-4). *Further suppose that* $\omega(s) = \frac{1}{2} = \omega'(s)$ *whenever* $s \in \mathbb{L}$ *satisfies* $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. *Then* $\omega \preceq_c \omega'$ *if and only if* $\omega(s) \leq \omega'(s)$ *for all* $s \in \mathbb{L}$ *with* ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$.
(Proof: A.54, p.223+)

The precondition of the theorem is e.g. satisfied by the models that propagate fuzziness. Based on this simplified criterion, it is now easy to prove the following results concerning specificity bounds.

Theorem 86 $\mathcal{F}_Z$ *is the most specific* $\mathcal{F}_\omega$*-DFS that propagates fuzziness in quantifiers.*
(Proof: A.55, p.224+)

Theorem 87 $\mathcal{F}_R$ *is the most specific* $\mathcal{F}_\omega$*-DFS that propagates fuzziness in arguments.*
(Proof: A.56, p.226+)

Theorem 88 $\mathcal{F}_M$ *is the most specific* $\mathcal{F}_\omega$*-DFS that propagates fuzziness both in quantifiers and arguments.*
(Proof: A.57, p.228+)

As concerns the issue of identifying the least specific model, we obtain the following result which confirms the special role of $\mathcal{M}_U$.

Theorem 89 $\mathcal{M}_U$ *is the least specific* $\mathcal{F}_\omega$*-DFS.*
(Proof: A.58, p.229+)

Finally let us consider continuity properties of $\mathcal{F}_\Omega$-DFSes. This investigation will help us to relate the new class of DFSes to its subclass of $\mathcal{F}_\xi$-DFSes. To this end, we introduce the following operation $^\Box$.

Definition 82 *For all* $S \in \mathbb{K}$, $S^\Box \in \mathbb{K}$ *is defined by*

$$S^\Box(\gamma) = [\inf S(\gamma), \sup S(\gamma)]$$

for all $\gamma \in \mathbf{I}$.

Note. It is apparent from Def. 52 that indeed $S^\Box \in \mathbb{K}$.

Theorem 90 *For all* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, $\mathcal{F}_\Omega$ *is an* $\mathcal{F}_\xi$*-QFM if and only if* Ω *is* $^\Box$*-invariant, i.e.* $\Omega(S) = \Omega(S^\Box)$ *for all* $S \in \mathbb{K}$.
(Proof: A.59, p.240+)

Utilizing this relationship, the following theorem is straightforward.

Theorem 91 *Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be an ‡-invariant mapping. If $\mathcal{F}_\Omega$ is Q-continuous, then it is an $\mathcal{F}_\xi$-QFM, i.e. there exists $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with $\mathcal{F}_\Omega = \mathcal{F}_\xi$. In particular, the theorem is applicable to all $\mathcal{F}_\Omega$-DFSes.*
(Proof: A.60, p.242+)

Hence all $\mathcal{F}_\Omega$-DFSes that are interesting from a practical perspective are already contained in the class of $\mathcal{F}_\xi$-QFMs.

Summarizing, this chapter was devoted to the definition and analysis of the full class of QFMs definable in terms of three-valued cuts. In order to carry out this generalisation, we first observed that the mappings $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ used to define $\mathcal{F}_\xi$-QFMs can be decomposed into subsequent application of the three-valued cut mechanism (which generates an ambiguity set of alternative interpretations for each cut level) followed by an aggregation step based on the infimum or supremum. In order to abstract from the concepts used to define $\mathcal{F}_\xi$, and to capture the full class of standard models that depend on three-valued cuts, it was straightforward to drop the $\sup/\inf$-based aggregation step and to start an investigation of those models that can be defined in terms of the 'raw' information obtained at the cut levels, i.e. in terms of the result sets $S_{Q,X_1,\ldots,X_n}(\gamma)$ which represent the ambiguity range of all possible interpretations of Q given the three-valued cuts of $X_1, \ldots, X_n$ at the cut levels γ. In order to develop the theory of these models, I first identified the precise range of possible mappings $S = S_{Q,X_1,\ldots,X_n}$ that can result from a choice of quantifier Q and fuzzy arguments $X_1, \ldots, X_n$. The resulting set $\mathbb{K}$ provides the proper domain to define aggregation operators $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$, from which QFMs can then be constructed in the apparent way, i.e. $\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) = \Omega(S_{Q,X_1,\ldots,X_n})$.
After introducing $\mathcal{F}_\Omega$-QFMs, I developed all formal machinery required to express the precise conditions on Ω that make $\mathcal{F}_\Omega$ a DFS. In particular, I have characterised the class of $\mathcal{F}_\Omega$-DFSes in terms of a set of necessary and sufficient conditions, and I have shown that these conditions are independent. This analysis also reveals that all $\mathcal{F}_\Omega$-DFSes are in fact standard DFSes, and hence fulfill the expectations on standard models of fuzzy quantification. In addition, the known class of $\mathcal{F}_\xi$-QFMs has been related to its apparent superclass of $\mathcal{F}_\Omega$-QFMs.
I then focused on an apparent subclass of $\mathcal{F}_\Omega$-QFMs, the class of $\mathcal{F}_\omega$-QFMs. These are obtained by defining coefficients $s_{Q,X_1,\ldots,X_n}(z) = \inf\{\gamma : z \in S_{Q,X_1,\ldots,X_n}(\gamma)$ which extract an important characteristic of the result sets $S_{Q,X_1,\ldots,X_n}(\gamma)$. Introducing this construction offers the advantage that we no longer need to work with *sets* of results, like it was the case with the $S_{Q,X_1,\ldots,X_n}(\gamma)$, which are subsets of the unit interval. By contrast, we can now focus on scalars $s_{Q,X_1,\ldots,X_n}$ in the unit range, and a subsequent aggregation by applying the chosen $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. Among others, this greatly simplifies the definition of models, and hence all examples of $\mathcal{F}_\Omega$-DFSes were presented in this succint format.
Noticing that the new coefficients $s_{Q,X_1,\ldots,X_n}$ are functions of $S_{Q,X_1,\ldots,X_n}$ which suppress some of the original information, the question then arises if some of the $\mathcal{F}_\Omega$-DFSes are lost under the new construction. To resolve this issue whether the $\mathcal{F}_\omega$-DFSes are a subclass proper, and to gain some insight into their structure, I have introduced

the concepts required to characterise adequate choices of ω. Building on these definitions, a set of independent conditions that precisely describe the $\mathcal{F}_\omega$-DFSes in terms of necessary and sufficient criteria on ω has been developed. In addition, the $\mathcal{F}_\omega$-QFMs have been related to their superclass of $\mathcal{F}_\Omega$-QFMs. This analysis revealed that the move from $\mathcal{F}_\Omega$-QFMs to $\mathcal{F}_\omega$-QFMs does not result in any loss of intended models, i.e. the classes of $\mathcal{F}_\Omega$-DFSes and $\mathcal{F}_\omega$-DFSes coincide.
Turning to examples of $\mathcal{F}_\Omega$-DFSes (or synonymously, $\mathcal{F}_\omega$-DFSes), the simplified format was utilized to define the four $\mathcal{F}_\omega$-DFSes $\mathcal{F}_M$, $\mathcal{F}_P$, $\mathcal{F}_Z$ and $\mathcal{F}_R$, all of which were shown to be 'genuine' members which go beyond the class of $\mathcal{F}_\xi$-QFMs. In order to gain more knowledge of these models, and to locate them precisely within the class of $\mathcal{F}_\omega$-DFSes, the full set of conditions on Ω and ω was then developed, that are required to investigate the characteristic properties of DFSes.

To this end, I first extended the specificity order to the case of $S \preceq_c S'$ and $s \preceq_c s'$. This allowed me to reduce $\mathcal{F}_\Omega$'s propagating fuzziness in quantifiers to the requirement that Ω propagate fuzziness, i.e. $S \preceq_c S'$ entails $\Omega(S) \preceq_c \Omega(S')$. Based on a different relation $S \Subset S'$ defined on $\mathbb{K}$, it was then possible to define a condition of propagating unspecificity on Ω, and to prove that $\mathcal{F}_\Omega$ propagates fuzziness in arguments if and only if Ω propagates unspecificity. In addition, I have shown that both the condition of propagating fuzziness and the condition of propagating unspecificity can be further simplified if the considered Ω is well-behaved (in particular if $\mathcal{F}_\Omega$ is a DFS). In this common case, a very elementary test on Ω is sufficient for detecting or rejecting these properties. All of these results have also been transferred to $\mathcal{F}_\omega$-QFMs, and hence turned into corresponding conditions on ω. After developing the formal apparatus required to investigate propagation of fuzziness in $\mathcal{F}_\Omega$- and $\mathcal{F}_\omega$-QFMs, the issue of most specific and least specific models was then discussed to some depth. Acknowledging its relevance to the existence of most specific models, I first extended the notion of specificity consistence to collections $\mathbb{\Omega}$ of of aggregation mappings Ω and proved that the resulting criterion on $\mathbb{\Omega}$ precisely captures specificity consistence of the class of QFMs $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathbb{\Omega}\}$. Hence the question whether $\mathbb{F}$ has a least upper specificity bound can be decided by looking at the aggregation mappings in $\mathbb{\Omega}$. I have also shown how the criterion can be simplified in common situations. Following this, the question was addressed how a specificity comparison $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ can be reformulated into a condition $\Omega \preceq_c \Omega'$ imposed on the aggregation mappings. Again, the condition for $\Omega \preceq_c \Omega'$ can be reduced to a very simple check in many typical situations. All of the above concepts and theorems were then adapted to $\mathcal{F}_\omega$-QFMs, in order to provide similar support for specificity comparison in those cases where the models of interest are defined in terms of an aggregation mapping ω.

Based on these preparations, it was easy to prove some results concerning propagation of fuzziness that elucidate the structure of the class of $\mathcal{F}_\omega$-DFSes, and that relate the examples of $\mathcal{F}_\omega$-DFSes to the class as a whole. First of all, the full class of $\mathcal{F}_\omega$-DFSes is not specificity consistent (because its subclass of $\mathcal{F}_\xi$-DFSes is known to violate specificity consistence), and hence a 'most specific $\mathcal{F}_\omega$-DFS' ceises to exist. However, the class of models that propagate fuzziness in quantifiers was shown to be specificity consistent, and the most specific $\mathcal{F}_\omega$-DFS with this property was also identified, and turned out to be $\mathcal{F}_Z$. Recalling that $\mathcal{F}_Z$ is not an $\mathcal{F}_\xi$-QFM, this demonstrates

that the class of $\mathcal{F}_\omega$-DFSes which propagate fuzziness in quantifiers is an extension proper of the class of $\mathcal{F}_\xi$-DFSes with the same behaviour. Turning to propagation of fuzziness in quantifiers, it was possible to prove a similar result. The corresponding class of $\mathcal{F}_\omega$-DFSes was shown to be specificity consistent, and $\mathcal{F}_R$ was established to be the most specific $\mathcal{F}_\omega$-DFS with this property. Again, we conclude from the fact that $\mathcal{F}_R$ is a 'genuine' $\mathcal{F}_\omega$-DFS that the $\mathcal{F}_\omega$-DFSes contain models which propagate fuzziness in arguments beyond those already known from the study of $\mathcal{F}_\xi$-DFSes. We then investigated those standard models that propagate fuzziness both in quantifiers and arguments. The model $\mathcal{F}_M$ was shown to be the most specific $\mathcal{F}_\omega$-DFS with these properties. The class of $\mathcal{F}_\xi$-DFSes that fulfill both conditions is known to coincide with the class of $\mathcal{M}_\mathcal{B}$-DFSes. Because $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, this proves that there are standard models beyond $\mathcal{M}_\mathcal{B}$-DFSes which propagate fuzziness both in quantifiers and arguments.
The problem of identifying the greatest lower specificity bound has also been addressed. In fact, the least specific $\mathcal{F}_\omega$-DFS was proven to coincide with one of the $\mathcal{M}_\mathcal{B}$-DFSes, namely $\mathcal{M}_U$, which was already known to be the least specific $\mathcal{M}_\mathcal{B}$- and $\mathcal{F}_\xi$-DFS.

Finally, I have addressed the continuity issue. It is indispensible for applications that the chosen QFM be robust against slight variations in the chosen quantifier and in its arguments, which might e.g. result from noise. In addition, both continuity conditions are desirable in order to account for imperfect knowledge of the precise interpretation of the involved NL quantifier and NL concepts in terms of numeric membership grades. Based on an auxiliary fill construction $S^\square$, it was then shown that every $\mathcal{F}_\Omega$-QFM which is continuous in quantifiers is in fact an $\mathcal{F}_\xi$-QFM. The class of $\mathcal{F}_\omega$-DFSes which are Q-continuous therefore collapses into the class of Q-continuous $\mathcal{F}_\xi$-DFSes, and those Q-continuous $\mathcal{F}_\omega$-DFSes which propagate fuzziness in quantifiers and arguments collapse into the class of $\mathcal{M}_\mathcal{B}$-DFSes. This proves that all *practical* models are already contained in the class of $\mathcal{F}_\xi$-DFSes, and those practical models which propagate fuzziness both in quantifiers and arguments are contained in the class of $\mathcal{M}_\mathcal{B}$-DFSes. This justifies the development and thorough analysis of these simpler classes in [9] and [11], every model of practical interest will belong to one of these classes. It can hence be expressed through constructions simpler than those used to define $\mathcal{F}_\Omega$- and $\mathcal{F}_\omega$-QFMs, which in turn permit a simpler check of the relevant formal properties, like being a DFS, propagation of fuzziness, specificity comparisons, and continuity, and which suggest simple algorithms for implementing quantifiers in the model.

5 The class of models based on the extension principle

In this chapter, an attempt is made to define DFSes from independent considerations, and to establish a new class of fuzzification mechanisms not constructed from three-valued cuts. Starting from a straightforward definition of argument similarity, we first introduce the full class of QFMs defined in terms of the similarity measure, the class of $\mathcal{F}_\psi$-QFMs. It encloses the interesting subclass of $\mathcal{F}_\varphi$-QFMs, i.e. the class of models defined through the standard extension principle (which serves to aggregate similarity grades). The necessary and sufficient conditions are then developed, which the aggregation mappings must satisfy in order to make the corresponding fuzzification mechanism a DFS. Based on this analysis, it becomes possible to prove the main result of this chapter, which states that the classes of $\mathcal{F}_\omega$-DFSes and $\mathcal{F}_\psi$-DFSes/$\mathcal{F}_\varphi$-DFSes coincide. Because the same class of models is obtained from independent considerations, this provides evidence that it indeed represents a natural class of standard models of fuzzy quantification.
To begin with, the similarity grade $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n)$ of the fuzzy arguments $(X_1,\ldots,X_n)$ to a choice of crisp arguments $(Y_1,\ldots,Y_n)$ can be defined as follows.

Definition 83 *Let $E \neq \varnothing$ be some base set and $Y \in \mathcal{P}(E)$. The mapping $\Xi_Y : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is defined by*

$$\Xi_Y(X) = \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\})$$

for all $X \in \widetilde{\mathcal{P}}(E)$. For n-tuples of arguments $Y_1,\ldots,Y_n \in \mathcal{P}(E)$, we define $\Xi^{(n)}_{Y_1,\ldots,Y_n} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$\Xi^{(n)}_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) = \bigwedge_{i=1}^{n} \Xi_{Y_i}(X_i)$$

for all $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. Whenever n is clear from context, we shall omit the superscript and write $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n)$ instead of $\Xi^{(n)}_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n)$.

At times, it will be convenient to use the following abbreviation. We recall the fuzzy equivalence connective $\leftrightarrow : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ defined by

$$x \leftrightarrow y = (x \wedge y) \vee (\neg x \wedge \neg y)$$

for all $x, y \in \mathbf{I}$. In the case that $y \in \{0,1\}$, this apparently becomes

$$x \leftrightarrow y = \begin{cases} x & : \quad y = 1 \\ \neg x & : \quad y = 0 \end{cases}$$

Now consider a base set $E \neq \varnothing$ and let $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$. We make use of the $\leftrightarrow$-connective to define $\delta_{X,Y} : E \longrightarrow \mathbf{I}$ by

$$\delta_{X,Y}(e) = (\mu_X(e) \leftrightarrow \chi_Y(e)) = \begin{cases} \mu_X(e) & : \quad e \in Y \\ 1 - \mu_X(e) & : \quad e \notin Y \end{cases} \tag{60}$$

for all $e \in E$. In terms of $\delta_{Y,E}$, we can now conveniently reformulate $\Xi_Y(X)$. In particular, we can express $\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n)$, where $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ and $Y_1,\dots,Y_n \in \mathcal{P}(E)$, by

$$\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) = \inf\{\delta_{X_i,Y_i}(e) : e \in E, i = 1,\dots,n\}\,. \qquad (61)$$

This succint notation will at times be used in the proofs. Next we define the set of all compatibility grades which corresponds to a given choice of fuzzy arguments $X_1,\dots,X_n$.

Definition 84 *Let $E \neq \varnothing$ be a given base set and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$, $n \geq 0$. Then $D^{(n)}_{X_1,\dots,X_n} \in \mathcal{P}(\mathbf{I})$ is defined by*

$$D^{(n)}_{X_1,\dots,X_n} = \{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) : Y_1,\dots,Y_n \in \mathcal{P}(E)\}\,.$$

Whenever this causes no ambiguity, the superscript (n) will be omitted, thus abbreviating $D_{X_1,\dots,X_n} = D^{(n)}_{X_1,\dots,X_n}$.

Note. The superscript is only needed to discern $D^{(0)}_\varnothing$ (which corresponds to the empty tuple) from $D^{(1)}_\varnothing$ (which corresponds to the empty set).

Definition 85 *By $\mathbb{D} \subseteq \mathcal{P}(\mathcal{P}(\mathbf{I}))$ we denote the set of all $D \in \mathcal{P}(\mathbf{I})$ with the following properties:*

1. *$D \cap [\frac{1}{2}, 1] = \{r_+\}$ for some $r_+ = r_+(D) \in [\frac{1}{2}, 1]$;*
2. *for all $D' \subseteq D$ with $D' \neq \varnothing$, $\inf D' \in D$;*
3. *if $r_+ > \frac{1}{2}$, then $\sup D \setminus \{r_+\} = 1 - r_+$.*

Theorem 92 *Suppose $E \neq \varnothing$ is some base set and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. Then $D_{X_1,\dots,X_n} \in \mathbb{D}$.*
(Proof: B.1, p.247+)

Hence $\mathbb{D}$ is large enough to contain all $D_{X_1,\dots,X_n}$. As we shall see later in Th-95, $\mathbb{D}$ is indeed the smallest possible subset of $\mathcal{P}(\mathcal{P}(\mathbf{I}))$ which contains all $D_{X_1,\dots,X_n}$. (The theorem has been delayed because it then becomes a corollary).
In order to define the new class of fuzzification mechanisms, we now relate the similarity information expressed by $\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n)$ to the behaviour of a quantifier on its arguments.

Definition 86 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given semi-fuzzy quantifier and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Then $A^{(n)}_{Q,X_1,\dots,X_n} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$A^{(n)}_{Q,X_1,\dots,X_n}(z) = \{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) : (Y_1,\dots,Y_n) \in Q^{-1}(z)\}$$

for all $z \in \mathbf{I}$. When n is clear from context, I usually omit the superscript (n), thus abbreviating $A_{Q,X_1,\dots,X_n} = A^{(n)}_{Q,X_1,\dots,X_n}$.

Note. Again, the superscript is only needed to eliminate the ambiguity between $A^{(0)}_{Q,\varnothing}$, where Q is a nullary quantifier and $\varnothing$ the empty tuple, and $A^{(1)}_{Q,\varnothing}$, where Q is a one-place quantifier and $\varnothing$ is the empty argument set.
Let us now describe the range of all possible $A_{Q,X_1,\ldots,X_n}$.

Definition 87 *By $\mathbb{A}$ we denote the set of all mappings $A : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ with the following properties:*

a. $\cup\{A(z) : z \in \mathbf{I}\} \in \mathbb{D}$;

b. for all $z, z' \in \mathbf{I}$, $\sup A(z) > \frac{1}{2}$ and $\sup A(z') > \frac{1}{2}$ entails that $z = z'$.

In the following, $D(A)$ denotes the set

$$D(A) = \cup\{A(z) : z \in \mathbf{I}\}\,. \tag{62}$$

In addition, r_+ abbreviates $r_+(A) = r_+(D(A))$. It is then apparent from Def. 87.**a** and Def. 85 that there exists $z_+ = z_+(A) \in \mathbf{I}$ with

$$r_+ \in A(z_+)\,. \tag{63}$$

In the following, z_+ is assumed to be an arbitrary but fixed choice of $z_+ \in \mathbf{I}$ which satisfies (63) for a considered $A \in \mathbb{A}$.

Theorem 93 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then $A_{Q,X_1,\ldots,X_n} \in \mathbb{A}$.*
(Proof: B.2, p.255+)

Hence $\mathbb{A}$ contains all $A_{Q,X_1,\ldots,X_n}$.

Theorem 94 *Let $A \in \mathbb{A}$ be given and $D(A) = \cup\{A(z) : z \in \mathbf{I}\}$.*

a. If $D(A) = \{1\}$, then $A = A^{(0)}_{Q,\varnothing}$, where $Q : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ is the constant quantifier $Q(\varnothing) = z_+$.

b. If $D(A) \neq \{1\}$, then we can choose some mapping $\zeta : D(A) \longrightarrow \mathbf{I}$ with

$$r \in A(\zeta(r)) \tag{64}$$

for all $r \in D(A)$. If $r_+ = r_+(A)$ equals $\frac{1}{2}$, then $r_+ \in D(A) \cap [0, 1 - r_+]$. If $r_+ > \frac{1}{2}$, then we recall from Def. 87 that $\sup D(A) \setminus \{r_+\} = 1 - r_+$. Because $D(A) \neq \{1\}$ by assumption, we hence know that there exists

$$r_- \in D(A) \cap [0, 1 - r_+] \tag{65}$$

and we shall assume an arbitrary choice of r_- with this property. Based on r_-, we define $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by

$$\mu_X(z, r) = \begin{cases} r & : \quad r \in A(z) \setminus \{r_+\} \\ r_- & : \quad r \notin A(z) \vee r = r_+ > \frac{1}{2} \\ \frac{1}{2} & : \quad r = r_+ = \frac{1}{2} \end{cases} \tag{66}$$

for all $z, r \in \mathbf{I}$*. For all* $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$*, we abbreviate*

$$r' = r'(Y) = \Xi_Y(X) \tag{67}$$
$$z' = z'(Y) = \inf\{z \in \mathbf{I} : (z, r') \in Y \text{ and } r' = r'(Y) \in A(z)\}\,. \tag{68}$$

Based on ζ*, we define* $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *by*

$$Q(Y) = \begin{cases} z' & : \ r' \in A(z') \\ \zeta(r') & : \ r' \notin A(z') \end{cases} \tag{69}$$

for all $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$*.*
Then $A = A_{Q,X}$*.*

(Proof: B.3, p.255+)

Let us also state the following corollary:

Theorem 95 *For all* $D \in \mathbb{D}$*,*

a. *If* $D = \{1\}$*, then* $D = D^{(0)}_{\varnothing}$*, where* $\varnothing$ *is the empty tuple* $\varnothing \in \mathcal{P}(\{*\})^0$*.*

b. *If* $D \neq \{1\}$*, then there exists* $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ *such that* $D = D_X$*.*

(Proof: B.4, p.261+)

Hence $\mathbb{D}$ is indeed the smallest subset of $\mathcal{P}(\mathcal{P}(\mathbf{I}))$ which contains all $D_{X_1,\dots,X_n}$.
In order to carry out the desired aggregation, which will turn the compatibility grades into a fuzzification mechanism, we now deploy mappings $\psi : \mathbb{A} \longrightarrow \mathbf{I}$. These can be used to define a QFM in the apparent way, by composing with the $A_{Q,X_1,\dots,X_n}$'s:

Definition 88 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given. The QFM* $\mathcal{F}_\psi$ *is defined by*

$$\mathcal{F}_\psi(Q)(X_1, \dots, X_n) = \psi(A_{Q,X_1,\dots,X_n})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$*.*

This definition spans the full class of QFMs definable in terms of argument similarity, and we will now investigate its well-behaved models. To this end, we need some more notation, for expressing the properties required from legal choices of ψ. As usual, the goal is that of characterising the new class of DFSes in terms of the necessary and sufficient conditions on the aggregation mapping. In order to describe the desired monotonicity properties, I first define a suitable pre-order on $\mathbb{A}$.

Definition 89 *For all* $A, A' \in \mathbb{A}$*, we say that* $A \sqsubseteq A'$ *if and only if the following conditions are satisfied by* A, A'*.*

a. *for all* $z \in \mathbf{I}$ *and all* $r \in A(z)$*, there exists* $z' \geq z$ *with* $r \in A'(z')$*;*

b. for all $z' \in \mathbf{I}$ *and all* $r \in A'(z')$*, there exists* $z \leq z'$ *with* $r \in A(z)$*.*

Next I introduce a 'cut/fill operator' $\square$ on $\mathbb{A}$, which will be essential to ensure that ψ satisfy (Z-4). In order to define this operator, I first describe its behaviour on the $A(z)$'s. $\square : \mathcal{P}(\mathbf{I}) \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by

$$\square\, B = \{r \in [0, \tfrac{1}{2}] : \text{there exists } r' \geq r \text{ with } r' \in B\} \tag{70}$$

for all $B \in \mathcal{P}(\mathbf{I})$. If $B = A(z)$ for some $A \in \mathbb{A}$, $z \in \mathbf{I}$, then it is apparent from Def. 87 and Def. 85 that

$$\square\, A(z) = \begin{cases} [0, s) & : \quad s \notin A(z) \\ [0, \min(s, \tfrac{1}{2})] & : \quad s \in A(z) \end{cases} \tag{71}$$

where $s = \sup A(z)$. Based on this definition of $\square\, A(z)$, we define $\square\, A$ element-wise for all $z \in \mathbf{I}$.

Definition 90 $\square : \mathbb{A} \longrightarrow \mathbb{A}$ *assigns to each* $A \in \mathbb{A}$ *the mapping* $\square\, A \in \mathbb{A}$ *defined by*

$$(\square\, A)(z) = \square(A(z))$$

for all $z \in \mathbf{I}$*.*

Note. It is apparent from (71) and Def. 87 that indeed $\square\, A \in \mathbb{A}$. We shall use this operator in some of the theorems to follow. An invariance condition with respect to $\square\, A$ will not be imposed on ψ, though. Instead, we require that ψ be invariant with respect to a stronger cut/fill operator, $\boxplus$, which suppresses even more information:

Definition 91 *For all* $A \in \mathbb{A}$, $\boxplus A \in \mathbb{A}$ *is defined by*

$$\boxplus A(z) = [0, \widehat{\boxplus} A(z)]\,, \tag{72}$$

where

$$\widehat{\boxplus} A(z) = \min(\sup A(z), \tfrac{1}{2}) \tag{73}$$

for all $z \in \mathbf{I}$*.*

Notes

- It is immediate from the definition of $\boxplus A$ that $\boxplus A \in \mathbb{A}$, see Def. 87.
- For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we abbreviate

$$\boxplus_{Q,X_1,\ldots,X_n} = \boxplus A_{Q,X_1,\ldots,X_n} \tag{74}$$

$$\widehat{\boxplus}_{Q,X_1,\ldots,X_n} = \widehat{\boxplus} A_{Q,X_1,\ldots,X_n}\,. \tag{75}$$

- In the above definition, $\widehat{\boxplus}$ has been used to express $\boxplus$. In fact, both are definable in terms of each others, because conversely

$$\widehat{\boxplus}A(z) = \sup \boxplus A(z)\,, \tag{76}$$

for all $A \in \mathbb{A}$ and $z \in \mathbf{I}$.

- It is immediate from Def. 90 that

$$\widehat{\boxplus}A(z) = \sup \square\, A(z) \tag{77}$$

for all $z \in \mathbf{I}$.

The cut/fill operator $\boxplus$ is of special relevance to the characterisation of $\mathcal{F}_\psi$-DFSes because $\boxplus$-invariance ensures that (Z-6) be valid.
In order to define the conditions on ψ succintly and to support the corresponding proofs, it is useful to introduce some additional abbreviations. For all $A \in \mathbb{A}$,

$$\mathrm{NV}(A) = \{z \in \mathbf{I} : A(z) \neq \varnothing\} \tag{78}$$

$$\mathrm{VL}(A) = \{z \in \mathbf{I} : A(z) \setminus \{0\} \neq \varnothing\} = \{z \in \mathbf{I} : A(z) \cap (0,1] \neq \varnothing\}\,. \tag{79}$$

We have now introduced all notation required to express the conditions on admissible choices of ψ.

Definition 92 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given. The conditions* (ψ-1)–(ψ-5) *are defined as follows. For all $A, A' \in \mathbb{A}$,*

If $D(A) = \{1\}$, then $\psi(A) = z_+$. (ψ-1)

If $A(z) = A'(1-z)$ for all $z \in \mathbf{I}$, then $\psi(A) = 1 - \psi(A')$. (ψ-2)

If $\mathrm{NV}(A) \subseteq \{0,1\}$ and $r_+ \in A(1)$,
then $\psi(A) = 1 - \sup A(0)$. (ψ-3)

If $A \sqsubseteq A'$, then $\psi(A) \leq \psi(A')$. (ψ-4)

$\psi(\boxplus A) = \psi(A)$. ($\psi$-5)

The proof that these conditions are sufficient and necessary for $\mathcal{F}_\psi$ to be a (standard) DFS has been split up into several theorems, in order to reveal the dependency structure and some of the constructions that were useful in accomplishing this task.

Theorem 96 *If $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-1) *and* (ψ-5)*, then $\mathcal{F}_\psi$ satisfies* (Z-1).
(Proof: B.5, p.261+)

Next we consider the behaviour of ψ on two-valued quantifiers. To this end, it is convenient to relate $A_{Q,X_1,\ldots,X_n}$ to the coefficient $s_{Q,X_1,\ldots,X_n}$ that was used to define $\mathcal{F}_\omega$-QFMs. As a by-product of these results, we will discover that all $\mathcal{F}_\omega$-QFMs are in fact $\mathcal{F}_\psi$-QFMs. In particular, all $\mathcal{F}_\Omega$ and $\mathcal{F}_\omega$-DFSes constitute a subclass of our new class of $\mathcal{F}_\psi$-DFSes.

Theorem 97 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $z \in \mathbf{I}$,*

$$s_{Q,X_1,\ldots,X_n}(z) = s(A_{Q,X_1,\ldots,X_n})(z)\,,$$

where $s(A) \in \mathbb{L}$, $A \in \mathbb{A}$ is defined by

$$s(A)(z) = \max(0, 1 - 2 \cdot \sup A(z)) \tag{80}$$

for all $z \in \mathbf{I}$.
(Proof: B.6, p.263+)

Theorem 98 *Every $\mathcal{F}_\omega$-QFM is an $\mathcal{F}_\psi$-QFM, i.e. for all $\omega : \mathbb{L} \longrightarrow \mathbf{I}$, there exists $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ with $\mathcal{F}_\omega = \mathcal{F}_\psi$. ψ is defined by*

$$\psi(A) = \omega(s(A)) \tag{81}$$

for all $A \in \mathbb{A}$.
(Proof: B.7, p.269+)

Theorem 99 *If $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-2) *and* (ψ-3)*, then $\mathcal{F}_\psi$ coincides with all standard DFSes on two-valued quantifiers, i.e. for every standard DFS $\mathcal{F}$ and two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \{0, 1\}$, it holds that $\mathcal{F}_\psi(Q) = \mathcal{F}(Q)$.*
(Proof: B.8, p.269+)

Theorem 100 *Suppose $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-2) *and* (ψ-3)*. Then $\mathcal{F}_\psi$ satisfies* (Z-2)*.*
(Proof: B.9, p.271+)

Theorem 101 *If $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-2) *and* (ψ-3)*, then $\mathcal{F}_\psi$ satisfies* (Z-3)*.*
(Proof: B.10, p.271+)

Let us now utilize the operator $\square$ defined above. We first notice that $\boxplus$-invariance implies $\square$-invariance.

Theorem 102 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-5)*. Then ψ is also $\square$-invariant, i.e. it holds that*

$$\psi(\square A) = \psi(A)\,, \tag{ψ-5$'$}$$

for all $A \in \mathbb{A}$. (Proof: B.11, p.273+)

Theorem 103 *Suppose $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-2), (ψ-3) *and* (ψ-5$'$)*. Then $\mathcal{F}_\psi$ satisfies* (Z-4)*.*
(Proof: B.12, p.273+)

As concerns the preservation of monotonicity properties of quantifiers in their arguments, we firstly observe:

Theorem 104 *If* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-4)*, then* $\mathcal{F}_\psi$ *is monotonic, i.e.* $\mathcal{F}_\psi(Q) \leq \mathcal{F}_\psi(Q')$ *for all* $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *with* $Q \leq Q'$*.*
(Proof: B.13, p.278+)

In the case that a few other conditions are valid for ψ, monotonicity of $\mathcal{F}_\psi$ can be shown to entail the desired preservation of monotonicity in arguments (Z-5):

Theorem 105 *If* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-2)*,* (ψ-3)*,* (ψ-4) *and* (ψ-5′)*, then* $\mathcal{F}_\psi$ *satisfies* (Z-5)*.*
(Proof: B.14, p.278+)

Theorem 106 *If* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-2)*,* (ψ-3) *and* (ψ-5)*, then* $\mathcal{F}_\psi$ *satisfies* (Z-6)*.*
(Proof: B.15, p.280+)

These results can be summarized as follows.

Theorem 107 *If* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-1)*–*(ψ-5)*, then* $\mathcal{F}_\psi$ *is a standard DFS.*
(Proof: B.16, p.286+)

Next I prove that the conditions imposed on ψ are necessary for $\mathcal{F}_\psi$ to be a DFS.

Theorem 108 *If* $\mathcal{F}_\psi$ *satisfies* (Z-1)*. then* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-1)*.*
(Proof: B.17, p.286+)

Theorem 109 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be a given mapping such that* $\mathcal{F}_\psi$ *satisfies* (Z-2)*. Then* $\mathcal{F}_\psi$ *induces the standard negation* $\neg x = 1 - x$*.*
(Proof: B.18, p.287+)

Theorem 110 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and suppose that* $\mathcal{F}_\psi$ *satisfies* (Z-1) *and* (Z-2)*. If* $\mathcal{F}_\psi$ *satisfies* (Z-3)*, then* ψ *satisfies* (ψ-2)*.*
(Proof: B.19, p.289+)

Theorem 111 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and suppose that* $\mathcal{F}_\psi$ *satisfies* (Z-2)*. If the induced disjunction is an* s*-norm, then* $\mathcal{F}_\psi$ *induces the standard fuzzy disjunction* $x \vee y = \max(x, y)$*.*
(Proof: B.20, p.290+)

Theorem 112 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and suppose that* $\mathcal{F}_\psi$ *induces the standard disjunction and the standard extension principle. If* ψ *satisfies* (Z-4) *and* (Z-6)*, then* $\psi(\square\, A) = \psi(A)$ *for all* $A \in \mathbb{A}$*. Hence* ψ *satisfies* (ψ-5′)*, i.e.* ψ *is* $\square$*-invariant.*
(Proof: B.21, p.293+)

This result can be extended to the case of invariance with respect to $\boxplus$:

Theorem 113 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and suppose that* $\mathcal{F}_\psi$ *induces the standard disjunction and the standard extension principle. If* $\mathcal{F}_\psi$ *satisfies* (Z-4) *and* (Z-6)*, then* $\psi(\boxplus A) = \psi(A)$ *for all* $A \in \mathbb{A}$*. Hence* (ψ-5) *is valid, i.e.* ψ *is* $\boxplus$*-invariant.*
(Proof: B.22, p.326+)

Based on these interrelationships, we can now state a theorem concerned with the necessity of condition (ψ-3).

Theorem 114 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is $\boxplus$-invariant, i.e. ψ satisfies* (ψ-5)*. If $\mathcal{F}_\psi$ satisfies* (Z-2)*, then ψ satisfies* (ψ-3)*.*
(Proof: B.23, p.334+)

The next theorem unveils the core reason, why (ψ-4) is necessary for $\mathcal{F}_\psi$ to be a DFS.

Theorem 115 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-5)*, i.e. ψ is $\boxplus$-invariant. If $\mathcal{F}_\psi$ is monotonic, i.e. $\mathcal{F}_\psi(Q_1) \leq \mathcal{F}_\psi(Q_2)$ for all semi-fuzzy quantifiers $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ such that $Q_1 \leq Q_2$, then ψ satisfies* (ψ-4)*.*
(Proof: B.24, p.336+)

To sum up, it has been shown that the conditions (ψ-1)–(ψ-5) imposed on ψ are necessary for $\mathcal{F}_\psi$ to be a DFS, as stated in the next theorem (actually, a corollary):

Theorem 116 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping. If $\mathcal{F}_\psi$ is a DFS, then ψ satisfies* (ψ-1)–(ψ-5)*.*
(Proof: B.25, p.341+)

In particular, every choice of ψ which makes $\mathcal{F}_\psi$ a DFS satisfies (ψ-5). As I will now show, this entails that the class of $\mathcal{F}_\psi$-QFMS, although considerably broader than the class of $\mathcal{F}_\omega$-QFMs, does not introduce any new DFSes compared to those that already belong to the class of $\mathcal{F}_\omega$-DFSes. To see this, we notice the following relationship between $s(A)$ and $\widehat{\boxplus} A$.

Theorem 117 *Let $A \in \mathbb{A}$ be given. Then*

$$\widehat{\boxplus} A(z) = \tfrac{1}{2} - \tfrac{1}{2} s(A)(z) \tag{82}$$

and

$$s(A)(z) = 1 - 2\widehat{\boxplus} A(z)\,, \tag{83}$$

for all $z \in \mathbf{I}$.
(Proof: B.26, p.342+)

Based on this relationship, it is then apparent that all $\boxplus$-invariant $\mathcal{F}_\psi$-QFMs are in fact $\mathcal{F}_\omega$-QFMs.

Theorem 118 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies* (ψ-5)*. Then $\mathcal{F}_\psi$ is an $\mathcal{F}_\omega$-QFM, i.e.*

$$\mathcal{F}_\psi = \mathcal{F}_\omega$$

provided we define

$$\omega(s) = \psi(A_s), \tag{84}$$

for all $s \in \mathbb{L}$, where

$$A_s(z) = [0, \tfrac{1}{2} - \tfrac{1}{2}s(z)] \tag{85}$$

for all $z \in \mathbf{I}$. In particular, all $\mathcal{F}_\psi$-DFSes are $\mathcal{F}_\omega$-DFSes.
(Proof: B.27, p.343+)

We have already shown in Th-98 how to relate the known class of $\mathcal{F}_\omega$-QFMs to the new class of $\mathcal{F}_\psi$-QFMs. We shall now proceed and relate the conditions (ω-1)–(ω-4) imposed on ω to the conditions (ψ-1)–(ψ-5) imposed on the corresponding ψ. This will permit us to prove the independence of the new set of conditions in terms of the known independence of the ω-conditions. We first notice that

Theorem 119 *The $\mathcal{F}_\omega$-QFMs are exactly those $\mathcal{F}_\psi$-QFMs that depend on a mapping $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ which satisfies (ψ-5).*
(Proof: B.28, p.344+)

In the following, we will hence assume that (ψ-5) be valid. Then

Theorem 120 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81). *Then*

a. ω satisfies (ω-1) *if and only if ψ satisfies* (ψ-1)

b. ω satisfies (ω-2) *if and only if ψ satisfies* (ψ-2)*;*

c. ω satisfies (ω-3) *if and only if ψ satisfies* (ψ-3)*;*

d. ω satisfies (ω-4) *if and only if ψ satisfies* (ψ-4)*;*

e. ψ satisfies (ψ-5)*.*

(Proof: B.29, p.345+)

Based on these results, it is now easy to prove a theorem concerning the independence of the ‘ψ-conditions’.

Theorem 121 *The conditions* (ψ-1)–(ψ-5) *are independent.*
(Proof: B.30, p.358+)

In the following, I will discuss a slight reformulation of the aggregation mechanism which shows that the $\mathcal{F}_\psi$-DFSes coincide with the models defined in terms of the standard extension principle. The discovered class of models is hence theoretically appealing, because it evolves from the fundamental principle that underlies fuzzy set theory. In order to define the class of those QFMs that depend on the extension principle, we consider the following basic construction.

Definition 93 *For all $A \in \mathbb{A}$, we denote by $f_A : \mathbf{I} \longrightarrow \mathbf{I}$ the mapping defined by*

$$f_A(z) = \sup A(z)$$

for all $z \in \mathbf{I}$.

It is apparent from (73) that

$$\widehat{\boxplus} A(z) = \min(f_A(z), \tfrac{1}{2}) , \tag{86}$$

for all $A \in \mathbb{A}$ and $z \in \mathbf{I}$.
It is then apparent from Def. 91 that $\boxplus A$ can be defined in terms of f_A, i.e. there exists g such that $\boxplus A = g(f_A)$ for all $A \in \mathbb{A}$. In turn, we conclude that every ψ which makes $\mathcal{F}_\psi$ a DFS, can be defined in terms of f_A because every such ψ is $\boxplus$-invariant by Th-113, and hence $\psi(A) = \psi(\boxplus A) = \psi(g(f_A))$. In other words, we do not loose any models of interest if we restrict attention to those QFMs that are a function of f_A.
I now introduce the constructions necessary to define the new class of QFMs.

Definition 94 *Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. By $f_{Q,X_1,\ldots,X_n} = f^{(n)}_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ we denote the mapping defined by*

$$f_{Q,X_1,\ldots,X_n} = f_{A_{Q,X_1,\ldots,X_n}} ,$$

i.e.

$$f_{Q,X_1,\ldots,X_n}(z) = \sup A_{Q,X_1,\ldots,X_n}(z) .$$

for all $z \in \mathbf{I}$.

Notes

- Again, the superscript (n) in $f^{(n)}_{Q,X_1,\ldots,X_n}$ is usually omitted when no ambiguity arises.
- $f_{Q,X_1,\ldots,X_n}(z)$ expresses a measure of the maximal similarity of $(X_1, \ldots, X_n)$ to those $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$ which are mapped to $Q(Y_1, \ldots, Y_n) = z$.

Next let us describe the range of all possible f_A.

Definition 95 *By $\mathbb{X} \in \mathcal{P}(\mathbf{I}^{\mathbf{I}})$ we denote the set of all mappings $f : \mathbf{I} \longrightarrow \mathbf{I}$ with the following properties:*

a. $\operatorname{Im} f \cap [\frac{1}{2}, 1] = \{r_+\}$ *for some* $r_+ = r_+(f) \geq \frac{1}{2}$;

b. If $z_+ = z_+(f) \in \mathbf{I}$ is chosen such that $f(z_+) = r_+$, then $f(z) \leq 1 - r_+$ for all $z \neq z_+$.

Theorem 122 *For all $A \in \mathbb{A}$, $f_A \in \mathbb{X}$. In particular, if $Q : \mathcal{P}(E)^n$ is a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, then $f_{Q,X_1,\dots,X_n} \in \mathbb{X}$.*
(Proof: B.31, p.369+)

Theorem 123 *For all $f \in \mathbb{X}$, there exists $A \in \mathbb{A}$ with $f = f_A$. In particular, there exist $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ with $f = f_{Q,X_1,\dots,X_n}$.*
(Proof: B.32, p.370+)

Hence $\mathbb{X}$ is indeed the range of all possible f_A and $f_{Q,X_1,\dots,X_n}$. We can therefore define the class of QFMs computable from $f_{Q,X_1,\dots,X_n}$, called $\mathcal{F}_\varphi$-QFMs, in the apparent way.

Definition 96 *Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given. The QFM $\mathcal{F}_\varphi$ is defined by*

$$\mathcal{F}_\varphi(Q)(X_1, \dots, X_n) = \varphi(f_{Q,X_1,\dots,X_n}),$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

The $\mathcal{F}_\varphi$-QFMs comprise the class of those fuzzification mechanisms which can be defined from the argument similarity grades by applying the extension principle. This is because $f_{Q,X_1,\dots,X_n}$ is obtained from the standard extension principle in the following way. We start from a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. By applying the extension principle, we obtain $\hat{\hat{Q}} : \widetilde{\mathcal{P}}(\mathcal{P}(E)) \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$. Hence for a given $V \in \widetilde{\mathcal{P}}(\mathcal{P}(E))$, $\hat{\hat{Q}}(V) \in \widetilde{\mathcal{P}}(\mathbf{I})$ is the fuzzy subset defined by

$$\mu_{\hat{\hat{Q}}(V)}(z) = \sup\{\mu_V(Y_1, \dots, Y_n) : (Y_1, \dots, Y_n) \in Q^{-1}(z)\}$$

for all $z \in \mathbf{I}$, see (3). Given a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, we now express $V = V_{X_1,\dots,X_n}$ in terms of argument similarity, viz

$$\mu_V(Y_1, \dots, Y_n) = \Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n),$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. It is then apparent from Def. 94 that

$$f_{Q,X_1,\dots,X_n}(z) = \mu_{\hat{\hat{Q}}(V)}(z)$$

for all $z \in \mathbf{I}$. Because $V = V_{X_1,\dots,X_n}$ represents argument similarity, $\hat{\hat{Q}}$ is obtained from Q by applying the standard extension principle, $f_{Q,X_1,\dots,X_n}$ is defined by composing $\hat{\hat{Q}}$ and $V_{X_1,\dots,X_n}$, and $\mathcal{F}_\varphi(Q)(X_1, \dots, X_n) = \varphi(f_{Q,X_1,\dots,X_n})$ is a function of $f_{Q,X_1,\dots,X_n}$, this proves our claim that every $\mathcal{F}_\varphi$ is defined from the argument similarity grades by applying the extension principle. Noticing that no additional assumptions were made in defining $f_{Q,X_1,\dots,X_n}$, which merely composes similarity assessment and the extended $\hat{\hat{Q}}$, this demonstrates that the $\mathcal{F}_\varphi$-QFMs are precisely the QFMs definable in terms of argument similarity and the standard extension principle. The $\mathcal{F}_\varphi$-QFMs hence constitute an interesting class of fuzzification mechanisms. In order to unveil the structure of its well-behaved models, we first make two observations, which relate $\mathcal{F}_\varphi$-QFMs and their apparent superclass of $\mathcal{F}_\psi$-QFMs.

Theorem 124 *All $\mathcal{F}_\varphi$-QFMs are $\mathcal{F}_\psi$-QFMs, i.e. $\mathcal{F}_\varphi = \mathcal{F}_\psi$, provided that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined in dependence on $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ by*

$$\psi(A) = \varphi(f_A)\,, \tag{87}$$

for all $A \in \mathbb{A}$.
(Proof: B.33, p.371+)

Conversely,

Theorem 125 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-5). Then $\mathcal{F}_\psi = \mathcal{F}_\varphi$, where $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ is defined by*

$$\varphi(f) = \psi(A_f)\,, \tag{88}$$

for all $f \in \mathbb{X}$, and

$$A_f(z) = \begin{cases} [0, f(z)] & : \quad f(z) \le \frac{1}{2} \\ [0, 1 - f(z)] \cup \{f(z)\} & : \quad f(z) > \frac{1}{2} \end{cases} \tag{89}$$

for all $z \in \mathbf{I}$.
(Proof: B.34, p.371+)

We shall now impose a number of conditions on admissable choices of φ. Let us first define a preorder on $\mathbb{X}$, again needed to express a monotonicity condition.

Definition 97 *For all $f, f' \in \mathbb{X}$, we write $f \sqsubseteq f'$ if and only if the following conditions are satisfied for f, f'.*

a. for all $z \in \mathbf{I}$, $\sup\{f'(z') : z' \ge z\} \ge f(z)$;

b. for all $z' \in \mathbf{I}$, $\sup\{f(z) : z \le z'\} \ge f'(z')$.

We can now state the conditions that must be obeyed by φ in order to make $\mathcal{F}_\varphi$ a DFS.

Definition 98 *Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given. The conditions (φ-1)–(φ-5) are defined as follows. For all $f, f' \in \mathbb{X}$,*

If $f^{-1}((0,1]) = \{z_+\}$ and $f(z_+) = 1$, then then $\varphi(f) = z_+$. (φ-1)

If $f'(z) = f(1-z)$ for all $z \in \mathbf{I}$, then $\varphi(f') = 1 - \varphi(f)$. ($\varphi$-2)

*If $f^{-1}((0,1]) \subseteq \{0,1\}$ and $f(1) \ge \frac{1}{2}$,
then $\varphi(f) = 1 - f(0)$.* (φ-3)

If $f \sqsubseteq f'$, then $\varphi(f) \le \varphi(f')$. ($\varphi$-4)

If $f'(z) = \min(f(z), \frac{1}{2})$ for all $z \in \mathbf{I}$, then $\varphi(f') = \varphi(f)$. (φ-5)

The proof that these conditions describe precisely the intended class of models, is greatly facilitated if we notice the close relationship between the 'φ-conditions' and corresponding 'ψ-conditions'.

Theorem 126 *Let* $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *be given and suppose that* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *is defined by* (87). *Then*

a. φ *satisfies* (φ-1) *if and only if* ψ *satisfies* (ψ-1)*;*

b. φ *satisfies* (φ-2) *if and only if* ψ *satisfies* (ψ-2)*;*

c. φ *satisfies* (φ-3) *if and only if* ψ *satisfies* (ψ-3)*;*

d. φ *satisfies* (φ-4) *if and only if* ψ *satisfies* (ψ-4)*;*

e. φ *satisfies* (φ-5) *if and only if* ψ *satisfies* (ψ-5)*.*

(Proof: B.35, p.372+)

The following theorems are then straightforward from the previous results on $\mathcal{F}_\psi$-QFMs:

Theorem 127 *If* $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *satisfies* (φ-1)–(φ-5)*, then* $\mathcal{F}_\varphi$ *is a standard DFS.*
(Proof: B.36, p.388+)

Theorem 128 *Consider* $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$*. If* $\mathcal{F}_\varphi$ *is a DFS, then* φ *satisfies* (φ-1)–(φ-5)*.*
(Proof: B.37, p.388+)

Theorem 129 *The conditions* (φ-1)–(φ-5) *are independent.*
(Proof: B.38, p.388+)

In [7, pp. 66-78], I have made a first attempt to define DFSes in terms of the extension principle. The construction of these models was motivated by the fuzzification mechanism proposed by Gaines [6] as a 'foundation of fuzzy reasoning'. This basic mechanism was then fitted to the purpose of defining DFSes. Because the resulting approach also relies on the extension principle, but utilizes a different notion of argument compatibility, the question arises how this 'Gainesian approach' relates to the $\mathcal{F}_\varphi$-QFMs defined in terms of the extension principle. In order to answer this question, I recall some concepts needed to define the new models.
First we define the compatibility $\theta(x, y)$ of a gradual truth value $x \in \mathbf{I}$ to a crisp truth value $y \in \mathbf{2} = \{0, 1\}$.

Definition 99 $\theta : \mathbf{I} \times \mathbf{2} \longrightarrow \mathbf{I}$ *is defined by*

$$\theta(x, y) = \begin{cases} 2x & : \quad x \leq \frac{1}{2}, y = 1 \\ 2 - 2x & : \quad x \geq \frac{1}{2}, y = 0 \\ 1 & : \quad \text{else} \end{cases}$$

for all $x \in \mathbf{I}$*,* $y \in \{0, 1\}$*.*

Hence a gradual truth value $x \leq \frac{1}{2}$ is considered fully compatible with 'false' ($y = 0$), but only gradually compatible with 'true' ($y = 1$), and a gradual truth value $x' \geq \frac{1}{2}$ is considered fully compatible with 'true' ($y = 1$), but only gradually compatible with 'false' ($y = 0$). θ can be applied to compare membership grades $\mu_X(e)$ ($X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E) with 'crisp' membership values $\chi_Y(e)$ (i.e. "Is $e \in Y$?", $Y \in \mathcal{P}(E)$ crisp), where $e \in E$ is some element of the universe. This suggests the following definition of the compatibility $\Theta(X, Y)$ of a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ to a crisp subset $Y \in \mathcal{P}(E)$.

Definition 100 *Let E be a nonempty set. The mapping $\Theta = \Theta_E : \widetilde{\mathcal{P}}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by*

$$\Theta(X, Y) = \inf\{\theta(\mu_X(e), \chi_Y(e)) : e \in E\},$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$.

The compatibility $\Theta(X, Y)$ of a fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ to a crisp set $Y \in \mathcal{P}(E)$ is therefore the minimal degree of element-wise compatibility of the membership function of X and the characteristic function of Y. Based on $\Theta(X, Y)$, I now define $\widetilde{Q}_z(X_1, \ldots, X_n)$, the compatibility of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to the gradual truth value $z \in \mathbf{I}$, given a choice $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ of fuzzy argument sets.

Definition 101 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $z \in \mathbf{I}$. The fuzzy quantifier $\widetilde{Q}_z : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$\widetilde{Q}_z(X_1, \ldots, X_n) = \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : Y = (Y_1, \ldots, Y_n) \in Q^{-1}(z)\},$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

In [7, p. 71], I have argued that the fuzzification mechanism proposed by Gaines can naturally be expressed in terms of $\widetilde{Q}_z$. In addition, three examples were developed which illustrate how DFSes can be defined from $(\widetilde{Q}_z(X_1, \ldots, X_n))_{z \in \mathbf{I}}$; these models have subsequently been shown to be $\mathcal{M}_{\mathcal{B}}$-DFSes, however. The next theorem establishes that all such models, which are defined as a function of $(\widetilde{Q}_z(X_1, \ldots, X_n))_{z \in \mathbf{I}}$, are in fact $\mathcal{F}_\varphi$-DFSes:

Theorem 130 *Consider a QFM $\mathcal{F}$. Then the following statements are equivalent:*

a. $\mathcal{F}$ is an $\mathcal{F}_\varphi$-QFM which satisfies (φ-5);

b. $\mathcal{F}$ is a function of the coefficients $\widetilde{Q}_z(X_1, \ldots, X_n)$.

(Proof: B.39, p.405+)

Hence the QFMs defined in terms of $\widetilde{Q}_z$ are exactly the $\mathcal{F}_\varphi$-QFMs which satisfy (φ-5). We conclude from Th-128 that the 'Gainesian' DFSes defined in terms of $\widetilde{Q}_z$ coincide with the models defined in terms of the extension principle, i.e. with the $\mathcal{F}_\varphi$-DFSes.

To sum up, this chapter has introduced a different construction of QFMs and developed the corresponding theory, in order to span a new class of models which is interesting for theoretical investigation because of its motivation from independent considerations. This departure from the three-valued cut scheme was necessary because this scheme has now been fully exploited by the introduction of $\mathcal{F}_\Omega$-QFMs. The models $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ defined in a previous publication on DFS theory [7] represent an earlier effort to accomplish the intended departure, which was inspired by the fuzzification mechanism proposed by Gaines [6]. These models, though, were subsequently shown to be $\mathcal{M}_\mathcal{B}$-DFSes, and no systematic attempt was made to extract the mechanism underlying these models and to develop a general class of models. In principle, the 'Gainesian' fuzzification mechanism is a good point of departure, due to its foundation in the extension principle of fuzzy set theory. However, the assumed compatibility measure (of a gradual to a crisp truth value; of a fuzzy subset to a crisp set) was considered somewhat awkward, and raised some concerns that the required definitions and theorems would become more complicated than necessary, to capture the target class of standard models. Consequently, I started by defining a simpler measure which quantizes the similarity of fuzzy subsets to given crisp sets $\Xi_Y(X)$, and corresponding tuples of arguments $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n)$. It was then necessary to introduce the set of similarity grades $D_{X_1,\ldots,X_n}$ that are generated from a choice of fuzzy subsets $X_1,\ldots,X_n$ under the similarity measure, and to characterize its range of possible values, $\mathbb{D}$. After that, the key construction was introduced, which to each potential quantification result assigns the set of similarity grades $A_{Q,X_1,\ldots,X_n}(z)$, which are generated by those choices of crisp $Y_1,\ldots,Y_n$ with $Q(Y_1,\ldots,Y_n) = z$. After characterising the range $\mathbb{A}$ of possible $A_{Q,X_1,\ldots,X_n}$, the class of QFMs definable in terms of argument similarity was introduced in the apparent way, based on aggregation mappings $\psi : \mathbb{A} \longrightarrow \mathbf{I}$. In order to express properties of the mappings ψ that are of relevance to the resulting QFMs $\mathcal{F}_\psi$, the required concepts were then developed, and subsequently applied to analyse the precise conditions on ψ under which the resulting QFM $\mathcal{F}_\psi$ becomes a DFS. The proposed system of conditions (ψ-1)–(ψ-5) was shown to be necessary and sufficient for $\mathcal{F}_\psi$ to be a DFS, and all $\mathcal{F}_\psi$-DFSes were proven to be standard models. In addition the independence of the criteria was established. Next I turned to the issue of relating the new class of $\mathcal{F}_\psi$-DFSes to the known class of $\mathcal{F}_\Omega$/$\mathcal{F}_\omega$-DFSes. It came as a surprise that every $\mathcal{F}_\psi$-DFS is in fact an $\mathcal{F}_\omega$-DFS (and vice versa), i.e. the class of $\mathcal{F}_\psi$-DFSes coincides with the class of $\mathcal{F}_\omega$-DFSes. Noticing that the two classes of models arose from constructions which are conceptually very different and motivated independently, this exciting finding confirms that the $\mathcal{F}_\omega$-DFSes (or synonymously, $\mathcal{F}_\psi$-DFSes) form a natural class of standard models of fuzzy quantification, that might even comprise the full class of standard DFSes. The latter hypothesis calls for the development of analytic tools for a deeper investigation of these models, in order to locate their precise place within the standard models.

The remainder of the chapter was concerned with the class of models defined in terms of the extension principle. To this end, a mapping f_A was derived from each

$A \in \mathbb{A}$. By composing these mappings with $A_{Q,X_1,\dots,X_n}$ I defined the new base construction, that of $f_{Q,X_1,\dots,X_n}$. For each potential quantification result z, $f_{Q,X_1,\dots,X_n}(z)$ expressed the maximum similarity of the fuzzy arguments $X_1, \dots, X_n$ to a choice of crisp arguments $Y_1, \dots, Y_n \in \widetilde{\mathcal{P}}(E)$ subject to the condition that $Q(Y_1, \dots, Y_n) = z$. Next the set $\mathbb{X}$ was introduced and shown to precisely describe the set of those mappings f that occur as $f = f_{Q,X_1,\dots,X_n}$ for a choice of Q and $X_1, \dots, X_n$. Hence $\mathbb{X}$ is the proper domain of aggregation operators $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ which span the new class of $\mathcal{F}_\varphi$-QFMs in the usual way, i.e. $\mathcal{F}_\varphi(Q)(X_1, \dots, X_n) = \varphi(f_{Q,X_1,\dots,X_n})$. I have explained that the resulting fuzzification mechanisms are exactly the QFMs definable in terms of the standard extension principle, which is applied to the similarity grades obtained for the quantifier's arguments. The move from the base construction $A_{Q,X_1,\dots,X_n}$ to the new construction $f_{Q,X_1,\dots,X_n}$ means a great simplification because we now deal with a single scalar $f_{Q,X_1,\dots,X_n}(z)$ in the unit range, rather than sets of such scalars $A_{Q,X_1,\dots,X_n}(z)$. It is hence worthwhile studying this subclass of models and elucidating their structure, although no new DFSes are introduced compared to the full class of $\mathcal{F}_\psi$-DFSes. Interestingly, the converse is also true, and in fact no models are *lost* when restricting attention to the subclass of $\mathcal{F}_\varphi$-DFSes. This is because every $\mathcal{F}_\psi$-DFS is known to satisfy (ψ-5), which entails that $\psi(A)$ can be computed from f_A, which underlies the definition of $\mathcal{F}_\varphi$-QFMs. It is hence of particular interest to develop conditions that fit this simpler presentation of $\mathcal{F}_\psi$-DFSes, which is offered by $f_{Q,X_1,\dots,X_n}$ and aggregation mappings φ. Due to the close relationship between $A_{Q,X_1,\dots,X_n}$ and the derived $f_{Q,X_1,\dots,X_n}$, the precise conditions on φ which make $\mathcal{F}_\varphi$ a DFS are apparent from the corresponding conditions (ψ-1)–(ψ-5) imposed on ψ. By adapting these conditions, it was easy to obtain a set of necessary and sufficient conditions (φ-1)–(φ-5) imposed on φ, and to prove that these conditions are independent.

Finally I have reviewed the fuzzification mechanism proposed by Gaines [6] and its reformulation into a base construction for QFMs proposed in [7]. In the course of this investigation, it was proven that all of the resulting QFMs are $\mathcal{F}_\varphi$-QFMs and hence definable in terms of argument similarity and the extension principle. Conversely, all 'reasonable' choices of $\mathcal{F}_\varphi$ where φ satisfies at least (φ-5), can be expressed as 'Gainesian' QFMs, and hence be reduced to a mechanism claimed to provide a 'foundation of fuzzy reasoning' [6].

6 Conclusion

In this report, an effort was made to boost the research into standard models of fuzzy quantification. The pivotal objective was to prospect new classes of such models within the DFS theory of fuzzy quantification, i.e. to explore novel constructions of potential models and to characterize the resulting classes and their relevant properties in terms of the exact conditions that must be imposed on the underlying constructions. In order to better understand the obtained classes and the structure of their models, it was hence necessary to develop the full set of theorems for investigating propagation of fuzziness, continuity and other adequacy properties. A related goal was to develop representative examples and to identify boundary cases of models within the classes (e.g. with respect to specificity). In the report, this basic strategy was implemented for two general constructions of fuzzification mechanisms and corresponding classes of models, which are discussed in chapter 4 ($\mathcal{F}_\Omega$-QFMs) and chapter 5 ($\mathcal{F}_\psi$-QFMs), respectively.

In chapter 4, the known construction of $\mathcal{F}_\xi$-QFMs, which form the broadest class of standard models developed in previous work on DFS theory, was extended to the full class of QFMs definable in terms of three-valued cuts of the argument sets, the class of $\mathcal{F}_\Omega$-QFMs. To this end, the underlying mechanism of $\mathcal{F}_\xi$-QFMs was decomposed into two stages, (a) determination of the 'raw' set of results obtained from the three-valued cut, and (b) subsequent computation of upper and lower bounds. By isolating the first step, which simply determines the ambiguity set $S_{Q,X_1,\ldots,X_n}(\gamma)$ obtained from the quantifier and arguments at the given cut-level γ, we then got grip of a construction which captures the full class of QFMs definable in terms of three-valued cuts. Embarking on the strategy outlined above, the structure of $\mathcal{F}_\Omega$-DFSes was exposed by formalizing the necessary and sufficient conditions on the aggregation mapping Ω that make $\mathcal{F}_\Omega$ a DFS. In addition, a number of theorems have been proven which reduce the test whether a given model $\mathcal{F}_\Omega$ fulfills additional adequacy properties (like propagation of fuzziness), to an easy check on the underlying aggregation mapping Ω. Subsequently the apparent subclass of $\mathcal{F}_\omega$-QFMs was introduced, which are based on a coefficient $s_{Q,X_1,\ldots,X_n}$ computed from $S_{Q,X_1,\ldots,X_n}$. Although the construction of $\mathcal{F}_\omega$-QFMs ignores part of the 'raw' data present in $S_{Q,X_1,\ldots,X_n}$, a formal investigation of $\mathcal{F}_\omega$-DFSes confirmed that no models of interest are lost compared to the construction based on Ω, i.e. the classes of $\mathcal{F}_\Omega$- and $\mathcal{F}_\omega$-DFSes coincide. The rationale for putting effort into $\mathcal{F}_\omega$-QFMs is that they connect the models defined in terms of three-valued cuts to the alternative construction of models in terms of the extension principle that was discussed later. In addition, the alternative format usually permits a simpler definition of models, as witnessed by the succint descriptions of the examples. The final investigation of $\mathcal{F}_\Omega$/$\mathcal{F}_\omega$-DFSes with respect to continuity properties also revealed an interesting result, by substantiating that all 'practical' models in the new class in fact belong to the known class of $\mathcal{F}_\xi$-DFSes. Noticing that the construction of this latter class is conceptually much simpler than that of $\mathcal{F}_\Omega$-DFSes, because only bounds on the quantification results in the cut ranges are considered, this justifies in retrospective the introduction and study of $\mathcal{F}_\xi$-QFMs as a separate class.

The subsequent chapter 5 was concerned with the definition and structure of $\mathcal{F}_\psi$-QFMs (fuzzification mechanisms definable in terms of argument similarity) and their subclass of $\mathcal{F}_\varphi$-QFMs (fuzzification mechanisms definable in terms of the extension principle). An investigation into these models was considered auspicious because the underlying construction is conceptually different from the three-valued cut mechanism deployed in the known classes of DFSes and motivated by independent considerations. The study of these models was therefore hoped to draw attention to some general characteristics not idiosyncratic to the three-valued cut construction. In order to develop this class of models, I first introduced a similarity measure on fuzzy arguments and discussed some of its properties. After showing how the novel $\mathcal{F}_\psi$-QFMs can be built from this construction, the 'intended models' of $\mathcal{F}_\psi$-DFSes were then characterised by stating the exact conditions on the aggregation mapping ψ which make $\mathcal{F}_\psi$ a DFS. In turn, the subclass of $\mathcal{F}_\varphi$-QFMs definable in terms of the extension principle was introduced, and it was shown that their simplified construction does not result in any loss of intended models. Hence all $\mathcal{F}_\psi$-DFSes are $\mathcal{F}_\varphi$-DFSes and vice versa. Again, the reformulation of the original class into the modified construction of $\mathcal{F}_\varphi$-DFSes is targeted at permitting simpler descriptions of models. In addition, the anchoring of this class into the extension principle is satisfying from a theoretical position, and acknowledges the foundational role that the extension principle plays to fuzzy set theory. It has also been remarked that the construction of $\mathcal{F}_\varphi$-QFMs embeds the 'Gainesian' approach which extends the fuzzification mechanism first described in [6] into a base construction for models of fuzzy quantification, see [7].

The main result of the report I consider the proof that the models definable in terms of three-valued cuts coincide with those defined in terms of argument similarity and the extension principle. This reveals that the $\mathcal{F}_\Omega$-DFSes/$\mathcal{F}_\omega$-DFSes and $\mathcal{F}_\psi$-DFSes/$\mathcal{F}_\varphi$-DFSes are merely different presentations of one and the same general class of target models. Its distinct constructions elucidate two complementary faces of the identified class of models:

- The presentation of this class in terms of the extension principle is theoretically appealing because of the unique role of the extension principle to the foundation of fuzzy logic. This reduces these DFSes to the fundamental principle underlying fuzzy set theory, and hence provides a theoretical justification for the use of three-valued cuts to model fuzzy quantification.
- Knowing that the considered class can be defined in terms of three-valued cuts is of great practical interest because algorithms that implement quantifiers in the models are easily derived from the cut-based presentation. Three-valued cuts lend themselves to similar procedures as the familiar two-valued cuts because every three-valued cut can be represented by a pair of α-cuts. This renders it possible to compute subresults on the few resulting layers, which are subsequently aggregated into the final interpretation of the quantifier. The utility of this general strategy has been confirmed by the successful implementation of common quantifiers in $\mathcal{M}$, $\mathcal{M}_{CX}$ and $\mathcal{F}_{\mathrm{Ch}}$ described in [10, 12]. No principled difficulty is expected in transferring these techniques to general $\mathcal{F}_\Omega$-DFSes as well.

Noticing the twofold justification of the new class of models from different constructions and motivated by independent considerations, there is evidence that it indeed forms a *natural* class of DFSes, which captures a broad range of standard models of fuzzy quantification. Future research must decide if it even spans the full class of *all* standard models. In order to encourage this investigation, let me briefly draw attention to a promising starting point for relating this class to the full class of standard DFSes, which might be provided by an analysis of upper and lower bounds on quantification results. Research into this topic has been initiated in [9, Chap. 8] and the techniques developed from this analysis soon approved themselves invaluable for deriving interesting results on DFSes. For example, it now became possible to substantiate the hypothesis that all standard models coincide on two-valued quantifiers. I suggest the use of these upper and lower bounds (or of straightforward variants) to introduce pairs of $\mathcal{F}_L, \mathcal{F}_U$ with $\mathcal{F}_L \preceq_c \mathcal{F} \preceq_c \mathcal{F}_U$, for each considered standard DFS $\mathcal{F}$. For example, one could define $\mathcal{F}_L$ and $\mathcal{F}_U$ by

$$\mathcal{F}_L(Q)(X_1, \ldots, X_n) = \mathrm{med}_{\frac{1}{2}}(\widetilde{Q}^L(X_1, \ldots, X_n), \widetilde{Q}^U(X_1, \ldots, X_n))$$

$$\mathcal{F}_U(Q)(X_1, \ldots, X_n) = \begin{cases} \widetilde{Q}^U(X_1, \ldots, X_n) & : \mathcal{F}(Q)(X_1, \ldots, X_n) > \frac{1}{2} \\ \widetilde{Q}^L(X_1, \ldots, X_n) & : \mathcal{F}(Q)(X_1, \ldots, X_n) < \frac{1}{2} \\ \frac{1}{2} & : \mathcal{F}(Q)(X_1, \ldots, X_n) = \frac{1}{2} \end{cases}$$

where $\widetilde{Q}^U$ and $\widetilde{Q}^L$ are the upper and lower bounds, respectively, as defined in [9, Def. 99, p. 70], but alternative definitions of $\mathcal{F}_L$ and $\mathcal{F}_U$ are also conceivable and might prove equally useful. The proposed strategy of resorting to specificity bounds alleviates the need for a direct assessment of fully general standard DFSes, the internal construction of which is not yet known.

Following these lines, and adopting other techniques that have already been developed in DFS theory, it might become feasible to characterise the full class of standard DFSes. Judging from today's knowledge, it is perfectly possible that this full class is indeed exhausted by the models introduced here. A thorough analysis is required to decide this matter and to locate the present class of $\mathcal{F}_\Omega$-DFSes (or equivalently, $\mathcal{F}_\psi$-DFSes) within the total class of models. This endeavour may take some time to develop, but it promises a number of amazing results that will anchor the models used for applications into an iron-clad theoretical foundation.

Appendix

Any proposition which occurs in the main text is called a *theorem*, and any proposition which only occurs in the proofs a *lemma*. Theorems are referred to as Th-n, where n is the number of the theorem, while lemmata are referred to as L-n, where n is the number of the lemma. Equations which are embedded in proofs are referred to as (n), where n is the number of the equation.

A Proof of theorems in chapter 4

A.1 Proof of Theorem 32

Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given.

a. By Def. 31, $(X_i)_0^{\min} = (X_i)_{>\frac{1}{2}}$ and $(X_i)_0^{\max} = (X_i)_{\geq\frac{1}{2}}$. Because $(X_i)_{>\frac{1}{2}} \subseteq (X_i)_{\geq\frac{1}{2}}$, each

$$\mathcal{T}_0(X_i) = \{Y : (X_i)_0^{\min} \subseteq Y \subseteq (X_i)_0^{\max}\} = \{Y : (X_i)_{>\frac{1}{2}} \subseteq Y \subseteq (X_i)_{\geq\frac{1}{2}}\}$$

is nonempty. Hence $\mathcal{T}_0(X_1, \dots, X_n)$ is nonempty as well, which entails that

$$\mathcal{S}_{Q,X_1,\dots,X_n}(0) = \{Q(Y_1, \dots, Y_n) : Y_1, \dots, Y_n \in \mathcal{T}_0(X_1, \dots, X_n)\} \neq \varnothing\,.$$

b. To see that the second claim of the theorem is valid, consider $\gamma, \gamma' \in \mathbf{I}$ with $\gamma \leq \gamma'$. Then

$$X_\gamma^{\min} \supseteq X_{>\frac{1}{2}+\frac{1}{2}\gamma} \supseteq X_{\geq\frac{1}{2}+\frac{1}{2}\gamma'} \supseteq X_{\gamma'}^{\min}\,.$$

Similarly

$$X_\gamma^{\max} \subseteq X_{\geq\frac{1}{2}-\frac{1}{2}\gamma} \subseteq X_{>\frac{1}{2}-\frac{1}{2}\gamma'} \subseteq X_{\gamma'}^{\max}$$

by Def. 29, Def. 30 and Def. 31. Hence for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} \mathcal{T}_\gamma(X_i) &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &\subseteq \{Y : (X_i)_{\gamma'}^{\min} \subseteq Y \subseteq (X_i)_{\gamma'}^{\max}\} = \mathcal{T}_{\gamma'}(X_i) \end{aligned} \tag{90}$$

because $(X_i)_{\gamma'}^{\min} \subseteq (X_i)_\gamma^{\min}$ and $(X_i)_\gamma^{\max} \subseteq (X_i)_{\gamma'}^{\max}$. In turn

$$\begin{aligned} \mathcal{S}_{Q,X_1,\dots,X_n}(\gamma) &= \{Q(Y_1, \dots, Y_n) : Y_1, \dots, Y_n \in \mathcal{T}_\gamma(X_1, \dots, X_n)\} && \text{by Def. 51} \\ &\subseteq \{Q(Y_1, \dots, Y_n) : Y_1, \dots, Y_n \in \mathcal{T}_{\gamma'}(X_1, \dots, X_n)\} && \text{by (90)} \\ &= \mathcal{S}_{Q,X_1,\dots,X_n}(\gamma')\,. && \text{by Def. 51} \end{aligned}$$

A.2 Proof of Theorem 33

We first observe that for $\gamma = 0$,

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing \tag{91}$$

$$X_0^{\max} = X_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1,0)\}\,, \tag{92}$$

this is apparent from (33) and Def. 31. Similarly for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq \frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{93}$$

$$X_\gamma^{\max} = X_{> \frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))\,. \tag{94}$$

In order to prove the theorem, I first show that

$$S(\gamma) \subseteq S_{Q,X}(\gamma)\,. \tag{95}$$

Hence let us consider a choice of $z \in S(\gamma)$. In the case that $\gamma = 0$, we clearly have $s(z) = 0$ by Def. 53 and hence $z \in S(s(z))$. Now we consider $Y = \{(0, z), (1, 0)\} \in \mathcal{T}_0(X)$, which has $\inf Y' = \inf\{z\} = z$ and $Y'' = \{0\}$, see (28) and (29). Hence $Q(Y) = Q_z(Y'') = Q_z(\{0\})$. Because $z \in S(s(z)) = S(0)$, equation (31) applies, i.e. $Q(Y) = Q_z(\{0\}) = z$.
Next we consider the case that $\gamma > 0$. We can then choose

$$Y = \{(0, z)\} \cup (\{1\} \times [0, \gamma))\,.$$

For this choice of $Y \in \mathcal{T}_\gamma(X)$, we obtain $\inf Y' = \inf\{z\} = z$ and $Y'' = [0, \gamma)$ by (28) and (29), i.e. $\sup Y'' = \gamma$. Hence $Q(Y) = Q_z([0, \gamma)) = z$ by (27) and (30), (31) because $z \in S(\gamma)$ by assumption.
It remains to be shown that $S_{Q,X}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. Let us first consider the case that $\gamma = 0$ and let $Y \in \mathcal{T}_0(X)$ be given. We abbreviate $z = \inf Y' \in \mathbf{I}$. It is apparent from (91) and (92) that we either have $Y'' = \varnothing$ or $Y'' = \{0\}$. In any case, $\sup Y'' = 0$. If $z \in S(\gamma) = S(0)$, then $0 = s(z)$ by Def. 53 and hence $z \in S(s(z))$. We then obtain from (31) that $Q(Y) = Q_z(Y'') = z \in S(0)$, as desired. If $z \notin S(\gamma) = S(0)$, then either $s(0) \geq 0$ and $z \notin S(s(z))$, i.e. (30) applies, or $s(0) > 0$ and $z \in S(s(z))$, i.e. (31) applies. In any case, we obtain that $Q(Y) = Q_z(Y'') = z_0$, and hence $Q(Y) \in S(0)$ by (32).
Finally in the case that $\gamma > 0$ we consider $Y \in \mathcal{T}_\gamma(X)$. Again we abbreviate $z = \inf Y' \in \mathbf{I}$. We also notice that by (93) and (94), $0 \leq \sup Y'' \leq \gamma$. If $z \in S(s(z))$ and $s(z) \leq \sup Y''$, we hence have $Q(Y) = Q_z(Y'') = z \in S(s(z)) \subseteq S(\sup Y'') \subseteq S(\gamma)$. Similarly if $z \notin S(s(z))$ and $s(z) < \sup Y''$, we obtain $Q(Y) = Q_z(Y'') = z \in S(\sup Y'') \subseteq S(\gamma)$ which is apparent from Def. 53 and $\gamma \geq \sup Y'' \geq s(z)$. Hence there are two cases left to prove. If $z \in S(s(z))$ and $\sup Y'' < s(z)$, then $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma)$, see (31) and (32). In the case that $z \notin S(s(z))$ and $\sup Y'' \leq s(z)$, we have $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma)$ by (30) and (32). This finishes the proof that $S_{Q,X}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. Combining this with (95), we obtain the desired $S_{Q,X} = S$.

A.3 Proof of Theorem 34

Lemma 1 *If $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-1), then*

$$\mathcal{U}(\mathcal{F}_\Omega(Q)) = Q$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Note. In particular, $\mathcal{F}_\Omega$ satisfies (Z-1), which weakens the lemma to the case that $n \leq 1$.

Proof Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-1). Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of *crisp* arguments $X_1, \ldots, X_n \in \mathcal{P}(E)$. We have to show that $\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n)$. To this end, we first observe that because the X_i are crisp sets, it holds that $(X_i)^{\min}_\gamma = (X_i)^{\max}_\gamma = X_i$ for all $i = 1, \ldots, n$ and $\gamma \in \mathbf{I}$, see Def. 31. Hence

$$\mathcal{T}_\gamma(X_i) = \{X_i\} \tag{96}$$

for $i = 1, \ldots, n$ and $\gamma \in \mathbf{I}$. In turn,

$$\begin{aligned} S_{Q,X_1,\ldots,X_n}(\gamma) &= \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\} && \text{by Def. 51} \\ &= \{Q(Y_1, \ldots, Y_n) : Y_1 \in \{X_1\}, \ldots, Y_n \in \{X_n\}\} && \text{by (96)} \\ &= \{Q(X_1, \ldots, X_n)\}\,, \end{aligned}$$

i.e.

$$S_{Q,X_1,\ldots,X_n}(\gamma) = \{Q(X_1, \ldots, X_n)\} \tag{97}$$

for all $\gamma \in \mathbf{I}$. From this we obtain the desired

$$\begin{aligned} \mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) &= \Omega(S_{Q,X_1,\ldots,X_n}) && \text{by Def. 55} \\ &= Q(X_1, \ldots, X_n)\,. && \text{by (97), (Ω-1)} \end{aligned}$$

Lemma 2 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$S_{\neg Q,X_1,\ldots,X_n}(\gamma) = \{1 - z : z \in S_{Q,X_1,\ldots,X_n}(\gamma)\}\,,$$

for all $\gamma \in \mathbf{I}$.

Proof Trivial. Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Recalling that $\neg x = 1 - x$ denotes the standard negation, we obtain

$$\begin{aligned} &S_{\neg Q,X_1,\ldots,X_n}(\gamma) \\ &= \{(\neg Q)(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\} && \text{by Def. 51} \\ &= \{1 - Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\} && \text{by Def. 9, } \neg x = 1 - x \\ &= \{1 - z : \\ &\qquad z \in \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\}\} \\ &= \{1 - z : z \in S_{Q,(X_1,\ldots,X_n)}(\gamma)\}\,, && \text{by Def. 51} \end{aligned}$$

for all $\gamma \in \mathbf{I}$.

Lemma 3 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-2) *and* (Ω-3). *Then* $\mathcal{F}_\Omega$ *coincides with* $\mathcal{M}$ *on two-valued quantifiers, i.e. whenever* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ *is a two-valued quantifier and* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ *are fuzzy arguments, then*

$$\mathcal{F}_\Omega(Q)(X_1, \dots, X_n) = \mathcal{M}(Q)(X_1, \dots, X_n)\,.$$

Note. Hence $\mathcal{F}_\Omega$ induces the standard negation $\neg x = 1 - x$, the standard conjunction $x \wedge y = \min(x, y)$, the standard disjunction $x \vee y = \max(x, y)$ and the standard extension principle $\widehat{\mathcal{F}}_\xi = (\hat{\hat{\bullet}})$. This is apparent because all of these are obtained from two-valued quantifiers, and $\mathcal{M}$ is known to be a standard DFS by Th-12.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ be a two-valued quantifier and let $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Then $S_{Q,X_1,\dots,X_n}(\gamma) \in \mathbf{2}$ for all $\gamma \in \mathbf{I}$. We also know from Th-32 that $S_{Q,X_1,\dots,X_n}(0) \neq \varnothing$. Hence there are two cases to consider.

a.: $1 \in S_{Q,X_1,\dots,X_n}(0)$**.** Then $\top_{Q,X_1,\dots,X_n}(\gamma) = \sup S_{Q,X_1,\dots,X_n}(\gamma) = 1$ for all $\gamma \in \mathbf{I}$, i.e. $\top_{Q,X_1,\dots,X_n} = \mathrm{c}_1$. In addition, $\bot_{Q,X_1,\dots,X_n}$ becomes

$$\bot_{Q,X_1,\dots,X_n}(\gamma) = \begin{cases} 1 & : \quad 0 \notin S_{Q,X_1,\dots,X_n}(\gamma) \\ 0 & : \quad 0 \in S_{Q,X_1,\dots,X_n}(\gamma) \end{cases}$$

for all $\gamma \in \mathbf{I}$, i.e. $\widehat{\bot}_{Q,X_1,\dots,X_n}(\mathbf{I}) \subseteq \mathbf{2}$. Because $\mathcal{M}$ is an $\mathcal{M}_\mathcal{B}$-DFS by Th-12, it is also an $\mathcal{F}_\xi$-DFS by Th-22. Hence $\mathcal{M}$ satisfies (X-3) by Th-23, and

$$\begin{aligned} \mathcal{M}(Q)(X_1, \dots, X_n) &= \tfrac{1}{2} + \tfrac{1}{2}(\bot_{Q,X_1,\dots,X_n})_*^0 && \text{by (X-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0) && \text{by (12), Def. 53} \\ &= \mathcal{F}_\Omega(Q)(X_1, \dots, X_n)\,. && \text{by Def. 55, } (\Omega\text{-3}) \end{aligned}$$

b.: $0 \in S_{Q,X_1,\dots,X_n}(0)$**.** Then $1 \in \{1 - z : z \in S_{Q,X_1,\dots,X_n}(0)\} = S_{\neg Q,X_1,\dots,X_n}(0)$ by L-2. We can hence reduce **b.** to the proof of **a.** as follows.

$$\begin{aligned} \mathcal{F}_\Omega(Q)(X_1, \dots, X_n) &= \Omega(S_{Q,X_1,\dots,X_n}) && \text{by Def. 55} \\ &= 1 - \Omega(S_{\neg Q,X_1,\dots,X_n}) && \text{by } (\Omega\text{-2}), \text{ L-2} \\ &= 1 - \mathcal{M}(\neg Q)(X_1, \dots, X_n) && \text{by part } \textbf{a.} \text{ of the lemma} \\ &= 1 - (1 - \mathcal{M}(Q)(X_1, \dots, X_n)) && \text{by Th-12, Th-2} \\ &= \mathcal{M}(Q)(X_1, \dots, X_n)\,. \end{aligned}$$

Lemma 4 *If* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-2) *and* (Ω-3), *then* $\mathcal{F}_\Omega$ *satisfies* (Z-2).

Proof This is now trivial. Let a choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given which satisfies (Ω-2) and (Ω-3). Now consider a base set $E \neq \varnothing$ and an element $e \in E$. By Def. 6,

$\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is two-valued. Hence $\mathcal{F}_\Omega(\pi_e) = \mathcal{M}(\pi_e)$ by L-3. In turn, we conclude from $\mathcal{M}$ being a DFS by Th-12 that $\mathcal{M}(\pi_e) = \widetilde{\pi}_e$. Hence $\mathcal{F}_\Omega(\pi_e) = \widetilde{\pi}_e$, i.e. $\mathcal{F}_\Omega$ satisfies (Z-2).

Lemma 5 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$. Then for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{S}_{Q\neg,X_1,\ldots,X_n} = \mathcal{S}_{Q,X_1,\ldots,X_{n-1},\neg X_n} ,$$

where $\neg X_n \in \widetilde{\mathcal{P}}(E)$ is the standard fuzzy complement $\mu_{X_n}(e) = 1 - \mu_{X_n}(e)$, for all $e \in E$.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given ($n > 0$) and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We already know from the proof of [7, L-22, p.127] ($\gamma > 0$) and [9, L-30, p.110] ($\gamma = 0$) that

$$\mathcal{T}_\gamma(\neg X_n) = \{\neg Y : Y \in \mathcal{T}_\gamma(X_n)\} , \tag{98}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
&\mathcal{S}_{Q\neg,X_1,\ldots,X_n}(\gamma) \\
&= \{Q(Y_1, \ldots, Y_{n-1}, \neg Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 51, Def. 10} \\
&= \{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), \\
&\qquad Y_n \in \mathcal{T}_\gamma(\neg X_n)\} && \text{by (98)} \\
&= \mathcal{S}_{Q,X_1,\ldots,X_{n-1},\neg X_n}(\gamma) && \text{by Def. 51}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, as desired.

Lemma 6 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-2) and (Ω-3). Then $\mathcal{F}_\Omega$ satisfies (Z-3).*

Proof Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ with the desired properties (Ω-2) and (Ω-3) be given. We know from L-3 that $\mathcal{F}_\Omega$ induces the standard negation $\neg x = 1 - x$. Now consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
&\mathcal{F}_\Omega(Q\square)(X_1, \ldots, X_n) \\
&= \Omega(\mathcal{S}_{Q\square,X_1,\ldots,X_n}) && \text{by Def. 55} \\
&= \Omega(\mathcal{S}_{\neg Q\neg,X_1,\ldots,X_n}) && \text{by Def. 11} \\
&= 1 - \Omega(\mathcal{S}_{Q\neg,X_1,\ldots,X_n}) && \text{by } (\Omega\text{-2}), \text{L-2} \\
&= 1 - \Omega(\mathcal{S}_{Q,X_1,\ldots,X_{n-1},\neg X_n}) && \text{by L-5} \\
&= \neg\mathcal{F}_\Omega(Q)(X_1, \ldots, X_{n-1}, \neg X_n) , && \text{by Def. 55}
\end{aligned}$$

i.e. $\mathcal{F}_\Omega$ satisfies (Z-3), as desired.

Lemma 7 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-2) and (Ω-3). Then $\mathcal{F}_\Omega$ satisfies (Z-4).*

Proof Consider a choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ for which (Ω-2) and (Ω-3) are valid. We then know from L-3 that $\mathcal{F}_\Omega$ induces the standard disjunction $x \vee y = \max(x, y)$ and the corresponding standard fuzzy union $\cup$.
Now let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and a choice of fuzzy arguments $X_1, \ldots, X_n, X_{n+1} \in \widetilde{\mathcal{P}}(E)$ be given. It has been shown in [11, p. 52, eq. (58)] that

$$\mathcal{T}_\gamma(X_n \cup X_{n+1}) = \{Y_n \cup Y_{n+1} : Y_n \in \mathcal{T}_\gamma(X_n), Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\}\,, \tag{99}$$

for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned}
&\mathcal{S}_{Q\cup, X_1,\ldots,X_{n+1}}(\gamma)\\
&= \{Q\cup(Y_1, \ldots, Y_{n+1}) : (Y_1, \ldots, Y_{n+1}) \in \mathcal{T}_\gamma(X_1, \ldots, X_{n+1})\} && \text{by Def. 51}\\
&= \{Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1}) : (Y_1, \ldots, Y_{n+1}) \in \mathcal{T}_\gamma(X_1, \ldots, X_{n+1})\} && \text{by Def. 12}\\
&= \{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_{n_1} \in \mathcal{T}_\gamma(X_{n-1}),\\
&\qquad Y_n \in \mathcal{T}_\gamma(X_n \cup X_{n+1})\} && \text{by (99)}\\
&= \mathcal{S}_{Q, X_1,\ldots,X_{n_1}, X_n \cup X_{n+1}}(\gamma)
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$\mathcal{S}_{Q\cup, X_1,\ldots,X_{n+1}} = \mathcal{S}_{Q, X_1,\ldots,X_{n_1}, X_n \cup X_{n+1}}\,. \tag{100}$$

In turn, we obtain the desired

$$\begin{aligned}
&\mathcal{F}_\Omega(Q\cup)(X_1, \ldots, X_{n+1})\\
&= \Omega(\mathcal{S}_{Q\cup, X_1,\ldots,X_{n+1}}) && \text{by Def. 55}\\
&= \Omega(\mathcal{S}_{Q, X_1,\ldots,X_{n-1}, X_n \cup X_{n+1}}) && \text{by (100)}\\
&= \mathcal{F}_\Omega(Q)(X_1, \ldots, X_{n-1}, X_n \cup X_{n+1})\,. && \text{by Def. 55}
\end{aligned}$$

Lemma 8 *If $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies* (Ω-5)*, then $\mathcal{F}_\Omega$ satisfies* (Z-5)*.*

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be nonincreasing in the n-th argument and consider $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ with $X_n \subseteq X'_n$.
Let $\gamma \in \mathbf{I}$. I first show that for all $z \in \mathcal{S}_{Q, X_1,\ldots,X_{n-1}, X'_n}(\gamma)$, there exists $z' \geq z$ such that $z' \in \mathcal{S}_{Q, X_1,\ldots,X_n}(\gamma)$. Hence let $z \in \mathcal{S}_{Q, X_1,\ldots,X_{n-1}, X'_n}(\gamma)$, i.e. by Def. 51 there exists a choice of $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_{n-1}, X'_n)$ with

$$z = Q(Y_1, \ldots, Y_n)\,. \tag{101}$$

Because $Y_n \in \mathcal{T}_\gamma(X'_n)$, it holds that ${X'_n}^{\min}_\gamma \subseteq Y_n \subseteq {X'_n}^{\max}_\gamma$. Because $X_n \subseteq X'_n$, we have ${X_n}^{\min}_\gamma \subseteq {X'_n}^{\min}_\gamma \subseteq Y_n$ by Def. 31. In turn, we conclude from ${X_n}^{\min}_\gamma \subseteq {X_n}^{\max}_\gamma$ that

$${X_n}^{\min}_\gamma = {X_n}^{\min}_\gamma \cap {X_n}^{\max}_\gamma \subseteq Y_n \cap {X_n}^{\max}_\gamma\,.$$

Noticing the apparent $Y_n \cap {X_n}_\gamma^{\max} \subseteq {X_n}_\gamma^{\max}$, this proves that $Y_n \cap {X_n}_\gamma^{\max} \in \mathcal{T}_\gamma(X_n)$. Because Q is assumed to be nonincreasing in the n-th argument, we obtain from $Y_n \cap {X_n}_\gamma^{\max} \subseteq Y_n$ that

$$\begin{aligned} z &= Q(Y_1, \ldots, Y_n) && \text{by (101)} \\ &\leq Q(Y_1, \ldots, Y_{n-1}, Y_n \cap {X_n}_\gamma^{\max}) && \text{because } Q \text{ noninc } n\text{-th arg} \\ &= z' \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\,. \end{aligned}$$

Next I prove that for all $z \in \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$, there exists $z' \leq z$ such that $z' \in \mathcal{S}_{Q,X_1,\ldots,X_{n-1},X'_n}(\gamma)$. Hence let $z \in \mathcal{S}_{Q,X_1,\ldots,X_n}$. By Def. 51, there exist $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$ with $z = Q(Y_1, \ldots, Y_n)$. In particular, ${X_n}_\gamma^{\min} \subseteq Y_n \subseteq {X_n}_\gamma^{\max}$. By similar reasoning as in the previous case, we can conclude from this and ${X_n}_\gamma^{\min} \subseteq {X'_n}_\gamma^{\min}$ as well as ${X_n}_\gamma^{\max} \subseteq {X'_n}_\gamma^{\max}$ that ${X'_n}_\gamma^{\min} \subseteq Y_n \cup {X'_n}_\gamma^{\min} \subseteq {X'_n}_\gamma^{\max}$, i.e. $Y_n \cup {X'_n}_\gamma^{\min} \in \mathcal{T}_\gamma(X'_n)$. In addition, we clearly have $Y_n \subseteq Y_n \cup {X'_n}_\gamma^{\min}$. Hence $z' = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup {X'_n}_\gamma^{\min}) \leq Q(Y_1, \ldots, Y_n) = z$ and $z' \in \mathcal{S}_{Q,X_1,\ldots,X_{n-1},X'_n}(\gamma)$, as desired.
Combining the first two results yields

$$\mathcal{S}_{Q,X_1,\ldots,X_{n-1},X'_n} \sqsubseteq \mathcal{S}_{Q,X_1,\ldots,X_n}\,. \tag{102}$$

Therefore

$$\begin{aligned} \mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) &= \Omega(\mathcal{S}_{Q,X_1,\ldots,X_n}) && \text{by Def. 55} \\ &\geq \Omega(\mathcal{S}_{Q,X_1,\ldots,X_{n-1},X'_n}) && \text{by (Ω-5) and (102)} \\ &= \mathcal{F}_\Omega(Q)(X_1, \ldots, X_{n-1}, X'_n)\,, \end{aligned}$$

i.e. $\mathcal{F}_\Omega(Q)$ is nonincreasing in the n-th whenever Q is nonincreasing in the n-th argument.

The following lemmata are required for the proof that the conjunction of the 'Ω-conditions' is sufficient for $\mathcal{F}_\Omega$ to satisfy (Z-6). The idea of the proof is the same as in [7, p. 132], [9, p. 116] and [11, p. 53], viz I introduce a modified definition of $\mathcal{S}_{Q,X_1,\ldots,X_n}$ which is apparently compatible with functional application (Z-6). I then show that the original definition produced the same results as the modified definition, thus inheriting its compliance with (Z-6).

Definition 102 *Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. $\mathcal{S}^\blacktriangledown_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$\mathcal{S}^\blacktriangledown_{Q,X_1,\ldots,X_n} = \{Q(Y_1, \ldots, Y_n) : Y \in \mathcal{T}^\blacktriangledown_\gamma(X_1, \ldots, X_n) \tag{103}$$

where

$$\mathcal{T}_\gamma^{\blacktriangledown}(X_1, \ldots, X_n) = \{(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma^{\blacktriangledown}(X_1), \ldots, Y_n \in \mathcal{T}_\gamma^{\blacktriangledown}(X_n)\} \tag{104}$$

$$\mathcal{T}_\gamma^{\blacktriangledown}(X) = \{Y : X_\gamma^{\blacktriangledown\min} \subseteq Y \subseteq X_\gamma^{\blacktriangledown\max}\} \tag{105}$$

$$X_\gamma^{\blacktriangledown\min} = X_{>\frac{1}{2}+\frac{1}{2}\gamma} \tag{106}$$

$$X_\gamma^{\blacktriangledown\max} = X_\gamma^{\max} = \begin{cases} X_{\geq\frac{1}{2}} & : \quad \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \quad \gamma > 0 \end{cases} \tag{107}$$

for all $\gamma \in \mathbf{I}$.

Lemma 9 *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier,* E' *is some non-empty base set,* $f_1, \ldots, f_n : E' \longrightarrow E$ *are mappings and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$*. Then for all* $\gamma \in (0, 1]$*,*

$$S^{\blacktriangledown}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \ldots, X_n}(\gamma) = S^{\blacktriangledown}_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}(\gamma)\,.$$

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $f_1, \ldots, f_n : E' \longrightarrow E$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$ be given and $\gamma \in (0, 1]$. We first recall that by [7, p.134, eq. (*)],

$$\mathcal{T}_\gamma^{\blacktriangledown}(\hat{\hat{f}}_i(X_i)) = \{\widehat{f}_i(Y) : Y \in \mathcal{T}_\gamma^{\blacktriangledown}(X_i)\} \tag{108}$$

for all $i \in \{1, \ldots, n\}$ in the assumed case that $\gamma > 0$. Therefore

$$\begin{aligned}
&S^{\blacktriangledown}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \ldots, X_n}(\gamma) \\
&= \{(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i)(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma^{\blacktriangledown}(X_1, \ldots, X_n)\} && \text{by Def. 51} \\
&= \{Q(f_1(Y_1), \ldots, f_n(Y_n)) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma^{\blacktriangledown}(X_1, \ldots, X_n)\} && \text{by (4)} \\
&= \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}^{\blacktriangledown}_{f_1(X_1), \ldots, f_n(X_n)}(Q)\gamma\} && \text{by (108)} \\
&= S^{\blacktriangledown}_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}(\gamma)\,.
\end{aligned}$$

Lemma 10 *For every semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,*

$$S^{\blacktriangledown}_{Q, X_1, \ldots, X_n} \in \mathbb{K}\,.$$

Proof Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We first observe that

$$X_0^{\blacktriangledown\min} = X_{>\frac{1}{2}} \subseteq X_{\geq\frac{1}{2}} = X_0^{\blacktriangledown\max}\,,$$

i.e. $X_0^{\blacktriangledown\min} \in \mathcal{T}_0^{\blacktriangledown}(X)$ for all $X = X_i$, $i = 1, \ldots, n$. Therefore

$$Q({X_1}_0^{\blacktriangledown\min}, \ldots, {X_n}_0^{\blacktriangledown\min}) \in \mathcal{S}_{Q,X_1,\ldots,X_n}(0)\,,$$

i.e. $\mathcal{S}_{Q,X_1,\ldots,X_n}(0) \neq \varnothing$.
Now consider $\gamma, \gamma' \in \mathbf{I}$ with $\gamma < \gamma'$ (the case $\gamma = \gamma'$ is trivial). For all $X = X_i$, $i = 1, \ldots, n$, we obtain from (106) and (107) and $\gamma < \gamma'$ that

$$X_{\gamma'}^{\blacktriangledown\min} = X_{>\frac{1}{2}+\frac{1}{2}\gamma'} \subseteq X_{>\frac{1}{2}+\frac{1}{2}\gamma} = X_{\gamma}^{\blacktriangledown\min}$$

and

$$X_{\gamma}^{\blacktriangledown\max} \subseteq X_{\geq\frac{1}{2}-\frac{1}{2}\gamma} \subseteq X_{>\frac{1}{2}-\frac{1}{2}\gamma'} \subseteq X_{\gamma'}^{\blacktriangledown\max}\,.$$

Hence by (105),

$$\mathcal{T}_{\gamma}^{\blacktriangledown}(X_1, \ldots, X_n) \subseteq \mathcal{T}_{\gamma'}^{\blacktriangledown}(X_1, \ldots, X_n) \tag{109}$$

and in turn,

$$\begin{aligned}
\mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown}(\gamma) &= \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma}^{\blacktriangledown}(X_1, \ldots, X_n)\} && \text{by (103)}\\
&\subseteq \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma'}^{\blacktriangledown}(X_1, \ldots, X_n)\} && \text{by (109)}\\
&= \mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown}(\gamma')\,. && \text{by (103)}
\end{aligned}$$

We conclude from Def. 52 that $\mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown} \in \mathbb{K}$.

Lemma 11 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in \mathbf{I}$,*

$$\begin{aligned}
&\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \subseteq \mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown}(\gamma)\\
&\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma') \supseteq \mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown}(\gamma) \qquad \textit{for all } \gamma' > \gamma\,.
\end{aligned}$$

Proof Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy subsets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We already know from [11, p. 55, eq. (69)] that for all $\gamma \in \mathbf{I}$ and $X = X_i$, $i \in \{1, \ldots, n\}$,

$$\mathcal{T}_{\gamma}(X) \subseteq \mathcal{T}_{\gamma}^{\blacktriangledown}(X)\,. \tag{110}$$

Therefore

$$\begin{aligned}
\mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) &= \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma}(X_1, \ldots, X_n)\} && \text{by Def. 51}\\
&\subseteq \{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma}^{\blacktriangledown}(X_1, \ldots, X_n)\} && \text{by (110)}\\
&= \mathcal{S}_{Q,X_1,\ldots,X_n}^{\blacktriangledown}(\gamma)\,. && \text{by (103)}
\end{aligned}$$

This proves the first claim of the lemma. As concerns the second claim, let $\gamma, \gamma' \in \mathbf{I}$ with $\gamma' > \gamma$. In this case, we recall [11, p. 55, eq. (70)] which states that

$$\mathcal{T}_{\gamma'}(X) \supseteq \mathcal{T}_{\gamma}^{\blacktriangledown}(X) \tag{111}$$

for all $X = X_i$, $i \in \{1, \ldots, n\}$. Therefore

$$\begin{aligned} S_{Q,X_1,\ldots,X_n}(\gamma') &= \{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}_{\gamma'}(X_1,\ldots,X_n)\} && \text{by Def. 51} \\ &\supseteq \{Q(Y_1,\ldots,Y_n) : (Y_1,\ldots,Y_n) \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1,\ldots,X_n)\} && \text{by (111)} \\ &= S^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma)\,, && \text{by (103)} \end{aligned}$$

as desired.

Lemma 12 *For all $S \in \mathbb{K}$, $S^{\sharp\sharp} = S^{\sharp}$.*

Proof Clearly $S^{\sharp\sharp}(1) = \mathbf{I} = S^{\sharp}(1)$ by Def. 56. In the remaining case that $\gamma < 1$,

$$S^{\sharp\sharp}(\gamma) = \bigcap_{\gamma' > \gamma} \bigcap_{\gamma'' > \gamma'} S(\gamma'') = \bigcap_{\gamma'' > \gamma} S(\gamma'') = S^{\sharp}(\gamma)\,.$$

Lemma 13 *For all $S \in \mathbb{K}$ and all $\gamma \in \mathbf{I}$, $S^{\flat}(\gamma) \subseteq S(\gamma) \subseteq S^{\sharp}(\gamma)$.*

Proof In the case that $\gamma = 0$,

$$\begin{aligned} S^{\flat}(0) &= S(0) && \text{by Def. 56} \\ &\subseteq \bigcap_{\gamma' > 0} S(\gamma') && \text{by Def. 52, } S(\gamma') \supseteq S(0) \\ &= S^{\sharp}(0)\,. && \text{by Def. 56} \end{aligned}$$

In the case that $\gamma \in (0, 1)$,

$$\begin{aligned} S^{\flat}(\gamma) &= \bigcup_{\gamma' < \gamma} S(\gamma') && \text{by Def. 56} \\ &\subseteq S(\gamma) && \text{because } S(\gamma') \subseteq S(\gamma) \text{ for } \gamma' < \gamma \text{ by Def. 52} \\ &\subseteq \bigcap_{\gamma' > \gamma} S(\gamma') && \text{because } S(\gamma) \subseteq S(\gamma') \text{ for } \gamma' > \gamma \text{ by Def. 52} \\ &= S^{\sharp}(\gamma)\,. \text{by Def. 56} \end{aligned}$$

Finally in the case that $\gamma = 1$,

$$\begin{aligned} S^{\flat}(1) &= \bigcup_{\gamma' < 1} S(\gamma') && \text{by Def. 56} \\ &\subseteq S(1) && \text{because } S(\gamma') \subseteq S(\gamma) \text{ for } \gamma' < \gamma \text{ by Def. 52} \\ &\subseteq \mathbf{I}\,. && \text{by Def. 52} \end{aligned}$$

Lemma 14 *For all $S, S' \in \mathbb{K}$, if $S(\gamma) \subseteq S'(\gamma)$ for all $\gamma \in \mathbf{I}$, then $S^{\sharp}(\gamma) \subseteq S'^{\sharp}(\gamma)$ for all $\gamma \in \mathbf{I}$.*

Proof Let $S, S' \in \mathbb{K}$ be given such that

$$S(\gamma) \subseteq S'(\gamma) \tag{112}$$

for all $\gamma \in \mathbf{I}$. The claim of the lemma holds trivially if $\gamma = 1$ where $S^\sharp(1) = \mathbf{I} = S^{\sharp\prime}(1)$ by Def. 56. Hence let $\gamma < 1$. Then

$$\begin{aligned} S^\sharp(\gamma) &= \bigcap_{\gamma'>\gamma} S(\gamma') && \text{by Def. 56} \\ &\subseteq \bigcap_{\gamma'>\gamma} S'(\gamma') && \text{by (112)} \\ &= S'^\sharp(\gamma)\,. && \text{by Def. 56} \end{aligned}$$

Lemma 15 *For all $S \in \mathbb{K}$, $S^{\flat\sharp} = S^\sharp$.*

Proof The case that $\gamma = 1$ is trivial because $S^{\flat\sharp}(1) = \mathbf{I} = S^\sharp(1)$ by Def. 56. The case $\gamma = 0$ is also trivial because $S^\flat(0) = S(0)$ by Def. 56, i.e. $S^{\flat\sharp}(0) = S^\sharp(0)$. Hence let $\gamma \in (0,1)$. We first observe that $S^\flat(\gamma') \subseteq S(\gamma')$ for all $\gamma' \in \mathbf{I}$ by L-13. Hence

$$S^{\flat\sharp}(\gamma) \subseteq S^\sharp(\gamma) \tag{113}$$

by L-14. It remains to be shown that $S^\sharp(\gamma) \subseteq S^{\flat\sharp}(\gamma)$. For all $\gamma' > \gamma$, I abbreviate

$$f(\gamma') = \tfrac{\gamma+\gamma'}{2} \tag{114}$$

Apparently

$$\bigcup_{\gamma''<\gamma'} S(\gamma'') \supseteq S(f(\gamma')) \tag{115}$$

because $f(\gamma') \in (\gamma, \gamma')$, i.e. $f(\gamma') < \gamma'$, and therefore $S(f(\gamma') \subseteq S(\gamma'')$ for $\gamma'' > f(\gamma')$. Hence

$$\begin{aligned} & S^\sharp(\gamma) \\ &= \bigcap_{\gamma'>\gamma} S(\gamma') && \text{by Def. 56} \\ &= \bigcap_{\gamma'\in(\gamma,\frac{\gamma+1}{2}]} S(\gamma') && \text{because } S(\tfrac{\gamma+1}{2}) \subseteq S(\gamma') \text{ for } \gamma' > \tfrac{\gamma+1}{2} \text{ by Def. 52} \\ &= \bigcap_{\gamma'>\gamma} S(f(\gamma')) && \text{by (114)} \\ &\subseteq \bigcap_{\gamma'>\gamma} \bigcup_{\gamma''<\gamma'} S(\gamma'') && \text{by (115)} \\ &= S^{\flat\sharp}(\gamma)\,. && \text{by Def. 56} \end{aligned}$$

Lemma 16 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-4) and consider $S, S' \in \mathbb{K}$ with $S^\flat(\gamma) \subseteq S'(\gamma) \subseteq S^\sharp(\gamma)$ for all $\gamma \in \mathbf{I}$. Then $\Omega(S') = \Omega(S)$.*

Note. In particular, the lemma shows that $\Omega(S^\flat) = \Omega(S) = \Omega(S^\sharp)$.

Proof Let us consider an arbitrary choice of $\gamma \in \mathbf{I}$. We first conclude from $S^\flat(\gamma) \subseteq S'(\gamma)$ and L-14 that

$$S^{\flat\sharp}(\gamma) \subseteq S'^{\sharp}(\gamma)\,.$$

Hence

$$S^\sharp(\gamma) = S^{\flat\sharp}(\gamma) \subseteq S'^{\sharp}(\gamma) \tag{116}$$

by L-15. On the other hand, we deduce from $S'(\gamma) \subseteq S^\sharp(\gamma)$, L-14 and L-12 that

$$S'^{\sharp}(\gamma) \subseteq S^{\sharp\sharp}(\gamma) = S^\sharp(\gamma)\,. \tag{117}$$

Combining (116) and (117), we obtain that $S'^{\sharp}(\gamma) = S^\sharp(\gamma)$. Because $\gamma \in \mathbf{I}$ was arbitrarily chosen, this proves that

$$S'^{\sharp} = S^\sharp\,. \tag{118}$$

Therefore

$$\begin{aligned} \Omega(S) &= \Omega(S^\sharp) && \text{by } (\Omega\text{-}4)\\ &= \Omega(S'^{\sharp}) && \text{by } (118)\\ &= \Omega(S')\,, && \text{by } (\Omega\text{-}4) \end{aligned}$$

as desired.

Lemma 17 *If* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-4) *and* (Ω-5), *then*

$$\Omega(S_{Q,X_1,\ldots,X_n}) = \Omega(S^{\blacktriangledown}_{Q,X_1,\ldots,X_n})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof Let $\gamma \in \mathbf{I}$ be given. We already know from L-11 that $S_{Q,X_1,\ldots,X_n}(\gamma) \subseteq S^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma)$. Hence by L-13,

$$(S_{Q,X_1,\ldots,X_n})^\flat(\gamma) \subseteq S_{Q,X_1,\ldots,X_n}(\gamma) \subseteq S^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma) \tag{119}$$

for all $\gamma \in \mathbf{I}$. I will now prove that

$$S^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma) \subseteq (S_{Q,X_1,\ldots,X_n})^\flat(\gamma) \tag{120}$$

for all $\gamma \in \mathbf{I}$. This is trivial if $\gamma = 1$, because in this case

$$S^{\blacktriangledown}_{S,X_1,\ldots,X_n}(1) \subseteq \mathbf{I} = (S_{Q,X_1,\ldots,X_n})^\sharp(1)$$

by Def. 56. Hence let $\gamma < 1$. In this case, we can utilize that

$$S^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma) \subseteq S_{Q,X_1,\ldots,X_n}(\gamma') \tag{121}$$

for all $\gamma' > \gamma$ by L-11. Therefore

$$\begin{aligned} S^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) &\subseteq \bigcap_{\gamma'>\gamma} S_{Q,X_1,\dots,X_n}(\gamma') && \text{by (121)} \\ &= (S_{Q,X_1,\dots,X_n})^{\flat}(\gamma)\,. && \text{by Def. 56} \end{aligned}$$

Hence (120) is valid. Combining this with (119), we notice that

$$(S_{Q,X_1,\dots,X_n})^{\flat}(\gamma) \subseteq S^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \subseteq (S_{Q,X_1,\dots,X_n})^{\sharp}(\gamma) \tag{122}$$

for all $\gamma \in \mathbf{I}$. From this we obtain the desired $\Omega(S_{Q,X_1,\dots,X_n}) = \Omega(S^{\blacktriangledown}_{Q,X_1,\dots,X_n})$ by applying lemma L-16.

Lemma 18 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-4). *Then* $\Omega(S) = \Omega(S')$ *whenever* $S, S' \in \mathbb{K}$ *coincide for all* $\gamma \in (0,1)$, *i.e. if* $S|_{(0,1)} = S'|_{(0,1)}$.

Proof Assume $S, S' \in \mathbb{K}$ satisfy

$$S(\gamma) = S'(\gamma) \tag{123}$$

for all $\gamma \in (0,1)$, as required by the lemma. Let us now show that $S^{\sharp} = S'^{\sharp}$. This is apparent for $\gamma = 1$, in which case $S^{\sharp}(1) = \mathbf{I} = S'^{\sharp}(1)$ by Def. 56. In the remaining case that $\gamma < 1$, we compute

$$\begin{aligned} S^{\sharp}(\gamma) &= \bigcap_{\gamma'>\gamma} S(\gamma') && \text{by Def. 56} \\ &= \bigcap_{1>\gamma'>\gamma} S(\gamma') && \text{because } S(\gamma') \subseteq S(1) \text{ for } \gamma' < 1 \text{ by Def. 52} \\ &= \cap 1 > \gamma' > \gamma S'(\gamma') && \text{by (123)} \\ &= \bigcap_{\gamma'>\gamma} S'(\gamma') && \text{because } S(\gamma') \subseteq S(1) \text{ for } \gamma' < 1 \text{ by Def. 52} \\ &= S'^{\sharp}(\gamma)\,. && \text{by Def. 56} \end{aligned}$$

Hence indeed $S^{\sharp} = S'^{\sharp}$ and

$$\begin{aligned} \Omega(S) &= \Omega(S^{\sharp}) && \text{by } (\Omega\text{-4}) \\ &= \Omega(S'^{\sharp}) && \text{because } S^{\sharp} = S'^{\sharp} \\ &= \Omega(S')\,. && \text{by } (\Omega\text{-4}) \end{aligned}$$

Lemma 19 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-2), (Ω-3) *and* (Ω-4). *Then* $\mathcal{F}_{\Omega}$ *satisfies* (Z-6).

Proof We first notice that by L-3, $\mathcal{F}_\Omega$ induces the standard extension principle. Now consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, a choice of mappings $f_i : E' \longrightarrow E$, $i = 1, \ldots, n$ where $E' \neq \varnothing$ and fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$. Then

$$\begin{aligned}
&\mathcal{F}_\Omega(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n) && \\
&= \Omega(S_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1, \ldots, X_n}) && \text{by Def. 55} \\
&= \Omega(S^{\blacktriangledown}_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1, \ldots, X_n}) && \text{by L-17} \\
&= \Omega(S^{\blacktriangledown}_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}) && \text{by L-9, L-18} \\
&= \Omega(S_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}) && \text{by L-17} \\
&= \mathcal{F}_\Omega(Q)(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n))\,, && \text{by Def. 55}
\end{aligned}$$

i.e. $\mathcal{F}_\Omega$ satisfies (Z-6).

Proof of Theorem 34

The claim of the theorem that $\mathcal{F}_\Omega$ is a DFS whenever $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-1)–(Ω-5) is now a corollary of L-1, L-4, L-6, L-7, L-8 and L-19. It is then an immediate consequence of L-3 and the fact that $\mathcal{M}$ is a standard DFS by Th-12 that $\mathcal{F}_\Omega$ is a standard DFS also.

A.4 Proof of Theorem 35

Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Z-5) and consider a choice of $S \in \mathbb{K}$. In the following, I show that $\Omega(S) = \Omega(S^\ddagger)$ by proving that $\Omega(S) \leq \Omega(S^\ddagger)$ (part **a.**) and that $\Omega(S) \geq \Omega(S^\ddagger)$ (part **b.**).

a.: $\Omega(S) \leq \Omega(S^\ddagger)$**.**
I define a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I})^2 \longrightarrow \mathbf{I}$ by

$$Q(Y_1, Y_2) = \begin{cases} Q''_{\inf Y'}(Y'') & : \quad Y_2 = \varnothing \\ Q'_{\inf Y'}(Y'') & : \quad Y_2 \neq \varnothing \end{cases} \tag{124}$$

for all $Y_1, Y_2 \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y_1\} \tag{125}$$
$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y_1\} \tag{126}$$

and the semi-fuzzy quantifiers $Q'_z, Q''_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ are defined as follows. We first choose an arbitrary $z_0 \in S(0)$, which is known to exist by Def. 52. Next we consider some $z \in \mathbf{I}$ and $y_s \in \mathbf{I}$. If $z \leq z_0$ and $S(y_s) \cap [0, z) \neq \varnothing$, then we can choose some $\lambda(z, y_s) \in S(y_s) \cap [0, z)$. In particular,

$$\lambda(z, y_s) < z\,. \tag{127}$$

In dependence on the choices of z_0 and of the $\lambda(z, y_s)$'s, we define Q'_z by

$$Q'_z(Y'') = \begin{cases} z & : \; z \in S(y_s) \\ \lambda(z, y_s) & : \; z \notin S(y_s), z \leq z_0, S(y_s) \cap [0, z) \neq \varnothing \\ z_0 & : \; \text{else} \end{cases} \tag{128}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, where I have abbreviated

$$y_s = \sup Y'' . \tag{129}$$

The quantifier Q''_z is defined in dependence on the same choice of z_0 by

$$Q''_z(Y'') = \begin{cases} z & : \; z \in S^{\ddagger}(y_s) \\ z_0 & : \; z \notin S^{\ddagger}(y_s) \end{cases} \tag{130}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$.

In order to relate Q with $\mathcal{F}_\Omega$'s fulfilling (Z-5), I first show that Q is nonincreasing in its second argument. Hence let Y_1 be given. It is obvious from (124) that the only interesting choices for Y_2, Y'_2 are $Y_2 = \varnothing$ and $Y'_2 \neq \varnothing$. We apparently have $\varnothing \subseteq Y'_2$, and it must be shown that $Q(Y_1, \varnothing) \geq Q(Y_1, Y'_2)$. In the following it is convenient to abbreviate $z = \inf Y''$. I discern four cases.

a. $z \in S(y_s)$.
Then $Q(Y_1, Y'_2) = Q'_z(Y'') = z$ by (124) and (128). It is clear from Def. 59 that $S(\gamma) \subseteq S^{\ddagger}(\gamma)$ for all $\gamma \in \mathbf{I}$. Hence $z \in S(y_s)$ implies that $z \in S^{\ddagger}(y_s)$ also. Therefore $Q(Y_1, \varnothing) = Q''_z(Y'') = z$ by (124) and (130). In particular, $Q(Y_1, \varnothing) \geq Q(Y_1, Y'_2)$.

b. $z \notin S(y_s)$, $z \leq z_0$, $S(y_s) \cap [0, z) \neq \varnothing$.
In this case, we have $Q(Y_1, Y'_2) = Q'_z(Y'') = \lambda(z, y_s)$ by (124) and (128). Because $z \leq z_0$, there exists $z'' \geq z$ with $z' \in S(y_s)$; we can choose $z'' = z_0$. In addition, $S(y_s) \cap [0, z)$ entails that there exists $z' \in S(y_s)$ with $z' < z$. Hence by Def. 59, $z \in S^{\ddagger}(y_s)$. We then conclude from (124) and (130) that $Q(Y_1, \varnothing) = Q''_z(Y'') = z$. Recalling (127), this proves that $Q(Y_1, \varnothing) = z > \lambda(z, y_s) = Q(Y_1, Y'_2)$.

c. $z \notin S(y_s)$ and $z > z_0$.
Then $Q(Y_1, Y'_2) = Q'_z(Y'') = z_0$ by (124) and (128). In addition, $Q(Y_1, \varnothing) = Q''_z(Y'') \in \{z_0, z\}$. Because $z > z_0$, this proves that $Q(Y_1, \varnothing) \geq z_0 = Q(Y_1, Y'_2)$.

d. $z \notin S(y_s)$ and $S(y_s) \cap [0, z) = \varnothing$.
Then $Q(Y_1, Y'_2) = Q'_z(Y'') = z_0$ by (124), (128). We notice that in this case, there does not exist a $z' \leq z$ with $z' \in S(y_s)$. Hence by Def. 59, $z \notin S^{\ddagger}(y_s)$. In turn, we obtain from (124) and (130) that $Q(Y_1, \varnothing) = Q''_z(Y'') = z_0$. In particular, $Q(Y_1, \varnothing) \geq Q(Y_1, Y'_2)$, as desired.

This finishes the proof that Q is nonincreasing in the second argument. Let us now investigate the behaviour of $S_{Q,X,\varnothing}$ and S_{Q,X,Y_2}, $Y_2 \neq \varnothing$, for the particular choice of $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ defined by

$$\mu_X(a, y) = \begin{cases} \frac{1}{2} & : \quad a = 0 \\ \frac{1}{2} - \frac{1}{2}y & : \quad a = 1 \end{cases} \tag{131}$$

for all $a \in \mathbf{2}$, $y \in \mathbf{I}$. We notice that X coincides with the fuzzy subset defined by (33). Hence by (91), (92), (93) and (94),

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing \tag{132}$$

$$X_0^{\max} = X_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1, 0)\} \tag{133}$$

and for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{134}$$

$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))\,. \tag{135}$$

Next we prove that $S_{Q,X,\varnothing} = S^\ddagger$. To this end, we first observe that by (124), $Q(Y_1, \varnothing) = Q''_{\inf Y'}(Y'')$ for all $Y_1 \in \mathcal{T}_\gamma(X) \subseteq \mathcal{P}(\mathbf{2} \times \mathbf{I})$. In addition, $Q''_{\inf Y'}(Y'')$ only depends on $z = \inf Y'$ and $y_s = \sup Y''$, see (130). Recalling equations (132)–(135), we know that $y_s \in [0, \gamma]$. For any such choice of y_s, one of the following cases applies.

1. $z \in S^\ddagger(y_s)$. Then by (130), $Q(Y_1, \varnothing) = Q''_z(Y'') = z$. From $z \in S^\ddagger(y_s)$, we conclude that $Q(Y_1, \varnothing) = z \in S^\ddagger(\gamma)$. This is because $S^\ddagger \in \mathbb{K}$; hence $z \in S^\ddagger(y_s)$ and $y_s \leq \gamma$ entails that $z \in S^\ddagger(\gamma)$, cf. Def. 52.

2. $z \notin S^\ddagger(y_s)$. Then $Q(Y_1, \varnothing) = Q''_z(Y'') = z_0 \in S^\ddagger(0)$. Because $\gamma \geq 0$, we again conclude that $Q(Y_1, \varnothing) = z_0 \in S^\ddagger(\gamma)$.

This proves that $S_{Q,X,\varnothing}(\gamma) \subseteq S^\ddagger(\gamma)$ for all $\gamma \in \mathbf{I}$. Let us now consider the converse inequation that $S^\ddagger \subseteq S_{Q,X,\varnothing}$. Hence let $z \in S^\ddagger(\gamma)$. We notice that $Y_1 = \{(0, z)\} \cup (\{(1)\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X)$. We then have $z = \inf Y'$ and $y_s = \sup Y'' = \gamma$. Because of the assumption that $z \in S^\ddagger(\gamma) = S^\ddagger(y_s)$, (124) and (130) result in $Q(Y_1, \varnothing) = Q''_z(Y'') = z$. Hence $S_{Q,X,\varnothing}(\gamma) \subseteq S^\ddagger(\gamma)$ for all $\gamma \in \mathbf{I}$. Combining both inequations, we obtain the desired

$$S_{Q,X,\varnothing} = S^\ddagger\,. \tag{136}$$

Finally we show that $S_{Q,X,Y_2} = S$. Recalling that Y_2 is an arbitrary crisp subset with $Y_2 \neq \varnothing$, we deduce from (124) that $Q(Y_1, \varnothing) = Q'_{\inf Y'}(Y'')$ for all $Y_1 \in \mathcal{T}_\gamma(X) \subseteq \mathcal{P}(\mathbf{2} \times \mathbf{I})$. Again, $Q'_{\inf Y'}(Y'')$ only depends on $z = \inf Y'$ and $y_s = \sup Y''$. This is obvious from (128). We also know from equations (132)–(135) that $y_s \in [0, \gamma]$. For any such choice of y_s, one of the following cases applies.

i. $z \in S(y_s)$.
Then $Q(Y_1, Y_2) = Q'_z(Y'') = z \in S(y_s) \subseteq S(\gamma)$ by (128) and because $y_s \leq \gamma$.

ii. $z \notin S(y_s)$, $z \leq z_0$, $S(y_s) \cap [0, z) \neq \varnothing$.
Then $Q(Y_1, Y_2) = Q'_z(Y'') = \lambda(z, y_s) \in S(y_s) \cap [0, z) \subseteq S(y_s) \subseteq S(\gamma)$ by (128) and by definition of the $\lambda(z, y_s)$'s.

iii. "else".
Then $Q(Y_1, Y_2) = Q'_z(Y'') = z_0 \in S(0) \subseteq S(\gamma)$ by (128).

Hence $S_{Q,X,Y_2}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. As concerns the converse inequation that $S \subseteq S_{Q,X,Y_2}$, we consider a choice of $z \in S(\gamma)$. As we observed above, $Y_1 = \{(0, z)\} \cup (\{(1)\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X)$ with $z = \inf Y'$ and $y_s = \sup Y'' = \gamma$. Because of the assumption that $z \in S(\gamma) = S(y_s)$, we obtain from (124) and (128) that $Q(Y_1, Y_2) = Q'_z(Y'') = z$. Hence $S_{Q,X,Y_2}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. Summarising these results, I have shown that

$$S_{Q,X,Y_2} = S\,. \tag{137}$$

Therefore

$$\begin{aligned}
\Omega(S) &= \Omega(S_{Q,X,Y_2}) && \text{by (137)}\\
&= \mathcal{F}_\Omega(Q)(X, Y_2) && \text{by Def. 55}\\
&\leq \mathcal{F}_\Omega(Q)(X, \varnothing) && \text{by (Z-5)}\\
&= \Omega(S_{Q,X,\varnothing}) && \text{by Def. 55}\\
&= \Omega(S^\ddagger)\,. && \text{by (136)}
\end{aligned}$$

b.: $\Omega(S) \geq \Omega(S^\ddagger)$. The proof of this case can be carried out in complete analogy to that of **a.** I hence only state the required changes. In this case, I define the semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I})^2 \longrightarrow \mathbf{I}$ by

$$Q(Y_1, Y_2) = \begin{cases} Q'_{\inf Y'}(Y'') & : \quad Y_2 = \varnothing \\ Q''_{\inf Y'}(Y'') & : \quad Y_2 \neq \varnothing \end{cases} \tag{138}$$

for all $Y_1, Y_2 \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where again

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y_1\} \tag{139}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y_1\} \tag{140}$$

and the semi-fuzzy quantifiers $Q'_z, Q''_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ are now defined as follows. We again choose an arbitrary $z_0 \in S(0)$, which is known to exist by Def. 52. Next we consider some $z \in \mathbf{I}$ and $y_s \in \mathbf{I}$. If $z \geq z_0$ and $S(y_s) \cap (z, 1] \neq \varnothing$, then we can choose some $\zeta(z, y_s) \in S(y_s) \cap (z, 1]$. In particular,

$$\zeta(z, y_s) > z\,. \tag{141}$$

In dependence on the choices of z_0 and of the $\zeta(z, y_s)$'s, we define Q'_z by

$$Q'_z(Y'') = \begin{cases} z & : \quad z \in S(y_s) \\ \zeta(z, y_s) & : \quad z \notin S(y_s), z \geq z_0, S(y_s) \cap (z, 1] \neq \varnothing \\ z_0 & : \quad \text{else} \end{cases} \tag{142}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, where I abbreviate as above

$$y_s = \sup Y'' \,. \tag{143}$$

The quantifier Q''_z is defined by (130), i.e. exactly as above. It is then apparent from the definition of Q in terms of Q'_z and Q''_z that Q is nonincreasing in its second argument. Based on the very same choice of the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ as in the case of **a.**, one shows that

$$S_{Q,X,\varnothing} = S \tag{144}$$

and

$$S_{Q,X,Y_2} = S^{\ddagger} \tag{145}$$

for an arbitrary crisp set $Y_2 \in \mathcal{P}(E)$, $Y_2 \neq \varnothing$. Based on these results, we then conclude that

$$\begin{aligned} \Omega(S) &= \Omega(S_{Q,X,\varnothing}) && \text{by (144)} \\ &= \mathcal{F}_\Omega(Q)(X, \varnothing) && \text{by Def. 55} \\ &\geq \mathcal{F}_\Omega(Q)(X, Y_2) && \text{by (Z-5)} \\ &= \Omega(S_{Q,X,Y_2}) && \text{by Def. 55} \\ &= \Omega(S^{\ddagger}) \,. && \text{by (145)} \end{aligned}$$

A.5 Proof of Theorem 36

Lemma 20 *If* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *does not satisfy* (Ω-1), *then* $\mathcal{F}_\Omega$ *does not satisfy* (Z-1).

Proof Suppose there exists $S \in \mathbb{K}$ and $a \in \mathbf{I}$ such that $S(\gamma) = \{a\}$ for all $\gamma \in \mathbf{I}$ and

$$\Omega(S) \neq a \,. \tag{146}$$

We consider the nullary quantifier $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ defined by $Q(\varnothing) = a$. We observe that by Def. 31, $\mathcal{T}_\gamma(\varnothing) = \{\varnothing\}$ for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned} S_{Q,\varnothing}(\gamma) &= \{Q(Y) : Y \in \{\varnothing\}\} && \text{by Def. 51} \\ &= \{Q(\varnothing)\} \\ &= a && \text{by definition of } Q \\ &= S(\gamma) \,, \end{aligned}$$

i.e.

$$S_{Q,\varnothing} = S\,. \tag{147}$$

Therefore

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(\varnothing) &= \Omega(S_{Q,\varnothing}) && \text{by Def. 55}\\
&= \Omega(S) && \text{by (147)}\\
&\neq a && \text{by (146)}\\
&= Q(\varnothing)\,,
\end{aligned}$$

i.e. (Z-1) fails, like we intended to show.

Lemma 21 *Consider $S, S' \in \mathbb{K}$ with $1 \in S(0)$, $1 \in S'(0)$, $S(\gamma) \subseteq \{0,1\}$ and $S'(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$. If $\mathcal{F}_\Omega$ satisfies* (Z-5) *and $S \sqsubseteq S'$, then $\Omega(S) \leq \Omega(S')$.*

Proof Let us first make some general observations on $S^* \in \{S, S'\}$. Because $1 \in S^*(0)$, we know from Def. 52 that $1 \in S^*(\gamma)$ for all $\gamma \in \mathbf{I}$. In addition, $0 \in S^*(\gamma)$ for some $\gamma \in \mathbf{I}$ entails that $0 \in S^*(\gamma')$ for all $\gamma' \geq \gamma$. Because $S^*(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$ by assumption, we conclude that $S^*(\gamma)$ has one of the following forms.

$$S^*(\gamma) = \begin{cases} \{1\} & : \quad \gamma < s^*(0) \\ \{0,1\} & : \quad \gamma \geq s^*(0) \end{cases}$$

or

$$S^*(\gamma) = \begin{cases} \{1\} & : \quad \gamma \leq s^*(0) \\ \{0,1\} & : \quad \gamma > s^*(0) \end{cases}$$

where $s^* : \mathbf{I} \longrightarrow \mathbf{I}$ is defined in terms of S^* according to Def. 53. It is then apparent from Def. 57 that in this case, $S \sqsubseteq S'$ entails that

$$S'(\gamma) \subseteq S(\gamma) \tag{148}$$

for all $\gamma \in \mathbf{I}$.

We now define a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{I})^2 \longrightarrow \mathbf{I}$ by

$$Q(Y_1, Y_2) = \begin{cases} 1 & : \quad \inf Y_1 = 0 \text{ and } 0 \notin Y_1 \\ 1 & : \quad Y_2 = \varnothing \text{ and } 0 \notin S'(\inf Y_1) \\ 1 & : \quad Y_2 \neq \varnothing \text{ and } 0 \notin S(\inf Y_1) \\ 0 & : \quad \text{else} \end{cases} \tag{149}$$

for all $Y_1, Y_2 \in \mathcal{P}(\mathbf{I})$. It is then apparent from (148) that Q is nonincreasing in the second argument.

Now consider the following choice of arguments $X_1, X_2, X_2' \in \widetilde{\mathcal{P}}(E)$: $X_2 = \mathbf{I}$, $X_2' = \varnothing$ and

$$\mu_{X_1}(v) = \tfrac{1}{2} + \tfrac{1}{2}v\,,$$

for all $v \in \mathbf{I}$. It is apparent from Def. 31 that

$$\mathcal{T}_\gamma(X_2) = \{\mathbf{I}\} \tag{150}$$

$$\mathcal{T}_\gamma(X_2') = \{\varnothing\} \tag{151}$$

for all $\gamma \in \mathbf{I}$. Considering $\mathcal{T}_\gamma(X_1)$, we obtain for $(X_1)_\gamma^{\min}$ and $(X_1)_\gamma^{\max}$ in the case that $\gamma = 0$,

$$(X_1)_0^{\min} = (X_1)_{>\frac{1}{2}} = (0, 1] \tag{152}$$

$$(X_1)_0^{\max} = (X_1)_{\geq\frac{1}{2}} = \mathbf{I} \tag{153}$$

and in the case that $\gamma > 0$,

$$(X_1)_\gamma^{\min} = (X_1)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = [\gamma, 1] \tag{154}$$

$$(X_1)_\gamma^{\max} = (X_1)_{>\frac{1}{2}-\frac{1}{2}\gamma} = \mathbf{I}\,. \tag{155}$$

In the following, assume a choice of $\gamma \in \mathbf{I}$. We first consider $\mathcal{S}_{Q,X_1,X_2}(\gamma)$. It is apparent from (152), (154) and (155) that $Y_1 = (0, 1] \in \mathcal{T}_\gamma(X_1)$. By (150), $\mathbf{I} = X_2 \in \mathcal{T}_\gamma(X_2)$. Hence

$$Q((0, 1], \mathbf{I}) = 1 \in \mathcal{S}_{Q,X_1,X_2}(\gamma)\,, \tag{156}$$

see (149). We now discern two cases.

- $0 \notin \mathcal{S}(\gamma)$.
  Noticing that $\inf Y_1 \leq \gamma$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$ by (152) and (154), we obtain from Def. 52 that $\mathcal{S}(\inf Y_1) \subseteq \mathcal{S}(\gamma)$ and hence $0 \notin \mathcal{S}(\inf Y_1)$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$. Hence $Q(Y_1, Y_2) = 1$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$ and $Y_2 \in \mathcal{T}_\gamma(X_2)$, i.e. $Y_2 = \mathbf{I}$ by (150) and $\mathcal{S}_{Q,X_1,X_2}(\gamma) = \{1\}$. Because $0 \notin \mathcal{S}(\gamma)$, we also have $\mathcal{S}(\gamma) = \{1\}$, i.e. $\mathcal{S}_{Q,X_1,X_2}(\gamma) = \mathcal{S}(\gamma)$, as desired.
- $0 \in \mathcal{S}(\gamma)$.
  We already know from (156) that $(Y_1, Y_2) \in \mathcal{T}_\gamma(X_1, X_2)$ exists with $Q(Y_1, Y_2) = 1$, i.e. $1 \in \mathcal{S}_{Q,X_1,X_2}(\gamma)$. Recalling (153) and (155), we also know that $Y_1 = [\gamma, 1] \in \mathcal{T}_\gamma(X_1)$. In addition, $Y_2 = \mathbf{I} \in \mathcal{T}_\gamma(X_2)$ by (150). Hence $Q(Y_1, Y_2) = 0 \in \mathcal{S}_{Q,X_1,X_2}(\gamma)$ by (149). We conclude that $\mathcal{S}_{Q,X_1,X_2}(\gamma) = \{0, 1\}$ because Q is two-valued. Considering $\mathcal{S}$, we know that $0 \in \mathcal{S}(\gamma)$ (by assumption of this case) and that $1 \in \mathcal{S}(\gamma)$ because $1 \in \mathcal{S}(0)$. In addition, $\mathcal{S}(\gamma) \subseteq \{0, 1\}$ by assumption of the lemma. Hence $\mathcal{S}(\gamma) = \{0, 1\} = \mathcal{S}_{Q,X_1,X_2}(\gamma)$.

Summarizing, I have shown that

$$\mathcal{S} = \mathcal{S}_{Q,X_1,X_2}\,. \tag{157}$$

Next we consider $\mathcal{S}_{Q,X_1,X_2'}$. We first observe that by similar reasoning based on (152), (154), (155),

$$Q((0, 1], \varnothing) = 1 \in \mathcal{S}_{Q,X_1,X_2'}(\gamma)\,, \tag{158}$$

by (149). Now we proceed as above and again discern the two cases.

- $0 \notin S'(\gamma)$.
Noticing that $\inf Y_1 \leq \gamma$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$ by (152) and (154), we obtain from Def. 52 that $S(\inf Y_1) \subseteq S(\gamma)$ and hence $0 \notin S(\inf Y_1)$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$. Hence $Q(Y_1, Y_2) = 1$ for all $Y_1 \in \mathcal{T}_\gamma(X_1)$ and $Y_2 \in \mathcal{T}_\gamma(X_2')$, i.e. $Y_2 = \varnothing$ by (151) and $S_{Q,X_1,X_2'}(\gamma) = \{1\}$. Because $0 \notin S'(\gamma)$, we also have $S'(\gamma) = \{1\}$, i.e. $S_{Q,X_1,X_2'}(\gamma) = S'(\gamma)$.

- $0 \in S'(\gamma)$.
In this case, we know from (158) that there exist $(Y_1, Y_2) \in \mathcal{T}_\gamma(X_1, X_2')$ with $Q(Y_1, Y_2) = 1$, i.e. $1 \in S_{Q,X_1,X_2'}(\gamma)$. Again recalling (153) and (155), we also know that $Y_1 = [\gamma, 1] \in \mathcal{T}_\gamma(X_1)$. In addition, $Y_2 = \varnothing \in \mathcal{T}_\gamma(X_2')$ by (151). Hence $Q(Y_1, Y_2) = 0 \in S_{Q,X_1,X_2'}(\gamma)$ by (149). We conclude that $S_{Q,X_1,X_2'}(\gamma) = \{0, 1\}$ because Q is two-valued. Considering S', we know that $0 \in S'(\gamma)$ (by assumption of this case) and that $1 \in S'(\gamma)$ because $1 \in S'(0)$. In addition, $S'(\gamma) \subseteq \{0, 1\}$ by assumption of the lemma. Hence $S'(\gamma) = \{0, 1\} = S_{Q,X_1,X_2'}(\gamma)$.

This proves that

$$S' = S_{Q,X_1,X_2'} . \tag{159}$$

Therefore

$$\begin{aligned}
\Omega(S') &= \Omega(S_{Q,X_1,X_2'}) && \text{by (159)} \\
&= \mathcal{F}_\Omega(Q)(X_1, X_2') && \text{by Def. 55} \\
&\geq \mathcal{F}_\Omega(Q)(X_1, X_2) && \text{by (Z-5)} \\
&= \Omega(S_{Q,X_1,X_2}) && \text{by Def. 55} \\
&= \Omega(S) . && \text{by (157)}
\end{aligned}$$

Lemma 22 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{F}_\Omega$ satisfies* (Z-5). *If Ω does not fulfill* (Ω-3), *then $\mathcal{F}_\Omega$ violates* (Z-2).

Proof Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ fails on (Ω-3). Then there exists $S \in \mathbb{K}$ such that $S(\gamma) \subseteq \{0, 1\}$ for all $\gamma \in \mathbf{I}$, $1 \in S(0)$ and

$$\Omega(S) \neq \tfrac{1}{2} + \tfrac{1}{2}s(0) . \tag{160}$$

We shall discern four cases.

a. $s(0) > 0$ **and** $0 \notin S(s(0))$**.** In this case, let $\{*\}$ be an arbitrary singleton set and define $X \in \mathcal{P}(\{*\})$ by

$$\mu_X(*) = \tfrac{1}{2} + \tfrac{1}{2}s(0) . \tag{161}$$

Because $s(0) > 0$, we have

$$X_0^{\min} = X_{>\frac{1}{2}} = \{*\}$$
$$X_0^{\max} = X_{\geq\frac{1}{2}} = \{*\}$$

and for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \begin{cases} \{*\} & : \quad \gamma \leq s(0) \\ \varnothing & : \quad \gamma > s(0) \end{cases}$$
$$X_\gamma^{\max} = X_{>\frac{1}{2}+\frac{1}{2}\gamma} = \{*\}\,.$$

Therefore

$$\begin{aligned} S_{\pi_*,X}(\gamma) &= \begin{cases} \{1\} & : \quad \gamma \leq s(0) \\ \{0,1\} & : \quad \gamma > s(0) \end{cases} \\ &= S(\gamma) \end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$S_{\pi_*,X} = S\,. \tag{162}$$

In turn

$$\begin{aligned} \widetilde{\pi}_* X &= \mu_X(*) && \text{by Def. 7} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0) && \text{by (161)} \\ &\neq \Omega(S) && \text{by (160)} \\ &= \Omega(S_{\pi_*,X}) && \text{bt (162)} \\ &= \mathcal{F}_\Omega(\pi_*)(X)\,, && \text{by Def. 55} \end{aligned}$$

i.e. Ω fails on (Z-2).

b. $s(0) = 0$ **and** $0 \in S(0)$**.**
In this case we define $X \in \widetilde{\mathcal{P}}(\{*\})$ by $\mu_X(*) = \frac{1}{2}$. Then by Def. 31,

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing$$
$$X_0^{\max} = X_{\geq\frac{1}{2}} = \{*\}$$

and for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing$$
$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{*\}\,.$$

Therefore

$$S_{\pi_*,X}(\gamma) = \{0,1\} = S(\gamma)$$

for all $\gamma \in \mathbf{I}$, i.e.

$$S_{\pi_*,X} = S \tag{163}$$

Hence again

$$\begin{aligned}
\widetilde{\pi}_* X &= \mu_X(*) && \text{by Def. 7}\\
&= \tfrac{1}{2} && \text{by choice of } X\\
&\neq \Omega(S) && \text{by (160)}\\
&= \Omega(S_{\pi_*,X}) && \text{bt (163)}\\
&= \mathcal{F}_\Omega(\pi_*)(X)\,, && \text{by Def. 55}
\end{aligned}$$

i.e. $\mathcal{F}_\Omega$ violates (Z-2).
Having shown that proper behaviour of Ω in the cases **a.** and **b.** is necessary for $\mathcal{F}_\Omega$ to satisfy (Z-2), we can now assume without loss of generality that

$$\Omega(S) = \tfrac{1}{2} + \tfrac{1}{2}s(0) \tag{164}$$

whenever $S \in \mathbb{K}$ satisfies $1 \in S(0)$, $S(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$, and it either holds that $s(0) > 0$ and $0 \notin S(s(0))$, or it holds that $s(0) = 0$ and $0 \in S(0)$.
In addition, Ω is known to fulfill the property stated in L-21 because it is supposed to satisfy (Z-5).

c. $s(0) > 0$ **and** $0 \in S(s(0))$**.**
In this case let $\varepsilon \in (0, s)$ and define $S', S'' \in \mathbb{K}$ by

$$S'(\gamma) = \begin{cases} \{1\} & : \quad \gamma \leq s(0) - \varepsilon \\ \{0,1\} & : \quad \gamma > s(0) - \varepsilon \end{cases} \tag{165}$$

$$S''(\gamma) = \begin{cases} \{1\} & : \quad \gamma \leq s(0) + \varepsilon \\ \{0,1\} & : \quad \gamma > s(0) + \varepsilon \end{cases} \tag{166}$$

It is apparent from Def. 57 that $S' \sqsubseteq S \sqsubseteq S''$. We also notice that $s'(0)$ and $s''(0)$, as defined by Def. 53 in terms of S' and S'', resp., are given by $s'(0) = s(0) - \varepsilon$ and $s''(0) = s(0) + \varepsilon$. In addition, S' and S'' apparently satisfy the conditions of case **a.**
Therefore

$$\begin{aligned}
\tfrac{1}{2} + \tfrac{1}{2}s(0) - \tfrac{1}{2}\varepsilon &= \Omega(S') && \text{by (164)}\\
&\leq \Omega(S) && \text{by L-21}\\
&\leq \Omega(S'') && \text{by L-21}\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0) + \tfrac{1}{2}\varepsilon\,.
\end{aligned}$$

Hence

$$\Omega(S) \in [\tfrac{1}{2} + \tfrac{1}{2}s(0) - \tfrac{1}{2}\varepsilon, \tfrac{1}{2} + \tfrac{1}{2}s(0) + \tfrac{1}{2}\varepsilon]\,.$$

$\varepsilon \to 0$ then yields $\Omega(S) = \frac{1}{2} + \frac{1}{2}s(0)$, as desired.

d. $s(0) = 0$ **and** $0 \notin S(0)$**.** In this case, consider $S' \in \mathbb{K}$ defined by

$$S'(\gamma) = \{0, 1\} \tag{167}$$

for all $\gamma \in \mathbf{I}$. By Def. 57, we have $S' \sqsubseteq S$. Hence by L-21, $\Omega(S) \geq \Omega(S') = \frac{1}{2}$. On the other hand, consider $S'' \in \mathbb{K}$ defined by

$$S''(\gamma) = \begin{cases} \{1\} & : \gamma \leq \varepsilon \\ \{0, 1\} & : \gamma > \varepsilon \end{cases} \tag{168}$$

for some $\varepsilon > 0$. Apparently $s''(0) = \varepsilon$ and $S \sqsubseteq S''$. We hence obtain

$$\begin{aligned} \tfrac{1}{2} + \tfrac{1}{2}\varepsilon &= \Omega(S'') && \text{by (164)} \\ a &\geq \Omega(S)\,. && \text{by L-21} \end{aligned}$$

$\varepsilon \to 0$ yields $\Omega(S) \leq \frac{1}{2}$. Combining this with the above inequation $\Omega(S) \geq \frac{1}{2}$ finishes the proof of case **d.**

Lemma 23 *Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given. If $\mathcal{F}_\Omega$ is a DFS, then $\mathcal{F}_\Omega$ induces the standard negation $\widetilde{\mathcal{F}_\Omega}(\neg) = \neg$.*

Proof Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{F}_\Omega$ is a DFS. Then Ω satisfies (Ω-3) by L-22. In the following, we shall abbreviate $\widetilde{\neg} = \widetilde{\mathcal{F}_\Omega}(\neg)$. In addition, let us recall that by Def. 8, $\widetilde{\neg}(x) = Q'(X)$ for all $x \in \mathbf{I}$, where $Q' : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$ is defined by

$$Q'(Y) = \neg\eta^{-1}(Y) \tag{169}$$

for all $Y \in \mathcal{P}(\{1\})$, and $X \in \widetilde{\mathcal{P}}(\{1\})$ is defined by $X = \widetilde{\eta}(x)$, i.e.

$$\mu_X(1) = x\,. \tag{170}$$

Now let $x \in [0, \frac{1}{2})$. By Def. 31,

$$\begin{aligned} X_0^{\min} &= X_{>\frac{1}{2}} = \varnothing \\ X_0^{\max} &= X_{\geq\frac{1}{2}} = \varnothing \end{aligned}$$

and for $\gamma > 0$,

$$\begin{aligned} X_\gamma^{\min} &= X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \\ X_\gamma^{\max} &= X_{>\frac{1}{2}-\frac{1}{2}\gamma} = \begin{cases} \varnothing & : \gamma \leq 1 - 2x \\ \{1\} & : \gamma > 1 - 2x \end{cases} \end{aligned}$$

Hence by Def. 51

$$S_{Q',X}(\gamma) = \begin{cases} \{1\} & : \gamma \leq 1 - 2x \\ \{0, 1\} & : \gamma > 1 - 2x \end{cases} \tag{171}$$

for all $\gamma \in \mathbf{I}$. Because Ω is assumed to satisfy (Ω-3), we conclude that

$$\begin{aligned}
\widetilde{\neg}\, x &= \mathcal{F}_\Omega(Q')(X) && \text{by Def. 8, (170)}\\
&= \Omega(S_{Q',X}) && \text{by Def. 55}\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0) && \text{by } (\Omega\text{-3})\\
&= \tfrac{1}{2} + \tfrac{1}{2}(1-2x) && \text{by (171), Def. 53}\\
&= 1 - x\,.
\end{aligned}$$

This proves that

$$\widetilde{\neg}\, x = 1 - x\,, \tag{172}$$

for all $x \in [0, \frac{1}{2})$. Now let $x \in (\frac{1}{2}, 1]$. By assumption, $\mathcal{F}_\Omega$ is a DFS, i.e. $\widetilde{\neg}$ is a strong negation operator by Th-1. In particular, $\widetilde{\neg}$ is an involutive bijection by Def. 18. Because $\widetilde{\neg}$ is involutive, it holds that $x = \widetilde{\neg}\,\widetilde{\neg}\, x$. On the other hand, $x \in (\frac{1}{2}, 1]$ implies that $1 - x \in [0, \frac{1}{2})$. Hence by (172), $x = 1 - (1 - x) = \widetilde{\neg}(1 - x)$. Combining both equations, we have $\widetilde{\neg}\,\widetilde{\neg}\, x = \widetilde{\neg}(1 - x)$. But $\widetilde{\neg}$ is an injection, i.e. we can cancel the leftmost $\widetilde{\neg}$ to obtain the desired $\widetilde{\neg}\, x = 1 - x$. This proves that $\widetilde{\neg}\, x = 1 - x$ for all $x \in \mathbf{I} \setminus \{\frac{1}{2}\}$. It is then apparent from the fact that $\widetilde{\neg}$ is a bijection that it fulfills $\widetilde{\neg}\, \frac{1}{2} = \frac{1}{2}$, which finishes the proof that $\widetilde{\neg} = \neg$.

Lemma 24 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{F}_\Omega$ induces the standard negation. If Ω does not satisfy* (Ω-2)*, then $\mathcal{F}_\Omega$ does not satisfy* (Z-3)*.*

Proof Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be a given mapping such that $\mathcal{F}_\Omega$ induces the standard negation $\widetilde{\mathcal{F}_\Omega}(\neg) = \neg$, $\neg x = 1x$. Further suppose that Ω violates (Ω-2), i.e. there exists $S \in \mathbb{K}$ such that

$$\Omega(S) \neq 1 - \Omega(S')\,, \tag{173}$$

where $S'(\gamma) = \{1 - z : z \in S(\gamma)\}$ for all $\gamma \in \mathbf{I}$. By Th-33 there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S_{Q,X} = S \tag{174}$$

Hence

$$\begin{aligned}
S_{Q\square,\neg X} &= S_{\neg Q\neg,\neg X} && \text{by Def. 11}\\
&= S'' && \text{by L-2 and L-5}
\end{aligned}$$

where

$$S''(\gamma) = \{1 - z : z \in S_{Q,X}(\gamma)\}$$

for all $\gamma \in \mathbf{I}$. Hence by (174), $S' = S''$, i.e.

$$S_{Q\square,\neg X} = S'\,. \tag{175}$$

Hence

$$\begin{aligned}
\mathcal{F}_\Omega(Q\square)(\neg X) &= \Omega(S_{Q\square,\neg X}) && \text{by Def. 55}\\
&= \Omega(S') && \text{by (175)}\\
&\neq 1 - \Omega(S) && \text{by (173)}\\
&= 1 - \Omega(S_{Q,X}) && \text{by (174)}\\
&= \neg\mathcal{F}_\Omega(Q)(X) && \text{by Def. 55}\\
&= \neg\mathcal{F}_\Omega(Q)(\neg\neg X) && \text{because } \neg\neg X = X\\
&= \mathcal{F}_\Omega(Q)\square(\neg X)\,, && \text{by Def. 11}
\end{aligned}$$

i.e. $\mathcal{F}_\Omega$ violates (Z-3).

Lemma 25 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is given. If* $\mathcal{F}_\Omega$ *satisfies* (Z-5), *then* Ω *satisfies* (Ω-5).

Proof Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given and suppose $\mathcal{F}_\Omega$ satisfies (Z-5). In order to show that Ω satisfies (Ω-5), we consider a choice of $S, S' \in \mathbb{K}$ such that $S \sqsubseteq S'$. It is then apparent from Def. 57 and Def. 59 that

$$S^\ddagger \sqsubseteq S'^\ddagger$$

as well. We define a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I})^2 \longrightarrow \mathbf{I}$ by

$$Q(Y_1, Y_2) = \begin{cases} Q'_{\inf Y'}(Y'') & : \quad Y_2 = \varnothing \\ Q''_{\inf Y'}(Y'') & : \quad Y_2 \neq \varnothing \end{cases} \tag{176}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y_1\} \tag{177}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y_1\}\,. \tag{178}$$

In order to define the semi-fuzzy quantifiers $Q'_z, Q''_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $z \in \mathbf{I}$, we first choose $z_0 \in S(0)$, $z'_0 \in S'(0)$. Because $S \sqsubseteq S'$, we can assume a choice of z_0, z'_0 such that

$$z_0 \leq z'_0\,. \tag{179}$$

Based on z_0 and z'_0, the quantifiers are then defined by

$$Q'_z(Y'') = \begin{cases} z & : \quad z \in S'^\ddagger(y_s) \\ z'_0 & : \quad z \notin S'^\ddagger(y_s) \end{cases} \tag{180}$$

$$Q''_z(Y'') = \begin{cases} z & : \quad z \in S^\ddagger(y_s) \\ z_0 & : \quad z \notin S^\ddagger(y_s) \end{cases} \tag{181}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, where

$$y_s = \sup Y''\,. \tag{182}$$

Let us now prove that Q is nonincreasing in its second argument. It is apparent from (176) that the only critical case is that of $Y_2 = \varnothing$, $Y_2' \neq \varnothing$. Hence let $Y_2' \neq \varnothing \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$ be given and let $Y_1 \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$. We abbreviate $z = \inf Y'$. It is obvious from (176) that

$$Q(Y_1, \varnothing) = Q_z'(Y'')$$
$$Q(Y_1, Y_2') = Q_z''(Y'')\,,$$

and I will repeatedly use these equations in the following. It is now convenient to discern four cases.

1. $z \notin {S'}^{\ddagger}(y_s)$ and $z \notin S^{\ddagger}(y_s)$.
   Then $Q(Y_1, \varnothing) = Q_z'(Y'') = z_0' \geq z_0 = Q_z''(Y'') = Q(Y_1, Y_2')$ by (180), (181) and (179).

2. $z \in {S'}^{\ddagger}(y_s)$ and $z \notin S^{\ddagger}(y_s)$.
   Then $z > v$ for all $v \in S^{\ddagger}(y_s)$ because $S^{\ddagger} \sqsubseteq {S'}^{\ddagger}$. In particular, $z > z_0$. Therefore $Q(Y_1, \varnothing) = Q_z''(Y'') = z > z_0 = Q_z'(Y'') = Q(Y_1, Y_2')$ by (180) and (181).

3. $z \in {S'}^{\ddagger}(y_s)$ and $z \in S^{\ddagger}(y_s)$.
   Then $Q(Y_1, \varnothing) = Q_z''(Y'') = z = Q_z'(Y'') = Q(Y_1, Y_2')$ by (180) and (181). In particular $Q(Y_1, \varnothing) \geq Q(Y_1, Y_2')$.

4. $z \notin {S'}^{\ddagger}(y_s)$ and $z \in S^{\ddagger}(y_s)$.
   Then $z < v$ for all $v \in {S'}^{\ddagger}(y_s)$ because $S^{\ddagger} \sqsubseteq {S'}^{\ddagger}$. In particular $z < z_0'$. Therefore $Q(Y_1, \varnothing) = Q_z''(Y'') = z_0' > z = Q_z'(Y'') = Q(Y_1, Y_2')$ by (180) and (181).

This finishes the proof that Q is nonincreasing in its second argument. Now consider the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ defined by

$$\mu_X(a, y) = \begin{cases} \frac{1}{2} & : \quad a = 0 \\ \frac{1}{2} - \frac{1}{2}y & : \quad a = 1 \end{cases}$$

for all $a \in \mathbf{2}$, $y \in \mathbf{I}$. We notice that this is the same choice of fuzzy set as used in the proof of Th-35, equation (131). In fact, it is now routine work to show that

$$S_{Q,X,\varnothing} = {S'}^{\ddagger} \tag{183}$$
$$S_{Q,X,Y_2'} = S^{\ddagger}\,. \tag{184}$$

(We simply need to recognize that the above cases are analogous to that of computing $Q(X, \varnothing)$ in part **a.** of the proof of Th-35.)

Therefore

$$\begin{aligned}
\Omega(S) &= \Omega(S^\ddagger) && \text{by Th-35} \\
&= \Omega(S_{Q,X,Y_2'}) && \text{by (184)} \\
&= \mathcal{F}_\Omega(Q)(X, Y_2') && \text{by Def. 55} \\
&\le \mathcal{F}_\Omega(Q)(X, \varnothing) && \text{by (Z-5)} \\
&= \Omega(S_{Q,X,\varnothing}) && \text{by Def. 55} \\
&= \Omega({S'}^\ddagger) && \text{by (183)} \\
&= \Omega(S')\,, && \text{by Th-35}
\end{aligned}$$

which finishes the proof of the lemma.

Lemma 26 *Consider a choice of $S \in \mathbb{K}$. Then*

$$S^\sharp(\gamma) \subseteq S(\gamma')$$

for all $\gamma, \gamma' \in \mathbf{I}$ with $\gamma < \gamma'$.

Proof Suppose $S \in \mathbb{K}$ and $\gamma, \gamma' \in \mathbf{I}$ with $\gamma < \gamma'$ are given. In particular $\gamma < 1$. Hence by Def. 56,

$$\begin{aligned}
S^\sharp(\gamma) &= \bigcap_{\gamma'' > \gamma} S(\gamma'') && \text{by Def. 56 and } \gamma < 1 \\
&= (\cap\{S(\gamma'') : \gamma'' > \gamma, \gamma'' \neq \gamma\}) \cap S(\gamma') \\
&\subseteq S(\gamma')\,.
\end{aligned}$$

Lemma 27 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies* (Ω-2) *and* (Ω-3)*. For a given choice of $S \in \mathbb{K}$, we define $S_1 \in \mathbb{K}$ by*

$$S_1(\gamma) = \begin{cases} S^\sharp(0) & : \quad \gamma = 0 \\ S(\gamma) & : \quad \gamma > 0 \end{cases} \tag{185}$$

for all $\gamma \in \mathbf{I}$. If $\mathcal{F}_\Omega$ satisfies (Z-6)*, then*

a. $\Omega(S_1) = \Omega(S^\sharp)$*;*

b. $\Omega(S) = \Omega(S_1)$.

Proof Consider a choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ which satisfies (Ω-2) and (Ω-3). We then know from L-3 that $\mathcal{F}_\Omega$ coincides with $\mathcal{M}$ for all two-valued quantifiers. We also know from Th-12 that $\mathcal{M}$ is a standard DFS. In particular, it induces the standard extension principle. We notice that the induced extension principle depends on two-valued quantifiers only, see Def. 16. Therefore $\mathcal{F}_\Omega$ induces the standard extension principle as well.

Now let us assume that (Z-6) is valid for Ω. We consider a choice of $S \in \mathbb{K}$ and assume that $S_1 \in \mathbb{K}$ is given by (185). Let us now define a mapping $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ by

$$g(c, z_1, z_2) = (c, z_1) \tag{186}$$

for all $c \in \mathbf{2}$, $z_1, z_2 \in \mathbf{I}$. In addition, let us define a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$ by

$$\mu_X(c, z_1, z_2) = \begin{cases} \frac{1}{2} & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2} z_2 & : \quad c = 1, z_2 < z_1 \\ \frac{1}{2} z_2 & : \quad c = 1, z_1 = 0, z_2 < 1 \\ 0 & : \quad \text{else} \end{cases} \tag{187}$$

for all $c \in \mathbf{2}$, $z_1, z_2 \in \mathbf{I}$. I now investigate some cut ranges. For $\gamma = 0$, we obtain from Def. 31 and (187) that

$$X_0^{\min} = X_{>\frac{1}{2}} = \{(1, z_1, z_2) : z_1 > z_2\}$$
$$X_0^{\max} = X_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I} \times \mathbf{I}) \cup \{(1, z_1, z_2) : z_1 > z_2\}\,.$$

Similarly for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{(1, z_1, z_2) : z_1 > z_2 \geq \gamma\}$$
$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I} \times \mathbf{I}) \cup \{(1, z_1, z_2) : z_1 > z_2\} \cup \{(1, 0, z_2) : z_2 > 1 - \gamma\}\,.$$

In turn by Def. 15 and (186),

$$\widehat{g}(X_0^{\min}) = \{1\} \times (0, 1] \tag{188}$$
$$\widehat{g}(X_0^{\max}) = (\{0\} \times \mathbf{I}) \cup (\{1\} \times (0, 1]) \tag{189}$$

and for $\gamma > 0$,

$$\widehat{g}(X_\gamma^{\min}) = \{1\} \times (\gamma, 1] \tag{190}$$
$$\widehat{g}(X_\gamma^{\max}) = (\{0\} \times \mathbf{I}) \cup (\{1\} \times \mathbf{I})\,. \tag{191}$$

In the following, I abbreviate $V = \hat{\hat{g}}(X)$. It is apparent from (3) that

$$\mu_V(c, z_1) = \begin{cases} \frac{1}{2} & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2} z_1 & : \quad c = 1 \end{cases} \tag{192}$$

for all $c \in \mathbf{2}$, $z_1 \in \mathbf{I}$. Hence by Def. 31,

$$V_0^{\min} = V_{>\frac{1}{2}} = \{1\} \times (0, 1] \tag{193}$$
$$V_0^{\max} = V_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times \mathbf{I}) \tag{194}$$

and for $\gamma > 0$,

$$V_\gamma^{\min} = V_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{1\} \times [\gamma, 1] \tag{195}$$
$$V_\gamma^{\max} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times \mathbf{I})\,. \tag{196}$$

We are now prepared to prove both parts of the lemma.

a.: $\Omega(S_1) = \Omega(S^\sharp)$**.**
In order to show this, we define a fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = Q_{\inf Y'}(Y'') \tag{197}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \tag{198}$$
$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} . \tag{199}$$

Based on an an arbitrary choice of $z_0 \in S(0)$, the semi-fuzzy quantifiers $Q_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $z \in \mathbf{I}$, are defined by

$$Q_z(Y'') = \begin{cases} z & : \quad y_\ell = 0, z \in S^\sharp(0) \\ z & : \quad y_\ell > 0, z \in S(y_\ell), y_\ell \in Y'' \\ z & : \quad y_\ell > 0, z \in S^\sharp(y_\ell), y_\ell \notin Y'' \\ z_0 & : \quad \text{else} \end{cases} \tag{200}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, where I have abbreviated

$$y_\ell = \inf Y'' . \tag{201}$$

In order to show that $S_{Q \circ \widehat{g}, X} = S^\sharp$, it is convenient to prove the following subsumption first, $S_{Q \circ \widehat{g}, X}(\gamma) \subseteq S^\sharp(\gamma)$ for all $\gamma \in \mathbf{I}$. In the case that $\gamma = 0$, we have $y_\ell = 0$ for all $Y \in \{\widehat{g}(Z) : Z \in \mathcal{T}_0(X)\}$, see (188), (189). Now consider $z = \inf Y'$. By (200), we have $Q(Y) = Q_z(Y'') = z$ if $z \in S^\sharp(0)$, i.e. $Q(Y) = z \in S^\sharp(0)$. In the remaining case that $z \notin S^\sharp(0)$, we obtain $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S^\sharp(0)$. Now consider $\gamma > 0$. Then $y_\ell \in [0, \gamma]$. Depending on $z = \inf Y'$, one of the following cases applies. If $y_\ell = 0$ and $z \in S^\sharp(0)$, then $Q(Y) = Q_z(Y'') = z \in S^\sharp(0) \subseteq S^\sharp(\gamma)$. If $y_\ell > 0$, $z \in S(y_\ell)$ and $y_\ell \in Y''$, then $Q(Y) = Q_z(Y'') = z \in S(y_\ell) \subseteq S(\gamma) \subseteq S^\sharp(\gamma)$. If $y_\ell > 0$, $z \in S^\sharp(y_\ell)$ and $y_\ell \notin Y''$, then $Q(Y) = Q_z(Y'') = z \in S^\sharp(y_\ell) \subseteq S^\sharp(\gamma)$. In the remaining cases, $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S^\sharp(\gamma)$.
Now we prove the converse subsumption, viz $S^\sharp(\gamma) \subseteq S_{Q \circ \widehat{g}, X}(\gamma)$ for all $\gamma \in \mathbf{I}$. Hence let $\gamma \in \mathbf{I}$ be given and consider $z \in S^\sharp(\gamma)$. It is apparent from (188) and (189) (in the case that $\gamma = 0$) or (190) and (191) (in the case that $\gamma > 0$) that $Y = \{(0, z)\} \cup (\{1\} \times (\gamma, 1]) \in \{\widehat{g}(Z) : Z \in \mathcal{T}_\gamma(X)\}$. We notice that by (198), $\inf Y' = \inf\{z\} = z$ and by (199), $y_\ell = \inf Y'' = \inf(\gamma, 1] = \gamma$. Because $y_\ell = \gamma \notin Y''$, we obtain from (197) and (200) that $Q(Y) = Q_z(Y'') = z$ because $z \in S^\sharp(\gamma) = S^\sharp(y_\ell)$. Summarizing, I have shown that

$$S_{Q \circ \widehat{g}, X} = S^\sharp . \tag{202}$$

Next I will show that $S_{Q, \hat{\hat{g}}(X)} = S_1$. Hence let $\gamma \in \mathbf{I}$. I first prove that $S_{Q, \hat{\hat{g}}(X)}(\gamma) \subseteq S_1(\gamma)$. If $\gamma = 0$, then $y_\ell = 0$ for all $Y \in \mathcal{T}_0(\hat{\hat{g}}(X)) = \mathcal{T}_0(V)$, see (193), (194) and (201). Let us abbreviate $z = \inf Y'$. If $z \in S^\sharp(0) = S_1(0)$, then $Q(Y) = Q_z(Y'') = z \in S_1(0)$ by (197) and (200). If $z \notin S^\sharp(0)$, then $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S^\sharp(0) = S_1(0)$ by (197) and (200). Now we consider the case that $\gamma > 0$. Then $y_\ell \in [0, \gamma]$ by (195) and (196). If $y_\ell < \gamma$, we either have $Q(Y) = Q_z(Y'') = z_0 \in$

$S(0) \subseteq S(\gamma) = S_1(\gamma)$, or $Q(Y) = Q_z(Y'') = z \in S(y_\ell) \subseteq S(\gamma) = S_1(\gamma)$, or $Q(Y) = Q_z(Y'') = z \in S^\sharp(y_\ell) \subseteq S(\gamma) = S_1(\gamma)$ by L-26 because $y_\ell < \gamma$. Finally if $y_\ell = \gamma$, then $y_\ell \in Y''$ by (195) and (196). Hence by (197) and (200), either $Q(Y) = Q_z(Y'') = z$, where $z \in S(\gamma) = S_1(\gamma)$, or $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma) = S_1(\gamma)$. This finishes the proof that $S_{Q,\hat{\hat{g}}(X)}(\gamma) \subseteq S_1(\gamma)$ for all $\gamma \in \mathbf{I}$. to see that the converse subsumption also holds, we consider a choice of $\gamma \in \mathbf{I}$ and $z \in S_0(\gamma)$. We notice that by (193) and (194) (in the case that $\gamma = 0$) or (195) and (196) (in the case that $\gamma > 0$), we can choose $Y = \{(0, z)\} \cup [\gamma, 1] \in \mathcal{T}_\gamma(\hat{\hat{g}}(X)) = \mathcal{T}_\gamma(V)$. For this choice of Y, we clearly obtain $\inf Y' = \inf\{z\} = z$ and $\inf Y'' = \inf[\gamma, 1] = \gamma$. By assumption, it holds that $z \in S_1(\gamma)$. Hence if $\gamma = 0$, then $Q(Y) = Q_z(Y'') = z$ because $z \in S_1(0) = S^\sharp(0)$ by (185). In the remaining case that $\gamma > 0$, we also obtain that $Q(Y) = Q_z(Y'') = z$ because $y_\ell \in Y''$ and $z \in S_1(\gamma) = S(\gamma)$, again by (185). Hence the converse subsumption relationship also holds, and we can summarize these results as

$$S_{Q,\hat{\hat{g}}(X)} = S_1 \,. \tag{203}$$

We conclude that

$$\begin{aligned}
\Omega(S_1) &= \Omega(S_{Q,\hat{\hat{g}}(X)}) && \text{by (203)}\\
&= \mathcal{F}_\Omega(Q)(\hat{\hat{g}}(X)) && \text{by Def. 55}\\
&= \mathcal{F}_\Omega(Q \circ \widehat{g})(X) && \text{by (Z-6)}\\
&= \Omega(S_{Q\circ\widehat{g},X}) && \text{by Def. 55}\\
&= \Omega(S^\sharp)\,, && \text{by (202)}
\end{aligned}$$

which finishes the proof of part **a.**

b.: $\Omega(S) = \Omega(S_1)$**.**
In order to proof the second part of the lemma, replace the definition of the Q_z's in part **a.** with the following modified definition:

$$Q_z(Y'') = \begin{cases} z & : \; y_\ell = 0, y_\ell \in Y'', z \in S^\sharp(0) \\ z & : \; y_\ell = 0, y_\ell \notin Y'', z \in S(0) \\ z & : \; y_\ell > 0, z \in S(y_\ell) \\ z_0 & : \; \text{else} \end{cases} \tag{204}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, assuming again some arbitrary choice of $z_0 \in S(0)$. The definition of Q in terms of these modified Q_z's and the definitions of Y', Y'' and y_ℓ remain unchanged, see (197), (198), (199) and (201), respectively.
In the following, I prove that $S_{Q,\hat{\hat{g}}(X)} = S_1$. To this end, I first consider the subsumption $S_{Q,\hat{\hat{g}}(X)}(\gamma) \subseteq S_1(\gamma)$. In the case that $\gamma = 0$, it is apparent from (193) and (194) that $y_\ell = 0$ for each choice of $Y \in \mathcal{T}_0(\hat{\hat{g}}(X)) = \mathcal{T}_0(V)$. Recalling (197) and (204), the following cases may apply. If $y_\ell \in Y''$ and $z \in S^\sharp(0)$, then $Q(Y) = Q_z(Y'') = z \in S^\sharp(0) = S_1(0)$ by (185). If $y_\ell \notin Y''$ and $z \in S(0)$, then

$Q(Y) = Q_z(Y'') = z \in S(0) \subseteq S^\sharp(0) = S_1(0)$. If neither of these conditions apply, then $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S^\sharp(0) = S_1(0)$. Next we consider the case that $\gamma > 0$. Then $y_\ell \in [0, \gamma]$. We already know that for $y_\ell = 0$, we obtain that $Q(Y) \in S_1(0) \subseteq S_1(\gamma)$. If $y_\ell \in (0, \gamma]$, then either $Q(Y) = Q_z(Y'') = z \in S(y_\ell) \subseteq S(\gamma) = S_1(\gamma)$ or $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma) = S_1(\gamma)$. Hence indeed $S_{Q,\hat{\hat{g}}(X)}(\gamma) \subseteq S_1(\gamma)$ for all $\gamma \in \mathbf{I}$. As concerns the converse inequation, consider a choice of $\gamma \in \mathbf{I}$ and $z \in S_1(\gamma)$. We notice that $Y = \{(0, z)\} \cup (\{1\} \times [\gamma, 1]) \in \mathcal{T}_\gamma(\hat{\hat{g}}(X)) = \mathcal{T}_\gamma(V)$ by (193) and (194) (in the case that $\gamma = 0$) or by (195) and (196) (in the case that $\gamma > 0$). We then obtain that $\inf Y' = \inf\{z\} = z$ and $y_\ell = \inf Y'' = \inf[\gamma, 1] = \gamma$. If $\gamma = 0$, we conclude from (197), (204) and $0 \in Y''$ that $Q(Y) = Q_z(Y'') = z$ because $z \in S_1(0) = S^\sharp(0) = S^\sharp(y_\ell)$. If $\gamma > 0$, then $Q(Y) = Q_z(Y'') = z$ because $z \in S_1(\gamma) = S(\gamma) = S(y_\ell)$. Hence the converse inequation also holds, which finishes the proof that

$$S_1 = S_{Q,\hat{\hat{g}}(X)} \,. \tag{205}$$

It remains to be shown that $S_{Q\circ\widehat{g},X} = S$. Again, I first prove that $S_{Q\circ\widehat{g},X}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. Hence let $\gamma \in \mathbf{I}$ be given. If $\gamma = 0$, then we obtain from (188) and (189) that $y_\ell = 0$. In addition, $y_\ell \notin Y'' = (0, 1]$ regardless of the choice of $Y \in \{\widehat{g}(Z) : Z \in \mathcal{T}_0(X)\}$. Hence if $z \in S(0)$, then $Q(Y) = Q_z(Y'') = z \in S(0)$ by (197) and (204). Otherwise we obtain $Q(Y) = Q_z(Y'') = z_0 \in S(0)$. Now we consider the case that $\gamma > 0$. Then $y_\ell \in [0, \gamma]$ by (190) and (191). If $\gamma = 0$, then we either have $Q(Y) = Q_z(Y'') = z \in S(0) \subseteq S(\gamma)$, or $Q(Y) = Q_z(Y'') = z \in S^\sharp(0) \subseteq S(\gamma)$ by L-26, or $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma)$. If $\gamma > 0$, we obtain from (197) and (204) that either $Q(Y) = Q_z(Y'') = z \in S(y_\ell) \subseteq S(\gamma)$ or $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S(\gamma)$. In any case, $Q(Y) \in S(\gamma)$, as desired. This proves that $S_{Q\circ\widehat{g},X}(\gamma) \subseteq S(\gamma)$ for all $\gamma \in \mathbf{I}$. To establish the converse inequation, we again consider $\gamma \in \mathbf{I}$ and a choice of $z \in S(\gamma)$. We notice that by (188) and (189) (in the case that $\gamma = 0$) or by (190) and (191) (in the case that $\gamma > 0$), we may choose $Y = \{(0, z)\} \cup (\{1\} \times (\gamma, 1]) \in \{\widehat{g}(Z) : Z \in \mathcal{T}_\gamma(X)\}$. Again, we obtain that $\inf Y' = \inf\{z\} = z$ and $Y'' = (\gamma, 1]$, i.e. $y_\ell = \inf(\gamma, 1] = \gamma$. In the case that $\gamma = 0$, we then obtain from (197) and (204) that $Q(Y) = Q_z(Y'') = z$ because $z \in S(0) = S(y_\ell)$. In the case that $\gamma > 0$, we similarly obtain that $Q(Y) = Q_z(Y'') = z$ because $z \in S(\gamma) = S(y_\ell)$. This finishes the proof of the converse inequation, and we can summarize these inequations as stating that

$$S = S_{Q\circ\widehat{g},X} \,. \tag{206}$$

Therefore

$$\begin{aligned}
\Omega(S) &= \Omega(S_{Q\circ\widehat{g},X}) && \text{by (206)}\\
&= \mathcal{F}_\Omega(Q \circ \widehat{g})(X) && \text{by Def. 55}\\
&= \mathcal{F}_\Omega(Q)(\hat{\hat{g}}(X)) && \text{by (Z-6)}\\
&= \Omega(S_{Q,\hat{\hat{g}}(X)}) && \text{by Def. 55}\\
&= \Omega(S_1)\,, && \text{by (205)}
\end{aligned}$$

i.e. part **b.** of the lemma holds, as desired.

Lemma 28 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is a given aggregation mapping which satisfies* (Ω-2) *and* (Ω-3). *If* $\mathcal{F}_\Omega$ *satisfies* (Z-6), *then* Ω *satisfies* (Ω-4).

Proof This is now trivial because

$$\begin{aligned} \Omega(S) &= \Omega(S_1) && \text{by L-27, part } \textbf{b.} \\ &= \Omega(S^\sharp)\,. && \text{by L-27, part } \textbf{a.} \end{aligned}$$

Proof of Theorem 36

The results of the preceeding lemmata can be summarized as stating that (Ω-1) to (Ω-5) are necessary for $\mathcal{F}_\Omega$ to be a DFS: (Ω-1) is known to be necessary for $\mathcal{F}_\Omega$ to satisfy (Z-1) by L-20. Lemma L-22 shows that (Ω-3) is necessary for $\mathcal{F}_\Omega$ to be a DFS. Lemma L-23 shows that $\mathcal{F}_\Omega$ can only be a DFS if it induces the standard negation, and L-24 hence proves that (Ω-2) is necessary for $\mathcal{F}_\Omega$ to be a DFS. Lemma L-25 shows that (Ω-5) is necessary for $\mathcal{F}_\Omega$ to satisfy (Z-5). Finally, lemma L-28 proves that (Ω-4) is necessary for $\mathcal{F}_\Omega$ to be a DFS, by relation this condition to (Z-6). This finishes the proof that (Ω-1)–(Ω-5) are necessary for $\mathcal{F}_\Omega$ to be a DFS.

A.6 Proof of Theorem 37

Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be given and define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ by (34), i.e.

$$\Omega(S) = \xi(\top_S, \bot_S)$$

for all $S \in \mathbb{K}$. Now consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We notice that

$$\begin{aligned} \top_{Q,X_1,\ldots,X_n}(\gamma) &= \sup\{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\} \\ &= \sup \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \end{aligned}$$

by Def. 44, Def. 51 and (35), and similarly

$$\begin{aligned} \bot_{Q,X_1,\ldots,X_n}(\gamma) &= \inf\{Q(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)\} \\ &= \inf \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \end{aligned}$$

for all $\gamma \in \mathbf{I}$ by (36), i.e.

$$\top_{Q,X_1,\ldots,X_n} = \sup \mathcal{S}_{Q,X_1,\ldots,X_n} \qquad (207)$$

$$\bot_{Q,X_1,\ldots,X_n} = \inf \mathcal{S}_{Q,X_1,\ldots,X_n}\,. \qquad (208)$$

Therefore

$$\begin{aligned} \mathcal{F}_\xi(Q)(X_1, \ldots, X_n) &= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by Def. 46} \\ &= \xi(\sup \mathcal{S}_{Q,X_1,\ldots,X_n}, \inf \mathcal{S}_{Q,X_1,\ldots,X_n}) && \text{by (207), (208)} \\ &= \Omega(\mathcal{S}_{Q,X_1,\ldots,X_n}) && \text{by (34)} \\ &= \mathcal{F}_\Omega(Q)(X_1, \ldots, X_n)\,. && \text{by Def. 55} \end{aligned}$$

A.7 Proof of Theorem 38

Lemma 29 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is given and $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ξ by* (34)*. For $(\top, \bot) \in \mathbb{T}$, we define $S_{(\top,\bot)} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by*

$$S_{(\top,\bot)}(\gamma) = [\bot(\gamma), \top(\gamma)]\,, \tag{209}$$

for all $\gamma \in \mathbf{I}$. Then $S_{(\top,\bot)} \in \mathbb{K}$, $\top_{S_{(\top,\bot)}} = \top$, $\bot_{S_{(\top,\bot)}} = \bot$ and hence $\xi(\top, \bot) = \Omega(S_{(\top,\bot)})$.

Proof By Def. 45, $\bot(0) \le \top(0)$, i.e. $S_{(\top,\bot)}(0) = [\bot(0), \top(0)] \neq \varnothing$. In addition, we have $\bot(\gamma') \le \bot(\gamma)$ and $\top(\gamma') \ge \top(\gamma)$ whenever $\gamma' > \gamma$, i.e. $S_{(\top,\bot)}(\gamma) = [\bot(\gamma), \top(\gamma)] \subseteq [\bot(\gamma'), \top(\gamma')] = S_{(\top,\bot)}(\gamma')$. Hence $S_{(\top,\bot)} \in \mathbb{K}$ by Def. 52. Now consider $\gamma \in \mathbf{I}$. We clearly have

$$\begin{aligned} \top_{S_{(\top,\bot)}}(\gamma) &= \sup S_{(\top,\bot)}(\gamma) && \text{by (35)} \\ &= \sup[\bot(\gamma), \top(\gamma)] && \text{by (209)} \\ &= \top(\gamma) \end{aligned}$$

and similarly

$$\begin{aligned} \bot_{S_{(\top,\bot)}}(\gamma) &= \inf S_{(\top,\bot)}(\gamma) && \text{by (36)} \\ &= \inf[\bot(\gamma), \top(\gamma)] && \text{by (209)} \\ &= \bot(\gamma)\,, \end{aligned}$$

i.e.

$$\top = \top_{S_{(\top,\bot)}} \tag{210}$$

$$\bot = \bot_{S_{(\top,\bot)}}\,. \tag{211}$$

Therefore

$$\begin{aligned} \xi(\top, \bot) &= \xi(\top_{S_{(\top,\bot)}}, \bot_{S_{(\top,\bot)}}) && \text{by (210), (211)} \\ &= \Omega(S_{(\top,\bot)})\,, && \text{by (34)} \end{aligned}$$

as desired.

Lemma 30 *Let $(\top, \bot) \in \mathbb{T}$ be given. Then*

$$\begin{aligned} \top^\sharp(\gamma) &= \sup\,(S_{(\top,\bot)})^\sharp(\gamma) \\ \bot^\sharp(\gamma) &= \inf\,(S_{(\top,\bot)})^\sharp(\gamma)\,, \end{aligned}$$

for all $\gamma \in [0, 1)$.

Proof For brevity, I write $S = S_{(\top,\bot)}$. Let us first consider $\top^\sharp$. Because $\gamma < 1$, we obtain

$$\begin{aligned} \top^\sharp(\gamma) &= \lim_{\gamma' \to \gamma^+} \top(\gamma) && \text{by Def. 35} \\ &= \inf\{\top(\gamma') : \gamma' > \gamma\} && \text{by [9, Th-43, p. 44]} \\ &= \inf\{\sup S(\gamma') : \gamma' > \gamma\} && \text{by L-29}\,, \end{aligned}$$

i.e.

$$\top^\sharp(\gamma) = \inf\{\sup S(\gamma') : \gamma' > \gamma\}\,. \tag{212}$$

Now I show that

$$\top^\sharp(\gamma) \geq \sup \cap\{S(\gamma') : \gamma' > \gamma\} \tag{213}$$

Consider $\varepsilon > 0$. Then there exists $z \in \cap\{S(\gamma') : \gamma' > \gamma\}$ such that

$$z > \sup \cap\{S(\gamma') : \gamma' > \gamma\} - \varepsilon\,. \tag{214}$$

Because $z \in \cap\{S(\gamma') : \gamma' > \gamma\}$, we apparently have

$$\sup S(\gamma') \geq z \tag{215}$$

for all $\gamma' > \gamma$, i.e.

$$\begin{aligned} \top^\sharp(\gamma) &= \inf \sup\{S(\gamma') : \gamma' > \gamma\} && \text{by (212)} \\ &\geq z && \text{by (215)} \\ &> \sup \cap\{S(\gamma') : \gamma' > \gamma\} - \varepsilon\,. && \text{by (214)} \end{aligned}$$

$\varepsilon \to 0$ proves the desired inequation (213). Let us now show that the reverse inequation also holds, i.e.

$$\top^\sharp(\gamma) \leq \sup \cap\{S(\gamma') : \gamma' > \gamma\}\,. \tag{216}$$

Consider $\alpha < \top^\sharp(\gamma)$. The proof is by contradiction; hence let us assume that

$$\sup \cap\{S(\gamma') : \gamma' > \gamma\} < \alpha\,. \tag{217}$$

Then for all $z \geq \alpha$, there exists $\gamma'_z > \gamma$ such that $z \notin S(\gamma'_z)$. Because $S(\gamma') \subseteq S(\gamma'_z)$ for all $\gamma < \gamma' \leq \gamma'_z$, this proves that $z \notin S(\gamma')$ for all $\gamma < \gamma' \leq \gamma'_z$. Recalling that by (209), $S(\gamma)$ is an interval $S(\gamma) = [\bot(\gamma), \top(\gamma)]$, we deduce that

$$z' \notin S(\gamma')$$

for all $z' \in [z, 1]$ and $\gamma < \gamma' \leq \gamma'_z$. Hence $\sup S(\gamma') \leq z$ for all $\gamma < \gamma' \leq \gamma'_z$ and for all $z \geq \alpha$, i.e.

$$\begin{aligned} \top^\sharp(\gamma) &= \inf\{\sup S(\gamma') : \gamma' > \gamma\} && \text{by (212)} \\ &\leq \alpha\,, \end{aligned}$$

which contradicts with the choice of $\alpha < \top^\sharp(\gamma)$. Hence the assumption (217) is false; instead it holds that

$$\sup \cap \{S(\gamma') : \gamma' > \gamma\} \geq \alpha \,.$$

Because $\alpha < \top^\sharp(\gamma)$ was arbitrarily chosen, this proves that $\sup \cap \{S(\gamma') : \gamma' > \gamma\} \geq \top^\sharp(\gamma)$, i.e. inequation (216) holds. Combining (213) and (216), we conclude that

$$\top^\sharp(\gamma) = \sup \cap \{S(\gamma') : \gamma' > \gamma\} = \sup S^\sharp(\gamma) \tag{218}$$

for all $\gamma \in [0, 1)$ by Def. 56. The proof that

$$\flat^\sharp(\gamma) = \inf S^\sharp(\gamma)$$

for all $\gamma \in [0, 1)$ is completely analogous.

Lemma 31 *Let* $(\top, \bot) \in \mathbb{T}$ *be given. Then*

$$\top^\flat(\gamma) = \sup \left(S_{(\top,\bot)}\right)^\flat(\gamma)$$
$$\bot^\flat(\gamma) = \inf \left(S_{(\top,\bot)}\right)^\flat(\gamma) \,,$$

for all $\gamma \in \mathbf{I}$.

Proof For brevity, I write $S = S_{(\top,\bot)}$. The case that $\gamma = 0$ is trivial because we then have

$$\top^\flat(0) = \top(0) = \sup S(0) = \sup S^\flat(0)$$

by Def. 35, L-29 and Def. 56. For the same reason we have

$$\bot^\flat(0) = \bot(0) = \inf S(0) = \inf S^\flat(0) \,.$$

Hence let $\gamma > 0$. Let us first consider $\top^\flat$. Because $\gamma > 0$, we obtain

$$\begin{aligned} \top^\flat(\gamma) &= \lim_{\gamma' \to \gamma^-} \top(\gamma) && \text{by Def. 35} \\ &= \sup\{\top(\gamma') : \gamma' < \gamma\} && \text{by [9, Th-43, p. 44]} \\ &= \sup\{\sup S(\gamma') : \gamma' < \gamma\} && \text{by L-29}\,, \end{aligned}$$

i.e.

$$\top^\flat(\gamma) = \sup\{\sup S(\gamma') : \gamma' < \gamma\} \,. \tag{219}$$

Now I show that

$$\top^\flat(\gamma) \geq \sup \cup \{S(\gamma') : \gamma' < \gamma\} \tag{220}$$

Consider $\varepsilon > 0$. Then there exists $z \in \cup\{S(\gamma') : \gamma' < \gamma\}$ with

$$z > \sup \cup \{S(\gamma') : \gamma' < \gamma\} - \varepsilon \,. \tag{221}$$

In particular, there exists $\gamma'' < \gamma$ with $z \in S(\gamma'')$. Hence

$$\begin{aligned}
\top^\flat(\gamma) &= \sup\{\sup S(\gamma') : \gamma' < \gamma\} && \text{by (219)} \\
&\geq \sup S(\gamma'') && \text{because } \gamma'' < \gamma \\
&\geq z && \text{because } z \in S(\gamma'') \\
&> \sup \cup\{S(\gamma') : \gamma' < \gamma\} - \varepsilon && \text{by (221)} .
\end{aligned}$$

$\varepsilon \to 0$ proves the desired inequation (220). Let us now show that the reverse inequation also holds, i.e.

$$\top^\flat(\gamma) \leq \sup \cup\{S(\gamma') : \gamma' < \gamma\} \,. \tag{222}$$

Hence let again $\varepsilon > 0$. Then there exists $\gamma'' < \gamma$ such that

$$\sup S(\gamma'') > \top^\flat(\gamma) - \tfrac{\varepsilon}{2} \,, \tag{223}$$

which is apparent from (219). In turn, there exists $z \in S(\gamma'')$ such that

$$z > \sup S(\gamma'') - \tfrac{\varepsilon}{2} \,. \tag{224}$$

We conclude that there exists $z \in \cup\{S(\gamma') : \gamma' < \gamma\} \supseteq S(\gamma'')$ with

$$\begin{aligned}
z &> \sup S(\gamma'') - \tfrac{\varepsilon}{2} && \text{by (224)} \\
&> \top^\flat(\gamma) - \varepsilon \,. && \text{by (223)}
\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, this proves that (222) holds. Combining (220) and (222), we conclude that

$$\top^\flat(\gamma) = \sup \cup\{S(\gamma') : \gamma' < \gamma\} = \sup S^\flat(\gamma)$$

for all $\gamma \in [0, 1)$ by Def. 56. The proof that

$$\bot^\flat(\gamma) = \inf S^\flat(\gamma)$$

for all $\gamma \in [0, 1)$ is analogous.

Lemma 32 *Let $S \in \mathbb{K}$ be given. Then*

$$\begin{aligned}
\top_S(\gamma) \leq \top_{S^\sharp}(\gamma) \leq (\top_S)^\sharp(\gamma) \\
(\bot_S)^\sharp(\gamma) \leq \bot_{S^\sharp}(\gamma) \leq \bot_S(\gamma) \,,
\end{aligned}$$

for all $\gamma \in [0, 1)$.

Proof Let us first consider the case of $\top_S$. Because $\gamma < 1$, $S^\sharp(\gamma)$ can be rewritten as

$$S^\sharp(\gamma) = \cap\{S(\gamma') : \gamma' > \gamma\} \tag{225}$$

by Def. 56. We know from Def. 52 that $S(\gamma) \subseteq S(\gamma')$ for all $\gamma' > \gamma$. Therefore

$$S(\gamma) \subseteq \cap\{S(\gamma') : \gamma' > \gamma\} = S^\sharp(\gamma)$$

and in turn,

$$\top_S(\gamma) = \sup S(\gamma) \leq \sup S^\sharp(\gamma) = \top_{S^\sharp}(\gamma)$$
$$\bot_S(\gamma) = \inf S(\gamma) \geq \inf S^\sharp(\gamma) = \bot_{S^\sharp}(\gamma)$$

by (35) and (36) because $S(\gamma) \subseteq S^\sharp(\gamma)$. As concerns the remaining two inequations to be proven, I first show that

$$(\top_S)^\sharp(\gamma) \geq \sup \cap\{S(\gamma') : \gamma' > \gamma\} \tag{226}$$

which proves that $\top_{S^\sharp}(\gamma) \leq (\top_S)^\sharp(\gamma)$ because of (225). Hence let us notice that for $\gamma < 1$, $(\top_S)^\sharp(\gamma)$ becomes

$$\begin{aligned}(\top_S)^\sharp(\gamma) &= \lim_{\gamma' \to \gamma^+} \top_S(\gamma) && \text{by Def. 35}\\ &= \inf\{\top_S(\gamma') : \gamma' > \gamma\} && \text{by [9, Th-43, p. 44]}\\ &= \inf\{\sup S(\gamma') : \gamma' > \gamma\} && \text{by L-29}\,.\end{aligned}$$

Therefore

$$(\top_S)^\sharp(\gamma) = \inf\{\sup S(\gamma') : \gamma' > \gamma\}\,. \tag{227}$$

To see that (226) holds, consider $\varepsilon > 0$. Then there exists $z \in \cap\{S(\gamma') : \gamma' > \gamma\}$ such that

$$z > \sup \cap\{S(\gamma') : \gamma' > \gamma\} - \varepsilon\,. \tag{228}$$

Because $z \in \cap\{S(\gamma') : \gamma' > \gamma\}$, we apparently have

$$\sup S(\gamma') \geq z \tag{229}$$

for all $\gamma' > \gamma$, i.e.

$$\begin{aligned}(\top_S)^\sharp(\gamma) &= \inf \sup\{S(\gamma') : \gamma' > \gamma\} && \text{by (227)}\\ &\geq z && \text{by (229)}\\ &> \sup \cap\{S(\gamma') : \gamma' > \gamma\} - \varepsilon\,. && \text{by (228)}\end{aligned}$$

$\varepsilon \to 0$ proves the desired inequation (226).
The proof that

$$(\bot_S)^\sharp(\gamma) \leq \bot_{S^\sharp}(\gamma)$$

for all $\gamma \in [0, 1)$ is completely analogous to that of (226).

Lemma 33 *Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be given and suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ξ according to* (34). *If*

$$\xi(\top, \bot) = \xi(\top', \bot') \tag{230}$$

for all $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$\top(\gamma) \leq \top'(\gamma) \leq \top^\sharp(\gamma) \tag{231}$$
$$\bot^\sharp(\gamma) \leq \bot'(\gamma) \leq \bot(\gamma) \tag{232}$$

for all $\gamma \in [0, 1)$, then Ω satisfies (Ω-4).

Proof Let $S \in \mathbb{K}$ be given. By L-32,

$$\top_S(\gamma) \leq \top_{S^\sharp}(\gamma) \leq (\top_S)^\sharp(\gamma) \tag{233}$$

$$(\bot_S)^\sharp(\gamma) \leq \bot_{S^\sharp}(\gamma) \leq \bot(\gamma) \tag{234}$$

for all $\gamma \in [0, 1)$, i.e. the conditions (231) and (232) are fulfilled and by the assumed property (230),

$$\xi(\top_{S^\sharp}, \bot_{S^\sharp}) = \xi(\top_S, \bot_S)\,. \tag{235}$$

Therefore

$$\begin{aligned} \Omega(S) &= \xi(\top_S, \bot_S) && \text{by Th-37} \\ &= \xi(\top_{S^\sharp}, \bot_{S^\sharp}) && \text{by (235)} \\ &= \Omega(S^\sharp)\,, && \text{by Th-37} \end{aligned}$$

i.e. Ω satisfies (Ω-4), as desired.

Lemma 34 *a. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nonincreasing, then*

$$f^\sharp \leq f \leq f^\flat\,.$$

b. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is a constant mapping, then $f^\sharp = f^\flat = f$.

c. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing, then

$$f^\flat \leq f \leq f^\sharp\,.$$

Proof See [9, L-39, p.117].

Lemma 35 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is given and $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ξ according to* (34). *If*

$$\xi(\top^\flat, \bot^\sharp) = \xi(\top^\sharp, \bot^\flat) \tag{236}$$

for all $(\top, \bot) \in \mathbb{T}$ and ξ satisfies (X-5), *then*

$$\xi(\top, \bot) = \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$\top^\flat \leq \top' \leq \top^\sharp \tag{237}$$

$$\bot^\sharp \leq \bot' \leq \bot^\flat\,. \tag{238}$$

Proof Let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given such that (237) and (238) hold. Then

$$\begin{aligned}
\xi(\top^\sharp, \bot^\flat) &= \xi(\top^\flat, \bot^\sharp) && \text{by (236)} \\
&\leq \xi(\top', \bot') && \text{by (X-5), (237), (238)} \\
&\leq \xi(\top^\sharp, \bot^\flat) \,, && \text{by (X-5), (237), (238)}
\end{aligned}$$

i.e.

$$\xi(\top', \bot') = \xi(\top^\sharp, \bot^\flat) \,. \tag{239}$$

We notice that $(\top, \bot)$ itself is a legal choice for $(\top', \bot')$ which satisfies (237) and (238), see L-34. Hence we obtain as a special case that

$$\xi(\top, \bot) = \xi(\top^\sharp, \bot^\flat) \,. \tag{240}$$

Combining (239) and (240) yields the desired $\xi(\top, \bot) = \xi(\top', \bot')$.

Lemma 36 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is given and $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ξ according to* (34)*. If*

$$\xi(\top^\flat, \bot^\sharp) = \xi(\top^\sharp, \bot^\flat) \tag{241}$$

for all $(\top, \bot) \in \mathbb{T}$ and ξ satisfies (X-5)*, then*

$$\xi(\top_1, \bot_1) = \xi(\top_2, \bot_2)$$

whenever $(\top_1, \bot_1), (\top_2, \bot_2) \in \mathbb{T}$ satisfy $\top_1|_{[0,1)} = \top_2|_{[0,1)}$ and $\bot_1|_{[0,1)} = \bot_2|_{[0,1)}$.

Proof Consider $(\top, \bot) \in \mathbb{T}$ and define $(\top', \bot') \in \mathbb{T}$ by

$$\top'(\gamma) = \begin{cases} \top(\gamma) & : \quad \gamma < 1 \\ 1 & : \quad \gamma = 1 \end{cases} \tag{242}$$

$$\bot'(\gamma) = \begin{cases} \bot(\gamma) & : \quad \gamma < 1 \\ 0 & : \quad \gamma = 1 \end{cases} \tag{243}$$

for all $\gamma \in \mathbf{I}$. We then obtain from Def. 35 and L-34 that ${\top'}^\flat = \top^\flat$, ${\bot'}^\flat = \bot^\flat$ and

$$\begin{aligned}
{\top'}^\flat \leq \top \leq {\top'}^\sharp \\
{\bot'}^\sharp \leq \bot \leq {\bot'}^\flat
\end{aligned}$$

Hence by L-35,

$$\xi(\top, \bot) = \xi(\top', \bot') \,. \tag{244}$$

Now let $(\top_1, \bot_1), (\top_2, \bot_2) \in \mathbb{T}$ with $\top_1|_{[0,1)} = \top_2|_{[0,1)}$ and $\bot_1|_{[0,1)} = \bot_2|_{[0,1)}$. We notice that in this case, the results of the construction (242) coincide for $\top_1$ and $\top_2$, i.e.

$${\top_1}' = {\top_2}' \,. \tag{245}$$

Similarly, the results of the construction (243) on $\bot_1$ and $\bot_2$ coincide, i.e.

$$\bot_1{}' = \bot_2{}' . \tag{246}$$

Therefore

$$\begin{aligned} \xi(\top_1, \bot_1) &= \xi(\top_1{}', \bot_1{}') && \text{by (244)} \\ &= \xi(\top_2{}', \bot_2{}') && \text{by (245), (246)} \\ &= \xi(\top_2, \bot_2) . && \text{by (244)} \end{aligned}$$

Lemma 37 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is given and* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is defined in terms of* ξ *according to* (34)*. If*

$$\xi(\top^\flat, \bot^\sharp) = \xi(\top^\sharp, \bot^\flat) \tag{247}$$

for all $(\top, \bot) \in \mathbb{T}$ *and* ξ *satisfies* (X-5)*, then* Ω *satisfies* (Ω-4)*.*

Proof Let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given with

$$\top(\gamma) \le \top'(\gamma) \le \top^\sharp(\gamma) \tag{248}$$

$$\bot^\sharp(\gamma) \le \bot'(\gamma) \le \bot(\gamma) \tag{249}$$

for all $\gamma \in [0, 1)$. Define $(\top'', \bot'') \in \mathbb{T}$ by

$$\top''(\gamma) = \begin{cases} \top(\gamma) & : \quad \gamma < 1 \\ \top'(\gamma) & : \quad \gamma = 1 \end{cases} \tag{250}$$

$$\bot''(\gamma) = \begin{cases} \bot(\gamma) & : \quad \gamma < 1 \\ \bot'(\gamma) & : \quad \gamma = 1 \end{cases} \tag{251}$$

for all $\gamma \in \mathbf{I}$. It is then apparent from (248), (249) and L-34 that

$$\top''^\flat \le \top' \le \top''^\sharp$$
$$\bot''^\sharp \le \bot' \le \bot''^\flat ,$$

i.e. L-35 is applicable. We also notice that $\top''|_{[0,1)} = \top|_{[0,1)}$ and $\bot''|_{[0,1)} = \bot|_{[0,1)}$, which is apparent from (250) and (251). Therefore L-36 is applicable. We may hence proceed as follows.

$$\begin{aligned} \xi(\top, \bot) &= \xi(\top'', \bot'') && \text{by L-36} \\ &= \xi(\top', \bot') . && \text{by L-35} \end{aligned}$$

This proves that ξ fulfills the property (230) required by L-33. The lemma is hence applicable, and we deduce that Ω satisfies (Ω-4).

Proof of Theorem 38

Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is given and $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ξ according to equation (34). We shall consider all entailments claimed by the theorem in turn, splitting each equivalence into two separate entailments.

(X-1) **entails** (Ω-1)**.**
Suppose ξ satisfies (X-1) and consider some $a \in \mathbf{I}$. Let $S \in \mathbb{K}$ be the mapping defined by $S(\gamma) = \{a\}$ for all $\gamma \in \mathbf{I}$. It is then apparent from (35) and (36) that $\top_S(\gamma) = \sup\{a\} = a$ and $\bot_S(\gamma) = \inf\{a\} = a$ for all $\gamma \in \mathbf{I}$, i.e.

$$\top_S = \bot_S = \mathrm{c}_a\,. \tag{252}$$

Therefore

$$\begin{aligned} \Omega(S) &= \xi(\top_S, \bot_S) && \text{by (34)} \\ &= \xi(\mathrm{c}_a, \mathrm{c}_a) && \text{by (252)} \\ &= a\,, && \text{by (X-1)} \end{aligned}$$

i.e. Ω satisfies (Ω-1).

(Ω-1) **entails** (X-1)
Suppose Ω satisfies (Ω-1) and consider $(\mathrm{c}_a, \mathrm{c}_a) \in \mathbb{T}$. In order to show that $\xi(\mathrm{c}_a, \mathrm{c}_a) = a$, we notice that $S_{(\mathrm{c}_a,\mathrm{c}_a)}$ as defined by (209) becomes

$$S_{(\mathrm{c}_a,\mathrm{c}_a)}(\gamma) = [\mathrm{c}_a(\gamma), \mathrm{c}_a(\gamma)] = [a, a] = \{a\} \tag{253}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} \xi(\mathrm{c}_a, \mathrm{c}_a) &= \Omega(S_{(\mathrm{c}_a,\mathrm{c}_a)}) && \text{by L-29} \\ &= a\,, && \text{by (Ω-1), (253)} \end{aligned}$$

i.e. (X-1) holds, as desired.

(X-2) **entails** (Ω-2)
Suppose ξ satisfies (X-2). Now consider a choice of $S, S' \in \mathbb{K}$ with

$$S'(\gamma) = \{1 - z : z \in S(\gamma)\} \tag{254}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} \top_{S'}(\gamma) &= \sup S'(\gamma) && \text{by (35)} \\ &= \sup\{1 - z : z \in S(\gamma)\} && \text{by (254)} \\ &= 1 - \inf S(\gamma) \\ &= 1 - \bot_S(\gamma) && \text{by (36)} \end{aligned}$$

and similarly

$$\begin{aligned}
\bot_{S'}(\gamma) &= \inf S'(\gamma) && \text{by (36)}\\
&= \inf\{1-z : z \in S(\gamma)\} && \text{by (254)}\\
&= 1-\sup S(\gamma)\\
&= 1-\top_S(\gamma)\,, && \text{by (35)}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$\top_{S'} = 1-\bot_S \tag{255}$$

$$\bot_{S'} = 1-\top_S\,. \tag{256}$$

Hence

$$\begin{aligned}
\Omega(S') &= \xi(\top_{S'},\bot_{S'}) && \text{by (34)}\\
&= \xi(1-\bot_S, 1-\top_S) && \text{by (255), (256)}\\
&= 1-\xi(\top_S,\bot_S) && \text{by (X-2)}\\
&= 1-\Omega(S)\,, && \text{by (34)}
\end{aligned}$$

i.e. Ω satisfies (Ω-2).

(Ω-2) **entails** (X-2)
Let us assume that Ω satisfies (Ω-2). We consider a choice of $(\top,\bot) \in \mathbb{T}$. Then

$$\begin{aligned}
S_{(1-\bot,1-\top)}(\gamma) &= [1-\top(\gamma), 1-\bot(\gamma)] && \text{by (209)}\\
&= \{1-z : z \in [\bot(\gamma),\top(\gamma)]\}\,,
\end{aligned}$$

i.e.

$$S_{(1-\bot,1-\top)}(\gamma) = \{1-z : z \in S_{(\top,\bot)}(\gamma)\} \tag{257}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
\xi(1-\bot, 1-\top) &= \Omega(S_{(1-\bot,1-\top)}) && \text{by L-29}\\
&= 1-\Omega(S_{(\top,\bot)}) && \text{by (Ω-2), (257)}\\
&= 1-\xi(\top,\bot)\,, && \text{by L-29}
\end{aligned}$$

which proves that ξ satisfies (X-2).

(X-3) **entails** (Ω-3)
Suppose ξ satisfies (X-3) and consider a choice of $S \in \mathbb{K}$ with $1 \in S(0)$ and $S(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$. It is then apparent from (35) that $\top_S(\gamma) = \sup S(\gamma) = 1$ for all $\gamma \in \mathbf{I}$, i.e.

$$\top_S = \mathrm{c}_1\,. \tag{258}$$

In addition, we notice that by (36), $\bot_S(\gamma) = \inf S(\gamma) \in \{0,1\}$ for all $\gamma \in \mathbf{I}$ because $S(\gamma) \subseteq \{0,1\}$. Hence

$$\widehat{\bot}_S(\mathbf{I}) \subseteq \{0,1\}\,. \tag{259}$$

Finally, we observe that

$$\begin{aligned}
(\bot_S)_*^0 &= \inf\{\gamma \in \mathbf{I} : \bot_S(\gamma) = 0\} && \text{by (12)}\\
&= \inf\{\gamma \in \mathbf{I} : \inf S(\gamma) = 0\} && \text{by (36)}\\
&= \inf\{\gamma \in \mathbf{I} : 0 \in S(\gamma)\} && \text{because } S(\gamma) \subseteq \{0,1\}\\
&= s(0)\,, && \text{by Def. 53}
\end{aligned}$$

i.e.

$$(\bot_S)_*^0 = s(0)\,. \tag{260}$$

Therefore

$$\begin{aligned}
\Omega(S) &= \xi(\top_S, \bot_S) && \text{by (34)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}(\bot_S)_*^0 && \text{by (258), (259) and (X-3)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,. && \text{by (260)}
\end{aligned}$$

This proves that Ω satisfies (Ω-3), as desired.

(Ω-3) **entails** (X-3) Let us assume that Ω satisfies (Ω-3). Now consider a choice of $(c_1, \bot) \in \mathbb{T}$ with $\widehat{\bot}(\mathbf{I}) \subseteq \{0,1\}$. We define

$$S(\gamma) = \{1\} \cup \{\bot(\gamma') : \gamma' \leq \gamma\} \tag{261}$$

for all $\gamma \in \mathbf{I}$. Then clearly $c_1 = \top_S$ and $\bot = \bot_S$ by (35) and (36). In addition, $1 \in S(0)$ and apparently $S(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$ because $\bot(\gamma') \in \{0,1\}$ for all $\gamma' \in \mathbf{I}$. Hence (Ω-3) applies to S. Finally

$$\begin{aligned}
s(0) &= \inf\{\gamma \in \mathbf{I} : 0 \in S(\gamma)\} && \text{by Def. 53}\\
&= \inf\{\gamma \in \mathbf{I} : \bot(\gamma) = 0\} && \text{by (261)}\\
&= \bot_*^0\,, && \text{by (12)}
\end{aligned}$$

i.e.

$$\bot_*^0 = s(0)\,. \tag{262}$$

Hence

$$\begin{aligned}
&\xi(\top, \bot)\\
&= \Omega(S) && \text{by Th-37 because } \top = \top_S,\ \bot = \bot_S\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0) && \text{by } (\Omega\text{-3})\\
&= \tfrac{1}{2} + \tfrac{1}{2}\bot_*^0\,. && \text{by (262)}
\end{aligned}$$

This finishes the proof that ξ satisfies (X-3).

The conjunction of (X-2)**,** (X-4) **and** (X-5) **entails** (Ω-4)
Let $(\top, \bot) \in \mathbb{T}$. We notice that

$$(1 - \bot)^{\sharp} = 1 - \bot^{\sharp}\,, \tag{263}$$

this is apparent from Def. 35 because for $\gamma < 1$,

$$\begin{aligned}
(1-\bot)^{\sharp}(\gamma) &= \lim_{\gamma' \to \gamma^+} (1 - \bot(\gamma)) && \text{by Def. 35} \\
&= 1 - \lim_{\gamma' \to \gamma^+} \bot(\gamma) \\
&= 1 - \bot^{\sharp}(\gamma)\,. && \text{by Def. 35}
\end{aligned}$$

(The case that $\gamma = 1$ is uncritical). Similarly

$$(1 - \bot)^{\flat} = 1 - \bot^{\flat}\,, \tag{264}$$

because for $\gamma > 0$,

$$\begin{aligned}
(1-\bot)^{\flat}(\gamma) &= \lim_{\gamma' \to \gamma^-} (1 - \bot(\gamma)) && \text{by Def. 35} \\
&= 1 - \lim_{\gamma' \to \gamma^-} \bot(\gamma) \\
&= 1 - \bot^{\flat}(\gamma)\,. && \text{by Def. 35}
\end{aligned}$$

(The case that $\gamma = 0$ is uncritical). Hence

$$\begin{aligned}
\xi(\top^{\flat}, \bot^{\sharp}) &= \xi(\top^{\sharp}, \bot^{\sharp}) && \text{by [11, L-20, p. 56]} \\
&= 1 - \xi(1 - \bot^{\sharp}, 1 - \top^{\sharp}) && \text{by (X-2)} \\
&= 1 - \xi((1 - \bot)^{\sharp}, 1 - \top^{\sharp}) && \text{by (263)} \\
&= 1 - \xi((1 - \bot)^{\flat}, 1 - \top^{\sharp}) && \text{by [11, L-20, p. 56]} \\
&= 1 - \xi(1 - \bot^{\flat}, 1 - \top^{\sharp}) && \text{by (264)} \\
&= \xi(\top^{\sharp}, \bot^{\flat})\,. && \text{by (X-2)}
\end{aligned}$$

This proves that ξ satisfies the precondition stated in lemma L-37. We may hence apply the lemma and deduce that Ω satisfies (Ω-4).

(Ω-4) **entails** (X-4)
Suppose Ω satisfies (Ω-4). Now consider $(\top, \bot) \in \mathbb{T}$. We abbreviate

$$S = S_{(\top, \bot)}\,,$$

see (209). We further choose some $z_0 \in S(0)$ and define $S', S'' : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$S'(\gamma) = \begin{cases} S(0) & : \quad \gamma = 0 \\ S^{\flat}(\gamma) \cup [\bot(\gamma), z_0] & : \quad \gamma > 0 \end{cases} \tag{265}$$

$$S''(\gamma) = \begin{cases} S^{\sharp}(\gamma) \cap [\bot(\gamma), 1] & : \quad \gamma < 1 \\ S(1) & : \quad \gamma = 1 \end{cases} \tag{266}$$

for all $\gamma \in \mathbf{I}$. It is apparent from the definition of S', S'' in terms of $S \in \mathbb{K}$ and the fact that $\bot$ is nondecreasing and $\bot \leq z_0$ that $S', S'' \in \mathbb{K}$ as well. I now show that $\top_{S'} = \top^\flat$, $\top_{S''} = \top^\sharp$ and $\bot_{S'} = \bot_{S''} = \bot$. To see that $\top_{S'} = \top^\flat$, let us first consider the case that $\gamma = 0$. We then have

$$\begin{aligned}
\top_{S'}(0) &= \sup S'(0) && \text{by (35)} \\
&= \sup S(0) && \text{by Def. 56} \\
&= \top(0) && \text{by L-29} \\
&= \top^\flat(0)\,. && \text{by Def. 35}
\end{aligned}$$

In the case that $\gamma > 0$, we recall that by L-31,

$$\top^\flat(\gamma) = \sup S^\flat(\gamma)\,. \tag{267}$$

We then have $S'(\gamma) \cap [z_0, 1] = S^\flat(\gamma) \cap [z_0, 1]$ from (265). Because $z_0 \in S(0) \subseteq S^\flat(\gamma)$ and $z_0 \in S'(\gamma)$, we conclude that

$$\begin{aligned}
\sup S^\flat(\gamma) &= \sup S^\flat(\gamma) \cap [z_0, 1] \\
&= \sup S'(\gamma) \cap [z_0, 1] \\
&= \sup S'(\gamma)\,.
\end{aligned}$$

Hence

$$\top_{S'}(\gamma) = \sup S'(\gamma) = \sup S^\flat(\gamma) = \top^\flat(\gamma)$$

by (267) and (35). Combining this with the case $\gamma = 0$, I have shown that

$$\top_{S'} = \top^\flat\,. \tag{268}$$

As concerns $\bot_{S'}$, we notice that in the nontrivial case that $\gamma > 0$,

$$\begin{aligned}
\bot_{S'}(\gamma) &= \inf S'(\gamma) && \text{by (36)} \\
&= \inf S^\flat(\gamma) \cup [\bot(\gamma), z_0] && \text{by (265)} \\
&= \min(\inf S^\flat(\gamma), \bot(\gamma)) \\
&= \bot(\gamma)\,,
\end{aligned}$$

because $S^\flat(\gamma) \subseteq S(\gamma)$ by L-13 and hence $\inf S^\flat(\gamma) \geq \inf S(\gamma) = \bot(\gamma)$ by L-29. Therefore

$$\bot_{S'} = \bot\,. \tag{269}$$

Next we consider $\top_{S''}$. The case that $\gamma = 1$ is trivial. Hence let us consider $\gamma < 1$. We notice that by L-30,

$$\top^\sharp(\gamma) = \top_{S^\sharp}(\gamma)\,. \tag{270}$$

Because $\perp(\gamma) \leq z_0$, we can then proceed as follows:

$$
\begin{aligned}
\top_{S''}(\gamma) &= \sup S^\sharp(\gamma) \cap [\perp(\gamma), 1] && \text{by (266)} \\
&= \sup S^\sharp(\gamma) \cap [z_0, 1] && \text{because } \perp(\gamma) \leq z_0 \\
&= \sup S^\sharp(\gamma) && \text{because } z_0 \in S^\sharp(\gamma) \\
&= \top_{S^\sharp}(\gamma) && \text{by (35)} \\
&= \top^\sharp(\gamma)\,. && \text{by (270)}
\end{aligned}
$$

This proves that

$$\top_{S''} = \top^\sharp\,. \tag{271}$$

Concerning $\perp_{S''}$, the case that $\gamma = 1$ is again trivial. For $\gamma < 1$, we deduce that

$$
\begin{aligned}
\perp_{S''}(\gamma) &= \inf S^\sharp(\gamma) \cap [\perp(\gamma), 1] && \text{by (266), (36)} \\
&= \inf S^\flat(\gamma) \cap [\perp(\gamma), \top(\gamma)] && \text{because } z_0 \in S(\gamma) \text{ and } z_0 \leq \top(\gamma) \\
&= \inf S^\sharp(\gamma) \cap S(\gamma) && \text{by (209)} \\
&= \inf S(\gamma) && \text{because } S^\flat(\gamma) \subseteq S(\gamma) \text{ by L-13} \\
&= \perp(\gamma)\,, && \text{by L-29}
\end{aligned}
$$

i.e. indeed

$$\perp_{S''} = \perp\,. \tag{272}$$

We further notice that $S^\flat(\gamma) \subseteq S'(\gamma) \subseteq S^\sharp(\gamma)$ and $S^\flat(\gamma) \subseteq S''(\gamma) \subseteq S^\sharp(\gamma)$. This is apparent from (265), (266) and L-13. We can hence apply L-16 to conclude that

$$\Omega(S') = \Omega(S) = \Omega(S'')\,. \tag{273}$$

Putting the pieces together,

$$
\begin{aligned}
\xi(\top^\flat, \perp) &= \xi(\top_{S'}, \perp_{S'}) && \text{by (268), (269)} \\
&= \Omega(S') && \text{by Th-37} \\
&= \Omega(S'') && \text{by (273)} \\
&= \xi(\top_{S''}, \perp_{S''}) && \text{by Th-37} \\
&= \xi(S^\sharp, \perp)\,. && \text{by (271) and (272)}
\end{aligned}
$$

Hence (X-4) holds, as desired.

(X-5) **entails** (Ω-5)
Suppose ξ satisfies (X-5) and let $S, S' \in \mathbb{K}$ be given where $S \sqsubseteq S'$. Now consider $\gamma \in \mathbf{I}$. For each $\varepsilon > 0$, there exists $z \in S(\gamma)$ such that

$$z > \sup S(\gamma) - \varepsilon\,. \tag{274}$$

Because $S \sqsubseteq S'$, there exists $z' \in S'(\gamma)$ with $z' \geq z$, see Def. 57. Hence

$$\sup S'(\gamma) \geq z' \geq z > \sup S(\gamma) - \varepsilon$$

by (274). $\varepsilon \to 0$ yields

$$\top_{S'}(\gamma) = \sup S'(\gamma) \geq \sup S(\gamma) = \top_S(\gamma) \tag{275}$$

by (35). By similar reasoning, we know that for each $\varepsilon > 0$ there exists $z' \in S'(\gamma)$ with

$$z' < \inf S'(\gamma) + \varepsilon\,. \tag{276}$$

Because $S \sqsubseteq S'$, we deduce from Def. 57 that there exists $z \in S(\gamma)$ with $z \leq z'$. Hence

$$\begin{aligned}
\bot_S(\gamma) &= \inf S(\gamma) && \text{by (36)}\\
&\leq z && \text{because } z \in S(\gamma)\\
&\leq z' && \text{by choice of } z\\
&< \inf S'(\gamma) + \varepsilon && \text{by (276)}\\
&= \bot_{S'}(\gamma) + \varepsilon\,. && \text{by (36)}
\end{aligned}$$

$\varepsilon \to 0$ yields

$$\bot_S(\gamma) \leq \bot_{S'}(\gamma)\,. \tag{277}$$

Therefore

$$\begin{aligned}
\Omega(S) &= \xi(\top_S, \bot_S) && \text{by Th-37}\\
&\leq \xi(\top_{S'}, \bot_{S'}) && \text{by (X-5), (275) and (277)}\\
&= \Omega(S')\,. && \text{by Th-37}
\end{aligned}$$

Hence Ω satisfies (Ω-5), as desired.

(Ω-5) **entails** (X-5)
Let us assume that Ω satisfies (Ω-5). Now consider $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\top \leq \top'$ and $\bot \leq \bot'$. For $\gamma \in \mathbf{I}$, we obtain from (209) that

$$S_{(\top,\bot)}(\gamma) = [\bot(\gamma), \top(\gamma)]$$

and

$$S_{(\top',\bot')}(\gamma) = [\bot'(\gamma), \top'(\gamma)]\,.$$

Consider $z \in S_{(\top,\bot)}(\gamma)$. Then clearly $z \leq \top(\gamma) \leq \top'(\gamma) \in S_{(\top',\bot')}(\gamma)$ because $\top \leq \top'$. On the other hand, for $z' \in S_{(\top',\bot')}(\gamma)$ we have $z' \geq \bot'(\gamma) \geq \bot(\gamma) \in S_{(\top,\bot)}(\gamma)$ because $\bot \leq \bot'$. Because $\gamma \in \mathbf{I}$ was arbitrarily chosen, this proves that

$$S_{(\top,\bot)} \sqsubseteq S_{(\top',\bot')} \tag{278}$$

by Def. 57. Therefore

$$\begin{aligned}
\xi(\top, \bot) &= \Omega(S_{(\top,\bot)}) && \text{by L-29} \\
&\leq \Omega(S_{(\top',\bot')}) && \text{by } (\Omega\text{-5}), (278) \\
&= \xi(\top', \bot')\,, && \text{by L-29}
\end{aligned}$$

i.e. (X-5) is valid for ξ.

A.8 Proof of Theorem 39

In order to prove that condition $(\Omega - i)$, $i \in \{1, \ldots, 5\}$, is independent of the remaining conditions, we need to show that there exists an Ω-QFM which validates all of $(\Omega - 1)$–$(\Omega - 5)$ except for $(\Omega - i)$. We can profit from Th-38 which permits us to reduce the independence proof to the independence proof of (X-1)–(X-5). These conditions have already been shown to be independent, see theorem Th-23.

(Ω-1) **is independent of the remaining conditions** We know from Th-23 that there exists a choice of $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ wich satisfies all 'X-conditions' except for (X-1). From Th-37, we know that $\mathcal{F}_\xi$ is an $\mathcal{F}_\Omega$-QFM, i.e. $\mathcal{F}_\xi = \mathcal{F}_\Omega$ if we define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ by (34). Now we utilize Th-38. By part a. of the theorem, (Ω-1) fails because (X-1) fails. In addition, we know from parts b., c., d.1, and e. of the theorem that (Ω-2), (Ω-3), (Ω-4) and (Ω-5) hold because (X-2), (X-3), (X-4) and (X-5) hold. Hence $\mathcal{F}_\Omega$ demonstrates that (Ω-1) is independent of the other conditions.

(Ω-2) **is independent of the remaining conditions** In this case, I recall the $\mathcal{M}_\mathcal{B}$-QFM used to prove the independence of the 'B-condition' (B-2) in [9, Th-66, p. 51]. It is defined in terms of the following $\mathcal{B}_{(\mathrm{B}-2)} : \mathbb{B} \longrightarrow \mathbf{I}$.

$$\mathcal{B}_{(\mathrm{B}-2)}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'_f(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}^{*\prime}(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{279}$$

for all $f \in \mathbb{B}$. This QFM is known to satisfy all 'B-conditions' except for (B-2). We recall that $\mathcal{M}_{\mathcal{B}_{(\mathrm{B}-2)}} = \mathcal{F}_\xi$ provided we define $\xi(\top, \bot) = \mathcal{B}_{(\mathrm{B}-2)}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))$ for all $(\top, \bot) \in \mathbb{T}$, see Th-22. We conclude from Th-24 that ξ satisfies all 'X-conditions' except for (X-2). Now we define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ in terms of ξ according to (34). Then Ω fails to satisfy (Ω-2) by part b. of Th-38. By parts a., c., and e. of the theorem, Ω is known to satisfy (Ω-1), (Ω-3) and (Ω-5) because ξ satisfies (X-1), (X-3) and (X-5), respectively. As concerns (Ω-4), we observe that ξ satisfies the preconditions of lemma L-37. Hence Ω satisfies (Ω-4). This finishes the independence proof for (Ω-2).

(Ω-3) **is independent of the remaining conditions** In this case, we recall that by Th-23 there exists a choice of $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ such that all 'x-conditions' except for (X-3) are satisfied. From Th-37, we know that $\mathcal{F}_\xi$ is an $\mathcal{F}_\Omega$-QFM, i.e. $\mathcal{F}_\xi = \mathcal{F}_\Omega$ for

$\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ defined by (34). Again we apply Th-38. By part c. of the theorem, (Ω-3) fails because (X-3) fails. By parts a., b., d.1, and e. of the theorem, we know that (Ω-1), (Ω-2), (Ω-4) and (Ω-5) hold because (X-1), (X-2), (X-4) and (X-5) hold. Hence $\mathcal{F}_\Omega$ proves that (Ω-3) is independent of the other conditions.

(Ω-4) **is independent of the remaining conditions** By Th-23, there exists $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which satisfies all 'x-conditions' except for (X-4). Because (X-4) fails, we obtain by contraposition from part d.2 of Th-38 that (Ω-4) fails. The remaining 'Ω-conditions' (Ω-1), (Ω-2), (Ω-3) and (Ω-5) are known to hold from parts a., b., c., and e. of the theorem, respectively. This proves the independence of (Ω-4).

(Ω-5) **is independent of the remaining conditions** To this end, we define a mapping $\mathcal{B}'_{(\mathrm{B}-5)} : \mathbb{H} \longrightarrow \mathbf{I}$ by

$$\mathcal{B}'_{(\mathrm{B}-5)}(f) = \begin{cases} f_*^0 & : \quad f_0^* = 1 \\ f_0^* & : \quad f_0^* < 1 \end{cases} \tag{280}$$

We further define $\mathcal{B}_{(\mathrm{B}-5)} : \mathbb{B} \longrightarrow \mathbf{I}$ in terms of $\mathcal{B}'_{(\mathrm{B}-5)}$ according to equation (18). $\mathcal{B}_{(\mathrm{B}-5)}$ is known to satisfy (B-1), (B-2), (B-3) and (B-4) and to violate (B-5). By Th-24, the mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ defined by $\xi(\top, \bot) = \mathcal{B}_{(\mathrm{B}-5)}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))$ for all $(\top, \bot) \in \mathbb{T}$ satisfies (X-1), (X-2), (X-3) and (X-4), but violates (X-5). As usual, we define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ in terms of ξ according to (34). By Th-38, Ω violates (Ω-5), but it satisfies (Ω-1), (Ω-2) and (Ω-3). We notice that Ω satisfies the precondition of lemma L-33. Hence Ω satisfies (Ω-4), i.e. condition (Ω-5) is indeed independent of (Ω-1)–(Ω-4), as desired.

A.9 Proof of Theorem 40

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and consider a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We know from Th-32.a that $S_{Q,X_1,\ldots,X_n}(0) \neq \varnothing$. Hence there exists a choice of $z_0 \in S_{Q,X_1,\ldots,X_n}(0)$. We notice that by Th-32.b, $z_0 \in S_{Q,X_1,\ldots,X_n}(\gamma)$ for all $\gamma \in \mathbf{I}$. We hence obtain that

$$\begin{aligned} & s_{Q,X_1,\ldots,X_n}(z_0) \\ &= \inf\{\gamma \in \mathbf{I} : z_0 \in S_{Q,X_1,\ldots,X_n}(\gamma)\} && \text{by Def. 54} \\ &= \inf \mathbf{I} && \text{because } z_0 \in S_{Q,X_1,\ldots,X_n}(\gamma) \text{ for all } \gamma \in \mathbf{I} \\ &= 0\,. \end{aligned}$$

Therefore $z_0 \in s_{Q,X_1,\ldots,X_n}{}^{-1}(0)$, i.e. $s_{Q,X_1,\ldots,X_n}{}^{-1}(0) \neq \varnothing$, as desired.

A.10 Proof of Theorem 41

Lemma 38 *Let $s \in \mathbb{L}$ be given and suppose that $S : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{281}$$

for all $\gamma \in \mathbf{I}$. Then $S \in \mathbb{K}$. Let us further denote the mapping defined in terms of S according to Def. 53 by $s' : \mathbf{I} \longrightarrow \mathbf{I}$. Then $s' = s$.

Proof Let $s \in \mathbb{L}$ be given and suppose S is defined by (281). I first show that $S \in \mathbb{K}$. By Def. 51, $s^{-1}(0) \neq \varnothing$, i.e. there exists $z_0 \in \mathbf{I}$ with $s(z_0) = 0$. By (281), $z_0 \in S(0)$, in particular $S(0) \neq \varnothing$. Now let $\gamma, \gamma' \in \mathbf{I}$ with $\gamma' \geq \gamma$. Then trivially

$$\{z \in \mathbf{I} : \gamma \geq s(z)\} \subseteq \{z \in \mathbf{I} : \gamma' \geq s(z)\} .$$

Hence $S(\gamma) \subseteq S(\gamma')$ by (281). This finishes the proof that $S \in \mathbb{K}$, see Def. 52.
Now suppose s' is defined from S by Def. 53. Then for all $z \in \mathbf{I}$,

$$\begin{aligned} s'(z) &= \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} && \text{by Def. 53} \\ &= \inf\{\gamma \in \mathbf{I} : \gamma \geq s(z)\} && \text{by (281)} \\ &= \inf[s(z), 1] \\ &= s(z) . \end{aligned}$$

This proves that $s' = s$.

Proof of Theorem 41

Consider $s \in \mathbb{L}$ and suppose that $S : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by (37). We know from L-38 that indeed $S \in \mathbb{K}$. Further suppose that $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ is defined by (27) and that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ is the fuzzy subset defined by (33). We then obtain from Th-33 that

$$S = S_{Q,X_1,\ldots,X_n} .$$

Hence by L-38 and Def. 54, $s = s_{Q,X}$.

A.11 Proof of Theorem 42

Lemma 39 *Let $S \in \mathbb{K}$ be given and suppose $s \in \mathbb{L}$ is defined in terms of S according to Def. 53. Further let $S' \in \mathbb{K}$ be defined in terms of s according to equation* (281). *Then $S' = S^\sharp$.*

Proof Immediate. Consider $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} S'(\gamma) &= \{z \in \mathbf{I} : \gamma \geq s(z)\} && \text{by (281)} \\ &= \{z \in \mathbf{I} : \gamma \geq \inf\{\gamma' \in \mathbf{I} : z \in S(\gamma')\}\} && \text{by Def. 53} \\ &= \{z \in \mathbf{I} : \text{ for all } \gamma' > \gamma, z \in S(\gamma')\} && \text{because } S(\gamma') \subseteq S(\gamma'') \text{ for } \gamma' \leq \gamma'' \\ &= \{z \in \mathbf{I} : z \in \cap\{S(\gamma') : \gamma' > \gamma\}\} \\ &= \cap\{S(\gamma') : \gamma' > \gamma\} \\ &= S^\sharp(\gamma) . && \text{by Def. 56} \end{aligned}$$

This is also correct for $\gamma = 1$ if we stipulate that $\cap\{S(\gamma') : \gamma' > 1\} = \bigcap_{\gamma' \in \varnothing} S(\gamma') = \mathbf{I}$.

Lemma 40 *Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given. Then $\mathcal{F}_\Omega = \mathcal{F}_{\Omega'}$ if and only if $\Omega = \Omega'$.*

Proof If $\Omega = \Omega'$, then trivially $\mathcal{F}_\Omega = \mathcal{F}_{\Omega'}$. It remains to be shown that $\Omega \neq \Omega'$ entails that $\mathcal{F}_\Omega \neq \mathcal{F}_{\Omega'}$. Hence suppose that $\Omega \neq \Omega'$. Then there exists a choice of $S \in \mathbb{K}$ such that

$$\Omega(S) \neq \Omega'(S)\,. \tag{282}$$

We now define $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by (27) and further define $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ by (33). Then by Th-33,

$$S = S_{Q,X}\,. \tag{283}$$

Therefore

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(X) &= \Omega(S_{Q,X}) && \text{by Def. 55}\\
&= \Omega(S) && \text{by (283)}\\
&\neq \Omega'(S) && \text{by (282)}\\
&= \Omega'(S_{Q,X}) && \text{by (283)}\\
&= \mathcal{F}_{\Omega'}(Q)(X)\,. && \text{by Def. 55}
\end{aligned}$$

Hence $\mathcal{F}_\Omega \neq \mathcal{F}_{\Omega'}$, as we intended to show.

Lemma 41 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ by*

$$\Omega(S) = \omega(s) \tag{284}$$

for all $S \in \mathbb{K}$, where s is defined in terms of S according to Def. 53. Then Ω satisfies (Ω-4).

Proof Expanding Def. 53, we obtain for the given $S \in \mathbb{K}$ that

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{285}$$

for all $z \in \mathbf{I}$. In the case of $S^\sharp$, I denote the mapping defined by Def. 53 by $s^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$. We then obtain

$$s^\sharp(z) = \inf\{\gamma \in \mathbf{I} : z \in S^\sharp(\gamma)\} \tag{286}$$

Let us recall that due to lemma L-13, $S(\gamma) \subseteq S^\sharp(\gamma)$ for all $\gamma \in \mathbf{I}$. It is hence apparent from (285) and (286) that

$$s^\sharp(z) \leq s(z) \tag{287}$$

for all $z \in \mathbf{I}$. As concerns the reverse inequation $s(z) \leq s^\sharp(z)$, we expand $S^\sharp$ in (286) according to Def. 56, thus

$$s^\sharp(z) = \inf\{\gamma \in \mathbf{I} : z \in \cap\{S(\gamma') : \gamma' > \gamma\}\} \tag{288}$$

Now consider $\varepsilon > 0$. By (288), there exists $\gamma \in \mathbf{I}$ such that

$$\gamma < s^\sharp(z) + \tfrac{\varepsilon}{3} \tag{289}$$

and for all $\gamma' > \gamma$, $z \in S(\gamma')$. Hence let $\gamma'' = \gamma + \frac{\varepsilon}{2}$. Then $\gamma'' > \gamma$, hence $z \in S(\gamma'')$. In addition, $\gamma'' < s^\sharp(z) + \varepsilon$ by (289). This proves that

$$\begin{aligned} s(z) &= \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} && \text{by (285)} \\ &\leq \gamma'' && \text{because } z \in S(\gamma'') \\ &< s^\sharp(z) + \varepsilon\,. \end{aligned}$$

$\varepsilon \to 0$ yields the desired $s(z) \leq s^\sharp(z)$ for all $z \in \mathbf{I}$. Combining this with (287) finishes the proof that $s = s^\sharp$.

Proof of Theorem 42

a. Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies (Ω-4) and further suppose that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is defined by (40). In order to prove that $\mathcal{F}_\omega = \mathcal{F}_\Omega$, we first define $\Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ by

$$\Omega'(S) = \omega(s) \tag{290}$$

for all $S \in \mathbb{K}$, where s is defined by Def. 53. It is then apparent from Def. 55 and Def. 61 that

$$\mathcal{F}_\omega = \mathcal{F}_{\Omega'}\,. \tag{291}$$

Now let $S \in \mathbb{K}$ be given, assume s is defined in terms of S by Def. 53 and further assume that $S' \in \mathbb{K}$ is defined in terms of s by (41). Then

$$\begin{aligned} \Omega'(S) &= \omega(s) && \text{by (290)} \\ &= \Omega(S') && \text{by (40)} \\ &= \Omega(S^\sharp) && \text{by L-39} \\ &= \Omega(S)\,. && \text{by } (\Omega\text{-4}) \end{aligned}$$

Hence $\Omega' = \Omega$ and $\mathcal{F}_\Omega = \mathcal{F}_\omega$ by (291).

b. Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ does not satisfy (Ω-4) and consider an arbitrary choice of $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. We define $\Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ by

$$\Omega'(S) = \omega(s)$$

for all $S \in \mathbb{K}$, where s is defined by Def. 53. We then know from Def. 55 and Def. 61 that

$$\mathcal{F}_\omega = \mathcal{F}_{\Omega'} \,. \tag{292}$$

In addition, Ω' is known to satisfy (Ω-4) by L-41. Now assume that $\mathcal{F}_\Omega = \mathcal{F}_\omega$. Then also $\mathcal{F}_\Omega = \mathcal{F}_{\Omega'}$ by (292). Applying L-40, we conclude that $\Omega = \Omega'$. But Ω is known to violate (Ω-4), while Ω' satisfies (Ω-4). Hence the assumption that $\mathcal{F}_\Omega = \mathcal{F}_\omega$ is false. Because ω was arbitrarily chosen, this proves that there is no $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ with $\mathcal{F}_\Omega = \mathcal{F}_\omega$, i.e. $\mathcal{F}_\Omega$ is not an $\mathcal{F}_\omega$-QFM.

A.12 Proof of Theorem 43

Consider a given $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ and suppose that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω according to (38). We consider the cases a.–e. of the theorem in turn.

a. Let us first show that Ω's satisfying (Ω-1) entails that ω satisfies (ω-1). Hence suppose that (Ω-1) holds for Ω and consider a choice of $s \in \mathbb{L}$ such that $s^{-1}([0,1)) = \{a\}$ for some $a \in \mathbf{I}$, i.e. $s(a) = a$ and $s(z) = 1$ for all $z \in \mathbf{I} \setminus \{a\}$ by Def. 60. We have to show that $\omega(s) = a$. To this end, we first notice that

$$\omega(s) = \Omega(S') \tag{293}$$

where $S' \in \mathbb{K}$ is defined by

$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} = \begin{cases} \{a\} & : \quad \gamma < 1 \\ \mathbf{I} & : \quad \gamma = 1 \end{cases}$$

This is apparent from L-38 and (38). We further notice that

$$S' = S^\sharp \tag{294}$$

for $S \in \mathbb{K}$ defined by

$$S(\gamma) = \{a\}$$

for all $\gamma \in \mathbf{I}$. This is immediate from Def. 56. Therefore

$$\begin{aligned} \omega(s) &= \Omega(S') && \text{by (293)} \\ &= \Omega(S^\sharp) && \text{by (294)} \\ &= \Omega(S) && \text{by L-41} \\ &= a \,, && \text{by (Ω-1)} \end{aligned}$$

which proves that ω satisfies (ω-1).
It remains to be shown that the converse entailment also holds. Hence suppose that ω satisfies (ω-1). Now consider a choice of $S \in \mathbb{K}$ such that there exists $a \in \mathbf{I}$ with $S(\gamma) = \{a\}$ for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} s(z) &= \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} && \text{by Def. 53} \\ &= \begin{cases} 0 & : \quad z = a \\ 1 & : \quad z \neq a \end{cases} \end{aligned}$$

for all $z \in \mathbf{I}$, i.e. $s^{-1}([0,1)) = \{a\}$. Hence $\Omega(S) = \omega(s) = a$ by (38) and (ω-1), i.e. Ω satisfies (Ω-1).

b. Again I prove the equivalence of (Ω-2) and (ω-2) by considering both implications separately. Hence let us assume that Ω satisfies (Ω-2); it must be shown that ω satisfies (ω-2). Consider $s, s' \in \mathbb{L}$ where

$$s'(z) = s(1-z) \tag{295}$$

for all $z \in \mathbf{I}$. In accordance with (281), I define

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{296}$$

$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s'(z)\} \tag{297}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} S'(\gamma) &= \{z \in \mathbf{I} : \gamma \geq s'(z)\} && \text{by (297)} \\ &= \{z \in \mathbf{I} : \gamma \geq s(1-z)\} && \text{by (295)} \\ &= \{1-z \in \mathbf{I} : \gamma \geq s(z)\} && \text{by substitution} \\ &= \{1-z : z \in S(\gamma)\} && \text{by (296)} \end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e. S and S' are related in the way required by (Ω-2). Therefore

$$\begin{aligned} \omega(s') &= \Omega(S') && \text{by L-38, (38)} \\ &= 1 - \Omega(S) && \text{by } (\Omega\text{-2}) \\ &= 1 - \omega(s)\,, && \text{by L-38, (38)} \end{aligned}$$

which proves that ω satisfies (ω-2).
To see that (ω-2) entails (Ω-2), suppose that ω satisfies (ω-2) and consider a choice of $S, S' \in \mathbb{K}$ with

$$S'(\gamma) = \{1-z : z \in S(\gamma)\} \tag{298}$$

for all $\gamma \in \mathbf{I}$. For the mappings s and s' defined by Def. 53 in terms of S and S' respectively, we then obtain

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{299}$$

and

$$\begin{aligned} s'(z) &= \inf\{\gamma \in \mathbf{I} : z \in S'(\gamma)\} && \text{by Def. 53} \\ &= \inf\{\gamma \in \mathbf{I} : 1-z \in S(\gamma)\} && \text{by (298)} \\ &= s(1-z)\,, && \text{by (299)} \end{aligned}$$

for all $z \in \mathbf{I}$. Hence s and s' are related in the way required by (ω-2). We conclude that

$$\begin{aligned} \Omega(S') &= \omega(s') && \text{by (38)} \\ &= 1 - \omega(s) && \text{by } (\omega\text{-2}) \\ &= 1 - \Omega(S)\,. && \text{by (38)} \end{aligned}$$

c. Next we prove the equivalence of Ω satisfying (Ω-3) and ω satisfying (ω-3). Hence suppose (Ω-3) is valid for Ω and consider a choice of $s \in \mathbb{L}$ with $s(1) = 0$ and $s^{-1}([0,1)) \subseteq \{0,1\}$, i.e. $s(z) = 1$ for all $z \in (0,1)$. Define $S \in \mathbb{K}$ by

$$S(\gamma) = \begin{cases} \{1\} & : \quad \gamma < s(0) \\ \{0,1\} & : \quad \gamma \geq s(0) \end{cases}$$

for all $\gamma \in \mathbf{I}$. Then the mapping defined in terms of S according to Def. 53 coincides with s. Therefore

$$\begin{aligned} \omega(s) &= \Omega(S) && \text{by (38)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,. && \text{by (Ω-3)} \end{aligned}$$

To see that (ω-3) entails (Ω-3), suppose that ω satisfies (ω-3) and consider $S \in \mathbb{K}$ with $1 \in S(0)$ and $S(\gamma) \subseteq \{0,1\}$ for all $\gamma \in \mathbf{I}$. Then the mapping s defined by Def. 53 in terms of S apparently satisfies $s(1) = 0$ and $s(z) = 1$ for all $z \in (0,1)$, i.e. $s^{-1}([0,1)) \subseteq \{0,1\}$. Hence s satisfies the requirements for application of (ω-3), and

$$\begin{aligned} \Omega(S) &= \omega(s) && \text{by (38)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,. && \text{by (ω-3)} \end{aligned}$$

d. The claim that every $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ defined in terms of some $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ according to equation (38) has already been proven in lemma L-41.

e. Finally we prove the equivalence of (Ω-5) and (ω-4), again by splitting it into two implications to be proven separately. Hence let us assume that Ω satisfies (Ω-5). We now consider a choice of $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$. In accordance with (281), I define

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{300}$$
$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s'(z)\} \tag{301}$$

for all $\gamma \in \mathbf{I}$. We now obtain from Def. 59 and (300)/(301) that

$$S^{\ddagger}(\gamma) = \{z \in \mathbf{I} : \text{there exist } z' \leq z \leq z'' \text{ with } \gamma \geq s(z') \text{ and } \gamma \geq s(z'')\} \tag{302}$$
$$S'^{\ddagger}(\gamma) = \{z \in \mathbf{I} : \text{there exist } z' \leq z \leq z'' \text{ with } \gamma \geq s'(z') \text{ and } \gamma \geq s'(z'')\}\,, \tag{303}$$

for all $\gamma \in \mathbf{I}$. Therefore $S^{\sharp}, S'^{\sharp} \in \mathbb{K}$ become

$$\begin{aligned} &S^{\ddagger\sharp}(\gamma) \\ &= \cap\{S^{\ddagger}(\gamma') : \gamma' > \gamma\} \\ &= \{z \in \mathbf{I} : \text{for all } \gamma' > \gamma, \text{ there exist } z' \leq z \leq z'' \text{ with } \gamma' \geq s(z') \text{ and } \gamma' \geq s(z'')\} \end{aligned} \tag{304}$$

$$\begin{aligned} &S'^{\ddagger\sharp}(\gamma) \\ &= \cap\{S^{\ddagger}(\gamma') : \gamma' > \gamma\} \\ &= \{z \in \mathbf{I} : \text{for all } \gamma' > \gamma, \text{ there exist } z' \leq z \leq z'' \text{ with } \gamma' \geq s'(z') \text{ and } \gamma' \geq s'(z'')\} \end{aligned} \tag{305}$$

for all $\gamma \in \mathbf{I}$, see Def. 56 and (302)/(303). Let us now show that $S^{\ddagger\sharp} \sqsubseteq S'^{\ddagger\sharp}$. We shall consider the two conditions in in turn which are imposed by Def. 57 on $S^{\ddagger\sharp}$ and $S'^{\ddagger\sharp}$ in order to have $S^{\ddagger\sharp} \sqsubseteq S'^{\ddagger\sharp}$. Hence let $\gamma \in \mathbf{I}$ be given and let $z \in S^{\ddagger\sharp}(\gamma)$. I will show that there exists $z' \in S'^{\ddagger\sharp}(\gamma)$ with $z' \geq z$. To this end, let us first recall that $S'^{\ddagger\sharp}(0) \supseteq S'^{\ddagger}(0) \supseteq S'(0) \neq \varnothing$ by L-13, Def. 59 and Def. 52. Let z'_0 denote an arbitrary element $z'_0 \in S'(0)$. If $z \leq z'_0$, then apparently $z'_0 \in S'(\gamma) \subseteq S'^{\ddagger}(\gamma) \subseteq S'^{\ddagger\sharp}(\gamma)$. Hence $z' = z'_0$ is an admissible choice of z' with $z' \geq z$ and $z' \in S'^{\ddagger\sharp}(\gamma)$. In the remaining case that $z > z'_0$, consider a choice of $\gamma' > \gamma$. Abbreviating $\gamma'' = (\gamma + \gamma')/2$, we apparently have

$$\gamma' > \gamma'' > \gamma\,. \tag{306}$$

From (304) and $\gamma'' > \gamma$, we deduce that $z \in S^{\ddagger}(\gamma'')$, i.e. there exist $z_1 \leq z \leq z_2$ with $z_1 \in S(\gamma'')$ and $z_2 \in S(\gamma'')$. Hence by (300) and (306),

$$\gamma' > \gamma'' \geq s(z_2)\,. \tag{307}$$

Now we notice that by Def. 62, $s \sqsubseteq s'$ entails that

$$\inf\{s'(z') : z' \geq z_2\} \leq s(z_2)\,. \tag{308}$$

Because $\gamma' > s(z_2)$ by (307), we conclude from (308) that there exists $z' \geq z_2$ with

$$s'(z') < \gamma'\,.$$

It is then immediate from (301) that $z' \in S'(\gamma')$. Now consider z. By assumption, $z > z'_0$ for $z'_0 \in S'(\gamma')$. In addition, $z \leq z_2 \leq z'$ for $z' \in S'(\gamma')$. Hence by Def. 59, $z \in S'^{\ddagger}(\gamma')$. Because $\gamma' > \gamma$ was arbitrary, this proves that $z \in S'^{\ddagger}(\gamma')$ for all $\gamma' > \gamma$, i.e. $z \in \cap\{S'^{\ddagger}(\gamma') : \gamma' > \gamma\} = S'^{\ddagger\sharp}(\gamma)$. Therefore $z' = z$ is an admissable choice of z' with $z' \geq z$ and $z' \in S'^{\ddagger\sharp}(\gamma)$.

Next I consider the second condition for $S^{\ddagger\sharp} \sqsubseteq S'^{\ddagger\sharp}$. Hence let $\gamma \in \mathbf{I}$ and let $z' \in S'^{\ddagger\sharp}(\gamma)$. I will show that there exists $z \in S^{\ddagger\sharp}(\gamma)$ with $z \leq z'$. Again, we first observe that $S^{\ddagger\sharp}(0) \supseteq S^{\ddagger}(0) \supseteq S(0) \neq \varnothing$, and assume a choice of some element $z_0 \in S(0)$. If $z' \geq z_0$, then $z = z_0$ is an admissible choice of z with $z \leq z'$ and $z_0 \in S(\gamma) \subseteq S^{\ddagger\sharp}(\gamma)$. In the remaining case that $z' < z_0$, we consider $\gamma' > \gamma$ and again abbreviate $\gamma'' = (\gamma + \gamma')/2$. We then deduce from (305) and $\gamma'' > \gamma$ that there exist $z_1 \leq z' \leq z_2$ with $z_1, z_2 \in S'(\gamma'')$. Hence by (301),

$$\gamma' > \gamma'' \geq s'(z_1)\,. \tag{309}$$

At this point we recall that by Def. 62, $s \sqsubseteq s'$ entails

$$\inf\{s(z) : z \leq z_1\} \leq s'(z_1) \tag{310}$$

From this we may conclude that there exists $z \leq z_1$ with

$$s(z) < \gamma'$$

because $\gamma' > s'(z_1)$ by (309). In turn, we obtain from (300) that $z \in S(\gamma')$. Now consider z'. We know that $z \leq z_1 \leq z'$ for $z \in S(\gamma')$. In addition, we have $z' \leq z_0$ for $z_0 \in S(\gamma)$ by assumption. Hence by Def. 59, $z' \in S^{\ddagger}(\gamma')$. Because $\gamma' > \gamma$ was arbitrary, $z' \in S^{\ddagger}(\gamma')$ for all $\gamma' > \gamma$, i.e. $z' \in \cap\{S^{\ddagger}(\gamma') : \gamma' > \gamma\} = S^{\ddagger^{\sharp}}(\gamma)$. Hence $z = z'$ is a legal choice of z with $z \leq z'$ and $z \in S^{\ddagger^{\sharp}}(\gamma)$. This finishes the proof that the defining conditions for

$$S^{\ddagger^{\sharp}} \sqsubseteq S'^{\ddagger^{\sharp}} \tag{311}$$

are satisfied, see Def. 57. Therefore

$$\begin{aligned} \omega(s) &= \Omega(S) && \text{by L-38, (38)} \\ &\leq \Omega(S') && \text{by } (\Omega\text{-5}), (311) \\ &= \omega(s')\,. && \text{by L-38, (38)} \end{aligned}$$

Hence ω satisfies (ω-4), i.e. (Ω-5) entails (ω-4), as desired.
Finally let us prove that ω's satisfying (ω-4) entails that Ω satisfies (Ω-5). Hence assume that (ω-4) is valid for ω and suppose that $S, S' \in \mathbb{K}$ satisfy $S \sqsubseteq S'$. In accordance with Def. 53, we define

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{312}$$

$$s'(z) = \inf\{\gamma \in \mathbf{I} : z \in S'(\gamma)\} \tag{313}$$

for all $z \in \mathbf{I}$. Now let $z \in \mathbf{I}$ and choose some $\varepsilon > 0$. We conclude from (312) that there exists

$$\gamma < s(z) + \varepsilon \tag{314}$$

such that $z \in S(\gamma)$. Because $S \sqsubseteq S'$, there exists $z'' \geq z$ with $z'' \in S'(\gamma)$, see Def. 57. Hence

$$\begin{aligned} s'(z'') &\leq \gamma && \text{by (313)} \\ &< s(z) + \varepsilon && \text{by (314)} \end{aligned}$$

and in turn,

$$\inf\{s'(z') : z' \geq z\} \leq s'(z'') < s(z) + \varepsilon\,.$$

$\varepsilon \to 0$ yields

$$\inf\{s'(z') : z' \geq z\} \leq s(z)\,. \tag{315}$$

To prove the second condition imposed in Def. 62 for $s \sqsubseteq s'$ to hold, consider $z' \in \mathbf{I}$ and choose some $\varepsilon > 0$. By (313), there exists

$$\gamma < s'(z') + \varepsilon \tag{316}$$

with $z' \in S'(\gamma)$. Because $S \sqsubseteq S'$, we obtain from Def. 57 that there exists $z'' \leq z'$ with $z'' \in S(\gamma)$. Hence

$$\begin{aligned} s(z'') &\leq \gamma && \text{by (312)} \\ &< s'(z') + \varepsilon\,. && \text{by (316)} \end{aligned}$$

In particular

$$\inf\{s(z) : z \leq z'\} \leq s(z'') < s'(z') + \varepsilon\,.$$

$\varepsilon \to 0$ yields

$$\inf\{s(z) : z \leq z'\} \leq s'(z')\,. \tag{317}$$

By Def. 62, (315) and (317) prove that $s \sqsubseteq s'$. Therefore

$$\begin{aligned} \Omega(S) &= \omega(s) && \text{by (38)} \\ &\leq \omega(s') && \text{by (ω-4)} \\ &= \Omega(S') && \text{by (38)}\,, \end{aligned}$$

i.e. (Ω-5) is indeed valid for Ω.

A.13 Proof of Theorem 44

Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies (ω-1)–(ω-4). Then $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ defined by (38) satisfies (Ω-1)–(Ω-5). We apply Th-34 and conclude that $\mathcal{F}_\Omega$ is a standard DFS. Finally, we notice that $\mathcal{F}_\omega = \mathcal{F}_\Omega$ by (39), i.e. $\mathcal{F}_\omega$ is a standard DFS, as desired.

A.14 Proof of Theorem 45

Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ violates one of (ω-1)–(ω-4). Then $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ as defined by (38) violates one of (ω-1), (ω-2), (ω-3) or (ω-4), see Th-43. Hence $\mathcal{F}_\Omega$ is not a DFS by Th-36. But $\mathcal{F}_\omega = \mathcal{F}_\Omega$ by (39), which proves that $\mathcal{F}_\omega$ is not a DFS.

A.15 Proof of Theorem 46

We know from Th-39 that (Ω-1)–(Ω-5) are independent. Hence for each choice of $i \in \{1, 2, 3, 5\}$, there exists a choice of $\Omega_i : \mathbb{K} \longrightarrow \mathbf{I}$ which violates ($\Omega - i$) and satisfies the remaining 'Ω-conditions', including (Ω-4). Because each Ω_i satisfies (Ω-4), we know that

$$\Omega_i(S) = \omega_i(s) \tag{318}$$

for all $S \in \mathbb{K}$ and s defined in terms of S according to Def. 53, where ω_i is defined in terms of Ω_i according to (40). This is apparent from (39), Th-42.a and L-40. Because (318) holds, we can apply Th-43 and conclude that each ω_1 satisfies all 'ω-conditions' except for (ω-1); ω_2 satisfies all conditions except for (ω-2); ω_3 satisfies all conditions except for (ω-3), and finally ω_5 satisfies all conditions except for (ω-4). Hence the conditions (ω-1)–(ω-4) are indeed independent.

A.16 Proof of Theorem 47

Let a choice of $s \in \mathbb{L}$ be given.

a. To see that $s^{\ddagger} \leq s$, consider $z \in \mathbf{I}$. Clearly

$$\inf\{s(z') : z' \leq z\} \leq s(z) \tag{319}$$
$$\inf\{s(z'') : z'' \geq z\} \leq s(z) \tag{320}$$

Hence

$$\begin{aligned} s^{\ddagger}(z) &= \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z'') : z'' \geq z\}) && \text{by Def. 65} \\ &\leq \max(s(z), s(z)) && \text{by (319), (320)} \\ &= s(z)\,. && \text{by idempotence of } \max \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $s^{\ddagger} \leq s$.

b. We know from Def. 60 that $s^{-1}(0) \neq \varnothing$, i.e. there exists $z_0 \in \mathbf{I}$ with $s(z_0) = 0$. By part a. of the theorem, $s^{\ddagger}(z_0) \leq s(z_0) = 0$, i.e. $s^{\ddagger}(z_0) = 0$. Hence ${s^{\ddagger}}^{-1}(0) \neq \varnothing$ and $s^{\ddagger} \in \mathbb{L}$ by Def. 60.

c. To see that $s^{\ddagger}$ is concave, consider $z_1 \leq z_2 \leq z_3$. Then

$$\inf\{s(z'') : z'' \geq z_2\} \leq \inf\{s(z'') : z'' \geq z_3\} \tag{321}$$

because $z_2 \leq z_3$, and

$$\inf\{s(z') : z' \leq z_1\} \geq \inf\{s(z') : z' \leq z_2\} \tag{322}$$

because $z_1 \leq z_2$. Recalling from Def. 65 that

$$s^{\ddagger}(z_2) = \max(\inf\{s(z') : z' \leq z_2\}, \inf\{s(z'') : z'' \geq z_2\})\,, \tag{323}$$

it is now convenient to discern two cases.

1. $\inf\{s(z') : z' \leq z_2\} \geq \inf\{s(z'') : z'' \geq z_2\}$. Then

$$\begin{aligned} s^{\ddagger}(z_2) &= \inf\{s(z') : z' \leq z_2\} && \text{by (323)} \\ &\leq \inf\{s(z') : z' \leq z_1\} && \text{by (322)} \\ &\leq \max(\inf\{s(z') : z' \leq z_1\}, \inf\{s(z'') : z'' \geq z_1\}) \\ &= s^{\ddagger}(z_1) && \text{by Def. 65} \\ &\leq \max(s^{\ddagger}(z_1), s^{\ddagger}(z_3))\,. \end{aligned}$$

2. $\inf\{s(z') : z' \leq z_2\} < \inf\{s(z'') : z'' \geq z_2\}$. In this case

$$\begin{aligned} s^\ddagger(z_2) &= \inf\{s(z'') : z'' \geq z_2\} && \text{by (323)} \\ &\leq \inf\{s(z'') : z'' \geq z_3\} && \text{by (321)} \\ &\leq \max(\inf\{s(z') : z' \leq z_3\}, \inf\{s(z'') : z'' \geq z_3\}) \\ &= s^\ddagger(z_3) && \text{by Def. 65} \\ &\leq \max(s^\ddagger(z_1), s^\ddagger(z_3))\,. \end{aligned}$$

This finishes the proof that $s^\ddagger$ is concave.

A.17 Proof of Theorem 48

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-4). We consider some choice of $s \in \mathbb{L}$. Because (ω-4) holds for ω, we can prove that $\omega(s) = \omega(s^\ddagger)$ by proving that $s \sqsubseteq s^\ddagger$ and $s^\ddagger \sqsubseteq s$. Let us first show that $s \sqsubseteq s^\ddagger$. Hence let $z \in \mathbf{I}$ be given. Then

$$\inf\{s^\ddagger(z') : z' \geq z\} \leq s^\ddagger(z) \leq s(z) \tag{324}$$

by Th-47, part a. This proves the first requirement for $s \sqsubseteq s^\ddagger$. To see that the second condition is also fulfilled, let $z' \in \mathbf{I}$. We have to prove that

$$\inf\{s(z) : z \leq z'\} \leq s^\ddagger(z')\,. \tag{325}$$

Let us denote by z_0 some element $z_0 \in s^{-1}(0) \neq \varnothing$, which is known to exist by Def. 60. If $z_0 \leq z'$, then (325) is trivially true because in this case

$$\inf\{s(z) : z \leq z'\} \leq s(z_0) = 0 \leq s^\ddagger(z')\,.$$

In the remaining case that $z_0 > z'$, we apparently have

$$\inf\{s(z'') : z'' \geq z'\} \leq s(z_0) = 0\,. \tag{326}$$

Therefore

$$\begin{aligned} s^\ddagger(z') &= \max(\inf\{s(z'') : z'' \geq z'\}, \inf\{s(z) : z \leq z'\}) && \text{by Def. 65} \\ &= \max(0, \inf\{s(z) : z \leq z'\}) && \text{by (326)} \\ &= \inf\{s(z) : z \leq z'\}\,. && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

In particular, (325) is valid. Combining (324) and (325) proves the desired $s \sqsubseteq s^\ddagger$, see Def. 65.
Next let us show that also $s^\ddagger \sqsubseteq s$. Firstly we notice that for all $z' \in \mathbf{I}$,

$$s^\ddagger(z') \leq s(z')\,,$$

again by Th-47, part a. Therefore

$$\inf\{s^\ddagger(z) : z \leq z'\} \leq s^\ddagger(z') \leq s(z)\,. \tag{327}$$

This proves the second requirement of Def. 62 for $s^\ddagger \sqsubseteq s$. To see that the first requirement also holds, consider $z \in \mathbf{I}$. It must be shown that

$$\inf\{s(z') : z' \geq z\} \leq s^\ddagger(z)\,. \tag{328}$$

Hence let $z_0 \in s^{-1}(0)$ be an arbitrary element with $s(z_0) = 0$, which is known to exist by Def. 60. If $z_0 \geq z$, then (328) holds because

$$\inf\{s(z') : z' \geq z\} \leq s(z_0) = 0 \leq s^\ddagger(z)\,.$$

In the remaining case that $z_0 < z$, we observe that

$$\inf\{s(z') : z' \leq z\} \leq s(z_0) = 0\,. \tag{329}$$

Therefore

$$\begin{aligned} s^\ddagger(z) &= \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z') : z' \geq z\}) && \text{by Def. 65} \\ &= \max(0, \inf\{s(z') : z' \geq z\}) && \text{by (329)} \\ &= \inf\{s(z') : z' \geq z\}\,. && \text{because } 0 \text{ identity of } \max \end{aligned}$$

In particular, (328) holds. It is then immediate from (327) and (328) that $s^\ddagger \sqsubseteq s$ by Def. 62. Hence $s \sqsubseteq s^\ddagger$, which entails that $\omega(s) \leq \omega(s^\ddagger)$ because ω is assumed to satisfy (ω-4), and $s^\ddagger \sqsubseteq s$, which entails that $\omega(s^\ddagger) \leq \omega(s)$. We conclude that $\omega(s) = \omega(s^\ddagger)$, as desired.

A.18 Proof of Theorem 49

Lemma 42 *Consider $s, s' \in \mathbb{L}$. If $s \trianglelefteq s'$, then it also holds that $s \sqsubseteq s'$.*

Proof Suppose that $s, s' \in \mathbb{L}$ satisfy $s \trianglelefteq s'$. We consider some $z \in \mathbf{I}$. We then know from Def. 64 that there exists $z'' \geq z$ with

$$s'(z'') \leq s(z)\,.$$

Therefore

$$\inf\{s'(z') : z' \geq z\} \leq s'(z'') \leq s(z)\,. \tag{330}$$

Now we consider some $z' \in \mathbf{I}$. Again by Def. 64, there exists $z'' \leq z'$ with

$$s(z'') \leq s(z')\,.$$

In turn,

$$\inf\{s(z) : z \leq z'\} \leq s(z'') \leq s(z')\,. \tag{331}$$

Hence both inequations (330) and (331) are valid, and we conclude from Def. 62 that $s \sqsubseteq s'$.

Lemma 43 *Let $s \in \mathbb{L}$ be given and let z_0 be an element $z_0 \in s^{-1}(0) \neq \varnothing$.*

a. If $z \geq z_0$, then $s^\ddagger(z) = \inf\{s(z') : z' \geq z\}$.

b. If $z \leq z_0$, then $s^\ddagger(z) = \inf\{s(z') : z' \leq z\}$.

Proof

a. Because $z \geq z_0$, we observe that

$$\begin{aligned} \inf\{s(z') : z' \leq z\} &\leq s(z_0) && \text{because } z_0 \leq z \\ &= 0\,, && \text{because } z_0 \in s^{-1}(0)\text{, i.e. } s(z_0) = 0 \end{aligned}$$

i.e.

$$\inf\{s(z') : z' \leq z\} = 0\,. \tag{332}$$

Therefore

$$\begin{aligned} s^{\ddagger}(z) &= \max(\inf\{s(z') : z' \geq z\}, \inf\{s(z') : z' \leq z\}) && \text{by Def. 65} \\ &= \max(\inf\{s(z') : z' \geq z\}, 0) && \text{by (332)} \\ &= \inf\{s(z') : z' \geq z\}\,. && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

b. In this case, we conclude from $z \leq z_0$ that

$$\begin{aligned} \inf\{s(z') : z' \geq z\} &\leq s(z_0) && \text{because } z_0 \geq z \\ &= 0\,. && \text{because } z_0 \in s^{-1}(0)\text{, i.e. } s(z_0) = 0 \end{aligned}$$

Hence in this case

$$\inf\{s(z') : z' \geq z\} = 0\,. \tag{333}$$

We then obtain

$$\begin{aligned} s^{\ddagger}(z) &= \max(\inf\{s(z') : z' \geq z\}, \inf\{s(z') : z' \leq z\}) && \text{by Def. 65} \\ &= \max(\inf\{s(z') : z' \leq z\}, 0) && \text{by (333)} \\ &= \inf\{s(z') : z' \leq z\}\,. && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

Lemma 44 *Consider $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$.*

a. Let z_0' be an element $z_0' \in {s'}^{-1}(0)$. Then ${s'}^{\ddagger}(z) \leq s(z)$ for all $z > z_0'$.

b. Let z_0 be an element $z_0 \in s^{-1}(0)$. Then $s^{\ddagger}(z) \leq s'(z)$ for all $z < z_0$.

Proof We recall from Def. 62 that $s \sqsubseteq s'$ entails that

$$\inf\{s'(z') : z' \geq z\} \leq s(z) \tag{334}$$

$$\inf\{s(z') : z' \leq z\} \leq s'(z)\,, \tag{335}$$

for all $z \in \mathbf{I}$. Now let us consider the two parts of the lemma.

a. In this case,

$$\begin{aligned} {s'}^{\ddagger}(z) &= \inf\{s'(z') : z' \geq z\} && \text{by L-43.a} \\ &\leq s(z) && \text{by (334)}\,. \end{aligned}$$

b. The proof of this part is analogous. Hence

$$\begin{aligned} s^{\ddagger}(z) &= \inf\{s(z') : z' \leq z\} && \text{by L-43.b} \\ &\leq s'(z)\,. && \text{by (335)} \end{aligned}$$

Lemma 45 *Let $s, s' \in \mathbb{L}$ be given with $s \sqsubseteq s'$.*

a. Let $z'_0 \in {s'}^{-1}(0)$ be given and suppose $z \geq z'_0$. Then $s^{\ddagger}(z) = \inf\{s(z') : z' \geq z\}$.

b. Let $z_0 \in s^{-1}(0)$ and suppose $z \leq z_0$. Then ${s'}^{\ddagger}(z) = \inf\{s'(z') : z' \leq z\}$.

Proof It is helpful to observe that the above inequations (334) and (335) are valid in this context, too. Now let us consider the parts of the lemma in turn.

a. Straightforward. We first notice that

$$\begin{aligned} \inf\{s(z') : z' \leq z\} &\leq \inf\{s(z') : z' \leq z'_0\} && \text{because } z'_0 \leq z \\ &\leq s'(z'_0) && \text{by (335)} \\ &= 0\,, && \text{because } z'_0 \in {s'}^{-1}(0) \end{aligned}$$

i.e.

$$\inf\{s(z') : z' \leq z\} = 0 \tag{336}$$

Therefore

$$\begin{aligned} & s^{\ddagger}(z) \\ &= \max(\inf\{s(z') : z' \geq z\}, \inf\{s(z') : z' \leq z\}) && \text{by Def. 65} \\ &= \max(\inf\{s(z') : z' \geq z\}, 0) && \text{by (336)} \\ &= \inf\{s(z') : z' \geq z\}\,, && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

as desired.

b. The proof of this case is analogous. Thus

$$\begin{aligned} \inf\{s'(z') : z' \geq z\} &\leq \inf\{s'(z') : z' \geq z_0\} && \text{because } z_0 \geq z \\ &\leq s(z_0) && \text{by (334)} \\ &= 0\,, && \text{because } z_0 \in s^{-1}(0) \end{aligned}$$

i.e.

$$\inf\{s'(z') : z' \geq z\} = 0 \tag{337}$$

Therefore

$$\begin{aligned}
&{s'}^{\ddagger}(z) \\
&= \max(\inf\{s'(z') : z' \leq z\}, \inf\{s'(z') : z' \geq z\}) && \text{by Def. 65} \\
&= \max(\inf\{s'(z') : z' \leq z\}, 0) && \text{by (337)} \\
&= \inf\{s'(z') : z' \leq z\}\,. && \text{because } 0 \text{ is identity of } \max
\end{aligned}$$

Lemma 46 *Suppose $s, s' \in \mathbb{L}$ satisfy $s \sqsubseteq s'$. Then $s^{\ddagger} \trianglelefteq {s'}^{\ddagger}$.*

Proof Let us first show that for all $z \in \mathbf{I}$, there exists $z' \geq z$ with

$${s'}^{\ddagger}(z') \leq s^{\ddagger}(z)\,. \tag{338}$$

To see this, choose some element $z'_0 \in {s'}^{-1}(0)$, which is known to exist from Def. 60. If $z \leq z'_0$, then ${s'}^{\ddagger}(z'_0) = 0 \leq s^{\ddagger}(z)$, i.e. $z' = z'_0$ is an admissible choice of z' with $z' \geq z$ and ${s'}^{\ddagger}(z') \leq s^{\ddagger}(z)$, which proves that (338) is valid in this case. In the remaining case that $z > z'_0$, consider the following chain of (in)equations.

$$\begin{aligned}
s^{\ddagger}(z) &= \inf\{s(z') : z' \geq z\} && \text{by L-45} \\
&\geq \inf\{\inf\{s'(z'') : z'' \geq z'\} : z' \geq z\} && \text{by (334)} \\
&= \inf\{s'(z') : z' \geq z\} \\
&= {s'}^{\ddagger}(z)\,. && \text{by L-43}
\end{aligned}$$

Hence $z' = z$ is an admissible choice of z' with $z' \geq z$ and ${s^{\ddagger}}'(z') \leq s^{\ddagger}(z)$, and (338) is valid in this case as well.
Next we show that for all $z' \in \mathbf{I}$, there exists $z \leq z'$ with

$$s^{\ddagger}(z) \leq {s'}^{\ddagger}(z')\,. \tag{339}$$

In this case, we choose some element $z_0 \in s^{-1}(0)$, which is again known to exist by Def. 60. If $z' \geq z_0$, then $s^{\ddagger}(z_0) = 0 \leq {s^{\ddagger}}'(z')$, i.e. $z = z_0$ is a legal choice of z with $z \leq z'$ and $s^{\ddagger}(z) \leq {s'}^{\ddagger}(z')$. Hence (339) is valid. In the remaining case that $z' < z_0$, we deduce that

$$\begin{aligned}
{s'}^{\ddagger}(z') &= \inf\{s'(z) : z \leq z'\} && \text{by L-45} \\
&\geq \inf\{\inf\{s(z'') : z'' \leq z\} : z \leq z'\} && \text{by (335)} \\
&= \inf\{s(z) : z \leq z'\} \\
&= s^{\ddagger}(z')\,. && \text{by L-43}
\end{aligned}$$

Hence $z = z'$ is a suitable choice of z with $z \leq z'$ and $s^{\ddagger}(z) \leq {s'}^{\ddagger}(z')$, i.e. (339) is valid in this case, too.
Finally, it is apparent from Def. 64 that (338) and (339) ensure the desired $s^{\ddagger} \trianglelefteq {s'}^{\ddagger}$.

Proof of Theorem 49

Let us first notice that the conditions **a.** and **b.** stated in the theorem are entailed by (ω-4). Hence suppose that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies (ω-4). Then for all $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$, we recall that by L-42, it also holds that $s \sqsubseteq s'$. Hence $\omega(s) \leq \omega(s')$ by (ω-4), i.e. condition **a.** is valid, as desired. As concerns condition **b.**, it has already been shown in Th-48 that ω's satisfying (ω-4) entails that $\omega(s) = \omega(s^\ddagger)$, i.e. condition **b.** holds as well. This finishes the proof that conditions **a.** and **b.** are entailed by (ω-4). Let us now prove the converse entailment. Hence suppose that **a.** and **b.** are valid and consider $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$. Then

$$\begin{aligned}
\omega(s) &= \omega(s^\ddagger) && \text{by property } \mathbf{b.} \\
&\leq \omega({s'}^\ddagger) && \text{by property } \mathbf{a.} \text{ and L-46} \\
&= \omega(s')\,. && \text{by property } \mathbf{b.}
\end{aligned}$$

Hence (ω-4) holds, as desired.

A.19 Proof of Theorem 50

Lemma 47 *Consider $s \in \mathbb{L}$ and suppose $s' \in \mathbb{L}$ is defined by $s'(z) = s(1 - z)$ for all $z \in \mathbf{I}$. Then it also holds that*

$${s'}^\ddagger(z) = s^\ddagger(1 - z)$$

for all $z \in \mathbf{I}$.

Proof Consider $z \in \mathbf{I}$. Then

$$\begin{aligned}
&{s'}^\ddagger(z) \\
&= \max(\inf\{s'(z') : z' \geq z\}, \inf\{s'(z') : z' \leq z\}) && \text{by Def. 65} \\
&= \max(\inf\{s(1 - z') : z' \geq z\}, \\
&\qquad \inf\{s(1 - z') : z' \leq z\}) && \text{by definition of } s' \\
&= \max(\inf\{s(z'') : z'' \leq 1 - z\}, \\
&\qquad \inf\{s(z'') : z'' \geq 1 - z\}) && \text{by substitution } z'' = 1 - z' \\
&= s^\ddagger(1 - z)\,, && \text{by Def. 65}
\end{aligned}$$

as desired.

Lemma 48 *For all $s, s' \in \mathbb{L}$ with $s'(z) = s(1 - z)$ for all $z \in \mathbf{I}$, it holds that*

a. ${s'}_*^{\top,0} = 1 - s_*^{\bot,0}$;

b. ${s'}_*^{\bot,0} = 1 - s_*^{\top,0}$.

Proof Suppose $s, s' \in \mathbb{L}$ satisfy

$$s'(z) = s(1-z) \tag{340}$$

for all $z \in \mathbf{I}$.

a. To see that the equation of case **a.** holds, we simply notice that

$$\begin{aligned}
{s'}_*^{\top,0} &= \sup {s'^{\ddagger}}^{-1}(0) && \text{by (42)} \\
&= \sup\{z \in \mathbf{I} : s'^{\ddagger}(z) = 0\} \\
&= \sup\{z \in \mathbf{I} : s^{\ddagger}(1-z) = 0\} && \text{by L-47 and (340)} \\
&= \sup\{1-z' \in \mathbf{I} : s^{\ddagger}(z') = 0\} && \text{by substitution } z' = 1-z \\
&= 1 - \inf\{z' \in \mathbf{I} : s^{\ddagger}(z') = 0\} \\
&= 1 - s_*^{\perp,0}\,. && \text{by (43)}
\end{aligned}$$

b. To see that the second equation also holds, we simply notice that (340) entails that $s(z) = s'(1-z)$ for all $z \in \mathbf{I}$, i.e. the 'roles' of s and s' are interchangeable. Therefore

$$\begin{aligned}
{s'}_*^{\perp,0} &= 1 - (1 - {s'}_*^{\perp,0}) \\
&= 1 - s_*^{\top,0}\,. && \text{by part } \textbf{a.} \text{ of the lemma}
\end{aligned}$$

Lemma 49 *For all $s, s' \in \mathbb{L}$ with $s'(z) = s(1-z)$ for all $z \in \mathbf{I}$, it holds that*

a. ${s'}_*^{\geq\frac{1}{2}} = s_*^{\leq\frac{1}{2}}$;

b. ${s'}_*^{\leq\frac{1}{2}} = s_*^{\geq\frac{1}{2}}$.

Proof Let us first prove that part **a.** of the lemma is valid. Hence suppose that s' is defined in terms of $s \in \mathbb{L}$ as stated in the lemma. Then

$$\begin{aligned}
{s'}_*^{\geq\frac{1}{2}} &= \inf\{s'(z) : z \geq \tfrac{1}{2}\} && \text{by (47)} \\
&= \inf\{s(1-z) : z \geq \tfrac{1}{2}\} && \text{by definition of } s' \\
&= \inf\{s(z') : z' \leq \tfrac{1}{2}\} && \text{by substitution } z' = 1-z \\
&= s_*^{\leq\frac{1}{2}}\,.
\end{aligned}$$

This also proves part **b.** because s and s' are interchangeable, i.e. it also holds that $s(z) = s'(1-z)$ for all $z \in \mathbf{I}$.

Lemma 50 *Suppose $s \in \mathbb{L}$ is concave, i.e. for all $z_1 \leq z_2 \leq z_3$, it holds that*

$$s(z_2) \leq \max(s(z_1), s(z_3))\,. \tag{341}$$

Then $s^{\ddagger} = s$.

Proof Let a choice of $s \in \mathbb{L}$ be given such that (341) holds. By Def. 60, $s^{-1}(0) \neq \varnothing$. We can hence choose some $z_0 \in \mathbf{I}$ with $s(z_0) = 0$. Now consider $z \in \mathbf{I}$. If $z \geq z_0$, then for all $z' \geq z$,

$$\begin{aligned} s(z) &\leq \max(s(z_0), s(z')) && \text{by (341)} \\ &\leq \max(0, s(z')) && \text{because } z_0 \in s^{-1}(0) \\ &= s(z')\,, && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

i.e.

$$s(z) \leq s(z')\,. \tag{342}$$

Hence

$$\begin{aligned} \inf\{s(z') : z' \geq z\} &\leq \inf\{s(z) : z' \geq z\} && \text{by (342)} \\ &= \inf\{s(z)\} \\ &= s(z)\,. \end{aligned}$$

On the other hand, $z \in \{z' \in \mathbf{I} : z' \geq z\}$ and hence $\inf\{s(z') : z' \geq z\} \leq s(z)$. Combining both inequations, we obtain that

$$\inf\{s(z') : z' \geq z\} = s(z)\,. \tag{343}$$

Therefore

$$\begin{aligned} s^{\ddagger}(z) &= \inf\{s(z') : z' \geq z\} && \text{by L-43} \\ &= s(z)\,. && \text{by (343)} \end{aligned}$$

In the remaining case that $z \leq z_0$, we can proceed analogously. Then for all $z' \leq z$, we again have

$$\begin{aligned} s(z) &\leq \max(s(z_0), s(z')) && \text{by (341)} \\ &\leq \max(0, s(z')) && \text{because } z_0 \in s^{-1}(0) \\ &= s(z')\,, && \text{because } 0 \text{ is identity of } \max \end{aligned}$$

i.e.

$$s(z) \leq s(z')\,. \tag{344}$$

Therefore

$$\begin{aligned} \inf\{s(z') : z' \leq z\} &\leq \inf\{s(z) : z' \leq z\} && \text{by (344)} \\ &= \inf\{s(z)\} \\ &= s(z)\,. \end{aligned}$$

We again notice that $z \in \{z' \in \mathbf{I} : z' \leq z\}$ and hence $\inf\{s(z') : z' \leq z\} \leq s(z)$. Combining both inequations proves that

$$\inf\{s(z') : z' \leq z\} = s(z)\,. \tag{345}$$

Therefore

$$\begin{aligned} s^{\ddagger}(z) &= \inf\{s(z') : z' \leq z\} && \text{by L-43} \\ &= s(z)\,. && \text{by (345)} \end{aligned}$$

Lemma 51 *For all $s \in \mathbb{L}$, ${s^{\ddagger}}^{\ddagger} = s^{\ddagger}$.*

Proof By Th-47.c, $s^{\ddagger}$ is concave. Hence by L-50, ${s^{\ddagger}}^{\ddagger} = s^{\ddagger}$, as desired.

Lemma 52 *For all $s \in \mathbb{L}$,*

a. ${s^{\ddagger}}_*^{\top,0} = s_*^{\top,0}$;

b. ${s^{\ddagger}}_*^{\perp,0} = s_*^{\perp,0}$.

Proof Consider some $s \in \mathbb{L}$. Concerning **a.**, we notice that

$$\begin{aligned} {s^{\ddagger}}_*^{\top,0} &= \sup {{s^{\ddagger}}^{\ddagger}}^{-1}(0) && \text{by (42)} \\ &= \sup {s^{\ddagger}}^{-1}(0) && \text{by L-51} \\ &= s_*^{\top,0}\,. && \text{by (42)} \end{aligned}$$

Similarly in the case of **b.**,

$$\begin{aligned} {s^{\ddagger}}_*^{\perp,0} &= \inf {{s^{\ddagger}}^{\ddagger}}^{-1}(0) && \text{by (43)} \\ &= \inf {s^{\ddagger}}^{-1}(0) && \text{by L-51} \\ &= s_*^{\perp,0}\,. && \text{by (43)} \end{aligned}$$

Lemma 53 *For all $s \in \mathbb{L}$,*

a. ${s^{\ddagger}}_*^{\geq\frac{1}{2}} = s_*^{\geq\frac{1}{2}}$;

b. ${s^{\ddagger}}_*^{\leq\frac{1}{2}} = s_*^{\leq\frac{1}{2}}$.

Proof Consider a choice of $s \in \mathbb{L}$ and denote by z_0 an arbitrary element $z_0 \in s^{-1}(0)$.
First I prove part **a.** of the lemma. This is trivial if $z_0 \geq \frac{1}{2}$. We then have

$$\begin{aligned} s_*^{\geq\frac{1}{2}} &= \inf\{s(z) : z \geq \tfrac{1}{2}\} && \text{by (47)} \\ &\leq s(z_0) && \text{because } z_0 \geq \tfrac{1}{2} \\ &= 0\,, && \text{by choice of } z_0 \end{aligned}$$

i.e.

$$s_*^{\geq \frac{1}{2}} = 0\,.$$

We recall that by Th-47.a, $s^\ddagger(z_0) \leq s(z_0) = 0$, i.e.

$$s^\ddagger(z_0) = 0\,. \tag{346}$$

Hence

$$\begin{aligned} s^\ddagger{}_*^{\geq \frac{1}{2}} &= \inf\{s^\ddagger(z) : z \geq \tfrac{1}{2}\} && \text{by (47)} \\ &\leq s^\ddagger(z_0) && \text{because } z_0 \geq \tfrac{1}{2} \\ &= 0\,, \end{aligned}$$

i.e.

$$s^\ddagger{}_*^{\geq \frac{1}{2}} = 0\,.$$

This finishes the proof that $s^\ddagger{}_*^{\geq \frac{1}{2}} = 0 = s_*^{\geq \frac{1}{2}}$. In the remaining case that $z_0 < \frac{1}{2}$, we notice that

$$s_*^{\geq \frac{1}{2}} = s^\ddagger(\tfrac{1}{2}) \tag{347}$$

by (47) and L-43. For similar reasons, it holds that

$$s^\ddagger{}_*^{\geq \frac{1}{2}} = s^{\ddagger\ddagger}(\tfrac{1}{2})\,; \tag{348}$$

this is apparent from L-43 and (346). Therefore

$$\begin{aligned} s_*^{\geq \frac{1}{2}} &= s^\ddagger(\tfrac{1}{2}) && \text{by (347)} \\ &= s^{\ddagger\ddagger}(\tfrac{1}{2}) && \text{by L-51} \\ &= s^\ddagger{}_*^{\geq \frac{1}{2}}\,. && \text{by (348)} \end{aligned}$$

Hence part **a.** of the lemma is valid. To see that part **b.** holds as well, let us define $s' \in \mathbb{L}$ by $s'(z) = s(1-z)$ for all $z \in \mathbf{I}$. We can then proceed as follows.

$$\begin{aligned} s_*^{\leq \frac{1}{2}} &= s'{}_*^{\geq \frac{1}{2}} && \text{by L-49} \\ &= s'^\ddagger{}_*^{\geq \frac{1}{2}} && \text{by part } \textbf{a.} \text{ of the lemma} \\ &= s^\ddagger{}_*^{\leq \frac{1}{2}}\,. && \text{by L-47 and L-49} \end{aligned}$$

Lemma 54 *For all $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$, it holds that*

a. $s_*^{\top,0} \leq s'{}_*^{\top,0}$*;*

b. $s_*^{\perp,0} \leq s'{}_*^{\perp,0}$.

Proof Let $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$ be given. We conclude from L-46 that $s^\ddagger \trianglelefteq s'^\ddagger$, i.e. for all $z \in \mathbf{I}$, there exists $z' \geq z$ with

$$s'^\ddagger(z') \leq s^\ddagger(z)\,, \tag{349}$$

and for all $z' \in \mathbf{I}$, there exists $z \leq z'$ with

$$s^\ddagger(z) \leq s'^\ddagger(z')\,. \tag{350}$$

Based on these inequations, the lemma is straightforward. I first consider case **a.** of the lemma. Now let $\varepsilon > 0$. Let us recall from (42) that $s_*^{\top,0} = \sup s^{\ddagger^{-1}}(0)$. Hence there exists $z \in s^{\ddagger^{-1}}(0)$ with

$$z > s_*^{\top,0} - \varepsilon\,. \tag{351}$$

By (349), there exists

$$z' \geq z \tag{352}$$

with $s'^\ddagger(z') \leq s^\ddagger(z)$. We notice that $s^\ddagger(z) \leq s(z)$ by Th-47.a and that $s(z) = 0$ because $z \in s^{-1}(0)$. Hence $s'^\ddagger(z') = 0$, i.e.

$$z' \in s'^{\ddagger^{-1}}(0)\,. \tag{353}$$

I conclude that

$$\begin{aligned}
s'^{\top,0}_* &= \sup s'^{\ddagger^{-1}}(0) && \text{by (42)}\\
&\geq z' && \text{because } z' \in s'^{\ddagger^{-1}}(0) \text{ by (353)}\\
&\geq z && \text{by (352)}\\
&> s_*^{\top,0} - \varepsilon\,. && \text{by (351)}
\end{aligned}$$

$\varepsilon \to 0$ proves the desired $s'^{\top,0}_* \geq s_*^{\top,0}$.
To see that part **b.** of the lemma also holds, we observe that $s \sqsubseteq s'$ entails that $\bar{s}' \sqsubseteq \bar{s}$, where $\bar{s}, \bar{s}' \in \mathbb{L}$ are defined by $\bar{s}(z) = s(1-z)$ and $\bar{s}'(z) = s'(1-z)$ for all $z \in \mathbf{I}$. Therefore

$$\begin{aligned}
s_*^{\perp,0} &= 1 - \bar{s}_*^{\top,0} && \text{by L-48}\\
&\leq 1 - \bar{s}'^{\top,0}_* && \text{by part } \textbf{a.} \text{ of the lemma}\\
&= s'^{\perp,0}_*\,. && \text{by L-48}
\end{aligned}$$

Lemma 55 *For all $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$, it holds that*

a. $s_*^{\geq\frac{1}{2}} \geq s'^{\geq\frac{1}{2}}_*$;

b. $s_*^{\leq\frac{1}{2}} \leq s'^{\leq\frac{1}{2}}_*$.

Proof Suppose that $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$ are given. We know from Def. 64 that for all $z \in \mathbf{I}$, there exists $z' \geq z$ with

$$s'(z') \leq s(z) \tag{354}$$

and that for all $z' \in \mathbf{I}$, there exists $z \leq z'$ with

$$s(z) \leq s'(z')\,. \tag{355}$$

As concerns part **a.** of the lemma, we first recall from (47) that

$$s_*^{\geq \frac{1}{2}} = \inf\{s(z) : z \geq \tfrac{1}{2}\}\,. \tag{356}$$

Now let $\varepsilon > 0$. We conclude from (356) that there exists $z \in [\frac{1}{2}, 1]$ with

$$s(z) < s_*^{\geq \frac{1}{2}} + \varepsilon\,. \tag{357}$$

By (354), there exists $z' \geq z$ with $s'(z') \leq s(z)$, hence

$$s'(z') < s_*^{\geq \frac{1}{2}} + \varepsilon \tag{358}$$

by (357). Therefore

$$\begin{aligned}
{s'}_*^{\geq \frac{1}{2}} &= \inf\{s'(z) : z \geq \tfrac{1}{2}\} && \text{by (47)}\\
&\leq s'(z') && \text{because } z' \geq z \geq \tfrac{1}{2}\\
&< s_*^{\geq \frac{1}{2}} + \varepsilon\,. && \text{by (358)}
\end{aligned}$$

$\varepsilon \to 0$ yields the desired ${s'}_*^{\geq \frac{1}{2}} \leq s_*^{\geq \frac{1}{2}}$.
Now let us consider part **b.** of the lemma. We define $\bar{s}, \bar{s}' \in \mathbb{L}$ by $\bar{s}(z) = s(1-z)$ and $\bar{s}'(z) = s'(1-z)$ for all $z \in \mathbf{I}$. It is then apparent from Def. 64 that $s \trianglelefteq s'$ entails that $\bar{s}' \trianglelefteq \bar{s}$. Therefore

$$\begin{aligned}
s_*^{\leq \frac{1}{2}} &= \bar{s}_*^{\geq \frac{1}{2}} && \text{by L-49}\\
&\leq {\bar{s}'}_*^{\geq \frac{1}{2}} && \text{by part } \textbf{a.} \text{ of the lemma}\\
&= {s'}_*^{\leq \frac{1}{2}}\,. && \text{by L-49}
\end{aligned}$$

Lemma 56 *For all $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$, it holds that*

a. $s_*^{\geq \frac{1}{2}} \geq {s'}_*^{\geq \frac{1}{2}}$;

b. $s_*^{\leq \frac{1}{2}} \leq {s'}_*^{\leq \frac{1}{2}}$.

Proof By L-55, $(\bullet)_*^{\geq\frac{1}{2}}$ and $(\bullet)_*^{\leq\frac{1}{2}}$ are monotonic with respect to $\trianglelefteq$. By L-53, $(\bullet)_*^{\geq\frac{1}{2}}$ and $(\bullet)_*^{\leq\frac{1}{2}}$ are ‡-invariant. We can hence apply Th-49 to deduce that $(\bullet)_*^{\geq\frac{1}{2}}$ and $(\bullet)_*^{\leq\frac{1}{2}}$ are monotonic with respect to $\sqsubseteq$, as desired.

Proof of Theorem 50

By Th-44, it is sufficient for $\mathcal{F}_M = \mathcal{F}_{\omega_M}$ to be a standard DFS (ω-1)–(ω-4) are valid for ω_M.

ω_M **satisfies** (ω-1) Let us consider a choice of $s \in \mathbb{L}$ such that

$$s^{-1}([0,1)) = \{a\} \tag{359}$$

for some $a \in \mathbf{I}$. Then

$$\begin{aligned} s_*^{\perp,0} &= \inf s^{\ddagger^{-1}}(0) && \text{by (43)} \\ &= \inf s^{-1}(0) && \text{by L-50 and } s \text{ concave} \\ &= \inf\{a\} && \text{by (359)} \\ &= a\,, \end{aligned}$$

i.e.

$$s_*^{\perp,0} = a\,. \tag{360}$$

By the same reasoning, we also obtain

$$s_*^{\top,0} = a\,. \tag{361}$$

Concerning the coefficients $s_*^{\leq\frac{1}{2}}$ and $s_*^{\geq\frac{1}{2}}$, we obtain from (46) and (359) that

$$s_*^{\leq\frac{1}{2}} = \inf\{s(z) : z \leq \tfrac{1}{2}\} = \begin{cases} 1 & : \quad a > \frac{1}{2} \\ 0 & : \quad a \leq \frac{1}{2} \end{cases} \tag{362}$$

Similarly by (47) and (359),

$$s_*^{\geq\frac{1}{2}} = \sup\{s(z) : z \geq \tfrac{1}{2}\} = \begin{cases} 1 & : \quad a < \frac{1}{2} \\ 0 & : \quad a \geq \frac{1}{2} \end{cases} \tag{363}$$

Hence if $a > \frac{1}{2}$, then $s_*^{\perp,0} = a > \frac{1}{2}$ and $s_*^{\leq\frac{1}{2}} = 1$ by (360) and (362). In turn, we obtain from Def. 67 that

$$\omega_M(s) = \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) = \min(a, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) = \min(a,1) = a\,.$$

If $a < \frac{1}{2}$, then $s_*^{\top,0} = a < \frac{1}{2}$ and $s_*^{\geq\frac{1}{2}} = 1$ by (361) and (363). In this case, we obtain from Def. 67 that

$$\omega_M(s) = \max(s_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\geq\frac{1}{2}}) = \max(a, \tfrac{1}{2} - \tfrac{1}{2}\cdot 1) = \max(a, 0) = a\,.$$

Finally if $a = \frac{1}{2}$, then $s_*^{\top,0} = s_*^{\perp,0} = \frac{1}{2}$ by (360) and (361). Therefore $\omega_M(s) = \frac{1}{2}$ by Def. 67.

ω_M **satisfies** (ω-2) Let $s \in \mathbb{L}$ be given and suppose that $s' \in \mathbb{L}$ is defined by $s'(z) = s(1-z)$. If ${s'}_*^{\perp,0} > \frac{1}{2}$, then

$$\begin{aligned}
\omega_M(s') &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}{s'}_*^{\leq\frac{1}{2}}) && \text{by Def. 67}\\
&= \min(1 - s_*^{\top,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\geq\frac{1}{2}}) && \text{by L-48, L-49}\\
&= \min(1 - s_*^{\top,0}, 1 - (\tfrac{1}{2} - \tfrac{1}{2}s_*^{\geq\frac{1}{2}}))\\
&= 1 - \max(s_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\geq\frac{1}{2}}) && \text{by De Morgan's law}\\
&= 1 - \omega_M(s)\,, && \text{by Def. 67}
\end{aligned}$$

where the last step holds because $s_*^{\top,0} = 1 - {s'}_*^{\perp,0} < \frac{1}{2}$ by L-48.
In the case that ${s'}_*^{\top,0} < \frac{1}{2}$, we notice that $s_*^{\perp,0} = 1 - {s'}_*^{\top,0} > \frac{1}{2}$ by L-48. Therefore

$$\begin{aligned}
\omega_M(s') &= \max({s'}_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}{s'}_*^{\geq\frac{1}{2}}) && \text{by Def. 67}\\
&= \max(1 - s_*^{\perp,0}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by L-48, L-49}\\
&= \max(1 - s_*^{\perp,0}, 1 - (\tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}))\\
&= 1 - \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by De Morgan's law}\\
&= 1 - \omega_M(s)\,. && \text{by Def. 67}
\end{aligned}$$

Finally if ${s'}_*^{\perp,0} \leq \frac{1}{2} \leq {s'}_*^{\top,0}$, then $s_*^{\perp,0} = 1 - {s'}_*^{\top,0} \leq \frac{1}{2} \leq 1 - {s'}_*^{\perp,0} = s_*^{\top,0}$ by L-48. Hence $\omega_M(s') = \frac{1}{2} = \omega_M(s)$ by Def. 67. In particular, $\omega_M(s') = 1 - \omega_M(s)$.

ω_M **satisfies** (ω-3) Consider a choice of $s \in \mathbb{L}$ with $s(1) = 0$ and $s^{-1}([0,1)) \subseteq \{0,1\}$, i.e. $s(z) = 1$ for all $z \in (0,1)$. We then obtain from Def. 65 that

$$s^\ddagger(z) = \begin{cases} 0 & : \quad z = 1 \\ s(0) & : \quad z < 1 \end{cases}$$

for all $z \in \mathbf{I}$. Therefore

$$s_*^{\top,0} = \sup {s^\ddagger}^{-1}(0) = 1 \tag{364}$$

$$s_*^{\perp,0} = \inf {s^\ddagger}^{-1}(0) = \begin{cases} 1 & : \quad s(0) > 0 \\ 0 & : \quad s(0) = 0 \end{cases} \tag{365}$$

by (42) and (43), resp. We further notice that

$$s_*^{\le\frac{1}{2}} = \inf\{s(z) : z \le \tfrac{1}{2}\} = s(0)\,, \tag{366}$$

which is apparent from (46) and the assumed properties of s. In the following, we discern two cases. If $s(0) = 0$, then $s_*^{\perp,0} = 0$ by (365) and $s_*^{\top,0} = 1$ by (364). Hence by Def. 67, $\omega_M(s) = \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{2} + \frac{1}{2}s(0)$, as desired. In the remaining case that $s(0) > 0$, we know from (365) that $s_*^{\perp,0} = 1$. Therefore

$$\begin{aligned}
\omega_M(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\le\frac{1}{2}}) && \text{by Def. 67}\\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by (365), (366)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,.
\end{aligned}$$

This proves that (ω-3) is indeed valid.

ω_M satisfies (ω-4) Let $s, s' \in \mathbb{L}$ be given with $s \sqsubseteq s'$. We know from L-54 and L-56 that

$$s_*^{\top,0} \le {s'}_*^{\top,0} \tag{367}$$

$$s_*^{\perp,0} \le {s'}_*^{\perp,0} \tag{368}$$

$$s_*^{\ge\frac{1}{2}} \ge {s'}_*^{\ge\frac{1}{2}} \tag{369}$$

$$s_*^{\le\frac{1}{2}} \le {s'}_*^{\le\frac{1}{2}}\,. \tag{370}$$

If ${s'}_*^{\perp,0} > \frac{1}{2}$ and $s_*^{\perp,0} \le \frac{1}{2}$, then $\omega_M(s') \ge \frac{1}{2} \ge \omega_M(s)$ by Def. 67. Similarly if ${s'}_*^{\top,0} \ge \frac{1}{2}$, $s_*^{\perp,0} \le \frac{1}{2}$ and $s_*^{\top,0} \ge \frac{1}{2}$, then $\omega_M(s') = \frac{1}{2} = \omega_M(s)$ by Def. 67. Hence there are only two critical cases, viz. ${s'}_*^{\perp,0} \ge s_*^{\perp,0} > \frac{1}{2}$ and $s_*^{\top,0} \le {s'}_*^{\top,0} < \frac{1}{2}$. It is sufficient to prove the monotonic behaviour of ω_M in the first case because the second case can be reduced to the first one through negation, noting that $s \sqsubseteq s'$ if and only if $\bar{s}' \sqsubseteq \bar{s}$, where $\bar{s}(z) = s(1-z)$ and $\bar{s}'(z) = s'(1-z)$ for all $z \in \mathbf{I}$. Hence let us consider the first case and assume that ${s'}_*^{\perp,0} \ge s_*^{\perp,0} > \frac{1}{2}$. Then

$$\begin{aligned}
\omega_M(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\le\frac{1}{2}}) && \text{by Def. 67}\\
&\le \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}{s'}_*^{\le\frac{1}{2}}) && \text{by (368), (370)}\\
&= \omega_M(s')\,. && \text{by Def. 67}
\end{aligned}$$

This proves that (ω-4) holds, as desired. I have hence shown that ω_M satisfies (ω-1)–(ω-4) and by Th-44, $\mathcal{F}_M$ is a standard DFS.

A.20 Proof of Theorem 51

To see that $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS, consider $S, S' \in \mathbb{K}$ defined by

$$S(\gamma) = \begin{cases} [\frac{3}{4}, 1] & : \quad \gamma \leq \frac{1}{4} \\ [\frac{1}{2}, 1] & : \quad \gamma > \frac{1}{4} \end{cases} \tag{371}$$

$$S'(\gamma) = \begin{cases} [\frac{3}{4}, 1] & : \quad \gamma \leq \frac{1}{4} \\ (\frac{1}{2}, 1] & : \quad \gamma > \frac{1}{4} \end{cases} \tag{372}$$

for all $\gamma \in \mathbf{I}$. We then obtain for the mappings $s, s' \in \mathbb{L}$ defined by Def. 53 in terms of S and S' that

$$s(z) = \begin{cases} 0 & : \quad z \geq \frac{3}{4} \\ \frac{1}{4} & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{373}$$

$$s'(z) = \begin{cases} 0 & : \quad z \geq \frac{3}{4} \\ \frac{1}{4} & : \quad z > \frac{1}{2} \\ 1 & : \quad z \leq \frac{1}{2} \end{cases} \tag{374}$$

for all $z \in \mathbf{I}$. Let us now consider the coefficients used in the definition of ω_M. We notice that s, s' are concave, i.e. $s^\ddagger = s$ and ${s'}^\ddagger = s'$ by L-50. Hence by (373), (374) and (43),

$$s_*^{\perp,0} = \inf {s^\ddagger}^{-1}(0) = \inf s^{-1}(0) = \inf[\tfrac{3}{4}, 1] = \tfrac{3}{4} \tag{375}$$

$${s'}_*^{\perp,0} = \inf {{s'}^\ddagger}^{-1}(0) = \inf {s'}^{-1}(0) = \inf[\tfrac{3}{4}, 1] = \tfrac{3}{4} \tag{376}$$

For $s_*^{\leq\frac{1}{2}}$ and ${s'}_*^{\leq\frac{1}{2}}$, we obtain from (373), (374) and (46) that

$$s_*^{\leq\frac{1}{2}} = \inf\{s(z) : z \leq \tfrac{1}{2}\} = \tfrac{1}{4} \tag{377}$$

$${s'}_*^{\leq\frac{1}{2}} = \inf\{s'(z) : z \leq \tfrac{1}{2}\} = 1\,. \tag{378}$$

Therefore

$$\omega_M(s) = \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) = \min(\tfrac{3}{4}, \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{4}) = \min(\tfrac{3}{4}, \tfrac{5}{8}) = \tfrac{5}{8} \tag{379}$$

by Def. 67, (375) and (377). Similarly, we have

$$\omega_M(s') = \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}{s'}_*^{\leq\frac{1}{2}}) = \min(\tfrac{3}{4}, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) = \min(\tfrac{3}{4}, 1) = \tfrac{3}{4} \tag{380}$$

by Def. 67 and (376), (378). Now let us recall that by Th-33, there exist $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S = S_{Q,X} \tag{381}$$

$$S' = S_{Q',X} \tag{382}$$

By Th-41, we then have

$$s = s_{Q,X} \tag{383}$$
$$s' = s_{Q',X}\,. \tag{384}$$

We hence have

$$\mathcal{F}_M(Q)(X) = \omega_M(s_{Q,X}) = \omega_M(s) = \tfrac{5}{8}$$

and

$$\mathcal{F}_M(Q')(X) = \omega_M(s_{Q',X}) = \omega_M(s') = \tfrac{3}{4}$$

by Def. 61, (383), (384), (379) and (380). In particular,

$$\mathcal{F}_M(Q)(X) \neq \mathcal{F}_M(Q')(X)\,. \tag{385}$$

Now consider an arbitrary mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. By Th-37, $\mathcal{F}_\xi = \mathcal{F}_\Omega$, where

$$\Omega(S) = \xi(\top_S, \bot_S)$$

for all $S \in \mathbb{K}$. In the present case of $S, S' \in \mathbb{K}$ defined by (371) and (372), we obtain from (35) and (36) that

$$\begin{aligned}
\top_S(\gamma) &= \sup S(\gamma) = 1\\
\bot_S(\gamma) &= \inf S(\gamma) = \begin{cases} \frac{3}{4} & : \quad \gamma \leq \frac{1}{4} \\ \frac{1}{2} & : \quad \gamma > \frac{1}{4} \end{cases}\\
\top_{S'}(\gamma) &= \sup S'(\gamma) = 1\\
\bot_{S'}(\gamma) &= \inf S'(\gamma) = \begin{cases} \frac{3}{4} & : \quad \gamma \leq \frac{1}{4} \\ \frac{1}{2} & : \quad \gamma > \frac{1}{4} \end{cases}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. In particular,

$$(\top_S, \bot_S) = (\top_{S'}, \bot_{S'})\,. \tag{386}$$

Therefore

$$\begin{aligned}
\mathcal{F}_\xi(Q)(X) &= \xi(\top_{S_{Q,X}}, \bot_{S_{Q,X}}) && \text{by Th-37}\\
&= \xi(\top_S, \bot_S) && \text{by (381)}\\
&= \xi(\top_{S'}, \bot_{S'}) && \text{by (386)}\\
&= \xi(\top_{S_{Q',X}}, \bot_{S_{Q',X}}) && \text{by (382)}\\
&= \mathcal{F}_\xi(Q')(X)\,. && \text{by Th-37}
\end{aligned}$$

Hence $\mathcal{F}_\xi(Q)(X) = \mathcal{F}_\xi(Q')(X)$ in every $\mathcal{F}_\xi$-QFM, but $\mathcal{F}_M(Q)(X) \neq \mathcal{F}_M(Q')(X)$ by (385). This proves that $\mathcal{F}_M$ is not an $\mathcal{F}_\xi$-DFS.

A.21 Proof of Theorem 52

Lemma 57 *For all $s, s' \in \mathbb{L}$ with $s'(z) = s(1-z)$ for all $z \in \mathbf{I}$, it holds that*

a. ${s'}_1^{\top,*} = 1 - s_1^{\perp,*}$;

b. ${s'}_1^{\perp,*} = 1 - s_1^{\top,*}$.

Proof Suppose that s' is defined in terms of $s \in \mathbb{L}$ as stated in the lemma. We then obtain that

$$\begin{aligned}
s_1^{\top,*} &= \sup {s'}^{-1}([0,1)) && \text{by (44)}\\
&= \sup\{z \in \mathbf{I} : s'(z) < 1\}\\
&= \sup\{z \in \mathbf{I} : s(1-z) < 1\} && \text{because } s'(z) = s(1-z)\\
&= \sup\{1-z' \in \mathbf{I} : s(z') < 1\} && \text{by substutution } z' = 1-z\\
&= 1 - \inf\{z' \in \mathbf{I} : s(z') < 1\}\\
&= 1 - \inf s^{-1}([0,1))\\
&= 1 - s_1^{\perp,*}. && \text{by (45)}
\end{aligned}$$

This proves part **a.** of the lemma. As concerns part **b.**, we notice that the roles of s and s' are interchangeable, i.e. it also holds that $s(z) = s'(1-z)$ for all $z \in \mathbf{I}$. Therefore

$$\begin{aligned}
s_1^{\perp,*} &= 1 - (1 - s_1^{\perp,*})\\
&= 1 - {s'}_1^{\top,*}. && \text{by part } \textbf{a.} \text{ of the lemma}
\end{aligned}$$

Lemma 58 *For all $s \in \mathbb{L}$,*

a. ${s^\ddagger}_1^{\top,*} = s_1^{\top,*}$;

b. ${s^\ddagger}_1^{\perp,*} = s_1^{\perp,*}$.

Proof I first prove part **a.** of the lemma. Hence let $s \in \mathbb{L}$ be given. Recalling that $s^\ddagger \leq s$ by Th-47, we deduce that

$$\begin{aligned}
s_1^{\top,*} &= \sup\{z \in \mathbf{I} : s(z) < 1\} && \text{by (44)}\\
&\leq \sup\{z \in \mathbf{I} : s^\ddagger(z) < 1\} && \text{because } s^\ddagger \leq s\\
&= {s^\ddagger}_1^{\top,*},
\end{aligned}$$

i.e.

$$s_1^{\top,*} \leq {s^\ddagger}_1^{\top,*}. \tag{387}$$

Let us choose some $z_0 \in s^{-1}(0)$. It is apparent from (44) that $z_0 \leq s_1^{\top,*}$; hence also $z_0 \leq {s^\ddagger}_1^{\top,*}$ by (387). In the following, it is therefore sufficient to consider $z \geq z_0$. Then

$$s^\ddagger(z) = \inf\{s(z') : z' \geq z\} \tag{388}$$

by L-43. Hence

$${s^\ddagger}_1^{\top,*} = \sup\{z \in \mathbf{I} : \inf\{s(z') : z' \geq z\} < 1\}\,. \tag{389}$$

Now let $\varepsilon > 0$. We conclude from (389) that there exists a choice of $z \in \mathbf{I}$ with

$$z > {s^\ddagger}_1^{\top,*} - \varepsilon \tag{390}$$

and

$$\inf\{s(z') : z' \geq z < 1\}\,. \tag{391}$$

It is then apparent from (391) that there exists $z' \in \mathbf{I}$ with $z' \geq z$ and $s(z') < 1$. Therefore

$$\begin{aligned} s_1^{\top,*} &= \sup\{z \in \mathbf{I} : s(z) < 1\} && \text{by (44)} \\ &\geq z' && \text{because } s(z') < 1 \\ &> {s^\ddagger}_1^{\top,*} - \varepsilon\,. && \text{by (390) and } z' \geq z \end{aligned}$$

$\varepsilon \to 0$ yields $s_1^{\top,*} \geq {s^\ddagger}_1^{\top,*}$. Recalling (387), we obtain the desired $s_1^{\top,*} = {s^\ddagger}_1^{\top,*}$, i.e. part **a.** of the lemma is valid. As concerns part **b.**, let us define $s' \in \mathbb{L}$ by $s'(z) = s(1-z)$ for all $z \in \mathbf{I}$. Then

$$\begin{aligned} s_1^{\perp,*} &= 1 - {s'}_1^{\top,*} && \text{by L-57} \\ &= 1 - {s'^\ddagger}_1^{\top,*} && \text{by part \textbf{a.} of the lemma} \\ &= {s^\ddagger}_1^{\perp,*}\,. && \text{by L-57 and L-47} \end{aligned}$$

Lemma 59 *For all $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$,*

a. $s_1^{\top,*} \leq {s'}_1^{\top,*}$*;*

b. $s_1^{\perp,*} \leq {s'}_1^{\perp,*}$.

Proof Suppose that $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$. I first show that part **a.** of the lemma is valid. Hence let $\varepsilon > 0$. Recalling that $s_1^{\top,*} = \sup s^{-1}([0,1))$ by (44), there exists $z \in \mathbf{I}$ with

$$s(z) < 1 \tag{392}$$

and

$$z > s_1^{\top,*} - \varepsilon\,. \tag{393}$$

It is then immediate from Def. 64 and $s \trianglelefteq s'$ that there exists $z' \geq z$ with $s'(z') \leq s(z)$. Hence $z' > s_1^{\top,*} - \varepsilon$ by (393) and $s'(z') < 1$ by (392), i.e. $z' \in {s'}^{-1}([0,1))$. We conclude that

$$\begin{aligned} {s'}_1^{\top,*} &= \sup {s'}^{-1}([0,1)) && \text{by (44)} \\ &\geq z' && \text{because } z' \in {s'}^{-1}([0,1)) \\ &> s_1^{\top,*} - \varepsilon\,. \end{aligned}$$

$\varepsilon \to 0$ proves the desired ${s'}_1^{\top,*} \geq s_1^{\top,*}$.
To see that part **b.** of the lemma holds as well, define $\bar{s}, \bar{s'} \in \mathbb{L}$ by $\bar{s}(z) = s(1-z)$, $\bar{s'}(z) = s'(1-z)$ for all $z \in \mathbf{I}$. It is then obvious from Def. 64 and $s \trianglelefteq s'$ that $\bar{s'} \trianglelefteq \bar{s}$. Therefore

$$\begin{aligned} s_1^{\perp,*} &= 1 - \bar{s}_1^{\top,*} && \text{by L-57} \\ &\leq 1 - \bar{s'}_1^{\top,*} && \text{by part } \textbf{a.} \text{ of present lemma} \\ &= {s'}_1^{\perp,*}\,. && \text{by L-57} \end{aligned}$$

Lemma 60 *For all $s, s' \in \mathbb{L}$ with $s \sqsubseteq s'$, it holds that*

a. $s_1^{\top,*} \geq {s'}_1^{\top,*}$;

b. $s_1^{\perp,*} \leq {s'}_1^{\perp,*}$.

Proof By L-59, $(\bullet)_1^{\top,*}$ and $(\bullet)_1^{\perp,*}$ are monotonic with respect to $\trianglelefteq$. By L-58, $(\bullet)_1^{\top,*}$ and $(\bullet)_1^{\perp,*}$ are also ‡-invariant. We can hence apply Th-49 and conclude that $(\bullet)_1^{\top,*}$ and $(\bullet)_1^{\perp,*}$ are monotonic with respect to $\sqsubseteq$, as stated in the lemma.

Proof of Theorem 52

By Th-44, it is sufficient for $\mathcal{F}_P = \mathcal{F}_{\omega_P}$ to be a standard DFS that ω_P satisfy (ω-1)–(ω-4). Hence let us consider these conditions in turn.

ω_P **satisfies** (ω-1) Let us consider a choice of $s \in \mathbb{L}$ such that

$$s^{-1}([0,1)) = \{a\} \tag{394}$$

for some $a \in \mathbf{I}$. Let us recall from the proof of Th-50 that

$$s_*^{\perp,0} = a\,. \tag{395}$$

$$s_*^{\top,0} = a\,, \tag{396}$$

see equations (360) and (361) above. Concerning the coefficients $s_*^{\leq \frac{1}{2}}$ and $s_*^{\geq \frac{1}{2}}$, we recall equations (362) and (363), viz

$$s_*^{\leq \frac{1}{2}} = \begin{cases} 1 & : \quad a > \frac{1}{2} \\ 0 & : \quad a \leq \frac{1}{2} \end{cases} \tag{397}$$

$$s_*^{\geq \frac{1}{2}} = \begin{cases} 1 & : \quad a < \frac{1}{2} \\ 0 & : \quad a \geq \frac{1}{2} \end{cases} \tag{398}$$

Finally we need to consider the coefficients $s_1^{\top,*}$ and $s_1^{\perp,*}$, thus

$$s_1^{\top,*} = \sup s^{-1}([0,1)) = \sup\{a\} = a \tag{399}$$

by (44) and (394), and

$$s_1^{\perp,*} = \inf s^{-1}([0,1)) = \inf\{a\} = a \tag{400}$$

by (45) and (394).

Hence if $a > \frac{1}{2}$, then $s_*^{\perp,0} = a > \frac{1}{2}$, $s_*^{\leq \frac{1}{2}} = 1$ and $s_1^{\top,*} = a$ by (395), (397) and (399), resp. In turn, we obtain from Def. 68 that

$$\omega_P(s) = \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) = \min(a, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) = \min(a, 1) = a\,.$$

If $a < \frac{1}{2}$, then $s_*^{\top,0} = a < \frac{1}{2}$, $s_*^{\geq \frac{1}{2}} = 1$ and $s_1^{\perp,*} = a$ by (396), (398) and (400), resp. In this case, we obtain from Def. 68 that

$$\omega_P(s) = \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) = \max(a, \tfrac{1}{2} - \tfrac{1}{2} \cdot 1) = \max(a, 0) = a\,.$$

Finally if $a = \frac{1}{2}$, then $s_*^{\top,0} = s_*^{\perp,0} = \frac{1}{2}$ by (395) and (396). Therefore $\omega_P(s) = \frac{1}{2}$ by Def. 68. This completes the proof that ω_P satisfies (ω-1).

ω_P satisfies (ω-2) Let $s \in \mathbb{L}$ be given and suppose that $s' \in \mathbb{L}$ is defined by $s'(z) = s(1-z)$. If ${s'}_*^{\perp,0} > \frac{1}{2}$, then

$$\begin{aligned}
\omega_P(s') &= \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq \frac{1}{2}}) && \text{by Def. 68} \\
&= \min(1 - s_1^{\perp,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by L-57, L-49} \\
&= \min(1 - s_1^{\perp,*}, 1 - (\tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}})) \\
&= 1 - \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by De Morgan's law} \\
&= 1 - \omega_P(s)\,, && \text{by Def. 68}
\end{aligned}$$

where the last step holds because $s_*^{\top,0} = 1 - {s'}_*^{\perp,0} < \frac{1}{2}$ by L-48.
In the case that ${s'}_*^{\top,0} < \frac{1}{2}$, we notice that $s_*^{\perp,0} = 1 - {s'}_*^{\top,0} > \frac{1}{2}$ by L-48. Therefore

$$
\begin{aligned}
\omega_P(s') &= \max({s'}_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2}{s'}_*^{\geq\frac{1}{2}}) && \text{by Def. 68} \\
&= \max(1 - s_1^{\top,*}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by L-57, L-49} \\
&= \max(1 - s_1^{\top,*}, 1 - (\tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}})) \\
&= 1 - \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by De Morgan's law} \\
&= 1 - \omega_P(s)\,. && \text{by Def. 68}
\end{aligned}
$$

Finally if ${s'}_*^{\perp,0} \leq \frac{1}{2} \leq {s'}_*^{\top,0}$, then $s_*^{\perp,0} = 1 - {s'}_*^{\top,0} \leq \frac{1}{2} \leq 1 - {s'}_*^{\perp,0} = s_*^{\top,0}$ by L-48. Hence $\omega_P(s') = \frac{1}{2} = \omega_P(s)$ by Def. 68. In particular, $\omega_P(s') = 1 - \omega_P(s)$.

ω_P **satisfies** (ω-3) Consider a choice of $s \in \mathbb{L}$ with $s(1) = 0$ and $s^{-1}([0,1)) \subseteq \{0,1\}$, i.e. $s(z) = 1$ for all $z \in (0,1)$. Let us notice that equations (364), (365) and (366) are valid in the present case, too. Hence

$$s_*^{\top,0} = 1 \tag{401}$$

$$s_*^{\perp,0} = \begin{cases} 1 & : \quad s(0) > 0 \\ 0 & : \quad s(0) = 0 \end{cases} \tag{402}$$

and

$$s_*^{\leq\frac{1}{2}} = s(0)\,. \tag{403}$$

As regards the coefficient $s_1^{\top,*}$, we observe that

$$s_1^{\top,*} = \sup s^{-1}([0,1)) = 1 \tag{404}$$

by (44) because $s(1) = 0$, i.e. $1 \in s^{-1}([0,1))$. In the following, we discern two cases. If $s(0) = 0$, then $s_*^{\perp,0} = 0$ by (402) and $s_*^{\top,0} = 1$ by (401). Hence by Def. 68, $\omega_P(s) = \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\cdot 0 = \frac{1}{2} + \frac{1}{2}s(0)$, as desired. In the remaining case that $s(0) > 0$, we know from (402) that $s_*^{\perp,0} = 1$. Therefore

$$
\begin{aligned}
\omega_P(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by Def. 68} \\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by (403), (404)} \\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,.
\end{aligned}
$$

This proves that (ω-3) is indeed valid.

ω_P **satisfies** (ω-4) Let $s, s' \in \mathbb{L}$ be given with $s \sqsubseteq s'$. We know from L-54, L-56 and L-60 that

$$s_*^{\top,0} \le {s'}_*^{\top,0} \tag{405}$$

$$s_*^{\perp,0} \le {s'}_*^{\perp,0} \tag{406}$$

$$s_*^{\ge\frac{1}{2}} \ge {s'}_*^{\ge\frac{1}{2}} \tag{407}$$

$$s_*^{\le\frac{1}{2}} \le {s'}_*^{\le\frac{1}{2}} \tag{408}$$

$$s_1^{\top,*} \le {s'}_1^{\top,*} \tag{409}$$

$$s_1^{\perp,*} \le {s'}_1^{\perp,*} . \tag{410}$$

If ${s'}_*^{\perp,0} > \frac{1}{2}$ and $s_*^{\perp,0} \le \frac{1}{2}$, then $\omega_P(s') \ge \frac{1}{2} \ge \omega_P(s)$ by Def. 68. Similarly if ${s'}_*^{\top,0} \ge \frac{1}{2}$, $s_*^{\perp,0} \le \frac{1}{2}$ and $s_*^{\top,0} \ge \frac{1}{2}$, then $\omega_P(s') = \frac{1}{2} = \omega_P(s)$ by Def. 68. Hence there are only two critical cases, viz. ${s'}_*^{\perp,0} \ge s_*^{\perp,0} > \frac{1}{2}$ and $s_*^{\top,0} \le {s'}_*^{\top,0} < \frac{1}{2}$. Like in the case of Th-50, it is sufficient to prove the monotonic behaviour of ω_P in the first case because the second case can be reduced to the first one through negation, again noting that $s \sqsubseteq s'$ if and only if $\bar{s}' \sqsubseteq \bar{s}$, where $\bar{s}(z) = s(1-z)$ and $\bar{s}'(z) = s'(1-z)$ for all $z \in \mathbf{I}$. Hence let us consider the first case and assume that ${s'}_*^{\perp,0} \ge s_*^{\perp,0} > \frac{1}{2}$. Then

$$\begin{aligned}
\omega_P(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\le\frac{1}{2}}) && \text{by Def. 68} \\
&\le \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\le\frac{1}{2}}) && \text{by (408), (409)} \\
&= \omega_P(s') . && \text{by Def. 68}
\end{aligned}$$

To sum up, ω_P satisfies all conditions (ω-1)–(ω-4). We can hence conclude from Th-44 that $\mathcal{F}_P$ is a standard DFS.

A.22 Proof of Theorem 53

To see that $\mathcal{F}_P$ is not an $\mathcal{F}_\xi$-DFS, we consider the same choice of $S, S' \in \mathbb{K}$, $s, s' \in \mathbb{L}$ and $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ as in the proof of Th-51. We already know that $\mathcal{F}_\xi(Q)(X) = \mathcal{F}_\xi(Q')(X)$ for all $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. Hence $\mathcal{F}_P$ is not an $\mathcal{F}_\xi$-DFS if we can show that $\mathcal{F}_P(Q)(X) \ne \mathcal{F}_P(Q')(X)$. We recall that the coefficients $s_*^{\perp,0}$, ${s'}_*^{\perp,0}$, $s_*^{\le\frac{1}{2}}$ and ${s'}_*^{\le\frac{1}{2}}$ are given by equations (375), (376), (377) and (378), respectively. For the coefficients $s_1^{\top,*}$ and ${s'}_1^{\top,*}$, we obtain from (44) and (373), (374) that

$$s_1^{\top,*} = \sup s^{-1}([0,1)) = 1 \tag{411}$$

$${s'}_1^{\top,*} = \sup s^{-1}([0,1)) = 1 \tag{412}$$

because $s(1) = s'(1) = 0$. Therefore

$$\begin{aligned}
\mathcal{F}_P(Q)(X) &= \omega_P(s_{Q,X}) && \text{by Def. 61}\\
&= \omega_P(s) && \text{by (383)}\\
&= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by Def. 68 and (375)}\\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{4}) && \text{by (411), (377)}\\
&= \tfrac{5}{8}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}_P(Q')(X) &= \omega_P(s_{Q',X}) && \text{by Def. 61}\\
&= \omega_P(s') && \text{by (384)}\\
&= \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq \frac{1}{2}}) && \text{by Def. 68 and (376)}\\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (412), (378)}\\
&= 1\,.
\end{aligned}$$

In particular $\mathcal{F}_P(Q)(X) \neq \mathcal{F}_P(Q')(X)$. Hence $\mathcal{F}_P$ cannot be an $\mathcal{F}_\xi$-DFS.

A.23 Proof of Theorem 54

Lemma 61 *Let $s \in \mathbb{L}$ and $z_0 \in {s^\ddagger}^{-1}(0)$ be given. Then*

a. If $z \geq z_0$, then $s^\ddagger(z) = \inf\{s(z') : z' \geq z\}$.

b. If $z \leq z_0$, then $s^\ddagger(z) = \inf\{s(z') : z' \leq z\}$.

Note. The lemma is very similar to L-43, but this time z_0 is chosen from ${s^\ddagger}^{-1}(0)$ and not from $s^{-1}(0)$.

Proof Let $s \in \mathbb{L}$ be given and suppose z_0 is a choice of $z_0 \in {s^\ddagger}^{-1}(0) \neq \varnothing$. We shall further choose some $z_0' \in s^{-1}(0) \neq \varnothing$. Now consider $z \in \mathbf{I}$.

a.: $z \geq z_0$. It is convenient to discern two cases. If $z \geq z_0'$, then

$$s^\ddagger(z) = \inf\{s(z') : z' \leq z\}$$

by L-43. In the remaining case that $z_0 \leq z < z_0'$, we notice that

$$\begin{aligned}
\inf\{s(z') : z' \leq z\} &\leq \inf\{s(z') : z' \leq z_0\}\\
&= s^\ddagger(z_0) && \text{by L-43}\\
&= 0\,, && \text{because } z_0 \in {s^\ddagger}^{-1}(0)
\end{aligned}$$

i.e.

$$\inf\{s(z') : z' \leq z\} = 0\,.$$

In addition, we know that $s^\ddagger$ is concave, see Th-47.c. Hence

$$\begin{aligned} s^\ddagger(z) &\leq \max(s^\ddagger(z_0), s^\ddagger(z_0')) && \text{by Th-47.c} \\ &= 0\,, && \text{because } s^\ddagger(z_0) = s^\ddagger(z_0') = 0 \end{aligned}$$

i.e.

$$s^\ddagger(z) = 0\,.$$

Hence $s^\ddagger(z) = 0 = \inf\{s(z') : z' \leq z\}$.

b.: $z \leq z_0$**.** Again we discern two cases. If $z \leq z_0'$, then we obtain from L-43 that

$$s^\ddagger(z) = \inf\{s(z') : z' \leq z\}\,,$$

as desired. In the remaining case that $z_0 \geq z > z_0'$, we observe that

$$\begin{aligned} \inf\{s(z') : z' \leq z\} &\leq s(z_0') && \text{because } z_0' < z \\ &= 0\,, && \text{because } z_0' \in s^{-1}(0) \end{aligned}$$

i.e. $\inf\{s(z') : z' \leq z\} = 0$. In addition, we have

$$\begin{aligned} s^\ddagger(z) &= \inf\{s(z') : z' \geq z\} && \text{by L-43} \\ &\leq \inf\{s(z') : z' \geq z_0\} && \text{because } z_0 \geq z \\ &= s^\ddagger(z_0) && \text{by L-43} \\ &= 0\,, && \text{because } z_0 \in {s^\ddagger}^{-1}(0) \end{aligned}$$

i.e. $s^\ddagger(z) = 0 = \inf\{s(z') : z' \leq z\}$.

Lemma 62 *For all $s \in \mathbb{L}$,*

$$\omega_Z(s) = \begin{cases} \min(s_1^{\top,*}, \frac{1}{2} + \frac{1}{2}s_*^{\leq\frac{1}{2}}) & : \quad {s^\ddagger}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing \\ \max(s_1^{\perp,*}, \frac{1}{2} - \frac{1}{2}s_*^{\geq\frac{1}{2}}) & : \quad {s^\ddagger}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing \end{cases}$$

Proof It is apparent from Def. 69 that the equation stated in the lemma holds in the case that ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$ or ${s^\ddagger}^{-1}(0) \subseteq [0, \frac{1}{2}]$. In the remaining case that both ${s^\ddagger}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and ${s^\ddagger}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, we conclude from the fact that $s^\ddagger$ is concave by Th-47.c that

$$s^\ddagger(\tfrac{1}{2}) = 0\,. \tag{413}$$

Hence

$$s_*^{\leq \frac{1}{2}} = \inf\{s(z) : z \leq \tfrac{1}{2}\} = s^{\ddagger}(\tfrac{1}{2}) = 0 \tag{414}$$

by (46), L-61 and (413). Similarly

$$s_*^{\geq \frac{1}{2}} = \inf\{s(z) : z \geq \tfrac{1}{2}\} = s^{\ddagger}(\tfrac{1}{2}) = 0 \tag{415}$$

by (46), L-61 and (413). We also notice that

$$\begin{aligned}
& s_1^{\top,*} \\
&= {s^{\ddagger}}_1^{\top,*} && \text{by L-58} \\
&= \sup {s^{\ddagger}}^{-1}([0,1)) && \text{by (44)} \\
&\geq \sup {s^{\ddagger}}^{-1}(0) && \text{by monotonicity of } \sup \\
&\geq \tfrac{1}{2}
\end{aligned}$$

because ${s^{\ddagger}}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, i.e.

$$s_1^{\top,*} \geq \tfrac{1}{2}\,. \tag{416}$$

By similar reasoning

$$\begin{aligned}
& s_1^{\perp,*} \\
&= {s^{\ddagger}}_1^{\perp,*} && \text{by L-58} \\
&= \inf {s^{\ddagger}}^{-1}([0,1)) && \text{by (45)} \\
&\leq \inf {s^{\ddagger}}^{-1}(0) && \text{by monotonicity of } \inf \\
&\leq \tfrac{1}{2}
\end{aligned}$$

because ${s^{\ddagger}}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Hence

$$s_1^{\perp,*} \leq \tfrac{1}{2}\,. \tag{417}$$

We conclude that

$$\begin{aligned}
\min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} \cdot 0) && \text{by (414)} \\
&= \tfrac{1}{2} && \text{by (416)} \\
&= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} \cdot 0) && \text{by (417)} \\
&= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}})\,, && \text{by (415)}
\end{aligned}$$

which coincides with the desired $\omega_Z(s) = \frac{1}{2}$ by Def. 69.

Lemma 63 *For all $s \in \mathbb{L}$, $\omega_Z(s^{\ddagger}) = \omega_Z(s)$.*

Proof It is obvious from L-62 that

$$\omega_Z(s) = f(s_1^{\top,*}, s_1^{\perp,*}, s_*^{\le\frac{1}{2}}, s_*^{\ge\frac{1}{2}}, s^{\ddagger^{-1}}(0)) \tag{418}$$

for all $s \in \mathbb{L}$, where $f : \mathbf{I}^4 \times \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ is defined by

$$f(a,b,c,d,e) = \begin{cases} \min(a, \frac{1}{2} + \frac{1}{2}c) & : \quad e \subseteq [\frac{1}{2}, 1] \\ \max(b, \frac{1}{2} - \frac{1}{2}e) & : \quad e \subseteq [0, \frac{1}{2}] \text{ and } e \neq \{\frac{1}{2}\} \\ \frac{1}{2} & : \quad \text{else} \end{cases} \tag{419}$$

for all $a, b, c, d \in \mathbf{I}$ and $e \in \mathcal{P}(\mathbf{I})$. We further notice that

$$(s^\ddagger)^{\ddagger^{-1}}(0) = s^{\ddagger^{-1}}(0) \tag{420}$$

by L-51. Hence for all $s \in \mathbb{L}$

$$\begin{aligned} \omega_Z(s^\ddagger) &= f((s^\ddagger)_1^{\top,*}, (s^\ddagger)_1^{\perp,*}, (s^\ddagger)_*^{\le\frac{1}{2}}, (s^\ddagger)_*^{\ge\frac{1}{2}}, (s^\ddagger)^{\ddagger^{-1}}(0)) && \text{by (419)} \\ &= f(s_1^{\top,*}, s_1^{\perp,*}, s_*^{\le\frac{1}{2}}, s_*^{\ge\frac{1}{2}}, s^{\ddagger^{-1}}(0)) && \text{by L-58, L-53 and (420)} \\ &= \omega_Z(s)\,, && \text{by (419)} \end{aligned}$$

as desired.

Lemma 64 *For all $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$, it holds that $\omega_Z(s) \le \omega_Z(s')$.*

Proof To see this, consider a choice of $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$. Then $s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ entails that ${s'}^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Utilizing L-62, it is hence sufficient to discern the following three cases.

$s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Then ${s'}^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ as well. Hence

$$\begin{aligned} \omega_Z(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\le\frac{1}{2}}) && \text{by L-62} \\ &\le \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\le\frac{1}{2}}) && \text{by L-59, L-55} \\ &= \omega_Z(s')\,. && \text{by L-62} \end{aligned}$$

$s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] = \varnothing$ and ${s'}^{\ddagger^{-1}}(0) \neq \varnothing$. Then $s^{\ddagger^{-1}} \cap [0, \frac{1}{2}] \neq \varnothing$, in particular

$$s_1^{\perp,*} = \inf s^{\ddagger^{-1}}(0) \le \tfrac{1}{2} \tag{421}$$

by (45). By similar reasoning, ${s'}^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ entails that

$${s'}_1^{\top,*} = \sup s^{\ddagger^{-1}}(0) \ge \tfrac{1}{2} \tag{422}$$

by (44). Therefore

$$
\begin{aligned}
\omega_Z(s) &= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\geq\frac{1}{2}}) && \text{by L-62}\\
&\leq \tfrac{1}{2} && \text{by (421)}\\
&\leq \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2}{s'}_*^{\leq\frac{1}{2}}) && \text{by (422)}\\
&= \omega_Z(s')\,. && \text{by L-62}
\end{aligned}
$$

${s'}^{\ddagger^{-1}}(0)\cap[\frac{1}{2},1] = \varnothing$. Then $s^{\ddagger^{-1}}(0)\cap[\frac{1}{2},1] = \varnothing$ as well because $s \trianglelefteq s'$. We conclude that $s^{\ddagger^{-1}}(0) \cap [0, \frac{1}{2}] \neq \varnothing$ and ${s'}^{\ddagger^{-1}}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Therefore

$$
\begin{aligned}
\omega_Z(s) &= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\geq\frac{1}{2}}) && \text{by L-62}\\
&\leq \max({s'}_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2}{s'}_*^{\geq\frac{1}{2}}) && \text{by L-59, L-55}\\
&= \omega_Z(s')\,. && \text{by L-62}
\end{aligned}
$$

Proof of Theorem 54

I utilize Th-44 in order to prove that $\mathcal{F}_Z$ is a standard DFS by showing that ω_Z satisfies (ω-1) to (ω-4).

ω_Z satisfies (ω-1). Let us consider a choice of $s \in \mathbb{L}$ such that

$$s^{-1}([0,1)) = \{a\} \tag{423}$$

for some $a \in \mathbf{I}$. As concerns the coefficients $s_*^{\leq\frac{1}{2}}$ and $s_*^{\geq\frac{1}{2}}$, we recall equations (362) and (363), viz

$$s_*^{\leq\frac{1}{2}} = \begin{cases} 1 & : \quad a > \frac{1}{2} \\ 0 & : \quad a \leq \frac{1}{2} \end{cases} \tag{424}$$

$$s_*^{\geq\frac{1}{2}} = \begin{cases} 1 & : \quad a < \frac{1}{2} \\ 0 & : \quad a \geq \frac{1}{2} \end{cases} \tag{425}$$

Finally as concerns the coefficients $s_1^{\top,*}$ and $s_1^{\perp,*}$, we recall from (399) and (400) that

$$s_1^{\top,*} = a \tag{426}$$

and

$$s_1^{\perp,*} = a\,. \tag{427}$$

Let us further observe that

$$s^{\ddagger} = s\,, \tag{428}$$

which is apparent from Def. 65 and (423). Therefore

$$s^{\ddagger^{-1}}(0) = s^{-1}(0) = \{a\} \tag{429}$$

by (428) and (423).
Hence if $a > \frac{1}{2}$, then $s^{\ddagger^{-1}}(0) = \{a\} \subseteq [0, \frac{1}{2}]$ by (429). Therefore

$$\begin{aligned}
\omega_Z(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by Def. 69}\\
&= \min(a, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (426) and (424)}\\
&= a\,.
\end{aligned}$$

In the case that $a = \frac{1}{2}$, $s^{\ddagger^{-1}}(0) = \{\frac{1}{2}\} \subseteq [\frac{1}{2}, 1]$ by (429) and hence

$$\begin{aligned}
\omega_Z(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by Def. 69}\\
&= \min(\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2} \cdot 0) && \text{by (426) and (424)}\\
&= \tfrac{1}{2}\,.
\end{aligned}$$

Finally in the case that $a < \frac{1}{2}$, we know from (429) that $s^{\ddagger^{-1}}(0) = \{a\} \subseteq [0, \frac{1}{2}]$. Therefore

$$\begin{aligned}
\omega_Z(s) &= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by Def. 69}\\
&= \max(a, \tfrac{1}{2} - \tfrac{1}{2} \cdot 1) && \text{by (427), (425)}\\
&= a\,.
\end{aligned}$$

ω_Z satisfies (ω-2). Let $s \in \mathbb{L}$ be given and suppose that $s' \in \mathbb{L}$ is defined by $s'(z) = s(1-z)$. If $s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $s'^{\ddagger^{-1}}(0) \cap [0, \frac{1}{2}] \neq \varnothing$ by L-47. Therefore

$$\begin{aligned}
\omega_Z(s') &= \min(s'^{\top,*}_1, \tfrac{1}{2} + \tfrac{1}{2} s'^{\leq \frac{1}{2}}_*) && \text{by L-62}\\
&= \min(1 - s_1^{\perp,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by L-57, L-49}\\
&= \min(1 - s_1^{\perp,*}, 1 - (\tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}))\\
&= 1 - \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by De Morgan's law}\\
&= 1 - \omega_Z(s)\,. && \text{by L-62}
\end{aligned}$$

In the remaining case that ${s^{\ddagger}}^{-1}(0) \cap [0, \frac{1}{2}] \neq \frac{1}{2}$, we again conclude from L-47 that ${s'^{\ddagger}}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Therefore

$$\begin{aligned}
\omega_Z(s') &= \max({s'}_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2}{s'}_*^{\geq\frac{1}{2}}) && \text{by L-62} \\
&= \max(1 - s_1^{\top,*}, \tfrac{1}{2} - \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by L-57, L-49} \\
&= \max(1 - s_1^{\top,*}, 1 - (\tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}})) && \\
&= 1 - \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by De Morgan's law} \\
&= 1 - \omega_Z(s)\,. && \text{by L-62}
\end{aligned}$$

ω_Z **satisfies** (ω-3). Consider a choice of $s \in \mathbb{L}$ with $s(1) = 0$ and $s^{-1}([0, 1)) \subseteq \{0, 1\}$, i.e. $s(z) = 1$ for all $z \in (0, 1)$. Let us notice that equation (366) is valid in the present case, too. Hence

$$s_*^{\leq\frac{1}{2}} = s(0)\,. \tag{430}$$

As regards the coefficient $s_1^{\top,*}$, we recall from (404) that

$$s_1^{\top,*} = 1\,. \tag{431}$$

Let us also notice that

$${s^{\ddagger}}^{-1}(0) = \begin{cases} \{1\} & : \quad s(0) > 0 \\ \mathbf{I} & : \quad s(0) = 0 \end{cases} \tag{432}$$

In the following, we discern two cases. If $s(0) = 0$, then ${s^{\ddagger}}^{-1}(0) \not\subseteq [\frac{1}{2}, 1]$ and ${s^{\ddagger}}^{-1}(0) \not\subseteq [0, \frac{1}{2}]$ by (432). Hence $\omega_Z(s) = \frac{1}{2} = \frac{1}{2} + \frac{1}{2}s(0)$ by Def. 69.
In the remaining case that $s(0) > 0$, we know from (432) that ${s^{\ddagger}}^{-1}(0) = \{1\} \subseteq [\frac{1}{2}, 1]$. Therefore

$$\begin{aligned}
\omega_Z(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2}s_*^{\leq\frac{1}{2}}) && \text{by Def. 69} \\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by (430), (431)} \\
&= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,.
\end{aligned}$$

This proves that (ω-3) is indeed valid.

ω_Z **satisfies** (ω-4). This is apparent from L-63 and L-64, recalling theorem Th-49.

A.24 Proof of Theorem 55

Lemma 65 *For all $s \in \mathbb{L}$,*

a. if $s_*^{\perp,0} > \frac{1}{2}$, then $\omega_P(s) = \omega_Z(s)$;

b. if $s_*^{\top,0} < \frac{1}{2}$, then $\omega_P(s) = \omega_Z(s)$.

Proof Let us first consider case **a.** of the lemma, i.e. $s_*^{\perp,0} > \frac{1}{2}$. By (43), this means that $s_*^{\perp,0} = \inf s^{\ddagger -1}(0) > \frac{1}{2}$. Hence $s^{\ddagger -1}(0) \subseteq [\frac{1}{2}, 1]$ and

$$\begin{aligned} \omega_P(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by Def. 68} \\ &= \omega_Z(s)\,. && \text{by Def. 69} \end{aligned}$$

Now we consider case **b.** of the lemma, i.e. $s_*^{\top,0} < \frac{1}{2}$. We observe that $s_*^{\top,0} = \sup s^{\ddagger -1}(0) < \frac{1}{2}$, in particular $s^{\ddagger -1}(0) \subseteq [0, \frac{1}{2}]$. Therefore

$$\begin{aligned} \omega_P(s) &= \max(s_1^{\perp,*}, \tfrac{1}{2} - \tfrac{1}{2} s_*^{\geq \frac{1}{2}}) && \text{by Def. 68} \\ &= \omega_Z(s)\,. && \text{by Def. 69} \end{aligned}$$

Proof of Theorem 55

The very same example as in the proof of Th-53 can be used to show that $\mathcal{F}_Z$ is not an $\mathcal{F}_\xi$-DFS, noticing that in the example, $s_*^{\perp,0} = {s'}_*^{\perp,0} = \frac{3}{4}$, i.e. $\omega_Z(s) = \omega_P(s)$ and $\omega_Z(s') = \omega_P(s')$ by L-65.

A.25 Proof of Theorem 56

Lemma 66 *For all $s \in \mathbb{L}$, $s^\ddagger(0) = s(0)$ and $s^\ddagger(1) = s(1)$.*

Proof To see this, consider $s \in \mathbb{L}$. Clearly

$$\begin{aligned} s^\ddagger(0) &= \inf\{s(z) : z \leq 0\} && \text{by L-43} \\ &= \inf\{s(0)\} \\ &= s(0)\,. \end{aligned}$$

Similarly

$$\begin{aligned} s^\ddagger(1) &= \inf\{s(z) : z \geq 1\} && \text{by L-43} \\ &= \inf\{s(1)\} \\ &= s(1)\,, \end{aligned}$$

as desired.

Lemma 67 *ω_R is ‡-invariant, i.e. $\omega_R(s) = \omega_R(s^\ddagger)$ for all $s \in \mathbb{L}$.*

Proof Define $f : \mathbf{I}^4 \longrightarrow \mathbf{I}$ by

$$f(a,b,c,d) = \begin{cases} \min(a, \frac{1}{2} + \frac{1}{2}c) & : \ b \geq a > \frac{1}{2} \\ \max(b, \frac{1}{2} - \frac{1}{2}d) & : \ a \leq b < \frac{1}{2} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

for all $a, b, c, d \in \mathbf{I}$. It is apparent from Def. 70 that

$$\mathcal{F}_R(s) = f(s_*^{\perp,0}, s_*^{\top,0}, s(0), s(1)) \tag{433}$$

for all $s \in \mathbb{L}$. Therefore

$$\begin{aligned}
\omega_R(s^{\ddagger}) &= f({s'}_*^{\perp,0}, {s'}_*^{\top,0}, s^{\ddagger}(0), s^{\ddagger}(1)) && \text{by (433)} \\
&= f(s_*^{\perp,0}, s_*^{\top,0}, s(0), s(1)) && \text{by L-52, L-66} \\
&= \omega_R(s) && \text{by (433)}
\end{aligned}$$

for all $s \in \mathbb{L}$.

Lemma 68 *ω_R is monotonic with respect to $\trianglelefteq$, i.e. whenever $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$, it holds that $\omega_R(s) \leq \omega_R(s')$.*

Proof Let $s, s' \in \mathbb{L}$ be given with $s \trianglelefteq s'$. Hence by Def. 64,

$$\text{for all } z \in \mathbf{I}, \text{ there exists } z' \geq z \text{ with } s'(z') \leq s(z) \tag{434}$$

$$\text{for all } z' \in \mathbf{I}, \text{ there exists } z \leq z \text{ with } s(z) \leq s'(z') \tag{435}$$

It is apparent from (434) that

$$s(1) \geq s'(1)\,, \tag{436}$$

and it is apparent from (435) that

$$s(0) \leq s'(0)\,. \tag{437}$$

Let us now recall that

$$s_*^{\perp,0} \leq {s'}_*^{\perp,0} \tag{438}$$

$$s_*^{\top,0} \leq {s'}_*^{\top,0} \tag{439}$$

by L-54 and L-42. It is hence sufficient to discern the following five cases.

a.: $s_*^{\perp,0} > \frac{1}{2}$. Then ${s'}_*^{\perp,0} > \frac{1}{2}$ as well by (438). Hence

$$\begin{aligned}
\omega_R(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by Def. 70} \\
&\leq \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by (438), (437)} \\
&= \omega_R(s')\,. && \text{by Def. 70}
\end{aligned}$$

b.: ${s'}_*^{\perp,0} > \frac{1}{2}$ and $s_*^{\perp,0} \leq \frac{1}{2}$. Then $\omega_R(s) \leq \frac{1}{2} \leq \omega_R(s')$ by Def. 70.

c.: ${s'}_*^{\top,0} < \frac{1}{2}$. Then $s_*^{\top,0} < \frac{1}{2}$ also by (439). Hence

$$\begin{aligned}
\omega_R(s) &= \max(s_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s(1)) && \text{by Def. 70} \\
&\leq \max({s'}_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s'(1)) && \text{by (439), (436)} \\
&= \omega_R(s')\,. && \text{by Def. 70}
\end{aligned}$$

d.: ${s'}_*^{\perp,0} \le \frac{1}{2}$ **and** ${s'}_*^{\top,0} \ge \frac{1}{2}$ **and** $s_*^{\top,0} < \frac{1}{2}$**.** Then $\omega_R(s) \le \frac{1}{2} = \omega_R(s')$ by Def. 70.

e.: ${s'}_*^{\perp,0} \le \frac{1}{2}$ **and** $s_*^{\top,0} \ge \frac{1}{2}$**.** Then $s_*^{\perp,0} \le \frac{1}{2}$ and ${s'}_*^{\top,0} \ge \frac{1}{2}$ as well, see (438) and (439). Hence $\omega_R(s) = \frac{1}{2} = \omega_R(s')$ by Def. 70, in particular $\omega_R(s) \le \omega_R(s')$, as desired.

Proof of Theorem 56

By Th-44, we can show that $\mathcal{F}_R = \mathcal{F}_{\omega_R}$ is a standard DFS by proving that (ω-1)–(ω-4) are valid for ω_R. Hence let us consider these conditions in turn.

ω_R **satisfies** (ω-1)**.** Let $x \in \mathbf{I}$ be given and define $s \in \mathbb{L}$ by

$$s(z) = \begin{cases} 0 & : \quad z = x \\ 1 & : \quad \text{else} \end{cases}$$

for all $z \in \mathbf{I}$. It is then apparent from L-43 that $s^\ddagger = s$. Therefore

$$s_*^{\perp,0} = \inf {s^\ddagger}^{-1}(0) = \inf s^{-1}(0) = \inf\{x\} = x \tag{440}$$

$$s_*^{\top,0} = \sup {s^\ddagger}^{-1}(0) = \sup s^{-1}(0) = \sup\{x\} = x \tag{441}$$

by (43) and (42). In addition, we observe that

$$s(0) = \begin{cases} 0 & : \quad x = 0 \\ 1 & : \quad x > 0 \end{cases} \tag{442}$$

$$s(1) = \begin{cases} 0 & : \quad x = 1 \\ 1 & : \quad x < 1\,. \end{cases} \tag{443}$$

Hence if $x > \frac{1}{2}$, then $s_*^{\perp,0} = x > \frac{1}{2}$, i.e.

$$\begin{aligned} \omega_R(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by Def. 70} \\ &= \min(x, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (440), (442)} \\ &= x\,. \end{aligned}$$

In the case that $x = \frac{1}{2}$, we obtain from (440) and (441) that $s_*^{\perp,0} = s_*^{\top,0} = \frac{1}{2}$, and hence $\omega_R(s) = \frac{1}{2} = x$ by Def. 70.
Finally in the case that $x < \frac{1}{2}$, we know from (441) that

$$\begin{aligned} \omega_R(s) &= \max(s_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s(1)) && \text{by Def. 70} \\ &= \max(x, \tfrac{1}{2} - \tfrac{1}{2} \cdot 1) && \text{by (441), (443)} \\ &= x\,, \end{aligned}$$

i.e. ω_R satisfies (ω-1), as desired.

ω_R **satisfies** (ω-2). Let $s \in \mathbb{L}$ be given and define $s' \in \mathbb{L}$ by

$$s'(z) = s(1 - z)\,, \tag{444}$$

for all $z \in \mathbf{I}$. In particular

$$s'(0) = s(1) \tag{445}$$
$$s'(1) = s(0)\,. \tag{446}$$

Let us also recall from L-48 that ${s'}_*^{\perp,0} = 1 - s_*^{\top,0}$ and ${s'}_*^{\top,0} = 1 - s_*^{\perp,0}$. Hence if $s_*^{\perp,0} > \frac{1}{2}$, then ${s'}_*^{\top,0} = 1 - s_*^{\perp,0} < \frac{1}{2}$ and therefore

$$\begin{aligned}
\omega_R(s') &= \max({s'}_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s'(1)) && \text{by Def. 70}\\
&= \max(1 - s_*^{\perp,0}, \tfrac{1}{2} - \tfrac{1}{2}s(0)) && \text{by L-48 and (446)}\\
&= \max(1 - s_*^{\perp,0}, 1 - (\tfrac{1}{2} + \tfrac{1}{2}s(0)))\\
&= 1 - \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by De Morgan's Law}\\
&= 1 - \omega_R(s)\,.
\end{aligned}$$

In the case that $s_*^{\top,0} < \frac{1}{2}$, it holds that ${s'}_*^{\perp,0} = 1 - s_*^{\top,0} > \frac{1}{2}$ and hence

$$\begin{aligned}
\omega_R(s') &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s'(0)) && \text{by Def. 70}\\
&= \min(1 - s_*^{\top,0}, \tfrac{1}{2} + \tfrac{1}{2}s(1)) && \text{by L-48 and (445)}\\
&= \min(1 - s_*^{\top,0}, 1 - (\tfrac{1}{2} - \tfrac{1}{2}s(1)))\\
&= 1 - \max(s_*^{\top,0}, \tfrac{1}{2} - \tfrac{1}{2}s(1)) && \text{by De Morgan's Law}\\
&= 1 - \omega_R(s)\,. && \text{by Def. 70}
\end{aligned}$$

Finally in the case that $s_*^{\top,0} \geq \frac{1}{2}$ and $s_*^{\perp,0} \leq \frac{1}{2}$, we obtain from L-48 that ${s'}_*^{\top,0} = 1 - s_*^{\perp,0} \geq \frac{1}{2}$ and ${s'}_*^{\perp,0} = 1 - s_*^{\top,0} \leq \frac{1}{2}$. Hence $\omega_R(s) = \frac{1}{2}$ and $\omega_R(s') = \frac{1}{2}$ by Def. 70. In particular $\omega_R(s') = 1 - \omega_R(s)$.

ω_R **satisfies** (ω-3). To see this, consider $s \in \mathbb{L}$ with $s^{-1}([0,1)) \subseteq \{0,1\}$ and $s(1) = 0$. We can choose $z_0 = 1 \in s^{-1}(0)$. Hence for all $z < 1$,

$$s^{\ddagger}(z) = \inf\{s(z') : z' \leq z\} = s(0)$$

by L-43, i.e.

$$s^{\ddagger}(z) = \begin{cases} 0 & : \quad z = 1 \\ s(0) & : \quad z < 1 \end{cases} \tag{447}$$

for all $z \in \mathbf{I}$, and

$${s^{\ddagger}}^{-1}(0) = \begin{cases} \mathbf{I} & : \quad s(0) = 0 \\ \{1\} & : \quad s(0) > 0 \end{cases} \tag{448}$$

Hence if $s(0) = 0$, then

$$\begin{aligned} s_*^{\perp,0} &= \inf s^{\ddagger^{-1}}(0) && \text{by (43)} \\ &= \inf \mathbf{I} && \text{by (448)} \\ &= 0 \\ &\le \tfrac{1}{2} \end{aligned}$$

and

$$\begin{aligned} s_*^{\top,0} &= \sup s^{\ddagger^{-1}}(0) && \text{by (43)} \\ &= \sup \mathbf{I} && \text{by (448)} \\ &= 1 \\ &\ge \tfrac{1}{2}\,, \end{aligned}$$

i.e. $\omega_R(s) = \frac{1}{2} = \frac{1}{2} + \frac{1}{2} \cdot 0 = \frac{1}{2} + \frac{1}{2}s(0)$ by Def. 70.
In the remaining case that $s(0) > 0$, we obtain from (43) and (448) that

$$s_*^{\perp,0} = \inf s^{\ddagger^{-1}}(0) = \inf\{1\} = 1\,. \tag{449}$$

In particular, $s_*^{\perp,0} > \frac{1}{2}$ and hence

$$\begin{aligned} \omega_R(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by Def. 70} \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by (449)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,, \end{aligned}$$

which completes the proof that ω_R satisfies (ω-3).

ω_R satisfies (ω-4). This is apparent from L-67, L-68 and Th-49.

A.26 Proof of Theorem 57

To see that $\mathcal{F}_R$ is not an $\mathcal{F}_\xi$-DFS, consider $S, S' \in \mathbb{K}$ defined by

$$S(\gamma) = \begin{cases} \{1\} & : \gamma < \frac{1}{2} \\ (0,1] & : \gamma \ge \frac{1}{2} \end{cases} \tag{450}$$

$$S'(\gamma) = \begin{cases} \{1\} & : \gamma < \frac{1}{2} \\ [0,1] & : \gamma \ge \frac{1}{2} \end{cases} \tag{451}$$

for all $\gamma \in \mathbf{I}$. We then obtain for the mappings $s, s' \in \mathbb{L}$ defined by Def. 53 in terms of S and S' that

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} = \begin{cases} 0 & : z = 1 \\ \frac{1}{2} & : z \in (0,1) \\ 1 & : z = 0 \end{cases} \tag{452}$$

$$s'(z) = \inf\{\gamma \in \mathbf{I} : z \in S'(\gamma)\} = \begin{cases} 0 & : z = 1 \\ \frac{1}{2} & : z < 1 \end{cases} \tag{453}$$

for all $z \in \mathbf{I}$. Let us now consider the coefficients used in the definition of ω_R. We first notice that s, s' are concave, i.e. $s^\ddagger = s$ and ${s'}^\ddagger = s'$ by L-50. Hence by (452), (453) and (43),

$$s_*^{\perp,0} = \inf {s^\ddagger}^{-1}(0) = \inf s^{-1}(0) = \inf\{1\} = 1 \tag{454}$$

$${s'}_*^{\perp,0} = \inf {{s'}^\ddagger}^{-1}(0) = \inf {s'}^{-1}(0) = \inf\{1\} = 1 \tag{455}$$

Therefore

$$\begin{aligned} \omega_R(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by Def. 70} \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (454) and (452)} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \omega_R(s') &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s'(0)) && \text{by Def. 70} \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2}) && \text{by (455) and (453)} \\ &= \tfrac{3}{4}\,, \end{aligned}$$

i.e.

$$\omega_R(s) = 1 \tag{456}$$

and

$$\omega_R(s') = \tfrac{3}{4}\,. \tag{457}$$

Now let us recall that by Th-33, there exist $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S = S_{Q,X} \tag{458}$$

$$S' = S_{Q',X} \tag{459}$$

By Th-41, we then have

$$s = s_{Q,X} \tag{460}$$

$$s' = s_{Q',X}\,. \tag{461}$$

We hence have

$$\mathcal{F}_R(Q)(X) = \omega_R(s_{Q,X}) = \omega_R(s) = 1$$

and

$$\mathcal{F}_R(Q')(X) = \omega_R(s_{Q',X}) = \omega_R(s') = \tfrac{3}{4}$$

by Def. 61, (460), (461), (456) and (457). In particular,

$$\mathcal{F}_R(Q)(X) \neq \mathcal{F}_R(Q')(X)\,. \tag{462}$$

Now consider an arbitrary mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. By Th-37, $\mathcal{F}_\xi = \mathcal{F}_\Omega$, where

$$\Omega(S) = \xi(\top_S, \bot_S)$$

for all $S \in \mathbb{K}$. In the present case of $S, S' \in \mathbb{K}$ defined by (450) and (451), we obtain from (35) and (36) that

$$\begin{aligned}
\top_S(\gamma) &= \sup S(\gamma) = 1 \\
\bot_S(\gamma) &= \inf S(\gamma) = \begin{cases} 1 & : \quad \gamma < \frac{1}{2} \\ 0 & : \quad \gamma \geq \frac{1}{2} \end{cases} \\
\top_{S'}(\gamma) &= \sup S'(\gamma) = 1 \\
\bot_{S'}(\gamma) &= \inf S'(\gamma) = \begin{cases} 1 & : \quad \gamma < \frac{1}{2} \\ 0 & : \quad \gamma \geq \frac{1}{2} \end{cases}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. In particular,

$$(\top_S, \bot_S) = (\top_{S'}, \bot_{S'})\,. \tag{463}$$

Therefore

$$\begin{aligned}
\mathcal{F}_\xi(Q)(X) &= \xi(\top_{S_{Q,X}}, \bot_{S_{Q,X}}) && \text{by Th-37} \\
&= \xi(\top_S, \bot_S) && \text{by (458)} \\
&= \xi(\top_{S'}, \bot_{S'}) && \text{by (463)} \\
&= \xi(\top_{S_{Q',X}}, \bot_{S_{Q',X}}) && \text{by (459)} \\
&= \mathcal{F}_\xi(Q')(X)\,. && \text{by Th-37}
\end{aligned}$$

Hence $\mathcal{F}_\xi(Q)(X) = \mathcal{F}_\xi(Q')(X)$ in every $\mathcal{F}_\xi$-QFM, but $\mathcal{F}_R(Q)(X) \neq \mathcal{F}_R(Q')(X)$ by (462). This proves that $\mathcal{F}_R$ is not an $\mathcal{F}_\xi$-DFS.

A.27 Proof of Theorem 58

Lemma 69 *For all $S, S' \in \mathbb{K}$, it holds that $S \preceq_c S'$ if and only if the following conditions are satisfied for all $\gamma \in \mathbf{I}$.*

for all $z' \in S'(\gamma) \cap [\frac{1}{2}, 1]$, there exists $z \in S(\gamma) \cap [\frac{1}{2}, z']$; (464)

for all $z \in S(\gamma) \cap (\frac{1}{2}, 1]$, there exists $z' \in S'(\gamma) \cap [z, 1]$; (465)

for all $z' \in S'(\gamma) \cap [0, \frac{1}{2}]$, there exists $z \in S(\gamma) \cap [z', \frac{1}{2}]$; (466)

for all $z \in S(\gamma) \cap [0, \frac{1}{2})$, there exists $z' \in S'(\gamma) \cap [0, z]$. (467)

Proof Let us first notice that (464)–(467) are entailed by $S \preceq_c S'$. To see this, consider a choice of $S, S' \in \mathbb{K}$ with $S \preceq_c S'$. For $z' \in S'(\gamma) \cap [\frac{1}{2}, 1]$, we obtain from (52) that there exists $z \in S(\gamma)$ with $z \preceq_c z'$. We conclude from (5) and $z' \geq \frac{1}{2}$ that $\frac{1}{2} \leq z \leq z'$, i.e. $z \in S(\gamma) \cap [\frac{1}{2}, z']$. This proves that equation (464) holds. Now consider $z \in S(\gamma) \cap (\frac{1}{2}, 1]$. We know from (53) that there exists $z' \in S'(\gamma)$ with $z \preceq_c z'$. It is then obvious from $z > \frac{1}{2}$ and (5) that $z' \geq z$. Hence $z' \in S'(\gamma) \cap [z, 1]$, and (465) is valid. As regards $z' \in S'(\gamma) \cap [0, \frac{1}{2}]$, we know from (52) that there exists $z \in S(\gamma)$ with $z \preceq_c z'$. We obtain from (5) and $z' \leq \frac{1}{2}$ that $z' \leq z \leq \frac{1}{2}$. Therefore $z \in S(\gamma) \cap [z', \frac{1}{2}]$. In particular, equation (466) holds. Finally consider $z \in S(\gamma) \cap [0, \frac{1}{2})$. By (53), there exists $z' \in S'(\gamma)$ with $z \preceq_c z'$. We then deduce from (5) and $z < \frac{1}{2}$ that $z' \leq z$. Hence $z' \in S'(\gamma) \cap [0, z]$, which proves that equation (467) is indeed valid. To sum up, I have shown that the conditions (464)–(467) are entailed by $S \preceq_c S'$.

Now let us consider the converse entailment. Hence suppose that $S, S' \in \mathbb{K}$ satisfy (464)–(467). It must be shown that $S \preceq_c S'$, i.e. (53) and (52) must hold, see Def. 71. I will first prove that (52) holds. Hence let $\gamma \in \mathbf{I}$ and consider $z' \in S'(\gamma)$. If $z' \in [\frac{1}{2}, 1]$, then there exists $z \in S(\gamma) \cap [\frac{1}{2}, z']$ by (464). From (5) and $\frac{1}{2} \leq z \leq z'$, we obtain that the given choice of $z \in S(\gamma)$ satisfies $z \preceq_c z'$. In the remaining case that $z' \in [0, \frac{1}{2}]$, we conclude from (52) that there exists $z \in S(\gamma) \cap [z', \frac{1}{2}]$. We then obtain from (5) and $z' \leq z \leq \frac{1}{2}$ that the given choice of $z \in S(\gamma)$ satisfies $z \preceq_c z'$. This proves that condition (52) is valid.

As concerns (53), let $z \in S(\gamma)$ be given. If $z \in (\frac{1}{2}, 1]$, then (465) is applicable, which states that there exists $z' \in S'(\gamma) \cap [z, 1]$. We then conclude from (5) and $\frac{1}{2} < z \leq z'$ that the given choice of $z' \in S'(\gamma)$ satisfies $z \preceq_c z'$. In the case that $z = \frac{1}{2}$, consider a choice of $z'_0 \in S'(0) \neq \varnothing$. By (5), $z = \frac{1}{2} \preceq_c z'_0 \in S'(\gamma)$. Finally consider the case that $z \in [0, \frac{1}{2})$. Then (467) states that there exists $z' \in S'(\gamma) \cap [0, z]$. We obtain from (5) and $z' \leq z < \frac{1}{2}$ that the given choice of $z' \in S'(\gamma)$ satisfies $z \preceq_c z'$. Hence condition (53) is also valid. We conclude from Def. 71 that $S \preceq_c S'$, as desired.

Lemma 70 *Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers with $Q \preceq_c Q'$. Then for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{S}_{Q,X_1,\ldots,X_n} \preceq_c \mathcal{S}_{Q',X_1,\ldots,X_n}$$

Proof Consider a choice of semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ with $Q \preceq_c Q'$. Further suppose that $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ is a choice of fuzzy arguments. I will now show that the conditions (464)–(467) are valid for all $\gamma \in \mathbf{I}$. Hence consider $z' \in \mathcal{S}_{Q',X_1,\ldots,X_n}(\gamma) \cap [\frac{1}{2}, 1]$. Because $z' \in \mathcal{S}_{Q',X_1,\ldots,X_n}(\gamma)$, we conclude from Def. 51 that there exists a choice of $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$ with $z' = Q'(Y_1, \ldots, Y_n) \geq \frac{1}{2}$. We then conclude from $Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n)$ and (5) that $Q(Y_1, \ldots, Y_n) \in [\frac{1}{2}, Q'(Y_1, \ldots, Y_n)] = [\frac{1}{2}, z']$. Hence $z = Q(Y_1, \ldots, Y_n)$ satisfies $z \in \mathcal{S}_{Q,(X_1,\ldots,X_n)}(\gamma)$ and $z \in [\frac{1}{2}, z']$, i.e. condition (464) holds.

Now I consider (465). Hence let $z \in \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma) \cap (\frac{1}{2}, 1]$. Because $z \in \mathcal{S}_{Q,X_1,\ldots,X_n}(\gamma)$, we obtain from Def. 51 that $z = Q(Y_1, \ldots, Y_n)$ for a choice of $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$. We conclude from $Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n)$ and (5) that $Q'(Y_1, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_n)$. Hence the considered $z' = Q'(Y_1, \ldots, Y_n)$ satis-

fies $z' \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$ and $z' \in [z, 1]$, i.e. condition (465) holds, as desired.
As concerns condition (466), we proceed analogously. Hence let $z' \in S_{Q',X_1,\ldots,X_n}(\gamma) \cap [0, \frac{1}{2}]$. Then there exists a choice of $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$ with $z' = Q'(Y_1, \ldots, Y_n)$. From $Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n)$ and (5), we then obtain that $Q(Y_1, \ldots, Y_n) \in [Q'(Y_1, \ldots, Y_n), \frac{1}{2}] = [z', \frac{1}{2}]$. Hence $z = Q(Y_1, \ldots, Y_n)$ satisfies $z \in S_{Q,X_1,\ldots,X_n}(\gamma)$ and $z \in [z', \frac{1}{2}]$, which proves that (466) is valid.
Finally I consider (467). Hence let $z \in S_{Q,X_1,\ldots,X_n}(\gamma) \cap [0, \frac{1}{2})$. There exists a choice of $(Y_1, \ldots, Y_n) \in \mathcal{T}_\gamma(X_1, \ldots, X_n)$ with $z = Q(Y_1, \ldots, Y_n)$. We conclude from $Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n)$ and (5) that $Q'(Y_1, \ldots, Y_n) \in [0, Q(Y_1, \ldots, Y_n)] = [0, z]$. Hence $z' = Q'(Y_1, \ldots, Y_n)$ satisfies $z' \in [0, z]$ and $z' \in S_{Q',\ldots}(\gamma)$, i.e. (467) holds.
Hence the conditions stated in lemma L-69 are satisfied. We conclude that $S_{Q,X_1,\ldots,X_n} \preceq_c S_{Q',X_1,\ldots,X_n}$.

Proof of Theorem 58

Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given and suppose $\mathcal{F}_\Omega$ is the QFM defined by Def. 55.

a.: If Ω propagates fuzziness, then $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers.
Hence let us assume that Ω propagates fuzziness. Now we consider a choice of semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ with $Q \preceq_c Q'$. For all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we then obtain that

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) &= \Omega(S_{Q,X_1,\ldots,X_n}) && \text{by Def. 55} \\
&\preceq_c \Omega(S_{Q',X_1,\ldots,X_n}) && \text{by Def. 72, L-70} \\
&= \mathcal{F}_\Omega(Q')(X_1, \ldots, X_n)\,. && \text{by Def. 55}
\end{aligned}$$

Hence $\mathcal{F}_\Omega(Q) \preceq_c \mathcal{F}_\Omega(Q')$, which proves that $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers.

b.: If $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers, then Ω propagates fuzziness.
Hence suppose that $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers and consider $S, S' \in \mathbb{K}$ with $S \preceq_c S'$. It must be shown that $\Omega(S) \preceq_c \Omega(S')$. To this end, we first notice that there exist $z_0 \in S(0)$, $z_0' \in S'(0)$ with

$$z_0 \preceq_c z_0'\,. \tag{468}$$

This is apparent from Def. 71 and Def. 52. We also notice that for all $v \in \mathbf{I}$ and all $z \in S'(v)$, we can choose some $\zeta_{z,v} \in S(v)$ with

$$\zeta_{z,v} \preceq_c z\,. \tag{469}$$

This is immediate from (5), (464) of L-69 and (466). For similar reasons, there exist choices of $\zeta'_{z,v} \in S'(v)$ for all $v \in \mathbf{I}$ and $z \in S(v)$, such that

$$z \preceq_c \zeta'_{z,v}\,. \tag{470}$$

We now define semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = \begin{cases} z & : \quad z \in S(v) \\ \zeta_{z,v} & : \quad z \notin S(v), z \in S'(v) \\ z_0 & : \quad z \notin S(v), z \notin S'(v) \end{cases} \tag{471}$$

$$Q'(Y) = \begin{cases} z & : \quad z \in S'(v) \\ \zeta'_{z,v} & : \quad z \notin S'(v), z \in S(v) \\ z'_0 & : \quad z \notin S'(v), z \notin S(v) \end{cases} \tag{472}$$

where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \tag{473}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \tag{474}$$

$$z = \inf Y' \tag{475}$$

$$v = \sup Y'' \tag{476}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$. Next we consider the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ defined by

$$\mu_X(a, y) = \begin{cases} \frac{1}{2} & : \quad a = 0 \\ \frac{1}{2} - \frac{1}{2}y & : \quad a = 1 \end{cases} \tag{477}$$

for all $a \in \mathbf{2}$, $y \in \mathbf{I}$. We recall from the proof of Th-33 that for $\gamma = 0$,

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing \tag{478}$$

$$X_0^{\max} = X_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1, 0)\} \tag{479}$$

by (91) and (92). Similarly for $\gamma > 0$,

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{480}$$

$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma)) \tag{481}$$

by (93) (94). Let us now use these cut ranges to prove that $S_{Q,X} = S$. Hence let $\gamma \in \mathbf{I}$. We first notice that

$$S(\gamma) \subseteq S_{Q,X}(\gamma)\,. \tag{482}$$

To see this, consider $z \in S(\gamma)$. If $\gamma = 0$, then $Y = \{(0, z), (1, 0)\} \in \mathcal{T}_0(X)$, see (478), (479). For this choice of Y, we obtain that $z = \inf\{z\} = \inf Y'$ by (473) and $v = \sup Y'' = \sup\{0\} = 0 = \gamma$ by (474). Hence $Q(Y) = z$ because $z \in S(v) = S(\gamma)$, see (471). If $\gamma > 0$, then we know from (480) and (481) that $Y = \{(0, z)\} \cup (\{1\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X)$. For this choice of Y, we obtain $z = \inf\{z\} = \inf Y'$ and $v = \sup Y'' = \sup[0, \gamma) = \gamma$ by equations (473) and (474). We then conclude from $z \in S(\gamma)$ and (471) that $Q(Y) = z$. This finishes the proof of (482). To see that the converse subsumption $S_{Q,X}(\gamma) \subseteq S(\gamma)$ also holds, consider a choice of $Y \in \mathcal{T}_\gamma(X)$. It is then apparent from (478) and (479) (if $\gamma = 0$) and (480), (481) (if $\gamma > 0$) that $v = \sup Y'' \in [0, \gamma]$. Let us again abbreviate $z = \inf Y'$. We observe from (471) that either $Q(Y) = z \in S(v) \subseteq S(\gamma)$, or $Q(Y) = \zeta_{z,v} \in S(v) \subseteq S(\gamma)$, or $Q(Y) = z_0 \in S(0) \subseteq S(\gamma)$. In any case, $Q(Y) \in S(\gamma)$. Hence $S_{Q,X}(\gamma) \subseteq S(\gamma)$ by Def. 51, as desired. Combining this with (482) proves that $S_{Q,X}(\gamma) = S(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e.

$$S_{Q,X} = S\,. \tag{483}$$

Next I show that $S_{Q',X} = S'$. Again let $\gamma \in \mathbf{I}$. In order to prove that

$$S'(\gamma) \subseteq S_{Q',X}(\gamma) \tag{484}$$

consider $z \in S'(\gamma)$. If $\gamma = 0$, then $Y = \{(0, z), (1, 0)\} \in \mathcal{T}_0(X)$ by (478), (479). We obtain $z = \inf\{z\} = \inf Y'$ by (473) and $v = \sup Y'' = \sup\{0\} = 0 = \gamma$ by (474). Hence $Q'(Y) = z$ because $z \in S'(v) = S'(\gamma)$ by (472). For $\gamma > 0$ we obtain from (480) and (481) that $Y = \{(0, z)\} \cup (\{1\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X)$. This choice of Y yields $z = \inf\{z\} = \inf Y'$ and $v = \sup Y'' = \sup[0, \gamma) = \gamma$ by (473) and (474). We deduce from $z \in S'(\gamma)$ and (472) that $Q'(Y) = z$. Hence (484) is valid. To see that $S_{Q',X}(\gamma) \subseteq S'(\gamma)$ also holds, consider a choice of $Y \in \mathcal{T}_\gamma(X)$. It is then apparent from (478) and (479) (if $\gamma = 0$) and (480), (481) (if $\gamma > 0$) that $v = \sup Y'' \in [0, \gamma]$. $z = \inf Y'$ can assume arbitrary values $z \in \mathbf{I}$. We observe from (472) that either $Q'(Y) = z \in S'(v) \subseteq S'(\gamma)$, or $Q'(Y) = \zeta'_{z,v} \in S'(v) \subseteq S'(\gamma)$, or $Q'(Y) = z'_0 \in S'(0) \subseteq S'(\gamma)$. In any case, $Q'(Y) \in S'(\gamma)$. Hence $S_{Q',X}(\gamma) \subseteq S'(\gamma)$ by Def. 51. Recalling (484), we have shown that $S_{Q',X}(\gamma) = S'(\gamma)$ for all $\gamma \in \mathbf{I}$, thus

$$S_{Q',X} = S'\,. \tag{485}$$

Finally we notice that $Q \preceq_c Q'$. To this end, we consider some $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$. If $z \in S(v)$ and $z \in S'(v)$, then $Q(Y) = z = Q'(Y)$ by (471) and (472). In particular, $Q(Y) \preceq_c Q'(Y)$. In the case that $z \in S(v)$ and $z \notin S'(V)$, then $Q(Y) = z \preceq_c \zeta'_{z,v} = Q'(Y)$ by (471), (472) and (470). In the case that $z \notin S(v)$ and $z \in S'(v)$, we obtain that $Q(Y) = \zeta_{z,v} \preceq_c z = Q'(Y)$ by (471), (472) and (469). In the remaining case that $z \notin S(v)$ and $z \notin S'(v)$, we conclude from (471), (472) and (468) that $Q(Y) = z_0 \preceq_c z'_0 = Q'(Y)$. Hence indeed $Q \preceq_c Q'$, as desired. In particular, because $\mathcal{F}_\Omega$ is assumed to propagate fuzziness in quantifiers, we have

$$\mathcal{F}_\Omega(Q)(X) \preceq_c \mathcal{F}_\Omega(Q')(X)\,. \tag{486}$$

Now we can put the pieces together.

$$\begin{aligned}
\Omega(S) &= \Omega(S_{Q,X}) && \text{by (483)}\\
&= \mathcal{F}_\Omega(Q)(X) && \text{by Def. 55}\\
&\preceq_c \mathcal{F}_\Omega(Q')(X) && \text{by (486)}\\
&= \Omega(S_{Q',X}) && \text{by Def. 55}\\
&= \Omega(S')\,. && \text{by (485)}
\end{aligned}$$

Because the choice of $S \preceq_c S'$ was arbitrary, this proves that Ω propagates fuzziness, as desired.

A.28 Proof of Theorem 59

Lemma 71 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *is a mapping such that* (Ω-2) *is valid. If*

$$\Omega(S) = \Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1]) \tag{487}$$

for all $S \in \mathbb{K}$ *with* $S(0) \subseteq [\frac{1}{2}, 1]$*, then it also holds that*

$$\Omega(S) = \Omega(S^{\ddagger} \cap [0, \tfrac{1}{2}])$$

for all $S \in \mathbb{K}$ *with* $S(0) \subseteq [0, \frac{1}{2}]$.

Proof Let $S \in \mathbb{K}$ be given with $S(0) \subseteq [0, \frac{1}{2}]$. We define $S' \in \mathbb{K}$ by

$$S'(\gamma) = \{1 - z : z \in S(\gamma)\} \tag{488}$$

for all $\gamma \in \mathbf{I}$. Then $S'(0) \subseteq [\frac{1}{2}, 1]$ and hence

$$\Omega(S') = \Omega({S'}^{\ddagger} \cap [\tfrac{1}{2}, 1])\,. \tag{489}$$

We further define ${S^{\ddagger}}'(\gamma) = \{1 - z : z \in S^{\ddagger}\}$ for all $\gamma \in \mathbf{I}$. It is then apparent from Def. 59 that

$${S^{\ddagger}}' = {S'}^{\ddagger}\,. \tag{490}$$

We further define $S'' \in \mathbb{K}$ by $S''(\gamma) = \{1 - z : z \in S^{\ddagger}(\gamma) \cap [0, \frac{1}{2}]\}$ for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} S''(\gamma) &= \{1 - z : z \in S^{\ddagger}(\gamma) \cap [0, \tfrac{1}{2}]\} \\ &= \{1 - z : z \in S^{\ddagger}(\gamma), z \in [0, \tfrac{1}{2}]\} \\ &= \{1 - z : z \in S^{\ddagger}(\gamma), 1 - z \in [\tfrac{1}{2}, 1]\} \\ &= \{z' : 1 - z' \in S^{\ddagger}(\gamma), z' \in [\tfrac{1}{2}, 1]\} && \text{by substitution } z' = 1 - z \\ &= \{z' : z' \in {S^{\ddagger}}'(\gamma), z' \in [\tfrac{1}{2}, 1]\} && \text{by definition of } {S^{\ddagger}}' \\ &= \{z' : z' \in {S'}^{\ddagger}(\gamma), z' \in [\tfrac{1}{2}, 1]\} && \text{by (490)} \\ &= {S'}^{\ddagger} \cap [\tfrac{1}{2}, 1]\,. \end{aligned}$$

Therefore

$$\Omega({S'}^{\ddagger} \cap [\tfrac{1}{2}, 1]) = 1 - \Omega(S \cap [0, \tfrac{1}{2}]) \tag{491}$$

by (Ω-2). In turn

$$\begin{aligned} \Omega(S \cap [0, \tfrac{1}{2}]) &= 1 - \Omega({S'}^{\ddagger} \cap [\tfrac{1}{2}, 1]) \\ &= 1 - \Omega(S') && \text{by (487)} \\ &= 1 - (1 - \Omega(S)) && \text{by (488), } (\Omega\text{-2}) \\ &= \Omega(S)\,, \end{aligned}$$

as desired.

Lemma 72 *Consider a choice of* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *which satisfies* (Ω-2), (Ω-4) *and* (Ω-5). *Further suppose that*

$$\Omega(S) = \Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1]) \tag{492}$$

for all $S \in \mathbb{K}$ *with* $S(0) \subseteq [\frac{1}{2}, 1]$. *Then*

$$\Omega(S) = \tfrac{1}{2}$$

whenever $S \in \mathbb{K}$ *is such that* $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$.

Proof Suppose $S(0) \in \mathbb{K}$ satisfies $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We define S', S'' by

$$S'(\gamma) = \begin{cases} S(0) \cap [\frac{1}{2}, 1] & : \quad \gamma = 0 \\ S(\gamma) & : \quad \gamma > 0 \end{cases} \tag{493}$$

$$S''(\gamma) = \begin{cases} S(0) \cap [0, \frac{1}{2}] & : \quad \gamma = 0 \\ S(\gamma) & : \quad \gamma > 0 \end{cases} \tag{494}$$

for all $\gamma \in \mathbf{I}$. By the assumption on S, there exist $z^-, z^+ \in S(0)$ with $z^- \leq \frac{1}{2} \leq z^+$. Hence $z^+ \in S'(0)$ and $z^- \in S''(0)$, which shows that $S', S'' \in \mathbb{K}$. For the same reason $S \cap [0, \frac{1}{2}] \in \mathbb{K}$ and $S \cap [\frac{1}{2}, 1] \in \mathbb{K}$, see Def. 52. We notice that

$$S'^{\sharp} = S^{\sharp} = S''^{\sharp}\,, \tag{495}$$

which is obvious from Def. 56. Now define $S_{1/2} \in \mathbb{K}$ by $S_{1/2}(\gamma) = \{\frac{1}{2}\}$ for all $\gamma \in \mathbf{I}$. Clearly

$$S'' \cap [0, \tfrac{1}{2}] \sqsubseteq S_{1/2} \sqsubseteq S' \cap [\tfrac{1}{2}, 1]\,. \tag{496}$$

Because $S_{1/2}(\gamma) = \{1 - z : z \in S_{1/2}(\gamma)\}$, we conclude from ($\Omega$-2) that $\Omega(S_{1/2}) = 1 - \Omega(S_{1/2})$, i.e.

$$\Omega(S_{1/2}) = \tfrac{1}{2}\,. \tag{497}$$

Therefore

$$\begin{aligned} \Omega(S) &= \Omega(S') && \text{by (495), (Ω-4)} \\ &= \Omega(S' \cap [\tfrac{1}{2}, 1]) && \text{by (492)} \\ &\geq \Omega(S_{1/2}) && \text{by (496)} \\ &= \tfrac{1}{2}\,. && \text{by (497)} \end{aligned}$$

By similar reasoning

$$\begin{aligned} \Omega(S) &= \Omega(S'') && \text{by (495), (Ω-4)} \\ &= \Omega(S'' \cap [0, \tfrac{1}{2}]) && \text{by L-71} \\ &\leq \Omega(S_{1/2}) && \text{by (496)} \\ &= \tfrac{1}{2}\,. && \text{by (497)} \end{aligned}$$

Hence $\Omega(S) = \frac{1}{2}$, as desired.

Lemma 73 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-2), (Ω-4) *and* (Ω-5). *If*

$$\Omega(S) = \Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1]) \tag{498}$$

for all $S \in \mathbb{K}$ *with* $S(0) \subseteq [\frac{1}{2}, 1]$, *then* Ω *propagates fuzziness.*

Proof Let $S, S' \in \mathbb{K}$ be given with $S \preceq_c S'$. We shall discern three cases.

a.: $S(0) \subseteq (\frac{1}{2}, 1]$**.** It is then apparent from $S \preceq_c S'$ and Def. 71 that $S'(0) \subseteq (\frac{1}{2}, 1]$ as well. To see this, consider $z' \in S'(0)$. Because $S \preceq_c S'$, there exists $z \in S(0)$ with $z \preceq_c z'$. But $S(0) \subseteq (\frac{1}{2}, 1]$, i.e. $z > \frac{1}{2}$. It follows from (5) that $z' \geq z > \frac{1}{2}$. We hence know from (498) that

$$\Omega(S) = \Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1]) \tag{499}$$

$$\Omega(S') = \Omega({S'}^{\ddagger} \cap [\tfrac{1}{2}, 1])\,. \tag{500}$$

I will now show that

$$S^{\ddagger} \cap [\tfrac{1}{2}, 1] \sqsubseteq {S'}^{\ddagger} \cap [\tfrac{1}{2}, 1]\,, \tag{501}$$

according to definition Def. 57. Hence let $\gamma \in \mathbf{I}$ and consider a choice of $z \in S^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$. By definition Def. 59, there exists $z' \geq z$ with $z' \in S(\gamma)$, in particular $z' \in S(\gamma) \cap [\frac{1}{2}, 1]$. In the case that $z; > \frac{1}{2}$ we proceed as follows. We know from $S \preceq_c S'$ and Def. 71 that there exists $z'' \in S'(\gamma)$ with $z' \preceq_c z''$. Because $z' > \frac{1}{2}$, we conclude from (5) that $z'' \geq z' \geq z$. In the remaining case that $z' = \frac{1}{2}$, $\frac{1}{2} \leq z \leq z'$ entails that $z = \frac{1}{2}$ as well. Then every $z'' \in S'(\gamma) \cap [\frac{1}{2}, 1] \neq \varnothing$ satisfies $z'' \geq z$ anyway. To sum up, I have shown that for all $z \in S^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$, there exists $z'' \in S'(\gamma) \cap [\frac{1}{2}, 1] \subseteq {S'}^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$ with $z \leq z''$. In order to prove (501), it remains to be shown that for all $z' \in {S'}^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$, there exists $z \in S^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$ such that $z \leq z'$. To this end, assume a choice of $z' \in {S'}^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$. By Def. 59, there exist $z'_1, z'_2 \in S'(\gamma)$ with $z'_1 \leq z' \leq z'_2$. Because $z'_2 \geq z' \geq \frac{1}{2}$, we conclude from $S \preceq_c S'$ and Def. 71 that there exists $z_2 \in S(\gamma)$ with $z_2 \preceq_c z'_2$, and hence $\frac{1}{2} \leq z_2 \leq z'_2$ by (5).
If $z_2 \leq z'$, then we have found a $z = z_2 \in S(\gamma) \cap [\frac{1}{2}, 1] \subseteq S^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$ with the desired properties. If $z_2 > z'$, we must proceed as follows.

- If $z'_1 \geq \frac{1}{2}$, then we utilize that by $S \preceq_c S'$ that there exists $z \in S(\gamma)$ with $z \preceq_c z'_1$. Hence $\frac{1}{2} \leq z \leq z'_1$ by (5). In particular, there exists $z' \in S(\gamma) \cap [\frac{1}{2}, 1] \subseteq S^{\ddagger}(\gamma) \cap [\frac{1}{2}, 1]$ with $z \leq z'_1 \leq z'$.
- If $z'_1 < \frac{1}{2}$, we again use that there exists $z_1 \in S(\gamma)$ with $z_1 \preceq_c z'_1$. In this case, (5) yields that $z'_1 \leq z_1 \leq \frac{1}{2}$ because $z'_1 < \frac{1}{2}$. Hence $z_1 \leq \frac{1}{2} \leq z' < z_2$, and we conclude from Def. 59 that $z' \in S^{\ddagger}(\gamma)$. Because $z' \geq \frac{1}{2}$, this proves that $z' \in S^{\ddagger} \cap [\frac{1}{2}, 1]$. Therefore the choice of $z = z'$ satisfies $z \in S^{\ddagger} \cap [\frac{1}{2}, 1]$ and $z \leq z'$, as desired.

Hence both conditions of Def. 57 are satisfied, and we conclude that (501) is valid. Hence by (Ω-5),

$$\Omega(S) \le \Omega(S') \,. \tag{502}$$

Next we consider the mapping $S'' \in \mathbb{K}$ defined by $S''(\gamma) = \{\frac{1}{2}\}$ for all $\gamma \in \mathbf{I}$. Clearly $S''(\gamma) = \{1 - z : z \in S''(\gamma)\}$ for all $\gamma \in \mathbf{I}$. Hence $\Omega(S'') = 1 - \Omega(S'')$ by (Ω-2), i.e. $\Omega(S'') = \frac{1}{2}$. It is further apparent from Def. 57 that $S'' \sqsubseteq S$. Therefore

$$\tfrac{1}{2} = \Omega(S'') \le \Omega(S) \tag{503}$$

by (Ω-5). Combining (502) and (503) yields $\frac{1}{2} \le \Omega(S) \le \Omega(S')$. Applying (5), we obtain the intended $\Omega(S) \preceq_c \Omega(S')$.

b.: $S(0) \subseteq [0, \frac{1}{2})$. Lemma L-71 shows that Ω satisfies a condition analogous to (498) that is required to prove case **b.** The proof is analogous to that of **a.**, based on the property described in the lemma.

c.: there exist $z^-, z^+ \in S(0)$ with $z^- \le \frac{1}{2} \le z^+$. In this case, we obtain from L-71 that $\Omega(S) = \frac{1}{2}$. Hence trivially $\Omega(S) \preceq_c \Omega(S')$, see (5).

Lemma 74 *Suppose $S_1, S_2 \in \mathbb{K}$ are defined by*

$$S_1(\gamma) = \{0, \tfrac{1}{2}\}$$
$$S_2(\gamma) = \{\tfrac{1}{2}, 1\}$$

for all $\gamma \in \mathbf{I}$. If $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ propagates fuzziness and satisfies (Ω-3)*, then $\Omega(S_1) = \Omega(S_2) = \frac{1}{2}$.*

Proof Define $S_3 \in \mathbb{K}$ by $S_3(\gamma) = \{0, 1\}$ for all $\gamma \in \mathbf{I}$. We observe that

$$\Omega(S_3) = \tfrac{1}{2} + \tfrac{1}{2} s_3(0) = \tfrac{1}{2} + \tfrac{1}{2} \cdot 0 = \tfrac{1}{2} \tag{504}$$

by (Ω-3). We notice that $S_1 \preceq_c S_3$ by Def. 71. Because Ω propagates fuzziness, we conclude that $\Omega(S_1) \preceq_c \Omega(S_3) = \frac{1}{2}$ by Def. 72 and (504). Hence $\Omega(S_1) = \frac{1}{2}$ by (5). By similar reasoning, we conclude from the apparent $S_2 \preceq_c S_3$ that $\Omega(S_2) \preceq_c \Omega(S_3) = \frac{1}{2}$. Hence $\Omega(S_2) = \frac{1}{2}$, as desired.

Lemma 75 *Suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ satisfies* (Ω-1), (Ω-3) *and* (Ω-5)*. If there exists $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$ and $\Omega(S) \ne \Omega(S^\ddagger \cap [\frac{1}{2}, 1]$, then Ω does not propagate fuzziness.*

Proof Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (Ω-1), (Ω-3) and (Ω-5). Further suppose that there exists $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$ and

$$\Omega(S) \ne \Omega(S^\ddagger \cap [\tfrac{1}{2}, 1] \,. \tag{505}$$

In order to proof that Ω does not propagate fuzziness, we first notice that

$$S^{\ddagger} \cap [\tfrac{1}{2}, 1] \preceq_c S \tag{506}$$

$$S \sqsubseteq S^{\ddagger} \cap [\tfrac{1}{2}, 1] \tag{507}$$

$$\Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1]) \geq \Omega(S_{1/2}) = \tfrac{1}{2}\,, \tag{508}$$

where $S_{1/2}(\gamma) = \{\frac{1}{2}\}$ for all $\gamma \in \mathbf{I}$. This is apparent from Def. 71, Def. 59, Def. 57, (Ω-1) and (Ω-5). In turn, we conclude from (507) and (Ω-5) that

$$\Omega(S) \leq \Omega(S^{\ddagger} \cap [\tfrac{1}{2}, 1])\,. \tag{509}$$

In the following, let us discern two cases.

a.: $\Omega(S) \geq \frac{1}{2}$**.** Then $\frac{1}{2} \leq \Omega(S) \leq \Omega(S^{\ddagger} \cap [\frac{1}{2}, 1]$ by (509), i.e. $\Omega(S) \preceq_c \Omega(S^{\ddagger} \cap [\frac{1}{2}, 1])$ by (5). The following proof is by contradiction. Hence assume that Ω propagates fuzziness. We can then conclude from (506) that $\Omega(S^{\ddagger} \cap [\frac{1}{2}, 1]) \preceq_c \Omega(S)$ as well. Because $\preceq_c$ is a partial order, this entails that $\Omega(S^{\ddagger} \cap [\frac{1}{2}, 1]) = \Omega(S)$. This contradicts (505). Hence the assumption that Ω propagates fuzziness is false, i.e. Ω does not propagate fuzziness.

b. $\Omega(S) < \frac{1}{2}$**.** In this case, consider $S' \in \mathbb{K}$ defined by $S'(\gamma) = \{0, \frac{1}{2}\}$ for all $\gamma \in \mathbf{I}$. Because $S(0) \subseteq [\frac{1}{2}, 1]$ and $S(0) \neq \varnothing$, we know that there exists $z_0 \in [\frac{1}{2}, 1]$ with $z_0 \in S(\gamma)$ for all $\gamma \in \mathbf{I}$. It is hence apparent that $S' \sqsubseteq S$ by Def. 57. The following argument is again by contradiction. Assume that Ω propagates fuzziness. Then $\Omega(S') = \frac{1}{2}$ by L-74, i.e. $\Omega(S) \geq \Omega(S') \geq \frac{1}{2}$. This conflicts with the assumption of case **b.** that $\Omega(S) < \frac{1}{2}$. Hence Ω does not propagate fuzziness.

Proof of Theorem 59

The condition on Ω is sufficient for Ω to propagate fuzziness by lemma L-73. It is necessary for Ω to propagate fuzziness by L-75.

A.29 Proof of Theorem 60

Lemma 76 *Consider a quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ with $X_i \preceq_c X'_i$ for $i = 1, \ldots, n$. Then $\mathcal{S}_{Q,X_1,\ldots,X_n} \Subset \mathcal{S}_{Q,X'_1,\ldots,X'_n}$.*

Proof It is known from [11, L-59, p. 105] that $X_i \preceq_c X'_i$, $i \in \{1, \ldots, n\}$ entails that

$$\mathcal{T}_\gamma(X_i) \supseteq \mathcal{T}_\gamma(X'_i) \tag{510}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
&\mathcal{S}_{Q,X_1,\dots,X_n}(\gamma) \\
&= \{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 51} \\
&\supseteq \{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X'_1),\dots,Y_n \in \mathcal{T}_\gamma(X'_n)\} && \text{by (510)} \\
&= \mathcal{S}_{Q,X'_1,\dots,X'_n}(\gamma)\,. && \text{by Def. 51}
\end{aligned}$$

Because $\gamma \in \mathbf{I}$ was arbitrary, we deduce from Def. 73 that $\mathcal{S}_{Q,X_1,\dots,X_n} \Subset \mathcal{S}_{Q,X'_1,\dots,X'_n}$.

Proof of Theorem 60

Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be given and suppose $\mathcal{F}_\Omega$ is the QFM defined by Def. 55.

a.: If Ω propagates unspecificity, then $\mathcal{F}_\Omega$ propagates fuzziness in arguments.
Hence let us assume that Ω propagates unspecificity. Now we consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and choices of arguments $X_1,\dots,X_n,X'_1,\dots,X'_n \in \widetilde{\mathcal{P}}(E)$ with $X_i \preceq_c X'_i$ for all $i = 1,\dots,n$.

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(X_1,\dots,X_n) &= \Omega(\mathcal{S}_{Q,X_1,\dots,X_n}) && \text{by Def. 55} \\
&\preceq_c \Omega(\mathcal{S}_{Q,X'_1,\dots,X'_n}) && \text{by Def. 74, L-76} \\
&= \mathcal{F}_\Omega(Q)(X'_1,\dots,X'_n)\,. && \text{by Def. 55}
\end{aligned}$$

Hence $\mathcal{F}_\Omega(Q)(X_1,\dots,X_n) \preceq_c \mathcal{F}_\Omega(Q)(X'_1,\dots,X'_n)$, which proves that $\mathcal{F}_\Omega$ propagates fuzziness in arguments.

b.: If $\mathcal{F}_\Omega$ propagates fuzziness in arguments, then Ω propagates unspecificity.
Hence suppose that $\mathcal{F}_\Omega$ propagates fuzziness in arguments and consider $S, S' \in \mathbb{K}$ with $S \Subset S'$, i.e.

$$S'(\gamma) \subseteq S(\gamma) \tag{511}$$

for all $\gamma \in \mathbf{I}$. We must prove that $\Omega(S) \preceq_c \Omega(S')$. To this end, we first choose some $z_0 \in S'(0) \neq \varnothing$. Noticing that $S \Subset S'$ entails $S(0) \supseteq S'(0)$, we also have $z_0 \in S(0)$. Based on this choice of z_0, I define a semi-fuzzy quantifier $Q : \mathcal{P}(\{*\} \cup (\mathbf{2} \times \mathbf{I})) \longrightarrow \mathbf{I}$ as follows, where $\{*\}$ is an arbitrary singleton set with $* \notin \mathbf{2} \times \mathbf{I}$. For all $Y \in \{*\} \cup (\mathbf{2} \times \mathbf{I})$,

$$Q(Y) = \begin{cases} z & : \quad z \in S(v), * \in Y \\ z & : \quad z \in S'(v), * \notin Y \\ z_0 & : \quad \text{else} \end{cases} \tag{512}$$

where

$$Y' = \{y \in \mathbf{I} : (0, y) \in Y\} \tag{513}$$
$$Y'' = \{y \in \mathbf{I} : (1, y) \in Y\} \tag{514}$$
$$z = \inf Y' \tag{515}$$
$$v = \sup Y''\,. \tag{516}$$

In addition, I define fuzzy subsets $X, X' \in \widetilde{\mathcal{P}}(\{*\} \cup (\mathbf{2} \times \mathbf{I}))$ by

$$\mu_X(e) = \begin{cases} \frac{1}{2} & : \quad e = * \\ \frac{1}{2} & : \quad e = (0, y) \text{ for some } y \in \mathbf{I} \\ \frac{1}{2} - \frac{1}{2}y & : \quad e = (1, y) \text{ for some } y \in \mathbf{I} \end{cases} \tag{517}$$

$$\mu_{X'}(e) = \begin{cases} 0 & : \quad e = * \\ \frac{1}{2} & : \quad e = (0, y) \text{ for some } y \in \mathbf{I} \\ \frac{1}{2} - \frac{1}{2}y & : \quad e = (1, y) \text{ for some } y \in \mathbf{I} \end{cases} \tag{518}$$

for all $e \in \{*\} \cup (\mathbf{2} \times \mathbf{I})$. It is immediate from these definitions that $\mu_X(e) \preceq_c \mu_{X'}(e)$ for all $e \in \{*\} \cup (\mathbf{2} \times \mathbf{I})$, i.e.

$$X \preceq_c X' .$$

We hence deduce from $\mathcal{F}_\Omega$ propagating fuzziness in arguments that

$$\mathcal{F}_\Omega(Q)(X) \preceq_c \mathcal{F}_\Omega(Q)(X') . \tag{519}$$

Next we investigate the cut ranges. For $\gamma = 0$, we obtain from Def. 31 and (517), (518) that

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing \tag{520}$$

$$X_0^{\max} = X_{\geq\frac{1}{2}} = \{*\} \cup (\{0\} \times \mathbf{I}) \cup \{(1, 0)\} \tag{521}$$

$$X'^{\min}_0 = X'_{>\frac{1}{2}} = \varnothing \tag{522}$$

$$X'^{\max}_0 = X'_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1, 0)\} . \tag{523}$$

Now we consider the case that $\gamma > 0$. We then obtain from Def. 31 and (517), (518) that

$$X_\gamma^{\min} = X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{524}$$

$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{*\} \cup (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma)) \tag{525}$$

$$X'^{\min}_\gamma = X'_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{526}$$

$$X'^{\max}_\gamma = X'_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma)) . \tag{527}$$

Based on these cut ranges, I now prove that $S = S_{Q,X}$. Hence let $\gamma \in \mathbf{I}$. Firstly let us observe that

$$S(\gamma) \subseteq S_{Q,X}(\gamma) . \tag{528}$$

To see this, consider $z \in S(\gamma)$. If $\gamma = 0$, then $Y = \{*, (0, z), (1, 0)\} \in \mathcal{T}_0(X)$, see (520), (521). For this choice of Y, we obtain that $z = \inf\{z\} = \inf Y'$ by (513) and $v = \sup Y'' = \sup\{0\} = 0 = \gamma$ by (514). Hence $Q(Y) = z$ because $z \in$

$S(v) = S(\gamma)$ and $* \in Y$, see (512). If $\gamma > 0$, then we know from (524) and (525) that $Y = \{*, (0, z)\} \cup (\{1\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X)$. For this choice of Y, we obtain $z = \inf\{z\} = \inf Y'$ and $v = \sup Y'' = \sup[0, \gamma) = \gamma$ by equations (513) and (514). We then conclude from $z \in S(\gamma)$, $* \in Y$ and (512) that $Q(Y) = z$. This finishes the proof of (528). To see that the converse subsumption $S_{Q,X}(\gamma) \subseteq S(\gamma)$ also holds, consider a choice of $Y \in \mathcal{T}_\gamma(X)$. It is then apparent from (520) and (521) (if $\gamma = 0$) and (524), (525) (if $\gamma > 0$) that $v = \sup Y'' \in [0, \gamma]$. Let us again abbreviate $z = \inf Y'$. We observe from (512) that either $Q(Y) = z \in S(v) \subseteq S(\gamma)$, or $Q(Y) = z \in S'(v) \subseteq S(v)$ by (511), or $Q(Y) = z_0 \in S(0) \subseteq S(\gamma)$. In any case, $Q(Y) \in S(\gamma)$. Hence $S_{Q,X}(\gamma) \subseteq S(\gamma)$ by Def. 51, as desired. Combining this with (528) proves that $S_{Q,X}(\gamma) = S(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e.

$$S_{Q,X} = S\,. \tag{529}$$

Next I prove that $S' = S_{Q,X'}$. Let again $\gamma \in \mathbf{I}$. In order to prove that

$$S'(\gamma) \subseteq S_{Q,X'}(\gamma) \tag{530}$$

consider $z \in S'(\gamma)$. If $\gamma = 0$, then $Y = \{(0, z), (1, 0)\} \in \mathcal{T}_0(X')$ by (522), (523). We obtain $z = \inf\{z\} = \inf Y'$ by (513) and $v = \sup Y'' = \sup\{0\} = 0 = \gamma$ by (514). Hence $Q(Y) = z$ because $z \in S'(v) = S'(\gamma)$ and $* \notin Y$, see (512). For $\gamma > 0$ we obtain from (526) and (527) that $Y = \{(0, z)\} \cup (\{1\} \times [0, \gamma)) \in \mathcal{T}_\gamma(X')$. This choice of Y yields $z = \inf\{z\} = \inf Y'$ and $v = \sup Y'' = \sup[0, \gamma) = \gamma$ by (513) and (514). We deduce from $z \in S'(\gamma)$, $* \notin Y$ and (512) that $Q(Y) = z$. Hence (530) is valid. To see that $S_{Q,X'}(\gamma) \subseteq S'(\gamma)$ also holds, consider a choice of $Y \in \mathcal{T}_\gamma(X')$. It is then apparent from (522) and (523) (if $\gamma = 0$) and (526), (527) (if $\gamma > 0$) that $v = \sup Y'' \in [0, \gamma]$ and $* \notin Y$; $z = \inf Y'$ can assume arbitrary values $z \in \mathbf{I}$. We observe from (512) and $* \notin Y$ that either $Q(Y) = z \in S'(v) \subseteq S'(\gamma)$ or $Q(Y) = z_0 \in S'(0) \subseteq S'(\gamma)$. In any case, $Q(Y') \in S'(\gamma)$. Hence $S_{Q,X'}(\gamma) \subseteq S'(\gamma)$ by Def. 51. Recalling (530), we have shown that $S_{Q,X'}(\gamma) = S'(\gamma)$ for all $\gamma \in \mathbf{I}$, thus

$$S_{Q,X'} = S'\,. \tag{531}$$

Based on these auxiliary results, we can now proceed as follows.

$$\begin{aligned} \Omega(S) &= \Omega(S_{Q,X}) && \text{by (529)} \\ &= \mathcal{F}_\Omega(Q)(X) && \text{by Def. 55} \\ &\preceq_c \mathcal{F}_\Omega(Q)(X') && \text{by (519)} \\ &= \Omega(S_{Q,X'}) && \text{by Def. 55} \\ &= \Omega(S')\,. && \text{by (531)} \end{aligned}$$

A.30 Proof of Theorem 61

Lemma 77 *Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be a given mapping which satisfies* (Ω-2). *If Ω satisfies condition* ***b.*** *of theorem Th-61, then Ω also satisfies the following condition. For all $s \in \mathbb{K}$ with $S(0) \subseteq [0, \frac{1}{2}]$, it holds that $\Omega(S) = \Omega(S')$, where $S' \in \mathbb{K}$ is defined by*

$$S'(\gamma) = \begin{cases} [0, z^*] & : \quad z^* \in S(\gamma) \\ [0, z^*) & : \quad z^* \notin S(\gamma) \end{cases} \tag{532}$$

for all $\gamma \in \mathbf{I}$*, where* $z^* = z^*(\gamma)$ *abbreviates*

$$z^* = \sup S(\gamma)\,. \tag{533}$$

Proof Suppose $S \in \mathbb{K}$ satisfies $S(0) \subseteq [0, \frac{1}{2}]$. We define $S_1 \in \mathbb{K}$ by

$$S_1(\gamma) = \{1 - z : z \in S(\gamma)\} \tag{534}$$

for all $\gamma \in \mathbf{I}$. We further define $S' \in \mathbb{K}$ according to (532), and $S'_1 \in \mathbb{K}$ according to (54). It is apparent from these equations and (534) that

$$S'(\gamma) = \{1 - z : z \in S'_1(\gamma)\} \tag{535}$$

for all $\gamma \in \mathbf{I}$. Noticing that $S_1(0) \subseteq [\frac{1}{2}, 1]$, we hence obtain

$$\begin{aligned} \Omega(S) &= 1 - \Omega(S_1) && \text{by } (\Omega\text{-}2) \\ &= 1 - \Omega(S'_1) && \text{by condition } \textbf{b.} \text{ of Th-61} \\ &= \Omega(S')\,, && \text{by (535)} \end{aligned}$$

as desired.

Lemma 78 *Suppose* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *satisfies* (Ω-2), (Ω-4) *and* (Ω-5). *If* Ω *satisfies condition* ***b.*** *of theorem Th-61, then* Ω *also satisfies the following condition. For all* $S \in \mathbb{K}$ *with* $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$*, it holds that* $\Omega(S) = \frac{1}{2}$.

Proof To see this, consider $S \in \mathbb{K}$ with $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We define $S^+, S^- : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$S^+(\gamma) = \begin{cases} S(0) \cap [\frac{1}{2}, 1] & : \quad \gamma = 0 \\ S(\gamma) & : \quad \gamma > 0 \end{cases} \tag{536}$$

$$S^-(\gamma) = \begin{cases} S(0) \cap [0, \frac{1}{2}] & : \quad \gamma = 0 \\ S(\gamma) & : \quad \gamma > 0 \end{cases} \tag{537}$$

for all $\gamma \in \mathbf{I}$. It is apparent from $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$ that $S^+, S^- \in \mathbb{K}$ by Def. 52. We further define ${S'}^+, {S'}^- \in \mathbb{K}$ by (54) and (532), respectively. Finally, we define $S'' \in \mathbb{K}$ by

$$S''(\gamma) = \mathbf{I} \tag{538}$$

for all $\gamma \in \mathbf{I}$. It is then apparent from (Ω-2) and $S''(\gamma) = \{1 - z : z \in S''(\gamma)\}$ for all $\gamma \in \mathbf{I}$ that

$$\Omega(S'') = \tfrac{1}{2}\,. \tag{539}$$

Noticing that $S'^{-} \sqsubseteq S''$, we hence obtain

$$\begin{aligned} \Omega(S) &= \Omega(S^{-}) && \text{by (537) and L-18} \\ &= \Omega(S'^{-}) && \text{by L-77} \\ &\leq \Omega(S'') && \text{by (Ω-5)} \\ &= \tfrac{1}{2}\,. && \text{by (539)} \end{aligned}$$

In a similar way, we obtain from $S'' \sqsubseteq S'^{+}$ that

$$\begin{aligned} \Omega(S) &= \Omega(S^{+}) && \text{by (536) and L-18} \\ &= \Omega(S'^{+}) && \text{by condition } \textbf{a.} \text{ of Th-61} \\ &\geq \Omega(S'') && \text{by (Ω-5)} \\ &= \tfrac{1}{2}\,. && \text{by (539)} \end{aligned}$$

We conclude that $\Omega(S) = \frac{1}{2}$.

Proof of Theorem 61

Consider a choice of $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ which satisfies (Ω-1), (Ω-2), (Ω-4) and (Ω-5).

Condition b. of the theorem entails condition a.: To see this, let us suppose that Ω satisfies condition **b.** of the theorem. We have to show that Ω propagates unspecificity. Hence let $S_1, S_2 \in \mathbb{K}$ be given with $S_1 \Subset S_2$, i.e.

$$S_2(\gamma) \subseteq S_1(\gamma) \tag{540}$$

for all $\gamma \in \mathbf{I}$. I will discern three cases in the proof that $\Omega(S_1) \preceq_c \Omega(S_2)$.
Firstly if $S_1(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S_1(0) \cap [0, \frac{1}{2}] \neq \varnothing$, then $\Omega(S_1) = \frac{1}{2} \preceq_c \Omega(S_2)$, which is apparent from L-78 and (5).
In the second case that $S_1(0) \subseteq [\frac{1}{2}, 1]$, we observe from (540) that $S_2(0) \subseteq [\frac{1}{2}, 1]$ as well. We define S'_1 and S'_2 according to equation (54) in terms of S_1 and S_2, respectively. It is then apparent from (540) and (54) that $S'_1 \sqsubseteq S'_2$ and hence

$$\Omega(S_1) = \Omega(S'_1) \leq \Omega(S'_2) = \Omega(S_2) \tag{541}$$

by (Ω-5) and the assumption that Ω satisfies condition **b.** of the theorem. We further define $S''_1 \in \mathbb{K}$ by

$$S''_1(\gamma) \cup \{\tfrac{1}{2}\}$$

for all $\gamma \in \mathbf{I}$. Because $S_1(0) \subseteq [\frac{1}{2}, 1]$ and $S_1(0) \neq \varnothing$, we conclude that $S_1(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Hence $S''_1 \sqsubseteq S_1$ by Def. 57. Therefore

$$\begin{aligned} \Omega(S_1) &\leq \Omega(S''_1) && \text{by (Ω-5)} \\ &= \tfrac{1}{2}\,. && \text{by L-78} \end{aligned}$$

Hence $\frac{1}{2} \leq \Omega(S_1) \leq \Omega(S_2)$ by (541), i.e. $\Omega(S_1) \preceq_c \Omega(S_2)$ by (5).

Finally in the case that $S(0) \subseteq [0, \frac{1}{2}]$, we define $S_1'(\gamma) = \{1 - z : z \in S_1(\gamma)\}$ and $S_2'(\gamma) = \{1 - z : z \in S_2(\gamma)\}$ for all $\gamma \in \mathbf{I}$. We can then conclude from the second case and (Ω-2) that

$$1 - \Omega(S_1) = \Omega(S_1') \preceq_c \Omega(S_2') = 1 - \Omega(S_2)\,.$$

It is then apparent from (5) that $\Omega(S_1) \preceq_c \Omega(S_2)$.

Condition a. of the theorem entails condition b.: To see that condition **b.** of the theorem holds whenever **a.** holds, let us assume that Ω propagates unspecificity. Now consider $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$ and define S' by (54). It is then apparent that $S(\gamma) \subseteq S'(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e. $S' \Subset S$ and hence

$$\Omega(S') \preceq_c \Omega(S)\,. \tag{542}$$

Because $S(0) \subseteq [\frac{1}{2}, 1]$, we can choose some $z_0 \geq \frac{1}{2}$ with $z_0 \in S(0)$. We define $S'' \in \mathbb{K}$ by $S''(\gamma) = \{z_0\}$ for all $\gamma \in \mathbf{I}$. Then clearly $S \Subset S''$ and hence $\Omega(S) \preceq_c \Omega(S'')$. In addition $\Omega(S'') = z_0 \geq \frac{1}{2}$ by (Ω-1). Hence $\Omega(S) \geq \frac{1}{2}$ by (5). Combining this with (542), we obtain that

$$\Omega(S') \leq \Omega(S)\,, \tag{543}$$

which is apparent from (5). We now notice from (54) and Def. 57 that $S \sqsubseteq S'$. Hence $\Omega(S) \leq \Omega(S')$. Combining this with (543) yields the desired $\Omega(S') = \Omega(S)$.

A.31 Proof of Theorem 62

Lemma 79 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is given and $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω by* (38). *If Ω propagates fuzziness, then ω propagates fuzziness.*

Proof Consider $s, s' \in \mathbb{L}$ with $s \preceq_c s'$. We define $S, S' \in \mathbb{K}$ by

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\}$$
$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s'(z)\}\,.$$

for all $\gamma \in \mathbf{I}$. Next let us prove that $S \preceq_c S'$. Hence let $\gamma \in \mathbf{I}$ and consider $z \in S(\gamma)$, i.e. $\gamma \geq s(z)$. We know from (56) that there exists $z' \in \mathbf{I}$ with $z \preceq_c z'$ and $s'(z') \leq s(z) \leq \gamma$. Hence $z' \in S'(\gamma)$ and $z \preceq_c z'$. This proves that condition (53) holds.
Now consider some $z' \in S'(\gamma)$, i.e. $\gamma \geq s'(z')$. We know from (57) that there exists $z \in \mathbf{I}$ with $z \preceq_c z'$ and $s(z) \leq s'(z')$. In particular, $\gamma \geq s(z)$; hence $z \in S(\gamma)$. This proves that condition (52) also holds; we conclude from Def. 71 that $S \preceq_c S'$. Because Ω is assumed to propagate fuzziness, $S \preceq_c S'$ entails that

$$\Omega(S) \preceq_c \Omega(S')\,. \tag{544}$$

Therefore

$$\begin{aligned} \omega(s) &= \Omega(S) && \text{by (39)} \\ &\preceq_c \Omega(S') && \text{by (544)} \\ &= \omega(s') \,. && \text{by (39)} \end{aligned}$$

Hence ω propagates fuzziness by Def. 76, as desired.

Lemma 80 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ has the following property. If $s, s' \in \mathbb{L}$ satisfy*

$$\text{for all } z \in \mathbf{I}, \inf\{s'(z') : z \preceq_c z'\} \leq s(z) \tag{545}$$

and

$$\text{for all } z' \in \mathbf{I}, \inf\{s(z) : z \preceq_c z'\} \leq s'(z'), \tag{546}$$

then $\omega(s) \preceq_c \omega(s')$. Further suppose that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined by (38)*. Then Ω propagates fuzziness.*

Proof Suppose that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ has the properties stated in the lemma. In order to prove that Ω propagates fuzziness, we consider a choice of $S, S' \in \mathbb{K}$ with $S \preceq_c S'$. We define $s, s' \in \mathbb{L}$ in terms of S, S', viz

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{547}$$

$$s'(z) = \inf\{\gamma \in \mathbf{I} : z \in S'(\gamma)\} \tag{548}$$

for all $z \in \mathbf{I}$. To see that (545) is satisfied, consider $z \in \mathbf{I}$ and let $\gamma > s(z)$. Then $z \in S(\gamma)$ by (547). By (53), there exists $z' \in S'(\gamma)$ with $z \preceq_c z'$. Because $z \in S'(\gamma)$, we conclude from (548) that $s'(z') \leq \gamma$. Because $\gamma > s(z)$ was arbitrary, this proves that

$$\inf\{s'(z') : z \preceq_c z'\} \leq s(z) \,,$$

i.e. (545) is satisfied. To see that (546) holds as well, consider $z' \in \mathbf{I}$. Then for all $\gamma > s'(z')$, $z' \in S'(\gamma)$ by (548). We hence know from (52) that there exists $z \in S(\gamma)$ with $z \preceq_c z'$. In particular, $z \in S(\gamma)$ entails that $s(z) \leq \gamma$. Because $\gamma > s'(z')$ was arbitrary, this proves that

$$\inf\{s(z) : z \preceq_c z'\} \leq s'(z') \,.$$

Hence (546) is valid, too. From the assumption on ω stated in the lemma we deduce that

$$\omega(s) \preceq_c \omega(s') \,. \tag{549}$$

Therefore

$$\begin{aligned} \Omega(S) &= \omega(s) && \text{by L-38, (38)} \\ &\preceq_c \omega(s') && \text{by (549)} \\ &= \Omega(S') \,. && \text{by L-38, (38)} \end{aligned}$$

This proves that Ω propagates fuzziness, as desired.

Lemma 81 *Suppose* $s, s' \in \mathbb{L}$ *are related by* (545) *and* (546). *Then there exist* $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z_0' \in {s'^{\ddagger}}^{-1}(0)$ *with* $z_0 \preceq_c z_0'$.

Proof Choose some $z_0 \in s^{-1}(0)$. Then also $z_0 \in {s^{\ddagger}}^{-1}(0)$ by Th-47.a. If $z_0 = \frac{1}{2}$ then any choice of $z_0' \in {s'^{\ddagger}}^{-1}(0) \neq \varnothing$ satisfies $z_0 = \frac{1}{2} \preceq_c z_0'$, see (5). In the case that $z_0 > \frac{1}{2}$, choose some $z_0'' \in {s'}^{-1}(0)$. If $z_0'' \geq z_0$, then z_0'' is a proper choice for z_0' because $s'^{\ddagger} \leq s'$ entails that $z_0'' \in {s'^{\ddagger}}^{-1}(0)$, and because $z_0'' \geq z_0 > \frac{1}{2}$ entails that $z_0 \preceq_c z_0''$ by (5). If $z_0'' < z_0$, however, we proceed as follows. we know that

$$\begin{aligned} 0 = s(z_0) & & \text{because } z_0 \in s^{-1}(0) \\ \geq \inf\{s'(z') : z_0 \preceq_c z'\} & & \text{by (545)} \\ = \inf\{s'(z') : z' \geq z_0\} & & \text{by } z_0 > \tfrac{1}{2} \text{ and (5)} \\ = {s^{\ddagger}}'(z_0)\,, & & \text{by L-43} \end{aligned}$$

i.e. ${s^{\ddagger}}'(z_0) = 0$. Hence $z_0' = z_0$ is a proper choice for z_0' with $z_0 \preceq_c z_0'$ and $s^{\ddagger}(z_0') = 0$. Finally let us consider the case that $z_0 < \frac{1}{2}$. Again choose some $z_0'' \in {s'}^{-1}(0)$. If $z_0'' \leq z_0$, then we are done because in this case $z_0' = z_0''$ is a proper choice of z_0' which satisfies $s'^{\ddagger}(z_0') = 0$ and $z_0 \preceq_c z_0'$. In the remaining case that $z_0'' > z_0$, we notice that

$$\begin{aligned} s'^{\ddagger}(z_0) &= \inf\{s'(z') : z' \leq z_0\} & & \text{by L-43} \\ &= \inf\{s'(z') : z_0 \preceq_c z'\} & & \text{by } z_0 < \tfrac{1}{2} \text{ and L-43} \\ &\leq s(z_0) & & \text{by (545)} \\ &= 0\,, & & \text{because } z_0 \in s^{-1}(0) \end{aligned}$$

i.e. $s'^{\ddagger}(z_0) = 0$. Hence $z_0' = z_0$ is an admissible choice of z_0' which satisfies $s'^{\ddagger}(z_0') = 0$ and $z_0 \preceq_c z_0'$, as desired.

Lemma 82 *Suppose* $s, s' \in \mathbb{L}$ *satisfy the condition* (545). *Further suppose that* $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z_0' \in {s'^{\ddagger}}^{-1}(0)$ *are given with* $z_0 \preceq_c z_0'$. *Then for all* $z \in \mathbf{I}$,

a. If $z > \frac{1}{2}$ *and* $z_0' < z$, *then* $s'^{\ddagger}(z) \leq s(z)$;

b. If $z > \frac{1}{2}$ *and* $z_0' \geq z$ *then* $z \preceq_c z_0'$ *and* $s'^{\ddagger}(z_0') \leq s^{\ddagger}(z)$;

c. If $z = \frac{1}{2}$, *then* $z \preceq_c z_0'$ *and* $s'^{\ddagger}(z_0') \leq s^{\ddagger}(z)$;

d. If $z < \frac{1}{2}$ *and* $z_0' \leq z$, *then* $z \preceq_c z_0'$ *and* $s'^{\ddagger}(z_0) \leq s^{\ddagger}(z)$;

e. If $z < \frac{1}{2}$ *and* $z_0' > z$ *then* $s'^{\ddagger}(z) \leq s(z)$.

Proof We know from L-81 that there exist $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z_0' \in {s'^{\ddagger}}^{-1}(0)$ with $z_0 \preceq_c z_0'$. Now let us consider some $z \in \mathbf{I}$.

a.: If $z > \frac{1}{2}$ and $z_0' < z$, then

$$\begin{aligned} s'^{\ddagger}(z) &= \inf\{s'(z') : z' \geq z\} && \text{by L-61} \\ &= \inf\{s'(z') : z \preceq_c z'\} && \text{by (5) and } z > \tfrac{1}{2} \\ &\leq s(z)\,. && \text{by (545)} \end{aligned}$$

b., c. and d.: Immediate from (5) and $s'^{\ddagger}(z_0') = 0$.

e.: If $z < \frac{1}{2}$ and $z_0' > z$ then

$$\begin{aligned} s'^{\ddagger}(z) &= \inf\{s'(z') : z' \leq z\} && \text{by L-61} \\ &= \inf\{s'(z') : z \preceq_c z'\} && \text{by (5) because } z < \tfrac{1}{2} \\ &\leq s(z)\,. && \text{by (545)} \end{aligned}$$

Lemma 83 *Suppose $s, s' \in \mathbb{L}$ satisfy the condition* (545)*. Further suppose that $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z_0' \in {s'^{\ddagger}}^{-1}(0)$ are given with $z_0 \preceq_c z_0'$. Then for all $z \in \mathbf{I}$,*

a. If $z > \frac{1}{2}$ and $z_0' < z$, then $s'^{\ddagger}(z) \leq s^{\ddagger}(z)$;

b. If $z < \frac{1}{2}$ and $z_0' > z$ then $s'^{\ddagger}(z) \leq s^{\ddagger}(z)$.

Proof Let $z \in \mathbf{I}$ be given.

a.: $z > \frac{1}{2}$ and $z_0' < z$. Suppose $z_0' \leq \frac{1}{2}$. We then deduce from $z_0 \preceq_c z_0'$ and (5) that $z_0 \in [z_0', \frac{1}{2}]$. In particular, $z_0 < z$. In the remaining case that $z_0' > \frac{1}{2}$, we obtain from $z_0 \preceq_c z_0'$ and (5) that $z_0 \in [\frac{1}{2}, z_0']$, and again $z_0 \leq z_0' < z$. Hence in both cases $z_0 < z$. Therefore

$$\begin{aligned} s^{\ddagger}(z) &= \inf\{s(z') : z' \geq z\} && \text{by L-61} \\ &\geq \inf\{s'^{\ddagger}(z') : z' \geq z\} && \text{by L-82.a} \\ &= {s'^{\ddagger}}^{\ddagger}(z) && \text{by L-43} \\ &= s'^{\ddagger}(z)\,. && \text{by L-51} \end{aligned}$$

b.: $z < \frac{1}{2}$ and $z_0' > z$. If $z_0' \in (z, \frac{1}{2}]$, then $z_0 \preceq_c z_0'$ entails that $z_0 \in [z_0', \frac{1}{2}]$ by (5). In particular $z_0 > z$. In the remaining case that $z_0' \in [\frac{1}{2}, 1]$, $z_0 \preceq_c z_0'$ entails that

$z_0 \in [\frac{1}{2}, z_0']$ by (5). In particular $z_0 \geq \frac{1}{2} > z$. Hence in both cases $z_0 > z$. Therefore

$$\begin{aligned} s^{\ddagger}(z) &= \inf\{s(z') : z' \leq z\} && \text{by L-61} \\ &\geq \inf\{{s'}^{\ddagger}(z') : z' \leq z\} && \text{by L-82.e} \\ &= {s'}^{\ddagger\ddagger}(z) && \text{by L-43} \\ &= {s'}^{\ddagger}(z)\,. && \text{by L-51} \end{aligned}$$

Lemma 84 *Suppose $s, s' \in \mathbb{L}$ satisfy the condition* (546)*. Further suppose that $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z_0' \in {s'^{\ddagger}}^{-1}(0)$ are given with $z_0 \preceq_c z_0'$. Then for all $z' \in \mathbf{I}$,*

a. If $z' \geq \frac{1}{2}$ and $z_0 \in (z', 1]$, then $s^{\ddagger}(z') \leq s'(z')$;

b. If $z' \geq \frac{1}{2}$ and $z_0 \in [\frac{1}{2}, z']$, then $z_0 \preceq_c z'$ and $s^{\ddagger}(z_0) \leq {s'}^{\ddagger}(z')$;

c. If $z' \geq \frac{1}{2}$ and $z_0 \in [0, \frac{1}{2})$, then $s^{\ddagger}(\frac{1}{2}) \leq s'(z')$;

d. If $z' < \frac{1}{2}$ and $z_0 \in [0, z')$, then $s^{\ddagger}(z') \leq s'(z')$;

e. If $z' < \frac{1}{2}$ and $z_0 \in [z', \frac{1}{2}]$, then $s^{\ddagger}(z_0) \leq {s'}^{\ddagger}(z')$;

f. If $z' < \frac{1}{2}$ and $z_0 \in (\frac{1}{2}, 1]$, then $s^{\ddagger}(\frac{1}{2}) \leq s'(z')$.

Proof

a.: $z' \geq \frac{1}{2}$ and $z_0 \in (z', 1]$. Then $z_0 > z' \geq \frac{1}{2}$. Therefore

$$\begin{aligned} s^{\ddagger}(z') &= \inf\{s(z) : z \leq z'\} && \text{by L-61} \\ &= \min(\inf\{s(z) : z \in [\tfrac{1}{2}, z']\}, \inf\{s(z) : z \in [0, \tfrac{1}{2})\}) \\ &\leq \inf\{s(z) : z \in [\tfrac{1}{2}, z']\} \\ &= \inf\{s(z) : z \preceq_c z'\} && \text{by (5) because } z' \geq \tfrac{1}{2} \\ &\leq s'(z')\,. && \text{by (546)} \end{aligned}$$

Hence indeed $s^{\ddagger}(z') \leq s'(z')$.

b.: $z' \geq \frac{1}{2}$ and $z_0 \in [\frac{1}{2}, z']$. We conclude from $\frac{1}{2} \leq z_0 \leq z'$ that $z_0 \preceq_c z'$ by (5). The claim of part **b.** is then immediate from $s^{\ddagger}(z_0) = 0 \leq {s'}^{\ddagger}(z')$.

c.: $z' \geq \frac{1}{2}$ and $z_0 \in [0, \frac{1}{2})$. Because $z_0 < \frac{1}{2}$, we may proceed as follows.

$$\begin{aligned} s^{\ddagger}(\tfrac{1}{2}) &= \inf\{s(z) : z \geq \tfrac{1}{2}\} && \text{by L-61} \\ &\leq \inf\{s(z) : z \in [\tfrac{1}{2}, z']\} \\ &= \inf\{s(z) : z \preceq_c z'\} && \text{by (5) because } z \geq \tfrac{1}{2} \\ &\leq s'(z')\,. && \text{by (546)} \end{aligned}$$

Hence $s^{\ddagger}(\frac{1}{2}) \leq s'(z')$ holds, as desired.

d.: $z' < \frac{1}{2}$ **and** $z_0 \in [0, z')$**.** Then $z'_0 \leq z_0 < z'$, see (5). Therefore

$$\begin{aligned} s^{\ddagger}(z') &= \inf\{s(z'') : z'' \geq z'\} && \text{by L-61} \\ &\leq \inf\{s(z'') : z'' \in [z', \tfrac{1}{2}]\} \\ &= \inf\{s(z'') : z'' \preceq_c z'\} && \text{by (5) because } z' < \tfrac{1}{2} \\ &\leq s'(z')\,. && \text{by (546)} \end{aligned}$$

This proves that indeed $s^{\ddagger}(z') \leq s'(z')$.

e.: $z' < \frac{1}{2}$ **and** $z_0 \in [z', \frac{1}{2}]$**.** Then $z' \leq z_0 \leq \frac{1}{2}$. Hence $z_0 \preceq_c z'$ by (5). The claim of part **e.** is then immediate from $s^{\ddagger}(z_0) = 0 \leq s'^{\ddagger}(z')$.

f.: $z' < \frac{1}{2}$ **and** $z_0 \in (\frac{1}{2}, 1]$ Because $z_0 > z'$, we can proceed as follows.

$$\begin{aligned} s^{\ddagger}(\tfrac{1}{2}) &= \inf\{s(z) : z \leq \tfrac{1}{2}\} && \text{by L-61} \\ &\leq \inf\{s(z) : z \in [z', \tfrac{1}{2}]\} \\ &= \inf\{s(z) : z \preceq_c z'\} && \text{by (5) because } z' < \tfrac{1}{2} \\ &\leq s'(z')\,. && \text{by (546)} \end{aligned}$$

Therefore $s^{\ddagger}(\frac{1}{2}) \leq s'(z')$, i.e. the claim of part **f.** is valid, as desired.

Lemma 85 *Suppose $s, s' \in \mathbb{L}$ satisfy the condition* (546)*. Further suppose that $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z'_0 \in {s'^{\ddagger}}^{-1}(0)$ are given with $z_0 \preceq_c z'_0$. Then for all $z' \in \mathbf{I}$,*

a. If $z' \geq \frac{1}{2}$ and $z_0 \in (z', 1]$, then $s^{\ddagger}(z') \leq s'^{\ddagger}(z')$;

b. If $z' \geq \frac{1}{2}$ and $z_0 \in [0, \frac{1}{2})$, then $s^{\ddagger}(\frac{1}{2}) \leq s'^{\ddagger}(z')$;

c. If $z' < \frac{1}{2}$ and $z_0 \in [0, z')$, then $s^{\ddagger}(z') \leq s'^{\ddagger}(z')$;

d. If $z' < \frac{1}{2}$ and $z_0 \in (\frac{1}{2}, 1]$, then $s^{\ddagger}(\frac{1}{2}) \leq s'^{\ddagger}(z')$.

Proof Consider $z' \in \mathbf{I}$.

a.: $z' \geq \frac{1}{2}$ **and** $z_0 \in (z', 1]$**.** We observe that by (5), $z_0 \preceq_c z'_0$ and $z_0 > z' \geq \frac{1}{2}$ implies that $z'_0 \geq z_0 > z'$. Therefore

$$
\begin{aligned}
{s'}^{\ddagger}(z') &= \inf\{s'(z'') : z'' \leq z'\} && \text{by L-61}\\
&= \min(\inf\{s'(z'') : z'' \in [\tfrac{1}{2}, z']\},\\
&\qquad \inf\{s'(z'') : z' \in [0, \tfrac{1}{2})\})\\
&\geq \min(\inf\{s^{\ddagger}(z'') : z'' \in [\tfrac{1}{2}, z']\}, s^{\ddagger}(\tfrac{1}{2})) && \text{by L-84.a and L-84.f}\\
&= \inf\{s^{\ddagger}(z'') : z'' \in [\tfrac{1}{2}, z']\}\\
&\geq \inf\{s^{\ddagger}(z'') : z'' \leq z'\}\\
&= s^{\ddagger\ddagger}(z') && \text{by L-43}\\
&= s^{\ddagger}(z')\,. && \text{by L-51}
\end{aligned}
$$

This proves the claim of part **b.** that $s^{\ddagger}(z') \leq {s'}^{\ddagger}(z')$.

b.: $z' \geq \frac{1}{2}$ **and** $z_0 \in [0, \frac{1}{2})$**.** In this case, we obtain from $z_0 \preceq_c z'_0$ and (5) that $z'_0 \leq z_0 < \frac{1}{2}$. In particular $z'_0 < z'$. Therefore

$$
\begin{aligned}
{s'}^{\ddagger}(z') &= \inf\{s'(z'') : z'' \geq z'\} && \text{by L-61}\\
&\geq \inf\{s^{\ddagger}(\tfrac{1}{2}) : z'' \geq z'\} && \text{by L-84.c}\\
&= \inf\{s^{\ddagger}(\tfrac{1}{2})\}\\
&= s^{\ddagger}(\tfrac{1}{2})\,.
\end{aligned}
$$

This proves the desired $s^{\ddagger}(\frac{1}{2}) \leq {s'}^{\ddagger}(z')$.

c.: $z' < \frac{1}{2}$ **and** $z_0 \in [0, z')$**.** In this case, we have $z'_0 \leq z_0 < z' \leq \frac{1}{2}$, which is apparent from $z_0 \preceq_c z'_0$ and (5). In particular $z'_0 \leq z'$. Therefore

$$
\begin{aligned}
{s'}^{\ddagger}(z') &= \inf\{s'(z'') : z'' \geq z'\} && \text{by L-61}\\
&= \min(\inf\{s'(z'') : z'' \in [z', \tfrac{1}{2})\},\\
&\qquad \inf\{s'(z'') : z'' \in [\tfrac{1}{2}, 1]\})\\
&\geq \min(\inf\{s^{\ddagger}(z'') : z'' \in [z', \tfrac{1}{2})\}, s^{\ddagger}(\tfrac{1}{2})\} && \text{by L-84.c, L-84.d}\\
&= \inf\{s^{\ddagger}(z'') : z'' \in [z', \tfrac{1}{2}]\}\\
&\geq \inf\{s^{\ddagger}(z'') : z'' \geq z'\}\\
&= s^{\ddagger\ddagger}(z') && \text{by L-43}\\
&= s^{\ddagger}(z')\,. && \text{by L-51}
\end{aligned}
$$

This proves that $s^{\ddagger}(z') \leq {s'}^{\ddagger}(z')$.

d.: $z' < \frac{1}{2}$ **and** $z_0 \in (\frac{1}{2}, 1]$**.** Then we know from $z_0 \preceq_c z'_0$ and (5) that $z'_0 \geq z_0 > \frac{1}{2}$. In particular, $z' < z'_0$. Therefore

$$\begin{aligned} {s'}^{\ddagger}(z') &= \inf\{s'(z'') : z'' \leq z'\} && \text{by L-61} \\ &\geq \inf\{s^{\ddagger}(\tfrac{1}{2}) : z'' \leq z'\} && \text{by L-84.f} \\ &= \inf\{s^{\ddagger}(\tfrac{1}{2})\} \\ &= s^{\ddagger}(\tfrac{1}{2})\,. \end{aligned}$$

Hence $s^{\ddagger}(\frac{1}{2}) \leq {s'}^{\ddagger}(z')$, as desired.

Lemma 86 *Suppose* $s, s' \in \mathbb{L}$ *satisfy conditions* (545) *and* (546)*. Then* $s^{\ddagger} \preceq_c {s'}^{\ddagger}$.

Proof We first notice that by L-81, there exist $z_0 \in {s^{\ddagger}}^{-1}(0)$, $z'_0 \in {{s'}^{\ddagger}}^{-1}(0)$ such that $z_0 \preceq_c z'_0$. We then obtain from L-82.b/c and L-83.a/b that (56) holds for $s^{\ddagger}, {s'}^{\ddagger}$. In addition, we obtain from L-84.b/e and L-85.a-d that (57) holds for $s^{\ddagger}, {s'}^{\ddagger}$. Hence $s^{\ddagger} \preceq_c {s'}^{\ddagger}$ by Def. 75, as desired.

Proof of Theorem 62

Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is ‡-invariant, i.e. $\omega(s^{\ddagger}) = \omega(s)$ for all $s \in \mathbb{L}$.

a.: If $\mathcal{F}_\omega$ propagates fuzziness in quantifiers, then ω propagates fuzziness. To see this, assume that $\mathcal{F}_\omega$ propagates fuzziness in quantifiers. Let us recall that $\mathcal{F}_\omega = \mathcal{F}_\Omega$ by (39), provided we define Ω by (38). We hence know from Th-58 that Ω propagates fuzziness. In turn, lemma L-79 permits us to conclude that ω propagates fuzziness, as desired.

b.: If ω propagates fuzziness, then $\mathcal{F}_\omega$ propagates fuzziness in quantifiers. Consider a choice of $s, s' \in \mathbb{L}$ which satisfies (545) and (546). Then $s^{\ddagger} \preceq_c {s'}^{\ddagger}$ by L-86, i.e.

$$\omega(s^{\ddagger}) \preceq_c \omega({s'}^{\ddagger}) \tag{550}$$

because ω is assumed to propagate fuzziness. Hence

$$\begin{aligned} \omega(s) &= \omega(s^{\ddagger}) && \text{by assumption that } \omega \text{ be } {}^{\ddagger}\text{-invariant} \\ &\preceq_c \omega({s'}^{\ddagger}) && \text{by (550)} \\ &= \omega(s')\,. && \text{by assumption that } \omega \text{ be } {}^{\ddagger}\text{-invariant} \end{aligned}$$

Hence ω fulfills the preconditons of lemma L-80. We conclude that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ defined by (38) propagates fuzziness. Hence $\mathcal{F}_\Omega$ propagates fuzziness in quantifiers by Th-58. But $\mathcal{F}_\omega = \mathcal{F}_\Omega$ by (39), i.e. $\mathcal{F}_\omega$ propagates fuzziness in quantifiers.

A.32 Proof of Theorem 63

Lemma 87 *Let $s \in \mathbb{L}$ be given with $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$ and suppose $s' \in \mathbb{L}$ is defined by*

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{551}$$

for all $z \in \mathbf{I}$. Abbreviating

$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s'(z)\} \tag{552}$$

$$S''(\gamma) = \{z \in \mathbf{I} : \gamma \geq s^{\ddagger}(z)\} \tag{553}$$

for all $\gamma \in \mathbf{I}$, it holds that

$$S'(\gamma) = S''(\gamma) \cap [\tfrac{1}{2}, 1]$$

for all $\gamma \in [0, 1)$.

Proof I first show that

$$S'(\gamma) \subseteq S''(\gamma) \cap [\tfrac{1}{2}, 1] \tag{554}$$

for all $\gamma < 1$. Hence let $\gamma < 1$ and $z \in S'(\gamma)$. Then

$$\gamma \geq s'(z) \tag{555}$$

which is apparent from (552). Because $s'(z') = 1 > \gamma$ for all $z' < \frac{1}{2}$ by (551), we conclude from (555) that

$$z \geq \tfrac{1}{2} \tag{556}$$

Therefore $s'(z) = s^{\ddagger}(z)$ by (551). We hence obtain from (555) that $\gamma \geq s^{\ddagger}(z)$. In turn, we conclude from (553) that $z \in S''(\gamma)$. This proves that $z \in S''(\gamma) \cap [\frac{1}{2}, 1]$ because $z \geq \frac{1}{2}$ by (556). Hence indeed $S'(\gamma) \subseteq S''(\gamma) \cap [\frac{1}{2}, 1]$, i.e. (554) holds, as desired.
To see that

$$S''(\gamma) \cap [\tfrac{1}{2}, 1] \subseteq S'(\gamma) \tag{557}$$

is also valid for all $\gamma < 1$, consider $z \in S''(\gamma) \cap [\frac{1}{2}, 1]$. Then $z \geq \frac{1}{2}$ and $z \in S''(\gamma)$, hence

$$\gamma \geq s^{\ddagger}(z)\,. \tag{558}$$

Because $z \geq \frac{1}{2}$, we conclude from (551) that $s'(z) = s^{\ddagger}(z)$, hence $\gamma \geq s'(z)$ by (558). In turn $z \in S'(\gamma)$ by (552), which finishes the proof of (557).
Combining (554) and (557), we finally obtain the desired $S'(\gamma) = S''(\gamma) \cap [\frac{1}{2}, 1]$ for all $\gamma < 1$.

Lemma 88 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is a given mapping which satisfies* (ω-1) *to* (ω-4)*. Further suppose that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω according to equation* (38)*. If Ω propagates fuzziness, then ω has the following property. For all $s \in \mathbb{L}$ with $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$, it holds that $\omega(s) = \omega(s')$, where*

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{559}$$

for all $z \in \mathbf{I}$.

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be the given mapping which satisfies (ω-1) to (ω-4) and suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω according to (38). In order to prove the claim of the lemma, we assume that Ω propagates fuzziness. Now consider a choice of $s \in \mathbb{L}$ with $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$ and define $s' : \mathbf{I} \longrightarrow \mathbf{I}$ by (559). It is then apparent from $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ that ${s'}^{-1}(0) \neq \varnothing$, i.e. $s' \in \mathbb{L}$. We define $S', S'', S^* \in \mathbb{K}$ by

$$S'(\gamma) = \{z \in \mathbf{I} : \gamma \geq s'(z)\} \tag{560}$$

$$S''(\gamma) = \{z \in \mathbf{I} : \gamma \geq s^{\ddagger}(z)\} \tag{561}$$

$$S^*(\gamma) = \begin{cases} S''(0) \cap [\frac{1}{2}, 1] & : \quad \gamma = 0 \\ S''(\gamma) & : \quad \gamma > 0 \end{cases} \tag{562}$$

for all $\gamma \in \mathbf{I}$. We notice that for all $\gamma \in \mathbf{I}$, $S^*(\gamma)$ is convex in the sense that $a \leq b \leq c$ and $a, c \in S^*(\gamma)$ entail that $b \in S^*(\gamma)$. Therefore

$$S^{*\ddagger} = S^*, \tag{563}$$

which is apparent from Def. 59. We also notice from (561) and (562) that

$$S^* \cap [\tfrac{1}{2}, 1] = S'' \cap [\tfrac{1}{2}, 1]. \tag{564}$$

Therefore

$$\begin{aligned}
\omega(s') &= \Omega(S') && \text{by (560), (38) and L-38} \\
&= \Omega(S'' \cap [\tfrac{1}{2}, 1]) && \text{by L-87} \\
&= \Omega(S^* \cap [\tfrac{1}{2}, 1]) && \text{by (564)} \\
&= \Omega(S^{*\ddagger} \cap [\tfrac{1}{2}, 1]) && \text{by (563)} \\
&= \Omega(S^*) && \text{by Th-59} \\
&= \Omega(S'') && \text{by (561), (562) and L-18} \\
&= \omega(s^{\ddagger}) && \text{by (561), (38) and L-38} \\
&= \omega(s). && \text{by Th-48}
\end{aligned}$$

Lemma 89 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is a given mapping which satisfies* (ω-1) *to* (ω-4)*. Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be defined in terms of ω according to* (38)*. If Ω propagates fuzziness, then $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.*

Proof To see this, define $S \in \mathbb{K}$ by

$$S(\gamma) = \{z \in \mathbf{I} : \gamma \geq s(z)\} \tag{565}$$

for all $\gamma \in \mathbf{I}$. Then $S(0) = s^{-1}(0)$, hence $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We conclude from L-72 that

$$\Omega(S) = \tfrac{1}{2}\,. \tag{566}$$

Therefore

$$\begin{aligned} \omega(s) &= \Omega(S) && \text{by (565), (38) and L-38} \\ &= \tfrac{1}{2}\,. && \text{by (566)} \end{aligned}$$

Lemma 90 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies (ω-1), (ω-2) and (ω-4). Then condition **1.** is equivalent to the conjunction of conditons **2.a** and **2.b**: are equivalent.*

1. *For all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega(s) = \omega(s')$, where*

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases}$$

   for all $z \in \mathbf{I}$;

2.a *For all $s \in \mathbb{L}$ with $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$, it holds that $\omega(s) = \omega(s')$, where $s' \in \mathbb{L}$ is defined as in **1.**;*

2.b *For all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, it holds that $\omega(s) = \frac{1}{2}$.*

Proof

1. entails 2.a: This is trivially the case because **2.a** is an apparent weakening of **1.**

1. entails 2.b: To see this, suppose ω fulfills **1.** and consider a choice of $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We define $s', s'' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases}$$

$$s''(z) = \begin{cases} 1 & : \quad z > \frac{1}{2} \\ s^{\ddagger}(z) & : \quad z \leq \frac{1}{2} \end{cases}$$

for all $z \in \mathbf{I}$. It is apparent from $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ that $s' \in \mathbb{L}$. Similarly, we conclude from $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$ that $s'' \in \mathbb{L}$. We notice that

$$\omega(s) = \omega(s') \tag{567}$$

by **1.**, and also

$$\omega(s) = \omega(s'') \tag{568}$$

which is apparent from **1.** and (ω-2). Now let us define $s_{\frac{1}{2}} \in \mathbb{L}$ by

$$s_{\frac{1}{2}}(z) = \begin{cases} 0 & : \quad z = \frac{1}{2} \\ 1 & : \quad z \neq \frac{1}{2} \end{cases}$$

for all $z \in \mathbf{I}$. Apparently $s_{\frac{1}{2}} \sqsubseteq s'$ and hence

$$\begin{aligned} \omega(s) &= \omega(s') && \text{by (567)} \\ &\geq \omega(s_{\frac{1}{2}}) && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,, && \text{by } (\omega\text{-1}) \end{aligned}$$

i.e. $\omega(s) \geq \frac{1}{2}$. By similar reasoning, we conclude from $s'' \sqsubseteq s_{\frac{1}{2}}$ that

$$\begin{aligned} \omega(s) &= \omega(s'') && \text{by (568)} \\ &\leq \omega(s_{\frac{1}{2}}) && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,, && \text{by } (\omega\text{-1}) \end{aligned}$$

i.e. $\omega(s) \leq \frac{1}{2}$. Combining this with the former inequation yields $\omega(s) = \frac{1}{2}$.

The conjunction of 2.a and 2.b entails 1.: Let us assume that both **2.a** and **2.b** are valid. Now consider $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} s^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{569}$$

for all $z \in \mathbf{I}$.
If $s^{-1} \cap [0, \frac{1}{2}) = \varnothing$, then $s^{-1} \subseteq [\frac{1}{2}, 1]$. Hence $\omega(s) = \omega(s')$ by **2.a**, as desired.
In the remaining case that $s^{-1}(0) \cap [0, \frac{1}{2}) \neq \varnothing$, we conclude from **2.b** that

$$\omega(s) = \tfrac{1}{2}\,. \tag{570}$$

In order to show that $\omega(s') = \frac{1}{2}$ as well, we consider $z = \frac{1}{2}$. We know that $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, hence there exists z^+ with $s(z^+) = 0$ and $z^+ \geq \frac{1}{2}$. Hence $\inf\{s(z) : z \geq \frac{1}{2}\} \leq s(z^+) = 0$, i.e.

$$\inf\{s(z) : z \geq \tfrac{1}{2}\} = 0 \tag{571}$$

By similar reasoning, we conclude from $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$ that there exists z^- with $s(z^-) = 0$ and $z^- \leq \frac{1}{2}$. Therefore $\inf\{s(z) : z \leq \frac{1}{2}\} \leq s(z^-) = 0$, i.e.

$$\inf\{s(z) : z \leq \tfrac{1}{2}\} = 0 \tag{572}$$

We conclude from (571), (572) and Def. 65 that $s^{\ddagger}(\frac{1}{2}) = 0$. By (569), $s'(\frac{1}{2}) = s^{\ddagger}(\frac{1}{2}) = 0$. Hence ${s'}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and ${s'}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, i.e. $\omega(s') = \frac{1}{2} = \omega(s)$ by **2.b** and (570).

Lemma 91 *Let $S \in \mathbb{K}$ be given. Define $s, s^* \in \mathbb{L}$ by*

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{573}$$

$$s^*(z) = \inf\{\gamma \in \mathbf{I} : z \in S^\ddagger(\gamma)\} \tag{574}$$

for all $z \in \mathbf{I}$. Then $s^\ddagger = s^*$.

Proof Consider $z \in \mathbf{I}$. I will show that $s^\ddagger(z) = s^*(z)$ by proving both inequations $s^\ddagger(z) \leq s^*(z)$ and $s^\ddagger(z) \geq s^*(z)$.

a.: $s^\ddagger(z) \leq s^*(z)$. Consider $\gamma' > s^*(z)$. Then $z \in S^\ddagger(\gamma')$ by (574). By Def. 59 there exist $z^-, z^+ \in S(\gamma')$ with

$$z^- \leq z \leq z^+ \,. \tag{575}$$

Because $z^-, z^+ \in S(\gamma')$, we conclude from (573) that

$$s(z^-) \leq \gamma' \tag{576}$$

$$s(z^+) \leq \gamma' \,. \tag{577}$$

We can hence proceed as follows.

$$\begin{aligned} \inf\{s(z') : z' \leq z\} &\leq s(z^-) && \text{by (575)} \\ &\leq \gamma' && \text{by (576)} \end{aligned}$$

and siimilarly

$$\begin{aligned} \inf\{s(z') : z' \geq z\} &\leq s(z^+) && \text{by (575)} \\ &\leq \gamma' \,. && \text{by (577)} \end{aligned}$$

Hence by Def. 65,

$$s^\ddagger(z) = \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z') : z' \geq z\}) \leq \gamma' \,.$$

Because $\gamma' > s^*(z)$ was arbitrarily chosen, this proves that $s^\ddagger(z) \leq s^*(z)$.

b.: $s^\ddagger(z) \geq s^*(z)$. Hence let $\gamma' > s^\ddagger(z)$. Recalling that $s^\ddagger(z) = \max(\inf\{s(z') : z' \leq z\}, \inf\{s(z') : z' \geq z\})$ by Def. 65, we conclude that there exist $z^-, z^+ \in \mathbf{I}$ with $s(z^-) < \gamma'$, $z^- \leq z$, $s(z^+) < \gamma'$ and $z^+ \geq z$. It is apparent from (573) that $s(z^-) < \gamma'$ and $s(z^+) < \gamma'$ entail that $z^-, z^+ \in S(\gamma')$. Hence by Def. 59, $z \in S^\ddagger(\gamma')$. In turn, we obtain from (574) that $s^*(z) \leq \gamma'$. Because $\gamma' > s^\ddagger(z)$ was arbitrarily chosen, this proves the desired $s^*(z) \leq s^\ddagger(z)$.

Lemma 92 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies (ω-1) to (ω-4). Further suppose that for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega(s) = \omega(s')$, where*

$$s'(z) = \begin{cases} s^\ddagger(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases}$$

for all $z \in \mathbf{I}$. Then the mapping $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ defined by (38) propagates fuzziness.

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a mapping with the properties stated in the lemma and suppose that Ω is defined in terms of ω according to equation (38). In order to prove that Ω propagates fuzziness, it is sufficient to show that $\Omega(S) = \Omega(S^\ddagger \cap [\frac{1}{2}, 1])$ whenever $S(0) \subseteq [\frac{1}{2}, 1]$, see Th-59. Hence let a choice of $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$ be given. We define $s \in \mathbb{L}$ by

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{578}$$

for all $z \in \mathbf{I}$. We further define $s', s^*, s^+ \in \mathbb{L}$ by

$$s'(z) = \begin{cases} s^\ddagger(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{579}$$

$$s^*(z) = \inf\{\gamma \in \mathbf{I} : z \in S^\ddagger(\gamma)\} \tag{580}$$

$$s^+(z) = \begin{cases} s^*(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{581}$$

for all $z \in \mathbf{I}$. We notice that

$$s^+(z) = \inf\{\gamma \in \mathbf{I} : z \in S^\ddagger(\gamma) \cap [\tfrac{1}{2}, 1]\} \tag{582}$$

for all $z \in \mathbf{I}$, which is apparent from (581) and (580). Therefore

$$\begin{aligned}
&\Omega(S) \\
&= \omega(s) && \text{by (38), (578)} \\
&= \omega(s') && \text{by (579) and assumed property of } \omega \\
&= \omega(s^+) && \text{by L-91} \\
&= \Omega(S^\ddagger \cap [\tfrac{1}{2}, 1])\,. && \text{by (582), (38)}
\end{aligned}$$

Proof of Theorem 63

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1)–(ω-4) and suppose $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is defined in terms of ω according to equation (38). I first prove that the condition stated in the theorem is sufficient for ω to propagate fuzziness. Hence suppose that the condition (58) holds for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. We may then conclude from lemma L-92 that Ω propagates fuzziness. In turn, we conclude from Th-58 that $\mathcal{F}_\omega = \mathcal{F}_\Omega$ propagates fuzziness in quantifiers. Finally, we conclude from Th-62 that ω propagates fuzziness, as desired. To see that the condition stated in the theorem is also necessary for ω to propagate fuzziness, suppose ω propagates fuzziness. Then Ω also propagates fuzziness by Th-62 and Th-58. We conclude from L-88, L-89 and L-90 that ω satisfies condition (58) for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$.

A.33 Proof of Theorem 64

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping. Then $\mathcal{F}_\omega = \mathcal{F}_\Omega$, provided we define $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ by equation (38). We already know from Th-60 that $\mathcal{F}_\omega = \mathcal{F}_\Omega$ propagates fuzziness in arguments if and only if Ω propagates unspecificity. Hence we can prove that $\mathcal{F}_\omega$ propagates fuzziness in arguments if and only if ω propagates unspecificity by showing that ω propagates unspecificity if and only if Ω propagates unspecificity.

a.: If Ω propagates unspecificity, then ω propagates unspecificity. Hence suppose that Ω propagates unspecificity, i.e. $S \Subset S'$ implies that $\Omega(S) \preceq_c \Omega(S')$, see Def. 74. Now consider $s, s' \in \mathbb{L}$ with $s \leq s'$. We abbreviate

$$\begin{aligned} S(\gamma) &= \{z \in \mathbf{I} : \gamma \geq s(z)\} \\ S'(\gamma) &= \{z \in \mathbf{I} : \gamma \geq s'(z)\} \end{aligned}$$

It is apparent from these definitions and $s(z) \leq s'(z)$ for all $z \in \mathbf{I}$ that $S(\gamma) \supseteq S'(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e. $S \Subset S'$ by Def. 73. Because Ω is assumed to propagate unspecificity, we obtain that

$$\Omega(S) \preceq_c \Omega(S') \,. \tag{583}$$

Therefore

$$\begin{aligned} \omega(s) &= \Omega(S) && \text{by (39)} \\ &\preceq_c \Omega(S') && \text{by (583)} \\ &= \omega(s') \,. && \text{by (39)} \end{aligned}$$

b.: If ω propagates unspecificity, then Ω propagates unspecificity. To see this, let us assume that ω propagates unspecificity, i.e. $\omega(s) \preceq_c \omega(s')$ whenever $s \leq s'$. Now consider $S, S' \in \mathbb{K}$ with $S \Subset S'$. We define $s, s' \in \mathbb{L}$ by

$$s(z) = \inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \tag{584}$$

$$s'(z) = \inf\{\gamma \in \mathbf{I} : z \in S'(\gamma)\} \tag{585}$$

for all $z \in \mathbf{I}$. Now we recall that by Def. 73, $S(\gamma) \supseteq S'(\gamma)$ for all $\gamma \in \mathbf{I}$. Hence for all $z \in \mathbf{I}$,

$$\{\gamma \in \mathbf{I} : z \in S(\gamma)\} \supseteq \{\gamma \in \mathbf{I} : z \in S'(\gamma)\} \,,$$

which proves that $s(z) \leq s'(z)$, see (584) and (585). Because ω is assumed to propagate unspecificity, we conclude from Def. 77 that

$$\omega(s) \preceq_c \omega(s') \,. \tag{586}$$

Therefore

$$\begin{aligned} \Omega(S) &= \omega(s) && \text{by L-38, (38)} \\ &\preceq_c \omega(s') && \text{by (586)} \\ &= \Omega(S') \,. && \text{by L-38, (38)} \end{aligned}$$

A.34 Proof of Theorem 65

Lemma 93 *Suppose $s \in \mathbb{L}$ is given and $s' \in \mathbb{L}$ is defined by* (59)*. Then*

$$s'(z) = \begin{cases} 0 & : \quad z \geq z_0 \\ s^{\ddagger}(z) & : \quad z < z_0 \end{cases}$$

for all $z \in \mathbf{I}$, where z_0 is an arbitrary element $z_0 \in s^{-1}(0)$.

Proof To see this, consider $z \geq z_0$. Then

$$\begin{aligned} s'(z) &= \inf\{s(z') : z' \leq z\} && \text{by (59)} \\ &\leq s(z_0) && \text{because } z_0 \leq z \\ &= 0\,, && \text{because } z_0 \in s^{-1}(0) \end{aligned}$$

i.e. indeed $s'(z) = 0$. In the remaining case that $z < z_0$, we obtain that

$$\begin{aligned} s'(z) &= \inf\{s(z') : z' \leq z\} && \text{by (59)} \\ &= s^{\ddagger}(z)\,, && \text{by L-43} \end{aligned}$$

as desired.

Lemma 94 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies* (ω-1)*. If ω propagates unspecificity, then $\omega(s) \geq \frac{1}{2}$ whenever $s \in \mathbb{L}$ satisfies $s^{-1} \cap [\frac{1}{2}, 1] \neq \varnothing$.*

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a mapping which satisfies (ω-1). Further suppose that ω propagates unspecificity. Now let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1]$ be given. We may hence choose some $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. We define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} 0 & : \quad z = z_0 \\ 1 & : \quad \text{else} \end{cases}$$

for all $z \in \mathbf{I}$. Then $s \leq s'$, i.e.

$$\omega(s) \preceq_c \omega(s') \tag{587}$$

because ω propagates unspecificity. In addition, we know that $\omega(s') = z_0 \geq \frac{1}{2}$ because ω satisfies (ω-1). In turn, we conclude from (5) and (587) that $\frac{1}{2} \leq \omega(s) \leq z_0$, in particular $\omega(s) \geq \frac{1}{2}$.

Lemma 95 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies* (ω-2) *and* (ω-4)*. If ω fulfills condition* ***b.*** *stated in Th-65, then $\omega(s) \geq \frac{1}{2}$ whenever $s \in \mathbb{L}$ satisfies $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$.*

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a mapping which satisfies (ω-2) as well as (ω-4) and also fulfills condition **b.** of Th-65. Now consider $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. We define $s' \in \mathbb{L}$ by (59), and we further define $s'' \in \mathbb{L}$ by $s''(z) = 0$ for all $z \in \mathbf{I}$. It is then apparent from $s''(z) = s''(1-z)$ for all $z \in \mathbf{I}$ and (ω-2) that

$$\omega(s'') = \tfrac{1}{2}\,. \tag{588}$$

It is further clear from Def. 62 that $s'' \sqsubseteq s'$. Hence

$$\begin{aligned} \omega(s) &= \omega(s') && \text{by assumed condition } \textbf{b.} \text{ of Th-65} \\ &\geq \omega(s'') && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,. && \text{by (588)} \end{aligned}$$

Proof of Theorem 65

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1), (ω-2) and (ω-4).

Condition b. entails condition a. of the theorem: To see that condition **b.** is sufficient for ω to propagate unspecificity, suppose that ω satisfies the condition and consider $s_1, s_2 \in \mathbb{L}$ with $s_1 \leq s_2$. In order to prove that $\omega(s_1) \preceq_c \omega(s_2)$, I discern two cases.
Firstly in the case that $s_2^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, we know from $s_1 \leq s_2$ that $s_1^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ as well. We shall define s_1', s_2' according to equation (59) in terms of s_1 and s_2, respectively. We notice that $s_1 \leq s_2$ entails that ${s_1}^{\ddagger} \leq {s_2}^{\ddagger}$, see Def. 65. It is hence clear from L-93 and Def. 62 that $s_1' \sqsubseteq s_2'$. In turn, we conclude from the assumed property **b.** and (ω-4) that

$$\omega(s_1) = \omega(s_1') \leq \omega(s_2') = \omega(s_2)\,.$$

On the other hand, we know from L-95 and $s_1^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ that $\omega(s_1) \geq \frac{1}{2}$. Hence $\frac{1}{2} \leq \omega(s_1) \leq \omega(s_2)$, i.e. $\omega(s_1) \preceq_c \omega(s_2)$ by (5).
In the remaining case that $s_2^{-1}(0) \cap [\frac{1}{2}, 1] = \varnothing$, we know that $s_2^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$ because $s_2^{-1}(0) \neq \varnothing$ by Def. 60. This case can hence be reduced to the proof of the previous case by utilizing condition (ω-2).

Condition b. is entailed by condition a. of the theorem: In order to prove that condition **b.** is also necessary for ω to propagate unspecificity, suppose that ω propagates unspecificity and consider $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Further suppose that $s' \in \mathbb{L}$ is defined by (59). It is apparent from (59) that $s' \leq s$ and hence

$$\omega(s') \preceq_c \omega(s) \tag{589}$$

because ω propagates unspecificity. We conclude from $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and L-94 that $\omega(s) \geq \frac{1}{2}$. Combining this with (589), it is then apparent from (5) that $\frac{1}{2} \leq \omega(s') \leq \omega(s)$, in particular

$$\omega(s') \leq \omega(s)\,. \tag{590}$$

We further notice L-93 and Def. 62 that $s^\ddagger \sqsubseteq s'$ and hence

$$\omega(s) = \omega(s^\ddagger) \leq \omega(s')$$

by (ω-4) and Th-48. Recalling the converse inequation (590), this proves that $\omega(s) = \omega(s')$.

A.35 Proof of Theorem 66

Lemma 96 *Let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and suppose that $s' \in \mathbb{L}$ is defined by* (58). *Then ${s'}_*^{\leq\frac{1}{2}} = s_*^{\leq\frac{1}{2}}$.*

Proof Straightforward. Because $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. Hence

$$\begin{aligned}
{s'}_*^{\leq\frac{1}{2}} &= \inf\{s'(z) : z \leq \tfrac{1}{2}\} && \text{by (46)} \\
&= \min(\inf\{s'(z) : z < \tfrac{1}{2}\}, s'(\tfrac{1}{2})) \\
&= \min(\inf\{1\}, s^\ddagger(\tfrac{1}{2})) && \text{by (58)} \\
&= s^\ddagger(\tfrac{1}{2}) \\
&= \inf\{s(z) : z \leq \tfrac{1}{2}\} && \text{by L-43 because } \tfrac{1}{2} \leq z_0 \\
&= s_*^{\leq\frac{1}{2}}\,. && \text{by (58)}
\end{aligned}$$

Lemma 97 *Let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and suppose $s' \in \mathbb{L}$ is defined in terms of s according to* (58). *Then $s' = {s'}^\ddagger$.*

Proof To see this, we choose some $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. We notice that $s'(z_0) = s^\ddagger(z_0) = 0$ by (58) and Th-47.a. Now for all $z \geq z_0$,

$$\begin{aligned}
{s'}^\ddagger(z) &= \inf\{s'(z') : z' \geq z\} && \text{by L-43} \\
&= \inf\{s^\ddagger(z') : z' \geq z\} && \text{by (58)} \\
&= {s^\ddagger}^\ddagger(z) && \text{by L-43} \\
&= s^\ddagger(z) && \text{by L-51} \\
&= s'(z)\,. && \text{by (58)}
\end{aligned}$$

In the case that $z \in [\frac{1}{2}, z_0)$, we first notice that

$$\begin{aligned}
&s^\ddagger(\tfrac{1}{2}) \\
&= {s^\ddagger}^\ddagger(\tfrac{1}{2}) && \text{by L-51} \\
&= \inf\{s^\ddagger(z) : z \leq \tfrac{1}{2}\}\,, && \text{by L-43}
\end{aligned}$$

i.e.

$$s^{\ddagger}(\tfrac{1}{2}) = \inf\{s^{\ddagger}(z) : z \leq \tfrac{1}{2}\}\,. \tag{591}$$

Therefore

$$\begin{aligned}
{s'}^{\ddagger}(z) &= \inf\{s'(z') : z' \leq z\} && \text{by L-43}\\
&= \min(\inf\{s'(z') : z' \in [\tfrac{1}{2}, z]\}, \inf\{s'(z') : z' \in [0, \tfrac{1}{2})\})\\
&= \min(\inf\{s^{\ddagger}(z') : z' \in [\tfrac{1}{2}, z]\}, \inf\{1\}) && \text{by (58)}\\
&= \inf\{s^{\ddagger}(z') : z' \in [\tfrac{1}{2}, z]\}\\
&= \min(\inf\{s^{\ddagger}(z') : z' \in (\tfrac{1}{2}, z]\}, s^{\ddagger}(\tfrac{1}{2}))\\
&= \min(\inf\{s^{\ddagger}(z') : z' \in (\tfrac{1}{2}, z]\}, \inf\{s^{\ddagger}(z') : z' \in [0, \tfrac{1}{2}]\}) && \text{by (591)}\\
&= \inf\{s^{\ddagger}(z') : z' \leq z\}\\
&= {s^{\ddagger}}^{\ddagger}(z) && \text{by L-43}\\
&= s^{\ddagger}(z) && \text{by L-51}\\
&= s'(z)\,. && \text{by (58)}
\end{aligned}$$

In the remaining case that $z < \frac{1}{2}$, we obtain that

$$\begin{aligned}
{s'}^{\ddagger}(z) &= \inf\{s'(z') : z' \leq z\} && \text{by L-43}\\
&= \inf\{1\} && \text{by (58)}\\
&= 1\\
&= s'(z)\,, && \text{by (58)}
\end{aligned}$$

which completes the proof of the lemma.

Lemma 98 *Let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and suppose that $s' \in \mathbb{L}$ is defined in terms of s according to* (58)*.*

a. If $s_*^{\perp,0} > \frac{1}{2}$, then ${s'}_*^{\perp,0} = s_*^{\perp,0}$;

b. If $s_*^{\perp,0} \leq \frac{1}{2}$, then ${s'}_*^{\perp,0} = \frac{1}{2}$.

Proof Suppose that s, s' are chosen as stated in the lemma. Then

$$\begin{aligned}
{s'}_*^{\perp,0} &= \inf {{s'}^{\ddagger}}^{-1}(0) && \text{by (43)}\\
&= \inf {s'}^{-1}(0) && \text{by L-97}\\
&= \min(\inf {s'}^{-1}(0) \cap [\tfrac{1}{2}, 1], \inf {s'}^{-1}(0) \cap [0, \tfrac{1}{2}))\\
&= \min(\inf {s^{\ddagger}}^{-1}(0) \cap [\tfrac{1}{2}, 1], \inf \varnothing) && \text{by (58)}\\
&= \min(\inf {s^{\ddagger}}^{-1}(0) \cap [\tfrac{1}{2}, 1], 1)
\end{aligned}$$

i.e.

$$s'^{\perp,0}_{*} = \inf s^{\ddagger^{-1}}(0) \cap [\tfrac{1}{2}, 1] \tag{592}$$

In case **a.** of the lemma we have $s^{\perp,0}_{*} > \frac{1}{2}$, i.e. $\inf s^{\ddagger^{-1}}(0) > \frac{1}{2}$ by (43). Hence $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$, in particular $s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] = s^{\ddagger^{-1}}(0)$. Combining this with (592) and (43) yields the desired $s'^{\perp,0}_{*} = \inf s^{\ddagger^{-1}}(0) = s^{\perp,0}_{*}$.

In case **b.** of the lemma, it holds that $s^{\perp,0}_{*} \leq \frac{1}{2}$, i.e. $\inf s^{\ddagger^{-1}}(0) \leq \frac{1}{2}$ by (43). Now let $\varepsilon > 0$. Then there exists $z_1 \in s^{\ddagger^{-1}}(0)$, i.e. $s^{\ddagger}(z_1) = 0$, with $z < \frac{1}{2} + \varepsilon$. In addition, we know that $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. We can hence choose $z_0 \geq \frac{1}{2}$ with $s^{\ddagger}(z_0) = s(z_0) = 0$. Without loss of generality, we may assume that $z_1 \leq z_0$. Now consider $z = \max(\frac{1}{2}, z_1)$. We conclude from $z_1 \leq z \leq z_0$ and Th-47.c that $s^{\ddagger}(z) \leq \max(s^{\ddagger}(z_1), s^{\ddagger}(z_0)) = \max(0, 0) = 0$. Hence $s^{\ddagger}(z) = 0$, i.e. $z \in s^{\ddagger^{-1}}(0)$. Because $z \geq \frac{1}{2}$, it also holds that $z \in s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1]$. We conclude from $z_1 < \frac{1}{2} + \varepsilon$ and $z = \max(z_1, \frac{1}{2})$ that $z < \frac{1}{2} + \varepsilon$ as well. Hence

$$\begin{aligned} s'^{\perp,0}_{*} &= \inf s^{\ddagger^{-1}}(0) \cap [\tfrac{1}{2}, 1] && \text{by (592)} \\ &\leq s && \text{because } s^{\ddagger}(z) = 0 \text{ and } z \geq \tfrac{1}{2} \\ &< \tfrac{1}{2} + \varepsilon\,. \end{aligned}$$

$\varepsilon \to 0$ yields $s'^{\perp,0}_{*} \leq \frac{1}{2}$. Noticing that

$$\begin{aligned} s'^{\perp,0}_{*} &= \inf s^{\ddagger^{-1}}(0) \cap [\tfrac{1}{2}, 1] && \text{by (592)} \\ &\geq \inf[\tfrac{1}{2}, 1] \\ &= \tfrac{1}{2}\,, \end{aligned}$$

we indeed obtain $s'^{\perp,0}_{*} = \frac{1}{2}$, as desired.

Lemma 99 *Let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and suppose that $s' \in \mathbb{L}$ is defined by* (58)*. Then $s'^{\top,0}_{*} = s^{\top,0}_{*}$.*

Proof Because $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$, in particular $z_0 \in s^{\ddagger^{-1}}(0)$. We now observe that

$$\begin{aligned} s'^{\ddagger^{-1}}(0) &= s'^{-1}(0) && \text{by L-97} \\ &= s^{\ddagger^{-1}}(0) \cap [\tfrac{1}{2}, 1]\,. && \text{by (58)} \end{aligned}$$

Hence $s'^{\ddagger^{-1}}(0) \subseteq s^{\ddagger^{-1}}(0)$, which entails that

$$s'^{\top,0}_{*} = \sup s'^{\ddagger^{-1}}(0) \leq \sup s^{\ddagger^{-1}}(0) = s^{\top,0}_{*}\,. \tag{593}$$

On the other hand

$$\begin{aligned}
s_*^{\top,0} &= \sup s^{\ddagger^{-1}}(0) && \text{by (42)}\\
&= \sup s^{\ddagger^{-1}}(0) \cap [z_0, 1] && \text{because } z_0 \in s^{\ddagger^{-1}}(0)\\
&\leq \sup s^{\ddagger^{-1}}(0) \cap [\tfrac{1}{2}, 1] && \text{because } \tfrac{1}{2} \leq z_0\\
&= \sup s'^{-1}(0) && \text{by (58)}\\
&= \sup s'^{\ddagger^{-1}}(0) && \text{by L-97}\\
&= {s'}_*^{\top,0}\,. && \text{by (42)}
\end{aligned}$$

Combining this with (593) yields the desired ${s'}_*^{\top,0} = s_*^{\top,0}$.

Proof of Theorem 66

We know from Th-50 that $\mathcal{F}_M$ is a DFS. Hence ω_M satisfies (ω-1)–(ω-4) by Th-45. In particular, ω_M is ‡-invariant by Th-48. Hence Th-62 and Th-63 are applicable, and we can show that $\mathcal{F}_M$ propagates fuzziness in quantifiers by proving that for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega_M(s) = \omega_M(s')$, where s' is defined by (58). Hence let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Then there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. Hence

$$\begin{aligned}
s_*^{\top,0} &= \sup s^{\ddagger^{-1}}(0) && \text{by (42)}\\
&\geq \sup s^{-1}(0) && \text{by Th-47.a}\\
&\geq z_0 && \text{because } z_0 \in s^{-1}(0)\\
&\geq \tfrac{1}{2}\,,
\end{aligned}$$

i.e. $s_*^{\top,0} \geq \frac{1}{2}$. Recalling Def. 67, it is hence sufficient to consider the following two cases.

$s_*^{\perp,0} > \frac{1}{2}$. Then

$$\begin{aligned}
\omega_M(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq\frac{1}{2}}) && \text{by Def. 67}\\
&= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq\frac{1}{2}}) && \text{by L-98.a and L-96}\\
&= \omega_M(s')\,. && \text{by (58)}
\end{aligned}$$

$s_*^{\perp,0} \leq \frac{1}{2}$. In this case, $\omega_M(s) = \frac{1}{2}$ by Def. 67. As concerns s', we know from L-98.b that ${s'}_*^{\perp,0} = \frac{1}{2}$, and we know from L-99 that ${s'}_*^{\top,0} = s_*^{\top,0} \geq \frac{1}{2}$. Hence $\omega_M(s') = \frac{1}{2} = \omega_M(s)$ by Def. 67.

A.36 Proof of Theorem 67

Lemma 100 *Let $s \in \mathbb{L}$ be given with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and suppose $s' \in \mathbb{L}$ is defined in terms of s according to* (59)*. Then ${s'}_*^{\le \frac{1}{2}} = s_*^{\le \frac{1}{2}}$.*

Proof Because $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, there exists $z_0 \ge \frac{1}{2}$ with $s(z_0) = 0$. Recalling that $s^\ddagger \le s$, we also have $s^\ddagger(z_0) = 0$. Therefore

$$\begin{aligned}
{s'}_*^{\le \frac{1}{2}} &= \inf\{s'(z) : z \le \tfrac{1}{2}\} && \text{by (46)} \\
&= \inf\{s^\ddagger(z) : z \le \tfrac{1}{2}\} && \text{by L-93} \\
&= s^{\ddagger\ddagger}(\tfrac{1}{2}) && \text{by L-43} \\
&= s^\ddagger(\tfrac{1}{2}) && \text{by L-51} \\
&= \inf\{s(z) : z \le \tfrac{1}{2}\} && \text{by L-43} \\
&= s_*^{\le \frac{1}{2}} \,.
\end{aligned}$$

Lemma 101 *Let $s \in \mathbb{L}$ be given and suppose $s' \in \mathbb{L}$ is defined in terms of s according to* (59)*. Then ${s'}^\ddagger = s'$.*

Proof Let $z_0 \in s^{-1}(0) \neq \varnothing$. Then $s'(z_0) = 0$ by L-93, i.e. $z_0 \in {s'}^{-1}(0)$. Hence for $z \ge z_0$,

$$\begin{aligned}
{s'}^\ddagger(z) &\le s'(z) && \text{by Th-47.a} \\
&= 0 \,, && \text{by L-93}
\end{aligned}$$

i.e. ${s'}^\ddagger(z) = 0 = s'(z)$. In the remaining case that $z < z_0$, we compute

$$\begin{aligned}
{s'}^\ddagger(z) &= \inf\{s'(z') : z' \le z\} && \text{by L-43} \\
&= \inf\{s^\ddagger(z') : z' \le z\} && \text{by L-93} \\
&= s^{\ddagger\ddagger}(z) && \text{by L-43} \\
&= s^\ddagger(z) && \text{by L-51} \\
&= s'(z) \,, && \text{by L-93}
\end{aligned}$$

which completes the proof of the lemma.

Lemma 102 *Let $s \in \mathbb{L}$ be given and suppose $s' \in \mathbb{L}$ is defined in terms of s according to* (59)*. Then ${s'}_*^{\perp,0} = s_*^{\perp,0}$.*

Proof To see this, consider $s \in \mathbb{L}$ and assume $s' \in \mathbb{L}$ is defined by (59). Let z_0 be some element $z_0 \in s^{-1}(0) \neq \varnothing$. In particular, $z_0 \in s^{\ddagger^{-1}}(0)$ by Th-47.a. In turn, we conclude that $z_0 \in s^{\ddagger^{-1}}(0) \cap [0, z_0]$ and hence

$$\inf s^{\ddagger^{-1}}(0) \leq z_0 \tag{594}$$

$$\inf s^{\ddagger^{-1}}(0) = \inf s^{\ddagger^{-1}}(0) \cap [0, z_0] \,. \tag{595}$$

Therefore

$$\begin{aligned}
{s'}_*^{\perp,0} &= \inf {s'}^{\ddagger^{-1}}(0) && \text{by (43)}\\
&= \inf {s'}^{-1}(0) && \text{by L-101}\\
&= \min(\inf {s'}^{-1}(0) \cap [z_0, 1], \inf {s'}^{-1}(0) \cap [0, z_0]) &&\\
&= \min(\inf [z_0, 1], \inf s^{\ddagger^{-1}}(0) \cap [0, z_0]) && \text{by L-93}\\
&= \min(z_0, \inf s^{\ddagger^{-1}}(0)) && \text{by (595)}\\
&= \inf s^{\ddagger^{-1}}(0) && \text{by (594)}\\
&= s_*^{\perp,0} \,. && \text{by (43)}
\end{aligned}$$

Lemma 103 *Let $s \in \mathbb{L}$ be given and suppose $s' \in \mathbb{L}$ is defined in terms of s according to* (59)*. Then ${s'}_*^{\top,0} = 1$.*

Proof Choose some $z_0 \in s^{-1}(0) \neq \varnothing$. Then

$$\begin{aligned}
s'(1) &= \inf\{s(z) : z \leq 1\} && \text{by (59)}\\
&\leq s(z_0) && \text{because } z_0 \leq 1\\
&= 0 \,, && \text{because } z_0 \in s^{-1}(0)
\end{aligned}$$

i.e. $s'(1) = 0$ and

$$1 \in {s'}^{-1}(0) \,. \tag{596}$$

Therefore

$$\begin{aligned}
{s'}_*^{\top,0} &= \sup {s'}^{\ddagger^{-1}}(0) && \text{by (42)}\\
&\geq 1 && \text{by (596)}\,,
\end{aligned}$$

i.e. $s_*^{\top,0} = 1$, as desired.

Proof of Theorem 67

We know from Th-50 that $\mathcal{F}_M$ is a DFS and hence satisfies (ω-1) to (ω-4) by Th-45. In particular, ω_M is ‡-invariant by Th-48. Hence Th-64 and Th-65 are applicable, which

allow us to show that $\mathcal{F}_M$ propagates fuzziness in arguments by proving that for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega_M(s) = \omega_M(s')$, where s' is defined by (59).
Hence let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and define s' in terms of s according to (59). Because $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. Hence

$$\begin{aligned} s_*^{\top,0} &= \sup s^{\ddagger^{-1}}(0) && \text{by (42)} \\ &\geq \sup s^{-1}(0) && \text{by Th-47.a} \\ &\geq z_0 && \text{because } z_0 \in s^{-1}(0) \\ &\geq \tfrac{1}{2}\,, \end{aligned}$$

i.e. $s_*^{\top,0} \geq \frac{1}{2}$. Recalling Def. 67, there are only two cases left to consider.

a.: $s_*^{\perp,0} > \frac{1}{2}$. Then

$$\begin{aligned} \omega_M(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq\frac{1}{2}}) && \text{by Def. 67} \\ &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq\frac{1}{2}}) && \text{by L-100 and L-102} \\ &= \omega_M(s')\,. && \text{by Def. 67} \end{aligned}$$

b.: $s_*^{\perp,0} \leq \frac{1}{2}$. In this case, we conclude from Def. 67 that $\omega_M(s) = \frac{1}{2}$ because $s_*^{\top,0} \geq \frac{1}{2}$. Considering s', we firstly know from L-102 that ${s'}_*^{\perp,0} = s_*^{\perp,0} \leq \frac{1}{2}$. In addition, we know from L-103 that $s_*^{\top,0} = 1$. Hence $\omega_M(s') = \frac{1}{2} = \omega_M(s)$ by Def. 67, which completes the proof.

A.37 Proof of Theorem 68

Lemma 104 *Let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ be given and suppose that $s' \in \mathbb{L}$ is defined in terms of s according to* (58). *Then ${s'}_1^{\top,*} = s_1^{\top,*}$.*

Proof To see this, we first notice that there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$, which is immediate from $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. In particular $z_0 \in s^{-1}([0, 1))$ and

$$z_0 \in s^{\ddagger^{-1}}([0, 1))\,. \tag{597}$$

Because $z_0 \geq \frac{1}{2}$, this of course entails that

$$z_0 \in s^{\ddagger^{-1}}([0, 1)) \cap [\tfrac{1}{2}, 1]\,. \tag{598}$$

Therefore

$$\begin{aligned}
s_1^{\top,*} &= (s^\ddagger)_1^{\top,*} && \text{by L-58}\\
&= \sup {s^\ddagger}^{-1}([0,1)) && \text{by (44)}\\
&= \sup {s^\ddagger}^{-1}([0,1)) \cap [\tfrac{1}{2},1] && \text{by (598)}\\
&= \max(\sup {s^\ddagger}^{-1}([0,1)) \cap [\tfrac{1}{2},1], \sup \varnothing)\\
&= \max(\sup {s'}^{-1}([0,1)) \cap [\tfrac{1}{2},1], \sup {s'}^{-1}([0,1)) \cap [0,\tfrac{1}{2})) && \text{by (58)}\\
&= \sup {s'}^{-1}([0,1))\\
&= {s'}_1^{\top,*}\,.
\end{aligned}$$

Proof of Theorem 68

We know from Th-52, Th-45 and Th-48 that ω_P satisfies (ω-1)–(ω-4) and that ω_P is ‡-invariant. Therefore Th-62 and Th-63 are applicable, i.e. we can show that $\mathcal{F}_P$ propagates fuzziness in quantifiers by proving that for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2},1] \neq \varnothing$, it holds that $\omega_P(s) = \omega_P(s')$, where s' is defined by (58).
Hence let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2},1] \neq \varnothing$. Then there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. By the same reasoning as in the proof of Th-66, this entails that $s_*^{\top,0} \geq \frac{1}{2}$. Recalling Def. 68, it is hence sufficient to consider the following two cases.

a.: $s_*^{\perp,0} > \frac{1}{2}$. Then

$$\begin{aligned}
\omega_P(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq\frac{1}{2}}) && \text{by Def. 68}\\
&= \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq\frac{1}{2}}) && \text{by L-104 and L-96}\\
&= \omega_P(s')\,. && \text{by Def. 68}
\end{aligned}$$

b.: $s_*^{\perp,0} \leq \frac{1}{2}$. In this case, we recall that by Def. 68, $s_*^{\perp,0} \leq \frac{1}{2}$ and $s_*^{\top,0} \geq \frac{1}{2}$ entail that $\omega_P(s) = \frac{1}{2}$. As concerns s', we know from L-98.b that ${s'}_*^{\perp,0} = \frac{1}{2}$, and we know from L-99 that ${s'}_*^{\top,0} = s_*^{\top,0} \geq \frac{1}{2}$. Hence $\omega_P(s') = \frac{1}{2} = \omega_P(s)$ by Def. 68.

A.38 Proof of Theorem 69

In order to prove that $\mathcal{F}_P$ does not propagate fuzziness in arguments, we can utilize that $\mathcal{F}_P$ is a DFS by Th-52 and hence satisfies (ω-1)–(ω-4) by Th-45. We can therefore apply theorems Th-64 and Th-65. In order to show that $\mathcal{F}_P$ does not propagate fuzziness in arguments it is hence sufficient to prove that there exists $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2},1] \neq \varnothing$ and $\omega_P(s) \neq \omega_P(s')$, where $s' \in \mathbb{L}$ is defined by (59). To this end, let us define $s \in \mathbb{L}$ by

$$s(z) = \begin{cases} 1 & : & z > \frac{4}{5} \\ 0 & : & z \in (\frac{3}{5}, \frac{4}{5}] \\ 1 & : & z \leq 35 \end{cases} \tag{599}$$

It is then immediate from (43), (44) and (46) that

$$\begin{aligned} s_*^{\perp,0} &= \inf s^{\ddagger^{-1}}(0) = \inf(\tfrac{3}{5}, \tfrac{4}{5}] = \tfrac{3}{5} \\ s_1^{\top,*} &= \sup s^{-1}([0,1)) = \sup(\tfrac{3}{5}, \tfrac{4}{5}] = \tfrac{4}{5} \\ s_*^{\leq\frac{1}{2}} &= \inf\{s(z) : z \leq \tfrac{1}{2}\} = \inf\{1\} = 1\,. \end{aligned}$$

We conclude from Def. 68 that

$$\omega_P(s) = \min(\tfrac{4}{5}, \tfrac{1}{2} + \tfrac{1}{2}1) = \min(\tfrac{4}{5}, 1) = \tfrac{4}{5}\,. \tag{600}$$

Now let us consider s' defined by (59), i.e.

$$s'(z) = \inf\{s(z') : z' \leq z\} = \begin{cases} 0 & : & z > \frac{3}{5} \\ 1 & : & z \leq \frac{3}{5} \end{cases}$$

for all $z \in \mathbf{I}$, which is straightforward from (599). In this case we obtain from (43), (44) and (46) that

$$\begin{aligned} {s'}_*^{\perp,0} &= \inf {s'}^{\ddagger^{-1}}(0) = \inf(\tfrac{3}{5}, 1] = \tfrac{3}{5} \\ {s'}_1^{\top,*} &= \sup s^{-1}([0,1)) = \sup(\tfrac{3}{5}, 1] = 1 \\ {s'}_*^{\leq\frac{1}{2}} &= \inf\{s(z) : z \leq \tfrac{1}{2}\} = \inf\{1\} = 1\,. \end{aligned}$$

Hence by Def. 68 and (600)

$$\omega_P(s') = \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) = 1 \neq \tfrac{4}{5} = \omega_P(s)\,.$$

We conclude from Th-65 and Th-64 that $\mathcal{F}_P$ does not propagate fuzziness in arguments.

A.39 Proof of Theorem 70

We know from Th-54, Th-45 and Th-48 that ω_Z satisfies (ω-1)–(ω-4) and that ω_P is ‡-invariant. Therefore Th-62 and Th-63 are applicable, i.e. we can show that $\mathcal{F}_Z$ propagates fuzziness in quantifiers by proving that for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, it holds that $\omega_P(s) = \omega_P(s')$, where s' is defined by (58).
Hence let $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Because $s^{-1}(0) \subseteq s^{\ddagger^{-1}}(0)$ by Th-47.a, we hence know that $s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Considering s' defined by (58), we recall from L-97 that ${s'}^{\ddagger} = s'$. We further notice that $s'(z) = z^\ddagger$ for all $z \in [\frac{1}{2}, 1]$. Hence

${s'}^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] = {s'}^{-1}(0) \cap [\frac{1}{2}, 1] = s^{\ddagger^{-1}}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Therefore

$$\begin{aligned}
\omega_Z(s) &= \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by L-62} \\
&= \min({s'}_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} {s'}_*^{\leq \frac{1}{2}}) && \text{by L-104 and L-96} \\
&= \omega_Z(s')\,, && \text{by L-62}
\end{aligned}$$

i.e. $\mathcal{F}_Z$ propagates fuzziness in quantifiers, as desired.

A.40 Proof of Theorem 71

By utilizing L-65, the very same example as in the proof of theorem Th-69 proves that ω_Z fails to propagate unspecificity, i.e. $\mathcal{F}_Z$ does not propagate fuzziness in arguments by Th-64.

A.41 Proof of Theorem 72

In order to prove that $\mathcal{F}_R$ does not propagate fuzziness in quantifiers, we recall that ω_R satisfies (ω-1)–(ω-4) by Th-56 and Th-45. In particular, ω_R is ‡-invariant by Th-48. Hence Th-62 and Th-63 are applicable, and we can prove that $\mathcal{F}_R$ fails to propagate fuzziness in quantifiers by showing that there exists $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $\omega_R(s) \neq \omega_R(s')$, where s' is defined in terms of s according to (58).
To see this, consider $s \in \mathbb{L}$ defined by

$$s(z) = \begin{cases} 0 & : \quad z = 1 \\ \frac{1}{2} & : \quad z < 1 \end{cases} \tag{601}$$

for all $z \in \mathbf{I}$. We observe that s is concave, i.e.

$$s^\ddagger = s \tag{602}$$

by L-50. Hence $s' \in \mathbb{L}$ as defined by (58) becomes

$$s'(z) = \begin{cases} 0 & : \quad z = 1 \\ \frac{1}{2} & : \quad \frac{1}{2} \leq z < 1 \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{603}$$

for all $z \in \mathbf{I}$. We notice that s' is concave as well, hence

$${s'}^\ddagger = s' \tag{604}$$

by L-50. We hence obtain for the coefficient $s_*^{\perp,0}$ that

$$s_*^{\perp,0} = \inf s^{\ddagger^{-1}}(0) = \inf s^{-1}(0) = \inf\{1\} = 1 \tag{605}$$

by (43), (602) and (601). Similarly, we conclude from (43), (604) and (603) that

$${s'}_*^{\perp,0} = \inf {s'}^{\ddagger^{-1}}(0) = \inf {s'}^{-1}(0) = \inf\{1\} = 1\,. \tag{606}$$

Hence

$$\begin{aligned}\omega_R(s) &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by Def. 70}\\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2}\cdot\tfrac{1}{2}) && \text{by (605), (601)}\\ &= \tfrac{3}{4}\end{aligned}$$

and

$$\begin{aligned}\omega_R(s') &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s'(0)) && \text{by Def. 70}\\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2}\cdot 1) && \text{by (606), (603)}\\ &= 1\,.\end{aligned}$$

Hence $\omega_R(s) = \frac{3}{4} \neq 1 = \omega_R(s')$, which completes the proof that $\mathcal{F}_R$ does not propagate fuzziness in quantifiers.

A.42 Proof of Theorem 73

By Th-64, we can reduce the proof of $\mathcal{F}_R$ propagating fuzziness in arguments to the proof that ω_R propagates unspecificity. In turn, Th-65 permits us to simplify the proof that ω_R propagates unspecificity to showing that $\omega_R(s) = \omega_R(s')$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, where $s' \in \mathbb{L}$ is defined by (59). To see that this condition is satisfied by ω_R, we first notice that $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ entails that there exists $z_0 \geq \frac{1}{2}$ with $s(z_0) = 0$. In particular

$$s_*^{\top,0} = \sup {s^{\ddagger}}^{-1}(0) \geq \sup s^{-1}(0) \geq z_0 \geq \tfrac{1}{2} \qquad (607)$$

by Th-47.a and (42). It is hence sufficient to discern the following two cases.

$s_*^{\perp,0} > \frac{1}{2}$. Then ${s'}_*^{\perp,0} = s_*^{\perp,0} > \frac{1}{2}$. We also notice that

$$\begin{aligned}s'(0) &= \inf\{s(z) : z \leq 0\} && \text{by (59)}\\ &= \inf\{s(0)\}\\ &= s(0)\,,\end{aligned}$$

i.e.

$$s'(0) = s(0)\,. \qquad (608)$$

Therefore

$$\begin{aligned}\omega_R(s') &= \min({s'}_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s'(0)) && \text{by Def. 70}\\ &= \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by L-102 and (608)}\\ &= \omega_R(s)\,. && \text{by Def. 70}\end{aligned}$$

$s_*^{\perp,0} \leq \frac{1}{2}$. Recalling that $s_*^{\top,0} \geq \frac{1}{2}$ by (607), we then conclude from Def. 70 that $\omega_R(s) = \frac{1}{2}$. As concerns s', we first notice that ${s'}_*^{\perp,0} = s_*^{\perp,0} \leq \frac{1}{2}$ by L-102. In addition, we know from L-103 that ${s'}_*^{\top,0} = 1 \geq \frac{1}{2}$. Hence $\omega_R(s') = \frac{1}{2} = \omega_R(s)$ by Def. 70.

A.43 Proof of Theorem 74

Let $\mathbb{\Omega}$ be a collection of mappings $\Omega \in \mathbb{\Omega}$, $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ and let

$$\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathbb{\Omega}\} \tag{609}$$

be the corresponding collection of QFMs.
To see that $\mathbb{\Omega}$ is specificity consistent whenever $\mathbb{F}$ is specificity consistent, suppose that $\mathbb{F}$ is specificity consistent and consider a choice of $S \in \mathbb{K}$. By Th-33, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S_{Q,X} = S\,. \tag{610}$$

Because $\mathbb{F}$ is specificity consistent, we know from Def. 25 that

$$\{\mathcal{F}_\Omega(Q)(X) : \mathcal{F}_\Omega \in \mathbb{F}\} \subseteq A \tag{611}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. Therefore

$$\begin{aligned}
\{\Omega(S) : \Omega \in \mathbb{\Omega}\} &= \{\Omega(S_{Q,X}) : \Omega \in \mathbb{\Omega}\} && \text{by (610)}\\
&= \{\mathcal{F}_\Omega(Q)(X) : \Omega \in \mathbb{\Omega}\} && \text{by Def. 55}\\
&= \{\mathcal{F}_\Omega(Q)(X) : \mathcal{F}_\Omega \in \mathbb{F}\} && \text{by (609)}\\
&\subseteq A
\end{aligned}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, see (611). Because $S \in \mathbb{K}$ was arbitrarily chosen, this proves that $\mathbb{\Omega}$ is specificity consistent according to Def. 78.
To see that $\mathbb{F}$ is specificity consistent whenever $\mathbb{\Omega}$ is specificity consistent, consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\{\Omega(S_{Q,X_1,\ldots,X_n}) : \Omega \in \mathbb{\Omega}\} \subseteq A \tag{612}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ according to Def. 78. Therefore

$$\begin{aligned}
&\{\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) : \mathcal{F}_\Omega \in \mathbb{F}\}\\
&= \{\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) : \Omega \in \mathbb{\Omega}\} && \text{by (609)}\\
&= \{\Omega(S_{Q,X_1,\ldots,X_n}) : \Omega \in \mathbb{\Omega}\} && \text{by Def. 55}\\
&\subseteq A
\end{aligned}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, see (612). Hence $\mathbb{F}$ is specificity consistent by Def. 25.

A.44 Proof of Theorem 75

Lemma 105 *Let* $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ *be a given mapping which satisfies* (Ω-5). *Further suppose that* $\Omega(S) = \frac{1}{2}$ *for all* $S \in \mathbb{K}$ *with* $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ *and* $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. *Then for all* $S \in \mathbb{K}$*,*

a. If $S(0) \subseteq [\frac{1}{2}, 1]$*, then* $\Omega(S) \geq \frac{1}{2}$*;*

b. If $S(0) \subseteq [0, \frac{1}{2}]$*, then* $\Omega(S) \leq \frac{1}{2}$*.*

Proof To see that **a.** holds, consider a choice of $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$ and define $S' \in \mathbb{K}$ by

$$S'(\gamma) = S(\gamma) \cup \{\tfrac{1}{2}\}$$

for all $\gamma \in \mathbf{I}$. Then

$$\Omega(S') = \tfrac{1}{2} \tag{613}$$

because $S'(0) \cap [\frac{1}{2}, 1] \supseteq \{\frac{1}{2}\} \neq \varnothing$ and $S'(0) \cap [0, \frac{1}{2}] \supseteq \{\frac{1}{2}\} \neq \varnothing$. Let us also notice that

$$S' \sqsubseteq S\,. \tag{614}$$

This is immediate from Def. 57: firstly $S(\gamma) \subseteq S'(\gamma)$ entails that for all $z \in S(\gamma)$, there exists $z' \in S'(\gamma)$ with $z' \leq z$ because $z' = z$ is a suitable choice. Secondly if $z' \in S'(\gamma)$, then there exists $z \in S(\gamma)$ with $z \geq z'$. This is apparent for $z' \neq \frac{1}{2}$, where again $z = z'$ is a suitable choice. In the remaining case that $z' = \frac{1}{2}$, we notice that $S(0) \subseteq [\frac{1}{2}, 1]$ and $S(0) \neq \varnothing$ ensures the existence of some $z_0 \in S(0)$ with $z_0 \geq \frac{1}{2} = z'$. Hence (614) is indeed valid. We conclude that

$$\begin{aligned} \Omega(S) &\geq \Omega(S') && \text{by } (\Omega\text{-5}), (614) \\ &= \tfrac{1}{2}\,, && \text{by } (613) \end{aligned}$$

as desired.

The proof of part **b.** of the lemma is completely analogous to that of part **a.**. In this case, we use the apparent inequation $S \sqsubseteq S'$, where $S(0) \subseteq [0, \frac{1}{2}]$, and S' is defined as above.

Proof of Theorem 75

Suppose that $\mathfrak{Q}$ is a collection of mappings $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ with the properties stated in the theorem and define $\mathbb{F} = \{\mathcal{F}_\Omega : \Omega \in \mathfrak{Q}\}$. To see that $\mathbb{F}$ is specificity consistent, we utilize theorem Th-74. It is hence sufficient to prove that $\mathfrak{Q}$ is specificity consistent. Hence let $S \in \mathbb{K}$ be given. We discern three cases.

$S(0) \subseteq [\frac{1}{2}, 1]$**.** Then $\Omega(S) \geq \frac{1}{2}$ for all $\Omega \in \mathfrak{Q}$ by part **a.** of L-105, i.e.

$$\{\Omega(S) : S \in \mathfrak{Q}\} \subseteq [\tfrac{1}{2}, 1]\,.$$

$S(0) \subseteq [0, \frac{1}{2}]$. Then $\Omega(S) \leq \frac{1}{2}$ for all $\Omega \in \mathbb{\Omega}$ by part **b.** of L-105, i.e.

$$\{\Omega(S) : S \in \mathbb{\Omega}\} \subseteq [0, \tfrac{1}{2}] \, .$$

$S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\Omega(S) = \frac{1}{2}$ for all $\Omega \in \mathbb{\Omega}$ by the assumption on $\mathbb{\Omega}$. In particular,

$$\{\Omega(S) : S \in \mathbb{\Omega}\} = \{\tfrac{1}{2}\} \subseteq [\tfrac{1}{2}, 1] \, .$$

Hence $\{\Omega(S) : S \in \mathbb{\Omega}\} \subseteq A$ for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. This proves that $\mathbb{\Omega}$ is specificity consistent by Def. 78.

A.45 Proof of Theorem 76

Let $\Omega, \Omega' \in \mathbb{K}$ be given. Let us first prove that $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ whenever $\Omega \preceq_c \Omega'$. Hence suppose that $\Omega \preceq_c \Omega'$ and consider some semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) &= \Omega(S_{Q,X_1,\ldots,X_n}) && \text{by Def. 55} \\
&\preceq_c \Omega'(S_{Q,X_1,\ldots,X_n}) && \text{because } \Omega \preceq_c \Omega' \\
&= \mathcal{F}_{\Omega'}(Q)(X_1, \ldots, X_n) \, .
\end{aligned}$$

To see that the reverse relationship also holds, suppose that $\mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'}$ and consider a choice of $S \in \mathbb{K}$. By Th-33, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S_{Q,X} = S \, . \tag{615}$$

Therefore

$$\begin{aligned}
\Omega(S) &= \Omega(S_{Q,X}) && \text{by (615)} \\
&= \mathcal{F}_\Omega(Q)(X) && \text{by Def. 55} \\
&\preceq_c \mathcal{F}_{\Omega'}(Q)(X) && \text{because } \mathcal{F}_\Omega \preceq_c \mathcal{F}_{\Omega'} \\
&= \Omega'(S_{Q,X}) && \text{by Def. 55} \\
&= \Omega'(S) \, . && \text{by (615)}
\end{aligned}$$

A.46 Proof of Theorem 77

Let $\Omega, \Omega' : \mathbb{K} \longrightarrow \mathbf{I}$ be given mappings which satisfy (Ω-2) and (Ω-5). Further suppose that $\Omega(S) = \frac{1}{2} = \Omega'(S)$ whenever $S \in \mathbb{K}$ has $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$. To see that $\Omega \preceq_c \Omega'$ if and only if $\Omega(S) \leq \Omega'(S)$ for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$, I first prove that the latter property is entailed by former. Hence suppose that $\Omega \preceq_c \Omega'$ and consider some $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$. Then

$$\Omega'(S) \geq \tfrac{1}{2}$$

by L-105. In addition

$$\Omega(S) \preceq_c \Omega'(S)$$

because $\Omega \preceq_c \Omega'$. We conclude from (5) that $\frac{1}{2} \leq \Omega(S) \leq \Omega'(S)$, in particular $\Omega(S) \leq \Omega'(S)$, as desired.
In order to prove the converse implication, let us assume that $\Omega(S) \leq \Omega'(S)$ for all $S \in \mathbb{K}$ with $S(0) \subseteq [\frac{1}{2}, 1]$. To see that $\Omega \preceq_c \Omega'$, consider a choice of $S \in \mathbb{K}$. If $S(0) \subseteq [\frac{1}{2}, 1]$, then $\frac{1}{2} \leq \Omega(S) \leq \Omega'(S)$ by L-105 and the assumed property of Ω and Ω'. Hence $\Omega(S) \preceq_c \Omega'(S)$ by (5). The case that $S(0) \subseteq [0, \frac{1}{2}]$ can be reduced to the previous case by means of (Ω-2). Finally if $S(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $S(0) \cap [0, \frac{1}{2}] \neq \varnothing$, then $\Omega(S) = \frac{1}{2}$ by the assumed property of Ω. In particular $\Omega(S) \preceq_c \Omega'(S)$ by (5).

A.47 Proof of Theorem 78

Let ϖ be a collection of mappings $\omega \in \varpi$, $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ and let

$$\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \varpi\} \tag{616}$$

be the corresponding collection of QFMs as defined by Def. 61.
To see that ϖ is specificity consistent whenever $\mathbb{F}$ is specificity consistent, suppose that $\mathbb{F}$ is specificity consistent and consider a choice of $s \in \mathbb{L}$. By Th-41, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$s_{Q,X} = s\,. \tag{617}$$

Because $\mathbb{F}$ is specificity consistent, we know from Def. 25 that

$$\{\mathcal{F}_\omega(Q)(X) : \mathcal{F}_\omega \in \mathbb{F}\} \subseteq A \tag{618}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. Therefore

$$\begin{aligned}
\{\omega(s) : \omega \in \varpi\} &= \{\omega(s_{Q,X}) : \omega in \varpi\} && \text{by (617)}\\
&= \{\mathcal{F}_\omega(Q)(X) : \omega \in \varpi\} && \text{by Def. 61}\\
&= \{\mathcal{F}_\omega(Q)(X) : \mathcal{F}_\omega \in \mathbb{F}\} && \text{by (616)}\\
&\subseteq A
\end{aligned}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, see (618). Because $s \in \mathbb{L}$ was arbitrarily chosen, this proves that ϖ is specificity consistent according to Def. 80.
To see that $\mathbb{F}$ is specificity consistent whenever ϖ is specificity consistent, consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\{\omega(s_{Q,X_1,\ldots,X_n}) : \omega \in \varpi\} \subseteq A \tag{619}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ according to Def. 80. Therefore

$$\begin{aligned}
&\{\mathcal{F}_\omega(Q)(X_1, \ldots, X_n) : \mathcal{F}_\omega \in \mathbb{F}\}\\
&= \{\mathcal{F}_\omega(Q)(X_1, \ldots, X_n) : \omega \in \varpi\} && \text{by (616)}\\
&= \{\omega(s_{Q,X_1,\ldots,X_n}) : \omega \in \varpi\} && \text{by Def. 61}\\
&\subseteq A
\end{aligned}$$

for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$, see (619). Hence $\mathbb{F}$ is specificity consistent by Def. 25.

A.48 Proof of Theorem 79

Lemma 106 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-4). Further suppose that $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then for all $s \in \mathbb{L}$,*

a. If $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$, then $\omega(s) \geq \frac{1}{2}$;

b. If $s^{-1}(0) \subseteq [0, \frac{1}{2}]$, then $\omega(s) \leq \frac{1}{2}$.

Proof Let a choice of $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given which satisfies (ω-4) and has the additional property stated in the lemma, i.e. $\omega(s) = \frac{1}{2}$ whenever $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$.
To see that **a.** holds, consider some $s \in \mathbb{L}$ with $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$ and define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} s(z) & : \quad z \neq \frac{1}{2} \\ 0 & : \quad z = \frac{1}{2} \end{cases} \tag{620}$$

for all $z \in \mathbf{I}$. Then ${s'}^{-1}(0) \cap [\frac{1}{2}, 1] \supseteq \{\frac{1}{2}\} \neq \varnothing$ and ${s'}^{-1}(0) \cap [0, \frac{1}{2}] \supseteq \{\frac{1}{2}\} \neq \varnothing$. Hence

$$\omega(s') = \tfrac{1}{2} \tag{621}$$

by the assumed special property of ω. We further notice that $s' \trianglelefteq s$. To see this, consider $z \in \mathbf{I}$. It is apparent from (620) that $z' = z$ is a legal choice of $z' \leq z$ with $s'(z') \leq s(z)$. Similarly, it holds that for all $z' \in \mathbf{I}$, there exists $z \geq z'$ with $s(z) \leq s'(z')$. This is apparent for $z' \neq \frac{1}{2}$, when $z = z'$ is a suitable choice for z. In the case that $z' = \frac{1}{2}$, we utilize that $s^{-1}(0) \subseteq [\frac{1}{2}, 1]$ and $s^{-1}(0) \neq \varnothing$. Hence there exists $z_0 \in s^{-1}(0)$ with $z_0 \geq \frac{1}{2} = z'$ and $s(z_0) = 0 \leq s'(z')$. We conclude that indeed $s' \trianglelefteq s$ by Def. 64. In turn we obtain from L-42 that $s' \sqsubseteq s$. We conclude that

$$\begin{aligned} \omega(s) &\geq \omega(s') && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,. && \text{by (621)} \end{aligned}$$

This completes the proof of part **a.** of the lemma. The proof of part **b.** is entirely analogous. In this case, we have $s^{-1}(0) \subseteq [0, \frac{1}{2}]$ and hence $s \sqsubseteq s'$, where s' is defined as above. This permits us to conclude that $\omega(s) \leq \omega(s') = \frac{1}{2}$, as desired.

Proof of Theorem 79

Suppose that $\mathfrak{Q}$ is a collection of mappings $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ with the properties stated in the theorem and define $\mathbb{F} = \{\mathcal{F}_\omega : \omega \in \mathfrak{Q}\}$. To see that $\mathbb{F}$ is specificity consistent, we utilize theorem Th-78. It is hence sufficient to prove that $\mathfrak{Q}$ is specificity consistent. Hence let $s \in \mathbb{L}$ be given. We discern three cases.

$s^{-1}(0) \subseteq [\frac{1}{2}, 1]$. Then $\omega(s) \geq \frac{1}{2}$ for all $\omega \in \varpi$ by part **a.** of L-106, i.e.

$$\{\omega(s) : s \in \varpi\} \subseteq [\tfrac{1}{2}, 1] .$$

$s^{-1}(0) \subseteq [0, \frac{1}{2}]$. Then $\omega(s) \leq \frac{1}{2}$ for all $\omega \in \varpi$ by part **b.** of L-106, i.e.

$$\{\omega(s) : s \in \varpi\} \subseteq [0, \tfrac{1}{2}] .$$

$s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then $\omega(s) = \frac{1}{2}$ for all $\omega \in \varpi$ by the assumption on ϖ. In particular,

$$\{\omega(s) : s \in \varpi\} = \{\tfrac{1}{2}\} \subseteq [\tfrac{1}{2}, 1] .$$

Hence $\{\omega(s) : s \in \varpi\} \subseteq A$ for a choice of $A \in \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$. This proves that ϖ is specificity consistent by Def. 80.

A.49 Proof of Theorem 80

Consider a choice of $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ which satisfies (ω-1)–(ω-4). Let us also suppose that the corresponding QFM $\mathcal{F}_\omega$ defined by Def. 61 propagates fuzziness in quantifiers, i.e. ω propagates fuzziness by Th-62.
Now let $s \in \mathbb{L}$ be given with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We define $s', s'', s''', s^+, s^- \in \mathbb{L}$ by

$$s'(z) = \begin{cases} 0 & : \quad z \in \{0, 1\} \\ 1 & : \quad \text{else} \end{cases} \tag{622}$$

$$s''(z) = \begin{cases} 0 & : \quad z \in \{\frac{1}{2}, 1\} \\ 1 & : \quad \text{else} \end{cases} \tag{623}$$

$$s'''(z) = \begin{cases} 0 & : \quad z \in \{0, \frac{1}{2}\} \\ 1 & : \quad \text{else} \end{cases} \tag{624}$$

$$s^+(z) = \begin{cases} s^\ddagger(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{625}$$

$$s^-(z) = \begin{cases} 1 & : \quad z > \frac{1}{2} \\ s^\ddagger(z) & : \quad z \leq \frac{1}{2} \end{cases} \tag{626}$$

for all $z \in \mathbf{I}$. We first observe that

$$\omega(s') = \tfrac{1}{2} \tag{627}$$

which is apparent from (622) and (ω-2). We then notice that $s'' \preceq_c s'$, see (622), (623) and Def. 75. Hence $\omega(s'') \preceq_c \omega(s')$ because ω propagates fuzziness. But $\omega(s') = \frac{1}{2}$, hence

$$\omega(s'') = \tfrac{1}{2} \tag{628}$$

by (5). We further notice that $s^+ \sqsubseteq s''$, which is apparent from Def. 64 and L-42. Hence

$$\begin{aligned} \omega(s) &= \omega(s^+) && \text{by Th-63} \\ &\le \omega(s'') && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,. && \text{by (628)} \end{aligned}$$

The remaining proof that $\omega(s) \le \frac{1}{2}$ is analogous. In this case, we first notice that $s''' \preceq_c s'$; hence $\omega(s''') \preceq_c \omega(s')$ because ω propagates fuzziness. We then conclude from $\omega(s') = \frac{1}{2}$ that

$$\omega(s''') = \tfrac{1}{2} \tag{629}$$

as well. We further notice that $s''' \sqsubseteq s^-$, which is apparent from Def. 64 and L-42. Hence

$$\begin{aligned} \omega(s) &= \omega(s^-) && \text{by Th-63 and } (\omega\text{-2}) \\ &\ge \omega(s''') && \text{by } (\omega\text{-4}) \\ &= \tfrac{1}{2}\,. && \text{by (629)} \end{aligned}$$

A.50 Proof of Theorem 81

The claim of the theorem is an immediate consequence of Th-80 and Th-79.

A.51 Proof of Theorem 82

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1)–(ω-4). Further suppose that the DFS $\mathcal{F}_\omega$ defined in terms of ω according to Def. 61 propagates fuzziness in arguments, i.e. ω propagates unspecificity by Th-64.
Now let $s \in \mathbb{L}$ be given with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then

$$s^{\ddagger}(\tfrac{1}{2}) = 0 \tag{630}$$

by Def. 65. We define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} 0 & : \quad z = \frac{1}{2} \\ 1 & : \quad z \neq \frac{1}{2} \end{cases}$$

for all $z \in \mathbf{I}$. It is then apparent from $s'(z) = s'(1-z)$ for all $z \in \mathbf{I}$ and (ω-2) that

$$\omega(s') = \tfrac{1}{2}\,. \tag{631}$$

We notice from (630) that $s^{\ddagger} \le s'$. Hence $\omega(s) \preceq_c \omega(s')$ because ω propagates unspecificity. But $\omega(s') = \frac{1}{2}$ by (631). Therefore $\omega(s) = \frac{1}{2}$ by (5), as desired.

A.52 Proof of Theorem 83

The claim of the theorem is an immediate consequence of Th-82 and Th-79.

A.53 Proof of Theorem 84

Let $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\omega$ and $\mathcal{F}_{\omega'}$ are defined by Def. 61.
I first prove that $\mathcal{F}_\omega \preceq_c \mathcal{F}_{\omega'}$ whenever $\omega \preceq_c \omega'$. Hence suppose that $\omega \preceq_c \omega'$ and consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\omega(Q)(X_1, \ldots, X_n) &= \omega(s_{Q,X_1,\ldots,X_n}) && \text{by Def. 61}\\
&\preceq_c \omega'(s_{Q,X_1,\ldots,X_n}) && \text{by Def. 81}\\
&= \mathcal{F}_{\omega'}(Q)(X_1, \ldots, X_n)\,, && \text{by Def. 61}
\end{aligned}$$

as desired.
To see that $\omega \preceq_c \omega'$ whenever $\mathcal{F}_\omega \preceq_c \mathcal{F}_{\omega'}$, suppose that the latter condition holds and consider $s \in \mathbb{L}$. By Th-41, there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$s_{Q,X} = s\,. \tag{632}$$

Hence

$$\begin{aligned}
\omega(s) &= \omega(s_{Q,X}) && \text{by (632)}\\
&= \mathcal{F}_\omega(Q)(X) && \text{by Def. 61}\\
&\preceq_c \mathcal{F}_{\omega'}(Q)(X) && \text{by Def. 81}\\
&= \omega'(s_{Q,X}) && \text{by Def. 61}\\
&= \omega'(s)\,. && \text{by (632)}
\end{aligned}$$

A.54 Proof of Theorem 85

Lemma 107 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a mapping which satisfies (ω-4). Further suppose that $\omega(s) = \frac{1}{2}$ for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. Then for all $s \in \mathbb{L}$,*

a. if ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$, then $\omega(s) \geq \frac{1}{2}$;

b. if ${s^\ddagger}^{-1}(0) \subseteq [0, \frac{1}{2}]$, then $\omega(s) \leq \frac{1}{2}$;

c. if ${s^\ddagger}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and ${s^\ddagger}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, then $\omega(s) = \frac{1}{2}$.

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which fulfills the requirements of the lemma.

a.: ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$**.** Then

$$\begin{aligned}
\omega(s) &= \omega(s^\ddagger) && \text{by Th-48}\\
&\geq \tfrac{1}{2}\,. && \text{by L-106.a}
\end{aligned}$$

b.: ${s^{\ddagger}}^{-1}(0) \subseteq [0, \frac{1}{2}]$**.** Then

$$\begin{aligned} \omega(s) &= \omega(s^{\ddagger}) && \text{by Th-48} \\ &\leq \tfrac{1}{2}\,. && \text{by L-106.b} \end{aligned}$$

c.: ${s^{\ddagger}}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ **and** ${s^{\ddagger}}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$**.** In this case, $\omega(s^{\ddagger}) = \frac{1}{2}$ by the assumed property of ω. Hence $\omega(s) = \omega(s^{\ddagger}) = \frac{1}{2}$ by Th-48.

Proof of Theorem 85

Let $\omega, \omega' : \mathbb{L} \longrightarrow \mathbf{I}$ be given mappings which satisfy (ω-2) and (ω-4). Further suppose that $\omega(s) = \frac{1}{2} = \omega'(s)$ whenever $s \in \mathbb{L}$ has $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. To see that $\omega \preceq_c \omega'$ if and only if $\omega(s) \leq \omega'(s)$ for all $s \in \mathbb{L}$ with ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$, I first prove that the latter property is entailed by former. Hence suppose that $\omega \preceq_c \omega'$ and consider some $s \in \mathbb{L}$ with ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$. Then

$$\omega'(s) \geq \tfrac{1}{2}$$

by L-107. In addition

$$\omega(s) \preceq_c \omega'(s)$$

because $\omega \preceq_c \omega'$. We conclude from (5) that $\frac{1}{2} \leq \omega(s) \leq \omega'(s)$, in particular $\omega(s) \leq \omega'(s)$, as desired.
In order to prove the converse implication, let us assume that $\omega(s) \leq \omega'(s)$ for all $s \in \mathbb{L}$ with ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$. To see that $\omega \preceq_c \omega'$, consider a choice of $s \in \mathbb{L}$. If ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$, then $\frac{1}{2} \leq \omega(s) \leq \omega'(s)$ by L-107 and the assumed property of ω and ω'. Hence $\omega(s) \preceq_c \omega'(s)$ by (5). The case that ${s^{\ddagger}}^{-1}(0) \subseteq [0, \frac{1}{2}]$ can be reduced to the previous case by means of (ω-2). Finally if ${s^{\ddagger}}^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and ${s^{\ddagger}}^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, then $\omega(s) = \frac{1}{2}$ by the assumed property of ω. In particular $\omega(s) \preceq_c \omega'(s)$ by (5).

A.55 Proof of Theorem 86

Lemma 108 *Suppose* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *propagates fuzziness and satisfies* (ω-1)–(ω-4). *Then* $\omega(s) \leq \frac{1}{2} + \frac{1}{2} s_*^{\leq \frac{1}{2}}$ *for all* $s \in \mathbb{L}$.

Proof Define $s_1, s_2 \in \mathbb{L}$ by

$$s_1(z) = \begin{cases} 0 & : \quad z = 1 \\ 1 & : \quad z \in (0,1) \\ s_*^{\leq \frac{1}{2}} & : \quad z = 0 \end{cases} \tag{633}$$

$$s_2(z) = \begin{cases} 0 & : \quad z = 1 \\ s_*^{\leq \frac{1}{2}} & : \quad z \in [\frac{1}{2}, 1) \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{634}$$

i.e.

$$s_2(z) = \begin{cases} {s_1}^{\ddagger}(z) & : \quad z \geq \frac{1}{2} \\ 1 & : \quad z < \frac{1}{2} \end{cases} \tag{635}$$

for all $z \in \mathbf{I}$. Let us now show that $s \sqsubseteq s_2$:

- Let $z \in \mathbf{I}$. Then $\inf\{s_2(z') : z' \geq z\} = 0 \leq s(z)$.
- Let $z' \in \mathbf{I}$. We must show that $\inf\{s(z) : z \leq z'\} \leq s_2(z')$. If $z' = 1$, then any choice of $z \in s^{-1}(0) \neq \varnothing$ satisfies $z \leq z'$ and $s(z) = 0$. Hence $\inf\{s(z) : z \leq 1\} = 0 \leq s_2(1)$. In the case that $z' \in [\frac{1}{2}, 1)$, we simply observe that

$$\begin{aligned} & \inf\{s(z) : z \leq z'\} \\ & \leq \inf\{s(z) : z \leq \tfrac{1}{2}\} && \text{because } z' \geq \tfrac{1}{2} \\ & = s_*^{\leq \frac{1}{2}} && \text{by (46)} \\ & = s_2(z)\,. && \text{by (634)} \end{aligned}$$

  Finally if $z < \frac{1}{2}$, then trivially $\inf\{s(z) : z \leq z'\} \leq 1 = s_2(z)$.

This proves that indeed $s \sqsubseteq s_2$ by Def. 62, and

$$\begin{aligned} \omega(s) & \leq \omega(s_2) && \text{by } (\omega\text{-4}) \\ & = \omega(s_1) && \text{by (635) and Th-63} \\ & = \tfrac{1}{2} + \tfrac{1}{2} s_1(0) && \text{by } (\omega\text{-3}) \\ & = \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}\,. && \text{by (633)} \end{aligned}$$

Lemma 109 *Suppose* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *satisfies* (ω-1) *and* (ω-4). *Then for all* $s \in \mathbb{L}$, $\omega(s) \leq s_1^{\top,*}$.

Proof Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ω-1) and (ω-4). Now consider $s \in \mathbb{L}$. We define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} 0 & : \quad z = s_1^{\top,*} \\ 1 & : \quad \text{else} \end{cases} \tag{636}$$

for all $z \in \mathbf{I}$. We first notice that $s \trianglelefteq s'$.

- Let $z \in \mathbf{I}$. If $z > s_1^{\top,*}$, then $s(z) = 1$, see (44). Hence $z' = z$ is a proper choice of z' with $z' \geq z$ and $s'(z') = 1 \leq 1 = s(z)$. In the remaining case that $z \leq s_1^{\top,*}$, $z' = s_1^{\top,*}$ is a choice of z' with $z' \geq z$ and $s'(z') = 0 \leq s(z)$.
- Now consider $z' \in \mathbf{I}$. If $z' > s_1^{\top,*}$, then $z = z'$ satisfies $z \leq z'$ and $s(z) = 1 \leq 1 = s'(z')$, see (44). In the case that $z' = s_1^{\top,*}$, we choose some $z \in s^{-1}(0) \neq \varnothing$. Clearly $z \leq \sup s^{-1}(0) \leq \sup s^{-1}([0,1)) = s_1^{\top,*} = z'$. In addition, $s(z) = 0 \leq s'(z')$. Finally if $z' < s_1^{\top,*}$, then $z = z'$ has the desired properties $z \leq z'$ and $s(z) \leq 1 = s'(z')$.

Hence indeed $s \trianglelefteq s'$, in particular $s \sqsubseteq s'$ by L-42. Therefore

$$\begin{aligned} \omega(s) &\leq \omega(s') && \text{by } (\omega\text{-4}) \\ &= s_1^{\top,*}\,. && \text{by } (\omega\text{-1}) \text{ and } (636) \end{aligned}$$

Proof of Theorem 86

It has been shown in Th-54 and Th-70 that $\mathcal{F}_Z$ is a DFS and propagates fuzziness in quantifiers. Hence ω_Z satisfies (ω-1)–(ω-4) by Th-45 and propagates fuzziness by Th-62. Now consider another $\mathcal{F}_\omega$-DFS which propagates fuzziness in quantifiers, i.e. ω satisfies (ω-1)–(ω-4) and propagates fuzziness. Utilizing Th-84, Th-80 and Th-85, we can prove that $\mathcal{F}_\omega \preceq_c \mathcal{F}_Z$ by showing that $\omega(s) \leq \omega_Z(s)$ for all $s \in \mathbb{L}$ with ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$. Hence let such a choice of s be given. Then

$$\begin{aligned} \omega(s) &\leq \min(s_1^{\top,*}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq\frac{1}{2}}) && \text{by L-109 and L-108} \\ &= \omega_Z(s)\,. && \text{by Def. 69} \end{aligned}$$

Hence $\omega(s) \leq \omega_Z(s)$ holds for all $s \in \mathbb{L}$ with ${s^{\ddagger}}^{-1}(0) \subseteq [\frac{1}{2}, 1]$, i.e. $\mathcal{F}_\omega \preceq_c \mathcal{F}_Z$. Because $\mathcal{F}_\omega$ was an arbitrary $\mathcal{F}_\omega$-DFS which propagates fuzziness in quantifiers, this proves that $\mathcal{F}_Z$ is indeed the most specific $\mathcal{F}_\omega$-DFS which propagates fuzziness in quantifiers.

A.56 Proof of Theorem 87

Lemma 110 *Suppose $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ satisfies (ω-3) and (ω-4). Then for all $s \in \mathbb{L}$, $\omega(s) \leq \frac{1}{2} + \frac{1}{2}s(0)$.*

Proof To see this, define $s' \in \mathbb{L}$ by

$$s'(z) = \begin{cases} 0 & : \quad z = 1 \\ 1 & : \quad z \in (0,1) \\ s(0) & : \quad z = 0 \end{cases} \tag{637}$$

Let us now notice that $s \trianglelefteq s'$:

- Let $z \in \mathbf{I}$. Then there exists $z' \geq z$ with $s'(z') \leq s(z)$, viz $z' = 1$ yields $s'(z') = 0 \leq s(z)$.
- Let $z' \in \mathbf{I}$. Then there exists $z \leq z'$ with $s(z) \leq s'(z')$. This is apparent for $z' = 1$, where $z_0 \in s^{-1}(0) \neq \varnothing$ is a suitable choice for z. For $z' \in [0,1)$, we can choose $z = 0$ because $s(0) \leq s'(z)$ by (637).

Hence $s \trianglelefteq s'$ by Def. 64. Recalling L-42, this proves that $s \sqsubseteq s'$. Therefore

$$\begin{aligned} \omega(s) &\leq \omega(s') && \text{by } (\omega\text{-}4) \\ &= \tfrac{1}{2} + \tfrac{1}{2}s'(0) && \text{by } (\omega\text{-}3) \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,. && \text{by } (637) \end{aligned}$$

Lemma 111 *Suppose* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *propagates unspecificity and further satisfies* (ω-1), (ω-2) *and* (ω-4)*. Then for all* $s \in \mathbb{L}$, $\omega(s) \leq s_*^{\perp,0}$.

Proof Consider $\varepsilon > 0$. Recalling (43), there exists $x \in {s^{\ddagger}}^{-1}(0)$ with

$$x < s_*^{\perp,0} + \varepsilon\,. \tag{638}$$

Now we define $s_1 \in \mathbb{L}$ by

$$s_1(z) = \begin{cases} 0 & : \quad z = x \\ 1 & : \quad \text{else} \end{cases} \tag{639}$$

for all $z \in \mathbf{I}$. We further define $s_2 \in \mathbb{L}$ by (59), i.e.

$$s_2(z) = \begin{cases} 0 & : \quad z \geq x \\ 1 & : \quad \text{else} \end{cases} \tag{640}$$

for all $z \in \mathbf{I}$. We notice that $s \sqsubseteq s_2$:

- Consider $z \in \mathbf{I}$. Then $\inf\{s_2(z') : z' \geq z\} \leq s_2(1) = 0$, i.e. $\inf\{s_2(z') : z' \geq z\} = 0 \leq s(z)$.
- Now let $z' \in \mathbf{I}$. In the case that $z' \geq x$, we utilize that $x \in {s^{\ddagger}}^{-1}(0)$, i.e.

$$\max(\inf\{s(z') : z' \geq x\}, \inf\{s(z') : z' \leq x\}) = s^{\ddagger}(x) = 0$$

by Def. 65. In particular $\inf\{s(z) : z \leq x\} = 0$. Because $z' \geq x$, we conclude that $\inf\{s(z) : z \leq z'\} \leq \inf\{s(z) : z \leq x\} = 0 \leq s'(z')$. In the remaining case that $z' < x$, it trivially holds that $\inf\{s(z) : z \leq z'\} \leq 1 = s_2(z')$ by (640).

Hence indeed $s \sqsubseteq s_2$ by Def. 65. We conclude that

$$\begin{aligned}\omega(s) &\leq \omega(s_2) && \text{by } (\omega\text{-4})\\ &= \omega(s_1) && \text{by Th-65}\\ &= x && \text{by } (\omega\text{-1})\\ &< s_*^{\perp,0} + \varepsilon\,. && \text{by (638)}\end{aligned}$$

$\varepsilon \to 0$ yields $\omega(s) \leq s_*^{\perp,0}$.

Proof of Theorem 87

We already know from Th-56 and Th-73 that $\mathcal{F}_R$ is a DFS and propagates fuzziness in arguments, i.e. ω_R satisfies (ω-1)–(ω-4) by Th-45 and propagates unspecificity by Th-64. Now consider another $\mathcal{F}_\omega$-DFS which propagates fuzziness in arguments, i.e. ω satisfies (ω-1)–(ω-4) and propagates unspecificity. Utilizing Th-84, Th-82 and Th-85, we can prove that $\mathcal{F}_\omega \preceq_c \mathcal{F}_R$ by showing that $\omega(s) \leq \omega_R(s)$ for all $s \in \mathbb{L}$ with ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$. Hence let such a choice of s be given. If $s_*^{\perp,0} > \frac{1}{2}$, then

$$\begin{aligned}\omega(s) &\leq \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2}s(0)) && \text{by L-111 and L-110}\\ &= \omega_R(s)\,. && \text{by Def. 70}\end{aligned}$$

In the case that $s_*^{\perp,0} = \frac{1}{2}$, we obtain that

$$\begin{aligned}\omega(s) &\leq s_*^{\perp,0} && \text{by L-111}\\ &= \tfrac{1}{2} && \text{by assumption}\\ &= \omega_R(s) && \text{by Def. 70}\end{aligned}$$

where the last equation holds because $s_*^{\top,0} \geq s_*^{\perp,0} = \frac{1}{2}$.
It is apparent that the case $s_*^{\perp,0} < \frac{1}{2}$ is not possible here because $s_*^{\perp,0} = \inf {s^\ddagger}^{-1}(0)$ by (43), but ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$ by assumption on s. Hence $\omega(s) \leq \omega_R(s)$ holds for all $s \in \mathbb{L}$ with ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$, i.e. $\mathcal{F}_\omega \preceq_c \mathcal{F}_R$. Because $\mathcal{F}_\omega$ was an arbitrary $\mathcal{F}_\omega$-DFS which propagates fuzziness in arguments, this proves that $\mathcal{F}_R$ is indeed the most specific $\mathcal{F}_\omega$-DFS which propagates fuzziness in arguments.

A.57 Proof of Theorem 88

We already know from Th-50, Th-66 and Th-67 that $\mathcal{F}_M$ is a DFS and propagates fuzziness both in quantifiers and arguments, i.e. ω_M satisfies (ω-1)–(ω-4) by Th-45, propagates fuzziness by Th-62 and propagates unspecificity by Th-64. Now consider another $\mathcal{F}_\omega$-DFS which propagates fuzziness both in quantifiers and arguments, i.e. ω satisfies (ω-1)–(ω-4), and propagates fuzziness as well as unspecificity. Utilizing Th-84, Th-82 and Th-85, we can prove that $\mathcal{F}_\omega \preceq_c \mathcal{F}_M$ by showing that $\omega(s) \leq \omega_M(s)$ for all $s \in \mathbb{L}$ with $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$. Hence let such a choice of s be given. If $s_*^{\perp,0} > \frac{1}{2}$, then

$$\begin{aligned} \omega(s) &\leq \min(s_*^{\perp,0}, \tfrac{1}{2} + \tfrac{1}{2} s_*^{\leq \frac{1}{2}}) && \text{by L-111 and L-108} \\ &= \omega_M(s)\,. && \text{by Def. 67} \end{aligned}$$

In the case that $s_*^{\perp,0} = \frac{1}{2}$, we obtain that

$$\begin{aligned} \omega(s) &\leq s_*^{\perp,0} && \text{by L-111} \\ &= \tfrac{1}{2} && \text{by assumption} \\ &= \omega_M(s) && \text{by Def. 67} \end{aligned}$$

where the last equation holds because $s_*^{\top,0} \geq s_*^{\perp,0} = \frac{1}{2}$.
The case $s_*^{\perp,0} < \frac{1}{2}$ is not possible here because $s_*^{\perp,0} = \inf s^{\ddagger^{-1}}(0)$ by (43), but $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$ by assumption on s. Hence $\omega(s) \leq \omega_M(s)$ holds for all $s \in \mathbb{L}$ with $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$, i.e. $\mathcal{F}_\omega \preceq_c \mathcal{F}_M$. Because $\mathcal{F}_\omega$ was an arbitrary $\mathcal{F}_\omega$-DFS which propagates fuzziness both in quantifiers and arguments, this proves that $\mathcal{F}_M$ is the most specific $\mathcal{F}_\omega$-DFS which propagates fuzziness both in quantifiers and arguments.

A.58 Proof of Theorem 89

In order to conduct the proof that $\mathcal{M}_U$ is the least specific $\mathcal{F}_\omega$-DFS, I first make explicit the exact shape of the mapping $\omega_U : \mathbb{L} \longrightarrow \mathbf{I}$ which corresponds to $\mathcal{B}'_U : \mathbb{H} \longrightarrow \mathbf{I}$, and hence results in $\mathcal{M}_U = \mathcal{F}_{\omega_U}$. We know that such mapping exists from Th-22, Th-37 and Th-42, where the last theorem is applicable by Th-13 and Th-36. The reformulation will be performed in a number of steps which take us from $\mathcal{B}'_U$ to $\mathcal{B}_U$, then to ξ_U, from there to Ω_U, and finally from Ω_U to the desired ω_U.

Lemma 112 *The DFS $\mathcal{M}_U$ can be rewritten as $\mathcal{M}_U = \mathcal{M}_{\mathcal{B}_U}$, where $\mathcal{B}_U : \mathbb{B} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}_U(f) = \begin{cases} \max(\frac{1}{2} + \frac{1}{2} f_*^{1\uparrow}, f_1^*) & : \quad f \in \mathbb{B}^+ \\ \min(\frac{1}{2} - \frac{1}{2} f_*^{0\uparrow}, f_1^*) & : \quad f \in \mathbb{B}^- \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \end{cases} \tag{641}$$

for all $f \in \mathbb{B}$.

Proof Consider $f \in \mathbb{B}$. In order to prove the claim of the theorem, we must show that $\mathcal{B}_U$ is the mapping defined by equation (18). It is convenient to discern three cases which correspond to the case distinction in (18).

a.: $f \in \mathbb{B}^+$. In this case, we first observe that

$$\begin{aligned}
&(2f-1)_*^{1\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : 2f(\gamma) - 1 = 1\} && \text{by (15)} \\
&= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{(apparent)} \\
&= f_*^{1\uparrow} && \text{by (15)}
\end{aligned}$$

and

$$\begin{aligned}
&(2f-1)_1^* \\
&= \lim_{\gamma \to 1^-} (2f(\gamma) - 1) && \text{by (14)} \\
&= 2(\lim_{\gamma \to 1^-} f(\gamma)) - 1 && \text{(apparent)} \\
&= 2f_1^* - 1\,, && \text{by (14)}
\end{aligned}$$

i.e.

$$(2f-1)_*^{1\uparrow} = f_*^{1\uparrow} \qquad (642)$$

$$(2f-1)_1^* = 2f_1^* - 1\,. \qquad (643)$$

Therefore

$$\begin{aligned}
\mathcal{B}_U(f) &= \max(\tfrac{1}{2} + \tfrac{1}{2} f_*^{1\uparrow}, f_1^*) && \text{by (641)} \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2} f_*^{1\uparrow}, \tfrac{1}{2} + \tfrac{1}{2}(2f_1^* - 1)) && \text{(apparent)} \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2}(2f-1)_*^{1\uparrow}, \tfrac{1}{2} + \tfrac{1}{2}(2f-1)_1^*) && \text{by (642), (643)} \\
&= \tfrac{1}{2} + \tfrac{1}{2} \max((2f-1)_*^{1\uparrow}, (2f-1)_1^*) && \text{(apparent)} \\
&= \tfrac{1}{2} + \tfrac{1}{2} \mathcal{B}'_U(2f-1)\,, && \text{by Def. 41}
\end{aligned}$$

i.e. $\mathcal{B}_U$ is defined in accordance with (18).

b.: $f \in \mathbb{B}^-$. This case can be treated analogously. We notice that

$$\begin{aligned}
&(1-2f)_*^{1\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : 1 - 2f(\gamma) = 1\} && \text{by (15)} \\
&= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{(apparent)} \\
&= f_*^{0\uparrow} && \text{by (16)}
\end{aligned}$$

and

$$\begin{aligned}
&(1-2f)_1^* \\
&= \lim_{\gamma \to 1^-} (1 - 2f(\gamma)) && \text{by (14)} \\
&= 1 - 2 \lim_{\gamma \to 1^-} f(\gamma) && \text{(apparent)} \\
&= 1 - 2f_1^*\,, && \text{by (14)}
\end{aligned}$$

i.e.

$$(1-2f)_*^{1\uparrow} = f_*^{0\uparrow} \tag{644}$$
$$(1-2f)_1^* = 2f_1^* - 1\,. \tag{645}$$

Consequently

$$\begin{aligned}
\mathcal{B}_U(f) &= \min(\tfrac{1}{2} - \tfrac{1}{2} f_*^{0\uparrow}, f_1^*) && \text{by (641)}\\
&= \min(\tfrac{1}{2} - \tfrac{1}{2} f_*^{0\uparrow}, \tfrac{1}{2} - \tfrac{1}{2}(1-2f_1^*)) && \text{(apparent)}\\
&= \min(\tfrac{1}{2} - \tfrac{1}{2}(1-2f)_*^{1\uparrow}, \tfrac{1}{2} - \tfrac{1}{2}(1-2f)_1^*) && \text{by (644), (645)}\\
&= \tfrac{1}{2} - \tfrac{1}{2}\max((1-2f)_*^{1\uparrow}, (1-2f)_1^*) && \text{(apparent)}\\
&= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}'_U(1-2f)\,. && \text{by Def. 41}
\end{aligned}$$

Hence we obtain the desired equation (18) in this case as well.

c.: $f \in \mathbb{B}^{\frac{1}{2}}$**.** In this case, we immediately look up $\mathcal{B}_U(f) = \frac{1}{2}$ from (641), which corresponds to the result required by (18).

Lemma 113 *$\mathcal{M}_U$ can be represented as $\mathcal{M}_U = \mathcal{F}_{\xi_U}$, where $\xi_U : \mathbb{T} \longrightarrow \mathbf{I}$ is defined by*

$$\xi_U(\top, \bot) = \begin{cases} \max(\frac{1}{2} + \frac{1}{2}\bot_*^{1\uparrow}, \bot_1^*) & : \quad \bot(0) > \frac{1}{2} \\ \min(\frac{1}{2} - \frac{1}{2}\top_*^{0\uparrow}, \top_1^*) & : \quad \top(0) < \frac{1}{2} \\ \frac{1}{2} & : \quad \bot(0) \le \frac{1}{2} \le \top(0) \end{cases} \tag{646}$$

for all $(\top, \bot) \in \mathbb{T}$.

Proof Recalling Th-22 and L-112, we simply need to show that

$$\xi_U(\top, \bot) = \mathcal{B}_U(f) \tag{647}$$

for all $(\top, \bot) \in \mathbb{T}$, where $f \in \mathbb{B}$ abbreviates

$$f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)\,. \tag{648}$$

Hence let us consider a choice of $(\top, \bot) \in \mathbb{T}$. It is useful to split the proof according to the cases discerned in the definition of ξ_U.

a.: $\bot(0) > \frac{1}{2}$**.** Then in particular $\top(0) \ge \bot(0) > \frac{1}{2}$ as well and by Def. 33,

$$f = \mathrm{med}_{\frac{1}{2}}(\top, \bot) \in \mathbb{B}^+\,. \tag{649}$$

Noticing that $\top(\gamma) \ge \top(0) > \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ because $\top$ is nondecreasing, we obtain from Def. 23 that in this case,

$$f(\gamma) = \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) = \max(\bot(\gamma), \tfrac{1}{2}) \tag{650}$$

for all $\gamma \in \mathbf{I}$. Let us now consider the coefficients used to define ξ_U. Firstly

$$\begin{aligned}
&\bot_*^{1\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : \bot(\gamma) = 1\} && \text{by (15)} \\
&= \sup\{\gamma \in \mathbf{I} : \max(\bot(\gamma), \tfrac{1}{2}) = 1\} \\
&= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (648) and (650)} \\
&= f_*^{1\uparrow} && \text{by (15)}
\end{aligned}$$

and

$$\begin{aligned}
&\max(\bot_1^*, \tfrac{1}{2}) \\
&= \max(\lim_{\gamma \to 1^-} \bot(\gamma), \tfrac{1}{2}) && \text{by (14)} \\
&= \lim_{\gamma \to 1^-} \max(\bot(\gamma), \tfrac{1}{2}) \\
&= f_1^* \,. && \text{by (650) and (14)}
\end{aligned}$$

To sum up,

$$\bot_*^{1\uparrow} = f_*^{1\uparrow} \tag{651}$$

$$\max(\bot_1^*, \tfrac{1}{2}) = f_1^* \,. \tag{652}$$

Based on these results, we obtain that

$$\begin{aligned}
&\xi_U(\top, \bot) \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2}\bot_*^{1\uparrow}, \bot_1^*) && \text{by (646)} \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2}\bot_*^{1\uparrow}, \max(\bot_1^*, \tfrac{1}{2})) && \text{because } \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{1\uparrow} \geq \tfrac{1}{2} \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2} f_*^{1\uparrow}, f_1^*) && \text{by (651) and (652)} \\
&= \mathcal{B}_U(f) \,. && \text{by (641), (649)}
\end{aligned}$$

Hence equation (647) is satisfied in case **a.**

b.: $\top(0) < \frac{1}{2}$**.** In this case, we can proceed in a similar way. We first notice that $\bot(0) \leq \top(0) < \frac{1}{2}$ as well and hence by Def. 33,

$$f = \operatorname{med}_{\frac{1}{2}}(\top, \bot) \in \mathbb{B}^- \,. \tag{653}$$

Observing that $\bot(\gamma) \leq \bot(0) < \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ because $\bot$ is nonincreasing, we hence obtain from Def. 23 that in case **b.**,

$$f(\gamma) = \operatorname{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) = \min(\top(\gamma), \tfrac{1}{2}) \tag{654}$$

for all $\gamma \in \mathbf{I}$. Again, we relate the coefficients used to define ξ_U to the coefficients used to define $\mathcal{B}_U$. In this case, the relevant coefficients are

$$\begin{aligned}
&\top_*^{0\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : \top(\gamma) = 0\} && \text{by (16)} \\
&= \sup\{\gamma \in \mathbf{I} : \min(\top(\gamma), \tfrac{1}{2}) = 0\} \\
&= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{by (648) and (654)} \\
&= f_*^{0\uparrow} && \text{by (16)}
\end{aligned}$$

and

$$\begin{aligned}
&\min(\top_1^*, \tfrac{1}{2}) \\
&= \min(\lim_{\gamma\to 1^-} \top(\gamma), \tfrac{1}{2}) && \text{by (14)} \\
&= \lim_{\gamma\to 1^-} \min(\top(\gamma), \tfrac{1}{2}) \\
&= \lim_{\gamma\to 1^-} f(\gamma) && \text{by (654)} \\
&= f_1^* . && \text{by (14)}
\end{aligned}$$

In other words

$$\top_*^{0\uparrow} = f_*^{0\uparrow} \tag{655}$$

$$\min(\top_1^*, \tfrac{1}{2}) = f_1^* . \tag{656}$$

From this it is immediate that

$$\begin{aligned}
&\xi_U(\top, \bot) \\
&= \min(\tfrac{1}{2} - \tfrac{1}{2}\top_*^{0\uparrow}, \top_1^*) && \text{by (646)} \\
&= \min(\tfrac{1}{2} - \tfrac{1}{2}\top_*^{0\uparrow}, \min(\top_1^*, \tfrac{1}{2})) && \text{because } \tfrac{1}{2} - \tfrac{1}{2}\top_*^{0\uparrow} \le \tfrac{1}{2} \\
&= \min(\tfrac{1}{2} - \tfrac{1}{2} f_*^{0\uparrow}, f_1^*) && \text{by (655) and (656)} \\
&= \mathcal{B}_U(f) . && \text{by (641), (653)}
\end{aligned}$$

Hence again equation (647) is satisfied, as desired.

c.: $\bot(0) \le \frac{1}{2} \le \top(0)$**.** In this case, we obtain from Def. 23 and (648) that $f(0) = \mathrm{med}_{\frac{1}{2}}(\top(0), \bot(0)) = \frac{1}{2}$. Hence by Def. 33, $f \in \mathbb{B}^{\frac{1}{2}}$. In turn, we obtain from (646) and (641) that $\xi_U(\top, \bot) = \frac{1}{2} = \mathcal{B}_U(f)$. Hence (647) is valid in case **c.** again, which completes the proof of the lemma.

In order to link the resulting Ω_U with a corresponding $\Omega_U : \mathbb{K} \longrightarrow \mathbf{I}$, we need two additional coefficients $s_*^{[0,1)}$ and $s_*^{(0,1]}$, which are computed from a given $s \in \mathbb{L}$ according to

$$s_*^{[0,1)} = \inf \widehat{s}([0,1)) \tag{657}$$

$$s_*^{(0,1]} = \inf \widehat{s}((0,1]) . \tag{658}$$

Based on these coefficients, we can now conveniently express the desired mapping $\Omega_U : \mathbb{K} \longrightarrow \mathbf{I}$.

Lemma 114 *$\mathcal{M}_U$ can be written as $\mathcal{M}_U = \mathcal{F}_{\Omega_U}$, where $\Omega_U : \mathbb{K} \longrightarrow \mathbf{I}$ is defined by*

$$\Omega_U(S) = \begin{cases} \max(\frac{1}{2} + \frac{1}{2}s_*^{[0,1)}, s_1^{\bot,*}) & : \inf S(0) > \frac{1}{2} \\ \min(\frac{1}{2} - \frac{1}{2}s_*^{(0,1]}, s_1^{\top,*}) & : \sup S(0) < \frac{1}{2} \\ \frac{1}{2} & : \inf S(0) \le \frac{1}{2} \le \sup S(0) \end{cases} \tag{659}$$

for all $l \in \mathbb{K}$, and s is defined from S according to Def. 53.

Proof Recalling theorem Th-37 and building on the previous lemma L-113, it is sufficient to show that Ω_U satisfies (34), i.e.

$$\Omega_U(S) = \xi_U(\top, \bot) \tag{660}$$

for all $S \in \mathbb{K}$, where $\top = \top_S$ and $\bot = \bot_S$ are defined by (35) and (36), respectively. Hence let $S \in \mathbb{K}$ be given. I first relate the coefficients used in the definition of ξ_U to those used for defining Ω_U. Firstly

$$\begin{aligned} & \bot_1^* \\ &= \lim_{\gamma \to 1^-} \bot(\gamma) && \text{by (14)} \\ &= \inf\{\bot(\gamma) : \gamma < 1\} && \text{because } \bot \text{ nonincreasing} \\ &= \inf\{\inf S(\gamma) : \gamma < 1\} && \text{by (36)} \\ &= \inf\{z \in \mathbf{I} : \text{there exists } \gamma < 1 \text{ s.th. } z \in S(\gamma)\} \\ &= \inf\{z \in \mathbf{I} : s(z) < 1\} && \text{apparent from Def. 53} \\ &= \inf s^{-1}([0,1)) \\ &= s_1^{\bot,*}, && \text{by (45)} \end{aligned}$$

and by similar reasoning

$$\begin{aligned} & \top_1^* \\ &= \lim_{\gamma \to 1^-} \top(\gamma) && \text{by (14)} \\ &= \sup\{\top(\gamma) : \gamma < 1\} && \text{because } \top \text{ nondecreasing} \\ &= \sup\{\sup S(\gamma) : \gamma < 1\} && \text{by (35)} \\ &= \sup\{z \in \mathbf{I} : \text{there exists } \gamma < 1 \text{ s.th. } z \in S(\gamma)\} \\ &= \sup\{z \in \mathbf{I} : s(z) < 1\} && \text{apparent from Def. 53} \\ &= \sup s^{-1}([0,1)) \\ &= s_1^{\top,*}. && \text{by (44)} \end{aligned}$$

We further notice that

$$\begin{aligned}
&\bot_*^{1\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : \bot(\gamma) = 1\} && \text{by (15)} \\
&= \inf\{\gamma \in \mathbf{I} : \bot(\gamma) < 1\} && \text{because } \bot \text{ is nonincreasing} \\
&= \inf\{\gamma \in \mathbf{I} : \inf S(\gamma) < 1\} && \text{by (36)} \\
&= \inf\{\gamma \in \mathbf{I} : \text{there exists } z < 1 \text{ s.th. } z \in S(\gamma)\} \\
&= \inf\{\inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} : z < 1\} \\
&= \inf\{s(z) : z < 1\} \\
&= \inf \widehat{s}([0,1)) && \text{by Def. 15} \\
&= s_*^{[0,1)}\,, && \text{by (657)}
\end{aligned}$$

and analogously

$$\begin{aligned}
&\top_*^{0\uparrow} \\
&= \sup\{\gamma \in \mathbf{I} : \top(\gamma) = 0\} && \text{by (16)} \\
&= \inf\{\gamma \in \mathbf{I} : \top(\gamma) > 0\} && \text{because } \top \text{ nondecreasing} \\
&= \inf\{\gamma \in \mathbf{I} : \sup S(\gamma) > 0\} && \text{by (35)} \\
&= \inf\{\gamma \in \mathbf{I} : \text{there exists } z > 0 \text{ s.th. } z \in S(\gamma)\} \\
&= \inf\{\inf\{\gamma \in \mathbf{I} : z \in S(\gamma)\} : z > 0\} \\
&= \inf\{s(z) : z > 0\} \\
&= \inf \widehat{s}((0,1]) && \text{by Def. 15} \\
&= s_*^{(0,1]}\,. && \text{by (658)}
\end{aligned}$$

To sum up,

$$\bot_1^* = s_1^{\bot,*} \tag{661}$$

$$\top_1^* = s_1^{\top,*} \tag{662}$$

$$\bot_*^{1\uparrow} = s_*^{[0,1)} \tag{663}$$

$$\top_*^{0\uparrow} = s_*^{(0,1]}\,. \tag{664}$$

In order to prove that (660) is valid, it is now convenient to discern three cases that parallel the definition of Ω_U.

a.: $\inf S(0) > \frac{1}{2}$**.** It is then immediate from (36) that $\bot(0) > \frac{1}{2}$ as well. Therefore

$$\begin{aligned}
\Omega_U(S) &= \max(\tfrac{1}{2} + \tfrac{1}{2}s_*^{[0,1)}, s_1^{\bot,*}) && \text{by (659)} \\
&= \max(\tfrac{1}{2} + \tfrac{1}{2}\bot_*^{1\uparrow}, \bot_1^*) && \text{by (661), (663)} \\
&= \xi_U(\top, \bot)\,, && \text{by (646)}
\end{aligned}$$

as desired.

b.: $\sup S(0) < \frac{1}{2}$**.** In this case, we obtain from (35) that $\top(0) < \frac{1}{2}$ as well. Consequently

$$\begin{aligned}
\Omega_U(S) &= \min(\tfrac{1}{2} - \tfrac{1}{2}s_*^{(0,1]}, s_1^{\top,*}) && \text{by (659)}\\
&= \min(\tfrac{1}{2} - \tfrac{1}{2}\top_*^{0\uparrow}, \top_1^*) && \text{by (662), (664)}\\
&= \xi_U(\top,\bot)\,. && \text{by (646)}
\end{aligned}$$

c.: $\inf S(0) \leq \frac{1}{2} \leq \sup S(0)$**.** In this remaining case, we observe from (35) and (36) that $\bot(0) \leq \frac{1}{2} \leq \top(0)$. Hence by (659) and (646), $\Omega_U(S) = \frac{1}{2} = \xi_U(\top,\bot)$.

The above definition of Ω_U can then easily be transformed into our target mapping $\omega_U : \mathbb{L} \longrightarrow \mathbf{I}$.

Lemma 115 *The DFS $\mathcal{M}_U$ can be expressed as $\mathcal{M}_U = \mathcal{F}_{\omega_U}$, where $\omega_U : \mathbb{L} \longrightarrow \mathbf{I}$ is defined by*

$$\omega_U(s) = \begin{cases} \max(\frac{1}{2} + \frac{1}{2}s_*^{[0,1)}, s_1^{\bot,*}) & : \inf S(0) > \frac{1}{2} \\ \min(\frac{1}{2} - \frac{1}{2}s_*^{(0,1]}, s_1^{\top,*}) & : \sup S(0) < \frac{1}{2} \\ \frac{1}{2} & : \inf S(0) \leq \frac{1}{2} \leq \sup S(0) \end{cases} \tag{665}$$

for all $s \in \mathbb{L}$, and S is defined according to equation (37)*, i.e. $S(\gamma) = \{z : \gamma \geq s(z)\}$ for all $\gamma \in \mathbf{I}$.*

Proof We first recall from the previous lemma L-114 that $\mathcal{M}_U = \mathcal{F}_{\Omega_U}$. It is hence sufficient to show that $\mathcal{F}_{\omega_U} = \mathcal{F}_{\Omega_U}$. It is now convenient to utilize theorem Th-42, which is applicable by Th-13 and Th-36. The theorem states that $\mathcal{F}_{\Omega_U} = \mathcal{F}_{\omega_U}$, provided that ω_U satisfies

$$\omega_U(s) = \Omega_U(S) \tag{666}$$

for all $s \in \mathbb{L}$, where $S \in \mathbb{K}$ is defined by (41). To see this, let $s \in \mathbb{L}$ and suppose that $S \in \mathbb{K}$ is defined by (41). Further suppose that $s' \in \mathbb{L}$ is defined in terms of S according to Def. 53. Then

$$\begin{aligned}
&\Omega_U(S)\\
&= \begin{cases} \max(\frac{1}{2} + \frac{1}{2}{s'}_*^{[0,1)}, {s'}_1^{\bot,*}) & : \inf S(0) > \frac{1}{2} \\ \min(\frac{1}{2} - \frac{1}{2}{s'}_*^{(0,1]}, {s'}_1^{\top,*}) & : \sup S(0) < \frac{1}{2} \\ \frac{1}{2} & : \inf S(0) \leq \frac{1}{2} \leq \sup S(0) \end{cases} && \text{by (659)}\\
&= \begin{cases} \max(\frac{1}{2} + \frac{1}{2}s_*^{[0,1)}, s_1^{\bot,*}) & : \inf S(0) > \frac{1}{2} \\ \min(\frac{1}{2} - \frac{1}{2}s_*^{(0,1]}, s_1^{\top,*}) & : \sup S(0) < \frac{1}{2} \\ \frac{1}{2} & : \inf S(0) \leq \frac{1}{2} \leq \sup S(0) \end{cases} && \text{by L-38, } s = s'\\
&= \omega_U(s)\,. && \text{by (665)}
\end{aligned}$$

Hence (666) is indeed valid, as desired.

This completes the chain of lemmata concerned with reformulations of $\mathcal{M}_U$, which finally lead to the explicit representation of $\mathcal{M}_U$ as an $\mathcal{F}_\omega$-DFS, based on the mapping ω_U defined by (665). In order to prove the theorem, I need some additional observations related to the greatest lower specificity bound on the $\mathcal{F}_\omega$-DFSes. It is convenient to introduce some abbreviations. The class of all $\mathcal{F}_\omega$-DFSes will be denoted $\mathbb{F}_\omega$. We know from Th-6 that $\mathbb{F}_\omega$ has a greatest lower specificity bound $\mathcal{F}_{\mathrm{glb}}$, which can be expressed as

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1,\ldots,X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_\omega(Q)(X_1,\ldots,X_n) : \mathcal{F}_\omega \in \mathbb{F}_\omega\}\,, \tag{667}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. We now introduce the mapping $\omega_{\mathrm{glb}} : \mathbb{L} \longrightarrow \mathbf{I}$, defined by

$$\omega_{\mathrm{glb}}(s) = \mathrm{m}_{\frac{1}{2}}\{\omega(s) : \omega \in \mathbf{I}^{\mathbb{L}} \text{ satisfies } (\omega\text{-1})\text{–}(\omega\text{-4})\} \tag{668}$$

for all $s \in \mathbb{L}$. It is then apparent that

$$\begin{aligned}
&\mathcal{F}_{\mathrm{glb}}(Q)(X_1,\ldots,X_n) \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_\omega(Q)(X_1,\ldots,X_n) : \mathcal{F}_\omega \in \mathbb{F}_\omega\} && \text{by (667)}\\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_\omega(Q)(X_1,\ldots,X_n) : \omega \in \mathbf{I}^{\mathbb{L}} \text{ satisfies } (\omega\text{-1})\text{–}(\omega\text{-4})\} && \text{by Th-45}\\
&= \mathrm{m}_{\frac{1}{2}}\{\omega(s_{Q,X_1,\ldots,X_n}) : \omega \in \mathbf{I}^{\mathbb{L}} \text{ satisfies } (\omega\text{-1})\text{–}(\omega\text{-4})\} && \text{by Def. 61}\\
&= \omega_{\mathrm{glb}}(s_{Q,X_1,\ldots,X_n}) && \text{by (668)}
\end{aligned}$$

and hence by Def. 61,

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1,\ldots,X_n) = \mathcal{F}_{\omega_{\mathrm{glb}}}(Q)(X_1,\ldots,X_n) \tag{669}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all choices of fuzzy arguments $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. This proves that the least specific $\mathcal{F}_\omega$-DFS is an $\mathcal{F}_\omega$-DFS itself, viz the DFS defined in terms of ω_{glb}.

Proof of Theorem 89

We already know from (669) that the least specific $\mathcal{F}_\omega$-DFS can be expressed as $\mathcal{F}_{\mathrm{glb}} = \mathcal{F}_{\omega_{\mathrm{glb}}}$. Recalling from L-115 that $\mathcal{M}_U = \mathcal{F}_{\omega_U}$, we can hence prove the theorem by showing that $\omega_U = \omega_{\mathrm{glb}}$. To this end, let us first notice that $\mathcal{F}_{\omega_U} \in \mathbb{F}_\omega$ by Th-13 and L-115. But $\mathcal{F}_{\mathrm{glb}}$ is the greatest lower specificity bound on $\mathbb{F}_\omega$, hence $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}_{\omega_U}$. By equation (669), this means that $\mathcal{F}_{\omega_{\mathrm{glb}}} \preceq_c \mathcal{F}_{\omega_U}$. Now utilizing Th-84, we deduce that

$$\omega_{\mathrm{glb}} \preceq_c \omega_U\,. \tag{670}$$

Observing that $\preceq_c$ is a partial order, it only remains to be shown that $\omega_U \preceq_c \omega_{\mathrm{glb}}$ as well. The proof is greatly simplified by Th-85, and it is worthwhile showing that the theorem is applicable. We first recall from Th-20 that $\mathcal{M}_U = \mathcal{F}_{\omega_U}$ propagates fuzziness in quantifiers. Hence by Th-80,

$$\omega_U(s) = \tfrac{1}{2} \tag{671}$$

for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$. We then conclude from the above $\omega_{\rm glb} \preceq_c \omega_U$ that

$$\omega_{\rm glb}(s) = \tfrac{1}{2} \tag{672}$$

for all $s \in \mathbb{L}$ with $s^{-1}(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $s^{-1}(0) \cap [0, \frac{1}{2}] \neq \varnothing$, because $\omega_{\rm glb}(s) \preceq_c \frac{1}{2} = \omega_U(s)$ is only possible for $\omega_{\rm glb}(s) = \frac{1}{2}$, see (5). This proves that the precondition of Th-85 is indeed satisfied. We can hence apply the theorem and reduce the proof of $\omega_U \preceq_c \omega_{\rm glb}$ to the proof that

$$\omega_U(s) \leq \omega_{\rm glb}(s) \tag{673}$$

for all $s \in \mathbb{L}$ with ${s^\ddagger}^{-1}(0) \subseteq [\frac{1}{2}, 1]$. Hence let such a choice of $s \in \mathbb{L}$ be given. Further suppose that $S \in \mathbb{K}$ is defined by (37), i.e. $S(\gamma) = \{z : \gamma \geq s(z)\}$ for all $\gamma \in \mathbf{I}$. We first notice that

$$\begin{aligned}
S(0) &= \{z \in \mathbf{I} : 0 \geq s(z)\} && \text{by (37)}\\
&= \{z \in \mathbf{I} : s(z) = 0\} && \text{because } z \in \mathbf{I}\\
&= s^{-1}(0) &&\\
&\subseteq {s^\ddagger}^{-1}(0) && \text{by Th-47.}\mathbf{a}\\
&\subseteq [\tfrac{1}{2}, 1]\,, &&
\end{aligned}$$

by the assumption on the choice of $s \in \mathbb{L}$. We hence know that $\frac{1}{2} \leq \inf S(0) \leq \sup S(0)$. This permits us to restrict the proof to the following two cases.

a.: $\inf S(0) = \frac{1}{2}$**.** Then $\omega_U(s) = \frac{1}{2}$ by (665). We conclude from (670) and (5) that $\omega_{\rm glb}(s) = \frac{1}{2}$ as well. In particular $\omega_U(s) \leq \omega_{\rm glb}(s)$, as desired.

b.: $\inf S(0) > \frac{1}{2}$**.** We then look up from (665) that $\omega_U(s) = \max(\frac{1}{2} + \frac{1}{2} s_*^{[0,1)}, s_1^{\perp,*})$. In order to prove the desired $\omega_U(s) \leq \omega_{\rm glb}(s)$, it is hence sufficient to show that both inequations $\omega_{\rm glb}(s) \geq s_1^{\perp,*}$ and $\omega_{\rm glb}(s) \geq \frac{1}{2} + \frac{1}{2} s_*^{[0,1)}$ are valid.
As concerns the first inequation, it is useful to introduce an additional mapping $s' \in \mathbb{L}$ defined by

$$s'(z) = \begin{cases} 0 & : \quad z = s_1^{\perp,*} \\ 1 & : \quad \text{else} \end{cases} \tag{674}$$

for all $z \in \mathbf{I}$. As I will now show, this choice of s' satisfies $s' \trianglelefteq s$. Hence let us consider the preconditions stated in Def. 64. In order to prove precondition **a.** for $s' \trianglelefteq s$, let $z \in \mathbf{I}$ be given; it must be verified that there exists $z' \geq z$ with $s(z') \leq s'(z)$.

- if $z \neq s_1^{\perp,*}$, then $s(z') \leq s'(z) = 1$ by (674). Hence $z' = z$ is a suitable choice of $z' \geq z$ with $s(z') \leq s'(z)$;

- if $z = s_1^{\perp,*}$, then

$$\begin{aligned} z &= \inf s^{-1}[0,1) && \text{by (45)} \\ &\leq \inf s^{-1}(0) \\ &\leq z' \end{aligned}$$

  for an arbitrary choice of $z' \in s^{-1}(0) \neq \varnothing$, see Def. 60. We then obtain $s(z') = 0 \leq s(z)$, as desired.

Next I prove precondition **b.** stated in Def. 64. Hence let $z' \in \mathbf{I}$ be given; it must be shown that there exists $z \leq z'$ with $s'(z) \leq s(z')$. This is apparent if $z' \geq s_1^{\perp,*}$; in this case $z = s_1^{\perp,*}$ is a suitable choice which results in $s(z) = 0 \leq s(z')$, see (674). In the remaining case that $z' < s_1^{\perp,*}$, we know from (45) that $z' < \inf s^{-1}([0,1))$. Hence $z' \notin s^{-1}([0,1))$, which proves that $s(z') = 1$. In other words, $z = z'$ is a suitable choice of $z \leq z'$ which results in $s'(z) \leq 1 = s(z')$. Because both of the preconditions are valid, we conclude from Def. 64 that indeed $s' \trianglelefteq s$. In turn, we conclude from L-42 that $s' \sqsubseteq s$. Because ω_{glb} is known to satisfy (ω-1) and (ω-4), this proves that

$$\omega_{\mathrm{glb}}(s) \geq \omega_{\mathrm{glb}}(s') = s_1^{\perp,*}\,, \tag{675}$$

i.e. the first inequation is valid.
In order to finish the proof, we must still show that the second inequation $\omega_{\mathrm{glb}}(s) \geq \frac{1}{2} + \frac{1}{2}s_*^{[0,1)}$ is valid. I will treat separately the following two cases.

- In the case that $s(1) > 0$, we conclude from $s^{-1}(0) \neq \varnothing$ that there exists $z_0 \in [0,1)$ with $s(z_0) = 0$. Consequently $s_*^{[0,1)} = \inf \widehat{s}([0,1)) \leq s(z_0) = 0$, i.e. $s_*^{[0,1)} = 0$. In particular $\frac{1}{2} + \frac{1}{2}s_*^{[0,1)} = \frac{1}{2}$. Now we recall that $\omega_U(s) = \max(\frac{1}{2} + \frac{1}{2}s_*^{[0,1)}, s_1^{\perp,*})$ by the assumed choice of s, and hence $\omega_U(s) \geq \frac{1}{2}$. In turn, we deduce from (670) and (5) that $\omega_{\mathrm{glb}}(s) \in [\frac{1}{2}, \omega_U(s)]$, in particular $s_*^{[0,1)} = \frac{1}{2} \leq \omega_{\mathrm{glb}}(s)$, as desired.

- In the remaining case that $s(1) = 0$, we consider the mapping $s' \in \mathbb{L}$ defined by

$$s'(z) = \begin{cases} 0 & : \quad z = 1 \\ 1 & : \quad z \in (0,1) \\ s_*^{[0,1)} & : \quad z = 0 \end{cases} \tag{676}$$

  for all $z \in \mathbf{I}$. As I will now show, it then holds that $s' \sqsubseteq s$. To prove this, we need to address the preconditions **a.** and **b.** for $s' \sqsubseteq s$ stated in Def. 62. Hence let $z \in \mathbf{I}$. Then $\inf\{s(z') : z' \geq z\} \leq s(1) = 0 \leq s'(z)$, which validates precondition **a.** As to the other condition, we assume a choice of $z' \in \int$. If $z' = 1$, then $\inf\{s'(z) : z \leq z'\} = \inf\{s'(z) : z \leq 1\} \leq s'(1) = 0 \leq s(1)$. In the second case that $z' < 1$, we obtain that $\inf\{s'(z) : z \leq z'\} \leq s'(0) = \inf\{s(z') : z' < 1\} \leq s(z')$. Hence both preconditions of Def. 62 are valid, and we conclude that $s' \sqsubseteq s$. Because ω_{glb} is known to satisfy (ω-3) and (ω-4), this proves that $\omega_{\mathrm{glb}}(s) \geq \omega_{\mathrm{glb}}(s') = \frac{1}{2} + \frac{1}{2}s'(0) = \frac{1}{2} + \frac{1}{2}s_*^{[0,1)}$.

This completes the proof that $\omega_U(s) \leq \omega_{\mathrm{glb}}(s)$ in the main case **b.** of the proof, i.e. in the case that $\inf S(0) > \frac{1}{2}$. Hence the desired inequation (673) is valid for all $s \in \mathbb{L}$ with $s^{\ddagger^{-1}}(0) \subseteq [\frac{1}{2}, 1]$, and application of Th-85 yields the desired $\omega_U \preceq_c \omega_{\mathrm{glb}}$. Combining this with (670), we hence conclude that $\omega_U = \omega_{\mathrm{glb}}$. Because $\mathcal{F}_{\omega_{\mathrm{glb}}}$ is known to be the least specific $\mathcal{F}_\omega$-DFS, and because $\mathcal{M}_U$ is known to coincide with $\mathcal{F}_{\omega_U}$, this proves that it is in fact the known model $\mathcal{M}_U$, which is the least specific $\mathcal{F}_\omega$-DFS.

A.59 Proof of Theorem 90

Lemma 116 *For all $S \in \mathbb{K}$,*

$$\top_S = \top_{S^\square}$$
$$\bot_S = \bot_{S^\square} .$$

Proof To see this, let $S \in \mathbb{K}$ be given and let $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
\top_{S^\square}(\gamma) &= \sup S^{\square}(\gamma) && \text{by (35)}\\
&= \sup[\inf S(\gamma), \sup S(\gamma)] && \text{by Def. 82}\\
&= \sup S(\gamma) &&\\
&= \top_S(\gamma) && \text{by (35)}
\end{aligned}$$

and similarly

$$\begin{aligned}
\bot_{S^\square}(\gamma) &= \inf S^{\square}(\gamma) && \text{by (36)}\\
&= \inf[\inf S(\gamma), \sup S(\gamma)] && \text{by Def. 82}\\
&= \inf S(\gamma) &&\\
&= \bot_S(\gamma) . && \text{by (36)}
\end{aligned}$$

Because $\gamma \in \mathbf{I}$ was arbitrary, this proves that $\top_{S^\square} = \top_S$ and $\bot_{S^\square} = \bot_S$, as desired.

Proof of Theorem 90

Let $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ be a given mapping and $\mathcal{F}_\Omega$ the QFM defined by Def. 55.

If $\mathcal{F}_\Omega$ is an $\mathcal{F}_\xi$-QFM, then Ω is $^\square$-invariant. Hence suppose that $\mathcal{F}_\Omega$ is an $\mathcal{F}_\xi$-QFM, i.e. there exists $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ with

$$\mathcal{F}_\Omega = \mathcal{F}_\xi . \tag{677}$$

Now consider $S \in \mathbb{K}$. By Th-33, there exist $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$S_{Q,X} = S \tag{678}$$

and

$$S_{Q',X} = S^{\Box}\,. \tag{679}$$

Hence

$$\begin{aligned}
\Omega(S) &= \Omega(S_{Q,X}) && \text{by (678)}\\
&= \mathcal{F}_\Omega(Q)(X) && \text{by Def. 55}\\
&= \mathcal{F}_\xi(Q)(X) && \text{by (677)}\\
&= \xi(\top_{Q,X}, \bot_{Q,X}) && \text{by Def. 46}\\
&= \xi(\top_{S_{Q,X}}, \bot_{S_{Q,X}}) && \text{by (25), (26), (35) and (36)}\\
&= \xi(\top_S, \bot_S) && \text{by (678)}\\
&= \xi(\top_{S^\Box}, \bot_{S^\Box}) && \text{by L-116}\\
&= \xi(\top_{S_{Q',X}}, \bot_{S_{Q',X}}) && \text{by (679)}\\
&= \xi(\top_{Q',X}, \bot_{Q',X}) && \text{by (25), (26), (35) and (36)}\\
&= \mathcal{F}_\xi(Q')(X) && \text{by Def. 46}\\
&= \mathcal{F}_\Omega(Q')(X) && \text{by (677)}\\
&= \Omega(S_{Q',X}) && \text{by Def. 55}\\
&= \Omega(S^{\Box})\,. && \text{by (679)}
\end{aligned}$$

Hence Ω is indeed $^\Box$-invariant.

If Ω is $^\Box$-invariant, then $\mathcal{F}_\Omega$ is an $\mathcal{F}_\xi$-QFM. Hence let us assume that Ω is $^\Box$-invariant. We define $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ by

$$\xi(\top, \bot) = \Omega(S) \tag{680}$$

for all $(\top, \bot) \in \mathbb{T}$, where

$$S(\gamma) = [\bot(\gamma), \top(\gamma)] \tag{681}$$

for all $\gamma \in \mathbf{I}$. We observe that for all $S \in \mathbb{K}$,

$$S^{\Box}(\gamma) = [\inf S(\gamma), \sup S(\gamma)] = [\bot_S(\gamma), \top_S(\gamma)]$$

for all $\gamma \in \mathbf{I}$. This is apparent from Def. 82, (35) and (36). Hence for all $S \in \mathbb{K}$,

$$\xi(\top, \bot) = \Omega(S^{\Box}) \tag{682}$$

by (680) and (681). To see that $\mathcal{F}_\Omega = \mathcal{F}_\xi$, consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\Omega(Q)(X_1, \ldots, X_n) &= \Omega(S_{Q,X_1,\ldots,X_n}) && \text{by Def. 55}\\
&= \Omega((S_{Q,X_1,\ldots,X_n})^{\Box}) && \text{because } \Omega \text{ is } ^\Box\text{-invariant}\\
&= \xi(\top_{S_{Q,X_1,\ldots,X_n}}, \bot_{S_{Q,X_1,\ldots,X_n}}) && \text{by (682)}\\
&= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by (25), (26), (35) and (36)}\\
&= \mathcal{F}_\xi(Q)(X_1, \ldots, X_n)\,. && \text{by Def. 46}
\end{aligned}$$

A.60 Proof of Theorem 91

Lemma 117 *For all $S \in \mathbb{K}$ and $\gamma \in \mathbf{I}$, $S^{\ddagger}(\gamma) \subseteq S^{\square}(\gamma)$.*

Proof To see this, let $S \in \mathbb{K}$ and $\gamma \in \mathbf{I}$ be given and consider some $z \in S^{\ddagger}(\gamma)$. Then by Def. 59, there exist $z', z'' \in S(\gamma)$ with $z' \leq z \leq z''$. Hence $\inf S(\gamma) \leq z' \leq z$ and $\sup S(\gamma) \geq z'' \geq z$. Because $z \in S^{\ddagger}(\gamma)$ was arbitrary, we conclude that $\inf S(\gamma) \leq z \leq \sup S(\gamma)$ for all $z \in S^{\ddagger}$, i.e. $\inf S(\gamma) \leq \inf S^{\ddagger}(\gamma)$ and $\sup S(\gamma) \geq \sup S^{\ddagger}(\gamma)$. Because $S^{\ddagger}(\gamma) \subseteq [\inf S^{\ddagger}(\gamma), \sup S^{\ddagger}(\gamma)]$, this permits us to conclude that $S^{\ddagger}(\gamma) \subseteq [\inf S^{\ddagger}(\gamma), \sup S^{\ddagger}(\gamma)] \subseteq [\inf S(\gamma), \sup S(\gamma)]$, i.e. $S^{\ddagger}(\gamma) \subseteq S^{\square}(\gamma)$ by Def. 82, as desired.

Lemma 118 *For all $S \in \mathbb{K}$ and all $\gamma \in \mathbf{I}$, $(\inf S(\gamma), \sup S(\gamma)) \subseteq S^{\ddagger}(\gamma)$.*

Proof Let $S \in \mathbb{K}$ and $\gamma \in \mathbf{I}$ be given. Now consider $z \in (\inf S(\gamma), \sup S(\gamma))$. Because $z > \inf S(\gamma)$, there exists $z' \in S(\gamma)$ with $\inf S(\gamma) \leq z' < z$. Similarly because $z < \sup S(\gamma)$, there exists $z'' \in S(\gamma)$ with $z < z'' \leq \sup S(\gamma)$. Hence $z \in S^{\ddagger}(\gamma)$ by Def. 59.

Lemma 119 *For all $S \in \mathbb{K}$ and $\gamma \in \mathbf{I}$, $S^{\square}(\gamma) \setminus S^{\ddagger}(\gamma) \subseteq \{\inf S(\gamma), \sup S(\gamma)\}$.*

Proof It is immediate from Def. 82 that $S^{\square}(\gamma) \setminus S^{\ddagger}(\gamma) \subseteq S^{\square}(\gamma) = [\inf S(\gamma), \sup S(\gamma)]$. In order to prove the lemma, it is hence sufficient to show that $(\inf S(\gamma), \sup S(\gamma)) \subseteq S^{\ddagger}(\gamma)$. This has already been proven in L-119.

Lemma 120 *Let $S \in \mathbb{K}$, $\gamma \in \mathbf{I}$ and $\delta > 0$ be given. Then*

a. there exists $z' \in S^{\ddagger}(\gamma)$ with $\sup S(\gamma) - z' \leq \frac{\delta}{2}$;

b. there exists $z' \in S^{\ddagger}(\gamma)$ with $z' - \inf S(\gamma) \leq \frac{\delta}{2}$.

Proof

a.: If $\inf S(\gamma) = \sup S(\gamma)$, then $z' = \sup S(\gamma)$ yields $\sup S(\gamma) - z' = 0 \leq \frac{\delta}{2}$. In addition, $z' \in S^{\ddagger}(\gamma)$. This is apparent because $\inf S(\gamma) = \sup S(\gamma) = z'$ entails that $S(\gamma) = \{z'\}$, and because $S(\gamma) \subseteq S^{\ddagger}(\gamma)$. In the case that $\inf S(\gamma) \neq \sup S(\gamma)$, we conclude from $S(\gamma) \supseteq S(0) \neq \varnothing$ that in fact $\inf S(\gamma) < \sup S(\gamma)$. Hence $(\inf S(\gamma), \sup S(\gamma))$ is nonempty, and we can choose $z' \in (\inf S(\gamma), \sup S(\gamma))$ with $\sup S(\gamma) - z' \leq \frac{\delta}{2}$. This completes the proof of part **a.** noticing that $z' \in S^{\ddagger}(\gamma)$ by L-118.

b.: If $\inf S(\gamma) = \sup S(\gamma)$, then $z' = \inf S(\gamma)$ yields $z' - \inf S(\gamma) = 0 \leq \frac{\delta}{2}$. In addition, $z' \in S^{\ddagger}(\gamma)$. In the remaining case that $\inf S(\gamma) \neq \sup S(\gamma)$, we again conclude from $S(\gamma) \supseteq S(0) \neq \varnothing$ that $\inf S(\gamma) < \sup S(\gamma)$. Hence $(\inf S(\gamma), \sup S(\gamma))$

is nonempty, and we can choose $z' \in (\inf S(\gamma), \sup S(\gamma))$ with $z' - \inf S(\gamma) \le \frac{\delta}{2}$. This proves **b.** because again $z' \in S^{\ddagger}(\gamma)$ by L-118.

Lemma 121 *Let $S \in \mathbb{K}$ and $\delta > 0$ be given and suppose that $Q, Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ are defined by*

$$Q(Y) = Q_{\inf Y'}(Y'') \tag{683}$$
$$Q'(Y) = Q'_{\inf Y'}(Y'') \tag{684}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where $Y', Y'' \in \mathcal{P}(\mathbf{I})$ are defined by (28) *and* (29)*, respectively. We assume an arbitrary but fixed choice of $z_0 \in S(0)$. The semi-fuzzy quantifiers $Q_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ are then defined by*

$$Q_z(Y'') = \begin{cases} z & : & z \in S^{\ddagger}(\sup Y'') \\ z' & : & z \in S^{\square}(\sup Y'') \setminus S^{\ddagger}(\sup Y'') \\ z_0 & : & z \notin S^{\square}(\sup Y'') \end{cases} \tag{685}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$, where the

$$z' \in S^{\ddagger}(\gamma) \tag{686}$$

are chosen such that

$$|z' - z| \le \tfrac{\delta}{2}\,, \tag{687}$$

which is possible by L-119 and L-120. The semi-fuzzy quantifiers $Q'_z : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ are defined by

$$Q'_z(Y'') = \begin{cases} z & : & z \in S^{\square}(\sup Y'') \\ z_0 & : & z \notin S^{\square}(\sup Y'') \end{cases} \tag{688}$$

for all $Y'' \in \mathcal{P}(\mathbf{I})$. We further suppose that $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ is defined as in (33)*.
Then $S_{Q,X} = S^{\ddagger}$, $S_{Q',X} = S^{\square}$ and $d(Q, Q') < \delta$.*

Proof

$S_{Q,X} = S^{\ddagger}$**.** We first recall equations (91), (92), i.e. for $\gamma = 0$,

$$X_0^{\min} = X_{>\frac{1}{2}} = \varnothing \tag{689}$$
$$X_0^{\max} = X_{\ge\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1, 0)\}\,. \tag{690}$$

In the case that $\gamma > 0$, we recall equations (93) and (94), viz

$$X_\gamma^{\min} = X_{\ge\frac{1}{2}+\frac{1}{2}\gamma} = \varnothing \tag{691}$$
$$X_\gamma^{\max} = X_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))\,. \tag{692}$$

In order to prove the claim that $S_{Q,X} = S^{\ddagger}$, I first show that

$$S^{\ddagger}(\gamma) \subseteq S_{Q,X}(\gamma)\,. \tag{693}$$

Hence let us consider a choice of $z \in S^{\ddagger}(\gamma)$. In the case that $\gamma = 0$, it hence holds that $z \in S^{\ddagger}(0)$. Now we consider $Y = \{(0,z),(1,0)\} \in \mathcal{T}_0(X)$, which has $\inf Y' = \inf\{z\} = z$ and $Y'' = \{0\}$, see (28) and (29). Therefore $Q(Y) = Q_z(Y'') = Q_z(\{0\})$. Because $z \in S^{\ddagger}(0) = S^{\ddagger}(\sup\{0\})$, equation (685) applies, i.e. $Q(Y) = Q_z(\{0\}) = z$.
Next we consider the case that $\gamma > 0$. We can then choose

$$Y = \{(0,z)\} \cup (\{1\} \times [0,\gamma))\,.$$

For this choice of $Y \in \mathcal{T}_\gamma(X)$, we obtain $\inf Y' = \inf\{z\} = z$ and $Y'' = [0,\gamma)$ by (28) and (29), i.e. $\sup Y'' = \gamma$. Hence $Q(Y) = Q_z([0,\gamma)) = z$ by (683) and (685) because $z \in S^{\ddagger}(\gamma)$ by assumption. This completes the proof of (693).
It remains to be shown that $S_{Q,X}(\gamma) \subseteq S^{\ddagger}(\gamma)$ for all $\gamma \in \mathbf{I}$.
Let us first consider the case that $\gamma = 0$. Hence let $Y \in \mathcal{T}_0(X)$ be given. We abbreviate $z = \inf Y' \in \mathbf{I}$. It is apparent from (689) and (690) that we either have $Y'' = \varnothing$ or $Y'' = \{0\}$. In any case, $\sup Y'' = 0$.

- If $z \in S^{\ddagger}(\gamma) = S^{\ddagger}(0)$, then $Q(Y) = Q_z(Y'') = z \in S^{\ddagger}(0)$ by (683) and (685).
- In the case that $z \in S^{\Box}(0) \setminus S^{\ddagger}(0)$, we obtain from (683), (685) and (686) that $Q(y) = Q_z(Y'') = z' \in S^{\ddagger}(0)$.
- Finally if $z \notin S^{\Box}(0)$, then $Q(Y) = Q_z(Y'') = z_0 \in S(0) \subseteq S^{\ddagger}(0)$ by (683) and (685).

In any case, we obtain that $Q(Y) \in S^{\ddagger}(0)$ for all $Y \in \mathcal{T}_0(X)$, i.e. $S_{Q,X}(0) \subseteq S^{\ddagger}(0)$ by Def. 51.
Let us now show that $S_{Q,X}(\gamma) \subseteq S^{\ddagger}(\gamma)$ also holds in the case that $\gamma > 0$. Hence let $\gamma > 0$ and consider some $Y \in \mathcal{T}_\gamma(X)$. Again we abbreviate $z = \inf Y' \in \mathbf{I}$. We also notice that by (691) and (692), $0 \leq \sup Y'' \leq \gamma$.

- If $z \in S^{\ddagger}(\sup Y'')$, then

$$\begin{aligned} Q(Y) &= Q_z(Y'') && \text{by (683)}\\ &= z && \text{by (685)}\\ &\in S^{\ddagger}(\sup Y'') && \text{by assumption}\\ &\subseteq S^{\ddagger}(\gamma)\,. && \text{because } \sup Y'' \leq \gamma \end{aligned}$$

- If $z \in S^{\Box}(\sup Y'') \setminus S^{\ddagger}(\sup Y'')$, then

$$\begin{aligned} Q(Y) &= Q_z(Y'') && \text{by (683)}\\ &= z' && \text{by (685)}\\ &\in S^{\ddagger}(\sup Y'') && \text{by (686)}\\ &\subseteq S^{\ddagger}(\gamma)\,. && \text{because } \sup Y'' \leq \gamma \end{aligned}$$

- If $z \notin S^{\Box}(\sup Y'')$, then

$$\begin{aligned}
Q(Y) &= Q_z(Y'') && \text{by (683)} \\
&= z_0 && \text{by (685)} \\
&\in S(0) && \text{by choice of } z_0 \\
&\subseteq S(\gamma) && \text{by Def. 52} \\
&\subseteq S^{\ddagger}(\gamma)\,. && \text{apparent from Def. 59}
\end{aligned}$$

Hence $Q(Y) \in S^{\ddagger}(\gamma)$ for all $Y \in \mathcal{T}_\gamma(X)$, i.e. $S_{Q,X} \subseteq S^{\ddagger}(\gamma)$. Combining this with (693), we obtain that $S_{Q,X}(\gamma) = S^{\ddagger}(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e. $S_{Q,X} = S^{\ddagger}$.

a.: $S_{Q',X} = S^{\Box}$. We observe that Q' and Q'_z as defined by (684) and (688) result from the very same construction that has been used in Th-33, equations (27) and (30)/(31), substituting $S^{\Box}$ for S. Hence by Th-33, $S_{Q',X} = S^{\Box}$, as desired.

b.: $d(Q, Q') < \delta$. To see this, consider some $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$ and let us abbreviate $z = \inf Y'$. Then $Q(Y) = Q_z(Y'')$ and $Q'(Y) = Q'_z(Y'')$ by (683) and (684).
If $z \in S^{\ddagger}(\sup Y'')$, then $z \in S^{\Box}(\sup Y'')$ as well, see L-117. Hence $Q(Y) = Q_z(Y'') = z = Q'_z(Y'') = Q'(Y)$ by (685) and (688), i.e. $|Q(Y) - Q'(Y)| = 0$.
In the case that $z \notin S^{\Box}(\sup Y'')$, we obtain from (685) and (688) that $Q(Y) = Q_z(Y'') = z_0 = Q'_z(Y'') = Q'(Y)$. Hence $|Q(Y) - Q'(Y)| = 0$ in this case, too.
In the remaining case that $z \in S^{\Box}(\gamma) \setminus S^{\ddagger}(\gamma)$, we obtain from (685) that $Q(Y) = Q_z(Y'') = z'$ and $Q'(Y) = Q'_z(Y'') = z$. Hence $|Q(Y) - Q'(Y)| = |z - z'| \leq \frac{\delta}{2}$ by (687).
We conclude that

$$\begin{aligned}
d(Q, Q') &= \sup\{|Q(Y) - Q'(Y)| : Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})\} && \text{by (7)} \\
&\leq \tfrac{\delta}{2} && \text{by above reasoning} \\
&< \delta\,. && \text{because } \delta > 0
\end{aligned}$$

Proof of Theorem 91

Suppose that $\Omega : \mathbb{K} \longrightarrow \mathbf{I}$ is an ‡-invariant mapping. The proof is by contraposition. Hence let us assume that $\mathcal{F}_\Omega$ is not an $\mathcal{F}_\xi$-QFM; it must be shown that $\mathcal{F}_\Omega$ is not Q-continuous.
Because $\mathcal{F}_\Omega$ is not an $\mathcal{F}_\xi$-QFM, we know from Th-90 that Ω is not $^{\Box}$-invariant. Hence there exists $S \in \mathbb{K}$ with $\Omega(S) \neq \Omega(S^{\Box})$. We may hence choose

$$\varepsilon = |\Omega(S) - \Omega(S^{\Box})| > 0\,. \tag{694}$$

Let us define the semi-fuzzy quantifier $Q' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ according to (684). According to Def. 28, we can prove that $\mathcal{F}_\Omega$ is not Q-continuous by proving that for all

$\delta > 0$, there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ with $d(Q, Q') < \delta$ but $d(\mathcal{F}_\Omega(Q), \mathcal{F}_\Omega(Q')) \geq \varepsilon$.
Hence let $\delta > 0$ and define $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by (683). We can then apply L-121 to conclude that $d(Q, Q') < \delta$ and

$$S_{Q,X} = S^{\ddagger} \tag{695}$$

$$S_{Q',X} = S^{\square} \tag{696}$$

for the choice of $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ defined by (33). Therefore

$$\begin{aligned}
&d(\mathcal{F}_\Omega(Q), \mathcal{F}_\Omega(Q')) \\
&= \sup\{|\mathcal{F}_\Omega(Q)(Z) - \mathcal{F}_\Omega(Q')(Z)| : Z \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})\} && \text{by (8)} \\
&\geq |\mathcal{F}_\Omega(Q)(X) - \mathcal{F}_\Omega(Q')(X)| && \text{because } X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I}) \\
&= |\Omega(S_{Q,X}) - \Omega(S_{Q',X})| && \text{by Def. 55} \\
&= |\Omega(S^{\ddagger}) - \Omega(S^{\square})| && \text{by (695), (696)} \\
&= |\Omega(S) - \Omega(S^{\square})| && \text{because } \Omega \text{ is } {}^{\ddagger}\text{-invariant} \\
&\geq \varepsilon\,. && \text{by (694)}
\end{aligned}$$

This completes the proof that $\mathcal{F}_\Omega$ fails to be Q-continuous.

B Proofs of theorems in chapter 5

B.1 Proof of Theorem 92

Lemma 122 *Let $E \neq \varnothing$ be some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, where $n > 0$. Further suppose that $X \in \widetilde{\mathcal{P}}(\{1, \ldots, n\} \times E)$ is defined by*

$$\mu_X(i, e) = \mu_{X_i}(e) \tag{697}$$

for all $i \in \{1, \ldots, n\}$ and $e \in E$. Then $D_{X_1,\ldots,X_n} = D_X$.

Proof Let us associate with each choice of $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$ a corresponding subset $Y \in \mathcal{P}(\{1, \ldots, n\} \times E)$,

$$Y = \{(i, e) : i \in \{1, \ldots, n\}, e \in Y_i\}\,. \tag{698}$$

It is apparent that $(Y_1, \ldots, Y_n) \mapsto Y$ is a bijection. In addition,

$$\begin{aligned}
&\Xi_{Y_1,\ldots,Y_n}((X_1, \ldots, X_n)) \\
&= \bigwedge_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in Y_i\}, \\
&\qquad \inf\{1 - \mu_{X_i}(e) : e \notin Y_i\}) && \text{by Def. 83} \\
&= \min(\min\{\inf \mu_{X_i}(e) : e \in Y_i\} : i \in \{1, \ldots, n\}\}, \\
&\qquad \min\{\inf\{1 - \mu_{X_i}(e) : e \notin Y_i\} : i \in \{1, \ldots, n\}\}) \\
&= \min(\inf\{\mu_{X_i}(e) : i \in \{1, \ldots, n\}, e \in Y_i\}, \\
&\qquad \inf\{1 - \mu_{X_i}(e) : i \in \{1, \ldots, n\}, e \notin Y_i\}) \\
&= \min(\inf\{\mu_X(i, e) : i \in \{1, \ldots, n\}, e \in Y_i\}, \\
&\qquad \inf\{1 - \mu_X(i, e) : i \in \{1, \ldots, n\}, e \notin Y_i\}) && \text{by (697)} \\
&= \min(\inf\{\mu_X(i, e) : (i, e) \in Y\}, \inf\{1 - \mu_X(i, e) : (i, e) \notin Y\}) && \text{by (698)} \\
&= \Xi_Y(X)\,,
\end{aligned}$$

i.e.

$$\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \Xi_Y(X)\,. \tag{699}$$

Therefore

$$\begin{aligned}
&D_{X_1,\ldots,X_n} \\
&= \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : Y_1, \ldots, Y_n \in \mathcal{P}(E)\} && \text{by Def. 84} \\
&= \{\Xi_Y(X) : (Y_1, \ldots, Y_n) \in \mathcal{P}(E)\} && \text{by (699)} \\
&= \{\Xi_Y(X) : Y \in \mathcal{P}(\{1, \ldots, n\} \times E)\} && \text{as } (Y_1, \ldots, Y_n) \mapsto Y \text{ bijection} \\
&= D_X\,, && \text{by Def. 84}
\end{aligned}$$

as desired.

Lemma 123 *Let $E \neq \varnothing$ be some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ where $n \geq 0$. We abbreviate*

$$Y_i^+ = X_{i_{\geq \frac{1}{2}}} \tag{700}$$

for $i = 1, \ldots, n$, and

$$r_+ = \Xi_{Y_1^+, \ldots, Y_n^+}(X_1, \ldots, X_n)\,. \tag{701}$$

Then $r_+ \geq \frac{1}{2}$.

Proof In the case that $n = 0$, there is only one possible choice of fuzzy arguments, viz the empty tuple $\varnothing$. Applying the α-cut at $\frac{1}{2}$ to the arguments in the empty tuple is a vacuous operation which again returns the empty tuple. We hence obtain that $r_+ = \Xi^{(0)}_{\varnothing}(\varnothing) = 1$. In particular $r_+ \geq \frac{1}{2}$, as desired.
As concerns the remaining cases that $n > 0$, it is apparent from L-122 and (700) that it is sufficient to consider the case $n = 1$ only. Hence let $E \neq \varnothing$ be a base set and $X \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
& r_+ \\
&= \min(\inf\{\mu_X(e) : e \in Y_i^+\}, \inf\{1 - \mu_X(e) : e \notin Y_i^+\}) && \text{by Def. 83, (701)} \\
&= \min(\inf\{\mu_X(e) : e \in X_{\geq \frac{1}{2}}\}, \\
&\qquad \inf\{1 - \mu_X(e) : e \notin X_{\geq \frac{1}{2}}\}) && \text{by (700)} \\
&= \min(\inf\{\mu_X(e) : e \in E, \mu_X(e) \geq \tfrac{1}{2}\}, \\
&\qquad \inf\{1 - \mu_X(e) : e \in E, \mu_X(e) < \tfrac{1}{2}\}) && \text{by Def. 29} \\
&\geq \min(\inf\{\tfrac{1}{2} : e \in E, \mu_X(e) \geq \tfrac{1}{2}\}, \\
&\qquad \inf\{\tfrac{1}{2} : e \in E, \mu_X(e) < \tfrac{1}{2}\}) \\
&= \tfrac{1}{2},
\end{aligned}$$

as desired.

Lemma 124 *Let $E \neq \varnothing$ be some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ where $n \geq 0$. Further assume that Y_i^+, $i = 1, \ldots, n$, and r_+ are defined by* (700) *and* (701)*, respectively. Then*

$$\Xi_{Y_1, \ldots, Y_n}(X_1, \ldots, X_n) \leq \tfrac{1}{2}$$

for all $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)$ with $(Y_1, \ldots, Y_n) \neq (Y_1^+, \ldots, Y_n^+)$. In particular, $\Xi_{Y_1, \ldots, Y_n}(X_1, \ldots, X_n) \leq r_+$.

Proof The case $n = 0$ is trivial. In this case, there is only one possible choice of fuzzy arguments, i.e. the empty tuple. The condition of the lemma then becomes

vacuous because there are no fuzzy arguments beyond that defined by (700).
In the remaining case that $n > 0$, we can again utilize L-122, which permits us to restrict attention to the case that indeed $n = 1$ (this is apparent from (697) and (700)).
Hence let $E \neq \varnothing$ be some base set and let $X \in \widetilde{\mathcal{P}}(E)$. Now consider some $Y \in \mathcal{P}(E)$, $Y \neq Y^+$. Then either $Y \setminus Y^+ \neq \varnothing$ or $Y^+ \setminus Y \neq \varnothing$.
In the case that $Y \setminus Y^+ \neq \varnothing$, choose some $e' \in Y \setminus Y^+$. Because $e' \notin Y^+$, we know from (700) and Def. 29 that

$$\mu_X(e') < \tfrac{1}{2}\,. \qquad (702)$$

Hence

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83}\\
&\leq \inf\{\mu_X(e) : e \in Y\}\\
&\leq \mu_X(e') && \text{because } e' \in Y\\
&< \tfrac{1}{2}\,. && \text{by (702)}
\end{aligned}$$

In the remaining case that $Y^+ \setminus Y \neq \varnothing$, let us choose some $e'' \in Y^+ \setminus Y \neq \varnothing$. Then

$$\mu_X(e'') \geq \tfrac{1}{2}\,. \qquad (703)$$

by (700) and Def. 29. Hence

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\})\\
&\leq \inf\{1 - \mu_X(e) : e \notin Y\}\\
&\leq 1 - \mu_X(e'') && \text{because } e'' \notin Y\\
&\leq \tfrac{1}{2}\,. && \text{by (703)}
\end{aligned}$$

The second claim of the lemma that $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) \leq r_+$ is then apparent from $\Xi_Y(X) \leq \frac{1}{2} \leq r_+$, see L-123.

Lemma 125 *Let $E \neq \varnothing$ be some base set, $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$, and suppose that Y_i^+, $i = 1,\ldots,n$, and r_+ are defined by* (700) *and* (701)*, respectively. If $r_+ > \frac{1}{2}$, then*

$$\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) < 1 - r_+ \qquad (704)$$

for all $(Y_1,\ldots,Y_n) \in \mathcal{P}(E)^n$ with $(Y_1,\ldots,Y_n) \neq (Y_1^+,\ldots,Y_n^+)$.

Proof The condition of the lemma again becomes vacuous if $n = 0$. Hence suppose that $n > 0$. By the same reasoning as above, we conclude from L-122 that it is sufficient to consider the case that $n = 1$. Hence let $E \neq \varnothing$ be some base set and let $X \in \widetilde{\mathcal{P}}(E)$. Further suppose that Y^+ and r_+ are defined by (700) and (701), respectively.
Now consider $Y \in \mathcal{P}(E)$, $Y \neq Y^+$. Then either $Y^+ \setminus Y \neq \varnothing$ or $Y \setminus Y^+ \neq \varnothing$.
In the former case, there exists $e' \in Y^+$ with $e' \notin Y$. Because $e' \in Y^+$,

$$\begin{aligned}
r_+ &= \min(\inf\{\mu_X(e) : e \in Y^+\}, \inf\{1 - \mu_X(e) : e \notin Y^+\}) && \text{by Def. 83, (701)}\\
&\leq \inf\{\mu_X(e) : e \in Y^+\}\\
&\leq \mu_X(e')\,, && \text{because } e' \in Y^+
\end{aligned}$$

i.e.

$$\mu_X(e') \geq r_+ \,. \tag{705}$$

Hence

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) \\
&\leq \inf\{1 - \mu_X(e) : e \notin Y\} \\
&\leq 1 - \mu_X(e') && \text{because } e' \notin Y \\
&\leq 1 - r_+ \,. && \text{by (705)}
\end{aligned}$$

In the remaining case that $Y \setminus Y^+ \neq \varnothing$, there exists $e'' \in Y$ with $e'' \notin Y^+$. Therefore

$$\begin{aligned}
r_+ &= \min(\inf\{\mu_X(e) : e \in Y^+\}, \inf\{1 - \mu_X(e) : e \notin Y^+\}) && \text{by Def. 83, (701)} \\
&\leq \inf\{1 - \mu_X(e) : e \notin Y^+\} \\
&\leq 1 - \mu_X(e'') \,, && \text{because } e'' \notin Y^+
\end{aligned}$$

i.e.

$$\mu_X(e'') \leq 1 - r_+ \,. \tag{706}$$

Hence

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\
&\leq \inf\{\mu_X(e) : e \in Y\} \\
&\leq \mu_X(e'') && \text{because } e'' \in Y \\
&\leq 1 - r_+ \,. && \text{by (706)}
\end{aligned}$$

This proves that indeed $\Xi_Y(X) \leq 1 - r_+$.

Lemma 126 *Suppose that $E \neq \varnothing$ is some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ where $n \geq 0$. Further let r_+ be defined by* (701)*. Then $D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1] = \{r_+\}$.*

Proof In the case that $n = 0$, we simply notice that the only possible choice of $X_1, \ldots, X_n$ is the empty tuple, which results in $D_\varnothing^{(0)} = \{1\}$ and hence $D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1] = \{1\} \cap [\frac{1}{2}, 1] = \{1\} = r_+$ in this case. For $n > 0$, then, we already know from L-123 that $r_+ \geq \frac{1}{2}$ and $r_+ = \Xi_{Y_1^+,\ldots,Y_n^+}(X_1, \ldots, X_n) \in D_{X_1,\ldots,X_n}$. Hence $\{r_+\} \subseteq D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1]$. Now consider $r \in D_{X_1,\ldots,X_n}$ with $r \neq r_+$, i.e. there exist $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ with $(Y_1, \ldots, Y_n) \neq (Y_1^+, \ldots, Y_n^+)$ and $r = \Xi_{Y_1,\ldots,Y_n}(Y_1, \ldots, Y_n)$.
In the case that $r_+ = \frac{1}{2}$, we apply L-124 to conclude that $r \leq \frac{1}{2}$. Because $r \neq r_+$ by assumption, it in fact holds that $r < \frac{1}{2}$. In particular, $r \notin D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1]$ and indeed $D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1] = \{r_+\}$.
In the remaining case that $r_+ > \frac{1}{2}$, we apply L-125 and conclude that

$$r = \Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \leq 1 - r_+ < \tfrac{1}{2} \,.$$

Hence again $r \notin D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1]$, which proves the desired $D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1] = \{r_+\}$.

Lemma 127 *Let $E \neq \varnothing$ be some base set, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and suppose that r_+ is defined by* (701). *If $r_+ > \frac{1}{2}$, then $\sup D_X \setminus \{r_+\} = 1 - r_+$.*

Proof In the case that $n = 0$, the claim of the lemma is apparent because $X_1, \ldots, X_n$ coincides with the empty tuple $D_{X_1,\ldots,X_n} = D_{\varnothing}^{(0)} = \{1\}$. In particular $r_+ = 1$ and $D_{\varnothing}^{(0)} \setminus \{r_+\} = \varnothing$. This proves that $\sup D_{\varnothing}^{(0)} \setminus \{r_+\} = \sup void = 0 = 1-1 = 1-r_+$, as desired. In the remaining cases that $n \neq 0$, it is again clear from L-122 that only the case $n = 1$ must be considered.
Hence let $E \neq \varnothing$ and $X \in \widetilde{\mathcal{P}}(E)$ be given and suppose that $r_+ > \frac{1}{2}$. We already know from L-125 that $D_X \setminus \{r_+\} \subseteq [0, 1 - r_+]$, and hence $\sup D_X \setminus \{r_+\} \leq 1 - r_+$. It remains to be shown that $\sup D_X \setminus \{r_+\} \geq 1 - r_+$. We now recall from (701) that either $r_+ = \inf\{\mu_X(e) : e \in Y^+\}$ or $r_+ = \inf\{1 - \mu_X(e) : e \notin Y^+\}$. It is hence useful to discern two cases.

a.: $r_+ = \inf\{\mu_X(e) : e \in Y^+\}$.
In the case that $Y^+ = \varnothing$, it trivially holds that

$$1 - r_+ = 1 - \inf \varnothing = 1 - 1 = 0 \leq \sup D_X \setminus \{r_+\}\,.$$

Hence suppose that $Y^+ \neq \varnothing$ and consider some $\varepsilon > 0$. Then there exists $e' \in Y^+$ with

$$r_+ \leq \mu_X(e') < r_+ + \varepsilon\,. \tag{707}$$

We abbreviate

$$Y = Y^+ \setminus \{e'\}\,. \tag{708}$$

It is then apparent that

$$\begin{aligned} r_+ &= \min(\inf\{\mu_X(e) : e \in Y^+\}, \inf\{1 - \mu_X(e) : e \notin Y^+\}) && \text{by (701), Def. 83} \\ &\leq \min(\inf\{\mu_X(e) : e \in Y^+ \setminus \{e'\}\}, \inf\{1 - \mu_X(e) : e \notin Y^+\})\,, \end{aligned}$$

i.e.

$$r_+ \leq \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \in Y^+\}) \tag{709}$$

by (708). Hence

$$\begin{aligned} 1 - r_+ &\geq \Xi_Y(X) && \text{by L-125} \\ &= \min(\inf\{\mu_X(e) : e \in Y\}, \\ &\qquad \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\ &= \min\{\inf\{\mu_X(e) : e \in Y\}, \\ &\qquad \inf\{1 - \mu_X(e) : e \notin Y, e \neq e'\}, 1 - \mu_X(e')\} && \text{because } e' \notin Y \text{ by (708)} \\ &= \min\{\inf\{\mu_X(e) : e \in Y\}, \\ &\qquad \inf\{1 - \mu_X(e) : e \notin Y^+\}, 1 - \mu_X(e)\}\,. && \text{by (708)} \end{aligned}$$

Because of (709), we conclude that $1 - \mu_X(e') \leq 1 - r_+$ and hence

$$\Xi_Y(X) = \mu_X(e') \,.$$

In turn, we obtain from (707) that $\Xi_Y(X) > 1 - r_+ - \varepsilon$. Because $\Xi_Y(X) \in D_X \setminus \{r_+\}$, this proves that $\sup D_X \setminus \{r_+\} \geq \Xi_Y(X) > 1 - r_+ + \varepsilon$. Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that $\sup D_X \setminus \{r_+\} \geq 1 - r_+$.

b.: $r_+ = \inf\{1 - \mu_X(e) : e \notin Y^+\}$**.**
In the case that $Y^+ = E$, it trivially holds that

$$1 - r_+ = 1 - \inf \varnothing = 1 - 1 = 0 \leq \sup D_X \setminus \{r_+\} \,.$$

Hence suppose that $Y^+ \neq E$ and consider some $\varepsilon > 0$. Then there exists $e' \in E \setminus Y^+$ with

$$r_+ \leq 1 - \mu_X(e') < r_+ + \varepsilon \,. \tag{710}$$

We abbreviate

$$Y = Y^+ \cup \{e'\} \,. \tag{711}$$

Then obviously

$$\begin{aligned}
&1 - r_+ \\
&< r_+ && \text{because } r_+ > \tfrac{1}{2} \\
&= \min(\inf\{\mu_X(e) : e \in Y^+\}, \inf\{1 - \mu_X(e) : e \notin Y^+\}) && \text{by (701), Def. 83} \\
&\leq \min(\inf\{\mu_X(e) : e \in Y^+\}, \inf\{1 - \mu_X(e) : e \notin Y^+, e \neq e'\}) \,,
\end{aligned}$$

i.e.

$$1 - r_+ < \min(\inf\{\mu_X(e) : e \in Y \setminus \{e\}\}, \inf\{1 - \mu_X(e) : e \notin Y\}) \tag{712}$$

by (711). We also notice that

$$\begin{aligned}
1 - r_+ &\geq \Xi_Y(X) && \text{by L-125} \\
&= \min(\inf\{\mu_X(e) : e \in Y\}, \\
&\qquad \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\
&= \min\{\inf\{\mu_X(e) : e \in Y \setminus \{e'\}\}, \\
&\qquad \inf\{1 - \mu_X(e) : e \notin Y\}, \mu_X(e')\} \,. && \text{because } e' \in Y \text{ by (711)}
\end{aligned}$$

Recalling (712), we conclude that that $\mu_X(e') \leq 1 - r_+$ and

$$\Xi_Y(X) = \mu_X(e') \,. \tag{713}$$

In turn, we obtain from (710) that $\Xi_Y(X) = \mu_X(e'') > 1 - r_+ - \varepsilon$. Because $\Xi_Y(X) \in D_X \setminus \{r_+\}$, we deduce that

$$\sup D_X \setminus \{r_+\} \geq \Xi_Y(X) > 1 - r_+ - \varepsilon \,.$$

$\varepsilon \to 0$ yields $\sup D_X \setminus \{r_+\} \geq 1 - r_+$, as desired.

Lemma 128 *Let $E \neq \varnothing$ be some base set and let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ where $n \geq 0$. Then for all $D' \subseteq D_{X_1,\ldots,X_n}$ with $D' \neq \varnothing$, $\inf D' \in D_{X_1,\ldots,X_n}$.*

Proof The claim is vacuous for $n = 0$, and for similar reasons as above, it is sufficient to restrict attention to $n = 1$ in order to cover the remaining cases. Hence let $X \in \widetilde{\mathcal{P}}(E)$ and suppose that $D' \subseteq D_X$, $D' \neq \varnothing$.
We first treat the case that $\inf D' \geq \frac{1}{2}$. Then $D' \subseteq [\frac{1}{2}, 1]$, i.e. $D' = D' \cap [\frac{1}{2}, 1] \subseteq D \cap [\frac{1}{2}, 1] = \{r_+\}$ by L-126. Because $D' \neq \varnothing$ by assumption, we conclude that $D' = \{r_+\}$ and $\inf D' = r_+ \in D_X$ by L-123.

In the remaining case that $\inf D' < \frac{1}{2}$, we abbreviate

$$r = \inf D' . \tag{714}$$

We now define $Y \in \mathcal{P}(E)$ by

$$Y = \{e \in E : \mu_X(e) \in [r, \tfrac{1}{2}) \cup (1 - r, 1]\} . \tag{715}$$

In order to prove that $r = \Xi_Y(X)$, let us first show that $\Xi_Y(X) \geq r$. We first notice that

$$\inf\{\mu_X(e) : \mu_X(e) > 1 - r\} \geq 1 - r > r \tag{716}$$

because $r = \inf D' < \frac{1}{2}$. For similar reasons

$$\inf\{\mu_X(e) : \mu_X(e) \in [r, \tfrac{1}{2})\} \geq r \tag{717}$$

$$\inf\{1 - \mu_X(e) : \mu_X(e) < r\} \geq 1 - r > r \tag{718}$$

$$\inf\{1 - \mu_X(e) : \mu_X(e) \in [\tfrac{1}{2}, 1 - r]\} \geq r . \tag{719}$$

Therefore

$$\begin{aligned}
&\Xi_Y(X) \\
&= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\
&= \min\{\inf\{\mu_X(e) : \mu_X(e) > 1 - r\}, \\
&\qquad \inf\{\mu_X(e) : \mu_X(e) \in [r, \tfrac{1}{2})\}, \\
&\qquad \inf\{1 - \mu_X(e) : \mu_X(e) < r\}, \\
&\qquad \inf\{1 - \mu_X(e) : \mu_X(e) \in [\tfrac{1}{2}, 1 - r]\}\} \\
&\geq r . && \text{by (716)–(719)}
\end{aligned}$$

It remains to be shown that $\Xi_Y(X) \leq r$. To this end, we consider a choice of $\varepsilon > 0$. Because $r < \frac{1}{2}$, $\frac{1}{2} - r > 0$. Without loss of generality, we can hence assume that ε is chosen small enough that

$$\varepsilon < \tfrac{1}{2} - r . \tag{720}$$

Recalling that $D' \neq \varnothing$ and $r = \inf D'$ by (714), there exists $r' \in D'$ with

$$r \leq r' < r + \tfrac{\varepsilon}{2} . \tag{721}$$

Because $r' \in D' \subseteq D_X$, there exists $Y' \in \mathcal{P}(E)$ with

$$r' = \Xi_{Y'}(X) . \tag{722}$$

We now notice that either $\Xi_{Y'}(X) = \inf\{\mu_X(e) : e \in Y'\}$ or $\Xi_{Y'}(X) = \inf\{1 - \mu_X(e) : e \notin Y'\}$, see Def. 83. It proves useful to treat these cases separately.

In the case that $\Xi_{Y'}(X) = \inf\{\mu_X(e) : e \in Y'\}$, there exists $e' \in Y'$ with $\mu_X(e') < \Xi_{Y'}(X) + \frac{\varepsilon}{2}$. Hence

$$r \leq \mu_X(e') < \Xi_{Y'}(X) + \tfrac{\varepsilon}{2} < r + \varepsilon < \tfrac{1}{2} \tag{723}$$

by (720)–(722). Hence $\mu_X(e') \in [r, \frac{1}{2})$, i.e. $e' \in Y$ by (715). Therefore

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\
&\leq \inf\{\mu_X(e) : e \in Y\} \\
&\leq \mu_X(e')\,, && \text{because } e' \in Y
\end{aligned}$$

i.e.

$$\Xi_Y(X) < r + \varepsilon \tag{724}$$

by (723).

In the remaining case that $\Xi_{Y'}(X) = \inf\{1 - \mu_X(e) : e \notin Y'\}$, there exists $e'' \in E \setminus Y'\}$ with $1 - \mu_X(e'') < \Xi_{Y'}(X) + \frac{\varepsilon}{2} < r + \varepsilon$, see (721) and (722). Hence

$$\mu_X(e'') > 1 - r - \varepsilon > \tfrac{1}{2} \tag{725}$$

by (720). On the other hand $r \leq \Xi_{Y'}(X) \leq 1 - \mu_X(e'')$ by (721), (722) and Def. 83. Consequently

$$\mu_X(e'') \leq 1 - r\,. \tag{726}$$

We conclude from (715), (725) and (726) that $e'' \notin Y$. Therefore

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) && \text{by Def. 83} \\
&\leq \inf\{1 - \mu_X(e) : e \notin Y\} \\
&\leq 1 - \mu_X(e'')\,, && \text{because } e' \in Y
\end{aligned}$$

i.e.

$$\Xi_Y(X) < r + \varepsilon \tag{727}$$

by (725).
Hence in both cases, $\Xi_Y(X) < r + \varepsilon$ by (724) and (727), respectively. $\varepsilon \to 0$ yields $\Xi_Y(X) \leq= \inf D'$. This completes the proof that there exists $Y \in \mathcal{P}(E)$ with $\Xi_Y(X) = \inf D'$, i.e. $\inf D' \in D_X$ by Def. 84.

Proof of Theorem 92

Let $E \neq \varnothing$ be some base set and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We know from L-126 that $D_{X_1,\ldots,X_n} \cap [\frac{1}{2}, 1] = \{r_+\}$, where r_+ is defined by (701). We further know from L-128 that $\inf D' \in D_{X_1,\ldots,X_n}$ for all $D' \subseteq D_{X_1,\ldots,X_n}$, $D' \neq \varnothing$. Finally in the case that $r_+ > \frac{1}{2}$, we know from L-127 that $\sup D_{X_1,\ldots,X_n} \setminus \{r_+\} = 1 - r_+$. Hence $D_{X_1,\ldots,X_n} \in \mathbb{D}$ by Def. 85, as desired.

B.2 Proof of Theorem 93

Lemma 129 *For all semi-fuzzy quantifiers* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,*

$$\cup\{A_{Q,X_1,\ldots,X_n}(z) : z \in \mathbf{I}\} = D_{X_1,\ldots,X_n} \,.$$

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Then

$$\begin{aligned}
&\cup\{A_{Q,X_1,\ldots,X_n}(z) : z \in \mathbf{I}\} \\
&= \cup\{\{\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) : Q(Y_1,\ldots,Y_n) = z\} : z \in \mathbf{I}\} && \text{by Def. 86} \\
&= \{\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) : Q(Y_1,\ldots,Y_n) = z \text{ for some } z \in \mathbf{I}\} \\
&= \{\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) : Y_1,\ldots,Y_n \in \mathcal{P}(E)\} \\
&= D_{X_1,\ldots,X_n} \,. && \text{by Def. 84}
\end{aligned}$$

Proof of Theorem 93

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier. Further suppose that fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given. To see that $A_{Q,X_1,\ldots,X_n} \in \mathbb{A}$, we first notice that

$$\cup\{A_{Q,X_1,\ldots,X_n}(z) : z \in \mathbf{I}\} = D_{X_1,\ldots,X_n} \in \mathbb{D} \tag{728}$$

by L-129 and Th-92. Recalling Def. 87, it remains to be shown that for all $z, z' \in \mathbf{I}$, $\sup A(z) > \frac{1}{2}$ and $\sup A(z') > \frac{1}{2}$ entails that $z = z'$.
Hence let us choose $Y_1^+, \ldots, Y_n^+ \in \mathcal{P}(E)$ according to (700) and suppose that r_+ is defined in terms of the Y_i^+ according to (701).
If $r_+ = \frac{1}{2}$, then $D_{X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ by L-124. Hence by (728), $\sup A_{Q,X_1,\ldots,X_n}(z) \leq \frac{1}{2}$ for all $z \in \mathbf{I}$, i.e. we are done because the above condition is vacuous in this case.
In the remaining case that $r_+ > \frac{1}{2}$, we know from L-123 and L-127 that $Y_1^+, \ldots, Y_n^+$ is the only choice of crisp subsets with $\Xi_{Y_1^+,\ldots,Y_n^+}(X_1, \ldots, X_n) = r_+$, and that for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ with $(Y_1, \ldots, Y_n) \neq (Y_1^+, \ldots, Y_n^+)$, $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \leq 1 - r_+ < \frac{1}{2}$. Hence $z = Q(Y_1^+, \ldots, Y_n^+)$ is the only choice of $z \in \mathbf{I}$ with

$$\sup A_{Q,X_1,\ldots,X_n}(z) = r_+ > \tfrac{1}{2} \,,$$

and for all $z' \neq z$, $\sup A_{Q,X_1,\ldots,X_n}(z) \leq 1 - r_+ < \frac{1}{2}$, as desired.

B.3 Proof of Theorem 94

Let a choice of $A \in \mathbb{A}$ be given and $D(A) = \cup\{A(z) : z \in \mathbf{I}\}$.

a. $D(A) = \{1\}$**.**
Then $A(z_+) = \{1\}$ and $A(z) = \varnothing$ for $z \neq z_+$, see Def. 87. Now we consider

$Q : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ defined by $Q(\varnothing) = z_+$, where $\varnothing$ is the empty tuple. Because Q is defined on the empty tuple only and $Q(\varnothing) = z_+$, we obtain

$$\begin{aligned} A^{(0)}_{Q,\varnothing}(z_+) &= \{\Xi^{(0)}_{\varnothing}(\varnothing)\} && \text{by Def. 86} \\ &= \{1\} && \text{by Def. 83} \\ &= A(z_+) \end{aligned}$$

and

$$A^{(0)}_{Q,\varnothing}(z) = \varnothing = A(z)$$

for $z \neq z_+$. Hence indeed $A = A^{(0)}_{A,\varnothing}$, as desired.

b. $D(A) \neq \{1\}$**.**
In this case, let us suppose that $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I} \in \mathbf{I})$ are defined by (69) and (66), respectively. In order to prove that $A = A_{Q,X}$, we consider some $z_0 \in \mathbf{I}$. Let us now prove in turn that $A_{Q,X}(z_0) \subseteq A(z_0)$ and $A(z_0) \subseteq A_{Q,X}(z_0)$.

a.: $A_{Q,X}(z_0) \subseteq A(z_0)$**.**
To see this, consider $r_0 \in A_{Q,X}(z_0)$. By Def. 86, there exists $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ with $r' = \Xi_Y(X) = r_0$ and $Q(Y) = z_0$. Now let $z' = \inf\{z \in \mathbf{I} : (z, r_0) \in Y, r_0 \in A(z)\}$ as in (68). We can then conclude from $Q(Y) = z_0$ and (69) that either $Q(Y) = \zeta(r_0)$, or $Q(Y) \neq \zeta(r_0)$, in which case $Q(Y) = z'$.

- In the former case we hence obtain $z_0 = Q(Y) = \zeta(r_0)$. It is then immediate from (64) that $r_0 \in A(\zeta(r_0)) = A(z_0)$.
- In the second case we conclude from (69) and $Q(Y) \neq \zeta(r_0)$ that $r_0 \in A(z')$ and $Q(Y) = z'$. Because $z_0 = Q(Y) = z'$, this proves the desired $r_0 \in A(z_0)$.

b.: $A(z_0) \subseteq A_{Q,X}(z_0)$**.**
To see this, consider $r_0 \in A(z_0)$. Recalling Def. 86, we must show that there exists $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ with $Q(Y) = z_0$ and $\Xi_Y(X) = r$. In the following, it is beneficial to discern two main cases.

b.1: $r_0 > \frac{1}{2}$**.**
In this case, we know from Def. 85 that

$$r_0 = r_+ \,. \tag{729}$$

Now consider $Y = \varnothing$. Then

$$\begin{aligned} r' &= \Xi_Y(X) && \text{by (67)} \\ &= \inf\{1 - \mu_X(z, r) : z, r \in \mathbf{I}\} && \text{by Def. 83} \\ &= \min(\inf\{1 - r : z \in \mathbf{I}, r \in A(z) \setminus \{r_+\}\}, \\ &\qquad \inf\{1 - r_- : z \in \mathbf{I}, r \notin A(z) \vee r = r_+\}) \end{aligned}$$

i.e.

$$r' = \min(\inf\{1 - r : z \in \mathbf{I}, r \in A(z) \setminus \{r_+\}\},\\ \inf\{1 - r_- : z \in \mathbf{I}, r \notin A(z) \vee r = r_+\}) \tag{730}$$

Now we recall from (729) and the assumption of case **b.1** that $r_+ = r_0 > \frac{1}{2}$. Therefore

$$\inf\{1 - r : z \in \mathbf{I}, r \in A(z) \setminus \{r_+\}\} = 1 - \sup D(A) \setminus \{r_+\} = 1 - (1 - r_+) = r_+ \tag{731}$$

by (62) and Def. 85. In addition

$$\inf\{1 - r_- : z \in \mathbf{I}, r \notin A(z) \vee r = r_+\}) = \inf\{1 - r_-\} = 1 - r_- \tag{732}$$

because there exists $z \in \mathbf{I}$ with $r_+ \in A(z)$. Combining these results,

$$\begin{aligned} r' &= \min(\inf\{1 - r : z \in \mathbf{I}, r \in A(z) \setminus \{r_+\}\}, \\ &\qquad \inf\{1 - r_- : z \in \mathbf{I}, r \notin A(z) \vee r = r_+\}) && \text{by (730)} \\ &= \min(r_+, 1 - r_-) \\ &= r_+ && \text{by (65)} \\ &= r_0 \,. && \text{by (729)} \end{aligned}$$

This finishes the proof that

$$r' = r_0 = r_+. \tag{733}$$

It remains to be shown that $Q(Y) = z_0$. Because $r_0 = r_+ \in A(z_0)$ and $r_+ > 0$, we conclude from Def. 87 that z_0 is the only choice of $z \in \mathbf{I}$ with

$$r_0 = r_+ \in A(z) \,. \tag{734}$$

Hence

$$\zeta(r_0) = z_0 \tag{735}$$

by (64). We first deduce from (68) and $Y = \varnothing$ that

$$z' = \inf\{z \in \mathbf{I} : (z, r') \in Y \text{ and } r' \in A(z)\} = \inf \varnothing = 1 \,.$$

Hence if $r_+ \in A(1)$, then

$$\begin{aligned} Q(Y) &= 1 && \text{by (69)} \\ &= z_0 \,. && \text{by (734)} \end{aligned}$$

In the remaining case that $r' \notin A(1)$, the result is

$$\begin{aligned} Q(Y) &= \zeta(r_0) && \text{by (69)} \\ &= z_0 \,. && \text{by (735)} \end{aligned}$$

Hence indeed $r' = r_0$ and $Q(Y) = z_0$, i.e. $r_0 \in A_{Q,X}(z_0)$, as desired.

b.2: $r_0 \leq \frac{1}{2}$.
In this case, we define $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ by

$$Y = \{(z_0, r_0)\} \cup \{(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} \,. \tag{736}$$

To see that $r_0 = \Xi_Y(X)$, we first consider $\inf\{\mu_X(z,r) : (z,r) \in Y\}$. Based on (736), we can rewrite this as

$$\inf\{\mu_X(z,r) : (z,r) \in Y\} = \min(\mu_X(z_0, r_0), \inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\}) \,. \tag{737}$$

As concerns $\mu_X(z_0, r_0)$, we reason as follows.

- If $r_0 = \frac{1}{2}$, then $r_0 = r_+ = \frac{1}{2}$, which is immediate from Def. 85. Consequently

$$\begin{aligned} \mu_X(z_0, r_0) &= \tfrac{1}{2} && \text{by (66)} \\ &= r_0 \,. && \text{by assumption on } r_0 \end{aligned}$$

- If $r_0 < \frac{1}{2}$, then $r_0 \neq r_+$ because $r_+ \geq \frac{1}{2}$. We can hence conclude from $r_0 \in A(z_0)$ that $r_0 \in A(z_0) \setminus \{r_+\}$. In turn, we obtain from (66) that $\mu_X(z_0, z_0) = r_0$.

Hence

$$\mu_X(z_0, r_0) = r_0 \,. \tag{738}$$

Let us now turn to $\inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \frac{1}{2}]\}$.

- If $r_0 = r_+ = \frac{1}{2}$, then $(r_0, \frac{1}{2}] = (\frac{1}{2}, \frac{1}{2}] = \varnothing$ and hence $\inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \frac{1}{2}]\} = \inf \varnothing = 1 \geq r_0$.
- If $r_0 < r_+ = \frac{1}{2}$, then

$$\begin{aligned} &\inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]) \\ &= \min(\inf\{\mu_X(z, \tfrac{1}{2}) : \tfrac{1}{2} \in A(z)\}, \\ &\qquad \inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \tfrac{1}{2})\}) \\ &= \min(\inf\{1 - r_- : \tfrac{1}{2} \in A(z)\}, \\ &\qquad \inf\{r : r \in A(z) \cap (r_0, \tfrac{1}{2})\}) && \text{by (66)} \\ &= \min(\inf\{\tfrac{1}{2} : \tfrac{1}{2} \in A(z)\}, \inf\{r : r \in A(z) \cap (r_0, \tfrac{1}{2})\}) && \text{by (65) and } r_+ = \tfrac{1}{2} \\ &\geq r_0 \,. && \text{because } r_0 < \tfrac{1}{2} \end{aligned}$$

- If $r_+ > \frac{1}{2}$, then $A(z) \cap [0, \frac{1}{2}] = A(z) \setminus \{r_+\}$, see Def. 85. Hence

$$\begin{aligned} &\inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} \\ &= \inf\{r : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} && \text{by (66)} \\ &\geq r_0 \,. \end{aligned}$$

This proves that

$$\inf\{\mu_X(z,r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} \geq r_0 \,. \tag{739}$$

Combining (738) and (739), we obtain from (737) that

$$\inf\{\mu_X(z,r) : (z,r) \in Y\} = r_0 \,. \tag{740}$$

Next we focus on

$$\begin{aligned}
&\inf\{1 - \mu_X(z,r) : (z,r) \notin Y\} \\
&= \inf\{1 - \mu_X(z,r) : r \notin A(z) \cap (r_0, \tfrac{1}{2}] \wedge (z,r) \neq (z_0, r_0)\}
\end{aligned}$$

by (736). Noticing that $r_0 \in A(z_0)$, we can further decompose this into

$$\begin{aligned}
\inf\{1 - \mu_X(z,r) : (z,r) \notin Y\} = \min(&\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r > \tfrac{1}{2}\}, \\
&\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r \leq r_0 \\
&\qquad \wedge (z,r) \neq (z_0, r_0)\}, \\
&\inf\{\mu_X(z,r) : r \notin A(z)\}\,.
\end{aligned} \tag{741}$$

Let us now consider the inf-subexpressions in turn. As concerns $\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r > \frac{1}{2}\}$, we obtain the following.

- if $r_+ = \frac{1}{2}$, then $D(A) \subseteq [0, \frac{1}{2}]$ by Def. 85 and hence $\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r > \frac{1}{2}\} = \inf \varnothing = 1 \geq r_0$.
- if $r_+ > \frac{1}{2}$, then $D(A) \cap [\frac{1}{2}, 1] = \{r_+\}$ and hence

$$\begin{aligned}
&\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r > \tfrac{1}{2}\} \\
&= \inf\{1 - r_-\} && \text{by (66)} \\
&\geq 1 - (1 - r_+) && \text{by (65)} \\
&= r_+ \\
&> \tfrac{1}{2} && \text{by assumption of this case} \\
&\geq r_0 \,.
\end{aligned}$$

Summarizing,

$$\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r > \tfrac{1}{2}\} \geq r_0\} \,. \tag{742}$$

Let us now turn to $\inf\{1 - \mu_X(z,r) : r \in A(z) \wedge r \leq r_0 \wedge (z,r) \neq (z_0, r_0)\}$.

- If $r = \frac{1}{2}$, then $r_+ = \frac{1}{2}$, which is apparent from Def. 85. Hence

$$\begin{aligned}
1 - \mu_X(z,r) &= 1 - \tfrac{1}{2} && \text{by (66)} \\
&= \tfrac{1}{2} \\
&= r && \text{by assumption of present case} \\
&\leq r_0 \,.
\end{aligned}$$

- If $r < \frac{1}{2}$, then $r < r_+$ because $r_+ \geq \frac{1}{2}$. Hence $r \in A(z)$ entails that $r \in A(z) \setminus \{r_+\}$, i.e. $\mu_X(z, r) = r \leq r_0$ by (66). In particular $1 - \mu_X(z, r) \geq r_0$.

These results can be summarized as

$$\inf\{1 - \mu_X(z, r) : r \in A(z) \wedge r \leq r_0 \wedge (z, r) \neq (z_0, r_0)\} \geq r_0 \,. \tag{743}$$

Finally we consider $\inf\{1 - \mu_X(z, r) : r \notin A(z)\}$. Hence let $z, r \in \mathbf{I}$ such that $r \notin A(z)$. Then

$$\begin{aligned}
1 - \mu_X(z, r) &= 1 - r_- && \text{by (66)}\\
&\geq 1 - (1 - r_+) && \text{by (65)}\\
&= r_+ \\
&\geq \tfrac{1}{2} && \text{by Def. 85}\\
&\geq r_0 \,.
\end{aligned}$$

Therefore

$$\inf\{1 - \mu_X(z, r) : r \notin A(z)\} \geq r_0 \,. \tag{744}$$

Recalling (741), we can now utilize inequations (742), (743) and (744) to conclude that

$$\inf\{1 - \mu_X(z, r) : (z, r) \notin Y\} \geq r_0 \,. \tag{745}$$

This finally proves the desired

$$\begin{aligned}
\Xi_Y(X) &= \min(\inf\{\mu_X(z, r) : (z, r) \in Y\}, \\
&\qquad \inf\{1 - \mu_X(z, r) : (z, r) \notin Y\}) && \text{by Def. 83}\\
&= r_0 \,, && \text{by (740), (745)}
\end{aligned}$$

i.e.

$$r' = \Xi_Y(X) = r_0 \tag{746}$$

by (67). It remains to be shown that $Q(Y) = z_0$. To this end, we simply notice that

$$\begin{aligned}
z' &= \inf\{z : (z, r') \in Y \wedge r' \in A(z)\} && \text{by (68)}\\
&= \inf\{z : (z, r_0) \in Y \wedge r_0 \in A(z)\} && \text{by (746)}\\
&= \inf\{z_0\} && \text{by (736)}\\
&= z_0 \,.
\end{aligned}$$

Hence $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ as defined by (736) indeed yields $\Xi_Y(X) = r_0$ and

$$Q(Y) = z_0, \tag{747}$$

i.e. $r_0 \in A_{Q,X}(z_0)$.

B.4 Proof of Theorem 95

Let $D \in \mathbb{D}$ be given.

a.: $D = \{1\}$**.**
In this case, consider the empty tuple $\varnothing \in \mathcal{P}(\{*\})^0$. Then

$$\begin{aligned} D_{\varnothing}^{(0)} &= \{\Xi_Y^{(0)}(\varnothing) : Y \in \mathcal{P}(\{*\})^0\} && \text{by Def. 84} \\ &= \{\Xi_Y^{(0)}(\varnothing) : Y \in \{\varnothing\}\} && \text{(unique empty tuple)} \\ &= \{\Xi_{\varnothing}^{(0)}(\varnothing)\} \\ &= \{1\}\,. && \text{by Def. 83} \end{aligned}$$

b.: $D \neq \{1\}$**.**
In this case we define $A : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$A(z) = \begin{cases} D & : \quad z = 1 \\ \varnothing & : \quad z \neq 1 \end{cases} \tag{748}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 87 that $A \in \mathbb{A}$. Now let us define $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by (66). Then by Th-94, $A = A_{Q,X}$. Hence

$$\begin{aligned} D &= \cup\{A(z) : z \in \mathbf{I}\} && \text{by (748)} && (749) \\ &= D(A) && \text{by (62)} && (750) \\ &= D(A_{Q,X}) && \text{by Th-94} && (751) \\ &= D_X\,, && \text{by L-129 and (62)} && (752) \end{aligned}$$

as desired.

B.5 Proof of Theorem 96

In order to prove the theorem, it is useful to introduce a slightly stronger condition on $\psi : \mathbb{A} \longrightarrow \mathbf{I}$, which states that for all $A \in \mathbb{A}$,

$$\text{If } \mathrm{VL}(A) = \{z\} \text{ for some } z \in \mathbf{I} \text{ and } r_+ = 1, \text{ then } \psi(A) = z. \tag{ψ-1$'$}$$

The new condition is apparently stronger than (ψ-1). Conversely, it is entailed by (ψ-1) in the case that (ψ-5) is valid as well.

Lemma 130 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-1) and (ψ-5). Then ψ also satisfies (ψ-1$'$).*

Proof To see this, consider $A \in \mathbb{A}$ with $\mathrm{VL}(A) = \{z'\}$ for some $z' \in \mathbf{I}$ and $r_+ = 1$. We define $A' \in \mathbb{A}$ by

$$A'(z) = \begin{cases} \{1\} & : \quad z = z' \\ \varnothing & : \quad \text{else} \end{cases}$$

for all $z \in \mathbf{I}$. It is then immediate from Def. 91 that

$$\boxplus A' = \boxplus A \,. \tag{753}$$

In addition, A' apparently has $D(A') = \{1\}$ and $z_+ = z_+(A') = z'$, see (63), i.e. (ψ-1) is applicable. Therefore

$$\begin{aligned} \psi(A) &= \psi(\boxplus A) && \text{by } (\psi\text{-5}) \\ &= \psi(\boxplus A') && \text{by (753)} \\ &= \psi(A') && \text{by } (\psi\text{-5}) \\ &= z' \,. && \text{by } (\psi\text{-1}) \end{aligned}$$

Lemma 131 *Let $E \neq \varnothing$ be given and $n \in \mathbb{N}$. Then for all $Y_1, \ldots, Y_n, Z_1, \ldots, Z_n \in \mathcal{P}(E)$,*

$$\Xi_{Y_1,\ldots,Y_n}(Z_1, \ldots, Z_n) = \begin{cases} 1 & : \quad Y_i = Z_i \textit{ for all } i \in \{1, \ \ldots, n\} \\ 0 & : \quad Y_i \neq Z_i \textit{ for some } i \in \{1, \ \ldots, n\}. \end{cases}$$

Proof To see this, we notice that for all $e \in E$,

$$\delta_{Z_i,Y_i}(e) = \begin{cases} 0 & : \quad e \in Y_i \text{ and } e \notin Z_i \\ 1 & : \quad e \in Y_i \text{ and } e \in Z_i \\ 1 & : \quad e \notin Y_i \text{ and } e \notin Z_i \\ 0 & : \quad e \notin Y_i \text{ and } e \in Z_i \end{cases} \tag{754}$$

by (60). Therefore

$$\begin{aligned} \Xi_{Y_1,\ldots,Y_n}(Z_1, \ldots, Z_n) &= \inf\{\delta_{Z_i,Y_i}(e) : e \in E, i = 1, \ldots, n\} && \text{by (61)} \\ &= \begin{cases} 1 & : \quad Z_i = Y_i \text{ for all } i \in \{1, \ \ldots, n\} \\ 0 & : \quad Z_i \neq Y_i \text{ for some } i \in \{1, \ \ldots, n\}, \end{cases} && \text{by (754)} \end{aligned}$$

as desired.

Lemma 132 *Suppose $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-1$'$). Then for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{U}(\mathcal{F}_\psi)(Q) = Q$.*

Proof Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of *crisp* arguments $Z_1, \ldots, Z_n \in \mathcal{P}(E)$ be given. We then know from L-131 that $\Xi_{Z_1,\ldots,Z_n}(Z_1, \ldots, Z_n) = 1$, in particular $1 \in A_{Q,Z_1,\ldots,Z_n}(z)$, where

$$z = Q(Z_1, \ldots, Z_n) \,, \tag{755}$$

and hence

$$r_+ = 1 \,. \tag{756}$$

We also know from L-131 that $\Xi_{Y_1,\ldots,Y_n}(Z_1,\ldots,Z_n) = 0$ for all $(Y_1,\ldots,Y_n) \neq (Z_1,\ldots,Z_n)$. In particular $A_{Q,Z_1,\ldots,Z_n}(z') \subseteq \{0\}$ for all $z' \in \mathbf{I} \setminus \{z\}$, i.e.

$$\mathrm{VL}(A_{Q,Z_1,\ldots,Z_n}) = \{z\} \tag{757}$$

Combining (756)–(757), we observe that $A_{Q,Z_1,\ldots,Z_n}$ satisfies the requirements of (ψ-1′). Hence

$$\psi(A_{Q,Z_1,\ldots,Z_n}) = z = Q(Z_1,\ldots,Z_n) \tag{758}$$

by (755). Therefore we obtain the desired

$$\begin{aligned} \mathcal{F}_\psi(Q)(Z_1,\ldots,Z_n) &= \psi(A_{Q,(Z_1,\ldots,Z_n)}) && \text{by Def. 88} \\ &= Q(Z_1,\ldots,Z_n)\,. && \text{by (758)} \end{aligned}$$

Proof of Theorem 96

We recall from L-130 that ψ also satisfies (ψ-1′). The theorem is hence entailed by L-132, because (Z-1) requires $\mathcal{U}(\mathcal{F}_\psi)(Q) = Q$ only in the case that $n \in \{0,1\}$.

B.6 Proof of Theorem 97

Lemma 133 *Let $E \neq \varnothing$ be some base set, $n \in \mathbb{N}$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $Y_1,\ldots,Y_n \in \mathcal{P}(E)$, the following are equivalent.*

a. $(Y_1,\ldots,Y_n) \in \mathcal{T}_0(X_1,\ldots,X_n)$;

b. $\Xi_{Y_1,\ldots,Y_n}(X_1,\ldots,X_n) \geq \frac{1}{2}$.

Proof

a.→b.:
Suppose that $(Y_1,\ldots,Y_n) \in \mathcal{T}_0(X_1,\ldots,X_n)$. Hence by Def. 31,

$$X_{i_{>\frac{1}{2}}} \subseteq Y_i \subseteq X_{i_{\geq\frac{1}{2}}}\,,$$

for $i = 1,\ldots,n$. Recalling Def. 29, we conclude that

$$\mu_{X_i}(e) \geq \tfrac{1}{2} \tag{759}$$

for all $e \in Y_i$. In addition, we conclude from $X_{i_{>\frac{1}{2}}} \subseteq Y_i$ and Def. 30 that $\mu_{X_i}(e) \leq \frac{1}{2}$, and hence

$$1 - \mu_{X_i}(e) \geq \tfrac{1}{2}\,, \tag{760}$$

for all $e \notin Y_i$. Therefore

$$\begin{aligned}
&\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) \\
&= \bigwedge_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in Y_i\}, 1 - \mu_{X_i}(e) : e \notin Y\}) && \text{by Def. 83} \\
&\geq \tfrac{1}{2}\,, && \text{by (759), (760)}
\end{aligned}$$

as desired.

b.→a.:
Suppose that $\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) \geq \frac{1}{2}$. Then $\Xi_{Y_i}(X_i) \geq \frac{1}{2}$ for all $i = 1,\dots,n$ (see Def. 83), which in turn yields

$$\inf\{\mu_{X_i}(e) : e \in Y_i\} \geq \tfrac{1}{2} \tag{761}$$

$$\inf\{1 - \mu_{X_i}(e) : e \notin Y_i\} \geq \tfrac{1}{2}\,. \tag{762}$$

Now consider some $e \in Y_i$. We then obtain from (761) that $\mu_{X_i}(e) \geq \frac{1}{2}$, i.e. $e \in X_{i_{\geq\frac{1}{2}}}$ by Def. 29. In particular $e \in X_{i\,0}^{\max}$ by Def. 31. This proves that

$$Y_i \subseteq X_{i\,0}^{\max}\,. \tag{763}$$

Next we consider $e \in X_{i\,0}^{\min}$, i.e. $e \in X_{i_{>\frac{1}{2}}}$ and $\mu_{X_i}(e) > \frac{1}{2}$ by Def. 31 and Def. 30. We hence know that

$$1 - \mu_{X_i}(e) < \tfrac{1}{2}\,. \tag{764}$$

The proof that $e \in Y_i$ is by contradiction. Hence suppose that $e \notin Y_i$. Then

$$\begin{aligned}
&\inf\{1 - \mu_{X_i}(e') : e' \notin Y_i\} \\
&\leq 1 - \mu_{X_i}(e) && \text{because } e \notin Y_i \text{ for the given } e \\
&< \tfrac{1}{2}\,. && \text{by (764)}
\end{aligned}$$

This contradicts (762). Hence the assumption $e \notin Y_i$ is false, i.e. $e \in Y_i$. This proves that

$$X_{i\,0}^{\min} \subseteq Y_i\,. \tag{765}$$

Combining (763) and (765), we obtain from Def. 31 that $Y_i \in \mathcal{T}_0(X_i)$. Because $i \in \{1,\ \dots,\ n\}$ was arbitrary, this finishes the proof of the desired $(Y_1,\dots,Y_n) \in \mathcal{T}_0(X_1,\dots,X_n)$.

Lemma 134 *Let $E \neq \varnothing$ be some base set, $n \in \mathbb{N}$ and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Further suppose that $Y_1,\dots,Y_n \in \mathcal{P}(E)$ is a choice of crisp subsets of E. We abbreviate*

$$\gamma = \max(0, 1 - 2\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n))\,. \tag{766}$$

Then

$$(Y_1,\dots,Y_n) \in \mathcal{T}_{\gamma'}(X_1,\dots,X_n)$$

for all $\gamma' > \gamma$.

Proof To see this, let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ be given. Let us now consider γ defined by equation (766).
In the case that $\gamma = 0$, we know from (766) that $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \geq \frac{1}{2}$. We now recall L-133, which states that $Y_1, \ldots, Y_n \in \mathcal{T}_0(X_1, \ldots, X_n)$ in this case. Noticing that $\mathcal{T}_0(X_1, \ldots, X_n) \subseteq \mathcal{T}_{\gamma'}(X_1, \ldots, X_n)$ for all $\gamma' \geq 0$, this proves the claim of the present lemma.
Now let us consider the remaining case that $\gamma > 0$. Then (766) can be simplified as follows.

$$\gamma = 1 - 2\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \,. \tag{767}$$

In other words, $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \frac{1}{2} - \frac{1}{2}\gamma$ and hence

$$\inf\{\mu_{X_i}(e) : e \in Y_i\} \geq \tfrac{1}{2} - \tfrac{1}{2}\gamma \tag{768}$$

$$\inf\{1 - \mu_{X_i}(e) : e \notin Y_i\} \geq \tfrac{1}{2} - \tfrac{1}{2}\gamma \,, \tag{769}$$

which is immediate from Def. 83.
Now let $\gamma' > \gamma$ be given. First we consider some $e \in Y_i$. Then $\mu_{X_i}(e) \geq \frac{1}{2} - \frac{1}{2}\gamma > \frac{1}{2} - \frac{1}{2}\gamma'$ by (768). Hence $e \in {X_i}_{> \frac{1}{2} - \frac{1}{2}\gamma'} = {X_i}^{\max}_{\gamma'}$ by Def. 30 and Def. 31. This proves that

$$Y_i \subseteq {X_i}^{\max}_{\gamma'} \,. \tag{770}$$

Finally we consider $e \in {X_i}^{\min}_{\gamma'}$, i.e. $e \in {X_i}_{\geq \frac{1}{2} + \frac{1}{2}\gamma'}$ and hence $\mu_{X_i}(e) \geq \frac{1}{2} + \frac{1}{2}\gamma'$ by Def. 31 and Def. 29. In particular

$$1 - \mu_{X_i}(e) \leq \tfrac{1}{2} - \tfrac{1}{2}\gamma' < \tfrac{1}{2} - \tfrac{1}{2}\gamma \,. \tag{771}$$

The proof that $e \in Y_i$ is by contradiction. Hence let us assume to the contrary that $e \notin Y_i$. Then

$$\begin{aligned}
\gamma &= 1 - 2\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) && \text{by (767)} \\
&\leq 1 - 2\inf\{1 - \mu_{X_i}(e') : e' \notin Y_i\} && \text{apparent from Def. 83} \\
&\leq 1 - 2(1 - \mu_{X_i}(e)) && \text{because } e \notin Y_i \text{ for the given } e \\
&< 1 - 2(\tfrac{1}{2} - \tfrac{1}{2}\gamma) && \text{by (771)} \\
&= 1 - 1 + \gamma \\
&= \gamma \,.
\end{aligned}$$

Hence $\gamma < \gamma$, a contradiction. This prove that the assumption $e \notin Y_i$ is false, in fact it holds that $e \in Y_i$. Because $e \in {X_i}^{\min}_{\gamma'}$ was arbitrarily chosen, this proves that

$${X_i}^{\min}_{\gamma'} \subseteq Y_i \,. \tag{772}$$

Combining equations (770) and (772), we obtain from Def. 31 that $Y_i \in \mathcal{T}_{\gamma'}(X_i)$. Because $i \in \{1, \ldots, n\}$ was arbitrary, this completes the proof that $(Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma'}(X_1, \ldots, X_n)$.

Lemma 135 *Suppose that* $E \neq \varnothing$ *is some base set and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $n \in \mathbb{N}$. *Further let* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ *be given. We again abbreviate*

$$\gamma = \max(0, 1 - 2\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)) . \tag{773}$$

Then

$$(Y_1, \ldots, Y_n) \notin \mathcal{T}_{\gamma'}(X_1, \ldots, X_n)$$

for all $\gamma' < \gamma$.

Proof The claim of the lemma is vacuous if $\gamma = 0$. Hence suppose that $\gamma > 0$ and consider a choice of $\gamma' < \gamma$. In this case, (773) reduces to

$$\gamma = 1 - 2\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) .$$

Hence

$$\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \tfrac{1}{2} - \tfrac{1}{2}\gamma < \tfrac{1}{2} - \tfrac{1}{2}\gamma' . \tag{774}$$

Let us also notice that

$$\begin{aligned}
\Xi_Y(X) &= \mathop{\wedge}_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in Y_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin Y_i\}) && \text{by Def. 83} \\
&= \min(\mathop{\wedge}_{i=1}^{n} \inf\{\mu_{X_i}(e) : e \in Y_i\}, \mathop{\wedge}_{i=1}^{n} \inf\{1 - \mu_{X_i}(e) : e \notin Y_i\}) .
\end{aligned}$$

It is hence sufficient to discern the following two cases.

a.: $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \mathop{\wedge}_{i=1}^{n} \inf\{\mu_{X_i}(e) : e \in Y_i\}$**.**
In this case, we can deduce from (774) that there exists $i \in \{1, \ldots, n\}$ and $e' \in Y_i$ with $\mu_{X_i}(e') < \frac{1}{2} - \frac{1}{2}\gamma'$. Hence $e' \notin X_{i_{\geq \frac{1}{2} - \frac{1}{2}\gamma'}} \supseteq X_{i\,\gamma'}^{\max}$ for the given $e' \in Y_i$, see Def. 29 and Def. 31. In particular $Y_i \not\subseteq X_{i\,\gamma'}^{\max}$. By Def. 31, this proves that $Y_i \notin \mathcal{T}_{\gamma'}(X_i)$, which results in $(Y_1, \ldots, Y_n) \notin \mathcal{T}_{\gamma'}(X_1, \ldots, X_n) = \mathcal{T}_{\gamma'}(X_i) \times \cdots \times \mathcal{T}_{\gamma'}(X_n)$, as desired.

b.: $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \mathop{\wedge}_{i=1}^{n} \inf\{1 - \mu_{X_i}(e) : e \notin Y_i\}$**.**
In this case, we conclude from (774) that there exists $i \in \{1, \ldots, n\}$ and $e' \in E \setminus Y_i$ with $1 - \mu_{X_i}(e') < \frac{1}{2} - \frac{1}{2}\gamma'$, i.e. $\mu_{X_i}(e') > \frac{1}{2} + \frac{1}{2}\gamma'$. Hence $e' \in X_{i_{> \frac{1}{2} + \frac{1}{2}\gamma'}} \subseteq X_{i\,\gamma'}^{\min}$ by Def. 30 and Def. 31. Because $e' \notin Y_i$, it witnesses the failure of $X_{i\,\gamma'}^{\min} \subseteq Y_i$. Hence $Y_i \notin \mathcal{T}_{\gamma'}(X_i)$ by Def. 31 and in turn, $(Y_1, \ldots, Y_n) \notin \mathcal{T}_{\gamma}(X_1, \ldots, X_n)$.

Lemma 136 *Let* $E \neq \varnothing$ *be some base set,* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ *and* $\gamma \in \mathbf{I}$. *If* $(Y_1, \ldots, Y_n) \in \mathcal{T}_{\gamma}(X_1, \ldots, X_n)$, *then* $\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \geq \frac{1}{2} - \frac{1}{2}\gamma$.

Proof Let $(Y_1, \dots, Y_n) \in \mathcal{T}_\gamma(X_1, \dots, X_n)$ be given. The proof is by contradiction. Hence let us assume that $\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) < \frac{1}{2} - \frac{1}{2}\gamma$, i.e. $\gamma < 1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n)$. This is only possible if $1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) > 0$, hence $1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) = \max(0, 1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n))$ in this case. We can hence apply L-135 and conclude that $(Y_1, \dots, Y_n) \notin \mathcal{T}_\gamma(X_1, \dots, X_n)$, a contradiction. Hence the assumption that $\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) < \frac{1}{2} - \frac{1}{2}\gamma$ is false, and it indeed holds that $\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) \geq \frac{1}{2} - \frac{1}{2}\gamma$.

Lemma 137 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ and $z \in \mathbf{I}$ be given. Then for all $\gamma' > s(A_{Q,X_1,\dots,X_n})(z)$, $z \in S_{Q,X_1,\dots,X_n}(\gamma')$.*

Proof

a.: $\sup A_{Q,X_1,\dots,X_n}(z) \leq \frac{1}{2}$.
Then (80) reduces to $s(A_{Q,X_1,\dots,X_n})(z) = 1 - 2\sup A_{Q,X_1,\dots,X_n}(z) \geq 0$, and $\gamma' > s(A_{Q,X_1,\dots,X_n})(z)$ can be reformulated into

$$\tfrac{1}{2} - \tfrac{1}{2}\gamma' < \sup A_{Q,X_1,\dots,X_n}(z)\,. \tag{775}$$

We first consider the special case that $A_{Q,X_1,\dots,X_n}(z) = \varnothing$. Then apparently

$$\sup A_{Q,X_1,\dots,X_n}(z) = 0 \qquad \text{and} \qquad s(A_{Q,X_1,\dots,X_n})(z) = 1\,,$$

i.e. the condition is vacuous because $\gamma' > 1$ is not possible for $\gamma' \in \mathbf{I}$.
Hence let us assume that $A_{Q,X_1,\dots,X_n}(z) \neq \varnothing$. Recalling (775), this entails that there exists $r \in A_{Q,X_1,\dots,X_n}(z)$ with $\frac{1}{2} - \frac{1}{2}\gamma' < r \leq \sup A_{Q,X_1,\dots,X_n}(z)$. By Def. 86, then, there exists $Y_1', \dots, Y_n' \in \mathcal{P}(E)$ with

$$Q(Y_1', \dots, Y_n') = z \tag{776}$$

$$\tfrac{1}{2} - \tfrac{1}{2}\gamma' < \Xi_{Y_1',\dots,Y_n'}(X_1, \dots, X_n) = r \leq \sup A_{Q,X_1,\dots,X_n}(z)\,. \tag{777}$$

We conclude from (777) that

$$\gamma' > 1 - 2\Xi_{Y_1',\dots,Y_n'}(X_1, \dots, X_n) = \max(0, 1 - 2\Xi_{Y_1',\dots,Y_n'}(X_1, \dots, X_n))\,.$$

Hence L-134 is applicable, from which we obtain that $(Y_1', \dots, Y_n') \in \mathcal{T}_{X_1,\dots,X_n}(\gamma')$. In turn, we conclude from Def. 51 that $z = Q(Y_1', \dots, Y_n') \in S_{Q,X_1,\dots,X_n}(\gamma')$, as desired.

b.: $\sup A_{Q,X_1,\dots,X_n}(z) > \frac{1}{2}$.
In this case, we know from (80) that

$$s(A_{Q,X_1,\dots,X_n})(z) = 0\,. \tag{778}$$

We then know from L-123 and L-124 that

$$\sup A_{Q,X_1,\dots,X_n}(z) = r_+ = \Xi_{Y_1^+,\dots,Y_n^+}(X_1, \dots, X_n) \tag{779}$$

and

$$Q(Y_1^+, \dots, Y_n^+) = z\,, \tag{780}$$

where $Y_1^+, \dots, Y_n^+ \in \mathcal{P}(E)$ are defined by (700). In particular, we notice from (779) that $\Xi_{Y_1^+,\dots,Y_n^+}(X_1, \dots, X_n) \geq \frac{1}{2}$. Hence

$$(Y_1^+, \dots, Y_n^+) \in \mathcal{T}_0(X_1, \dots, X_n) \subseteq \mathcal{T}_{\gamma'}(X_1, \dots, X_n) \tag{781}$$

by L-133 and Def. 31. Combining (780) and (781), we obtain from Def. 51 that $z = Q(Y_1^+, \dots, Y_n^+) \in S_{Q,X_1,\dots,X_n}(\gamma')$, as desired.

Lemma 138 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ and $z \in \mathbf{I}$ be given. If $\gamma' < s(A_{Q,X_1,\dots,X_n})(z)$, then $z \notin S_{Q,X_1,\dots,X_n}(\gamma')$.*

Proof The claim of the lemma is vacuous if $s(A_{Q,X_1,\dots,X_n})(z) = 0$. Hence suppose that $s(A_{Q,X_1,\dots,X_n})(z) > 0$, i.e.

$$s(A_{Q,X_1,\dots,X_n})(z) = 1 - 2 \cdot \sup A_{Q,X_1,\dots,X_n}(z) > 0 \tag{782}$$

by (80). Now let

$$\gamma' < s(A_{Q,X_1,\dots,X_n}) = 1 - 2 \sup A_{Q,X_1,\dots,X_n}(z)\,. \tag{783}$$

I will show that for all $(Y_1, \dots, Y_n) \in Q^{-1}(z)$, $(Y_1, \dots, Y_n) \notin \mathcal{T}_{\gamma'}(X_1, \dots, X_n)$. Hence consider a choice of $(Y_1, \dots, Y_n) \in Q^{-1}(z)$. Then

$$\begin{aligned}
&\sup A_{Q,X_1,\dots,X_n} \\
&= \sup\{\Xi_{Y_1',\dots,Y_n'}(X_1, \dots, X_n) : (Y_1', \dots, Y_n') \in Q^{-1}(z)\} && \text{by Def. 86} \\
&\geq \Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n)\,.
\end{aligned}$$

Hence $1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) \geq 1 - 2 \sup A_{Q,X_1,\dots,X_n}(z)$ and in turn,

$$\begin{aligned}
\max(0, 1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n)) &\geq \max(0, 1 - 2 \sup A_{Q,X_1,\dots,X_n}(z)) \\
&= 1 - 2 \sup A_{Q,X_1,\dots,X_n}(z)
\end{aligned}$$

by (80) and (782). We then obtain from (783) that

$$\gamma' < \max(0, 1 - 2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n))\,.$$

Hence by L-135,

$$(Y_1, \dots, Y_n) \notin \mathcal{T}_{\gamma'}(X_1, \dots, X_n)\,.$$

Because $(Y_1, \dots, Y_n) \in Q^{-1}(z)$ was arbitrary, this proves that $z \notin S_{Q,X_1,\dots,X_n}(\gamma')$, see Def. 51.

Proof of Theorem 97

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Further let $z \in \mathbf{I}$. Let us now apply the results of the previous lemmata. We know from L-137 that

$$\inf\{\gamma \in \mathbf{I} : z \in S_{Q,X_1,\dots,X_n}(\gamma)\} \leq s(A_{Q,X_1,\dots,X_n})(z)\,. \tag{784}$$

We further known from L-138 that

$$\inf\{\gamma \in \mathbf{I} : z \in S_{Q,X_1,\dots,X_n}(\gamma)\} \geq s(A_{Q,X_1,\dots,X_n})(z)\,. \tag{785}$$

Therefore

$$\begin{aligned} s_{Q,X_1,\dots,X_n}(z) &= \inf\{\gamma \in \mathbf{I} : z \in S_{Q,X_1,\dots,X_n}(\gamma)\} && \text{by Def. 54} \\ &= s(A_{Q,X_1,\dots,X_n})(z)\,, && \text{by (784), (785)} \end{aligned}$$

as desired.

B.7 Proof of Theorem 98

Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by (81). In order to prove that $\mathcal{F}_\omega = \mathcal{F}_\psi$, we consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned} \mathcal{F}_\omega(Q)(X_1, \dots, X_n) &= \omega(s_{Q,X_1,\dots,X_n}) && \text{by Def. 61} \\ &= \omega(s(A_{Q,X_1,\dots,X_n})) && \text{by Th-97} \\ &= \psi(A_{Q,X_1,\dots,X_n}) && \text{by (81)} \\ &= \mathcal{F}_\psi(Q)(X_1, \dots, X_n)\,. && \text{by Def. 88} \end{aligned}$$

B.8 Proof of Theorem 99

Lemma 139 *Consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$A_{\neg Q,X_1,\dots,X_n}(z) = A_{Q,X_1,\dots,X_n}(1-z)\,.$$

for all $z \in \mathbf{I}$.

Proof Straightforward.

$$\begin{aligned} & A_{\neg Q,X_1,\dots,X_n}(z) \\ &= \{\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) : (Y_1, \dots, Y_n) \in \neg Q^{-1}(z)\} && \text{by Def. 86} \\ &= \{\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) : (Y_1, \dots, Y_n) \in Q^{-1}(1-z)\} && \text{Def. 9, } \neg x = 1-x \\ &= A_{Q,X_1,\dots,X_n}(1-z)\,. && \text{by Def. 86} \end{aligned}$$

Proof of Theorem 99

We first notice that by Th-5, all standard DFSes coincide on two-valued quantifiers. It is hence sufficient to show that $\mathcal{F}_\psi$ coincide with an arbitrary standard DFS on two-valued quantifiers. In the following, it will be convenient to show that $\mathcal{F}_\psi$ coincides with $\mathcal{F}_P$.

Now let a two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ and a choice of $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Because Q is two-valued, we know that

$$A_{Q,X_1,\ldots,X_n}(z) = \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : (Y_1, \ldots, Y_n) \in Q^{-1}(z)\} = \varnothing \quad (786)$$

for all $z \in (0, 1)$. Hence either $r_+ \in A_{Q,X_1,\ldots,X_n}(1)$ or $r_+ \in A_{Q,X_1,\ldots,X_n}(0)$. We shall consider these cases in turn.

a.: $r_+ \in A_{Q,X_1,\ldots,X_n}(1)$**.**

Recalling (786), we observe that (ψ-3) is applicable, which lets us deduce that

$$\psi(A_{Q,X_1,\ldots,X_n}) = 1 - \sup A_{Q,X_1,\ldots,X_n}(0)\,. \quad (787)$$

We further notice that $r_+ \in A_{Q,X_1,\ldots,X_n}(1)$ entails that $\sup A_{Q,X_1,\ldots,X_n}(0) \leq \frac{1}{2}$, see L-124. Therefore

$$1 - 2\sup A_{Q,X_1,\ldots,X_n}(0) = \max(0, 1 - 2\sup A_{Q,X_1,\ldots,X_n}(0)) = s(A_{Q,X_1,\ldots,X_n})(0)\,, \quad (788)$$

see (80). We now proceed as follows.

$$\begin{aligned}
\mathcal{F}_\psi(Q)(X_1, \ldots, X_n) &= \psi(A_{Q,X_1,\ldots,X_n}) && \text{by Def. 88}\\
&= 1 - \sup A_{Q,X_1,\ldots,X_n}(0) && \text{by (787)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}(1 - 2\sup A_{Q,X_1,\ldots,X_n}(0)) &&\\
&= \tfrac{1}{2} + \tfrac{1}{2}s(A_{Q,X_1,\ldots,X_n})(0) && \text{by (788)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}s_{Q,X_1,\ldots,X_n}(0) && \text{by Th-97}\\
&= \omega_P(s_{Q,X_1,\ldots,X_n}) && \text{by (ω-3), Th-52}\\
&= \mathcal{F}_P(Q)(X_1, \ldots, X_n)\,, && \text{by Def. 61}
\end{aligned}$$

as desired.

b. $r_+ \in A_{Q,X_1,\ldots,X_n}(0)$**.**

In this case, we consider the standard negation $\neg Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ of Q. Clearly $A_{Q,X_1,\ldots,X_n}(z) = \varnothing$ for $z \in (0, 1)$ because $\neg Q$ is two-valued. In addition, $r_+ \in A_{Q,X_1,\ldots,X_n}(0)$ entails that $r_+ \in A_{\neg Q,X_1,\ldots,X_n}(1)$, see L-139. It is then apparent from the proof of part **a.** of the present lemma that

$$\mathcal{F}_\psi(\neg Q)(X_1, \ldots, X_n) = \mathcal{F}_P(\neg Q)(X_1, \ldots, X_n)\,. \quad (789)$$

Therefore

$$\begin{aligned}
\mathcal{F}_\psi(Q)(X_1,\dots,X_n) &= \psi(A_{Q,X_1,\dots,X_n}) && \text{by Def. 88}\\
&= 1-\psi(A_{\neg Q,X_1,\dots,X_n}) && \text{by (ψ-2) and L-139}\\
&= 1-\mathcal{F}_\psi(\neg Q)(X_1,\dots,X_n) && \text{by Def. 88}\\
&= 1-\mathcal{F}_P(\neg Q)(X_1,\dots,X_n) && \text{by (789)}\\
&= 1-(1-\mathcal{F}_P(Q)(X_1,\dots,X_n)) && \text{by Th-52, Th-2}\\
&= \mathcal{F}_P(Q)(X_1,\dots,X_n)\,.
\end{aligned}$$

B.9 Proof of Theorem 100

Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-2) and (ψ-3). Now let $E \neq \varnothing$ be some base set and consider some $e \in E$. Then $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is a two-valued quantifier, see Def. 6. Hence

$$\begin{aligned}
\mathcal{F}_\psi(\pi_e) &= \mathcal{F}_P(\pi_e) && \text{by Th-99, Th-52 and Th-5}\\
&= \widetilde{\pi}_e\,. && \text{by Th-52 and (Z-2)}
\end{aligned}$$

B.10 Proof of Theorem 101

Lemma 140 *Let $E \neq \varnothing$ be some base set, $X \in \widetilde{\mathcal{P}}(E)$ and $Y \in \mathcal{P}(E)$. Then*

$$\Xi_{\neg Y}(\neg X) = \Xi_Y(X)\,.$$

Proof To see this, consider the following chain of equations.

$$\begin{aligned}
&\Xi_Y(X)\\
&= \min(\inf\{\mu_X(e) : e \in Y\},\\
&\qquad \inf\{\mu_X(e) : e \notin Y\}) && \text{by Def. 83}\\
&= \min(\inf\{1-\mu_{\neg X}(e) : e \in Y\},\\
&\qquad \inf\{\mu_{\neg X}(e) : e \notin Y\}) && \text{by def. of fuzzy complement } \neg X\\
&= \min(\inf\{1-\mu_{\neg X}(e) : e \notin \neg Y\},\\
&\qquad \inf\{\mu_{\neg X}(e) : e \in \neg Y\}) && \text{by def. of crisp complement } \neg Y\\
&= \Xi_{\neg Y}(\neg X)\,. && \text{by Def. 83}
\end{aligned}$$

Lemma 141 *Let $E \neq \varnothing$ be some base set, $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ and $Y_1,\dots,Y_n \in \mathcal{P}(E)$ where $n > 0$. Then*

$$\Xi_{Y_1,\dots,Y_{n-1},\neg Y_n}(X_1,\dots,X_{n-1},\neg X_n) = \Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n)\,.$$

Proof Straightforward:

$$\begin{aligned}
&\Xi_{Y_1,\dots,Y_{n-1},\neg Y_n}(X_1,\dots,X_{n-1},\neg X_n) \\
&= \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \Xi_{\neg Y_n}(\neg X_n)) && \text{by Def. 83} \\
&= \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \Xi_{Y_n}(X_n)) && \text{by L-140} \\
&= \Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n)\,. && \text{by Def. 83}
\end{aligned}$$

Lemma 142 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier of arity $n > 0$ and let $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$\mathcal{A}_{Q\neg,X_1,\dots,X_n} = \mathcal{A}_{Q,X_1,\dots,X_{n-1},\neg X_n}\,.$$

Proof Let us first observe that for a given $z \in \mathbf{I}$,

$$\begin{aligned}
&\mathcal{A}_{Q\neg,X_1,\dots,X_n}(z) \\
&= \{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) : (Y_1,\dots,Y_n) \in (Q\neg)^{-1}(z)\} && \text{by Def. 86} \\
&= \{\Xi_{Y_1,\dots,Y_{n-1},\neg Y_n}(X_1,\dots,X_n) : (Y_1,\dots,Y_n) \in Q^{-1}(z)\} && \text{by Def. 10} \\
&= \{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_{n-1},\neg X_n) : (Y_1,\dots,Y_n) \in Q^{-1}(z)\} && \text{by L-141} \\
&= \mathcal{A}_{Q,X_1,\dots,X_{n-1},\neg X_n}(z)\,. && \text{by Def. 86}
\end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $\mathcal{A}_{Q\neg,X_1,\dots,X_n} = \mathcal{A}_{Q,X_1,\dots,X_{n-1},\neg X_n}$, as desired.

Proof of Theorem 101

Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-2) and (ψ-3). We then know from Th-99 that ψ induces the standard negation. Now let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier of arity $n > 0$ and let $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\psi(Q\Box)(X_1,\dots,X_n) &= \psi(\mathcal{A}_{Q\Box,X_1,\dots,X_n}) && \text{by Def. 88} \\
&= \psi(\mathcal{A}_{\neg Q\neg,X_1,\dots,X_n}) && \text{by Def. 11} \\
&= 1 - \psi(\mathcal{A}_{Q\neg,X_1,\dots,X_n}) && \text{by L-139 and (ψ-2)} \\
&= 1 - \psi(\mathcal{A}_{Q,X_1,\dots,X_{n-1},\neg X_n}) && \text{by L-142} \\
&= 1 - \mathcal{F}_\psi(Q)(X_1,\dots,X_{n-1},\neg X_n) && \text{by Def. 88} \\
&= \neg\mathcal{F}_\psi(Q)(X_1,\dots,X_{n-1},\neg X_n)\,.
\end{aligned}$$

This completes the proof that $\mathcal{F}_\psi$ satisfies (Z-3).

B.11 Proof of Theorem 102

Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-5). In order to see that (ψ-5′) is valid, we consider $A \in \mathbb{A}$. We notice that for all $z \in \mathbf{I}$,

$$\sup \Box A(z) = \min(\sup A(z), \tfrac{1}{2}) = \boxplus A(z)$$

and hence

$$\begin{aligned} \boxplus\Box A(z) &= \min(\sup \Box A(z), \tfrac{1}{2}) \\ &= \min(\sup A(z), \tfrac{1}{2}), \tfrac{1}{2}) \\ &= \min(\sup A(z), \tfrac{1}{2}) \\ &= \boxplus A(z)\,, \end{aligned}$$

which is apparent from Def. 90 and Def. 91. Because $z \in \mathbf{I}$ was arbitrary, this proves that

$$\boxplus\Box A = \boxplus A\,.$$

Hence $\psi(A) = \psi(\boxplus A) = \psi(\boxplus\Box A) = \psi(\Box A)$, because ψ satisfies (ψ-5).

B.12 Proof of Theorem 103

Lemma 143 *Let $E \neq \varnothing$ be some base set, $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ and $Y_1, Y_2 \in \mathcal{P}(E)$. Further abbreviate $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$. Then*

$$\Xi_Y(X) \geq \Xi_{Y_1,Y_2}(X_1, X_2)\,.$$

Proof Let us recall from (61) that

$$\begin{aligned} &\Xi_Y(X) = \inf\{\delta_{X,Y}(e) : e \in E\} \\ &\Xi_{Y_1,Y_2}(X_1, X_2) = \inf\{\min(\delta_{X_1,Y_1}(e), \delta_{X_2,Y_2}(e)) : e \in E\}\,. \end{aligned}$$

It is hence sufficient to show that $\delta_{X,Y}(e) \geq \min(\delta_{X_1,Y_1}(e), \delta_{X_2,Y_2}(e))$ for all $e \in E$. Hence consider $e \in E$. It is convenient to discern the following four cases.

a.: $e \notin Y_1$ **and** $e \notin Y_2$**.**
Hence $e \notin Y = Y_1 \cup Y_2$. Therefore

$$\begin{aligned} \delta_{X,Y}(e) &= 1 - \mu_X(e) && \text{by (60) and } e \notin Y \\ &= 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) && \text{because } X = X_1 \cup X_2 \\ &= \min(1 - \mu_{X_1}(e), 1 - \mu_{X_2}(e)) && \text{by De Morgan's law} \\ &= \min(\delta_{X_1,Y_1}(e), \delta_{X_2,Y_2}(e))\,. && \text{by (60), } e \notin X_1 \text{ and } e \notin X_2 \end{aligned}$$

b.: $e \in Y_1$ **and** $e \notin Y_2$**.**
In this case, $e \in Y = Y_1 \cup Y_2$. Therefore

$$\begin{aligned}
\delta_{X,Y}(e) &= \mu_X(e) && \text{by (60) and } e \in Y \\
&= \max(\mu_{X_1}(e), \mu_{X_2}(e)) && \text{because } X = X_1 \cup X_2 \\
&\geq \mu_{X_1}(e) \\
&\geq \min(\mu_{X_1}(e), 1 - \mu_{X_2}(e)) \\
&= \min(\delta_{X_1,Y_1}(e), \delta_{X_2,Y_2}(e))\,. && \text{by (60), } e \in Y_1 \text{ and } e \notin Y_2
\end{aligned}$$

c.: $e \notin Y_1$ **and** $e \in Y_2$**.**
The proof of this case is analogous to that of **b.**, exchanging the roles of X_1, Y_1 and X_2, Y_2.

d.: $e \in Y_1$ **and** $e \in Y_2$**.**
In this case, $e \in Y = Y_1 \cup Y_2$. Hence

$$\begin{aligned}
\delta_{X,Y}(e) &= \mu_X(e) && \text{by (60), } e \in Y \\
&= \max(\mu_{X_1}(e), \mu_{X_2}(e)) && \text{because } X = X_1 \cup X_2 \\
&\geq \min(\mu_{X_1}(e), \mu_{X_2}(e)) \\
&= \min(\delta_{X_1,Y_1}(e), \delta_{X_2,Y_2}(e))\,, && \text{by (60), } e \in Y_1 \text{ and } e \in Y_2
\end{aligned}$$

which completes the proof of the lemma.

Lemma 144 *Let $E \neq \varnothing$ be some base set, $Y \in \mathcal{P}(E)$ and $X = X_1 \cup X_2$, where $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Furter suppose that $\Xi_Y(X) \geq r$ for a given $r \in [0, \frac{1}{2}]$. Then there exist $Y_1, Y_2 \in \mathcal{P}(E)$ with $Y = Y_1 \cup Y_2$ and $\Xi_{Y_1,Y_2}(X_1, X_2) \geq r$.*

Proof Let us abbreviate

$$\begin{aligned}
Y_1 &= \{e \in Y : \mu_{X_1}(e) \geq \mu_{X_2}(e) \vee \mu_{X_1}(e) \geq \tfrac{1}{2}\} \\
&= \{e \in Y : \mu_{X_1}(e) \geq \min(\mu_{X_2}(e), \tfrac{1}{2})\}
\end{aligned} \tag{790}$$

$$\begin{aligned}
Y_2 &= \{e \in Y : \mu_{X_2}(\geq)\mu_{X_1}(e) \vee \mu_{X_2}(e) \geq \tfrac{1}{2}\} \\
&= \{e \in Y : \mu_{X_2}(e) \geq \min(\mu_{X_1}(e), \tfrac{1}{2})\}\,.
\end{aligned} \tag{791}$$

Clearly $Y_1 \subseteq Y$ and $Y_2 \subseteq Y$, hence $Y_1 \cup Y_2 \subseteq Y$. Now consider $e \in Y$. If $\mu_{X_1}(e) \geq \mu_{X_2}(e)$, then $e \in Y_1$ and hence $e \in Y_1 \cup Y_2$. If $\mu_{X_1}(e) < \mu_{X_2}(e)$, then $e \in Y_2$ and hence $e \in Y_1 \cup Y_2$. This proves that $Y \subseteq Y_1 \cup Y_2$. Combining this with the above $Y_1 \cup Y_2 \subseteq Y$, we obtain the desired $Y = Y_1 \cup Y_2$. It remains to be shown that $\Xi_{Y_1,Y_2}(X_1, X_2) \geq r$. To this end, let us first prove that $\Xi_{Y_1}(X_1) \geq r$. By (61), it is sufficient to show that $\delta_{X_1,Y_1}(e) \geq r$ for all $e \in E$. Hence let $e \in E$. We notice from $\Xi_Y(X) \geq r$ and (61) that

$$\delta_{X,Y}(e) \geq r\,. \tag{792}$$

We shall discern the following cases.

a.: $e \in Y$ **and** $\mu_{X_1}(e) \geq \mu_{X_2}(e)$**.**
In this case we know from (790) that $e \in Y_1$ and hence

$$\begin{aligned}
\delta_{X_1,Y_1}(e) &= \mu_{X_1}(e) && \text{by (60)} \\
&= \max(\mu_{X_1}(e), \mu_{X_2}(e)) && \text{by assumption of case } \textbf{a.} \\
&= \mu_X(e) && \text{because } X = X_1 \cup X_2 \\
&= \delta_{X,Y}(e) && \text{by (60) and } e \in Y \\
&\geq r\,. && \text{by (792)}
\end{aligned}$$

b.: $e \in Y$ **and** $\mu_{X_1}(e) \geq \frac{1}{2}$**.**
Again, we know from (790) that $e \in Y_1$. Hence

$$\begin{aligned}
\delta_{X_1,Y_1}(e) &= \mu_{X_1}(e) && \text{by (60)} \\
&\geq \tfrac{1}{2} && \text{by assumption of case } \textbf{b.} \\
&\geq r\,,
\end{aligned}$$

because $r \leq \frac{1}{2}$ by assumption of the lemma.

c.: $e \in Y$**,** $\mu_{X_1}(e) < \mu_{X_2}(e)$ **and** $\mu_{X_1}(e) < \frac{1}{2}$**.**
Then $e \notin Y_1$ by (790). Therefore

$$\begin{aligned}
\delta_{X_1,Y_1}(e) &= 1 - \mu_{X_1}(e) && \text{by (60)} \\
&> \tfrac{1}{2} && \text{by assumption of case } \textbf{c.} \\
&\geq r\,,
\end{aligned}$$

again recalling that $r \leq \frac{1}{2}$ by assumption of the lemma.

d.: $e \notin Y$**.**
In this case, we obtain from (790) that $e \notin Y_1$. Hence

$$\begin{aligned}
\delta_{X_1,Y_1}(e) &= 1 - \mu_{X_1}(e) && \text{by (60)} \\
&\geq \min(1 - \mu_{X_1}(e), 1 - \mu_{X_2}(e)) \\
&= 1 - \max(\mu_{X_1}(e), \mu_{X_2}(e)) && \text{by De Morgan's law} \\
&= 1 - \mu_X(e) && \text{because } X = X_1 \cup X_2 \\
&= \delta_{X,Y}(e) && \text{by (60) and } e \notin Y \\
&\geq r\,. && \text{by (792)}
\end{aligned}$$

This proves that indeed $\delta_{X_1,Y_1}(e) \geq r$ for all $e \in E$, and hence $\Xi_{Y_1}(X_1) \geq r$ by (61). By the very same reasoning, it can be shown that $\Xi_{Y_2}(X_2) \geq r$ as well, noticing the apparent symmetry between (790) and (791). Hence $\Xi_{Y_1,Y_2}(X_1, X_2) = \min(\Xi_{Y_1}(X_1), \Xi_{Y_2}(X_2)) \geq r$, as desired.

Lemma 145 *Suppose* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-2), (ψ-3) *and* (ψ-5′). *Then for all* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *and all* $X_1, \ldots, X_{n+1}$*,*

$$\Box A_{Q\cup,X_1,\ldots,X_{n+1}} = \Box A_{Q,X_1,\ldots,X_{n-1},X_n \cup X_{n+1}}\,.$$

Proof To see this, let $z \in \mathbf{I}$. We can then show that

$$\Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z) = \Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)$$

by proving that the set on the left hand side of the equation is contained in the set on the right hand side and vice versa.

a.: $\Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z) \subseteq \Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)$**.**
We consider $r \in \Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z)$. We then know from Def. 90 that $r \leq \frac{1}{2}$ and that there exists $r' \in A_{Q\cup,X_1,\ldots,X_{n+1}}(z)$ with $r \leq r'$. In turn, we obtain from Def. 86 that there exist $(Y_1,\ldots,Y_{n+1}) \in Q\cup^{-1}(z)$ with $\Xi_{Y_1,\ldots,Y_{n+1}}(X_1,\ldots,X_{n+1}) = r' \geq r$. Now we set $Y'_n = Y_n \cup Y_{n+1}$ and $X'_n = X_n \cup X_{n+1}$. Then

$$\begin{aligned} Q(Y_1,\ldots,Y_{n-1},Y'_n) &= Q(Y_1,\ldots,Y_{n-1},Y_n \cup Y_{n+1}) \\ &= Q\cup(Y_1,\ldots,Y_{n+1}) \\ &= z \end{aligned}$$

because $(Y_1,\ldots,Y_{n+1})$ has been chosen from $Q\cup^{-1}(z)$. We conclude that

$$(Y_1,\ldots,Y_{n-1},Y'_n) \in Q^{-1}(z)\,.$$

Let us now investigate $r'' = \Xi_{Y_1,\ldots,Y_{n-1},Y'_n}(X_1,\ldots,X_{n-1},X'_n)$. We proceed as follows.

$$\begin{aligned} r'' &= \Xi_{Y_1,\ldots,Y_{n-1},Y'_n}(X_1,\ldots,X_{n-1},X'_n) \\ &= \min(\bigwedge_{i=1}^{n-1} Y_iX_i, \Xi_{Y'_n}(X'_n)) && \text{by Def. 83} \\ &\geq \min(\bigwedge_{i=1}^{n-1} Y_iX_i, \min(\Xi_{Y_n}(X_n), \Xi_{Y_{n+1}}(X_{n+1}))) && \text{by L-143} \\ &= \Xi_{Y_1,\ldots,Y_{n+1}}(X_1,\ldots,X_{n+1})\,. && \text{by Def. 83} \end{aligned}$$

Hence there exists $r'' \in A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)$ with $r'' \geq r' \geq r$. Because $r \leq \frac{1}{2}$, we conclude from Def. 90 that $r \in \Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)$. Because $r \in \Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z)$ was arbitrarily chosen, this proves that indeed

$$\Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z) \subseteq \Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)\,.$$

b.: $\Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z) \subseteq \Box A_{Q\cup,X_1,\ldots,X_{n+1}}(z)$**.**
Again, we abbreviate $X'_n = X_n \cup X_{n+1}$. Now consider $r \in \Box A_{Q,X_1,\ldots,X_{n-1},X_n\cup X_{n+1}}(z)$. We then know from Def. 90 that $r \leq \frac{1}{2}$ and that there exists $(Y_1,\ldots,Y_{n-1},Y'_n) \in Q^{-1}(z)$ with

$$r' = \Xi_{Y_1,\ldots,Y_{n-1},Y'_n}(X_1,\ldots,X_{n-1},X'_n) \geq r\,. \tag{793}$$

In particular,

$$\begin{aligned} \Xi_{Y'_n}(X'_n) &\geq \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \Xi_{Y'_n}(X'_n)) \\ &= \Xi_{Y_1,\ldots,Y_{n-1},Y'_n}(X_1,\ldots,X_{n-1},X'_n) && \text{by Def. 83} \\ &\geq r\,. \end{aligned}$$

We can now apply lemma L-144, which establishes the existence of $Y_n, Y_{n+1} \in \mathcal{P}(E)$ with $Y'_n = Y_n \cup Y_{n+1}$ and

$$\Xi_{Y_n,Y_{n+1}}(X_n, X_{n+1}) \geq r\,. \tag{794}$$

Therefore

$$\begin{aligned}
&\Xi_{Y_1,\dots,Y_{n+1}}(X_1,\dots,X_{n+1}) \\
&= \bigwedge_{i=1}^{n+1} \Xi_{Y_i}(X_i) && \text{by Def. 83} \\
&= \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \bigwedge_{i=n}^{n+1} Y_i X_i) \\
&= \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \Xi_{Y_n,Y_{n+1}}(X_n, X_{n+1})) && \text{by Def. 83} \\
&\geq \min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), r) && \text{by (794)} \\
&\geq \min(\min(\bigwedge_{i=1}^{n-1} \Xi_{Y_i}(X_i), \Xi_{Y'_n}(X'_n)), r) \\
&= \min(\Xi_{Y_1,\dots,Y_{n-1},Y'_n}(X_1,\dots,X_{n-1},X'_n), r) && \text{by Def. 83} \\
&= r\,. && \text{by (793)}
\end{aligned}$$

Hence $r'' = \Xi_{Y_1,\dots,Y_{n+1}}(\geq)r$. We further notice that

$$\begin{aligned}
Q\cup(Y_1,\dots,Y_{n-1}) &= Q(Y_1,\dots,Y_{n-1},Y_n \cup Y_{n+1}) \\
&= Q(Y_1,\dots,Y_{n-1},Y'_n) = z\,,
\end{aligned}$$

i.e. $(Y_1,\dots,Y_{n+1}) \in Q\cup^{-1}(z)$. It is then apparent from Def. 86 that

$$r'' \in A_{Q\cup,X_1,\dots,X_{n+1}}(z)\,.$$

Because $r'' \geq r$ and $r \leq \frac{1}{2}$, this proves the desired $r \in \Box\, A_{Q\cup,X_1,\dots,X_{n+1}}(z)$.

Proof of Theorem 103

Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-2), (ψ-3) and (ψ-5′). We then know from Th-99 that $\mathcal{F}_\psi$ induces the standard fuzzy disjunction $\widetilde{\mathcal{F}}_\psi(\vee) = \vee$, where $x \vee y = \max(x,y)$. Now consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and a choice of fuzzy arguments $X_1,\dots,X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\psi(Q\cup)(X_1,\dots,X_{n+1}) &= \psi(A_{Q\cup,X_1,\dots,X_{n+1}}) && \text{by Def. 88} \\
&= \psi(\Box\, A_{Q\cup,X_1,\dots,X_{n+1}}) && \text{by (ψ-5′)} \\
&= \psi(\Box\, A_{Q,X_1,\dots,X_{n-1},X_n\cup X_{n+1}}) && \text{by L-145} \\
&= \psi(A_{Q,X_1,\dots,X_{n-1},X_n\cup X_{n+1}}) && \text{by (ψ-5′)} \\
&= \mathcal{F}_\psi(Q)(X_1,\dots,X_{n-1},X_n \cup X_{n+1})\,. && \text{by Def. 88}
\end{aligned}$$

Hence $\mathcal{F}_\psi$ satisfies (Z-4), as desired.

B.13 Proof of Theorem 104

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a mapping which satisfies (ψ-4). Now consider a choice of semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ such that $Q \leq Q'$. In order to prove that $\mathcal{F}_\psi(Q) \leq \mathcal{F}_\psi(Q')$, let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be a choice of fuzzy arguments. We now compare $A_{Q,X_1,\ldots,X_n}$ and $A_{Q',X_1,\ldots,X_n}$.
Hence let $z \in \mathbf{I}$ and consider some $r \in A_{Q,X_1,\ldots,X_n}(z)$. By Def. 86, there exists a choice of $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ with $r = \Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)$ and $z = Q(Y_1, \ldots, Y_n)$. Because $Q \leq Q'$, we know that $z' = Q'(Y_1, \ldots, Y_n) \geq z$. Further noticing that $r = \Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)$, we obtain from Def. 86 that $r \in A_{Q',X_1,\ldots,X_n}(z')$ for $z' \geq z$. Hence condition **a.** of Def. 89 is satisfied.
To see that condition **b.** of Def. 89 is also satisfied, consider some $z' \in \mathbf{I}$ and $r \in A_{Q',X_1,\ldots,X_n}(z')$. By Def. 86, there exist $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ with $z' = Q'(Y_1, \ldots, Y_n)$ and $r = \Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n)$. Because $Q \leq Q'$, we obtain for $z = Q(Y_1, \ldots, Y_n)$ that $z \leq z'$. Recalling Def. 86, $r \in A_{Q,X_1,\ldots,X_n}(z)$.
Hence both conditions stated in Def. 89 are valid, and

$$A_{Q,X_1,\ldots,X_n} \sqsubseteq A_{Q',X_1,\ldots,X_n} . \tag{795}$$

Therefore

$$\begin{aligned} \mathcal{F}_\psi(Q)(X_1, \ldots, X_n) &= \psi(A_{Q,X_1,\ldots,X_n}) && \text{by Def. 88} \\ &\leq \psi(A_{Q',X_1,\ldots,X_n}) && \text{by (795) and } (\psi\text{-4}) \\ &= \mathcal{F}_\psi(Q')(X_1, \ldots, X_n) . && \text{by Def. 88} \end{aligned}$$

B.14 Proof of Theorem 105

Lemma 146 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $\beta : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ a permutation. Further suppose that $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1, \ldots, Y_n) = Q(Y_{\beta(1)}, \ldots, Y_{\beta(n)}) \tag{796}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Then

$$\mathcal{F}_\psi(Q')(X_1, \ldots, X_n) = \mathcal{F}_\psi(Q)(X_{\beta(1)}, \ldots, X_{\beta(n)})$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof We first notice that

$$\begin{aligned} &\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) \\ &= \min\{\Xi_{Y_i}(X_i) : i \in \{1, \ldots, n\}\} && \text{by Def. 83} \\ &= \min\{\Xi_{Y_{\beta(i)}}(X_{\beta(i)}) : i \in \{1, \ldots, n\}\} && \beta \text{ is permutation of } \{1, \ldots, n\} \\ &= \Xi_{Y_{\beta(1)},\ldots,Y_{\beta(n)}}(X_{\beta(1)}, \ldots, X_{\beta(n)}) , && \text{by Def. 83} \end{aligned}$$

i.e.

$$\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) = \Xi_{Y_{\beta(1)},\ldots,Y_{\beta(n)}}(X_{\beta(1)}, \ldots, X_{\beta(n)}) \tag{797}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Now let $z \in \mathbf{I}$. In order to prove that

$$A_{Q',X_1,\dots,X_n}(z) \subseteq A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}(z)\,, \tag{798}$$

let $r \in A_{Q',X_1,\dots,X_n}(z)$. By Def. 86, there exist $Y_1, \dots, Y_n \in \mathcal{P}(E)$ with

$$Q'(Y_1, \dots, Y_n) = z \qquad \text{and} \qquad \Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) = r\,.$$

Now consider $(Y_{\beta(1)}, \dots, Y_{\beta(n)}) \in \mathcal{P}(E)^n$. We know from (796) and $Q'(Y_1, \dots, Y_n) = z$ that $Q(Y_{\beta(1)}, \dots, Y_{\beta(n)}) = z$. In addition, we know from (797) that

$$\Xi_{Y_{\beta(1)},\dots,Y_{\beta(n)}}(X_{\beta(1)}, \dots, X_{\beta(n)}) = \Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) = r\,.$$

Hence indeed $r \in A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}(z)$, which proves that (798) is valid. It remains to be shown that

$$A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}(z) \subseteq A_{Q',X_1,\dots,X_n}(z)\,. \tag{799}$$

Hence let $r \in A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}(z)$. By Def. 86, there exist $Y_1, \dots, Y_n \in \mathcal{P}(E)$ with $Q(Y_1, \dots, Y_n) = z$ and $\Xi_{Y_1,\dots,Y_n}(X_{\beta(1)}, \dots, X_{\beta(n)}) = r$. We now define $Z_1, \dots, Z_n \in \mathcal{P}(E)$ by

$$Z_j = Y_{\beta^{-1}(j)} \tag{800}$$

for $j \in \{1, \dots, n\}$. In particular

$$Y_i = Y_{\beta^{-1}(\beta(i))} = Z_{\beta(i)} \tag{801}$$

for all $i \in \{1, \dots, n\}$. Therefore

$$\begin{aligned} Q'(Z_1, \dots, Z_n) &= Q(Z_{\beta(1)}, \dots, Z_{\beta(n)}) && \text{by (796)} \\ &= Q(Y_1, \dots, Y_n) && \text{by (801)} \\ &= z\,. \end{aligned}$$

We also notice that

$$\begin{aligned} \Xi_{Z_1,\dots,Z_n}(X_1, \dots, X_n) &= \Xi_{Z_{\beta(1)},\dots,Z_{\beta(n)}}(X_{\beta(1)}, \dots, X_{\beta(n)}) && \text{by (797)} \\ &= \Xi_{Y_1,\dots,Y_n}(X_{\beta(1)}, \dots, X_{\beta(n)}) && \text{by (801)} \\ &= r\,. \end{aligned}$$

Hence $r \in A_{Q',X_1,\dots,X_n}(z)$, and (799) holds, as desired. Combining (798) and (799) then proves the equation

$$A_{Q',X_1,\dots,X_n} = A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}\,, \tag{802}$$

noticing that $z \in \mathbf{I}$ was arbitrary. Therefore

$$\begin{aligned} \mathcal{F}_\psi(Q')(X_1, \dots, X_n) &= \psi(A_{Q',X_1,\dots,X_n}) && \text{by Def. 88} \\ &= \psi(A_{Q,X_{\beta(1)},\dots,X_{\beta(n)}}) && \text{by (802)} \\ &= \mathcal{F}_\psi(Q)(X_{\beta(1)}, \dots, X_{\beta(n)})\,. && \text{by Def. 88} \end{aligned}$$

Lemma 147 *Suppose* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *satisfies* (ψ-2), (ψ-3) *and* (ψ-5′). *Then for all* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$, $\mathcal{F}_\psi(Q\cap) = \mathcal{F}_\psi(Q)\cap$.

Proof To see this, we observe that

$$Q\cap = Q\neg\cup\neg\tau_n\neg\tau_n$$

and

$$\mathcal{F}_\psi(Q)\cap = \mathcal{F}_\psi(Q)\neg\cup\neg\tau_n\neg\tau_n\,.$$

Therefore $\mathcal{F}_\psi(Q\cap) = \mathcal{F}_\psi(Q\neg\cup\neg\tau_n\neg\tau_n) = \mathcal{F}_\psi(Q)\neg\cup\neg\tau_n\neg\tau_n = \mathcal{F}_\psi(Q)\cap$, where the middle equation is known to hold from Th-103, Th-99, L-142 and L-146.

Proof of Theorem 105

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given andf suppose that (ψ-2), (ψ-3), (ψ-4) and (ψ-5′) are valid. In order to show that $\mathcal{F}_\psi$ satisfies (Z-5), let us consider a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Further suppose that Q is nonincreasing in its n-th argument. It has to be shown that $\mathcal{F}_\psi(Q)$ is nonincreasing in its n-th argument as well. Hence let $X_1, \dots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ with $X_n \subseteq X_{n+1}$. In particular

$$X_n = X_n \cap X'_n \tag{803}$$

$$X_{n+1} = X_n \cup X'_n\,. \tag{804}$$

Let us also notice that Q's being nonincreasing in the n-th argument entails that

$$Q\cap \geq Q\cup \tag{805}$$

which is apparent from Def. 14. Therefore

$$\begin{aligned}
\mathcal{F}_\psi(Q)(X_1,\dots,X_n) &= \mathcal{F}_\psi(X_1,\dots,X_{n-1},X_n\cap X'_n) && \text{by (803)}\\
&= \mathcal{F}_\psi(Q)\cap(X_1,\dots,X_{n-1},X_n,X'_n) && \text{by Def. 12}\\
&= \mathcal{F}_\psi(Q\cap)(X_1,\dots,X_{n-1},X_n,X'_n) && \text{by L-147}\\
&\geq \mathcal{F}_\psi(Q\cup)(X_1,\dots,X_{n-1},X_n,X'_n) && \text{by Th-104 and (805)}\\
&= \mathcal{F}_\psi(Q)\cup(X_1,\dots,X_{n-1},X_n,X'_n) && \text{by Th-103}\\
&= \mathcal{F}_\psi(Q)(X_1,\dots,X_{n-1},X_n\cup X'_n) && \text{by Def. 12}\\
&= \mathcal{F}_\psi(Q)(X_1,\dots,X_n)\,. && \text{by (804)}
\end{aligned}$$

B.15 Proof of Theorem 106

Lemma 148 *Let $E, E' \neq \varnothing$ be given base sets and $f : E \longrightarrow E'$. Then*

$$\Xi_Y(X) \leq \Xi_{\widehat{f}(Y)}(\hat{\hat{f}}(X))$$

for all $X \in \widetilde{\mathcal{P}}(E)$ and $Y \in \mathcal{P}(E)$.

Proof We first observe that for all $e' \in \widehat{f}(Y)$, there exists $e_0 \in f^{-1}(e')$ such that $e_0 \in Y$. Therefore

$$\begin{aligned}
\mu_{\hat{\hat{f}}(X)}(e') &= \sup\{\mu_X(e) : e \in f^{-1}(e')\} && \text{by (3)}\\
&\geq \mu_X(e_0) && \text{because } e_0 \in f^{-1}(e')\\
&= \delta_{X,Y}(e_0) && \text{by (60) and } e_0 \in Y
\end{aligned}$$

and hence

$$\mu_{\hat{\hat{f}}(X)}(e') \geq \Xi_X(Y)\,, \tag{806}$$

recalling (61).
Next we consider the case that $e' \in E'$, $e \notin \widehat{f}(Y)$. We then know from Def. 15 that $f^{-1}(e') \cap Y = \varnothing$, and hence

$$\delta_{X,Y}(e) = 1 - \mu_X(e) \tag{807}$$

for all $e \in f^{-1}(e')$, see (60). Therefore

$$\begin{aligned}
1 - \mu_{\hat{\hat{f}}(X)}(e') &= 1 - \sup\{\mu_X(e) : e \in f^{-1}(e')\} && \text{by (3)}\\
&= \inf\{1 - \mu_X(e) : e \in f^{-1}(e')\} && \text{by De Morgan's law}\\
&= \inf\{\delta_{X,Y}(e) : e \in f^{-1}(e')\} && \text{by (807)}\\
&\geq \inf\{\delta_{X,Y}(e) : e \in E\}\,,
\end{aligned}$$

and by (61),

$$1 - \mu_{\hat{\hat{f}}(X)}(e') \geq \Xi_Y(X)\,. \tag{808}$$

Finally

$$\begin{aligned}
&\Xi_{\widehat{f}(Y)}(\hat{\hat{f}}(X))\\
&= \min(\inf\{\mu_{\hat{\hat{f}}(X)}(e') : e' \in \widehat{f}(Y)\},\\
&\qquad \inf\{1 - \mu_{\hat{\hat{f}}(X)}(e') : e' \notin \widehat{f}(Y)\}) && \text{by Def. 83}\\
&\geq \Xi_Y(X)\,. && \text{by (806) and (808)}
\end{aligned}$$

Lemma 149 *Let $E, E' \neq \varnothing$ be given base sets, $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$, $f_1, \ldots, f_n : E \longrightarrow E'$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$\widehat{\boxplus}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \ldots, X_n}(z) \leq \widehat{\boxplus}_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}(z)$$

for all $z \in \mathbf{I}$.

Proof Let $r \in A_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \dots, X_n}(z)$. Then there exist $Y_1, \dots, Y_n \in \mathcal{P}(E)$ with $Q(\widehat{f}_1(Y_1), \dots, \widehat{f_n}(X_n)) = z$ and $\Xi_{Y_1, \dots, Y_n}(X_1, \dots, X_n) = r$. Hence $Z_i = \widehat{f}_i(Y_i)$ $(i = 1, \dots, n)$ satisfy $Q(Z_1, \dots, Z_n) = z$. From L-148 and Def. 83, we know that $r' = \Xi_{Z_1, \dots, Z_n}(\hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)) \geq \Xi_{Y_1, \dots, Y_n}(X_1, \dots, X_n) = r$. Hence there exists $r' \in A_{Q, \hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)}(z)$ with $r' \geq r$. Because $r \in A_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \dots, X_n}(z)$ was chosen arbitrarily, this proves that

$$\sup A_{Q, \hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)}(z) \geq \sup A_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \dots, X_n}(z) \,. \tag{809}$$

Therefore

$$\begin{aligned}
&\widehat{\boxplus}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \dots, X_n}(z) \\
&= \min(\sup A_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \dots, X_n}(z), \tfrac{1}{2}) && \text{by (73) and (75)} \\
&\leq \min(A_{Q, \hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)}(z), \tfrac{1}{2}) && \text{by (809)} \\
&= \widehat{\boxplus}_{Q, \hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)}(z) \,. && \text{by (73) and (75)}
\end{aligned}$$

Lemma 150 *Let $E, E' \neq \varnothing$, $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $Z \in \mathcal{P}(E')$. Then*

$$\min(\Xi_Z(\hat{\hat{f}}(X)), \tfrac{1}{2}) \leq \sup\{\Xi_Y(X) : \widehat{f}(Y) = Z\} \,.$$

Proof Suppose that $Z \not\subseteq \operatorname{Im} f$, i.e. there exists $e' \in Z$ with $f^{-1}(e) = \varnothing$. Then

$$\begin{aligned}
\Xi_Z(\hat{\hat{f}}(X)) &\leq \mu_{\hat{\hat{f}}(X)}(e') && \text{by Def. 83} \\
&= \sup\{\mu_X(e) : e \in f^{-1}(e')\} && \text{by (3)} \\
&= \sup \varnothing && \text{because } f^{-1}(e) = \varnothing \\
&= 0 \,,
\end{aligned}$$

i.e.

$$\Xi_Z(\hat{\hat{f}}(X)) = 0 \,. \tag{810}$$

Hence trivially

$$\sup\{\Xi_Y(X) : \widehat{f}(Y) = Z\} \geq 0 = \min(0, \tfrac{1}{2}) = \min(\Xi_Z(\hat{\hat{f}}(X)), \tfrac{1}{2}) \,.$$

Now let us consider the remaining case that $Z \subseteq \operatorname{Im} f = \{f(e) : e \in E\}$, i.e. for all $e' \in Z$,

$$f^{-1}(e') \neq \varnothing. \tag{811}$$

It is then sufficient to show that for all $\varepsilon > 0$ there exists $Y \in \mathcal{P}(E)$ with $\widehat{f}(Y) = Z$ and $\Xi_Y(X) \geq r - \varepsilon$, where $r = \min(\Xi_Z(\hat{\hat{f}}(X)), \frac{1}{2})$.
To see this, we first introduce abbreviations

$$V = f^{-1}(Z) = \{e \in E : f(e) \in Z\} \tag{812}$$
$$Y = \{e \in V : \mu_X(e) \geq \tfrac{1}{2} \vee \mu_X(e) \geq \mu_{\hat{\hat{f}}(X)}(f(e)) - \varepsilon\}\,. \tag{813}$$

It is immediate from (812) and Def. 15 that $\widehat{f}(V) = Z$. We then obtain from $Y \subseteq V$ that $\widehat{f}(Y) \subseteq Z$. Now consider $e' \in Z$. By (811), $f^{-1}(e') \neq \varnothing$. In particular $\{\mu_X(e) : e \in f^{-1}(e')\} \neq \varnothing$. We may hence conclude that there exists $e_0 \in f^{-1}(e')$ with $\mu_X(e_0) > \sup\{\mu_X(e) : e \in f^{-1}(e')\} - \varepsilon$. Hence by (3), $\mu_X(e_0) > \mu_{\hat{\hat{X}}}(()e') - \varepsilon$. This proves that $e_0 \in Y$, see (813). Recalling that $e_0 \in f^{-1}(e')$, i.e. $f(e_0) = e'$, we then obtain that $e' \in \widehat{f}(Y)$. Because $e' \in Z$ was arbitrary, this proves that $Z \subseteq \widehat{f}(Y)$. Combining this with the above inequation yields the desired $\widehat{f}(Y) = Z$.
Now consider $e \in Y$. We discern two cases.

a. If $\mu_X(e) \geq \frac{1}{2}$, then

$$\delta_{X,Y}(e) = \mu_X(e) \geq \tfrac{1}{2} \geq r > r - \varepsilon$$

by (60) and above definition of r.

b. If $\mu_X(e) \geq \mu_{\hat{\hat{f}}(X)}(f(e)) - \varepsilon$, then

$$\begin{aligned}
\delta_{X,Y}(e) &= \mu_X(e) && \text{by (60) because } e \in Y \\
&\geq \mu_{\hat{\hat{f}}(X)}(f(e)) - \varepsilon && \text{by assumption of case b.} \\
&\geq \Xi_Z(\hat{\hat{f}}(X)) - \varepsilon\,,
\end{aligned}$$

which is apparent from Def. 83 and (3) because $e \in Y$ entails that $f(e) \in \widehat{f}(Y) = Z$. We then proceed as follows.

$$\begin{aligned}
\delta_{X,Y}(e) &\geq \Xi_Z(\hat{\hat{f}}(X)) - \varepsilon \\
&\geq \min(\Xi_Z(\hat{\hat{f}}(X)), \tfrac{1}{2}) - \varepsilon \\
&= r - \varepsilon\,.
\end{aligned}$$

Hence indeed

$$\delta_{X,Y}(e) \geq r - \varepsilon \tag{814}$$

for all $e \in Y$.
Now suppose that $e \notin Y$. Again, it is convenient to discern two cases.

- $e \notin V$.
  Then in particular $f(e) \notin Z$, recalling equation (812). Abbreviating $e' = f(e)$, we obtain

$$\begin{aligned}
\delta_{X,Y}(e) &= 1 - \mu_X(e) && \text{by (60)} \\
&\geq \inf\{1 - \mu_X(e'') : e'' \in f^{-1}(e')\} && \text{because } f(e) = e' \\
&= 1 - \sup\{\mu_X(e'') : e'' \in f^{-1}(e')\} && \text{by De Morgan's law} \\
&= 1 - \mu_{\hat{\hat{f}}(X)}(()e') && \text{by (3)} \\
&= \delta_{\hat{\hat{f}}(X),Z}(e') && \text{by (60) and } e' \notin V \\
&\geq \Xi_Z(\hat{\hat{f}}(X)) && \text{by (61)} \\
&\geq \min(\Xi_Z(\hat{\hat{f}}(X)), \tfrac{1}{2}) \\
&= r \\
&> r - \varepsilon\,.
\end{aligned}$$

- $e \in V$, $\mu_X(e) < \frac{1}{2}$ and $\mu_X(e) < \mu_{\hat{\hat{f}}(X)}(f(e)) - \varepsilon$.
  This case is trivial because

$$\delta_{X,Y}(e) = 1 - \mu_X(e) > \tfrac{1}{2} \geq r > r - \varepsilon\,.$$

We have thus shown that

$$\delta_{X,Y}(e) \geq r - \varepsilon \tag{815}$$

for all $e \in E \setminus Y$. Therefore

$$\begin{aligned}
\Xi_X(Y) &= \inf\{\delta_{X,Y}(e) : e \in E\} && \text{by (61)} \\
&\geq r - \varepsilon\,, && \text{by (814), (815)}
\end{aligned}$$

as desired.

Lemma 151 *Let $E, E' \neq \varnothing$ be given base sets, $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$, $f_1, \ldots, f_n : E \longrightarrow E'$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$\widehat{\boxplus}_{Q,\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n)}(z) \leq \widehat{\boxplus}_{Q \circ \times_{i=1}^{n} \hat{f}_i, X_1,\ldots,X_n}(z)$$

for all $z \in \mathbf{I}$.

Proof We notice that

$$\begin{aligned}
\widehat{\boxplus}_{Q,\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n)}(z) &= \min(\sup A_{Q,\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n)}(z), \tfrac{1}{2}) && \text{by (73), (75)} \\
&= \sup\{\min(r, \tfrac{1}{2}) : r \in A_{Q,\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n)}(z)\}\,.
\end{aligned}$$

The claim of the lemma can hence be proven by showing that

$$\widehat{\boxplus}_{Q\circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\dots,X_n}(z) \geq \min(r, \tfrac{1}{2})$$

for all $r \in A_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}(z)\}$. Hence consider such choice of r. By Def. 86, there exist $Z_1,\dots,Z_n \in \mathcal{P}(E')$ with $QZ_1,\dots,Z_n = z$ and $\Xi_{Z_1,\dots,Z_n}(\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)) = r$. Now

$$\begin{aligned}
&\sup A_{Q\circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\dots,X_n}(z) \\
&= \sup\{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) : \\
&\qquad Q(\widehat{f_1}(Y_1),\dots,\widehat{f_n}(Y_n)) = z\} \\
&\geq \sup\{\Xi_{Y_1,\dots,Y_n}(X_1,\dots,X_n) : \\
&\qquad \widehat{f_1}(Y_1) = Z_1 \dots, \widehat{f_n}(Y_n) = Z_n\} \\
&= \sup\{\overset{n}{\underset{i=1}{\wedge}} \Xi_{Y_i}(X_i) : \\
&\qquad \widehat{f_1}(Y_1) = Z_1 \dots, \widehat{f_n}(Y_n) = Z_n\} \\
&= \overset{n}{\underset{i=1}{\wedge}} \sup\{\Xi_{Y_i}(X_i) : \widehat{f_i}(Y_i) = Z_i\} && \text{because the } Y_i \text{ can be} \\
& && \text{chosen independently} \\
&\geq \overset{n}{\underset{i=1}{\wedge}} \Xi_{Z_i}(\hat{\hat{f}}_i(X_i)) && \text{by L-150} \\
&= \Xi_{Z_1,\dots,Z_n}(\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)) \,. && \text{by Def. 83}
\end{aligned}$$

Proof of Theorem 106

Let $E, E' \neq \varnothing$ be given base sets, $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$, $f_1,\dots,f_n : E \longrightarrow E'$ and and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. The theorem is now a corollary of L-149 and L-151, which state that

$$\widehat{\boxplus}_{Q\circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\dots,X_n} = \widehat{\boxplus}_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)} \,.$$

Hence by (74) and Def. 91,

$$\boxplus_{Q\circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\dots,X_n} = \boxplus_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)} \,. \tag{816}$$

Finally

$$
\begin{aligned}
\mathcal{F}_\psi(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i)(X_1, \ldots, X_n) &= \psi(A_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \ldots, X_n}) && \text{by Def. 88} \\
&= \psi(\boxplus_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1, \ldots, X_n}) && \text{by } (\psi\text{-5}), (74) \\
&= \psi(\boxplus_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}) && \text{by } (816) \\
&= \psi(A_{Q, \hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)}) && \text{by } (\psi\text{-5}), (74) \\
&= \mathcal{F}_\psi(Q)(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n))\,. && \text{by Def. 88}
\end{aligned}
$$

B.16 Proof of Theorem 107

Suppose $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-1)–(ψ-5). Then by Th-102, ψ satisfies (ψ-5′) as well. We obtain from Th-96, Th-100, Th-101, Th-103, Th-105 and Th-106 that $\mathcal{F}_\psi$ satisfies (Z-1), (Z-2), (Z-3), (Z-4), (Z-5) and (Z-6), respectively. By Def. 17, $\mathcal{F}_\psi$ is a DFS. The proof is completed by recalling Th-99, from which we obtain that $\mathcal{F}_\psi$ is indeed a standard DFS.

B.17 Proof of Theorem 108

I will prove the theorem by contraposition. Hence let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that ψ does not satisfy (ψ-1). Then there exists $A \in \mathbb{A}$ with $D(A) = \{1\}$ and $\psi(A) \neq z_+$. Because $D(A) = \cup\{A(z) : z \in \mathbf{I}\} = \{1\}$, we know that $r_+ = 1$. By Th-93 and Def. 87, then, there exists a unique $z_+ \in \mathbf{I}$ with $1 = r_+ \in A(z_+)$, and $A(z) \cap (1 - r_+, 1] = A(z) \cap (0, 1]$ for $z \neq z_+$. Because $D(A) = \{1\}$, this proves that $A(z) = \varnothing$ for $z \neq z_+$.
We now define $Q : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ by $Q(\varnothing) = z_+$, where $\varnothing$ is the empty tuple. It is then apparent from Def. 86 that $A^{(0)}_{Q,\varnothing}(z_+) = \{1\}$ and $A^{(0)}_{Q,\varnothing}(z) = \varnothing$ for $z \neq z_+$. Hence $A^{(0)}_{Q,\varnothing} = A$. We can then proceed as follows.

$$
\begin{aligned}
\mathcal{F}_\psi(Q)(\varnothing) &= \psi(A^{(0)}_{Q,\varnothing}) && \text{by Def. 88} \\
&= \psi(A) && \text{because } A = A^{(0)}_{Q,\varnothing} \\
&\neq z_+ && \text{by assumption} \\
&= Q(\varnothing)\,.
\end{aligned}
$$

This proves that $\mathcal{F}_\psi$ does not satisfy (Z-1).

B.18 Proof of Theorem 109

Lemma 152 *Suppose that* $\mathcal{F}_\psi$ *satisfies* (Z-2) *and let* $x \in \mathbf{I}$*. Further suppose that* $A \in \mathbb{A}$ *is defined by*

$$A(z) = \begin{cases} \{x\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0,1) \\ \{1-x\} & : \quad z = 0 \end{cases}$$

for all $z \in \mathbf{I}$*. Then* $\psi(A) = x$*.*

Proof Let us consider the quantifier $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{I}$. We define a fuzzy argument set, $X \in \widetilde{\mathcal{P}}(\{*\})$, by $\mu_X(*) = x$. Because $\mathcal{P}(\{*\}) = \{\varnothing, \{*\}\}$ and $\pi_*(\varnothing) = 0$, $\pi_*(\{*\}) = 1$ by Def. 6, we obtain the following results for $A_{\pi_*,X}$.

$$\begin{aligned} A_{\pi_*,X}(1) &= \{\Xi_Y(X) : \pi_*(Y) = 1\} && \text{by Def. 86} \\ &= \{\Xi_{\{*\}}(X)\} && \text{by Def. 6} \\ &= \{\mu_X(*)\} && \text{by Def. 83} \\ &= x && \text{by definition of } X \\ &= A(1)\,. && \text{by assumption} \end{aligned}$$

For $z = 0$, we have

$$\begin{aligned} A_{\pi_*,X}(0) &= \{\Xi_Y(X) : \pi_*(Y) = 0\} && \text{by Def. 86} \\ &= \{\Xi_\varnothing(X)\} && \text{by Def. 6} \\ &= \{1 - \mu_X(*)\} && \text{by Def. 83} \\ &= 1 - x && \text{by definition of } X \\ &= A(0)\,. && \text{by assumption} \end{aligned}$$

Finally if $z \in (0,1)$, then

$$\begin{aligned} A_{\pi_*,X}(z) &= \{\Xi_Y(X) : \pi_*(Y) = z\} && \text{by Def. 86} \\ &= \varnothing && \text{because } \pi_* \text{ two-valued} \\ &= A(z)\,. && \text{by assumption of the lemma} \end{aligned}$$

To sum up, I have shown that

$$A = A_{\pi_*,X}\,. \tag{817}$$

Therefore

$$\begin{aligned} \psi(A) &= \psi(A_{\pi_*,X}) && \text{by (817)} \\ &= \mathcal{F}_\psi(\pi_*)(X) && \text{by Def. 88} \\ &= \widetilde{\pi}_*(X) && \text{by (Z-2)} \\ &= \mu_X(*) && \text{by Def. 7} \\ &= x\,, && \text{by definition of } X \end{aligned}$$

as desired.

Proof of Theorem 109

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ satisfies (Z-2). In order to show that $\mathcal{F}_\psi$ induces the standard negation, let us first recall the definition of induced fuzzy truth functions, i.e. $\widetilde{\mathcal{F}_\psi}(\neg)(x) = \mathcal{F}_\psi(Q_\neg)(\widetilde{\eta}(x))$ for all $x \in \mathbf{I}$, see Def. 8. Abbreviating $Q = Q_\neg$ and defining $X \in \widetilde{\mathcal{P}}(\{1\})$ by

$$\mu_X(1) = x \tag{818}$$

for the given $x \in \mathbf{I}$, it is hence sufficient to show that $\mathcal{F}_\psi(Q)(X) = 1 - x$, where $Q : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$ is given by

$$Q(Y) = \begin{cases} 1 & : \quad Y = \varnothing \\ 0 & : \quad Y = \{1\} \end{cases} \tag{819}$$

for all $Y \in \mathcal{P}(\{1\})$, see again Def. 8. Let us now consider $\mathcal{A}_{Q,X}$. It is convenient to abbreviate

$$x' = 1 - x\,. \tag{820}$$

We then obtain for $z = 1$,

$$\begin{aligned} \mathcal{A}_{Q,X}(1) &= \{\Xi_Y(X) : Q(Y) = 1\} && \text{by Def. 86} \\ &= \{\Xi_\varnothing(X)\} && \text{see (819)} \\ &= \{1 - \mu_X(1)\} && \text{by Def. 83} \\ &= \{1 - x\} && \text{by (818)} \\ &= \{x'\}\,. && \text{by (820)} \end{aligned}$$

For $z = 0$, we have

$$\begin{aligned} \mathcal{A}_{Q,X}(0) &= \{\Xi_Y(X) : Q(Y) = 0\} && \text{by Def. 86} \\ &= \{\Xi_{\{1\}}(X)\} && \text{see (819)} \\ &= \{\mu_X(1)\} && \text{by Def. 83} \\ &= \{x\} && \text{by (818)} \\ &= \{1 - (1 - x)\} && \text{because } 1 - x \text{ involutive} \\ &= \{1 - x'\}\,. && \text{by (820)} \end{aligned}$$

In the remaining case that $z \in (0, 1)$,

$$\begin{aligned} \mathcal{A}_{Q,X}(z) &= \{\Xi_Y(X) : Q(Y) = z\} && \text{by Def. 86} \\ &= \varnothing\,. && \text{because } Q \text{ two-valued} \end{aligned}$$

To sum up,

$$\mathcal{A}_{Q,X}(z) = \begin{cases} \{x'\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0, 1) \\ \{1 - x'\} & : \quad z = 0 \end{cases} \tag{821}$$

for all $z \in \mathbf{I}$. Therefore

$$\begin{aligned}
\widetilde{\mathcal{F}}_\psi(\neg)(x) &= \mathcal{F}_\psi(Q)(X) && \text{by Def. 8} \\
&= \psi(A_{Q,X}) && \text{by Def. 88} \\
&= x' && \text{by L-152 and (821)} \\
&= 1 - x\,. && \text{by (820)}
\end{aligned}$$

B.19 Proof of Theorem 110

Lemma 153 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ satisfies* (Z-1) *and* (Z-2). *If $\mathcal{F}_\psi$ violates* (ψ-2), *then there exists $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ with $\mathcal{F}_\psi(\neg Q) \neq \neg\mathcal{F}_\psi(Q)$.*

Proof To see this, suppose that ψ does not satisfy (ψ-2). Then there exist $A, A' \in \mathbb{A}$ with $A'(z) = A(1-z)$ for all $z \in \mathbf{I}$ and

$$\psi(A') \neq 1 - \psi(A)\,. \tag{822}$$

Let us assume that $D(A) = \{1\}$. Then $D(A') = \{1\}$ as well and $\mathrm{NV}(A) = \{z\}$, $\mathrm{NV}(A') = \{1-z\}$ for some $z \in \mathbf{I}$. Let us now observe that ψ satisfies (ψ-1), which is known from Th-108. Therefore $\psi(A) = z = 1 - (1-z) = 1 - \psi(A')$, which contradicts (825). Hence $D(A) \neq \{1\}$. By Th-94, then, there exist $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ with

$$A_{Q,X} = A\,. \tag{823}$$

Now consider $\neg Q$. Then $A_{\neg Q,X}(z) = A_{Q,X}(1-z) = A(1-z)$ by L-139 and (823). Hence

$$A_{\neg Q,X} = A' \tag{824}$$

We now proceed as follows.

$$\begin{aligned}
\mathcal{F}_\psi(\neg Q)X &= \psi(A_{\neg Q,X}) && \text{by Def. 88} \\
&= \psi(A') && \text{by (824)} \\
&\neq 1 - \psi(A) && \text{by (822)} \\
&= 1 - \psi(A_{Q,X}) && \text{by (823)} \\
&= 1 - \mathcal{F}_\psi(Q)X\,, && \text{by Def. 88}
\end{aligned}$$

as desired.

Proof of Theorem 110

Consider a choice of $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ such that $\mathcal{F}_\psi$ satisfies (Z-1) and (Z-2). The claim of the theorem will be proven by contraposition. Hence suppose that ψ does not satisfy (ψ-2); it must then be shown that $\mathcal{F}_\psi$ does not satisfy (Z-3). More specifically, it must

be shown that $\mathcal{F}_\psi$ is not compatible with dualisation based on the standard negation, because $\mathcal{F}_\psi$ is known to induce the standard negation, see Th-109.
Hence suppose that ψ violates (ψ-2). By L-153, there exists $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ with

$$\mathcal{F}_\psi(\neg Q)(X) \neq \neg\mathcal{F}_\psi(Q)(X)\,. \tag{825}$$

Hence

$$\begin{aligned}
\mathcal{F}_\psi(Q\square)(\neg X) &= \psi(A_{Q\square,\neg X}) && \text{by Def. 88}\\
&= \psi(A_{\neg Q\neg,\neg X} && \text{by Def. 11}\\
&= \psi(A_{\neg Q,\neg\neg X}) && \text{by L-142}\\
&= \psi(A_{\neg Q,X}) && \text{because } \neg \text{ involutive}\\
&\neq \neg\psi(A_{Q,X}) && \text{by (825)}\\
\\
&= \neg\mathcal{F}_\psi(Q)(X) && \text{by Def. 88}\\
&= \neg\mathcal{F}_\psi(Q)(\neg\neg X) && \text{because } \neg \text{ involutive}\\
&= \mathcal{F}_\psi(Q)\square(\neg X)\,. && \text{by Def. 11}
\end{aligned}$$

Hence $\mathcal{F}_\psi(Q\square) \neq \mathcal{F}_\psi(Q)\square$, which completes the proof that $\mathcal{F}_\psi$ does not satisfy (Z-3).

B.20 Proof of Theorem 111

Lemma 154 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ and $x \in \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ satisfies* (Z-2)*. Define $A \in \mathbb{A}$ by*

$$A(z) = \begin{cases} \{x, \min(x, 1-x)\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0,1) \\ \{1-x\} & : \quad z = 0 \end{cases}$$

for all $z \in \mathbf{I}$. Then $\psi(A) = x$.

Proof

a.: $a \geq \frac{1}{2}$.
To see that $\psi(A) = x$, we consider the projection quantifier $\pi_a : \{a, b\} \longrightarrow \mathbf{2}$ and the fuzzy subset $X \in \widetilde{\mathcal{P}}(\{a, b\})$ defined by

$$\mu_X(a) = \mu_X(b) = x\,. \tag{826}$$

We notice that $\mathcal{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ and $\pi_a(\varnothing) = \pi_a(\{b\}) = 0$, $\pi_a(\{a\}) = \pi_a(\{a, b\}) = 1$ by Def. 6. Therefore

$$\pi_a{}^{-1}(0) = \{\varnothing, \{b\}\} \tag{827}$$

$$\pi_a{}^{-1}(1) = \{\{a\}, \{a, b\}\} \tag{828}$$

and

$$\pi_a{}^{-1}(z) = \varnothing \tag{829}$$

In turn

$$\begin{aligned}
&A_{\pi_a,X}(0) \\
&= \{\Xi_Y(X) : Y \in \pi_a{}^{-1}(0)\} && \text{by Def. 86} \\
&= \{\Xi_\varnothing(X), \Xi_{\{b\}}(X)\} && \text{by (827)} \\
&= \{\min(1-\mu_X(a), 1-\mu_X(b)), \\
&\qquad \min(1-\mu_X(a), \mu_X(b))\} && \text{by Def. 83} \\
&= \{\min(1-x, 1-x), \min(1-x, x)\} && \text{by (826)} \\
&= \{1-x, \min(x, 1-x)\} \\
&= \{1-x\}\,. && \text{by assumption } x \geq \tfrac{1}{2}
\end{aligned}$$

For $z = 1$, we obtain

$$\begin{aligned}
&A_{\pi_a,X}(1) \\
&= \{\Xi_Y(X) : Y \in \pi_a{}^{-1}(1)\} && \text{by Def. 86} \\
&= \{\Xi_{\{a\}}(X), \Xi_{\{a,b\}}(X)\} && \text{by (828)} \\
&= \{\min(\mu_X(a), 1-\mu_X(b)), \min(\mu_X(a), \mu_X(b))\} && \text{by Def. 83} \\
&= \{\min(x, 1-x), \min(x, x)\} && \text{by (826)} \\
&= \{x, \min(x, 1-x)\}
\end{aligned}$$

Finally if $z \in (0, 1)$, then

$$\begin{aligned}
A_{\pi_a,X}(z) &= \{\Xi_Y(X) : Y \in \pi_a{}^{-1}(z)\} && \text{by Def. 86} \\
&= \varnothing\,. && \text{by (829)}
\end{aligned}$$

Hence

$$A_{\pi_a,X}(z) = \begin{cases} \{x, \min(x, 1-x)\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0,1) \\ \{1-x\} & : \quad z = 0 \end{cases} = A(z) \tag{830}$$

for all $z \in \mathbf{I}$. In turn

$$\begin{aligned}
\psi(A) &= \psi(A_{\pi_a,X}) && \text{by (830)} \\
&= \mathcal{F}_\psi(\pi_a)(X) && \text{by Def. 88} \\
&= \widetilde{\pi}_a(X) && \text{by (Z-2)} \\
&= \mu_X(a) && \text{by Def. 7} \\
&= x\,. && \text{by (826)}
\end{aligned}$$

b.: $x < \frac{1}{2}$**.** In this case, $\min(x, 1-x) = x$. Hence A as defined in the lemma becomes

$$A(z) = \begin{cases} \{x\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0,1) \\ \{1-x\} & : \quad z = 0 \end{cases}$$

for all $z \in \mathbf{I}$. The desired $\psi(A) = x$ is then already known from L-152.

Proof of Theorem 111

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given. We shall assume that ψ satisfies (Z-2) and that the induced disjunction $\widetilde{\vee} = \widetilde{\mathcal{F}_\psi}(\vee)$ is an s-norm. Now consider $x \in \mathbf{I}$. We abbreviate $X = \widetilde{\eta}(x, x)$, i.e.

$$\mu_X(1) = \mu_X(2) = x\,. \tag{831}$$

Further suppose that $Q = Q_\vee : \mathcal{P}(\{1,2\}) \longrightarrow \mathbf{2}$ is defined as in Def. 8. Then

$$x \mathbin{\widetilde{\vee}} x = \mathcal{F}_\psi(Q)(X) = \psi(A_{Q,X})\,. \tag{832}$$

As concerns $A_{Q,X}$, it is apparent that

$$\begin{aligned} A_{Q,X}(1) &= \{\Xi_Y(X) : Q(Y) = 1\} && \text{by Def. 86} \\ &= \{\Xi_{\{1\}}(X), \Xi_{\{2\}}(X), \Xi_{\{1,2\}}(X)\} && \text{by definition of } Q \\ &= \{\min(\mu_X(1), 1-\mu_X(2)), \\ &\qquad \min(1-\mu_X(1), \mu_X(2)), \\ &\qquad \min(\mu_X(1), \mu_X(2))\} && \text{by Def. 83} \\ &= \{\min(x, 1-x), \min(1-x, x), \min(x, x)\} && \text{by (831)} \\ &= \{x, \min(x, 1-x)\}\,. \end{aligned}$$

For $z = 0$, we obtain

$$\begin{aligned} A_{Q,X}(0) &= \{\Xi_Y(X) : Q(Y) = 0\} && \text{by Def. 86} \\ &= \{\Xi_\varnothing(X)\} && \text{by definition of } Q \\ &= \{\min(1-\mu_X(1), 1-\mu_X(2))\} && \text{by Def. 83} \\ &= \{\min(1-x, 1-x)\} && \text{by (831)} \\ &= \{1-x\} \end{aligned}$$

In the remaining case that $z \in (0,1)$, it is apparent from Def. 86 and the fact that Q is two-valued that $A_{Q,X}(z) = \varnothing$. Hence

$$A_{Q,X}(z) = \begin{cases} \{x, \min(x, 1-x)\} & : \quad z = 1 \\ \varnothing & : \quad z \in (0,1) \\ \{1-x\} & : \quad z = 0 \end{cases}$$

for all $z \in \mathbf{I}$. It is now immediate from (832) and L-154 that $x \mathbin{\widetilde{\vee}} x = \psi(A_{Q,X}) = x$. Hence the induced fuzzy disjunction $\widetilde{\vee}$ is an idempotent s-norm. It is well-known that $\vee = \max$ is the only idempotent s-norm, see e.g. [13, Th-3.14, p.77]. Hence $\mathcal{F}_\psi$ induces the standard disjunction $x \mathbin{\widetilde{\vee}} y = x \vee y = \max(x, y)$ for all $x, y \in \mathbf{I}$, as desired.

B.21 Proof of Theorem 112

Lemma 155 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and* $A \in \mathbb{A}$ *with* $D(A) = \{1\}$*. Further suppose that some* $N \in \mathcal{P}(\mathbf{I})$ *is given such that*

$$\inf Y \in N \tag{833}$$

for all $Y \subseteq N$ *with* $Y \neq \varnothing$*. If* $\mathcal{F}_\psi$ *satisfies* (Z-1)*, then* $\psi(A) = \psi(A')$*, where* $A' \in \mathbb{A}$ *is defined by*

$$A'(z) = \begin{cases} A(z) & : \quad z \notin N \\ A(z) \cup \{0\} & : \quad z \in N \end{cases} \tag{834}$$

for all $z \in \mathbf{I}$.

Proof To see this, we define $Q : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ by $Q(\varnothing) = z_+$. Then

$$A^{(0)}_{Q,\varnothing} = A \tag{835}$$

by Th-94, and in turn

$$\begin{aligned} \psi(A) &= \psi(A^{(0)}_{Q,\varnothing}) && \text{by Def. 86} \\ &= \mathcal{F}_\psi(Q)(\varnothing) && \text{by Def. 88} \\ &= Q(\varnothing) && \text{by (Z-1)}\,, \end{aligned}$$

i.e.

$$\psi(A) = z_+ \,. \tag{836}$$

We further define $Q' : \mathcal{P}(N) \longrightarrow \mathbf{I}$ by

$$Q'(Y) = \begin{cases} \nu & : \quad Y \neq \varnothing \\ z_+ & : \quad Y = \varnothing \end{cases} \tag{837}$$

for all $Y \in \mathcal{P}(N)$, where

$$\nu = \nu(Y) = \inf Y \,. \tag{838}$$

Then $Q'\varnothing = z_+$ and $\Xi^{(1)}_\varnothing(\varnothing) = 1$, i.e.

$$1 \in A^{(1)}_{Q',\varnothing}(z_+) \,.$$

For $Y \neq \varnothing$, $Q'(Y) = \nu$ by (837) and (833). In addition $\Xi^{(1)}_Y(\varnothing) = 0$, see Def. 83. Therefore

$$0 \in A^{(1)}_{Q',\varnothing}(\nu) \,.$$

These results can be summarized as stating that

$$A^{(()}_{Q',\varnothing}z) = \begin{cases} \{0,1\} & : \quad z = z_+, z_+ \notin N \\ \{1\} & : \quad z = z_+, z_+ \in N \\ \{0\} & : \quad z \in N, z \neq z_+ \\ \varnothing & : \quad z \notin N, z \neq z_+ \end{cases}$$

for all $z \in \mathbf{I}$, in other words:

$$A^{(1)}_{Q',\varnothing} = A' \tag{839}$$

and hence

$$\begin{aligned} \psi(A') &= \psi(A^{(1)}_{Q',\varnothing}) && \text{by (839)} \\ &= \mathcal{F}_\psi(Q')(\varnothing) && \text{by Def. 88} \\ &= Q'(\varnothing) && \text{by (Z-1)} \\ &= z_+ && \text{by (837)} \\ &= \psi(A)\,, && \text{by (836)} \end{aligned}$$

as desired.

Lemma 156 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping such that $\mathcal{F}_\psi$ induces the standard extension principle and satisfies* (Z-6). *Further suppose that some $N \in \mathcal{P}(\mathbf{I})$ is given such that*

$$\inf Y \in N \tag{840}$$

for all $Y \subseteq N$, $Y \neq \varnothing$. Then for all $A \in \mathbb{A}$, $\psi(A) = \psi(A')$, where $A' \in \mathbb{A}$ is defined by

$$A'(z) = \begin{cases} A(z) & : \quad z \notin N \\ A(z) \cup \{0\} & : \quad z \in N \end{cases} \tag{841}$$

for all $z \in \mathbf{I}$.

Proof The case that $D(A) = \{1\}$ has already been considered in lemma L-155. Hence suppose that $D(A) \neq \{1\}$. Then

$$A = A_{Q,X} \tag{842}$$

where $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$, $E = \mathbf{I} \times \mathbf{I}$ is defined by (69), and $X \in \widetilde{\mathcal{P}}(E)$ is defined by (66), see Th-94.
Now let $E' = (\{1\} \times E) \cup (\{2\} \times N)$. We define $Q' : \mathcal{P}(E') \longrightarrow \mathbf{I}$ by

$$Q'(Y') = \begin{cases} Q(Y) & : \quad Y' \cap (\{2\} \times N) = \varnothing \\ \inf\{z \in N : (2, z) \in Y'\} & : \quad Y' \cap (\{2\} \times N) \neq \varnothing \end{cases} \tag{843}$$

for all $Y' \in \mathcal{P}(E')$, where $Y \in \mathcal{P}(E)$ abbreviates

$$Y = \{(z,r) \in E : (1,z,r) \in Y'\}\,. \tag{844}$$

We denote by $j : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ the injection

$$j(z,r) = (1,z,r) \tag{845}$$

for all $z, r \in \mathbf{I}$. Let us notice that for a given $Y \in \mathcal{P}(E)$, Y and $Y' = \widehat{j}(Y)$ are related by (844). This proves that

$$Q = Q' \circ \widehat{j}\,. \tag{846}$$

Now we consider $\hat{\hat{j}}(X) \in \widetilde{\mathcal{P}}(E')$, based on our previous choice of $X \in \widetilde{\mathcal{P}}(E)$ according to equation (66). Recalling (3), $\hat{\hat{j}}(X)$ is the fuzzy subset defined by

$$\mu_{\hat{\hat{j}}(X)}(c,e) = \begin{cases} \mu_X(z,r) & : \quad c = 1, e = (z,r) \\ 0 & : \quad c = 2 \end{cases} \tag{847}$$

for all $(c,e) \in E'$.
Now consider a choice of $Y' \in \mathcal{P}(E')$ and let Y be defined by (844). It is then apparent from (845) that

$$(z,r) \in Y \Leftrightarrow (1,z,r) \in Y'\,,$$

for all $z, r \in \mathbf{I}$. Recalling (60) and (847), this proves that

$$\delta_{\hat{\hat{j}}(X),Y'}(1,z,r) = \delta_{X,Y}(z,r) \tag{848}$$

for all $z, r \in \mathbf{I}$. We notice that for all $Y' \in \widetilde{\mathcal{P}}(E')$ with $Y' \cap (\{2\} \times N) = \varnothing$,

$$\begin{aligned}
&\Xi_{Y'}(\hat{\hat{j}}(X)) \\
&= \inf\{\delta_{\hat{\hat{j}}(X),Y'}(c,e) : (c,e) \in E'\} && \text{by (61)} \\
&= \min(\inf\{\delta_{\hat{\hat{j}}(X),Y'}(1,z,r) : (1,z,r) \in E'\}, \\
&\qquad \inf\{\delta_{\hat{\hat{j}}(X),Y'}(2,e) : (2,e) \in E'\}) && \text{because } E' = (\{1\} \times E) \cup (\{2\} \times N) \\
&= \min(\inf\{\delta_{X,Y}(z,r) : (z,r) \in E\}, \\
&\qquad \inf\{1 - \mu_{\hat{\hat{j}}(X)}(2,e) : e \in N\}) && \text{by (848) and (60)} \\
&= \min(\inf\{\delta_{X,Y}(z,r) : (z,r) \in E\}, \\
&\qquad \inf\{1 - 0 : e \in N\}) && \text{by (847)} \\
&= \min(\inf\{\delta_{X,Y}(z,r) : (z,r) \in E\}, 1) \\
&= \inf\{\delta_{X,Y}(z,r) : (z,r) \in E\}\,,
\end{aligned}$$

i.e.

$$\Xi_{Y'}(\hat{\hat{j}}(X)) = \Xi_Y(X)\,. \tag{849}$$

In the remaining case that $Y' \cap (\{2\} \times N) \neq \varnothing$, there exists $e_0 \in N$ with $(2, e_0) \in Y'$. Hence

$$\begin{aligned}
\Xi_{Y'}(\hat{\hat{j}}(X)) &= \inf\{\delta_{\hat{j}(X),Y'}(c,e) : (c,e) \in E'\} && \text{by (61)} \\
&\leq \delta_{\hat{j}(X),Y'}(2, e_0) && \text{because } (2, e_0) \in E' \\
&= \mu_{\hat{j}(X)}(2, e_0) && \text{by (60) because } (2, e_0) \in Y' \\
&= 0\,. && \text{by (847)}
\end{aligned}$$

We conclude that

$$\Xi_{Y'}(\hat{\hat{j}}(X)) = 0 \tag{850}$$

in this case.

Now let $z_0 \in \mathbf{I}$. Based on the above observations, it is now easy to prove that $A_{Q',\hat{j}(X)}(z_0) = A'(z_0)$. To see that $A'(z_0) \subseteq A_{Q',\hat{j}(X)}(z_0)$, consider $r_0 \in A'(z_0)$.

a. If $r_0 \in A(z_0)$, then we know from (842) that there exists $Y \in \mathcal{P}(E)$ with $Q(Y) = z_0$ and $\Xi_Y(X) = r_0$. Abbreviating $Y' = \widehat{Y}$, we notice that Y' and Y are related by equation (844). In addition $Y' \cap (\{2\} \times N) = \varnothing$. Hence $\Xi_{Y'}(\hat{\hat{j}}(X)) = \Xi_Y(X) = r_0$ by (849) and $Q'(Y') = Q(Y) = z_0$ by (846). From Def. 86, then, we obtain that $r_0 \in A_{Q',\hat{j}(X)}(z_0)$.

b. If $r_0 \notin A(z_0)$, then $r_0 = 0$ by (841) and $z_0 \in N$. Now consider $Y' = \{(2, z_0)\}$. We then have $Q'(Y') = z_0$ by (843). In addition, $\Xi_{Y'}(\hat{\hat{j}}(X)) = 0 = r_0$ by (850). Hence indeed $r_0 \in A_{Q',\hat{j}(X)}(z_0)$.

It remains to be shown that $A_{Q',\hat{j}(X)}(z_0) \subseteq A'(z_0)$. Hence consider $r_0 \in A_{Q',\hat{j}(X)}(z_0)$. By Def. 86, there exists $Y' \in \mathcal{P}(E')$ with $\Xi_{Y'}(\hat{\hat{j}}(X)) = r_0$ and $Q'(Y') = z_0$.

a. If $Y' \cap (\{2\} \times N) = \varnothing$, then $z_0 = Q(Y') = Q(Y)$ by (844) and $r_0 = \Xi_{Y'}(\hat{\hat{j}}(X)) = \Xi_Y(X)$ by (849), i.e. $r_0 \in A_{Q,X}(z_0) = A(z_0) \subseteq A'(z_0)$, see (842) and (841). Hence $r_0 \in A'(z_0)$.

b. If $Y' \cap (\{2\} \times N) \neq \varnothing$, then $z_0 = Q(Y') = \inf Z$, where $Z = \{z \in N : (2, z) \in Y'$, see (843). Clearly $Z \subseteq N$ and $Z \neq \varnothing$ because $Y' \cap (\{2\} \times N) \neq \varnothing$. Hence by (840), $\inf Z \in N$. Because $z_0 = Q(Y') = \inf Z$, this proves that $z_0 \in N$. We further notice that $r_0 = \Xi_{Y'}(\hat{\hat{j}}(X)) = 0$ by (850). Equation (841) then shows that indeed $r_0 = 0 \in A'(z_0)$ because $z_0 \in N$.

These results can be summarized as stating that $A_{Q',\hat{j}(X)}(z_0) \subseteq A'(z_0)$. Combining this with the earlier result that $A'(z_0) \subseteq A_{Q',\hat{j}(X)}(z_0)$ and noticing that $z_0 \in \mathbf{I}$ was arbitrary, we obtain the desired

$$A' = A_{Q', \hat{\hat{j}}(X)} \,. \tag{851}$$

Now we proceed as follows.

$$\begin{aligned}
\psi(A) &= \psi(A_{Q,X}) && \text{by (842)} \\
&= \psi(A_{Q' \circ \widehat{j}, X}) && \text{by (846)} \\
&= \mathcal{F}_\psi(Q' \circ \widehat{j})(X) && \text{by Def. 88} \\
&= \mathcal{F}_\psi(Q')(\hat{\hat{j}}(X)) && \text{by (Z-6)} \\
&= \psi(A_{Q', \hat{\hat{j}}(X)}) && \text{by Def. 88} \\
&= \psi(A') \,, && \text{by (851)}
\end{aligned}$$

which completes the proof of the lemma.

In order to prove the theorem Th-112, we further need an observation how $A_{Q,X_1,\ldots,X_n}$ behaves with respect to the symmetric difference $\triangle$. For a given base set E and fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, $X_1 \triangle X_2 \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_{X_1 \triangle X_2}(e) = \max(\min(\mu_{X_1}(e), 1 - \mu_{X_2}(e)), \min(1 - \mu_{X_1}(e), \mu_{X_2}(e)))$$

for all $e \in E$. Here we shall only need the case that the second argument is crisp. Hence consider $X \in \widetilde{\mathcal{P}}(E)$ and $A \in \mathcal{P}(E)$. In this case, $X \triangle A$ becomes

$$\mu_{X \triangle A}(e) = \begin{cases} \mu_X(e) & : \quad e \notin A \\ 1 - \mu_X(e) & : \quad e \in A \end{cases} \tag{852}$$

for all $e \in E$. Let us now state that $A_{Q,X_1,\ldots,X_n}$ is compatible to symmetric set difference in this special case.

Lemma 157 *Let $E \neq \varnothing$ be some base set, $X \in \widetilde{\mathcal{P}}(E)$ and $Y, A \in \mathcal{P}(E)$. Then $\Xi_{Y \triangle A}(X \triangle A) = \Xi_Y(X)$.*

Proof Straightforward.

$$\begin{aligned}
&\Xi_{Y\triangle A}(X\triangle A) \\
&= \min(\inf\{\mu_{X\triangle A}(e) : e \in Y\triangle A\}, \\
&\qquad \inf\{1-\mu_{X\triangle A}(e) : e \notin Y\triangle A\}) && \text{by Def. 83} \\
&= \min\{\inf\{\mu_X(e) : e \notin A, e \in Y\triangle A\}, \\
&\qquad \inf\{1-\mu_X(e) : e \in A, e \in Y\triangle A\}, \\
&\qquad \inf\{1-\mu_X(e) : e \notin A, e \notin Y\triangle A\}, \\
&\qquad \inf\{\mu_X(e) : e \in A, e \notin Y\triangle A\}\} && \text{by (852)} \\
&= \min\{\inf\{\mu_X(e) : e \notin A, e \in Y\}, \\
&\qquad \inf\{1-\mu_X(e) : e \in A, e \notin Y\}, \\
&\qquad \inf\{1-\mu_X(e) : e \notin A, e \notin Y\}, \\
&\qquad \inf\{\mu_X(e) : e \in A, e \in Y\}\} && \text{by def. of crisp set difference} \\
&= \min(\inf\{\mu_X(e) : e \in Y\}, \\
&\qquad \inf\{1-\mu_X(e) : e \notin Y\}) && \text{because for all } e, e \in A \text{ or } e \notin A \\
&= \Xi_Y(X)\,. && \text{by Def. 83}
\end{aligned}$$

Lemma 158 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a quantifier of arity $n > 0$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $A \in \mathcal{P}(E)$. Further suppose that $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n\triangle A) \tag{853}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Then

$$\mathcal{A}_{Q',X_1,\ldots,X_n} = \mathcal{A}_{Q,X_1,\ldots,X_n\triangle A}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof To see this, let $z \in \mathbf{I}$. Then

$$\begin{aligned}
&\mathcal{A}_{Q',X_1,\ldots,X_n}(z) \\
&= \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : Q'(Y_1, \ldots, Y_n) = z\} && \text{by Def. 86} \\
&= \{\Xi_{Y_1,\ldots,Y_n}(X_1, \ldots, X_n) : Q(Y_1, \ldots, Y_n\triangle A) = z\} && \text{by (853)} \\
&= \{\Xi_{Y_1,\ldots,Y_n\triangle A}(X_1, \ldots, X_n\triangle A) : Q(Y_1, \ldots, Y_n\triangle A) = z\} && \text{by Def. 83, L-157} \\
&= \{\Xi_{Z_1,\ldots,Z_n}(X_1, \ldots, X_n\triangle A) : Q(Z_1, \ldots, Z_n) = z\}
\end{aligned}$$

where the last equation is obtained from the substitution $Z_1 = Y_1, \ldots, Z_{n-1} = Y_{n-1}, Z_n = Y_n\triangle A$. This step is valid because $Y_n \mapsto Y_n\triangle A = Z_n$ is a bijection, with obvious inverse $Z_n \mapsto Z_n\triangle A = Y_n$. Recalling Def. 86, this proves that $\mathcal{A}_{Q',X_1,\ldots,X_n}(z) = \mathcal{A}_{Q,X_1,\ldots,X_n\triangle A}(z)$. Because $z \in \mathbf{I}$ was arbitrary, we conclude that $\mathcal{A}_{Q',X_1,\ldots,X_n} = \mathcal{A}_{Q,X_1,\ldots,X_n\triangle A}$, as desired.

Lemma 159 *Let* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ *be given and suppose that* $\mathcal{F}_\psi$ *induces the standard fuzzy disjunction and the standard extension principle. If* $\mathcal{F}_\psi$ *satisfies* (Z-4) *and* (Z-6)*, then*

$$\psi(A') = \psi(A)$$

for all $A \in \mathbb{A}$*, where*

$$A'(z) = \begin{cases} A(z) & : \quad z \neq z_+ \\ A(z_+) \cup \{1 - r_+\} & : \quad z = z_+ \end{cases} \tag{854}$$

for all $z \in \mathbf{I}$*.*

Proof The claim of the lemma is trivial if $1 - r_+ \in A(z_+)$, in which case $A' = A$ and hence $\psi(A) = \psi(A')$. In particular, this covers the case that $r_+ = \frac{1}{2}$ where $1 - r_+ = 1 - \frac{1}{2} = \frac{1}{2} \in A(z_+)$.
Hence let us assume that $r_+ > \frac{1}{2}$ and $1 - r_+ \notin A(z_+)$. It is apparent from L-156 that without loss of generality, we may further assume that

$$0 \in A(z) \tag{855}$$

for all $z \in \mathbf{I}$. In particular $0 \in D(A)$ and hence $D(A) \neq \{1\}$. In addition, we obtain that $0 \in A(z_+)$. Because $1 - r_+ \notin A(z_+)$ by assumption, we only need to consider the case that $r_+ \neq 1$.

In order to prove the claim of the lemma, we introduce a suitable choice of semi-fuzzy quantifiers and fuzzy arguments, which permit us to reduce the lemma to fulfillment of (Z-4).
To this end, we shall suppose that $\zeta : D(A) \longrightarrow \mathbf{I}$ and $\zeta' : D(A') \longrightarrow \mathbf{I}$ are chosen such that

$$r \in A(\zeta(r)) \qquad \text{for all } r \in D(A) \tag{856}$$

$$r \in A'(\zeta'(r)) \qquad \text{for all } r \in D(A'). \tag{857}$$

It is apparent from (854) and (62) that $D(A') \setminus \{1 - r_+\} \subseteq D(A) \subseteq D(A')$. We can hence assume that

$$\zeta(r) = \zeta'(r) \qquad \text{for all } r \in D(A) \tag{858}$$

In the following, we abbreviate $r_- = 0$. It is then clear from (855) and $r_+ \neq 1$ that

$$r_- = 0 \in D(A) \cap [0, 1 - r_+) \,. \tag{859}$$

In particular, condition (65) is satisfied by r_-. Based on $r_- = 0$, we now define fuzzy subsets $X, X' \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by

$$\mu_X(z, r) = \begin{cases} r & : \quad r \in A(z) \setminus \{r_+\} \\ r_- & : \quad r \notin A(z) \\ 1 - r_- & : \quad r = r_+, z = z_+ \end{cases} \tag{860}$$

$$\mu_{X'}(z, r) = \begin{cases} r & : \quad r \in A(z) \setminus \{r_+\} \\ r_- & : \quad r \notin A(z) \\ r_+ & : \quad r = r_+, z = z_+ \end{cases} \tag{861}$$

i.e.

$$\mu_X(z,r) = \begin{cases} r & : & r \in A(z) \setminus \{r_+\} \\ 0 & : & r \notin A(z) \\ 1 & : & r = r_+, z = z_+ \end{cases} \tag{862}$$

$$\mu_{X'}(z,r) = \begin{cases} r & : & r \in A(z) \setminus \{r_+\} \\ 0 & : & r \notin A(z) \\ r_+ & : & r = r_+, z = z_+ \end{cases} \tag{863}$$

because $r_- = 0$. Given $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, we abbreviate

$$r' = \Xi_Y(X) \tag{864}$$

$$z' = \inf\{z \in \mathbf{I} : (z, r') \in Y \triangle \{(z_+, r_+)\}, r' = r'(Y) \in A(z)\} \tag{865}$$

Based on z' and r', we then define $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = \begin{cases} z' & : & r' \in A(z') \\ \zeta(r') & : & r' \notin A(z') \end{cases} \tag{866}$$

for all $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$. In the following, let us suppose that $X^* \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ is defined by (66), choosing $r_- = 0$ as above. We then know from Th-94 that

$$\mathcal{A}_{Q^*,X^*} = A \tag{867}$$

because $D(A) \neq \{1\}$. Observing that $X^* = X \triangle \{(z_+, r_+)\}$, and conversely $X = X^* \triangle \{(z_+, r_+)\}$, we can then conclude from L-158 that

$$\mathcal{A}_{Q,X} = \mathcal{A}_{Q^*,X^*} = A\,. \tag{868}$$

Next we prove that $\mathcal{A}_{Q\cup,X,X'} = A'$. To see this, let $z_0 \in \mathbf{I}$ be given.

a. We first consider the subsumption $A'(z_0) \subseteq \mathcal{A}_{Q\cup,X,X'}(z_0)$. Hence let $r_0 \in A'(z_0)$. We shall discern three cases.

a.1 If $r_0 = r_+$ then $z_0 = z_+$ by Def. 87 because $r_+ > \frac{1}{2}$. Now consider $Y^+ = X_{\geq\frac{1}{2}}$ and $Y^{+\prime} = X'_{\geq\frac{1}{2}}$. Then $Y^+ = \{(z_+, r_+)\}$ and $Y^{+\prime} = \{(z_+, r_+)\} = Y^+$, which is apparent from (862), (863), Def. 29 and Def. 87. We notice that

$$\begin{aligned} & \inf\{1 - r : r \in A(z) \setminus \{r_+\}\} & \\ & = 1 - \sup\{r : r \in A(z) \setminus \{r_+\}\} & \text{by De Morgan's law} \\ & = 1 - \sup D(A) \setminus \{r_+\} & \text{by (62)} \\ & = 1 - (1 - r_+)\,, & \text{by Def. 85} \end{aligned}$$

i.e.

$$\inf\{1 - r : r \in A(z) \setminus \{r_+\}\} = r_+\,. \tag{869}$$

Hence

$$\begin{aligned}
&\Xi_{Y^+}(X) \\
&= \min(\mu_X(z_+, r_+), \inf\{1 - \mu_X(z, r) : (z, r) \neq (z_+, r_+)\}) && \text{by Def. 83} \\
&= \min\{1, \inf\{1 - r : r \in A(z) \setminus \{r_+\}\}, \inf\{1 - 0 : r \notin A(z)\}\} && \text{by (862)} \\
&= \inf\{1 - r : r \in A(z) \setminus \{r_+\}\}\,,
\end{aligned}$$

i.e.

$$\Xi_{Y^+}(X) = r_+ \tag{870}$$

by (869). By similar reasoning,

$$\begin{aligned}
&\Xi_{Y^{+\prime}}(X') \\
&= \min(\mu_{X'}(z_+, r_+), \inf\{1 - \mu_{X'}(z, r) : (z, r) \neq (z_+, r_+)\}) && \text{by Def. 83} \\
&= \min\{r_+, \inf\{1 - r : r \in A(z) \setminus \{r_+\}\}, \inf\{1 - 0 : r \notin A(z)\}\} && \text{by (863)} \\
&= \min(r_+, \inf\{1 - r : r \in A(z) \setminus \{r_+\}\}) \\
&= \min(r_+, r_+)\,, && \text{by (869)}
\end{aligned}$$

i.e.

$$\Xi_{Y^{+\prime}}(X') = r_+\,.$$

Combining this with (870), we obtain from Def. 83 that

$$\Xi_{Y^+, Y^{+\prime}}(X, X') = \min(\Xi_{Y^+}(X), \Xi_{Y^{+\prime}}(X')) = \min(r_+, r_+) = r_+\,. \tag{871}$$

As concerns the quantification result, we notice that $Q\cup(Y^+, {Y^+}') = Q(Y^+ \cup {Y^+}') = Q(Y^+)$ because $Y^+ = {Y^+}'$. Now let ${Y^+}^* = {X^*}_{\geq \frac{1}{2}}$. Then

$$\begin{aligned}
z_+ &= z_+(A) \\
&= z_+(A_{Q^*, X^*}) && \text{by (868)} \\
&= Q^*({Y^+}^*) && \text{by L-123} \\
&= Q(Y^+)\,,
\end{aligned}$$

i.e.

$$Q(Y^+) = z_+ \tag{872}$$

which is apparent from the definitions of Q and Q^*, noticing that

$${Y^+}^* = Y^+ \triangle \{(z_+, r_+)\}\,.$$

Hence indeed

$$Q\cup(Y^+, {Y^+}') = Q(Y^+) = z_+ = z_0 \tag{873}$$

and $\Xi_{Y^+, Y^{+\prime}}(X, X') = r_+ = r_0$, i.e. $r_0 \in A_{Q\cup, X, X'}(z_0)$, as desired.

a.2 Now suppose that $z_0 = z_+$ and $r_0 = 1 - r_+$; we then know from (854) that $r_0 \in A'(z_0)$. In this case we let $Y = Y^+ = \{(z_+, r_+)\}$ as above and $Y' = \varnothing$. Then

$$\begin{aligned}
\Xi_\varnothing(X') &= \inf\{1 - \mu_{X'}(z, r) : z, r \in \mathbf{I}\} && \text{by Def. 83} \\
&= \min(\inf\{1 - r : r \in A(z) \setminus \{r_+\}\}, \\
&\qquad \inf\{1 - 0 : r \notin A(z)\}, 1 - r_+\} && \text{by (863)} \\
&= \min\{r_+, 1, 1 - r_+\}\,, && \text{by (869)}
\end{aligned}$$

i.e.

$$\Xi_\varnothing(X') = 1 - r_+ \tag{874}$$

because $r_+ > \frac{1}{2}$. Therefore

$$\begin{aligned}
\Xi_{Y^+,\varnothing}(X.X') &= \min(\Xi_{Y^+}(X), \Xi_\varnothing(X')) && \text{by Def. 83} \\
&= \min(r_+, 1 - r_+) && \text{by (870), (874)} \\
&= 1 - r_+\,, && \text{because } r_+ > \tfrac{1}{2}
\end{aligned}$$

i.e. $\Xi_{Y^+,\varnothing}(X.X') = r_0$. As concerns the quantification result, we notice that

$$Q\cup(Y^+, \varnothing) = Q(Y^+ \cup \varnothing) = Q(Y^+) = z_+ = z_0\,,$$

see (872). This proves that $r_0 \in A_{Q\cup,X,X'}(z_0)$, see Def. 86.

a.3 In the remaining case that $(z_0, r_0) \neq (z_+, r_+)$ and $(z_0, r_0) \neq (z_+, 1 - r_+)$, let

$$Y = \{(z_0, r_0), (z_+, r_+)\} \cup \{(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\}\,. \tag{875}$$

We notice that $Y \triangle \{(z_+, r_+)\} = Y^*$, where Y^* is defined by (736). Therefore

$$\begin{aligned}
\Xi_Y(X) &= \Xi_{Y^*}(X^*) && \text{by L-157} \\
&= r_0\,. && \text{by (746)}
\end{aligned}$$

Hence

$$\begin{aligned}
& r_0 \\
&= \Xi_Y(X) \\
&= \inf\{\delta_{X,Y}(z, r) : z, r \in \mathbf{I}\} && \text{by (61)} \\
&= \min(\inf\{\delta_{X,Y}(z, r) : z, r \in \mathbf{I}, (z, r) \neq (z_+, r_+)\}, \\
&\qquad \delta_{X,Y}(z_+, r_+)) \\
&= \min(\inf\{\delta_{X,Y}(z, r) : z, r \in \mathbf{I}, (z, r) \neq (z_+, r_+)\}, 1)\,. && \text{by (60), (875), (862)}
\end{aligned}$$

We conclude that in fact

$$r_0 = \inf\{\delta_{X,Y}(z, r) : z, r \in \mathbf{I}, (z, r) \neq (z_+, r_+)\}\,. \tag{876}$$

As concerns X', we compute

$$\begin{aligned}
&\Xi_Y(X') \\
&= \inf\{\delta_{X',Y}(z,r) : z, r \in \mathbf{I}\} && \text{by (61)} \\
&= \min(\inf\{\delta_{X',Y}(z,r) : z, r \in \mathbf{I}, (z,r) \neq (z_+, r_+)\}, \\
&\qquad \delta_{X',Y}(z_+, r_+)) \\
&= \min(\inf\{\delta_{X,Y}(z,r) : z, r \in \mathbf{I}, (z,r) \neq (z_+, r_+)\}, \\
&\qquad r_+) && \text{by (862), (863)} \\
&= \min(r_0, r_+) && \text{by (876)} \\
&= r_0\,.
\end{aligned}$$

Therefore $\Xi_{Y,Y}(X, X') = \min(\Xi_Y(X), \Xi_Y(X')) = \min(r_0, r_0) = r_0$. We further notice that $Q\cup(Y, Y) = Q(Y \cup Y) = Q(Y) = Q^*(Y^*) = z_0$ by (747). Hence again $r_0 \in A_{Q\cup,X,X'}$.
This completes the proof of claim **a.** that

$$A'(z_0) \subseteq A_{Q\cup,X,X'}(z_0)\,. \tag{877}$$

b. Now let us prove the converse subsumption, $A_{Q\cup,X,X'}(z_0) \subseteq A'(z_0)$, by considering a choice of $r_0 \in A_{Q\cup,X,X'}$. By Def. 86, then, there exist $Y, Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ with $Q\cup(Y, Y') = z_0$ and $\Xi_{Y,Y'}(X, X') = r_0$.

b.1 If $Y = Y^+ = \{(z_+, r_+)\}$ and $Y' = Y^{+'} = \{(z_+, r_+)\}$, then

$$r_0 = \Xi_{Y,Y'}(X, X') = r_+$$

by (871) and $z_0 = Q\cup(Y, Y') = z_+$ by (873). $r_+ \in A'(z_+)$ entails the desired $r_0 \in A'(z_0)$.

b.2 If $Y = Y^+ = \{(z_+, r_+)\}$ and $Y' = \varnothing$, then $\Xi_Y(X) = r_+$ by (870) and $\Xi_{Y'}(X') = 1 - r_+$ because

$$\begin{aligned}
&\Xi_{Y'}(X') \\
&= \Xi_\varnothing(X') && \text{by assumption } Y' = \varnothing \\
&= \inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\} && \text{by Def. 83} \\
&= \min\{\inf\{1 - \mu_{X'}(z,r) : r \in A(z) \setminus \{r_+\}\}, \\
&\qquad \inf\{1 - \mu_{X'}(z,r) : r \notin A(z)\}, \\
&\qquad 1 - \mu_{X'}(z_+, r_+)\} && \text{by splitting inf-expression} \\
&= \min\{\inf\{1 - r : r \in A(z) \setminus \{r_+\}\}, \\
&\qquad \inf\{1 - 0 : r \notin A(z)\}, 1 - r_+\} && \text{by (863)} \\
&= \min\{r_+, 1, 1 - r_+\} && \text{by (869)} \\
&= 1 - r_+\,. && \text{because } r_+ > \tfrac{1}{2}
\end{aligned}$$

Hence $r_0 = \Xi_{Y,Y'}(X, X') = \min(r_+, 1 - r_+) = 1 - r_+$, again noticing that $r_+ > \frac{1}{2}$. As concerns the quantification result, $z_0 = Q\cup(Y, Y') = Q\cup(Y^+, \varnothing) = Q(Y^+ \cup \varnothing) = Q(Y^+) = z_+$ by (872). But $r_0 = 1 - r_+ \in A'(z_+) = A'(z_0)$ is apparent from (854).

b.3 If $Y = \varnothing$ and $Y' = {Y^+}' = \{(z_+, r_+)\}$, then

$$
\begin{aligned}
&\Xi_Y(X) \\
&= \Xi_\varnothing(X) && \text{by assumption } Y = \varnothing \\
&= \inf\{1 - \mu_X(z, r) : z, r \in \mathbf{I}\} && \text{by Def. 83} \\
&= \min\{\inf\{1 - \mu_X(z, r) : r \in A(z) \setminus \{r_+\}\}, \\
&\qquad \inf\{1 - \mu_X(z, r) : r \notin A(z)\}, \\
&\qquad 1 - \mu_X(z_+, r_+)\} && \text{by splitting inf-expression} \\
&= \min\{\inf\{1 - r : r \in A(z) \setminus \{r_+\}\}, \\
&\qquad \inf\{1 - 0 : r \notin A(z)\}, 1 - 1\} && \text{by (862)} \\
&= \min\{r_+, 1, 0\} && \text{by (869)} \\
&= 0\,.
\end{aligned}
$$

In particular $r_0 = \Xi_{Y,Y'}(X, X') = \min(\Xi_Y(X), \Xi_{Y'}(X')) = \min(0, \Xi_{Y'}(X')) = 0$. But $r_0 = 0 \in A(z_0)$ is known from (855), and $A(z_0) \subseteq A'(z_0)$ is apparent from (854). Hence indeed $r_0 \in A'(z_0)$.

b.4 For all other choices of $Y, Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, we know that $Y \cup Y' \neq \{(z_+, r_+)\} = Y^+ = {Y^+}'$. Consequently

$$\Xi_{Y \cup Y'}(X) \leq 1 - r_+ \tag{878}$$

by L-125 because $\Xi_{Y^+}(X) = r_+$ for $Y^+ = X_{\geq \frac{1}{2}}$. In addition, we then know that either $Y \neq Y^+$ or $Y' \neq {Y^+}'$ and hence

$$r_0 = \Xi_{Y,Y'}(X, X') \leq 1 - r_+\,, \tag{879}$$

again by L-125. We shall utilize these inequations in a minute. In the following, it is convenient to discern two subcases.

b.4.i If $(z_+, r_+) \notin Y$, then

$$\begin{aligned}
\Xi_{Y,Y'}(X, X') &= \min(\Xi_Y(X), \Xi_{Y'}(X')) && \text{by Def. 83} \\
&\le \Xi_Y(X) \\
&= \inf\{\delta_{X,Y}(z, r) : z, r \in \mathbf{I}\} && \text{by (61)} \\
&\le \delta_{X,Y}(z_+, r_+) \\
&= 1 - \mu_X(z_+, r_+) && \text{by (60)} \\
&= 1 - 1 && \text{by (862)} \\
&= 0\,.
\end{aligned}$$

We conclude that $r_0 = \Xi_{Y,Y'}(X, X') = 0$. Hence $r_0 \in A'(z_0)$ by (855).

b.4.ii In the case that $(z_+, r_+) \in Y$, we know that $(z_+, r_+) \in Y \cup Y'$. Now consider $z, r \in \mathbf{I}$. For $(z, r) = (z_+, r_+)$, we then obtain from $(z_+, r_+) \in Y \cup Y'$ that

$$\delta_{X,Y\cup Y'}(z, r) = \delta_{X,Y\cup Y'}(z_+, r_+) = \mu_X(z_+, r_+) = 1 \tag{880}$$

see (862). In turn

$$\begin{aligned}
&\min(\delta_{X,Y}(z, r), \delta_{X',Y'}(z, r)) \\
&= \min(\delta_{X,Y}(z_+, r_+), \delta_{X',Y'}(z_+, r_+)) && \text{by assumption } (z, r) = (z_+, r_+) \\
&= \min(1, \delta_{X',Y'}(z_+, r_+)) && \text{by (880)} \\
&= \delta_{X',Y'}(z_+, r_+) \\
&\ge \min(\mu_{X'}(z_+, r_+), 1 - \mu_{X'}(z_+, r_+)) && \text{by (60)} \\
&= \min(r_+, 1 - r_+) && \text{by (863)} \\
&= 1 - r_+\,, && \text{because } r_+ > \tfrac{1}{2}
\end{aligned}$$

i.e.

$$\min(\delta_{X,Y}(z, r), \delta_{X',Y'}(z, r)) \ge 1 - r_+\,. \tag{881}$$

Now consider $(z, r) \ne (z_+, r_+)$. Then

$$\mu_X(z, r) = \mu_{X'}(z, r)\,, \tag{882}$$

see (862) and (863). We further notice that

$$\mu_X(z, r) = \mu_{X'}(z, r) \le 1 - r_+ < \tfrac{1}{2} \tag{883}$$

in this case, which is apparent from (882), (863), Def. 87 and $r_+ > \frac{1}{2}$. In the following, it is convenient to discern four cases for the given $(z, r) \ne (z_+, r_+)$.

— If $(z, r) \in Y$ and $(z, r) \in Y'$, then $(z, r) \in Y \cup Y'$ and

$$\begin{aligned}
\delta_{X,Y\cup Y'}(z, r) &= \mu_X(z, r) && \text{by (60)} \\
&= \min(\mu_X(z, r), \mu_X(z, r)) && \text{by idempotence of } \min \\
&= \min(\mu_X(z, r), \mu_{X'}(z, r))\,, && \text{by (882)}
\end{aligned}$$

i.e.

$$\delta_{X,Y\cup Y'}(z,r) = \min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \tag{884}$$

by (60).

— If $(z,r) \in Y$ and $(z,r) \notin Y'$, then $(z,r) \in Y \cup Y'$ and hence

$$\begin{aligned}
\delta_{X,Y\cup Y'}(z,r) &= \mu_X(z,r) && \text{by (60)} \\
&= \min(\mu_X(z,r), 1-\mu_X(z,r)) && \text{by (883)} \\
&= \min(\mu_X(z,r), 1-\mu_{X'}(z,r))\,, && \text{by (882)}
\end{aligned}$$

and again

$$\delta_{X,Y\cup Y'}(z,r) = \min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \tag{885}$$

by (60).

— If $(z,r) \notin Y$ and $(z,r) \in Y'$, then $(z,r) \in Y \cup Y'$. Therefore

$$\begin{aligned}
\delta_{X,Y\cup Y'}(z,r) &= \mu_X(z,r) && \text{by (60)} \\
&= \min(1-\mu_X(z,r), \mu_X(z,r)) && \text{by (883)} \\
&= \min(1-\mu_X(z,r), \mu_{X'}(z,r))\,. && \text{by (882)}
\end{aligned}$$

By (60), then,

$$\delta_{X,Y\cup Y'}(z,r) = \min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r))\,. \tag{886}$$

— Finally if $(z,r) \notin Y$ and $(z,r) \notin Y'$, then $(z,r) \notin Y \cup Y'$ as well. Hence

$$\begin{aligned}
\delta_{X,Y\cup Y'}(z,r) &= 1-\mu_X(z,r) && \text{by (60)} \\
&= \min(1-\mu_X(z,r), 1-\mu_X(z,r)) && \text{by idempotence of } \min \\
&= \min(1-\mu_X(z,r), 1-\mu_{X'}(z,r))\,, && \text{by (882)}
\end{aligned}$$

i.e.

$$\delta_{X,Y\cup Y'}(z,r) = \delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \tag{887}$$

by (60). Therefore

$$\begin{aligned}
&\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+, r_+)\}
\end{aligned} \tag{888}$$

by (884)–(887).
We can now put the pieces together, in order to handle the remaining case **b.4.ii**.

$$\begin{aligned}
&\Xi_{Y\cup Y'}(X) \\
&= \inf\{\delta_{X,Y\cup Y'}(z,r) : z,r \in \mathbf{I}\} && \text{by (61)} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+,r_+)\}, \\
&\qquad \delta_{X,Y\cup Y'}(z_+,r_+)) && \text{by splitting } \inf\text{-expression} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+,r_+)\}, 1) && \text{by (880)} \\
&= \inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+,r_+)\}\,, && \text{because } 1 \text{ is identity of } \min
\end{aligned}$$

i.e.

$$\Xi_{Y\cup Y'}(X) = \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+,r_+)\} \tag{889}$$

by (888) and in particular

$$\inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+,r_+)\} \leq 1 - r_+ \tag{890}$$

by (878). Therefore

$$\begin{aligned}
&\Xi_{Y,Y'}(X,X') \\
&= \min(\Xi_Y(X), \Xi_{Y'}(X')) && \text{by Def. 83} \\
&= \min(\inf\{\delta_{X,Y}(z,r) : z,r \in \mathbf{I}\}, \inf\{\delta_{X',Y'}(z,r) : z,r \in \mathbf{I}\}) && \text{by (61)} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : z,r \in \mathbf{I}\} \\
&= \min(\inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+,r_+)\}, \\
&\qquad \min(\delta_{X,Y}(z_+,r_+), \delta_{X',Y'}(z_+,r_+))) && \text{by splitting } \inf\text{-expression} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+,r_+)\}\,. && \text{by (881), (890)}
\end{aligned}$$

Hence

$$r_0 = \Xi_{Y,Y'}(X,X') = \Xi_{Y\cup Y'}(X) \tag{891}$$

by (889). Because also $z_0 = Q\cup(Y,Y') = Q(Y \cup Y')$, we know from Def. 86 that $r_0 \in A_{Q,X}$. But $A = A_{Q,X}$ by (868), hence $r_0 \in A(z_0)$. We further notice from (854) that $A'(z_0) \subseteq A(z_0)$. Hence indeed $r_0 \in A'(z_0)$, which completes the proof of case **b.4.ii**.
Noticing that $r_0 \in A_{Q\cup,X,X'}(z_0)$ was arbitrarily chosen, we hence know that

$$A_{Q\cup,X,X'}(z_0) \subseteq A'(z_0)\,.$$

Combining this with (877), we obtain that $A_{Q\cup,X,X'}(z_0) = A'(z_0)$. Because $z_0 \in \mathbf{I}$ was arbitrary, we conclude that

$$A_{Q\cup,X,X'} = A'\,. \tag{892}$$

Based on these preparations, the claim of the lemma is now apparent from the following reasoning.

$$\begin{aligned}
\psi(A') &= \psi(A_{Q\cup,X,X'}) && \text{by (892)}\\
&= \mathcal{F}_\psi(Q\cup)(X,X') && \text{by Def. 88}\\
&= \mathcal{F}_\psi(Q)(X\cup X') && \text{by (Z-4)}\\
&= \mathcal{F}_\psi(Q)(X) && \text{because } X'\subseteq X\text{, see (862) and (863)}\\
&= \psi(A_{Q,X}) && \text{by Def. 88}\\
&= \psi(A)\,, && \text{by (868)}
\end{aligned}$$

as desired.

Lemma 160 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ induces the standard fuzzy disjunction and the standard extension principle. If $\mathcal{F}_\psi$ satisfies* (Z-4) *and* (Z-6)*, then*

$$\psi(A') = \psi(A)$$

for all $A \in \mathbb{A}$, where

$$A'(z) = \begin{cases} A(z) & : \quad z \neq z_+ \\ (A(z_+) \setminus \{r_+\}) \cup \{\frac{1}{2}\} & : \quad z = z_+ \end{cases} \tag{893}$$

for all $z \in \mathbf{I}$.

Proof Consider a choice of $A \in \mathbb{A}$ and suppose that $A' \in \mathbb{A}$ is defined by (893). By L-159, we can assume without loss of generality that

$$1 - r_+ \in A(z_+)\,, \tag{894}$$

where r_+ refers to $r_+ = r_+(A)$ and $z_+ = z_+(A)$ as usual. In particular, we then know that $1 - r_+ \in D(A)$ and in turn, $D(A) \neq \{1\}$. We can further assume without loss of generality that $r_+ > \frac{1}{2}$. This is apparent because (893) yields $A = A'$ in the case that $r_+ = \frac{1}{2}$, and hence the claim of the lemma is trivially fulfilled.
In the following, we assume a choice of $\zeta : D(A) \longrightarrow \mathbf{I}$ and $\zeta' : D(A') \longrightarrow \mathbf{I}$ such that

$$r \in A(\zeta(r)) \qquad \text{for all } r \in D(A) \tag{895}$$

$$r \in A'(\zeta'(r)) \qquad \text{for all } r \in D(A') \tag{896}$$

$$\zeta(r) = \zeta'(r) \qquad \text{for all } r \in D(A) \cap D(A')\,. \tag{897}$$

It is apparent from (62) and (893) that suitable ζ, ζ' exist. Let us now define $X, X' \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by

$$\mu_X(z,r) = \begin{cases} r & : \quad r \in A(z) \\ 1 - r_+ & : \quad r \notin A(z) \end{cases} \tag{898}$$

$$\mu_{X'}(z,r) = \begin{cases} \mu_X(z,r) & : \quad (z,r) \neq (z_+, r_+) \\ \frac{1}{2} & : \quad (z,r) = (z_+, r_+) \end{cases} \tag{899}$$

for all $z, r \in \mathbf{I}$. We define $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = \begin{cases} z' & : \quad r' \in A(z') \\ \zeta(r') & : \quad r' \notin A(z') \end{cases} \tag{900}$$

where

$$r' = r'(Y) = \Xi_Y(X) \tag{901}$$
$$z' = z'(Y) = \inf\{z \in \mathbf{I} : (z, r') \in Y, r' = r'(Y) \in A(z)\} \tag{902}$$

for all $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$.
Now we consider $A_{Q,X\cup X'}$ and $A_{Q\cup,X,X'}$. As concerns $A_{Q,X\cup X'}$, we first notice from (898) and (899) that

$$X' \subseteq X\,, \tag{903}$$

in particular $X = X \cup X'$ and $A_{Q,X\cup X'} = A_{Q,X}$. We also know that $D(A) \neq \{1\}$. We hence notice that Q and X are defined in accordance with the corresponding definitions in Th-94.**b.**, which yields the desired

$$A_{Q,X\cup X'} = A_{Q,X} = A\,. \tag{904}$$

It remains to be shown that $A_{Q\cup,X,X'} = A'$. Hence consider $z_0 \in \mathbf{I}$.

a. I first prove the subsumption

$$A'(z_0) \subseteq A_{Q\cup,X,X'}(z_0)\,. \tag{905}$$

To see this, let $r_0 \in A'(z_0)$.

a.1 In the case that $r_0 = r_+(A') = \frac{1}{2}$, we know that $z_0 = z_+$. We now consider

$$(Y, Y') = (Y^+, Y'^+) = (X_{\geq\frac{1}{2}}, X'_{\geq\frac{1}{2}})\,, \tag{906}$$

see (700). It is then apparent from (898), (899), $r_+ = r_+(A) > \frac{1}{2}$ and L-125 that

$$Y = \{(z_+, r_+)\} = Y'\,. \tag{907}$$

Therefore

$$\begin{aligned}
&\Xi_Y(X) \\
&= \min(\inf\{\mu_X(z_+, r_+)\}, \\
&\qquad \inf\{1 - \mu_X(z, r) : (z, r) \neq (z_+, r_+)\}) && \text{by Def. 83 and (907)} \\
&\geq \min(r_+, \inf\{1 - \max(r, 1 - r_+) : (z, r) \neq (z_+, r_+)\}) && \text{by (898)} \\
&= \min(r_+, \inf\{\min(1 - r, r_+) : (z, r) \neq (z_+, r_+)\})\,, && \text{by De Morgan's law}
\end{aligned}$$

i.e.

$$\Xi_Y(X) = r_+ \tag{908}$$

by L-125; and

$$\begin{aligned}
&\Xi_{Y'}(X') \\
&= \min(\inf\{\mu_{X'}(z_+, r_+)\}, \\
&\qquad \inf\{1 - \mu_{X'}(z, r) : (z, r) \neq (z_+, r_+)\}) && \text{by Def. 83 and (907)} \\
&\geq \min(\tfrac{1}{2}, \inf\{1 - \max(r, 1 - r_+) : (z, r) \neq (z_+, r_+)\}) && \text{by (899)} \\
&= \min(\tfrac{1}{2}, \inf\{\min(1 - r, r_+) : (z, r) \neq (z_+, r_+)\}) && \text{by De Morgan's law} \\
&= \tfrac{1}{2} && \text{by L-125}
\end{aligned}$$

i.e. $\Xi_{Y,Y'}(X, X') = \min(\Xi_Y(X), \Xi_{Y'}(X')) \geq \min(r_+, \frac{1}{2}) = \frac{1}{2}$. Because

$$\Xi_{Y,Y'}(X, X') \leq \mu_{X'}(z_+, r_+) = \tfrac{1}{2}$$

by Def. 83 and (899), this proves that

$$\Xi_{Y'}(X') = \tfrac{1}{2} \tag{909}$$

and in turn,

$$\Xi_{Y,Y'}(X, X') = \tfrac{1}{2}\,. \tag{910}$$

Hence indeed $r_0 = \frac{1}{2} = r_+(A') \in \Xi_{Y,Y'}(X, X')$. In addition

$$\begin{aligned}
Q\cup(Y, Y') &= Q(Y \cup Y') && \text{by Def. 12} \\
&= Q(X_{\geq \frac{1}{2}} \cup X'_{\geq \frac{1}{2}}) && \text{by definition of } Y, Y' \\
&= Q(X \cup X'_{\geq \frac{1}{2}}) && \text{(property of } \alpha\text{-cuts)} \\
&= Q(X_{\geq \frac{1}{2}}) && \text{by (903)} \\
&= Q(Y) && \text{by definition of } Y \\
&= z_+\,, && \text{by (900)}
\end{aligned}$$

because in this case, $r' = r_+$ by (901) and (906), and hence $z' = z_+$. To sum up, I have shown that $r_0 = \frac{1}{2} \in \Xi_{Y,Y'}(X, X')$ and $Q\cup(Y, Y') = z_+ = z_0$, i.e. $\frac{1}{2} \in A_{Q\cup,X,X'}(z_0)$ by Def. 86, as desired.

a.2 Now suppose that $r_0 \neq r_+$. We can then conclude from $r_0 \in A'(z_0)$ and (893) that $r_0 \in A(z_0)$ as well. In addition,

$$r_0 \leq 1 - r_+ \tag{911}$$

by (893) and L-125, recalling the abbreviation $r_+ = r_+(A)$. We now define $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ by

$$Y = \{(z_0, r_0)\} \cup \{(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\}\,. \tag{912}$$

Let us first show that $\Xi_Y(X) = r_0$.

a.2.i Let us begin with the case that $(z, r) \in Y$. We observe that

$$\mu_X(z_0, r_0) = r_0 \tag{913}$$

by (898) because $r_0 \in A(z_0)$. In addition,

$$\begin{aligned}&\inf\{\mu_X(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} \\ &= \inf\{r : \text{there exists } z \in \mathbf{I} \text{ s.t. } r \in A(z) \cap (r_0, \tfrac{1}{2}]\}\,, && \text{by (898)}\end{aligned}$$

i.e.

$$\inf\{\mu_X(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\} \geq r_0\,. \tag{914}$$

Recalling that

$$\begin{aligned}&\inf\{\mu_X(z, r) : (z, r) \in Y\} \\ &= \min(\{\mu_X(z_0, r_0), \inf\{\mu_X(z, r) : r \in A(z) \cap (r_0, \tfrac{1}{2}]\})\,, && \text{by (912)}\end{aligned}$$

we hence obtain from (913) and (914) that

$$\inf\{\mu_X(z, r) : (z, r) \in Y\} = r_0\,. \tag{915}$$

a.2.ii Now let us consider the case that $(z, r) \notin Y$. Then

$$\begin{aligned}&\inf\{1 - \mu_X(z, r) : (z, r) \notin Y\} \\ &= \min\{\inf\{1 - \mu_X(z, r) : r \notin A(z)\}, \\ &\quad \inf\{1 - \mu_X(z, r) : r \in A(z) \cap [0, r_0]\}, \\ &\quad \inf\{1 - \mu_X(z, r) : r \in A(z) \cap [\tfrac{1}{2}, 1]\}\}\,, && \text{by (912)}\end{aligned}$$

i.e.

$$\begin{aligned}&\inf\{1 - \mu_X(z, r) : (z, r) \notin Y\} \\ &= \min\{\inf\{1 - \mu_X(z, r) : r \notin A(z)\}, \\ &\quad \inf\{1 - \mu_X(z, r) : r \in A(z) \cap [0, r_0]\}, \\ &\quad 1 - \mu_X(z_+, r_+)\}\,.\end{aligned} \tag{916}$$

We shall consider these subexpressions in turn.

— Firstly

$$\begin{aligned}&\inf\{1 - \mu_X(z, r) : r \notin A(z)\} \\ &= \inf\{1 - (1 - r_+) : r \notin A(z)\} && \text{by (898)} \\ &= \inf\{r_+ : r \notin A(z)\} \\ &\geq r_+ \\ &> \tfrac{1}{2}\,,\end{aligned}$$

in particular

$$\inf\{1 - \mu_X(z, r) : r \notin A(z)\} \geq r_0\,. \tag{917}$$

— Secondly,

$$
\begin{aligned}
&\inf\{1-\mu_X(z,r) : r \in A(z)\cap[0,r_0]\} \\
&= \inf\{1-r : r\in A(z)\cap[0,r_0)\} && \text{by (898)}\\
&\geq \inf\{1-r : r\in[0,r_0)\}\\
&= \inf(1-r_0,1]\\
&= 1-r_0\\
&\geq 1-(1-r_+) && \text{by (911)}\\
&= 1-r_+\\
&\geq r_0\,, && \text{by (911)}
\end{aligned}
$$

i.e.

$$\inf\{1-\mu_X(z,r) : r\in A(z)\cap[0,r_0]\} \geq r_0\,. \tag{918}$$

— Finally

$$1-\mu_X(z_0,r_0) = 1-r_0 \geq r_0 \tag{919}$$

by (898) and $r_0 \in A(z_0)$, and noticing that $r_0 < \frac{1}{2}$.
These results can be summarized as stating that

$$\inf\{1-\mu_X(z,r) : (z,r)\notin Y\} \geq r_0 \tag{920}$$

in case **a.2.ii**, which is straightforward from equations (916)–(919). The above results can then be combined to yield

$$\Xi_Y(X) = \min(\inf\{\mu_X(z,r) : (z,r)\in Y\}, \inf\{1-\mu_X(z,r) : (z,r)\notin Y\}) = r_0\,, \tag{921}$$

see Def. 83, (915), and (920).
Next we prove that $\Xi_Y(X') = r_0$ as well.

a.2.iii In order to treat the case that $(z,r)\in Y$, we first notice that

$$\mu_{X'}(r_0,z_0) = \mu_X(r_0,z_0) = r_0 \tag{922}$$

because $(z_0,r_0)\neq(z_+,r_+)$, see (899) and (913). In addition

$$
\begin{aligned}
&\inf\{\mu_{X'}(z,r) : r\in A(z)\cap(r_0,\tfrac{1}{2}]\}\\
&= \inf\{\mu_X(z,r) : r\in A(z)\cap(r_0,\tfrac{1}{2}]\}
\end{aligned}
$$

by (899) because $r\in(r_0,\frac{1}{2}]$ entails that $r<r_+$. Hence from (914),

$$\inf\{\mu_{X'}(z,r) : r\in A(z)\cap(r_0,\tfrac{1}{2}]\} \geq r_0\,. \tag{923}$$

In turn, (912), (922) and (923) prove that

$$
\begin{aligned}
&\inf\{\mu_{X'}(z,r) : (z,r)\in Y\}\\
&= \min(\mu_X(z_0,r_0), \inf\{\mu_{X'}(z,r) : r\in A(z)\cap(r_0,\tfrac{1}{2}]\})\\
&= r_0\,.
\end{aligned} \tag{924}
$$

a.2.iv We now consider the case that $(z, r) \notin Y$. Then

$$\begin{aligned}
&\inf\{1 - \mu_{X'}(z, r) : (z, r) \notin Y\} \\
&= \min(\inf\{1 - \mu_{X'}(z, r) : r \notin A(z)\}, \\
&\qquad \inf\{1 - \mu_{X'}(z, r) : r \in A(z) \cap [0, r_0)\}, \\
&\qquad \inf\{1 - \mu_{X'}(z, r) : r \in A(z) \cap [\tfrac{1}{2}, 1]\}\} && \text{by (912)} \\
&= \min(\inf\{1 - \mu_X(z, r) : r \notin A(z)\}, \\
&\qquad \inf\{1 - \mu_X(z, r) : r \in A(z) \cap [0, r_0)\}, \\
&\qquad 1 - \mu_{X'}(z_+, r_+)\} && \text{by (899), L-125} \\
&= \min(\inf\{1 - \mu_X(z, r) : r \notin A(z)\}, \\
&\qquad \inf\{1 - \mu_X(z, r) : r \in A(z) \cap [0, r_0)\}, \\
&\qquad \tfrac{1}{2}\} && \text{by (899)}
\end{aligned}$$

and hence

$$\inf\{1 - \mu_{X'}(z, r) : (z, r) \notin Y\} \geq r_0 \tag{925}$$

by (917), (918) and recalling that $r_0 < \frac{1}{2}$.
It is now apparent from Def. 83, (924) and (925) that

$$\Xi_Y(X') = r_0 \,.$$

Combining this with (921) yields

$$\Xi_{Y,Y}(X, X') = \min(r_0, r_0) = r_0 \,. \tag{926}$$

It remains to be shown that $Q\cup(Y, Y) = z_0$. To see this, we first observe that $r' = \Xi_Y(X) = r_0$ by (901) and (921). Hence

$$\begin{aligned}
z' &= \inf\{z : (z, r') \in Y, r' \in A(z)\} && \text{by (902)} \\
&= \inf\{z : (z, r_0) \in Y, r_0 \in A(z)\} && \text{because } r' = r_0 \\
&= \inf\{z_0\} \,, && \text{by (912)}
\end{aligned}$$

i.e.

$$z' = z_0 \,. \tag{927}$$

Consequently

$$\begin{aligned}
Q\cup(Y, Y) &= Q(Y \cup Y) && \text{by Def. 12} \\
&= Q(Y) \\
&= z' \,, && \text{by (900) and } r' = r_0 \in A(z_0)
\end{aligned}$$

i.e.

$$Q\cup(Y, Y) = z_0 \tag{928}$$

by (927). Recalling Def. 86, (926) and (928) prove that $r_0 \in A_{Q\cup,X,X'}(z_0)$. Hence (905) is valid, as desired.

b. Next we consider the converse subsumption,

$$A_{Q\cup,X,X'}(z_0) \subseteq A'(z_0)\,. \tag{929}$$

Hence let $r_0 \in A_{Q\cup,X,X'}(z_0)$. By Def. 86, then, there exist $Y, Y' \in \mathcal{P}(\mathbf{I}\times\mathbf{I})$ with $Q\cup(Y,Y') = z_0$ and $\Xi_{Y,Y'}(X,X') = r_0$. In the following, it is convenient to discern a number of cases.

b.1 If $Y = Y^+ = X_{\geq\frac{1}{2}}$ and $Y' = {Y^+}' = X'_{\geq\frac{1}{2}}$, i.e. $Y = Y' = \{(z_+, r_+)\}$ by (907), then $Y \cup Y' = \{(z_+, r_+)\} = Y^+$. Hence

$$\begin{aligned} \Xi_{Y\cup Y'}(X) &= \Xi_{Y^+}(X) && \text{because } Y^+ = Y\cup Y' \\ &= r_+ && \text{by (908)} \end{aligned}$$

for $r_+ = r_+(A_{Q,X})$, and hence $z_0 = z_+(A_{Q,X}) = z_+(A) = z_+$ by L-123 and L-124 because $r_+ > \frac{1}{2}$, and recalling equation (904). We further recall that $r_0 = \Xi_{Y,Y'}(X,X') = \frac{1}{2}$ by (910). But $r_0 = \frac{1}{2} \in A'(z_+)$ is immediate from (893).

b.2 The next case to consider is $Y = Y^+ = \{(z_+, r_+)\}$ and $Y' = \varnothing$. Then again $Y \cup Y' = \{(z_+, r_+)\} = Y^+$ and hence $z_0 = Q\cup(Y,Y') = Q(Y\cup Y') = Q(Y^+) = z_+$ by the same reasoning. In this case, we have $\Xi_Y(X) = r_+$ and

$$\begin{aligned} &\Xi_{Y'}(X') \\ &= \Xi_{\varnothing}(X') \\ &= \inf\{\delta_{X',\varnothing}(z,r) : z, r \in \mathbf{I}\} && \text{by (61)} \\ &= \inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\} && \text{by (60)} \\ &= \min(\inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\}, 1 - \mu_{X'}(z_+, r_+)) && \text{by splitting inf-expression} \\ &= \min(\inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\}, 1 - \tfrac{1}{2}) && \text{by (899)} \\ &= \min(\inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\}, \tfrac{1}{2}) \\ &= \min(\inf\{1 - \mu_{X'}(z,r) : z, r \in \mathbf{I}\}, \mu_{X'}(z_+, r_+)) && \text{by (899)} \\ &= \min(\inf\{\delta_{X',V}(z,r) : z, r \in \mathbf{I}\}, \delta_{X',V}(z_+, r_+)) && \text{by (60), } V = \{(z_+, r_+)\} \\ &= \inf\{\delta_{X',V}(z,r) : z, r \in \mathbf{I}\} \\ &= \Xi_{X'}(V) && \text{by (61)} \\ &= \tfrac{1}{2} && \text{by (909)} \end{aligned}$$

Hence $r_0 = \Xi_{Y,Y'}(X,X') = \min(r_+, \frac{1}{2}) = \frac{1}{2}$ and $z_0 = z_+$ in this case. We again notice from (893) that $r_0 = \frac{1}{2} \in A'(z_+) = A'(z_0)$ in this case.

b.3 Now we consider the case that $Y = \varnothing$ and $Y' = Y^{+\prime} = \{(z_+, r_+)\}$. We first observe that

$$\begin{aligned}
&\inf\{1 - \mu_X(z,r) : (z,r) \neq (z_+, r_+)\} \\
&= \inf\{\delta_{X,Y^+}(z,r) : (z,r) \neq (z_+, r_+)\} && \text{by (60), } Y^+ = \{(z_+, r_+)\} \\
&\geq \inf\{\delta_{X,Y^+}(z,r) : z, r \in \mathbf{I}\} \\
&= \Xi_{Y^+}(X) && \text{by (61)} \\
&= r_+\,, && \text{by (908)}
\end{aligned}$$

i.e.

$$\inf\{1 - \mu_X(z,r) : (z,r) \neq (z_+, r_+)\} \geq r_+\,. \tag{930}$$

Therefore

$$\begin{aligned}
&\Xi_\varnothing(X) \\
&= \inf\{\delta_{X,\varnothing}(z,r) : z, r \in \mathbf{I}\} && \text{by (61)} \\
&= \min(\inf\{1 - \mu_X(z,r) : (z,r) \neq (z_+, r_+)\}, \\
&\qquad 1 - \mu_X(z_+, r_+)) && \text{by (60)} \\
&= \min(\inf\{1 - \mu_X(z,r) : (z,r) \neq (z_+, r_+)\}, 1 - r_+) && \text{by (898)} \\
&= 1 - r_+\,. && \text{by (930) because } 1 - r_+ < r_+
\end{aligned}$$

Let us now recall from (909) that $\Xi_{Y'}(X') = \frac{1}{2}$. Hence $r_0 = \Xi_{Y,Y'}(X, X') = \min(\Xi_Y(X), \Xi_{Y'}(X')) = \min(1 - r_+, \frac{1}{2}) = 1 - r_+$ in this case. We notice that again $z_0 = Q\cup(Y, Y') = Q(Y^+) = z_+$, as in the previous cases. We can now utilize (894) to conclude that $r_0 = 1 - r_+ \in A(z_+) = A(z_0)$. It is then immediate from (893) that $r_0 \in A'(z_0)$ as well.

b.4 Finally we consider the case that $Y \cup Y' \neq \{(z_+, r_+)\} = Y^+$. We first need some observation on $\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)$, for given $z, r \in \mathbf{I}$. Hence let $z, r \in \mathbf{I}$.

b.4.i If $(z,r) = (z_+, r_+)$, then

$$\begin{aligned}
&\min(\delta_{X,Y}(z_+, r_+), \delta_{X',Y'}(z_+, r_+)) \\
&= \min(\delta_{X,Y}(z_+, r_+), \tfrac{1}{2}) && \text{by (60), (899)} \\
&\geq \min(\min(\mu_X(z_+, r_+), 1 - \mu_X(z_+, r_+)), \tfrac{1}{2}) && \text{by (60)} \\
&= \min(\min(r_+, 1 - r_+), \tfrac{1}{2})\,, && \text{by (898)}
\end{aligned}$$

i.e.

$$\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \geq 1 - r_+ \tag{931}$$

in this case because $r_+ > \frac{1}{2}$ by assumption.

b.4.ii In remaining case that $(z,r) \neq (z_+,r_+)$, we first observe that

$$\mu_X(z,r) = \mu_{X'}(z,r) \tag{932}$$

for $(z,r) \neq (z_+,r_+)$, which is immediate from (898) and (899). It is now convenient to discern four subcases.

— Firstly if $(z_+,r_+) \in Y$ and $(z_+,r_+) \in Y'$, then $(z_+,r_+) \in Y \cup Y'$ as well, and

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(\mu_X(z,r), \mu_{X'}(z,r)) && \text{by (60)} \\
&= \min(\mu_X(z,r), \mu_X(z,r)) && \text{by (932)} \\
&= \mu_X(z,r) && \text{by idempotence of } \min \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

— Secondly if $(z,r) \in Y$ and $(z,r) \notin Y'$, then again $(z,r) \in Y \cup Y'$ and

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(\mu_X(z,r), 1-\mu_{X'}(z,r)) && \text{by (60)} \\
&\min(\mu_X(z,r), 1-\mu_X(z,r)) && \text{by (932)} \\
&= \mu_X(z,r) && \text{because } \mu_X(z,r) \leq \tfrac{1}{2} \text{ by (898)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

— Next if $(z,r) \notin Y$ and $(z,r) \in Y'$, then in particular $(z,r) \in Y \cup Y'$ and hence

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(1-\mu_X(z,r), \mu_{X'}(z,r)) && \text{by (60)} \\
&\min(1-\mu_X(z,r), \mu_X(z,r)) && \text{by (932)} \\
&= \mu_X(z,r) && \text{because } \mu_X(z,r) \leq \tfrac{1}{2} \text{ by (898)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

— Finally if $(z,r) \notin Y$ and $(z,r) \notin Y'$, then $(z,r) \notin Y \cup Y'$ as well. Therefore

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(1-\mu_X(z,r), 1-\mu_{X'}(z,r)) && \text{by (60)} \\
&= 1-\max(\mu_X(z,r), \mu_{X'}(z,r)) && \text{by De Morgan's law} \\
&= 1-\mu_{X\cup X'}(z,r) \\
&= 1-\mu_X(z,r) && \text{by (903)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

These findings can be summarized as stating that

$$\begin{aligned}
&\inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : (z,r) \neq (z_+,r_+)\} \\
&= \inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+,r_+)\}\,.
\end{aligned} \tag{933}$$

We are now prepared to finish the proof that $A_{Q\cup,X,X'}(z_0) \subseteq A'(z_0)$. Hence consider the remaining case **b.4** that $Y \cup Y' \neq \{(z_+, r_+)\} = Y^+$.

— If $(z_+, r_+) \notin Y \cup Y'$, then we obtain from a simple computation that

$$\begin{aligned}
&\Xi_{Y,Y'}(X, X') \\
&= \min(\inf\{\delta_{X,Y}(z,r) : z, r \in \mathbf{I}\}, \\
&\qquad \inf\{\delta_{X',Y'}(z,r) : z, r \in \mathbf{I}\}) && \text{by Def. 83 and (60)} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : z, r \in \mathbf{I}\} \\
&= \min(\inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&\qquad\qquad : (z,r) \neq (z_+, r_+)\}, && \text{by splitting inf-expression} \\
&\qquad \min(\delta_{X,Y}(z_+, r_+), \delta_{X',Y'}(z_+, r_+))) \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \\
&\qquad \min(1 - \mu_X(z_+, r_+), 1 - \mu_{X'}(z_+, r_+))) && \text{by (933) and (60)}
\end{aligned}$$

and hence

$$\begin{aligned}
&\Xi_{Y,Y'}(X, X') \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \\
&\qquad 1 - \max(\mu_X(z_+, r_+), \mu_{X'}(z_+, r_+))) && \text{by De Morgan's law} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \\
&\qquad 1 - \mu_X(z_+, r_+)) && \text{by (903)} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \\
&\qquad \delta_{X,Y\cup Y'}(z_+, r_+)) && \text{by (60) because } (z_+, r_+) \notin Y \cup Y' \\
&= \inf\{\delta_{X,Y\cup Y'}(z,r) : z, r \in \mathbf{I}\} \\
&= \Xi_{Y\cup Y'}(X)\,. && \text{by (61)}
\end{aligned}$$

— In the remaining case that $Y \cup Y' \neq Y^+$ and $(z_+, r_+) \in Y \cup Y'$, we conclude from $\Xi_{Y^+}(X) = r_+$ and $Y \cup Y' \neq Y^+$ that

$$\Xi_{Y\cup Y'}(X) \leq 1 - r_+\,. \tag{934}$$

Therefore

$$\begin{aligned}
1 - r_+ \geq \Xi_{Y\cup Y'}(X) \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \delta_{X,Y\cup Y'}(z_+, r_+)) && \text{by (61)} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, \mu_X(z_+, r_+)) && \text{by (60)} \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}, r_+)\,. && \text{by (898)}
\end{aligned}$$

Because $r_+ > \frac{1}{2} > 1 - r_+$, this proves that

$$\Xi_{Y\cup Y'}(X) = \inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+, r_+)\}\,. \tag{935}$$

Hence

$$
\begin{aligned}
&\Xi_{Y,Y'}(X,X') \\
&= \min(\inf\{\delta_{X,Y}(z,r) : z,r \in \mathbf{I}\}, \\
&\qquad \inf\{\delta_{X',Y'}(z,r) : z,r \in \mathbf{I}\}) && \text{by Def. 83 and (61)} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : z,r \in \mathbf{I}\} \\
&= \min(\inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&\qquad\qquad : (z,r) \neq (z_+, r_+)\}, && \text{by splitting } \inf\text{-expression} \\
&\qquad \min(\delta_{X,Y}(z_+,r_+), \delta_{X',Y'}(z_+,r_+))) \\
&= \min(\inf\{\delta_{X,Y\cup Y'}(z,r) : (z,r) \neq (z_+,r_+)\}, \\
&\qquad \min(\delta_{X,Y}(z_+,r_+), \delta_{X',Y'}(z_+,r_+))) && \text{by (933)} \\
&= \min(\Xi_{Y\cup Y'}(X), \min(\delta_{X,Y}(z_+,r_+), \delta_{X',Y'}(z_+,r_+)) && \text{by (935)} \\
&= \Xi_{Y\cup Y'}(X)\,. && \text{by (931) and (934)}
\end{aligned}
$$

We conclude that $\Xi_{Y\cup Y'}(X) = \Xi_{Y,Y'}(X,X') = r_0$ in the considered case that $Y \cup Y' \neq Y^+$. In addition, $Q(Y \cup Y') = Q\cup(Y,Y') = z_0$. Hence $r_0 \in A_{Q,X}(z_0) = A(z_0)$ by Def. 86 and (904). Because $r_0 \neq r_+$, we obtain from (893) that $r_0 \in A'(z_0)$ as well.

This finishes the proof of part **b.** that $A_{Q\cup,X,X'}(z_0) \subseteq A'(z_0)$ for all $z_0 \in \mathbf{I}$. Combining this with (905), we obtain the desired

$$A' = A_{Q\cup,X,X'}\,. \tag{936}$$

To see how this proves the lemma, consider the following reasoning.

$$
\begin{aligned}
\psi(A') &= \psi(A_{Q\cup,X,X'}) && \text{by (936)} \\
&= \mathcal{F}_\psi(Q\cup)(X,X') && \text{by Def. 88} \\
&= \mathcal{F}_\psi(Q)(X \cup X') && \text{by (Z-4)} \\
&= \mathcal{F}_\psi(Q)(X) && \text{by (903)} \\
&= \psi(A_{Q,X}) && \text{by Def. 88} \\
&= \psi(A)\,. && \text{by (904)}
\end{aligned}
$$

Hence $\psi(A') = \psi(A)$, as desired.

Proof of Theorem 112

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping and suppose that $\mathcal{F}_\psi$ induces the standard fuzzy disjunction and the standard extension principle. Further suppose that (Z-4) and (Z-6) are valid. We now consider some $A_0 \in \mathbb{A}$. In order to prove that $\psi(A_0) = \psi(\Box\, A_0)$, it is convenient to define a special choice of $A \in \mathbb{A}$ in terms of A_0, viz

$$A(z) = \begin{cases} A_0(z) & : \quad z \neq z_+ \\ (A_0(z_+) \setminus \{r_+\}) \cup \{\frac{1}{2}\} & : \quad z = z_+ \end{cases} \tag{937}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 90 that

$$\Box A_0 = \Box A\,. \tag{938}$$

In addition, A_0 and A are related by (893), and hence

$$\psi(A_0) = \psi(A) \tag{939}$$

by L-160. We further notice that $r_+(A) = \frac{1}{2}$, and hence

$$A(z) \subseteq \Box A(z)$$

for all $z \in \mathbf{I}$, which is immediate from Def. 90.
We now recall Th-94. Noticing that $\frac{1}{2} = r_+(A) \in D(A)$, we clearly have $D(A) \neq \{1\}$. Hence by part **b.** of the theorem, we can construct $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ and $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ with

$$\mathcal{A}_{Q,X} = A\,. \tag{940}$$

Recalling this construction, we can choose $r_- = \frac{1}{2}$ here, see (65). Hence the definition of $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ becomes

$$\mu_X(z, r) = \begin{cases} r & : \quad r \in A(z) \\ \frac{1}{2} & : \quad r \notin A(z) \end{cases} \tag{941}$$

for all $z, r \in \mathbf{I}$, see (66). We further assume a choice of $\zeta : D(A) \longrightarrow \mathbf{I}$ which satisfies property (64). For a given $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, we shall then assume the usual definition of $r' = r'(Y)$, $z' = z'(Y)$ and $Q(Y)$, as stated by (67), (68) and (69), respectively. In the following, we introduce a second fuzzy subset $X' \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$, which we define by

$$\mu_{X'}(z, r) = \begin{cases} r & : \quad r \in \Box A(z) \\ \frac{1}{2} & : \quad r \notin \Box A(z) \end{cases} \tag{942}$$

for all $z, r \in \mathbf{I}$. We know that $r_+(A) = \frac{1}{2}$ and hence $r \in \Box A(z)$ entails that $r \le r_+ = \frac{1}{2}$. Therefore

$$\mu_X(z, r) \le \tfrac{1}{2} \tag{943}$$

$$\mu_{X'}(z, r) \le \tfrac{1}{2} \tag{944}$$

for all $z, r \in \mathbf{I}$. In addition, $r \le r_+ = \frac{1}{2}$ for all $r \in D(\Box A)$ entails that $X' \subseteq X$ and in turn,

$$X \cup X' = X\,. \tag{945}$$

We are now interested in $\mathcal{A}_{Q\cup,X,X'}$. Hence let $z_0 \in \mathbf{I}$. In order to prove that

$$\mathcal{A}_{Q\cup,X,X'}(z_0) = \Box A(z_0)\,,$$

a. I first consider the subsumption

$$\Box A(z_0) \subseteq \mathcal{A}_{Q\cup,X,X'}(z_0) \,. \tag{946}$$

Hence let $r_0 \in \Box A(z_0)$. Then there exists $r_1 \in A(z)$ with $r_0 \le r_1$. We define $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ by

$$Y = \{(z_0, r_0), (z_0, r_1)\} \cup \{(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} \,. \tag{947}$$

We now consider $r' = r'(Y)$. Some preparations are necessary to determine the precise result obtained for r'.

a.1 Let us begin with the case of $(z, r) \in Y$. We first notice that

$$\begin{aligned}
&\inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} \\
&\ge \inf\{r : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} && \text{by (941) and } D(A) \subseteq [0, \tfrac{1}{2}] \\
&\ge \inf(r_1, \tfrac{1}{2}] \,,
\end{aligned}$$

i.e.

$$\inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} \ge r_1 \,. \tag{948}$$

It is now convenient to discern two subcases.

a.1.i If $r_0 \in A(z_0)$, then

$$\begin{aligned}
&\inf\{\mu_X(z, r) : (z, r) \in Y\} \\
&= \min\{\mu_X(z_0, r_0), \mu_X(z_0, r_1), \\
&\qquad \inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\} && \text{by (947)} \\
&= \min\{r_0, r_1, \inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\} \,, && \text{by (941) and } r_0, r_1 \in A(z_0)
\end{aligned}$$

i.e.

$$\inf\{\mu_X(z, r) : (z, r) \in Y\} = r_0 \tag{949}$$

by (948) and recalling that $r_0 \le r_1$.

a.1.ii In the remaining case that $r_0 \notin A(z_0)$,

$$\begin{aligned}
&\inf\{\mu_X(z, r) : (z, r) \in Y\} \\
&= \min\{\mu_X(z_0, r_0), \mu_X(z_0, r_1), \\
&\qquad \inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\} && \text{by (947)} \\
&= \min\{\tfrac{1}{2}, r_1, \\
&\qquad \inf\{\mu_X(z, r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\} \,, && \text{by (941) and } r_0 \notin A(z_0), r_1 \in A(z_0)
\end{aligned}$$

i.e.

$$\inf\{\mu_X(z, r) : (z, r) \in Y\} = r_1 \tag{950}$$

by (948) and noticing that $r_1 \le \frac{1}{2} = r_+(A)$.

a.2 Now we consider the case that $(z, r) \notin Y$. We need some preparations.

a.2.i.: $(z, r) \notin Y$ and $r \notin A(z)$.
If $r > \frac{1}{2}$, then $r \notin A(z)$ for any $z \in \mathbf{I}$ because $r \in A(z)$ entails that $r \leq r_+ = \frac{1}{2}$. Hence for $r > \frac{1}{2}$, $\mu_X(z, r) = \frac{1}{2}$. Because such (z, r) exists, we know that

$$\sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} \geq \tfrac{1}{2}\,.$$

On the other hand, $\mu_X(z, r) \leq \max(r, \frac{1}{2}) = \frac{1}{2}$ for $r \leq \frac{1}{2}$, which is clear from (941) and $r_+(A) = \frac{1}{2}$. Hence

$$\sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} \leq \tfrac{1}{2}\,,$$

and we may summarize this as

$$\sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} = \tfrac{1}{2}\,. \tag{951}$$

a.2.ii.: $(z, r) \notin Y$ and $r \in A(z)$.
Next we consider the case that $(z, r) \notin \{(z_0, r_0), (z_0, r_1)\}$, $r \notin (r_1, \frac{1}{2}]$ and $r \in A(z)$. Because $r \in A(z) \subseteq D(A) \subseteq [0, r_+] = [0, \frac{1}{2}]$, we know that $r \leq \frac{1}{2}$ and hence $\mu_X(z, r) = r \leq \frac{1}{2}$ by (941). This proves that

$$\sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}], r \in A(z)\} \leq \tfrac{1}{2}\,. \tag{952}$$

In turn we combine cases **a.2.i** and **a.2.ii**, and thus obtain

$$\begin{aligned}
&\sup\{\mu_X(z,r) : (z,r) \notin Y\}\\
&= \max(\sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\},\\
&\qquad \sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}], r \in A(z)\}) \quad \text{by (947)}
\end{aligned}$$

i.e.

$$\sup\{\mu_X(z,r) : (z,r) \notin Y\} = \tfrac{1}{2}$$

by (951), (952). In particular

$$\inf\{1 - \mu_X(z,r) : (z,r) \notin Y\} = 1 - \sup\{\mu_X(z,r) : (z,r) \notin Y\} = \tfrac{1}{2} \tag{953}$$

by De Morgan's law. Considering $\Xi_Y(X)$, we recall that

$$\begin{aligned}
\Xi_Y(X) = \min(&\inf\{\mu_X(z,r) : (z,r) \in Y\},\\
&\inf\{1 - \mu_X(z,r) : (z,r) \notin Y\}) \qquad \text{by Def. 83}
\end{aligned}$$

and hence

$$r' = \Xi_Y(X) = \begin{cases} r_0 & : \quad r_0 \in A(z_0) \\ r_1 & : \quad r_0 \notin A(z_0) \end{cases} \tag{954}$$

by (67), (949), (950) and (953).
Next we focus on $z' = z'(Y)$, as defined by (68). If $r_0 \in A(z_0)$, then $r' = r_0$ by (954). We hence know from (947) that $\{(z,r') : (z,r') \in Y\} = \{(z,r_0) : (z,r_0) \in Y\} = \{(z_0,r_0)\}$, because $r_1 \geq r_0$. Hence in this case,

$$z' = \inf\{z \in \mathbf{I} : (z,r') \in Y, r' = r'(Y) \in A(z)\} = \inf\{z_0\} = z_0$$

by (68). In the remaining case that $r_0 \notin A(z_0)$, we recall that $r' = r_1$ by (954). Therefore (947) yields $\{(z,r') : (z,r') \in Y\} = \{(z,r_1) : (z,r_1) \in Y\} = \{(z_0,r_1)\}$. Hence again

$$z' = \inf\{z \in \mathbf{I} : (z,r') \in Y, r' = r'(Y) \in A(z)\} = \inf\{z_0\} = z_0$$

by (68). This completes the proof that in both cases,

$$z' = z_0 \,. \tag{955}$$

Next we consider $\Xi_Y(X')$.

a.3 Let us start with the case of $(z,r) \in Y$. We observe that

$$\begin{aligned}&\inf\{\mu_{X'}(z,r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\\ &= \inf\{\mu_X(z,r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} \qquad \text{by (941), (942)}\end{aligned}$$

because $A(z) \subseteq \Box A(z)$. Therefore

$$\inf\{\mu_{X'}(z,r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\} \geq r_1 \tag{956}$$

by (948). We can hence compute

$$\begin{aligned}&\inf\{\mu_{X'}(z,r) : (z,r) \in Y\}\\ &= \min\{\mu_{X'}(z_0,r_0), \mu_{X'}(z_0,r_1),\\ &\qquad \inf\{\mu_{X'}(z,r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\} \qquad \text{by (947)}\\ &= \min\{r_0, r_1, \inf\{\mu_{X'}(z,r) : r \in A(z) \cap (r_1, \tfrac{1}{2}]\}\}\,, \qquad \text{by (942)}\end{aligned}$$

i.e.

$$\inf\{\mu_{X'}(z,r) : (z,r) \in Y\} = r_0 \tag{957}$$

by (956) and recalling that $r_0 \leq r_1$.

a.4 Now we treat the case that $(z,r) \notin Y$. We shall split the proof into two subcases.

a.4.i.: $(z,r) \notin Y$ and $r \notin A(z)$.
We recall that $D(A) \cap (\frac{1}{2}, 1] = \varnothing$ because $r_+ = \frac{1}{2}$. Hence by Def. 90, $D(\Box A) \cap (\frac{1}{2}, 1] = \varnothing$ as well, i.e. if $r > \frac{1}{2}$, then $r \notin \Box A(z)$, regardless of $z \in \mathbf{I}$. By (942), then, we conclude that $\mu_{X'}(z,r) = \frac{1}{2}$ whenever $r > \frac{1}{2}$. In particular

$$\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} \geq \mu_{X'}(1,1) = \tfrac{1}{2}\,.$$

On the other hand, $\mu_{X'}(z,r) \leq \max(r, \frac{1}{2}) = \frac{1}{2}$ for $r \leq \frac{1}{2}$. Hence

$$\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} \leq \tfrac{1}{2}\,.$$

Combining this with the former result yields

$$\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\} = \tfrac{1}{2}\,. \tag{958}$$

a.4.ii.: $(z,r) \notin Y$ and $r \notin A(z)$.
In the remaining case that $(z,r) \notin \{(z_0,r_0),(z_0,r_1)\}$, $r \notin (r_1, \frac{1}{2}]$ and $r \in A(z)$, we know from (941) that $\mu_X(z,r) = r$. We conclude from $A(z) \subseteq \Box A(z)$ that $r \in \Box A(z)$ as well; hence $\mu_{X'}(z,r) = r = \mu_X(z,r)$ and

$$\begin{aligned}
&\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}], r \in A(z)\}\\
&= \sup\{\mu_X(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}], r \in A(z)\}\,.
\end{aligned}$$

Recalling (952), then, this becomes

$$\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}], r \in A(z)\} \leq \tfrac{1}{2}\,. \tag{959}$$

Now we combine the results of **a.4.i** and **a.4.ii**. Thus

$$\begin{aligned}
&\sup\{\mu_{X'}(z,r) : (z,r) \notin Y\}\\
&= \max(\sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin A(z)\},\\
&\qquad \sup\{\mu_{X'}(z,r) : (z,r) \notin \{(z_0,r_0),(z_0,r_1)\}, r \notin (r_1, \tfrac{1}{2}],\\
&\qquad\qquad r \in A(z)\}) && \text{by (947)}\\
&= \tfrac{1}{2} && \text{by (958), (959)}
\end{aligned}$$

and by De Morgan's law,

$$\inf\{1 - \mu_{X'}(z,r) : (z,r) \notin Y\} = 1 - \sup\{\mu_{X'}(z,r) : (z,r) \notin Y\} = \tfrac{1}{2}\,. \tag{960}$$

In order to compute $\Xi_Y(X')$, we recall that

$$\begin{aligned}
\Xi_Y(X') = \min(&\inf\{\mu_{X'}(z,r) : (z,r) \in Y\},\\
&\inf\{1 - \mu_{X'}(z,r) : (z,r) \notin Y\})\,. && \text{by Def. 83}
\end{aligned}$$

Therefore

$$\Xi_Y(X') = r_0\,, \tag{961}$$

which is apparent from (957) and (960). Based on this result, we now obtain

$$\begin{aligned}
\Xi_{Y,Y}(X,X') &= \min(\Xi_Y(X), \Xi_Y(X')) && \text{by Def. 83}\\
&= \min(\Xi_Y(X), r_0) && \text{by (961)}
\end{aligned}$$

and hence

$$\Xi_{Y,Y}(X,X') = r_0\,, \tag{962}$$

recalling that $r_0 \leq r_1$ and $\Xi_Y(X) \geq r_0$ by (954).
Finally we consider the quantification result $Q(Y)$. We notice that $r' \in A(z_0) = A(z')$, see (954) and (955). Hence $Q\cup(Y,Y) = Q(Y \cup Y) = Q(Y) = z_0$ by (69). Because $\Xi_{Y,Y}(X,X') = r_0$ by (962), we conclude from Def. 86 that $r_0 \in A_{Q\cup,X,X'}(z_0)$. Noticing that $r_0 \in \Box A(z_0)$ was arbitrarily chosen, this proves that the subsumption (946) is indeed valid.

b. It remains to be shown that

$$A_{Q\cup,X,X'}(z_0) \subseteq \Box A(z_0)\,. \tag{963}$$

To see this, let $r_0 \in A_{Q\cup,X,X'}(z_0)$. By Def. 86, there exist $Y, Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ with $z_0 = Q\cup(Y,Y')$ and $r_0 = \Xi_{Y,Y'}(X,X')$. In the following, I will show that $r_0 \leq r' = r'(Y \cup Y')$.
The proof requires some preparations, and we must relate $\min(\delta_{X,Y}(e), \delta_{X',Y'}(z,r)$ to $\delta_{X,Y\cup Y'}(z,r)$ for arbitrary $z, r \in \mathbf{I}$.
Hence let $z, r \in \mathbf{I}$. It is convenient to discern four cases.

b.1 If $(z,r) \in Y$ and $(z,r) \in Y'$, then $(z,r) \in Y \cup Y'$ and hence

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(\mu_X(z,r), \mu_{X'}(z,r)) && \text{by (60)} \\
&= \mu_{X'}(z,r) && \text{apparent from (945)} \\
&\leq \mu_X(z,r) && \text{by (945)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

b.2 In the second case that $(z,r) \in Y$ and $(z,r) \notin Y'$, we again have $(z,r) \in Y \cup Y'$ and hence

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(\mu_X(z,r), 1 - \mu_{X'}(z,r)) && \text{by (60)} \\
&= \mu_X(z,r) && \text{by (943), (944)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

b.3 In the third case that $(z,r) \notin Y$ and $(z,r) \in Y'$, it again holds that $(z,r) \in Y \cup Y'$. Therefore

$$\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(1 - \mu_X(z,r), \mu_{X'}(z,r)) && \text{by (60)} \\
&= \mu_{X'}(z,r) && \text{by (943), (944)} \\
&\leq \mu_X(z,r) && \text{by (945)} \\
&= \delta_{X,Y\cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}$$

b.4 Finally if $(z, r) \notin Y$ and $(z, r) \notin Y'$, then $(z, r) \notin Y \cup Y'$ as well. Therefore

$$
\begin{aligned}
&\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \\
&= \min(1 - \mu_X(z,r), 1 - \mu_{X'}(z,r)) && \text{by (60)} \\
&= 1 - \max(\mu_X(z,r), \mu_{X'}(z,r)) && \text{by De Morgan's law} \\
&= 1 - \mu_X(z,r) && \text{by (945)} \\
&= \delta_{X,Y \cup Y'}(z,r)\,. && \text{by (60)}
\end{aligned}
$$

The results obtained for these four cases can be summarized as

$$\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) \leq \delta_{X,Y \cup Y'}(z,r) \tag{964}$$

for all $z, r \in \mathbf{I}$. Therefore

$$
\begin{aligned}
r_0 &= \Xi_{Y,Y'}(X, X') && \text{by assumption on } Y, Y' \\
&= \min(\inf\{\delta_{X,Y}(z,r) : z, r \in \mathbf{I}\}, \\
&\qquad \inf\{\delta_{X',Y'}(z,r) : z, r \in \mathbf{I}\}) && \text{by Def. 83 and (61)} \\
&= \inf\{\min(\delta_{X,Y}(z,r), \delta_{X',Y'}(z,r)) : z, r \in \mathbf{I}\} \\
&\leq \inf\{\delta_{X,Y \cup Y'}(z,r) : z, r \in \mathbf{I}\} && \text{by (964)} \\
&= \Xi_{Y \cup Y'}(X)\,, && \text{by (61)}
\end{aligned}
$$

i.e.

$$r_0 = \Xi_{Y,Y'}(X, X') \leq \Xi_{Y \cup Y'}(X) = r'\,, \tag{965}$$

where r' abbreviates $r' = r'(Y \cup Y')$, see (67). Let us further notice that

$$Q(Y \cup Y') = Q\cup(Y, Y') = z_0\,, \tag{966}$$

by assumption on $Y, Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$. Now consider

$$z' = \inf\{z \in \mathbf{I} : r' \in A(z), (z, r') \in Y \cup Y'\}\,,$$

as defined by (68). It is convenient to discern two cases, in accordance with the definition of Q, see (69).

— In the first case that $z' \in A(r')$, we obtain from (69) that $z_0 = Q(Y \cup Y') = z'$. Hence $r' \in A(z_0)$. Because $r_0 \leq r'$ by (965), we conclude from Def. 90 that indeed $r_0 \in \Box A(z_0)$.

— In the remaining case that $z' \notin A(r')$, we know from (69) that $z_0 = Q(Y \cup Y') = \zeta(r')$. We then deduce from (64) that $r' \in A(\zeta(r')) = A(z_0)$. Because $r_0 \leq r'$ by (965), Def. 90 again proves that $r_0 \in \Box A(z_0)$, as desired.

Because $r_0 \in A_{Q\cup,X,X'}(z_0)$ was arbitrarily chosen, this proves that subsumption (963) of part **b.** is indeed valid. Combining this with our earlier result (946) of part **a.**, we have proven that $\Box A(z_0) = A_{Q\cup,X,X'}(z_0)$ for all $z_0 \in \mathbf{I}$, i.e.

$$A_{Q\cup,X,X'} = \Box A\,. \tag{967}$$

Based on this findings, it is now easy to prove the claim of the theorem. We simply notice that

$$\begin{aligned}
\psi(A_0) &= \psi(A) && \text{by (939)}\\
&= \psi(A_{Q,X}) && \text{by (940)}\\
&= \psi(A_{Q,X\cup X'}) && \text{by (945)}\\
&= \mathcal{F}_\psi(Q)(X\cup X') && \text{by Def. 88}\\
&= \mathcal{F}_\psi(Q\cup)(X,X') && \text{by (Z-4)}\\
&= \psi(A_{Q\cup,X,X'}) && \text{by Def. 88}\\
&= \psi(\Box\, A) && \text{by (967)}\\
&= \psi(\Box\, A_0)\,. && \text{by (938)}
\end{aligned}$$

Hence indeed $\psi(A_0) = \psi(\Box\, A_0)$, as desired.

B.22 Proof of Theorem 113

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ induces the standard disjunction and the standard extension principle. Further suppose that ψ satisfies (Z-4) and (Z-6). In order to prove that (ψ-5) is valid, we consider $A_0 \in \mathbb{A}$. In terms of A_0, we define $A_1 \in \mathbb{A}$ by

$$A_1(z) = A_0(z) \cup \{0\} \tag{968}$$

for all $z \in \mathbf{I}$. We further define $A \in \mathbb{A}$ by

$$A = \Box\, A_1\,. \tag{969}$$

It is then apparent from Def. 91 that

$$\boxplus A_0 = \boxplus A\,. \tag{970}$$

In addition, it is clear from (969), Def. 90 and $0 \in A_1(z)$ for all $z \in \mathbf{I}$, that

$$0 \in A(z) \tag{971}$$

for all $z \in \mathbf{I}$ as well. We further notice that

$$r_+ = r_+(A) = \tfrac{1}{2} \tag{972}$$

and

$$A(z) \in \{[0, \widehat{\boxplus} A(z)), [0, \widehat{\boxplus} A(z)]\} \tag{973}$$

which is apparent from (969) and Def. 90. In particular, because $\frac{1}{2} = r_+ \in A(z_+)$, this entails that

$$A(z_+) = \boxplus A(z_+) = [0, \tfrac{1}{2}]\,. \tag{974}$$

It is further immediate from Def. 91 that

$$A(z) \subseteq \boxplus A(z) \tag{975}$$

for all $z \in \mathbf{I}$. Recalling that $r_+(A) = r_+(\boxplus A) = \frac{1}{2}$, we further know from Def. 87 that

$$D(A) = [0, \tfrac{1}{2}] \tag{976}$$

and

$$D(\boxplus A) = [0, \tfrac{1}{2}]\,. \tag{977}$$

In dependence on A, we now define $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I} \times \mathbf{I})$ by

$$\mu_X(z, r, s) = \begin{cases} s & : \quad r \in \boxplus A(z), s < r \\ r & : \quad r \in A(z), s = r \\ 0 & : \quad \text{else} \end{cases} \tag{978}$$

for all $z, r, s \in \mathbf{I}$. We define $f : \mathbf{I}^3 \longrightarrow \mathbf{I}^2$ by

$$f(z, r, s) = (z, r) \tag{979}$$

for all $z, r, s \in \mathbf{I}$. We shall further abbreviate

$$X' = \hat{\hat{f}}(X) \tag{980}$$

As I will now show, it then holds that

$$\mu_{X'}(z, r) = \begin{cases} r & : \quad r \in \boxplus A(z) \\ 0 & : \quad r \notin \boxplus A(z) \end{cases} \tag{981}$$

for all $z, r \in \mathbf{I}$. To see this, consider $z, r \in \mathbf{I}$. Then

$$\begin{aligned}
&\mu_{X'}(z, r) \\
&= \sup\{\mu_X(z, r, s) : s \in \mathbf{I}\} && \text{by (980), (3) and (979)} \\
&= \max\{\sup\{\mu_X(z, r, s) : r \in \boxplus A(z), s < r\}, \\
&\qquad \sup\{\mu_X(z, r, r) : r \in A(z)\} \\
&\qquad \sup\{\mu_X(z, r, s) : s > r \vee (r \notin A(z) \wedge s = r) \\
&\qquad\qquad \vee r \notin \boxplus A(z)\} && \text{by splitting } \inf\text{-expression} \\
&= \max\{\sup\{s : r \in \boxplus A(z), s < r\}, \sup\{r : r \in A(z)\}, 0\}\,, && \text{by (978)}
\end{aligned}$$

i.e.

$$\mu_{X'}(z, r) = \max(\sup\{s : r \in \boxplus A(z), s < r\}, \sup\{r : r \in A(z)\})\,. \tag{982}$$

Now in the case that $r \in A(z)$, then $r \in \boxplus A(z)$ also and hence

$$\begin{aligned}
\mu_{X'}(z, r) &= \max(\sup[0, r), r) && \text{by (982)} \\
&= \max(r, r) \\
&= r\,.
\end{aligned}$$

In the case that $r \in \boxplus A(z) \setminus A(z)$, we obtain that

$$\begin{aligned} \mu_{X'}(z,r) &= \max(\sup[0,r), \sup\varnothing) && \text{by (982)} \\ &= \max(r,0) \\ &= r\,. \end{aligned}$$

Finally if $r \notin \boxplus A(z)$, then $r \notin A(z)$ either. Hence

$$\begin{aligned} \mu_{X'}(z,r) &= \max(\sup\varnothing, \sup\varnothing) && \text{by (982)} \\ &= \max(0,0) \\ &= 0 \end{aligned}$$

This completes the proof that (981) is indeed valid.
Let us now return to the task of defining a suitable quantifier. For a given $Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, we abbreviate

$$r' = r'(Y') = \Xi_{Y'}(X')\,, \tag{983}$$

$$Z = Z(Y') = \{z \in \mathbf{I} : (z,r') \in Y', r' \in \boxplus A(z)\} \tag{984}$$

and

$$z' = z'(Y') = \inf Z = \inf\{z \in \mathbf{I} : (z,r') \in Y', r' \in \boxplus A(z)\}\,. \tag{985}$$

We further notice that because of the specific properties of A, the $\zeta : D(\boxplus A) \longrightarrow \mathbf{I}$ as known from the construction used in Th-94.**b** can be replaced by the constant z_+, because

$$r \in A(z_+) = \boxplus A(z_+) \tag{986}$$

for all $r \in D(\boxplus A)$, see (974), (976) and (977). We now define $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y') = \begin{cases} z' & : \quad r' \in \boxplus A(z') \text{ and } z' \in Z \\ \zeta(r') & : \quad r' \notin \boxplus A(z') \text{ or } z' \notin Z \end{cases} \tag{987}$$

for all $Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$. Let us now show that

$$A_{Q,X'} = \boxplus A\,. \tag{988}$$

Hence let $z_0 \in \mathbf{I}$. We first consider the subsumption

$$\boxplus A(z_0) \subseteq A_{Q,X'}(z_0)\,. \tag{989}$$

To see this, we consider $r_0 \in \boxplus A(z_0)$. In dependence on r_0, we define

$$Y' = \{(z_0, r_0)\}\,. \tag{990}$$

Let us now notice that

$$\mu_{X'}(z,r) \leq \tfrac{1}{2} \tag{991}$$

for all $z, r \in \mathbf{I}$, which is clear from (981) and (977). Therefore

$$\begin{aligned}
&\Xi_{Y'}(X') \\
&= \min(\mu_{X'}(z_0, r_0), \inf\{1 - \mu_{X'}(z, r) : (z, r) \neq (z_0, r_0)\} && \text{by Def. 83 and (990)} \\
&= \min(\underbrace{r_0}_{\leq \frac{1}{2}}, \inf\{\underbrace{1 - \mu_{X'}(z, r)}_{\geq \frac{1}{2}} : (z, r) \neq (z_0, r_0)\}) && \text{by (991)} \\
&= r_0 \,,
\end{aligned}$$

i.e.

$$r' = r_0 \tag{992}$$

by (983). As concerns $Z = Z(Y')$, we hence obtain that

$$\begin{aligned}
Z &= \{z \in \mathbf{I} : (z, r') \in Y', r' \in \boxplus A(z)\} && \text{by (984)} \\
&= \{z \in \mathbf{I} : (z, r_0) \in \{(z_0, r_0)\}, r_0 \in \boxplus A(z)\} \,, && \text{by (990) and (992)}
\end{aligned}$$

i.e.

$$Z = \{z_0\} \,. \tag{993}$$

Therefore $z' = \inf Z = \inf\{z_0\} = z_0$. Because $r' = r_0 \in \boxplus A(z_0) = \boxplus A(z')$ by assumption and $z' = z_0 \in \{z_0\} = Z$, we conclude from (987) that $Q(Y') = z' = z_0$. Hence indeed $r_0 \in A_{Q,X'}(z_0)$ by Def. 86, i.e. (989) holds, as desired.
In order to prepare the proof of the converse subsumption

$$A_{Q,X'}(z_0) \subseteq \boxplus A(z_0) \,, \tag{994}$$

we first need to show that

$$r_+(A_{Q,X'}) = \tfrac{1}{2} \,. \tag{995}$$

To this end, we observe that

$$\begin{aligned}
&X'_{\geq \frac{1}{2}} \\
&= \{(z, r) \in \mathbf{I}^2 : \mu_{X'}(z, r) \geq \tfrac{1}{2}\} && \text{by Def. 29} \\
&= \{(z, r) \in \mathbf{I}^2 : \mu_{X'}(z, r) \geq \tfrac{1}{2}\} \,, && \text{by (991)}
\end{aligned}$$

i.e.

$$X'_{\geq \frac{1}{2}} = \{(z, \tfrac{1}{2}) : \tfrac{1}{2} \in \boxplus A(z)\} \tag{996}$$

by (981). Therefore

$$\begin{aligned}
&r_+(A_{Q,X'}) \\
&= \Xi_{X'_{\geq \frac{1}{2}}}(X') && \text{by L-124} \\
&= \min(\inf\{\mu_{X'}(z, \tfrac{1}{2}) : \tfrac{1}{2} \in \boxplus A(z)\}, \\
&\qquad \inf\{1 - \mu_{X'}(z, r) : \tfrac{1}{2} \notin \boxplus A(z) \vee r \neq \tfrac{1}{2}\}) && \text{by Def. 83 and (996)} \\
&= \min(\tfrac{1}{2}, \inf\{1 - \mu_{X'}(z, r) : \tfrac{1}{2} \notin \boxplus A(z) \vee r \neq \tfrac{1}{2}\}) && \text{because } \tfrac{1}{2} \in \boxplus A(z_+) \\
&= \tfrac{1}{2} \,,
\end{aligned}$$

noticing that $\inf\{1 - \mu_{X'}(z,r) : \frac{1}{2} \notin \boxplus A(z) \vee r \neq \frac{1}{2}\} \geq \frac{1}{2}$ by (991). This proves that (995) is valid, and we can now focus on (994).
Hence let $r_0 \in A_{Q,X'}(z_0)$. By Def. 86, there exists $Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ such that

$$r_0 = \Xi_{Y'}(X') \tag{997}$$

$$z_0 = Q(Y') \tag{998}$$

Let us abbreviate $r' = r'(Y')$, $Z = Z(Y')$ and $z' = z'(Y')$ as usual. Then

$$r' = \Xi_{Y'}(X') = r_0 \tag{999}$$

by (983) and (997). We now consider two cases: either $r' \in \boxplus A(z')$ and $z' \in Z$; or $r' \notin A(z')$ or $z' \notin Z$.
In the case that $r' \in \boxplus A(z')$ and $z' \in Z$, we obtain from (987) and (998) that $z_0 = Q(Y') = z'$. Hence $r' \in \boxplus A(z')$ entails that $r' \in \boxplus A(z_0)$ as well. Finally we apply (999), which yields the desired result $r_0 = r' \in \boxplus A(z_0)$.
In the remaining case that $r' \notin \boxplus A(z')$ or $z' \notin Z$, we obtain from (987) and (998) that $z_0 = Q(Y') = z_+$. We then observe from (995) and L-124 that $r_0 \leq r_+(A_{Q,X'}) = \frac{1}{2}$. Hence $r_0 \in [0, \frac{1}{2}] = \boxplus A(z_+) = \boxplus A(z_0)$ by (974). This completes the proof that (994) is valid. Recalling our former result (989), it is then obvious that equation (988) holds, as desired.
Next we prove that $A_{Q \circ \widehat{f}, X} = A$. To see this, let $z_0 \in \mathbf{I}$ be given. We first consider the subsumption

$$A(z_0) \subseteq A_{Q \circ \widehat{f}, X}(z_0)\,. \tag{1000}$$

Hence let $r_0 \in A(z_0)$. We define

$$Y = \{(z_0, r_0, r_0)\}\,. \tag{1001}$$

and

$$Y' = \widehat{f}(Y) = \{(z_0, r_0)\}\,. \tag{1002}$$

We now observe that

$$\mu_X(z, r, s) \leq \tfrac{1}{2}\,, \tag{1003}$$

for all $z, r, s \in \mathbf{I}$, which is apparent from (978), (976) and (977). Therefore

$$\begin{aligned}
&\Xi_Y(X) \\
&= \min(\overbrace{\mu_X(z_0, r_0, r_0)}^{\leq \frac{1}{2}}, \inf\{\overbrace{1 - \mu_X(z, r, s)}^{\geq \frac{1}{2}} : \\
&\qquad\qquad (z, r, s) \neq (z_0, r_0, r_0)\}) \qquad \text{by Def. 83 and (1001)} \\
&= \mu_X(z_0, r_0, s_0)\,,
\end{aligned}$$

i.e.

$$\Xi_Y(X) = r_0 \tag{1004}$$

because $\mu_X(z_0, r_0, r_0) = r_0$ by (978), and $r_0 \in A(z_0)$ by assumption. Recalling (1003), it is now apparent that

$$\begin{aligned}
&\Xi_{Y'}(X') \\
&= \min(\underbrace{\mu_{X'}(z_0, r_0)}_{\leq \frac{1}{2}}, \inf\{\underbrace{1 - \mu_{X'}(z, r)}_{\geq \frac{1}{2}} : (z, r) \neq (z_0, r_0)\}) \quad \text{by Def. 83 and (1002)} \\
&= \mu_{X'}(z_0, r_0)\,,
\end{aligned}$$

i.e.

$$r' = r'(Y') = \Xi_{Y'}(X') = r_0 \tag{1005}$$

by (983), (981) and noticing that $r_0 \in \boxplus A(z)$ by (975). We then obtain that

$$\begin{aligned}
z' &= z'(Y') \\
&= \inf\{z \in \mathbf{I} : (z, r') \in Y', r' \in A(z)\} && \text{by (985)} \\
&= \inf\{z \in \mathbf{I} : (z, r_0) \in \{(z_0, r_0)\}, r_0 \in A(z)\} && \text{by (1005) and (1002)} \\
&= \inf\{z_0\}\,,
\end{aligned}$$

and hence

$$z' = z_0\,. \tag{1006}$$

Because $r' = r_0 \in A(z_0) = A(z') \subseteq \boxplus A(z')$ by (1005), (1006), (975) and the assumption that $r_0 \in A(z_0)$, we conclude from (987) that $(Q \circ \widehat{f})(Y) = Q(\widehat{f}(Y)) = Q(Y') = z' = z_0$. Combining this with the above result (1004) that $\Xi_Y(X) = r_0$, we obtain from Def. 86 that $r_0 \in A_{Q\circ\widehat{f},X}(z_0)$, as desired. Because $r_0 \in A(z_0)$ was arbitrary, this proves that subsumption (1000) is indeed valid.
It remains to be shown that the converse subsumption is also valid, i.e.

$$A_{Q\circ\widehat{f},X}(z_0) \subseteq A(z_0)\,. \tag{1007}$$

I will first show that

$$r_+(A_{Q\circ\widehat{f},X}) = \tfrac{1}{2}\,. \tag{1008}$$

Hence let us observe that

$$\begin{aligned}
&X_{\geq\frac{1}{2}} \\
&= \{(z, r, s) \in \mathbf{I}^3 : \mu_X(z, r, s) \geq \tfrac{1}{2}\} && \text{by Def. 29} \\
&= \{(z, r, s) \in \mathbf{I}^3 : \mu_X(z, r, s) = \tfrac{1}{2}\}\,, && \text{by (1003)}
\end{aligned}$$

i.e.

$$X_{\geq\frac{1}{2}} = \{(z, \tfrac{1}{2}, \tfrac{1}{2}) : \tfrac{1}{2} \in A(z)\} \tag{1009}$$

by (981). Therefore

$$\begin{aligned}
&r_+(A_{Q\circ\widehat{f},X}) \\
&= \Xi_{X_{\geq\frac{1}{2}}}(X) && \text{by L-124} \\
&= \min(\inf\{\mu_X(z,\tfrac{1}{2},\tfrac{1}{2}) : \tfrac{1}{2} \in A(z)\}, \\
&\qquad \inf\{1-\mu_X(z,r,s) : \tfrac{1}{2} \notin A(z) \vee r \neq \tfrac{1}{2} \vee s \neq \tfrac{1}{2}\}) && \text{by Def. 83 and (1009)} \\
&= \min(\tfrac{1}{2}, \inf\{1-\mu_X(z,r,s) : \tfrac{1}{2} \notin A(z) \vee r \neq \tfrac{1}{2} \vee s \neq \tfrac{1}{2}\}) && \text{by (978) and} \\
& && \tfrac{1}{2} = r_+ \in A(z_+) \\
&= \tfrac{1}{2}\,,
\end{aligned}$$

noticing that $\inf\{1-\mu_X(z,r,s) : \frac{1}{2} \notin A(z) \vee r \neq \frac{1}{2} \vee s \neq \frac{1}{2}\} \geq \frac{1}{2}$ by (1003). This proves that (1008) is valid. We can hence turn attention to (1007).
To see that (1007) holds, consider $r_0 \in A_{Q\circ\widehat{f},X}(z_0)$. By Def. 86, then, there exists $Y \in \mathcal{P}(\mathbf{I}\times\mathbf{I}\times\mathbf{I})$ such that

$$\Xi_Y(X) = r_0 \tag{1010}$$

$$Q(Y') = (Q\circ\widehat{f})(Y) = z_0\,, \tag{1011}$$

where Y' abbreviates

$$Y' = \widehat{f}(Y)\,. \tag{1012}$$

In order to prove the desired (1007), we now make some observations how $\delta_{X',Y'}(z,r)$ relates to $\inf\{\delta_{X',Y'}(z,r,s) : s \in \mathbf{I}\}$ for $(z,r) \in Y'$.
Hence let $z,r \in \mathbf{I}$ be given and suppose that $(z,r) \in Y'$. Then there exists $\hat{s} \in \mathbf{I}$ with $(z,r,\hat{s}) \in Y$, see (1012). Hence

$$\inf\{\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \in Y\} \leq \mu_X(z,r,\hat{s})$$

because $(z,r,\hat{s}) \in Y$, which proves that

$$\inf\{\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \in Y\} \leq \tfrac{1}{2}\,, \tag{1013}$$

see (1003). We further notice that

$$\inf\{1-\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \notin Y\} \geq \tfrac{1}{2}\,, \tag{1014}$$

which is apparent from (1003). Therefore

$$\begin{aligned}
&\inf\{\delta_{X,Y}(z,r,s) : s \in \mathbf{I}\} \\
&= \min(\inf\{\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \in Y\}, \\
&\qquad \inf\{1-\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \notin Y\}) && \text{by (60)} \\
&= \inf\{\mu_X(z,r,s) : s \in \mathbf{I}, (z,r,s) \in Y\}\,, && \text{by (1013) and (1014)}
\end{aligned}$$

i.e.

$$\inf\{\delta_{X,Y}(z,r,s) : s \in \mathbf{I}\} \leq \mu_X(z,r,\hat{s}) \tag{1015}$$

because $(z, r, \hat{s}) \in Y$ by assumption. It is now convenient to discern three subcases. If $r \in A(z)$ and $\hat{s} = r$, then $\mu_X(z, r, \hat{s}) = \mu_X(z, r, r) = r$ by (978). Hence

$$\begin{aligned} &\inf\{\delta_{X,Y}(z, r, s) : s \in \mathbf{I}\} \\ &\leq \mu_X(z, r, \hat{s}) && \text{by (1015)} \\ &= r\,. \end{aligned}$$

If $r \in \Box\, A(z)$ and $\hat{s} < r$, then $\mu_X(z, r, \hat{s}) = \hat{s}$ by (978). Therefore

$$\inf\{\delta_{X,Y}(z, r, s) : s \in \mathbf{I}\} \tag{1016}$$
$$\leq \mu_X(z, r, \hat{s}) \quad \text{by (1015)} \tag{1017}$$
$$= \hat{s} \quad \text{by (978)} \tag{1018}$$
$$< r\,. \quad \text{by assumption} \tag{1019}$$

In the remaining case that $r \notin \Box\, A(z)$, or $r \in \Box\, A(z) \setminus A(z)$ and $\hat{s} \geq r$, or $r \in A(z)$ and $\hat{s} > r$, we notice that $\mu_X(z, r, \hat{s}) = 0$ by (978). This proves that

$$\begin{aligned} &\inf\{\delta_{X,Y}(z, r, s) : s \in \mathbf{I}\} \\ &\leq \mu_X(z, r, \hat{s}) && \text{by (1015)} \\ &= 0\,. && \text{by (978)} \end{aligned}$$

We can summarize these results as stating that

$$\inf\{\delta_{X,Y}(z, r, s) : s \in \mathbf{I}\} \in \begin{cases} [0, r] & : \quad r \in A(z) \\ [0, r) & : \quad r \in \boxplus A(z) \setminus A(z) \\ \{0\} & : \quad \text{else} \end{cases} \tag{1020}$$

In particular,

$$\inf\{\delta_{X,Y}(z, r, s) : s \in \mathbf{I}\} \in A(z) \tag{1021}$$

by (971) and noticing that $r \in \boxplus A(z) \setminus A(z)$ entails that $r = \widehat{\boxplus} A(z)$, because $A(z)$ satisfies (973).

Now let us return to the original goal of proving (1007). In the following, I will use the usual abbreviations $r' = r'(Y')$, $Z = Z(Y')$ and $z' = z'(Y')$. It is now convenient to discern two cases, in parallel with the two cases in the definition of Q by (987).

In the first case that $r' \in \boxplus A(z')$ and $z' \in Z$, we conclude from (984) that $(z', r') \in Y'$ and $r' \in \boxplus A(z')$. In turn, we conclude from $(z', r') \in Y'$ and (1021) that

$$\inf\{\delta_{X,Y}(z', r', s) : s \in \mathbf{I}\} \in A(z')\,.$$

Noticing that

$$\begin{aligned} r_0 &= \Xi_Y(X) && \text{by (1010)} \\ &= \inf\{\delta_{X,Y}(z, r, s) : z, r, s \in \mathbf{I}\} && \text{by (61)} \\ &\leq \inf\{\delta_{X,Y}(z', r', s) : s \in \mathbf{I}\}\,, \end{aligned}$$

we hence obtain from (973) that

$$r_0 \in A(z') \tag{1022}$$

as well. Due to the assumption of the present case that $r' \in \boxplus A(z')$ and $z' \in Z$, we look up from (987) that $z_0 = Q(Y') = z'$ by (1011). Substituting this into $r_0 \in A(z')$ yields the desired $r_0 \in A(z_0)$.
It remains to be shown that $r_0 \in A(z_0)$ in the case that $r' \notin \boxplus A(z')$ or $z' \notin Z$. We then obtain from (987) and (1011) that $z_0 = Q(Y') = z_+$. Noticing that $r_0 \leq r_+(A_{Q\circ\widehat{f},X}) = \frac{1}{2}$ by (1008), it is then immediate from (974) that $r_0 \in A(z_+) = A(z_0)$. This completes the proof of (1007). Recalling the former result stated in (1000), we have hence shown that

$$A_{Q\circ\widehat{f},X} = A\,. \tag{1023}$$

Based on these preparations, the proof of the theorem now reduces to the simple computation

$$\begin{aligned}
\psi(\boxplus A_0) &= \psi(\boxplus A) && \text{by (970)}\\
&= \psi(A_{Q,X'}) && \text{by (988)}\\
&= \mathcal{F}_\psi(Q)(X') && \text{by Def. 88}\\
&= \mathcal{F}_\psi(Q)(\hat{\hat{f}}(X)) && \text{by (980)}\\
&= \mathcal{F}_\psi(Q\circ\widehat{f})(X) && \text{by (Z-6)}\\
&= \psi(A_{Q\circ\widehat{f},X}) && \text{by Def. 88}\\
&= \psi(A) && \text{by (1023)}\\
&= \psi(A_1) && \text{by (969) and Th-112}\\
&= \psi(A_0)\,. && \text{by (968) and L-156}
\end{aligned}$$

Hence we have succeeded in proving the claim of the theorem, and it indeed holds that $\psi(\boxplus A_0) = \psi(A_0)$ for all $A_0 \in \mathbb{A}$.

B.23 Proof of Theorem 114

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ψ-5). Further suppose that $\mathcal{F}_\psi$ satisfies (Z-2). To see that given these properties, ψ satisfies (ψ-3) as well, we consider a choice of $A_0 \in \mathbb{A}$ with $\mathrm{NV}(A_0) \subseteq \{0,1\}$ and $r_+ \in A_0(1)$, i.e. $z_+ = z_+(A_0) = 1$. In particular, we then know from L-124 that

$$\sup A_0(0) \leq \tfrac{1}{2} \tag{1024}$$

and hence

$$\widehat{\boxplus} A_0(0) = \min(\sup A_0(0), \tfrac{1}{2}) = \sup A_0(0) \tag{1025}$$

by (73). Therefore

$$\boxplus A_0(z) = \begin{cases} [0, \sup A_0(0)] & : \quad z = 0 \\ \{0\} & : \quad z \in (0,1) \\ [0, \frac{1}{2}] & : \quad z = 1 \end{cases} \tag{1026}$$

for all $z \in \mathbf{I}$, which is apparent from Def. 91 and (1025), $\mathrm{NV}(A_0) \subseteq \{0,1\}$, and $z_+(A_0) = 1$, i.e. $\sup A_0(1) = r_+ \geq \frac{1}{2}$. We now define $A \in \mathbb{A}$ by

$$A(z) = \begin{cases} \{\sup A_0(0)\} & : \quad z = 0 \\ \varnothing & : \quad z \in (0,1) \\ \{\frac{1}{2}\} & : \quad z = 1 \end{cases} \tag{1027}$$

for all $z \in \mathbf{I}$. It is then clear from Def. 91 that

$$\boxplus A(z) = \begin{cases} [0, \sup A_0(0)] & : \quad z = 0 \\ \{0\} & : \quad z \in (0,1) \\ [0, \frac{1}{2}] & : \quad z = 1 \end{cases}$$

for all $z \in \mathbf{I}$, and hence

$$\boxplus A = \boxplus A_0 \,. \tag{1028}$$

We now consider a two-element set $\{a, b\}$. Let us define a fuzzy subset $X \in \widetilde{\mathcal{P}}(\{a, b\})$ by

$$\mu_X(a) = 1 - \sup A_0(0) \tag{1029}$$

$$\mu_X(b) = \tfrac{1}{2} \,. \tag{1030}$$

We notice that

$$\Xi_{\varnothing}(X) = \min(1 - (1 - \sup A_0(0)), 1 - \tfrac{1}{2}) = \sup A_0(0) \tag{1031}$$

$$\Xi_{\{a\}}(X) = \min(1 - \sup A_0(0), \tfrac{1}{2}) = \tfrac{1}{2} \tag{1032}$$

$$\Xi_{\{b\}}(X) = \min(1 - (1 - \sup A_0(0)), \tfrac{1}{2}) = \sup A_0(0) \tag{1033}$$

$$\Xi_{\{a,b\}}(X) = \min(1 - \sup A_0(0), \tfrac{1}{2}) = \tfrac{1}{2} \tag{1034}$$

by Def. 83, (1024), (1029) and (1030).
Now we consider the projection quantifier $\pi_a : \mathcal{P}(\{a, b\}) \longrightarrow \mathbf{2}$. It is apparent from Def. 6 that $(\pi_a)^{-1}(0) = \{\varnothing, \{b\}\}$. Hence by Def. 86,

$$A_{\pi_a,X}(0) = \{\Xi_{\varnothing}(X), \Xi_{\{b\}}(X)\} = \{\sup A_0(0)\} = A(0)$$

by (1031), (1033) and (1027). We further notice that $(\pi_a)^{-1}(1) = \{\{a\}, \{a, b\}\}$. Therefore

$$A_{\pi_a,X}(1) = \{\Xi_{\{a\}}(X), \Xi_{\{a,b\}}(X)\} = \{\tfrac{1}{2}\} = A(1) \,,$$

see (1032), (1034) and (1027). Finally if $z \in (0,1)$, then $(\pi_a)^{-1}(z) = \varnothing$ because π_a is two-valued. Therefore $A_{\pi_a,X}(z) = \varnothing = A(z)$ in this case, recalling Def. 86 and (1027). To sum up, we have shown that

$$A_{\pi_a,X} = A \,. \tag{1035}$$

We can hence proceed as follows.

$$\begin{aligned}
\psi(A_0) &= \psi(\boxplus A_0) && \text{by } (\psi\text{-5})\\
&= \psi(\boxplus A) && \text{by (1028)}\\
&= \psi(A) && \text{by } (\psi\text{-5})\\
&= \psi(A_{\pi_a,X}) && \text{by (1035)}\\
&= \mathcal{F}_\psi(\pi_a)(X) && \text{by Def. 88}\\
&= \widetilde{\pi}_a(X) && \text{by (Z-2)}\\
&= \mu_X(a) && \text{by Def. 7}\\
&= 1 - \sup A_0(0)\,, && \text{by (1029)}
\end{aligned}$$

i.e. $\psi(A_0) = 1 - \sup A_0(0)$, as desired. Because A_0 with NV$(A_0) \subseteq \{0,1\}$ and $z_+ = 1$ was arbitrary, this proves that (ψ-3) is indeed valid.

B.24 Proof of Theorem 115

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ψ-5). Further suppose that $\mathcal{F}_\psi$ is monotonic. Hence for all semi-fuzzy quantifiers $Q_0, Q_1 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\mathcal{F}_\psi(Q_0) \leq \mathcal{F}_\psi(Q_1) \tag{1036}$$

provided that $Q_0 \leq Q_1$. To see that ψ satisfies (ψ-4), we consider a choice of $A_0, A_1 \in \mathbb{A}$ such that

$$A_0 \sqsubseteq A_1\,. \tag{1037}$$

It is then apparent from Def. 91 and Def. 89 that

$$\boxplus A_0 \sqsubseteq \boxplus A_1 \tag{1038}$$

as well. In particular, we hence know from Def. 89 that for all $z' \in \mathbf{I}$ and $r \in \boxplus A_1(z')$, there exists $z \in \mathbf{I}$ with $z \leq z'$ and $r \in \boxplus A_0(z)$. In other words, there exists a mapping $\kappa : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ such that

$$\kappa(z, r) \leq z \tag{1039}$$

and

$$r \in \boxplus A_0(\kappa(z, r)) \tag{1040}$$

for all $z, r \in \mathbf{I}$ with $r \in \boxplus A_1(z)$. We shall utilize the mapping κ in a minute.
Next we notice that $r_+(\boxplus A_0) = r_+(\boxplus A_1) = \frac{1}{2}$. Therefore $D(\boxplus A_c) \neq \{1\}$, $c \in \{0,1\}$. We can hence define $X_0, X_1 \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by (66) and in accordance with part **b.** of Th-94, viz

$$\mu_{X_c}(z, r) = \begin{cases} r & : \quad r \in \boxplus A_c(z) \\ 0 & : \quad \text{else} \end{cases} \tag{1041}$$

for $c \in \{0,1\}$, noticing that $r_- = 0$ is a legal choice of r_- in (65) because $0 \in D(\boxplus A_c)$. We recall the coefficients (67) and (68). For X_c, $c \in \{0,1\}$, and $Y_c \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$, these become

$$r'_c = r'_c(Y_c) = \Xi_{Y_c}(X_c) \tag{1042}$$

$$z'_c = z'_c(Y_c) = \inf\{z \in \mathbf{I} : (z, r'_c) \in Y_c, r'_c \in \boxplus A_c(z)\}\,. \tag{1043}$$

We now notice that the constant $z_{+c} = z_+(A_c)$ is a legal choice for ζ_c in (64) because $\boxplus A_c(z_{+c}) = [0, r_+(\boxplus A_c)] = [0, \frac{1}{2}]$ and $D(\boxplus A_c) = [0, \frac{1}{2}]$. We can hence define $Q'_c : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ in accordance to (69) by

$$Q'_c(Y_c) = \begin{cases} z'_c & : \quad r'_c \in \boxplus A_c(z'_c) \\ z_{+c} & : \quad \text{else} \end{cases} \tag{1044}$$

for all $Y_c \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ and $c \in \{0,1\}$. Hence by Th-94.**b**,

$$A_{Q'_c, X_c} = \boxplus A_c \tag{1045}$$

for $c \in \{0,1\}$. Based on X_0 and X_1, we now define $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$ by

$$\mu_X(z, r, c) = \mu_{X_c}(z, r) \tag{1046}$$

for all $z, r \in \mathbf{I}$ and $c \in \{0,1\}$. For a given $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$, we denote by $Y_0 = Y_0(Y)$, $Y_1 = Y_1(Y) \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ the crisp sets defined by

$$Y_c = \{(z, r) : (z, r, c) \in Y\} \tag{1047}$$

for $c \in \{0,1\}$. We further abbreviate

$$Y_c^+ = X_{c_{\geq \frac{1}{2}}} = \{(z, \tfrac{1}{2}) : \tfrac{1}{2} \in \boxplus A_c(z)\} \tag{1048}$$

for $c \in \{0,1\}$, see (1041).
We now define semi-fuzzy quantifiers $Q_0, Q_1 : \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2}) \longrightarrow \mathbf{I}$ by

$$Q_1(Y) = Q'_1(Y_1) \tag{1049}$$

$$Q_0(Y) = \begin{cases} Q'_0(Y_0) & : \quad Y_1 = Y_1^+ \\ \kappa(Q_1(Y), \Xi_Y(X)) & : \quad \text{else} \end{cases} \tag{1050}$$

for all $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$. As I will now show, it holds that

$$A_{Q_c, X} = \boxplus A_c \tag{1051}$$

for $c \in \{0,1\}$. Hence let $c \in \{0,1\}$ and $\neg c = 1 - c$. Recalling (1045), it is sufficient to show that

$$A_{Q_c, X} = A_{Q'_c, X_c}\,.$$

Now let $z_0 \in \mathbf{I}$. I first show that

$$A_{Q'_c, X_c}(z_0) \subseteq A_{Q_c, X}(z_0)\,. \tag{1052}$$

Hence let $r_0 \in \boxplus Q'_c X_c(z_0)$. By Def. 86, there exists $Y' \in \mathcal{P}(\mathbf{I} \times \mathbf{I})$ such that

$$z_0 = Q'_c(Y') \tag{1053}$$
$$r_0 = \Xi_{Y'}(X_c)\,. \tag{1054}$$

Now consider $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$, defined by

$$Y = \{(z, r, c) : (z, r) \in Y'\} \cup \{(z, r, \neg c) : (z, r) \in Y^+_{\neg c}\}\,. \tag{1055}$$

It is then apparent from (1047) that

$$Y' = Y_c = Y_c(Y)\,. \tag{1056}$$

Therefore

$$\begin{aligned}
&\inf\{\delta_{X,Y}(z, r, c) : z, r \in \mathbf{I}\} \\
&= \min(\inf\{\mu_X(z, r, c) : z, r \in \mathbf{I}, (z, r, c) \in Y\}, \\
&\qquad \inf\{1 - \mu_{z,r,c}(:)z, r \in \mathbf{I}, (z, r, c) \notin Y\}) && \text{by (60)} \\
&= \min(\inf\{\mu_{X_c}(z, r) : (z, r) \in Y_c\}, \\
&\qquad \inf\{1 - \mu_{X_c}(z, r) : (z, r) \notin Y_c\}) && \text{by (1046), (1055)} \\
&= \inf\{\delta_{X_c,Y_c}(z, r) : z, r \in \mathbf{I}\} && \text{by (60)} \\
&= \Xi_{Y_c}(X_c) && \text{by Def. 83} \\
&= \Xi_{Y'}(X_c) && \text{by (1056)}
\end{aligned}$$

i.e.

$$\inf\{\delta_{X,Y}(z, r, c) : z, r \in \mathbf{I}\} = r_0 \tag{1057}$$

by (1054). We further notice that

$$\begin{aligned}
&\inf\{\delta_{X,Y}(z, r, \neg c) : z, r \in \mathbf{I}\} \\
&= \min(\inf\{\mu_X(z, r, \neg c) : z, r \in \mathbf{I}, (z, r, \neg c) \in Y\}, \\
&\qquad \inf\{1 - \mu_X(z, r, \neg c) : z, r \in \mathbf{I}, (z, r, \neg c) \notin Y\}) && \text{by (60)} \\
&= \min(\inf\{\mu_{X_{\neg c}}(z, r) : (z, r) \in X_{\neg c_{\geq \frac{1}{2}}}\}, \\
&\qquad \inf\{1 - \mu_{X_{\neg c}}(z, r) : (z, r) \notin X_{\neg c_{\geq \frac{1}{2}}}\}) && \text{by (1046), (1055)} \\
&= \min(\inf\{\mu_{X_{\neg c}}(z, r) : z, r \in \mathbf{I}, \mu_{X_{\neg c}}(z, r) \geq \tfrac{1}{2}\}, \\
&\qquad \inf\{1 - \mu_{X_{\neg c}}(z, r) : z, r \in \mathbf{I}, \mu_{X_{\neg c}}(z, r) < \tfrac{1}{2}\}) && \text{by Def. 29}
\end{aligned}$$

and hence

$$\inf\{\delta_{X,Y}(z, r, \neg c) : z, r \in \mathbf{I}\} \geq \tfrac{1}{2}\,. \tag{1058}$$

Therefore

$$\begin{aligned}
&\Xi_X(Y) \\
&= \inf\{\delta_{X,Y}(z, r, v) : z, r \in \mathbf{I}, v \in \mathbf{2}\} && \text{by (61)} \\
&= \min(\inf\{\delta_{X,Y}(z, r, c) : z, r \in \mathbf{I}\}, \\
&\qquad \inf\{\delta_{X,Y}(z, r, \neg c) : z, r \in \mathbf{I}\}) && \text{by splitting } \inf\text{-expression} \\
&= \min(r_0, \inf\{\delta_{X,Y}(z, r, \neg c) : z, r \in \mathbf{I}\})\,, && \text{by (1057)}
\end{aligned}$$

which proves that

$$\Xi_X(Y) = r_0 \tag{1059}$$

because $r_0 \in A_{Q'_c,X_c}(z_0) = \boxplus A_c(z_0) \subseteq D(\boxplus A_c) = [0, \frac{1}{2}]$ by (1045), (62) and Def. 91. Hence $r \leq \frac{1}{2} \leq \inf\{\delta_{X,Y}(z, r, \neg c) : z, r \in \mathbf{I}\}$ by (1058). In the case that $c = 1$, we directly obtain from (1049) that $Q_c(Y) = Q_1(Y) = Q'_1(Y_1) = Q'_c(Y_c)$. In the case that $c = 0$, we first observe that $Y_1 = Y_1(Y) = Y_1^+$, which is apparent from (1047), (1055) and (1048). Therefore $Q_c(Y) = Q_0(Y) = Q'_0(Y_0) = Q'_c(Y_c)$ by (1050), i.e. $Q_c(Y) = Q'_c(Y_c)$ is valid for both choices of $c \in \{0, 1\}$. In turn, $Q_c(Y) = Q'_c(Y_c) = Q'_c(Y') = z_0$ by (1056) and (1053). Hence indeed $r_0 \in A_{Q_c,X}(z_0)$ by Def. 86.

It remains to be shown that the converse subsumption $A_{Q_c,X}(z_0) \subseteq A_{Q'_c,X_c}(z_0)$ is also valid. Hence let $r_0 \in A_{Q_c,X}(z_0)$. By Def. 86, then, there exists $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$ such that

$$r_0 = \Xi_Y(X) \tag{1060}$$

$$z_0 = Q_c(Y)\,. \tag{1061}$$

In the following, it is convenient to discern two cases. In the first case that either $c = 1$, or $c = 0$ and $Y_1 = Y_1^+$, we obtain from (1049) and (1050) resp. that $Q_c(Y) = Q'_c(Y_c)$. Recalling (1061), this proves that

$$z_0 = Q_c(Y) = Q'_c(Y_c)\,, \tag{1062}$$

where $Y_c = Y_c(Y)$, see (1047). Let us abbreviate $r_* = \Xi_{Y_c}(X_c)$. It is apparent from (1062) and Def. 86 that $r_* \in A_{Q'_c,X_c}(z_0)$. In turn, we obtain from (1045) that $r_* \in \boxplus A_c$. We now observe that that

$$\begin{aligned}
r_* &= \Xi_{Y_c}(X_c)\\
&= \min(\inf\{\mu_{X_c}(z,r) : (z,r) \in Y_c\},\\
&\qquad \inf\{\,1 - \mu_{X_c}(z,r) : (z,r) \notin Y_c\}) && \text{by Def. 83}\\
&\geq \min\{\inf\{\mu_{X_c}(z,r) : (z,r) \in Y_c\},\\
&\qquad \inf\{\,1 - \mu_{X_c}(z,r) : (z,r) \notin Y_c\},\\
&\qquad \inf\{\mu_{X_{\neg c}}(z,r) : (z,r) \in Y_{\neg c}\},\\
&\qquad \inf\{\,1 - \mu_{X_{\neg c}}(z,r) : (z,r) \notin Y_{\neg c}\}\}\\
&= \min\{\inf\{\mu_X(z,r,c) : z,r \in \mathbf{I}, (z,r,c) \in Y\},\\
&\qquad \inf\{1 - \mu_X(z,r,c) : z,r \in \mathbf{I}, (z,r,c) \notin Y\},\\
&\qquad \inf\{\mu_X(z,r,\neg c) : z,r \in \mathbf{I}, (z,r,\neg c) \in Y\},\\
&\qquad \inf\{1 - \mu_X(z,r,\neg c) : z,r \in \mathbf{I}, (z,r,\neg c) \notin Y\}\} && \text{by (1046), (1047)}\\
&= \min(\inf\{\mu_X(z,r,v) : (z,r,v) \in Y\},\\
&\qquad \inf\{1 - \mu_X(z,r,v) : (z,r,v) \notin Y\})\\
&= \Xi_Y(X)\,, && \text{by Def. 83}
\end{aligned}$$

i.e. $r_* \geq r_0$ by (1060). Recalling Def. 91, we conclude from $r_* \in \boxplus A_c(z_0)$ and $r_0 \leq r_*$ that $r_0 \in \boxplus A_0(z_0)$ as well. Because of (1045), this proves that $r_0 \in A_{Q'_c,X_c}(z_0)$, as desired. In particular, we have shown that

$$A_{Q_c,X}(z_0) \subseteq A_{Q'_c,X_c}(z_0) = \boxplus A_c(z_0) \tag{1063}$$

in the case that either $c = 1$, or $c = 0$ and $Y_1 = Y_1^+$, again recalling (1045).
Now we consider the remaining case that $c = 0$ and $Y_1 \neq Y_1^+$. We abbreviate

$$z_1 = Q_1(Y)\,. \tag{1064}$$

We then know from (1063) that

$$r_0 = \Xi_Y(X) \in \boxplus A_1(z_1)\,. \tag{1065}$$

Therefore

$$\begin{aligned} Q_c(Y) &= Q_0(Y) && \text{because } c = 0 \\ &= \kappa(Q_1(Y), \Xi_Y(X))\,, && \text{by (1050) because } Y_1 \neq Y_1^+ \end{aligned}$$

i.e.

$$z_0 = Q_c(Y) = \kappa(z_1, r_0) \tag{1066}$$

by (1060), (1061) and (1064). In particular,

$$\begin{aligned} r_0 &\in \boxplus A_0(\kappa(z_1, r_0)) && \text{by (1040) and (1065)} \\ &= \boxplus A_0(z_0) && \text{by (1066)} \\ &= A_{Q'_0,(}(z_0) && \text{by (1045)} \\ &= A_{Q'_c,(}(z_0)\,. \end{aligned}$$

This finishes the proof that the subsumption $A_{Q_c,X}(z_0) \subseteq A_{Q'_c,X_c}(z_0)$ is valid in the second case as well; it hence holds unconditionally for for $c \in \{0,1\}$ and arbitrary $z_0 \in \mathbf{I}$. Combining this with our former result stated in (1052) and recalling that $A_{Q'_c,X_c} = \boxplus A_c$ by (1045), we hence obtain the desired $A_{Q_c,X} = \boxplus A_c$ for $c \in \{0,1\}$, i.e. (1051) is indeed valid.
Let us now notice that $Q_0 \leq Q_1$. To see this, consider $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$. We will treat separately two cases. Firstly if $Y_1(Y) = Y_1^+$, then

$$\begin{aligned} Q_1(Y) &= Q'_1(Y_1) && \text{by (1049)} \\ &= Q'_1(Y_1^+) && \text{by assumption that } Y_1 = Y_1^+ \\ &= z_+(A_{Q'_1,X}) && \text{by (1048) and L-124} \\ &= z_+(\boxplus A_1)\,, && \text{by (1045)} \end{aligned}$$

and hence

$$Q_1(Y) = \tfrac{1}{2} \tag{1067}$$

because $z_+(\boxplus A_1) = \frac{1}{2}$, see (63) and Def. 91. As concerns $Q_0(Y)$, then, we obtain that

$$\begin{aligned} Q_0(Y) &= Q_0'(Y_0) && \text{by (1050)} \\ &\leq z_+(A_{Q_0',X}) && \text{by Th-93 and Def. 87} \\ &= z_+(\boxplus A_0) && \text{by (1045)} \\ &= \tfrac{1}{2} && \text{see (63) and Def. 91} \\ &= Q_1(Y)\,, && \text{by (1067)} \end{aligned}$$

and hence indeed $Q_0(Y) \leq Q_1(Y)$. In the remaining case that $Y_1(Y) \neq Y_1^+$, we notice that

$$\begin{aligned} Q_0(Y) &= \kappa(Q_1(Y), \Xi_Y(X)) && \text{by (1050)} \\ &\leq Q_1(Y)\,, \end{aligned}$$

where the last step is apparent from (1039), because $\Xi_Y(X) \in A_{Q_1,X}(Q_1(Y)) = \boxplus A_1(Q_1(Y))$ by Def. 86 and (1045). To sum up, I have shown that $Q_0(Y) \leq Q_1(Y)$ regardless of $Y \in \mathcal{P}(\mathbf{I} \times \mathbf{I} \times \mathbf{2})$, and hence

$$Q_0 \leq Q_1\,. \tag{1068}$$

Therefore

$$\begin{aligned} \psi(A_0) &= \psi(\boxplus A_0) && \text{by } (\psi\text{-5}) \\ &= \psi(A_{Q_0,X}) && \text{by (1051)} \\ &= \mathcal{F}_\psi(Q_0)(X) && \text{by Def. 88} \\ &\leq \mathcal{F}_\psi(Q_1)(X) && \text{by (1036) and (1068)} \\ &= \psi(A_{Q_1,X}) && \text{by Def. 88} \\ &= \psi(\boxplus A_1) && \text{by (1051)} \\ &= \psi(A_1)\,. && \text{by } (\psi\text{-5}) \end{aligned}$$

Hence $A_0 \sqsubseteq A_1$ entails that $\psi(A_0) \leq \psi(A_1)$, and the desired property (ψ-4) is indeed valid.

B.25 Proof of Theorem 116

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi$ is a DFS. We then know from Def. 17 that $\mathcal{F}_\psi$ satisfies (Z-1)–(Z-6). Now let us show that ψ satisfies (ψ-1)–(ψ-5). We shall consider these conditions in turn.

ψ satisfies (ψ-1).
This is apparent from Th-108 because $\mathcal{F}_\psi$ satisfies (Z-1).

ψ satisfies (ψ-2).
This claim is immediate from Th-110 because $\mathcal{F}_\psi$ is known to satisfy (Z-1), (Z-2) and (Z-3).

ψ satisfies (ψ-5).
To this end, we recall from Th-1 that the induced disjunction of the DFS $\mathcal{F}_\psi$ is an s-norm. In turn, we obtain from Th-111 that $\mathcal{F}_\psi$ induces the standard disjunction $x \vee y = \max(x, y)$. It is then apparent from earlier work [9, Th-17.a, p. 20 and Th-25, p. 25] that $\mathcal{F}_\psi$ also induces the standard extension principle. Because $\mathcal{F}_\psi$ satisfies (Z-4) and (Z-6), we can hence conclude with Th-113 that ψ satisfies (ψ-5).

ψ satisfies (ψ-3).
Knowing that ψ satisfies (ψ-5), we can now apply Th-114 and conclude from the fact that $\mathcal{F}_\psi$ satisfies (Z-2) that ψ indeed satisfies (ψ-3).

ψ satisfies (ψ-4).
To see this, we again utilize that ψ satisfies (ψ-5). In addition, we know from Th-3 that the DFS $\mathcal{F}_\psi$ is monotonic. Hence Th-115 is applicable, an ψ indeed satisfies (ψ-4).

To sum up, I have shown that if $\mathcal{F}_\psi$ is a DFS, then ψ satisfies (ψ-1)–(ψ-5). Hence (ψ-1)–(ψ-5) are indeed necessary for $\mathcal{F}_\psi$ to be a DFS.

B.26 Proof of Theorem 117

Consider $A \in \mathbb{A}$ and let $z \in \mathbf{I}$. Then

$$\begin{aligned}
&\widehat{\boxplus} A(z) \\
&= \min(\sup A(z), \tfrac{1}{2}) && \text{by (73)} \\
&= \tfrac{1}{2} \cdot \min(2 \cdot \sup A(z), 1) \\
&= \tfrac{1}{2} \cdot \min(1 - (1 - 2 \cdot \sup A(z)), 1 - (1 - 1)) \\
&= \tfrac{1}{2}(1 - \max(1 - 2 \cdot \sup A(z), 0)) && \text{by De Morgan's law} \\
&= \tfrac{1}{2} - \tfrac{1}{2} \cdot \max(0, 1 - 2 \cdot \sup A(z)) \\
&= \tfrac{1}{2} - \tfrac{1}{2} \cdot s(A)(z)\,, && \text{by (80)}
\end{aligned}$$

which proves (82). As concerns (83), we discern two cases. Firstly if $\sup A(z) \leq \frac{1}{2}$, then

$$\begin{aligned}
&s(A)(z) \\
&= \max(0, 1 - 2 \cdot \sup A(z)) && \text{by (80)} \\
&= 1 - 2 \cdot \sup A(z) && \text{because } \sup A(z) \leq \tfrac{1}{2} \\
&= 1 - 2 \cdot \min(\sup A(z), \tfrac{1}{2}) && \text{because } \sup A(z) \leq \tfrac{1}{2} \\
&= 1 - 2 \cdot \widehat{\boxplus} A(z)\,. && \text{by (73)}
\end{aligned}$$

In the remaining case that $\sup A(z) > \frac{1}{2}$,

$$\begin{aligned}
&s(A)(z)\\
&= \max(0, 1 - 2 \cdot \sup A(z)) && \text{by (80)}\\
&= 0 && \text{because } \sup A(z) > \tfrac{1}{2}\\
&= 1 - 2 \cdot \tfrac{1}{2}\\
&= 1 - 2 \cdot \min(\sup A(z), \tfrac{1}{2}) && \text{because } \sup A(z) > \tfrac{1}{2}\\
&= 1 - 2 \cdot \widehat{\boxplus} A(z)\,. && \text{by (73)}
\end{aligned}$$

Hence indeed $s(A)(z) = 1 - 2 \cdot \widehat{\boxplus} A(z)$, regardless of $\sup A(z)$. This completes the proof that (83) is valid, as desired.

B.27 Proof of Theorem 118

Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that ψ satisfies (ψ-5). Further assume that $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ is defined by (84). We further define $\psi' : \mathbb{A} \longrightarrow \mathbf{I}$ according to (81), i.e.

$$\psi'(A) = \omega(s(A)) \tag{1069}$$

for all $A \in \mathbb{A}$. Now consider $z \in \mathbf{I}$. Then

$$\begin{aligned}
A_{s(A)}(z) &= [0, \tfrac{1}{2} - \tfrac{1}{2} s(A)(z)] && \text{by (85)}\\
&= [0, \tfrac{1}{2} - \tfrac{1}{2}(1 - 2\widehat{\boxplus} A(z))] && \text{by (83)}\\
&= [0, \tfrac{1}{2} - \tfrac{1}{2} + \widehat{\boxplus} A(z)]\\
&= [0, \widehat{\boxplus} A(z)]\\
&= \boxplus A(z)\,. && \text{by Def. 91}
\end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that

$$A_{s(A)} = \boxplus A\,. \tag{1070}$$

We notice that

$$\begin{aligned}
&\psi'(A)\\
&= \omega(s(A)) && \text{by (1069)}\\
&= \psi(A_{s(A)}) && \text{by (84)}\\
&= \psi(\boxplus A)\,, && \text{by (1070)}
\end{aligned}$$

and hence

$$\psi' = \psi \tag{1071}$$

by (ψ-5), noticing that $A \in \mathbb{A}$ was arbitrarily chosen. Therefore

$$\begin{aligned}
\mathcal{F}_\psi &= \mathcal{F}_{\psi'} && \text{by (1071)}\\
&= \mathcal{F}_\omega\,, && \text{by Th-98}
\end{aligned}$$

as desired.
The claim of the theorem that all $\mathcal{F}_\psi$-DFSes are $\mathcal{F}_\omega$-DFSes is then apparent from Th-116, because ψ satisfies (ψ-5) whenever $\mathcal{F}_\psi$ is a DFS.

B.28 Proof of Theorem 119

Lemma 161 *For all $A \in \mathbb{A}$, $\widehat{\boxplus}\boxplus A = \widehat{\boxplus} A$ and $\boxplus\boxplus A = \boxplus A$.*

Proof To see this, consider $A \in \mathbb{A}$ and let $z \in \mathbf{I}$. Then

$$\begin{aligned} \sup \boxplus A(z) &= \widehat{\boxplus} A(z) && \text{by (76)} \\ &= \min(\sup A(z), \tfrac{1}{2})\,, && \text{by (73)} \end{aligned}$$

i.e.

$$\sup \boxplus A(z) \leq \tfrac{1}{2}\,. \tag{1072}$$

Therefore

$$\begin{aligned} \widehat{\boxplus}\boxplus A(z) &= \min(\sup \boxplus A(z), \tfrac{1}{2}) && \text{by (73)} \\ &= \sup \boxplus A(z) && \text{by (1072)} \\ &= \widehat{\boxplus} A(z)\,, && \text{by (76)} \end{aligned}$$

and hence

$$\widehat{\boxplus}\boxplus A = \widehat{\boxplus} A\,, \tag{1073}$$

i.e. the first claim of the lemma is valid. As regards the second claim, we notice that

$$\begin{aligned} \boxplus\boxplus A(z) &= [0, \widehat{\boxplus}\boxplus A(z)] && \text{by Def. 91} \\ &= [0, \widehat{\boxplus} A(z)] && \text{by (1073)} \\ &= \boxplus A(z)\,. && \text{by Def. 91} \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $\boxplus\boxplus A = \boxplus A$, as desired.

Lemma 162 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. Then ψ satisfies* (ψ-5)*.*

Proof To see this, consider $A \in \mathbb{A}$. Further let $z \in \mathbf{I}$. Then

$$\begin{aligned} s(A)(z) &= 1 - 2\widehat{\boxplus} A(z) && \text{by (83)} \\ &= 1 - 2\widehat{\boxplus}\boxplus A(z) && \text{by L-161} \\ &= s(\boxplus A)(z) && \text{by (83)}\,. \end{aligned}$$

Because z was arbitrary, we conclude that

$$s(A) = s(\boxplus A)\,. \tag{1074}$$

In turn

$$\begin{aligned} \psi(A) &= \omega(s(A)) && \text{by (81)} \\ &= \omega(s(\boxplus A)) && \text{by (1074)} \\ &= \psi(\boxplus A)\,. && \text{by (81)} \end{aligned}$$

Hence $\psi(\boxplus A) = \psi(A)$ for all $A \in \mathbb{A}$, which proves that (ψ-5) is indeed valid.

Lemma 163 *Let $\psi, \psi' : \mathbb{A} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\psi = \mathcal{F}_{\psi'}$. Then $\psi = \psi'$.*

Proof To see this, consider $A \in \mathbb{A}$. By Th-94, there exists $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ with

$$A = A_{Q,X_1,\ldots,X_n}\,. \tag{1075}$$

Therefore

$$\begin{aligned} \psi(A) &= \psi(A_{Q,X_1,\ldots,X_n}) && \text{by (1075)} \\ &= \mathcal{F}_\psi(Q)(X_1, \ldots, X_n) && \text{by Def. 88} \\ &= \mathcal{F}_{\psi'}(Q)(X_1, \ldots, X_n) && \text{by assumption that } \mathcal{F}_\psi = \mathcal{F}_{\psi'} \\ &= \psi'(A_{Q,X_1,\ldots,X_n}) && \text{by Def. 88} \\ &= \psi'(A)\,. && \text{by (1075)} \end{aligned}$$

Hence indeed $\psi(A) = \psi(A')$. Because $A \in \mathbb{A}$ was arbitrary, this proves that $\psi = \psi'$.

Proof of Theorem 119

Consider $\omega : \mathbb{L} \longrightarrow \mathbf{I}$. We then know from Th-98 that $\mathcal{F}_\omega = \mathcal{F}_\psi$, provided we define $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ by (81). In addition, we know from L-162 that this particular choice of ψ satisfies (ψ-5). We now recall from L-163 that ψ is only mapping $\psi' : \mathbb{A} \longrightarrow \mathbf{I}$ which results in $\mathcal{F}_{\psi'} = \mathcal{F}_\omega$. This proves that every $\mathcal{F}_\omega$-QFM is an $\mathcal{F}_\psi$-QFM based on a mapping $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ which satisfies (ψ-5).
To see that the converse subsumption also holds, consider a choice of $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ which satisfies (ψ-5). We then obtain from Th-118 that $\mathcal{F}_\psi$ is an $\mathcal{F}_\omega$-QFM. Hence all $\mathcal{F}_\psi$-QFMs based on a mapping ψ that satisfies (ψ-5) are indeed $\mathcal{F}_\omega$-QFMs, as desired.

B.29 Proof of Theorem 120

Lemma 164 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ω satisfies* (ω-1)*, then ψ satisfies* (ψ-1)*.*

Proof In order to prove that ψ satisfies (ψ-1), we consider a choice of $A \in \mathbf{I}$ with $D(A) = \{1\}$. It is then immediate from Def. 87 that

$$A(z) = \begin{cases} \{1\} & : \quad z = z_+ \\ \varnothing & : \quad z \neq z_+\,. \end{cases} \tag{1076}$$

As concerns $s(A)$, we hence obtain in the case that $z = z_+$,

$$\begin{aligned} s(A)(z_+) &= \max(0, 1 - 2 \cdot \sup A(z_+)) && \text{by (80)} \\ &= \max(0, 1 - 2 \cdot \sup\{1\}) && \text{by (1076)} \\ &= \max(0, 1 - 2 \cdot 1) \\ &= \max(0, -1) \\ &= 0\,, \end{aligned}$$

and in the case that $z \neq z_+$,

$$\begin{aligned} s(A)(z_+) &= \max(0, 1 - 2 \cdot \sup A(z)) && \text{by (80)} \\ &= \max(0, 1 - 2 \cdot \sup \varnothing) && \text{by (1076)} \\ &= \max(0, 1 - 2 \cdot 0) \\ &= \max(0, 1) \\ &= 1\,. \end{aligned}$$

This proves that

$$s(A)(z) = \begin{cases} 0 & : \quad z = z_+ \\ 1 & : \quad z \neq z_+ \end{cases}$$

i.e.

$$(s(A))^{-1}([0, 1)) = \{z_+\}\,. \tag{1077}$$

Therefore

$$\begin{aligned} \psi(A) &= \omega(s(A)) && \text{by (81)} \\ &= z_+\,. && \text{by (ω-1) and (1077)} \end{aligned}$$

Because A with $D(A) = \{1\}$ was arbitrary, this completes the proof that ψ satisfies (ψ-1).

Lemma 165 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ω satisfies* (ω-2)*, then ψ satisfies* (ψ-2).

Proof Hence let $A, A' \in \mathbb{A}$ be given and suppose that

$$A(z) = A'(1 - z) \tag{1078}$$

for all $z \in \mathbf{I}$. Then apparently

$$\sup A(z) = \sup A'(1 - z) \tag{1079}$$

for all $z \in \mathbf{I}$. Therefore

$$\begin{aligned} s(A)(z) &= \max(0, 1 - 2 \cdot \sup A(z)) && \text{by (80)} \\ &= \max(0, 1 - 2 \cdot \sup A'(1 - z))\,, && \text{by (1079)} \end{aligned}$$

and hence

$$s(A)(z) = s(A')(1 - z) \tag{1080}$$

by (80). The claim of the lemma is then apparent from the following reasoning.

$$\begin{aligned} \psi(A) &= \omega(s(A)) && \text{by (81)} \\ &= 1 - \omega(s(A')) && \text{by (ω-2) and (1080)} \\ &= 1 - \psi(A')\,. && \text{by (81)} \end{aligned}$$

Hence (ψ-2) is indeed valid, as desired.

Lemma 166 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ω satisfies* (ω-3)*, then ψ satisfies* (ψ-3)*.*

Proof To see this, consider a choice of $A \in \mathbb{A}$ with $\mathrm{NV}(A) \subseteq \{0, 1\}$ and $r_+ \in A(1)$. We then know from (78) that

$$A(z) = \varnothing \tag{1081}$$

for all $z \in (0, 1)$. In addition, $r_+ \in A(1)$ and $r_+ \geq \frac{1}{2}$ entails that $z_+ = 1$ and

$$\sup A(1) \geq \tfrac{1}{2}\,. \tag{1082}$$

Therefore

$$\begin{aligned} s(A)(1) &= \max(0, 1 - 2 \cdot \sup A(1)) && \text{by (80)} \\ &= 0\,. && \text{by (1082)} \end{aligned}$$

In addition, we know that for $z \in (0, 1)$,

$$\begin{aligned} s(A)(z) &= \max(0, 1 - 2 \cdot \sup A(z)) && \text{by (80)} \\ &= \max(0, 1 - 2 \cdot \sup \varnothing) && \text{by (1081)} \\ &= \max(0, 1 - 2 \cdot 0) \\ &= \max(0, 1) \\ &= 1\,. \end{aligned}$$

To sum up, I have shown that $s(A)(1) = 0$ and $(s(A))^{-1}([0, 1)) \subseteq \{0, 1\}$, as required by ($\omega$-3). We further notice that

$$\sup A(0) \leq \tfrac{1}{2} \tag{1083}$$

because $0 \neq 1 = z_+$ and hence $r \leq 1 - r_+ \leq \frac{1}{2}$ for all $r \in A(0)$, see Def. 87. Therefore

$$\begin{aligned} \psi(A) &= \omega(s(A)) && \text{by (81)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} s(A)(0) && \text{by (ω-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \cdot \max(0, 1 - 2 \cdot \sup A(0)) && \text{by (80)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}(1 - 2 \cdot \sup A(0)) && \text{by (1083)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} - \sup A(0) \\ &= 1 - \sup A(0)\,. \end{aligned}$$

Hence ψ satisfies (ψ-3), as desired.

Lemma 167 *Let* $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ *be given and suppose that* $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ *is defined by* (81)*. If* ω *satisfies* (ω-4)*, then* ψ *satisfies* (ψ-4)*.*

Proof Let $A, A' \in \mathbb{A}$ be given and suppose that

$$A \sqsubseteq A' . \tag{1084}$$

As I will now prove, this entails that $s(A) \sqsubseteq s(A')$. By Def. 62, then, it must be shown that

a. for all $z \in \mathbf{I}$, $\inf\{s(A')(z') : z' \geq z\} \leq s(A)(z)$;

b. for all $z' \in \mathbf{I}$, $\inf\{s(A)(z) : z \leq z'\} \leq s(A')(z')$.

We shall consider these conditions in turn. As concerns condition **a.**, we recall from (80) that $s(A)(z) = \max(0, 1-2\cdot\sup A(z))$. In the case that $A(z) = \varnothing$, $\sup A(z) = 0$ and in turn, $s(A)(z) = \max(0, 1 - 2 \cdot 0) = 1$. Hence trivially $\inf\{s(A')(z') : z' \geq z\} \leq s(A)(z)$ because $s(A')(z') \leq 1$ for all $z' \in \mathbf{I}$, i.e. $\inf\{s(A')(z') : z' \geq z\} \leq 1$. In the remaining case that $A(z) \neq \varnothing$, we consider $\varepsilon > 0$. Because $A(z) \neq \varnothing$, there exists $r_0 \in A(z)$ such that

$$r > \sup A(z) - \tfrac{\varepsilon}{2} . \tag{1085}$$

By Def. 89, we conclude from $A \sqsubseteq A'$ that there exists $z_0 \geq z$ with $r_0 \in A'(z_0)$. In particular,

$$\sup A'(z_0) \geq r_0 > \sup A(z) - \tfrac{\varepsilon}{2} \tag{1086}$$

and hence

$$\begin{aligned} & s(A')(z_0) \\ & = \max(0, 1 - 2 \cdot \sup A'(z_0)) \\ & \leq \max(0, 1 - 2(\sup A(z) - \tfrac{\varepsilon}{2})) \\ & = \max(0, 1 - 2 \cdot \sup A(z) + \varepsilon) \\ & \leq \max(\varepsilon, 1 - 2 \cdot \sup A(z) + \varepsilon) \\ & = \max(0, 1 - 2 \cdot \sup A(z)) + \varepsilon \\ & = s(A)(z) + \varepsilon \end{aligned}$$

Therefore

$$\begin{aligned} \inf\{s(A')(z') : z' \geq z\} &\leq s(A')(z_0) && \text{because } z_0 \geq z \\ &\leq s(A)(z) + \varepsilon . \end{aligned}$$

$\varepsilon \to 0$ then yields

$$\inf\{s(A')(z') : z' \geq z\} \leq s(A)(z) ,$$

i.e. condition **a.** is valid. To see that condition **b.** is valid as well, consider $z' \in \mathbf{I}$. We recall from (80) that $s(A)'(z') = \max(0, 1 - 2 \cdot \sup A'(z'))$. In the following, we again discern two cases. Firstly if $A'(z') = \varnothing$, then $\sup A'(z') = 0$ and hence $s(A)'(z') = \max(0, 1 - 2 \cdot 0) = 1$. We then obtain from $s(A)(z) \leq 1$ for all $z \in \mathbf{I}$ that $\inf\{s(A)(z) : z \leq z'\} \leq 1 = s(A)'(z')$, as desired. In the remaining case that $A'(z') \neq \varnothing$, we consider $\varepsilon > 0$. Because $A'(z') \neq \varnothing$, there exists $r_0 \in A'(z')$ such that

$$r_0 > \sup A'(z') - \tfrac{\varepsilon}{2}\,.$$

Because $A \sqsubseteq A'$, we now obtain from Def. 89 that there exists $z_0 \leq z'$ with $r_0 \in A(z_0)$. In particular

$$\sup A(z_0) \geq r_0 > \sup A(z') - \tfrac{\varepsilon}{2}\,, \tag{1087}$$

for the given $z_0 \leq z'$. Therefore

$$\begin{aligned}
&\inf\{s(A)(z) : z \leq z'\} \\
&\leq s(A)(z_0) && \text{because } z_0 \leq z' \\
&= \max(0, 1 - 2 \cdot \sup A(z_0)) && \text{by (80)} \\
&\leq \max(0, 1 - 2 \cdot (\sup A'(z') - \tfrac{\varepsilon}{2})) && \text{by (1088)} \\
&= \max(0, 1 - 2 \cdot \sup A'(z') + \varepsilon) \\
&\leq \max(\varepsilon, 1 - 2 \cdot \sup A'(z') + \varepsilon) \\
&= \max(0, 1 - 2 \cdot \sup A'(z')) + \varepsilon\,,
\end{aligned}$$

i.e.

$$\inf\{s(A)(z) : z \leq z'\} \leq s(A')(z') + \varepsilon$$

by (80). Because $\varepsilon > 0$ was arbitrarily chosen, we conclude that

$$\inf\{s(A)(z) : z \leq z'\} \leq s(A')(z')\,.$$

To sum up, both condition **a.** and **b.** of Def. 62 are satisfied, which proves that

$$s(A) \sqsubseteq s(A')\,. \tag{1088}$$

The claim of the lemma is then obvious from the following computation.

$$\begin{aligned}
\psi(A) &= \omega(s(A)) && \text{by (81)} \\
&\leq \omega(s(A')) && \text{by (ω-4) and (1088)} \\
&= \psi(A')\,. && \text{by (81)}
\end{aligned}$$

Lemma 168 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ and $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given. Further suppose that ψ satisfies* (ψ-5)*. Then the following conditions are equivalent:*

a. ψ is defined in terms of ω according to (81)*;*

b. ω is defined in terms of ψ according to (84)*.*

Proof Suppose that ψ is defined in terms of ω according to (81), i.e. condition **a.** is satisfied.

To see that **b.** is valid as well, let us consider $\omega' : \mathbb{L} \longrightarrow \mathbf{I}$, defined by (84) in terms of ψ. We can then show that ω is defined in terms of ψ according to (84) by proving that $\omega = \omega'$. Hence let $s \in \mathbb{L}$. By Th-41, there exists $Q : \mathcal{P}(\mathbf{I} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ such that

$$s = s_{Q,X} . \tag{1089}$$

We can hence proceed as follows.

$$\begin{aligned}
\omega(s) &= \omega(s_{Q,X}) && \text{by (1089)} \\
&= \mathcal{F}_\omega(Q)(X) && \text{by Def. 61} \\
&= \mathcal{F}_\psi(Q)(X) && \text{by Th-98} \\
&= \mathcal{F}_{\omega'}(Q)(X) && \text{by Th-118 and } (\psi\text{-5}) \\
&= \omega'(s_{Q,X}) && \text{by Def. 61} \\
&= \omega'(s) . && \text{by (1089)}
\end{aligned}$$

Now let us show that condition **b.** entails condition **a.** Hence suppose that ω is defined in terms of ψ according to (84). We now consider $\psi' : \mathbb{A} \longrightarrow \mathbf{I}$, defined according to (81) in terms of ω. Apparently, we can show that **a.** holds by proving that $\psi = \psi'$. Hence let $A \in \mathbb{A}$. Then by Th-94, there exists $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ with

$$A = A_{Q,X_1,\ldots,X_n} . \tag{1090}$$

Therefore

$$\begin{aligned}
\psi(A) &= \psi(A_{Q,X_1,\ldots,X_n}) && \text{by (1090)} \\
&= \mathcal{F}_\psi(Q)(X_1, \ldots, X_n) && \text{by Def. 88} \\
&= \mathcal{F}_\omega(Q)(X_1, \ldots, X_n) && \text{by Th-118, } (\psi\text{-5}) \\
&= \mathcal{F}_{\psi'}(Q)(X_1, \ldots, X_n) && \text{by Th-98} \\
&= \psi'(A_{Q,X_1,\ldots,X_n}) && \text{by Def. 88} \\
&= \psi'(A) . && \text{by (1090)}
\end{aligned}$$

Hence $\psi = \psi'$, which completes the proof that condition **b.** is entailed by condition **a.**

Lemma 169 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ψ satisfies* (ψ-1)*, then ω satisfies* (ω-1).

Proof To see this, let $s \in \mathbb{L}$ be given with $s^{-1}([0, 1)) = \{a\}$ for some $a \in \mathbf{I}$. In other words,

$$s(z) = 1 \tag{1091}$$

for all $z \in \mathbf{I}$ with $z \neq a$. We can then conclude from Def. 60 that

$$s(a) = 0\,. \tag{1092}$$

Therefore

$$\begin{aligned} A_s(z) &= [0, \tfrac{1}{2} - \tfrac{1}{2}s(z)] && \text{by (85)} \\ &= \begin{cases} [0, \frac{1}{2} - \frac{1}{2} \cdot 0] & : \quad z = a \\ [0, \frac{1}{2} - \frac{1}{2} \cdot 1] & : \quad z \neq a\,, \end{cases} && \text{by (1091) and (1092)} \end{aligned}$$

i.e.

$$A_s(z) = \begin{cases} [0, \frac{1}{2}] & : \quad z = a \\ \{0\} & : \quad z \neq a \end{cases} \tag{1093}$$

for all $z \in \mathbf{I}$. Now consider $A \in \mathbb{A}$, defined by

$$A(z) = \begin{cases} \{1\} & : \quad z = a \\ \varnothing & : \quad z \neq a \end{cases} \tag{1094}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 91, (1093) and (1094) that

$$A_s = \boxplus A\,. \tag{1095}$$

In addition, $D(A) = \{1\}$ and $z_+(A) = a$. Therefore

$$\begin{aligned} \omega(s) &= \psi(A_s) && \text{by Th-118 and L-168} \\ &= \psi(\boxplus A) && \text{by (1095)} \\ &= \psi(A) && \text{by } (\psi\text{-5}) \\ &= z_+ && \text{by } (\psi\text{-1}) \\ &= a\,. \end{aligned}$$

Hence ω indeed satisfies (ω-1).

Lemma 170 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ψ satisfies* (ψ-2)*, then ω satisfies* (ω-2).

Proof Hence consider $s, s' \in \mathbb{L}$ with

$$s'(z) = s(1 - z) \tag{1096}$$

for all $z \in \mathbf{I}$. We then obtain from (85) that

$$A_{s'}(z) = [0, \tfrac{1}{2} - \tfrac{1}{2}s'(z)] = [0, \tfrac{1}{2} - \tfrac{1}{2}s(1 - z)] = A_s(1 - z) \tag{1097}$$

for all $z \in \mathbf{I}$. Therefore

$$\begin{aligned} \omega(s') &= \psi(A_{s'}) && \text{by Th-118 and L-168} \\ &= 1 - \psi(A_s) && \text{by } (\psi\text{-2}) \\ &= 1 - \omega(s)\,. && \text{by Th-118 and L-168} \end{aligned}$$

Lemma 171 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81). *If ψ satisfies* (ψ-3)*, then ω satisfies* (ω-3).

Proof Hence let $s \in \mathbb{L}$ be given. Further suppose that

$$s(1) = 0 \tag{1098}$$

and $s^{-1}([0,1)) \subseteq \{0,1\}$. Then in particular

$$s(z) = 1 \tag{1099}$$

for all $z \in (0,1)$. Therefore

$$\begin{aligned} A_s(z) &= [0, \tfrac{1}{2} - \tfrac{1}{2}s(z)] && \text{by (85)} \\ &= \begin{cases} [0, \tfrac{1}{2} - \tfrac{1}{2}s(0)] & : \quad z = 0 \\ [0, \tfrac{1}{2} - \tfrac{1}{2} \cdot 1] & : \quad z \in (0,1) \\ [0, \tfrac{1}{2} - \tfrac{1}{2} \cdot 0] & : \quad z = 1 \end{cases} && \text{by (1098), (1099)} \end{aligned}$$

and hence

$$A_s(z) = \begin{cases} [0, \tfrac{1}{2} - \tfrac{1}{2}s(0)] & : \quad z = 0 \\ \{0\} & : \quad z \in (0,1) \\ [0, \tfrac{1}{2}] & : \quad z = 1 \end{cases} \tag{1100}$$

for all $z \in \mathbf{I}$. We now define $A \in \mathbb{A}$ by

$$A(z) = \begin{cases} \{\tfrac{1}{2} - \tfrac{1}{2}s(0)\} & : \quad z = 0 \\ \varnothing & : \quad z \in (0,1) \\ \{\tfrac{1}{2}\} & : \quad z = 1 \end{cases} \tag{1101}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 91, (1100) and (1101) that

$$A_s = \boxplus A \,. \tag{1102}$$

We further notice that $\mathrm{NV}(A) \subseteq \{0,1\}$ and $r_+ = \frac{1}{2} \in A(1)$, i.e. ($\psi$-3) is applicable. The claim of the lemma can hence be proven as follows.

$$\begin{aligned} \omega(s) &= \psi(A_s) && \text{by Th-118 and L-168} \\ &= \psi(\boxplus A) && \text{by (1102)} \\ &= \psi(A) && \text{by } (\psi\text{-5}) \\ &= 1 - \sup A(0) && \text{by } (\psi\text{-3}) \\ &= 1 - (\tfrac{1}{2} - \tfrac{1}{2}s(0)) && \text{by (1101)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}s(0)\,, \end{aligned}$$

i.e. ω satisfies (ω-3).

Lemma 172 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (ψ-4). Consider some $A \in \mathbb{A}$ and suppose that $A' : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$A'(z) = \begin{cases} [0, \alpha(z)] & : \quad \alpha(z) \in \boxplus A(z) \\ [0, \alpha(z)) & : \quad \alpha(z) \notin \boxplus A(z) \end{cases} \tag{1103}$$

where

$$\alpha(z) = \alpha_A(z) = \min(\sup\{\widehat{\boxplus}A(z') : z' \leq z\}, \sup\{\widehat{\boxplus}A(z') : z' \geq z\})\,, \tag{1104}$$

for all $z \in \mathbf{I}$. Then $A' \in \mathbb{A}$ and $\psi(A') = \psi(\boxplus A)$.

Proof I first prove that $A' \in \mathbb{A}$. To this end, we notice that $r_+(\boxplus A) = \frac{1}{2}$, which is apparent from Def. 91. Hence there exists $z_+ \in \mathbf{I}$ with $\frac{1}{2} \in \boxplus A(z_+)$, and $\boxplus A(z) \subseteq [0, \frac{1}{2}]$ for all $z \in \mathbf{I}$. In turn, we obtain from (76) that $\widehat{\boxplus}A(z_+) = \frac{1}{2}$ and $\widehat{\boxplus}A(z) \leq \frac{1}{2}$ for all $z \in \mathbf{I}$. Recalling (1104), it is then apparent that $\alpha(z_+) = \frac{1}{2}$ and $\alpha(z) \leq \frac{1}{2}$ for all $z \in \mathbf{I}$. In particular,

$$A'(z_+) = [0, \tfrac{1}{2}] \tag{1105}$$

and

$$A'(z) \subseteq [0, \tfrac{1}{2}] \tag{1106}$$

for all $z \in \mathbf{I}$. We hence observe from (1103) that $D(A') \cap [\frac{1}{2}, 1] = \{\frac{1}{2}\}$, i.e. $r_+(A') = \frac{1}{2}$. In addition, we conclude from (1105) and (1106) that $D(A') = [0, \frac{1}{2}]$. In particular, if $D' \subseteq D(A') = [0, \frac{1}{2}]$ and $D' \neq \varnothing$, then $\inf D' \in [0, \frac{1}{2}]$ and hence $\inf D' \in D(A')$. We conclude from Def. 85 that $D(A') \in \mathbb{D}$. Because $r_+(A') = \frac{1}{2}$, this directly proves that $A' \in \mathbb{A}$, see Def. 87.
It remains to be shown that $\psi(A') = \psi(\boxplus A)$. To this end, I first prove that $A' \sqsubseteq \boxplus A$. By Def. 89, we must now show that

a. for all $z \in \mathbf{I}$ and all $r \in A'(z)$, there exists $z' \geq z$ with $r \in \boxplus A(z')$;

b. for all $z' \in \mathbf{I}$ and all $r \in \boxplus A(z')$, there exists $z \leq z'$ with $r \in A'(z)$.

Let us first consider condition **a.**; hence let $z \in \mathbf{I}$ and $r \in A'(z)$. It is now useful to discern two cases. If $\alpha(z) \in \boxplus A(z)$ then $A'(z) = [0, \alpha(z)]$ by (1103) and hence $r \leq \alpha(z)$. On the other hand, $\boxplus A(z) = [0, \widehat{\boxplus}A(z)]$ by Def. 91; hence $\alpha(z) \in \boxplus A(z)$ entails that $\alpha(z) \leq \widehat{\boxplus}A(z)$. Hence $r \leq \alpha(z) \leq \widehat{\boxplus}A(z)$, which proves that $r \in [0, \widehat{\boxplus}A(z)] = \boxplus A(z)$. This proves that $z = z'$ is a valid choice for $z' \geq z$ with $r \in \boxplus A(z')$, i.e. condition **a.** holds.
In the remaining case that $\alpha(z) \notin \boxplus A(z)$, we know from (1103) that $A'(z) = [0, \alpha(z))$. Hence $r \in A'(z)$ entails that $r < \alpha(z)$. In particular, $\alpha(z) > 0$. We now recall from (1104) that $\alpha(z) = \min(\sup\{\widehat{\boxplus}A(z') : z' \leq z\}, \sup\{\widehat{\boxplus}A(z') : z' \geq z\})$. In particular, $\sup\{\widehat{\boxplus}A(z') : z' \geq z\} \geq \alpha(z) > 0$, and hence

$$\{\widehat{\boxplus}A(z') : z' \geq z\} \neq \varnothing\,. \tag{1107}$$

In addition, $\sup\{\widehat{\boxplus}A(z') : z' \geq z\} \geq \alpha(z) > r$. Recalling (1107), then, we conclude that there exists $z' \geq z$ with $\widehat{\boxplus}A(z') > r$. Hence $r \in [0, \widehat{\boxplus}A(z')] = \boxplus A(z')$ for the given $z' \geq z$, see Def. 91. This proves that condition **a.** is valid in the second case as well, i.e. it holds unconditionally.

Next we turn to condition **b.** stated above. I first show that

$$\boxplus A(z) \subseteq A'(z) \tag{1108}$$

for all $z \in \mathbf{I}$. This is apparent if we notice from (1104) that $\alpha(z) \geq \widehat{\boxplus}A(z)$. In the case that $\alpha(z) > \widehat{\boxplus}A(z)$, we hence obtain from Def. 91 and (1103) that $\alpha(z) \notin [0, \widehat{\boxplus}A(z)] = \boxplus A(z)$ and hence $\boxplus A(z) = [0, \widehat{\boxplus}A(z)] \subseteq [0, \alpha(z)) = A'(z)$ because $\alpha(z) > \widehat{\boxplus}A(z)$. In the remaining case that $\alpha(z) = \widehat{\boxplus}A(z)$, we clearly have $\alpha(z) = \widehat{\boxplus}A(z) \in [0, \widehat{\boxplus}A(z)] = \boxplus A(z)$, see Def. 91. Hence $A'(z) = [0, \alpha(z)]$ by (1103) and in turn, $\boxplus A(z) = [0, \widehat{\boxplus}A(z)] = [0, \alpha(z)] = A'(z)$ by Def. 91, in particular $\boxplus A(z) \subseteq A'(z)$. This proves that (1108) is indeed valid.

Based on this result, it is now trivial to show that condition **b.** is satisfied. Hence let $z' \in \mathbf{I}$ and $r \in \boxplus A(z')$. Then $r \in A'(z')$ by (1108). Hence $z = z'$ is a valid choice for $z \leq z'$ which results in $r \in A'(z)$.

I have shown that both preconditions **a.** and **b.** for $A' \sqsubseteq \boxplus A$ are fulfilled. Hence

$$A' \sqsubseteq \boxplus A$$

by Def. 89. We can then conclude from (ψ-4) that

$$\psi(A') \leq \psi(\boxplus A)\,. \tag{1109}$$

Next I prove that $\boxplus A \sqsubseteq A'$. Again by Def. 89, this can be shown by proving that

a. for all $z \in \mathbf{I}$ and all $r \in \boxplus A(z)$, there exists $z' \geq z$ with $r \in A'(z')$;

b. for all $z' \in \mathbf{I}$ and all $r \in A'(z')$, there exists $z \leq z'$ with $r \in \boxplus A(z)$.

Condition **a.** is again trivial. To see this, consider $z \in \mathbf{I}$ and $r \in \boxplus A(z)$. By (1108), then, we know that $r \in A'(z)$. Hence $z' = z$ is a valid choice for $z' \geq z$ with $r \in A'(z')$.

Now we consider condition **b.**; hence let $z' \in \mathbf{I}$ and $r \in A'(z')$. It is convenient to discern two cases. Firstly if $\alpha(z') \in \boxplus A(z')$, then we know from (1103) that $A'(z') = [0, \alpha(z')]$ and hence $r \in A'(z')$ entails that $r \leq \alpha(z')$. In turn, we know from Def. 91 that $\boxplus A(z') = [0, \widehat{\boxplus}A(z')]$; hence $\alpha(z') \in \boxplus A(z')$ and $r \leq \alpha(z')$ entails that $r \leq \widehat{\boxplus}A(z')$ and hence $r \in \boxplus A(z')$ as well. This proves that $z = z'$ is a legal choice for $z \leq z'$ with $r \in \boxplus A(z')$, and condition **b.** holds.

In the remaining case that $\alpha(z') \notin \boxplus A(z)$, we conclude from $\boxplus A(z) = [0, \widehat{\boxplus}A(z)]$ that $\alpha(z') > \widehat{\boxplus}A(z)$, see Def. 91. In particular, $\alpha(z') > 0$. Recalling from (1104) that $\alpha(z') = \min(\sup\{\widehat{\boxplus}A(z) : z \leq z'\}, \sup\{\widehat{\boxplus}A(z) : z \geq z'\})$, we hence know that $\sup\{\widehat{\boxplus}A(z) : z \leq z'\} \geq \alpha(z') > 0$, and hence

$$\{\widehat{\boxplus}A(z) : z \leq z'\} \neq \varnothing\,. \tag{1110}$$

In addition,

$$\sup\{\widehat{\boxplus}A(z) : z \leq z'\} \geq \alpha(z) > r\,, \tag{1111}$$

because $r \in A'(z') = [0, \alpha(z'))$, which is apparent from (1103) and the assumption $\alpha(z') \notin \boxplus A(z')$. Recalling (1110) and (1111), then, we conclude that there exists $z \leq z'$ with $\widehat{\boxplus}A(z) > r$. Hence $r \in [0, \widehat{\boxplus}A(z)] = \boxplus A(z)$ for the given $z \leq z'$, see Def. 91. This proves that condition **b.** is valid in the second case as well.
We hence conclude from Def. 89 that

$$\boxplus A \sqsubseteq A'\,.$$

In turn, we conclude from (ψ-4) that

$$\psi(\boxplus A) \leq \psi(A') \tag{1112}$$

Combining inequations (1109) and (1112), we then obtain that $\psi(A') = \psi(\boxplus A)$, as desired.

Lemma 173 *Let $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ be a given mapping which satisfies* (ψ-4) *and* (ψ-5)*. Consider some $A \in \mathbb{A}$ and suppose that $A^\sharp : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ is defined by*

$$A^\sharp(z) = [0, \alpha(z)] \tag{1113}$$

for all $z \in \mathbf{I}$, where $\alpha(z)$ is defined by (1104)*. Then $A^\sharp \in \mathbb{A}$ and $\psi(A^\sharp) = \psi(\boxplus A)$.*

Proof Let $A' \in \mathbb{A}$ be defined by (1103). Then apparently $A^\sharp = \boxplus A'$. In particular, $A^\sharp \in \mathbb{A}$, and

$$\begin{aligned} \psi(A^\sharp) &= \psi(\boxplus A') && \text{because } A^\sharp = \boxplus A' \\ &= \psi(A') && \text{by } (\psi\text{-5}) \\ &= \psi(A)\,. && \text{by L-172} \end{aligned}$$

Lemma 174 *For all $s \in \mathbb{L}$, $(A_s)^\sharp = A_{(s^\ddagger)}$.*

Proof To see this, let $s \in \mathbb{L}$ and $z \in \mathbf{I}$. We notice that

$$\begin{aligned} \widehat{\boxplus}A_s(z) &= \min(\sup A_s(z), \tfrac{1}{2}) && \text{by (73)} \\ &= \min(\sup[0, \tfrac{1}{2} - \tfrac{1}{2}s(z)], \tfrac{1}{2}) && \text{by (85)} \\ &= \min(\tfrac{1}{2} - \tfrac{1}{2}s(z), \tfrac{1}{2})\,, \end{aligned}$$

and hence

$$\widehat{\boxplus}A_s(z) = \tfrac{1}{2} - \tfrac{1}{2}s(z) \tag{1114}$$

because $s(z) \in \mathbf{I}$. Therefore

$$\begin{aligned}
\alpha_{A_s}(z) &= \min(\sup\{\widehat{\boxplus} A_s(z') : z' \le z\}, \sup\{\widehat{\boxplus} A_s(z') : z' \ge z\}) && \text{by (1104)} \\
&= \min(\sup\{\tfrac{1}{2} - \tfrac{1}{2}s(z') : z' \le z\}, \sup\{\tfrac{1}{2} - \tfrac{1}{2}s(z') : z' \ge z\}) && \text{by (1114)} \\
&= \min(\tfrac{1}{2} - \tfrac{1}{2}\inf\{s(z') : z' \le z\}, \tfrac{1}{2} - \tfrac{1}{2}\inf\{s(z') : z' \ge z\}) \\
&= \tfrac{1}{2} - \tfrac{1}{2}\max(\inf\{s(z') : z' \le z\}, \inf\{s(z') : z' \ge z\})\,,
\end{aligned}$$

i.e.

$$\alpha_{A_s}(z) = \tfrac{1}{2} - \tfrac{1}{2}s^\ddagger(z)\,. \tag{1115}$$

by Def. 65. In turn

$$\begin{aligned}
(A_s)^\sharp(z) &= [0, \alpha_{A_s}(z)] && \text{by (1113)} \\
&= [0, \tfrac{1}{2} - \tfrac{1}{2}s^\ddagger(z)] && \text{by (1115)} \\
&= A_{(s^\ddagger)}(z)\,. && \text{by (85)}
\end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrarily chosen, this proves that $(A_s)^\sharp = A_{(s^\ddagger)}$, as desired.

Lemma 175 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ψ satisfies* (ψ-4)*, then ω is ‡-invariant, i.e. $\omega(s) = \omega(s^\ddagger)$ for all $s \in \mathbb{L}$.*

Proof We recall from L-162 that ψ is $\boxplus$-invariant, i.e. ψ satisfies (ψ-5). Because ψ also satisfies (ψ-4) by assumption, we know that lemma L-173 is applicable. The proof of the lemma hence reduces to the following simple computation.

$$\begin{aligned}
\omega(s) &= \psi(A_s) && \text{by Th-118 and L-168} \\
&= \psi(\boxplus A_s) && \text{by } (\psi\text{-5}) \\
&= \psi((A_s)^\sharp) && \text{by L-173} \\
&= \psi(A_{(s^\ddagger)}) && \text{by L-174} \\
&= \omega(s^\ddagger)\,. && \text{by Th-118 and L-168}
\end{aligned}$$

Because $s \in \mathbb{L}$ was arbitrary, this proves that ω is indeed ‡-invariant.

Lemma 176 *Let $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ be given and suppose that $\psi = \psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (81)*. If ψ satisfies* (ψ-4)*, then ω satisfies* (ω-4)*.*

Proof We already know from L-175 that ω is ‡-invariant. Recalling Th-49, we can hence reduce the proof of (ω-4) to the proof of the simpler condition that

$$\omega(s) \le \omega(s')$$

for every choice of $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$.
Hence consider $s, s' \in \mathbb{L}$ with $s \trianglelefteq s'$. Let us recall from (85) that $A_s, A_{s'} \in \mathbb{A}$ are

defined by

$$A_s(z) = [0, \tfrac{1}{2} - \tfrac{1}{2}s(z)] \tag{1116}$$

$$A_{s'}(z) = [0, \tfrac{1}{2} - \tfrac{1}{2}s'(z)] \tag{1117}$$

for all $z \in \mathbf{I}$. Let us now prove that $A_s \sqsubseteq A_{s'}$. We must hence show that

a. for all $z \in \mathbf{I}$ and all $r \in A_s(z)$, there exists $z' \geq z$ with $r \in A_{s'}(z')$;

b. for all $z' \in \mathbf{I}$ and all $r \in A_{s'}(z')$, there exists $z \leq z'$ with $r \in A_s(z)$;

see Def. 89. I first prove that condition **a.** is satisfied. Hence let $z \in \mathbf{I}$ and $r \in A_s(z)$. Then $r \leq \frac{1}{2} - \frac{1}{2}s(z)$ by (1116). We now recall that by Def. 64, $s \trianglelefteq s'$ entails that there exists $z' \geq z$ with $s'(z') \leq s(z)$. In particular, $\frac{1}{2} - \frac{1}{2}s'(z') \geq \frac{1}{2} - \frac{1}{2}s(z) \geq r$ and hence $r \in [0, \frac{1}{2} - \frac{1}{2}s'(z')] = A_{s'}(z')$ for the given $z' \geq z$, see (1117). This completes the proof that condition **a.** is satisfied. As concerns **b.**, we consider some $z' \in \mathbf{I}$ and $r \in A_{s'}(z')$. Hence by (1117), $r \in [0, \frac{1}{2} - \frac{1}{2}s'(z')]$, in particular $r \leq \frac{1}{2} - \frac{1}{2}s'(z')$. By Def. 64, we can conclude from $s \trianglelefteq s'$ that there exists $z \leq z'$ with $s(z) \leq s'(z')$. Hence $\frac{1}{2} - \frac{1}{2}s(z) \geq \frac{1}{2} - \frac{1}{2}s'(z') \geq r$ and hence $r \in [0, \frac{1}{2} - \frac{1}{2}s(z)] = A_s(z)$ for the given $z \leq z'$, see (1116). This proves that condition **b.** is valid as well. Because both preconditions stated in Def. 89 are satisfied, we conclude that

$$A_s \sqsubseteq A_{s'} \,. \tag{1118}$$

Therefore

$$\begin{aligned} \omega(s) &= \psi(A_s) && \text{by Th-118 and L-168} \\ &\leq \psi(A_{s'}) && \text{by } (\psi\text{-4}) \text{ and } (1118) \\ &= \omega(s') \,. && \text{by Th-118 and L-168} \end{aligned}$$

This proves that $\omega(s) \leq \omega(s')$ whenever $s \trianglelefteq s'$. Recalling that ω is also ‡-invariant, we can apply Th-49 and conclude that $\omega(s) \leq \omega(s')$ whenever $s \sqsubseteq s'$. Hence ω satisfies (ω-4), as desired.

Proof of Theorem 120

Consider $\omega : \mathbb{L} \longrightarrow \mathbf{I}$ and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by (81). We then know from L-162 that ψ satisfies (ψ-5). In particular, part **e.** of the theorem is valid. As concerns parts **a.**–**d.**, the claimed equivalences are apparent from the following lemmata:

a. L-164 and L-169;

b. L-165 and L-170;

c. L-166 and L-171;

d. L-167 and L-176,

which are applicable because ψ is known to satisfy (ψ-5).

B.30 Proof of Theorem 121

The only condition that still requires some work is (ψ-5); the independence of the other conditions is clear from Th-46 and Th-120. In order to prove that (ψ-5) is independent of (ψ-1)–(ψ-4), we consider the following choice of $\psi_{\boxplus} : \mathbb{A} \longrightarrow \mathbf{I}$, defined by

$$\psi_{\boxplus}(A) = \begin{cases} \ell(A) & : \quad \ell(A) > \frac{1}{2} \\ u(A) & : \quad u(A) < \frac{1}{2} \\ \frac{1}{2} & : \quad \ell(A) \leq \frac{1}{2} \leq u(A) \end{cases} \tag{1119}$$

where

$$\ell(A) = \begin{cases} \max(\inf \mathrm{NV}(A), 1 - \sup \cup\{A(z) : z < 1\}) & : \quad A(1) \cap [\frac{1}{2}, 1] \neq \varnothing \\ \max(\inf \mathrm{NV}(A), \sup A(1)) & : \quad A(1) \cap [\frac{1}{2}, 1] = \varnothing \end{cases} \tag{1120}$$

$$u(A) = \begin{cases} \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) & : \quad A(0) \cap [\frac{1}{2}, 1] \neq \varnothing \\ \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) & : \quad A(0) \cap [\frac{1}{2}, 1] = \varnothing \end{cases} \tag{1121}$$

for all $A \in \mathbb{A}$.
To see that $\psi_{\boxplus}$ is well-defined, we first make some observations on how the computation of $\ell(A)$ and $u(A)$ can be simplified.

Lemma 177 *For all $A \in \mathbb{A}$,*

a. If $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $\ell(A) = \sup A(1)$;

b. If $A(0) \neq \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] = \varnothing$, then $\ell(A) = \sup A(1)$;

c. If $A(0) \neq \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $\ell(A) = 1 - \sup \cup\{A(z) : z < 1\}$;

d. If $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ and $A(1) = \varnothing$, then $\ell(A) = \inf \mathrm{NV}(A)$.

Proof We first consider case **a.** Hence let $A \in \mathbb{A}$ be given with $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. In particular, $A(0) \neq \varnothing$. We hence know from (78) that

$$\inf \mathrm{NV}(A) = 0\,. \tag{1122}$$

In addition, Def. 87 permits us to conclude from $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ that $A(0) \cap [\frac{1}{2}, 1] = \{r_+\}$.

- If $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $A(1) \cap [\frac{1}{2}, 1] = \{r_+\}$ as well. We hence conclude from Def. 87 that $r_+ = \frac{1}{2}$. In particular, $\frac{1}{2} = r_+ \in A(0)$, $\frac{1}{2} = r_+ \in A(1)$ and $A(z) \subseteq [0, \frac{1}{2}]$ for all $z \in \mathbf{I}$. This entails that

$$\sup A(1) = \tfrac{1}{2}\,, \tag{1123}$$

and also that

$$\sup \cup\{A(z) : z < 1\} = \tfrac{1}{2}\,. \tag{1124}$$

In turn,

$$\begin{aligned}
\ell(A) &= \max(\inf \mathrm{NV}(A), 1 - \sup \cup\{A(z) : z < 1\}) && \text{by (1120)}\\
&= \max(0, \tfrac{1}{2}) && \text{by (1122) and (1124)}\\
&= \tfrac{1}{2}\\
&= \sup A(1)\,. && \text{by (1123)}
\end{aligned}$$

- If $A(1) \cap [\frac{1}{2}, 1] = \varnothing$, then

$$\begin{aligned}
\ell(A) &= \max(\inf \mathrm{NV}(A), \sup A(1)) && \text{by (1120)}\\
&= \max(0, \sup A(1)) && \text{by (1122)}\\
&= \sup A(1)\,.
\end{aligned}$$

Now we consider case **b.** Hence suppose that $A(0) \neq \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] = \varnothing$. Because $A(0) \neq \varnothing$, we know from (78) that $\inf \mathrm{NV}(A) = 0$. It is then immediate from (1120) that $\ell(A) = \max(\inf \mathrm{NV}(A), \sup A(1)) = \max(0, \sup A(1)) = \sup A(1)$, as desired.
As concerns part **c.**, suppose that $A(0) \neq \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$. Then in particular $\inf \mathrm{NV}(A) = 0$ by (78) and hence

$$\begin{aligned}
\ell(A) &= \max(\inf \mathrm{NV}(A), 1 - \sup \cup\{A(z) : z < 1\}) && \text{by (1120)}\\
&= \max(0, 1 - \sup \cup\{A(z) : z < 1\}) && \text{because } \inf \mathrm{NV}(A) = 0\\
&= 1 - \sup \cup\{A(z) : z < 1\}\,.
\end{aligned}$$

Finally let us prove that part **d.** of the lemma is also valid. Hence suppose that $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ and $A(1) = \varnothing$. Then clearly $\sup A(1) = 0$ and $A(1) \cap [\frac{1}{2}, 1] = \varnothing$. It is hence immediate from (1120) that $\ell(A) = \max(\inf \mathrm{NV}(A), \sup A(1)) = \max(\inf \mathrm{NV}(A), 0) = \inf \mathrm{NV}(A)$.

Lemma 178 *For all $A \in \mathbb{A}$,*

a. If $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $u(A) = 1 - \sup A(0)$;

b. If $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ and $A(1) \neq \varnothing$, then $u(A) = 1 - \sup A(0)$;

c. If $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $A(1) \neq \varnothing$, then $u(A) = \sup\{A(z) : z > 0\}$;

d. If $A(0) = \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] = \varnothing$, then $u(A) = \sup \mathrm{NV}(A)$.

Proof Analogous to the proof of L-177. As concerns case **a.**, we consider $A \in \mathbb{A}$ with $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$. In particular, $A(1) \neq \varnothing$. We hence know from (78) that

$$\sup \mathrm{NV}(A) = 1\,. \tag{1125}$$

In addition, we observe from Def. 87 that $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$ entails that $A(1) \cap [\frac{1}{2}, 1] = \{r_+\}$.

- If $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $A(0) \cap [\frac{1}{2}, 1] = \{r_+\}$ as well. We hence obtain from Def. 87 that $r_+ = \frac{1}{2}$. In particular, $\frac{1}{2} = r_+ \in A(0)$, $\frac{1}{2} = r_+ \in A(1)$ and $A(z) \subseteq [0, \frac{1}{2}]$ for all $z \in \mathbf{I}$. Again, this entails that

  $$\sup A(0) = \tfrac{1}{2}\,, \tag{1126}$$

  and also that

  $$\sup \cup\{A(z) : z > 0\} = \tfrac{1}{2}\,. \tag{1127}$$

  In turn,

  $$\begin{aligned} u(A) &= \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) && \text{by (1121)}\\ &= \min(1, \tfrac{1}{2}) && \text{by (1125) and (1127)}\\ &= \tfrac{1}{2}\\ &= 1 - \sup A(0)\,. && \text{by (1126)} \end{aligned}$$

- If $A(0) \cap [\frac{1}{2}, 1] = \varnothing$, then

  $$\begin{aligned} u(A) &= \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) && \text{by (1121)}\\ &= \min(1, 1 - \sup A(0)) && \text{by (1125)}\\ &= 1 - \sup A(0)\,. \end{aligned}$$

Now we address part **b.** of the lemma. Hence suppose that $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ and $A(1) \neq \varnothing$. Because $A(1) \neq \varnothing$, we know from (78) that $\sup \mathrm{NV}(A) = 1$. It is then immediate from (1121) that $u(A) = \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) = \min(1, 1 - \sup A(0)) = 1 - \sup A(0)$.
In order to prove part **c.** of the lemma, suppose that $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $A(1) \neq \varnothing$. Then $1 \in \mathrm{NV}(A)$ by (78) and hence $\sup \mathrm{NV}(A) = 1$. Therefore

$$\begin{aligned} u(A) &= \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) && \text{by (1121)}\\ &= \min(1, \sup \cup\{A(z) : z > 0\}) && \text{because } \sup \mathrm{NV}(A) = 1\\ &= \sup \cup\{A(z) : z > 0\}\,. \end{aligned}$$

It remains to be shown that claim **d.** of the lemma is also valid. Hence assume that $A(0) = \varnothing$ and $A(1) \cap [\frac{1}{2}, 1] = \varnothing$. Then $1 - \sup A(0) = 1$ and $A(0) \cap [\frac{1}{2}, 1] = \varnothing$. We therefore obtain from (1121) that $u(A) = \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) = \min(\sup \mathrm{NV}(A), 1) = \sup \mathrm{NV}(A)$, as desired.

Based on these results concerning $\ell(A)$ and $u(A)$, it is now easy to prove that $\psi_{\boxplus}$ is well-defined. Recalling (1119), we simply need to show that

Lemma 179 *For all $A \in \mathbb{A}$, $\ell(A) \leq u(A)$.*

Proof To see this, consider $A \in \mathbb{A}$. It is then convenient to discern four main cases.

a.: $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ **and** $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$**.**
In this case, we know from Def. 87 that $r_+ = \frac{1}{2}$ and

$$\sup A(0) = \sup A(1) = \tfrac{1}{2}\,. \tag{1128}$$

Therefore

$$\begin{aligned} \ell(A) &= \sup A(1) && \text{by L-177.a} \\ &= \tfrac{1}{2} && \text{by (1128)} \\ &= 1 - \sup A(0) && \text{by (1128)} \\ &= u(A)\,. && \text{by L-178.a} \end{aligned}$$

b.: $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ **and** $A(1) \cap [\frac{1}{2}, 1] = \varnothing$**.**
We discern two subcases. Firstly if $A(1) \neq \varnothing$, then

$$\begin{aligned} \ell(A) &= \sup A(1) && \text{by L-177.a} \\ &\leq \sup \cup\{A(z) : z > 0\} \\ &= u(A)\,. && \text{by L-178.c} \end{aligned}$$

In the remaining case that $A(1) = \varnothing$, we simply notice that

$$\begin{aligned} \ell(A) &= \sup A(1) && \text{by L-177.a} \\ &= 0 && \text{by assumption that } A(1) = \varnothing \\ &\leq u(A)\,. \end{aligned}$$

c: $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ **and** $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$**.**
We again discern two subcases. Firstly if $A(0) \neq \varnothing$, then

$$\begin{aligned} \ell(A) &= 1 - \sup \cup\{A(z) : z < 1\} && \text{by L-177.c} \\ &\leq 1 - \sup A(0) \\ &= u(A)\,. && \text{by L-178.a} \end{aligned}$$

In the remaining case that $A(0) = \varnothing$, we simply notice that

$$\begin{aligned} \ell(A) &\leq 1 \\ &= 1 - \sup A(0) && \text{by assumption that } A(0) = \varnothing \\ &= u(A)\,. && \text{by L-178.a} \end{aligned}$$

d.: $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ **and** $A(1) \cap [\frac{1}{2}, 1] = \varnothing$**.**
In order to handle this case, we will consider four subcases.

Firstly if $A(0) = \varnothing$ and $A(1) = \varnothing$, then

$$\begin{aligned}
\ell(A) &= \inf \mathrm{NV}(A) && \text{by L-177} \\
&\leq \sup \mathrm{NV}(A) && \text{because } \mathrm{NV}(A) \neq \varnothing \\
&= u(A)\,. && \text{by L-178.d}
\end{aligned}$$

Secondly if $A(0) = \varnothing$ and $A(1) \neq \varnothing$, then

$$\begin{aligned}
\ell(A) &\leq 1 && \text{because } \ell(A) \in \mathbf{I} \text{ by (1120)} \\
&= 1 - \sup A(0) && \text{because } A(0) = \varnothing \\
&= u(A)\,. && \text{by L-178.b}
\end{aligned}$$

Thirdly if $A(0) \neq \varnothing$ and $A(1) = \varnothing$, then

$$\begin{aligned}
\ell(A) &= \sup A(1) && \text{by L-177.b} \\
&= 0 && \text{by assumption that } A(1) = \varnothing \\
&\leq u(A)\,. && \text{because } u(A) \in \mathbf{I} \text{ by (1121)}
\end{aligned}$$

Finally if $A(0) \neq \varnothing$ and $A(1) \neq \varnothing$, then

$$\begin{aligned}
\ell(A) &= \sup A(1) && \text{by L-177.b} \\
&\leq \tfrac{1}{2} && \text{because } A(1) \cap [\tfrac{1}{2}, 1] = \varnothing \\
&\leq 1 - \sup A(0) && \text{because } A(0) \cap [\tfrac{1}{2}, 1] = \varnothing \\
&= u(A)\,, && \text{by L-178.b}
\end{aligned}$$

which completes the proof that $\ell(A) \leq u(A)$.

I now state some lemmata which facilitate the proof that $\psi_{\boxplus}$ satisfies all 'ψ-conditions' except for (ψ-5).

Lemma 180 *Let $A \in \mathbb{A}$ be given and suppose that $\ell(A) = u(A)$. Then $\psi_{\boxplus}(A) = \ell(A)$.*

Proof It is convenient to discern three cases. Firstly if $\ell(A) > \frac{1}{2}$, then

$$\psi_{\boxplus}(A) = \ell(A)$$

by (1119). In the second case that $\ell(A) < \frac{1}{2}$, we know that $u(A) = \ell(A) < \frac{1}{2}$ as well. Hence

$$\psi_{\boxplus}(A) = u(A) = \ell(A)$$

by (1119). In the remaining case that $\ell(A) = u(A) = \frac{1}{2}$, we obtain from (1119) that

$$\psi_{\boxplus}(A) = \tfrac{1}{2} = \ell(A)\,,$$

which completes the proof that $\psi_{\boxplus}(A) = \ell(A)$, as desired.

Lemma 181 *For all $A, A' \in \mathbb{A}$ with $A \sqsubseteq A'$, $\ell(A) \leq \ell(A')$ and $u(A) \leq u(A')$.*

Proof To see this, consider $A, A' \in \mathbb{A}$ with $A \sqsubseteq A'$. It is then clear from Def. 89 and $\text{NV}(A) = \{z \in \mathbf{I} : A(z) \neq \varnothing\}$, $\text{NV}(A') = \{z \in \mathbf{I} : A'(z) \neq \varnothing\}$ that

$$\inf \text{NV}(A) \leq \inf \text{NV}(A') \tag{1129}$$

and

$$\sup \text{NV}(A) \leq \sup \text{NV}(A') \,. \tag{1130}$$

In addition, $A \sqsubseteq A'$ entails that

$$\begin{aligned} &A(1) \subseteq A'(1) \\ &A(0) \supseteq A'(0) \\ &\cup\{A(z) : z > 0\} \subseteq \cup\{A'(z) : z > 0\} \end{aligned}$$

and

$$\cup\{A(z) : z < 1\} \supseteq \cup\{A'(z) : z < 1\} \,.$$

Therefore

$$\sup A(1) \leq \sup A'(1) \tag{1131}$$
$$1 - \sup A(0) \leq 1 - \sup A'(0) \tag{1132}$$
$$\sup \cup\{A(z) : z > 0\} \leq \sup \cup\{A'(z) : z > 0\} \tag{1133}$$

and

$$1 - \sup \cup\{A(z) : z < 1\} \leq 1 - \sup \cup\{A'(z) : z < 1\} \,. \tag{1134}$$

Now we consider $\ell(A)$ vs. $\ell(A')$. We notice that due to the above result $A(1) \subseteq A'(1)$, $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$ entails that $A'(1) \cap [\frac{1}{2}, 1] \neq \varnothing$. It is hence sufficient to consider the following three cases.

a. $A(1) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $A'(1) \cap [\frac{1}{2}, 1] \neq \varnothing$. Then

$$\begin{aligned} \ell(A) &= \max(\inf \text{NV}(A), 1 - \sup \cup\{A(z) : z < 1\}) && \text{by (1120)} \\ &\leq \max(\inf \text{NV}(A'), 1 - \sup \cup\{A'(z) : z < 1\}) && \text{by (1129) and (1134)} \\ &= \ell(A') \,. && \text{by (1120)} \end{aligned}$$

b. $A(1) \cap [\frac{1}{2}, 1] = \varnothing$ and $A'(1) \cap [\frac{1}{2}, 1] \neq \varnothing$. In this case, we notice that $r_+(A') \in A'(1)$ entails that $A'(z) \subseteq [0, 1 - r_+(A')] \subseteq [0, \frac{1}{2}]$ for all $z < 1$. Therefore $\sup\{A'(z) : z < 1\} \leq \frac{1}{2}$, and in turn $1 - \sup\{A'(z) : z < 1\} \geq \frac{1}{2}$. On the other hand, $A(1) \cap [\frac{1}{2}, 1] = \varnothing$ entails that $\sup A(1) \leq \frac{1}{2}$. We can hence combine these results, which yields

$$\sup A(1) \leq \tfrac{1}{2} \leq 1 - \sup\{A'(z) : z < 1\} \,. \tag{1135}$$

Therefore

$$\begin{aligned}
\ell(A) &= \max(\inf \mathrm{NV}(A), \sup A(1)) && \text{by (1120)}\\
&\le \max(\inf \mathrm{NV}(A'), 1 - \sup \cup\{A'(z) : z < 1\}) && \text{by (1129) and (1135)}\\
&= \ell(A')\,. && \text{by (1120)}
\end{aligned}$$

c. $A(1) \cap [\frac{1}{2}, 1] = \varnothing$ and $A'(1) \cap [\frac{1}{2}, 1] = \varnothing$. Then

$$\begin{aligned}
\ell(A) &= \max(\inf \mathrm{NV}(A), \sup A(1)) && \text{by (1120)}\\
&\le \max(\inf \mathrm{NV}(A'), \sup A'(1)) && \text{by (1129) and (1131)}\\
&= \ell(A')\,. && \text{by (1120)}
\end{aligned}$$

This completes the proof that $\ell(A) \le \ell(A')$. We now turn to $u(A)$ as compared to $u(A')$. It is then worth noticing that the above result $A(0) \supseteq A'(0)$ ensures that $A'(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ only if $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. It is hence sufficient to consider the following three cases.

a. $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $A'(0) \cap [\frac{1}{2}, 1] \neq \varnothing$. Then

$$\begin{aligned}
u(A) &= \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) && \text{by (1121)}\\
&\le \min(\sup \mathrm{NV}(A'), \sup \cup\{A'(z) : z > 0\}) && \text{by (1130), (1133)}\\
&= u(A')\,. && \text{by (1121)}
\end{aligned}$$

b. $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ and $A'(0) \cap [\frac{1}{2}, 1] = \varnothing$. In this case, we know that $r_+(A) \in A(0)$ and hence $A(z) \subseteq [0, 1 - r_+(A)] \subseteq [0, \frac{1}{2}]$ for all $z > 0$. In particular $\sup \cup\{A(z) : z > 0\} \le \frac{1}{2}$. We further observe that $1 - \sup A'(0) \ge \frac{1}{2}$ because $A'(0) \cap [\frac{1}{2}, 1] = \varnothing$ and hence $\sup A'(0) \le \frac{1}{2}$. These findings can be summarized by

$$\sup \cup\{A(z) : z > 0\} \le \tfrac{1}{2} \le 1 - \sup A'(0)\,. \tag{1136}$$

Therefore

$$\begin{aligned}
u(A) &= \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) && \text{by (1121)}\\
&\le \min(\sup \mathrm{NV}(A'), 1 - \sup A(0)) && \text{by (1130) and (1136)}\\
&= u(A')\,. && \text{by (1121)}
\end{aligned}$$

c. $A(0) \cap [\frac{1}{2}, 1] = \varnothing$ and $A'(0) \cap [\frac{1}{2}, 1] = \varnothing$. Then

$$\begin{aligned}
u(A) &= \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) && \text{by (1121)}\\
&\le \min(\sup \mathrm{NV}(A'), 1 - \sup A'(0)) && \text{by (1130) and (1132)}\\
&= u(A')\,. && \text{by (1121)}
\end{aligned}$$

Hence $u(A) \le u(A')$ whenever $A \sqsubseteq A'$, as desired.

Lemma 182 *Let $A \in \mathbb{A}$ be given and suppose that $A' \in \mathbb{A}$ is defined by*

$$A'(z) = A(1-z) \tag{1137}$$

for all $z \in \mathbf{I}$. Then

$$\ell(A') = 1 - u(A)$$

and

$$u(A') = 1 - \ell(A)\,.$$

Proof We first consider the claim of the lemma that $\ell(A') = 1 - u(A)$. To this end, we observe that

$$\begin{aligned}
\mathrm{NV}(A') &= \{z \in \mathbf{I} : A'(z) \neq \varnothing\} && \text{by (78)}\\
&= \{z \in \mathbf{I} : A(1-z) \neq \varnothing\} && \text{by (1137)}\\
&= \{1-z' : z' \in \mathbf{I}, A(z') \neq \varnothing\} && \text{by substitution } z' = 1-z\\
&= \{1-z' : z' \in \mathrm{NV}(A)\}\,. && \text{by (78)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\inf \mathrm{NV}(A') &= \inf\{1-z' : z' \in \mathrm{NV}(A)\}\\
&= 1 - \sup\{z' : z' \in \mathrm{NV}(A)\}\,,
\end{aligned}$$

i.e.

$$\inf \mathrm{NV}(A') = 1 - \sup \mathrm{NV}(A)\,. \tag{1138}$$

In addition,

$$1 - \sup \cup\{A'(z) : z < 1\} = 1 - \sup \cup\{A(1-z) : z < 1\}$$

by (1137), and hence

$$1 - \sup \cup\{A'(z) : z < 1\} = 1 - \sup \cup\{A(z) : z > 0\}\,. \tag{1139}$$

We further notice that

$$\sup A'(1) = \sup A(0) = 1 - (1 - \sup A(0))\,, \tag{1140}$$

again by (1137), which also results in $A'(1) \cap [\frac{1}{2}, 1] = A(0) \cap [\frac{1}{2}, 1]$, and hence

$$A'(1) \cap [\tfrac{1}{2}, 1] \neq \varnothing \iff A(0) \cap [\tfrac{1}{2}, 1] \neq \varnothing\,. \tag{1141}$$

It is therefore sufficient to prove the following two cases. If $A'(1) \cap [\frac{1}{2}, 1] \neq \varnothing$, then $A(0) \cap [\frac{1}{2}, 1] \neq \varnothing$ by (1141). Therefore

$$\begin{aligned}
\ell(A') &= \max(\inf \mathrm{NV}(A'), 1 - \sup \cup\{A'(z) : z < 1\}) && \text{by (1120)}\\
&= \max(1 - \sup \mathrm{NV}(A), 1 - \sup \cup\{A(z) : z > 0\}) && \text{by (1138) and (1139)}\\
&= 1 - \min(\sup \mathrm{NV}(A), \sup \cup\{A(z) : z > 0\}) && \text{by De Morgan's law}\\
&= 1 - u(A)\,. && \text{by (1121)}
\end{aligned}$$

In the second case that $A'(1)\cap[\frac{1}{2},1] = \varnothing$, we know from (1141) that $A(0)\cap[\frac{1}{2},1] = \varnothing$ as well. Therefore

$$\begin{aligned}
\ell(A') &= \max(\inf \mathrm{NV}(A'), \sup A'(1)) && \text{by (1120)} \\
&= \max(1 - \sup \mathrm{NV}(A), 1 - (1 - \sup A(0))) && \text{by (1138) and (1140)} \\
&= 1 - \min(\sup \mathrm{NV}(A), 1 - \sup A(0)) && \text{by De Morgan's law} \\
&= 1 - u(A)\,. && \text{by (1121)}
\end{aligned}$$

This finishes the proof that $\ell(A') = u(A)$. The second claim that $u(A') = 1 - \ell(A)$ is reducible to the first one, noticing that $A(z) = A(1-(1-z)) = A'(1-z)$ for all $z \in \mathbf{I}$. Hence

$$\ell(A) = 1 - u(A')$$

by the first claim of the lemma, which in turn proves the desired $u(A') = 1 - \ell(A)$.

Lemma 183 *For all $A \in \mathbb{A}$,*

a. If $u(A) \geq \frac{1}{2}$, then $\psi_{\boxplus}(A) \geq \frac{1}{2}$.

b. If $\ell(A) \leq \frac{1}{2}$, then $\psi_{\boxplus}(A) \leq \frac{1}{2}$.

Proof As concerns part **a.** of the lemma, we make use of the inequation $\ell(A) \leq u(A)$ proven in L-179. It is hence sufficient to discern the following cases. If $\frac{1}{2} < \ell(A) \leq u(A)$, then $\psi_{\boxplus}(A) = \ell(A) > \frac{1}{2}$ by (1119). In the remaining case that $u(A) \geq \frac{1}{2}$ and $\ell(A) \leq \frac{1}{2}$, we obtain from (1119) that $\psi_{\boxplus}(A) = \frac{1}{2}$. Hence **a.** is indeed valid. As concerns part **b.**, we can now profit from L-182, which permits us to reduce the proof of **b.** to that of part **a.** by means of negation.

Lemma 184 *The condition* (ψ-5) *is independent of* (ψ-1)–(ψ-4).

Proof I will show that (ψ-5) is independent of the remaining conditions by proving that $\psi_{\boxplus} : \mathbb{A} \longrightarrow \mathbf{I}$, defined by (1119), satisfies (ψ-1)–(ψ-4), and violates (ψ-5).

$\psi_{\boxplus}$ **satisfies** (ψ-1).

To see this, consider a choice of $A \in \mathbb{A}$ with $D(A) = \{1\}$. Then $r_+ = 1$ and by Def. 87, $D(A)(z) \subseteq [0, 1 - z_+] = \{0\}$ for all $z \neq z_+$. Because $0 \notin D(A)$, we conclude that

$$A(z) = \begin{cases} \{1\} & : \quad z = z_+ \\ \varnothing & : \quad z \neq z_+ \end{cases}$$

for all $z \in \mathbf{I}$. Hence by (78), $\mathrm{NV}(A) = \{z_+\}$ and in turn,

$$\inf \mathrm{NV}(A) = \sup \mathrm{NV}(A) = z_+\,. \tag{1142}$$

Therefore

$$\begin{aligned}
z_+ &= \inf \mathrm{NV}(A) && \text{by (1142)}\\
&\leq \ell(A) && \text{by (1120)}\\
&\leq u(A) && \text{by L-179}\\
&\leq \sup \mathrm{NV}(A) && \text{by (1121)}\\
&= z_+ \,. && \text{by (1142)}
\end{aligned}$$

In other words, $\ell(A) = u(A) = z_+$. Therefore $\psi_\boxplus(A) = z_+$ by L-180, i.e. (ψ-1) holds, as desired.

$\psi_\boxplus$ **satisfies** (ψ-2)
Hence let $A \in \mathbb{A}$ be given and define $A' \in \mathbb{A}$ by $A'(z) = A(1-z)$ for all $z \in \mathbf{I}$. Then $\ell(A') = 1 - u(A)$ and $u(A') = 1 - \ell(A)$, see L-182. Hence if $\ell(A) > \frac{1}{2}$, then $u(A') = 1 - \ell(A) < \frac{1}{2}$ and $\psi_\boxplus(A') = u(A') = 1 - \ell(A) = 1 - \psi_\boxplus(A)$ by (1119). In the second case that $u(A) < \frac{1}{2}$, we know that $\ell(A') = 1 - u(A) > \frac{1}{2}$ and hence $\psi_\boxplus(A') = \ell(A') = 1 - u(A) = 1 - \psi_\boxplus(A)$ by (1119). Finally if $\ell(A) \leq \frac{1}{2} \leq u(A)$, then $1 - u(A) \leq \frac{1}{2} \leq 1 - \ell(A)$ and hence $\ell(A') \leq \frac{1}{2} \leq u(A')$ as well, see L-182. Therefore $\psi_\boxplus(A') = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \psi_\boxplus(A)$ by (1119). This completes the proof that $\psi_\boxplus(A') = 1 - \psi_\boxplus(A)$, i.e. ($\psi$-2) is indeed valid.

$\psi_\boxplus$ **satisfies** (ψ-3)**.**
In order to prove this, we consider $A \in \mathbb{A}$ with $\mathrm{NV}(A) \subseteq \{0, 1\}$ and $z_+ = 1$, i.e. $A(1) \cap [0, \frac{1}{2}] \neq \varnothing$. In particular, $1 \in \mathrm{NV}(A)$ and hence $\sup \mathrm{NV}(A) = 1$. We now consider two cases.

- $A(0) = \varnothing$.
  Then $\mathrm{NV}(A) = \{1\}$, i.e.

  $$\inf \mathrm{NV}(A) = 1\,, \tag{1143}$$

  and

  $$\cup\{A(z) : z < 1\} = \varnothing\,,$$

  in particular

  $$1 - \sup \cup\{A(z) : z < 1\} = 1\,. \tag{1144}$$

  Therefore

  $$\begin{aligned}
  \ell(A) &= \max(\inf \mathrm{NV}(A), 1 - \sup \cup\{A(z) : z < 1\}) && \text{by (1120)}\\
  &= \max(1, 1) && \text{by (1143) and (1144)}\\
  &= 1\,.
  \end{aligned}$$

  As far as $u(A)$ is concerned, we obtain from L-178 that $u(A) = 1 - \sup A(0) = 1 - \sup \varnothing = 1$. Hence $\ell(A) = u(A) = 1$ and by L-180, $\psi_\boxplus(A) = 1 = 1 - \sup A(0)$, as desired.

- $A(0) \neq \varnothing$.
  Then $\mathrm{NV}(A) = \{0, 1\}$, and hence
  $$\inf \mathrm{NV}(A) = 0\,. \qquad (1145)$$
  As concerns $\ell(A)$, we then obtain that
  $$\begin{aligned} \ell(A) &= 1 - \sup \cup\{A(z) : z < 1\} && \text{by L-177.c} \\ &= 1 - \sup A(0)\,, \end{aligned}$$
  because $\mathrm{NV}(A) = \{0, 1\}$ and hence $\cup\{A(z) : z < 1\} = A(0) \cup \cup\{\varnothing : z \in (0, 1)\} = A(0)$.
  Now let us consider $u(A)$. In this case, $u(A) = 1 - \sup A(0)$ is immediate from part a. of lemma L-178. Hence again $\ell(A) = u(A) = 1 - \sup A(0)$ and by L-180, $\psi_{\boxplus}(A) = \ell(A) = 1 - \sup A(0)$.

This proves that $\psi_{\boxplus}(A) = 1 - \sup A(0)$ in both possible cases, i.e. $\psi_{\boxplus}$ indeed satisfies (ψ-3).

$\psi_{\boxplus}$ satisfies (ψ-4).
Hence let $A, A' \in \mathbb{A}$ be given and suppose that $A \sqsubseteq A'$. We then know from L-181 that $\ell(A) \leq \ell(A')$ and $u(A) \leq u(A')$. It is hence sufficient to consider the following cases.

a. If $\ell(A) > \frac{1}{2}$, then $\ell(A)' > \frac{1}{2}$ as well. Hence
$$\begin{aligned} \psi_{\boxplus}(A) &= \ell(A) && \text{by (1119)} \\ &\leq \ell(A') && \text{by L-181} \\ &= \psi_{\boxplus}(A')\,. && \text{by (1119)} \end{aligned}$$

b. If $u(A) < \frac{1}{2}$ and $u(A') \geq \frac{1}{2}$, then
$$\begin{aligned} \psi_{\boxplus}(A) &= u(A) && \text{by (1119)} \\ &< \tfrac{1}{2} && \text{by assumption of case } \textbf{b.} \\ &\leq \psi_{\boxplus}(A')\,. && \text{by L-183.a} \end{aligned}$$

c. If $u(A) < \frac{1}{2}$ and $u(A') < \frac{1}{2}$, then
$$\begin{aligned} \psi_{\boxplus}(A) &= u(A) && \text{by (1119)} \\ &\leq u(A') && \text{by L-181} \\ &= \psi_{\boxplus}(A')\,. && \text{by (1119)} \end{aligned}$$

d. Finally if $\ell(A) \leq \frac{1}{2} \leq u(A)$, then in particular $\frac{1}{2} \leq u(A)'$ by L-181. Therefore
$$\begin{aligned} \psi_{\boxplus}(A) &= \tfrac{1}{2} && \text{by (1119)} \\ &\leq \psi_{\boxplus}(A')\,. && \text{by L-183.a} \end{aligned}$$

To sum up, I have shown that $\psi_{\boxplus}(A) \leq \psi_{\boxplus}(A')$ whenever $A, A' \in \mathbb{A}$ satisfy $A \sqsubseteq A'$. This proves that $\psi_{\boxplus}$ satisfies (ψ-4).

$\psi_{\boxplus}$ violates (ψ-5).
Hence consider $A \in \mathbb{A}$, defined by

$$A(z) = \begin{cases} \{1\} & : \quad z = \frac{2}{3} \\ \varnothing & : \quad z \neq \frac{2}{3} \end{cases}$$

for all $z \in \mathbf{I}$. Because $\psi_{\boxplus}$ is already known to satisfy (ψ-1), we conclude that

$$\psi_{\boxplus}(A) = \tfrac{2}{3}\,.$$

As concerns $\boxplus A$, we observe from Def. 91 that

$$\boxplus A(z) = \begin{cases} [0,1] & : \quad z = \frac{2}{3} \\ \{0\} & : \quad z \neq \frac{2}{3} \end{cases}$$

for all $z \in \mathbf{I}$. Hence $\mathrm{NV}(\boxplus A) = \mathbf{I}$ and

$$\inf \mathrm{NV}(\boxplus A) = 0\,,$$
$$\sup \mathrm{NV}(\boxplus A) = 1\,.$$

In addition, $\boxplus A(1) \cap [\frac{1}{2}, 1] = \varnothing$ and consequently

$$\ell(\boxplus A) = \max(\inf \mathrm{NV}(\boxplus A), \sup \boxplus A(1)) = \max(0, \sup\{0\}) = 0$$

by (1120). Because $\boxplus A(0) \cap [\frac{1}{2}, 1] = \varnothing$ as well, we obtain from (1121) that

$$u(\boxplus A) = \min(\sup \mathrm{NV}(\boxplus A), 1 - \sup \boxplus A(0)) = \min(1, 1 - \sup\{0\}) = 1\,.$$

Hence $0 = \ell(\boxplus A) \leq \frac{1}{2} \leq 1 = u(\boxplus A)$ and by (1119),

$$\psi_{\boxplus}(\boxplus A) = \tfrac{1}{2}\,.$$

To sum up, I have shown that $\psi_{\boxplus}(A) = \frac{2}{3} \neq \frac{1}{2} = \psi_{\boxplus}(\boxplus A)$, i.e. A witnesses the failure of $\psi_{\boxplus}$ with respect to (ψ-5).

Proof of Theorem 121

It is apparent from the independence of (ω-1)–(ω-4), as stated in Th-46, and theorem Th-120, that each of (ψ-1), (ψ-2), (ψ-3) and (ψ-4) is independent of the remaining four conditions in (ψ-1)–(ψ-5). As concerns (ψ-5), it has already been shown in L-184 that this condition is independent of (ψ-1)–(ψ-4).

B.31 Proof of Theorem 122

Consider $A \in \mathbb{A}$. To see that $f_A \in \mathbb{X}$, we discern the cases that either $r_+ = \frac{1}{2}$ or $r_+ > \frac{1}{2}$.
Firstly in the case that $r_+ = \frac{1}{2}$, then we know from Def. 87 that $D(A) \cap [\frac{1}{2}, 1] = \{\frac{1}{2}\}$.

By (62), there exists $z_+ \in \mathbf{I}$ with $\frac{1}{2} \in A(z_+)$, and $A(z) \subseteq [0, \frac{1}{2}]$ for all $z \in \mathbf{I}$. Hence $f_A(z_+) = \sup A(z_+) = \frac{1}{2}$ for $z = z_+$, and $f_A(z) \leq \sup[0, \frac{1}{2}] = \frac{1}{2}$ for all other $z \in \mathbf{I}$. Hence conditions **a.** and **b.** of Def. 95 are satisfied, and indeed $f_A \in \mathbb{X}$.
In the remaining case that $r_+ > \frac{1}{2}$, we know from Def. 87 that $r_+ \in A(z_+)$ for a unique choice of $z_+ \in \mathbf{I}$. Noticing that $A(z_+) \cap [\frac{1}{2}, 1] \subseteq D(A) \cap [\frac{1}{2}, 1] = \{r_+\}$ by Def. 87, (62) and Def. 85, this proves that $f_A(z_+) = \sup A(z) = r_+$, and condition **a.** stated in Def. 95 is satisfied. Because $r_+ > \frac{1}{2}$, we further know from Def. 87 that $r_+ \notin A(z)$ for $z \in \mathbf{I} \setminus \{z_+\}$ and hence $A(z) \subseteq D(A) \setminus \{r_+\} \subseteq [0, 1 - r_+]$ by (62) and Def. 85. In particular $f_A(z) = \sup A(z) \leq \sup[0, 1 - r_+] = 1 - r_+$. Hence condition **b.** stated in Def. 95 is satisfied as well‘, and $f_A \in \mathbb{X}$, as desired.
Of course, this also proves that $f_{Q,X_1,\ldots,X_n} \in \mathbb{X}$ for a given semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, because $f_{Q,X_1,\ldots,X_n} = f_{A_{Q,X_1,\ldots,X_n}}$ by Def. 94, and $A_{Q,X_1,\ldots,X_n} \in \mathbb{A}$ by Th-93.

B.32 Proof of Theorem 123

Consider $f \in \mathbb{X}$ and let $z_+ \in \mathbf{I}$ be chosen such that $r_+ = f(z_+)$. We define $A_f : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$A_f(z) = \begin{cases} [0, 1 - r_+] \cup \{r_+\} & : \quad z = z_+ \\ [0, f(z)] & : \quad z \neq z_+ \end{cases} \tag{1146}$$

for all $z \in \mathbf{I}$.

In order to show that $A_f \in \mathbb{A}$, we first consider $D(A_f)$. Because $f(z) \leq 1 - r_+$ for all $z \neq z_+$ by condition **b.** of Def. 95, we know that $A_f(z) \subseteq [0, 1 - r_+] \subseteq A_f(z_+)$ for all $z \neq z_+$, see (1146). Hence by (62), $D(A_f) = \cup\{A_f(z) : z \in \mathbf{I}\} = A_f(z_+) = [0, 1-r_+] \cup \{r_+\}$. We notice that $D(A_f) \cap [\frac{1}{2}, 1] = \{r_+\}$ because $r_+ \geq \frac{1}{2}$. We further notice that $\inf D' \in D$ for all $D' \subseteq D(A_f)$ with $D' \neq \varnothing$, which is apparent because $D(A_f)$ is a union of closed intervals. In addition, if $r_+ > \frac{1}{2}$, then $\sup D(A_f) \setminus \{r_+\} = \sup([0, 1 - r_+] \cup \{r_+\}) \setminus \{r_+\} = \sup[0, 1 - r_+] = 1 - r_+$. Hence $D(A_f) \in \mathbb{D}$ by Def. 85.
In order to prove that $A_f \in \mathbb{A}$, suppose that $z, z' \in \mathbf{I}$ satisfy $\sup A_f(z) > \frac{1}{2}$ and $\sup A_f(z') > \frac{1}{2}$. By (1146), this means that $f(z) > \frac{1}{2}$ and $f(z') > \frac{1}{2}$. We then conclude from condition **a.** of Def. 95 that $f(z) = r_+ > \frac{1}{2}$. In addition, we conclude from part **b.** of Def. 95 that either $z' = z$ or $f(z') \leq 1 - r_+ < \frac{1}{2}$. But we know that $f(z') > \frac{1}{2}$, hence $z = z'$, and condition **b.** of Def. 87 is satisfied, as desired. This completes the proof that $A_f \in \mathbb{A}$.
Next I prove that this particular choice of $A_f \in \mathbb{A}$ results in $f = f_{A_f}$. To see this, consider $z \in \mathbf{I}$. It is then apparent from (1146) that $\sup A_f(z) = f(z)$, and hence $f_{A_f}(z) = f(z)$ by Def. 93. Because $z \in \mathbf{I}$ was arbitrary, this proves that $f = f_{A_f}$ for $A_f \in \mathbb{A}$ as defined by (1146).
Note that this also proves the second claim of the theorem concerning the existence of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ with $f = f_{Q,X_1,\ldots,X_n}$. This is because by Th-94, there exist Q and $X_1, \ldots, X_n$ with $A_f = A_{Q,X_1,\ldots,X_n}$. Hence $f = f_{A_f} = f_{A_{Q,X_1,\ldots,X_n}} = f_{Q,X_1,\ldots,X_n}$ by Def. 94.

B.33 Proof of Theorem 124

Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by (87). Then for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
\mathcal{F}_\varphi(Q)(X_1, \dots, X_n) &= \varphi(f_{Q,X_1,\dots,X_n}) && \text{by Def. 96} \\
&= \varphi(f_{A_{Q,X_1,\dots,X_n}}) && \text{by Def. 94} \\
&= \psi(A_{Q,X_1,\dots,X_n}) && \text{by (87)} \\
&= \mathcal{F}_\psi(Q)(X_1, \dots, X_n) && \text{by Def. 88}
\end{aligned}$$

Because Q and $X_1, \dots, X_n$ were arbitrary, this proves that $\mathcal{F}_\varphi = \mathcal{F}_\psi$, as desired.

B.34 Proof of Theorem 125

Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-5) and let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be defined by (88). Now consider a choice of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Further let $z \in \mathbf{I}$. We now consider equation (89). In the case that $f_A(z) \le \frac{1}{2}$, this equation yields

$$A_{f_A}(z) = [0, f_A(z)]$$

and hence $\sup A_{f_A}(z) = f_A(z) \le \frac{1}{2}$. In turn,

$$\widehat{\boxplus} A_{f_A}(z) = \min(\sup A_{f_A}(z), \tfrac{1}{2}) = \min(f_A(z), \tfrac{1}{2}) = f_A(z) \tag{1147}$$

by (73). In addition, we conclude from $f_A(z) \le \frac{1}{2}$ and $f_A(z) = \sup A(z) \le \frac{1}{2}$ by Def. 93 that

$$\widehat{\boxplus} A(z) = \min(\sup A(z), \tfrac{1}{2}) = \sup A(z) = f_A(z)\,. \tag{1148}$$

Therefore

$$\begin{aligned}
\boxplus A_{f_A}(z) &= [0, \widehat{\boxplus} A_{f_A}(z)] && \text{by Def. 91} \\
&= [0, f_A(z)] && \text{by (1147)} \\
&= [0, \widehat{\boxplus} A(z)] && \text{by (1148)} \\
&= \boxplus A(z)\,. && \text{by Def. 91}
\end{aligned}$$

In the remaining case that $\sup A(z) > \frac{1}{2}$, we obtain from (89) and Def. 93 that

$$A_{f_A}(z) = [0, 1 - \sup A(z)] \cup \{\sup A(z)\} \tag{1149}$$

because $f_A(z) = \sup A(z) > \frac{1}{2}$. Therefore

$$\begin{aligned}
\boxplus A_{f_A}(z) &= \boxplus([0, 1 - \sup A(z)] \cup \{\sup A(z)\}) && \text{by psi.ablo.phi.-1} \\
&= [0, \tfrac{1}{2}] && \text{because } \sup A(z) > \tfrac{1}{2} \\
&= \boxplus A(z)\,. && \text{by Def. 91 because } \sup A(z) \ge \tfrac{1}{2}
\end{aligned}$$

Recalling our above result concerned with the case that $\sup A(z) \leq \frac{1}{2}$, we have succeeded to show that $\boxplus A_{f_A}(z) = \boxplus A(z)$ for all $z \in \mathbf{I}$, and hence

$$\boxplus A_{f_A} = \boxplus A\,. \tag{1150}$$

Therefore

$$\begin{aligned}
\mathcal{F}_\psi(Q)(X_1,\ldots,X_n) &= \psi(A_{Q,X_1,\ldots,X_n}) && \text{by Def. 88}\\
&= \psi(\boxplus A_{Q,X_1,\ldots,X_n}) && \text{by } (\psi\text{-5})\\
&= \psi(\boxplus A_{f_{A_{Q,X_1,\ldots,X_n}}}) && \text{by (1150)}\\
&= \psi(A_{f_{A_{Q,X_1,\ldots,X_n}}}) && \text{by } (\psi\text{-5})\\
&= \psi(A_{f_{Q,X_1,\ldots,X_n}}) && \text{by Def. 94}\\
&= \varphi(f_{Q,X_1,\ldots,X_n}) && \text{by (88)}\\
&= \mathcal{F}_\varphi(Q)(X_1,\ldots,X_n)\,. && \text{by Def. 96}
\end{aligned}$$

B.35 Proof of Theorem 126

Lemma 185 *Consider $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined in terms of φ according to* (87)*. Then φ is defined in terms ψ according to* (88)*.*

Proof To see this, consider $f \in \mathbb{X}$. I will now prove that

$$f = f_{A_f}\,, \tag{1151}$$

where $A_f \in \mathbb{A}$ is defined by (89). Hence consider $z \in \mathbf{I}$. If $f(z) > \frac{1}{2}$, then

$$\begin{aligned}
f_{A_f}(z) &= \sup A_f(z) && \text{by Def. 93}\\
&= \sup[0, 1 - f(z)] \cup \{f(z)\} && \text{by (89)}\\
&= f(z)\,,
\end{aligned}$$

because $f(z) > \frac{1}{2}$ by assumption. In the remaining case that $f(z) \leq \frac{1}{2}$, we compute

$$\begin{aligned}
f_{A_f}(z) &= \sup A_f(z) && \text{by Def. 93}\\
&= \sup[0, f(z)] && \text{by (89)}\\
&= f(z)\,.
\end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $f_{A_f} = f$, i.e. equation (1151) is indeed valid. In turn

$$\begin{aligned}
\varphi(f) &= \varphi(f_{A_f}) && \text{by (1151)}\\
&= \psi(A_f)\,, && \text{by (87)}
\end{aligned}$$

i.e. equation (88) holds, as desired.

In order to relate the monotonicity conditions (ψ-4) and (φ-4), we need a convexification construction on $\mathbb{A}$. For all $A \in \mathbb{A}$, let us hence define $A^{\ddagger} : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$A^{\ddagger}(z) = (\cup\{A(z') : z' \leq z\}) \cap (\cup\{A(z') : z' \geq z\}), \tag{1152}$$

for all $z \in \mathbf{I}$. It is apparent from Def. 87 that indeed $A^{\ddagger} \in \mathbb{A}$. In addition, it is obvious from (1152) that

$$A(z) \subseteq A^{\ddagger}(z) \tag{1153}$$

for all $z \in \mathbf{I}$.

Lemma 186 *For all $A \in \mathbb{A}$, it both holds that $A \sqsubseteq A^{\ddagger}$ and $A^{\ddagger} \sqsubseteq A$.*

Proof To see that $A \sqsubseteq A^{\ddagger}$, we consider both conditions of Def. 89. Firstly let $z \in \mathbf{I}$ and $r \in A(z)$. Then $z' = z$ is an admissable choice for $z' \geq z$ with $r \in A^{\ddagger}(z')$ because $r \in A(z) \subseteq A^{\ddagger}(z) = A^{\ddagger}(z')$ by (1153). As concerns the second condition, let $z' \in \mathbf{I}$ and $r \in A^{\ddagger}(z')$. We now observe from (1152) that $A^{\ddagger}(z') \subseteq \cup\{A(z) : z \leq z'\}$. Therefore $r \in A(z')$ entails that $r \in \cup\{A(z) : z \leq z'\}$. In turn, this proves that there exists $z \leq z'$ with $r \in A(z)$. Because both preconditions of Def. 89 are satisfied, we conclude that $A \sqsubseteq A^{\ddagger}$.
It remains to be shown that $A^{\ddagger} \sqsubseteq A$. Again we consider both preconditions of Def. 89. As concerns the first condition, let $z \in \mathbf{I}$ and $r \in A^{\ddagger}$. It is then immediate from (1152) that $A^{\ddagger}(z) \subseteq \cup\{A(z') : z' \geq z\}$. Hence $r \in A^{\ddagger}(z)$ entails that there exists $z' \geq z$ with $r \in A(z')$. In other words, condition **a.** of Def. 89 is satisfied. Now let us address the second precondition. We consider $z' \in \mathbf{I}$ and $r \in A(z')$. Then $z = z'$ is a legal choice for $z \leq z'$ which satisfies $r \in A(z') = A(z) \subseteq A^{\ddagger}(z)$, i.e. $r \in A^{\ddagger}(z)$ by (1153), as desired. Because both preconditions of Def. 89 are valid, we deduce that in fact $A^{\ddagger} \sqsubseteq A$.

Lemma 187 *Suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ satisfies (ψ-4). Then ψ is ‡-invariant, i.e.*

$$\psi(A^{\ddagger}) = \psi(A)$$

for all $A \in \mathbb{A}$.

Proof We know from L-186 that $A \sqsubseteq A^{\ddagger}$ and $A^{\ddagger} \sqsubseteq A$. Hence $\psi(A) \leq \psi(A^{\ddagger})$ and $\psi(A^{\ddagger}) \leq \psi(A)$ by (ψ-4), i.e. $\psi(A^{\ddagger}) = \psi(A)$, as desired.

Lemma 188 *Let $f, f' \in \mathbb{X}$ be given and suppose that $f \sqsubseteq f'$. Then $r_+(f) = r_+(f')$.*

Proof Let $f, f' \in \mathbb{X}$ be given and suppose that $r_+(f) \neq r_+(f')$. I will show that $f \sqsubseteq f'$ is not possible in this case.
Suppose that $r_+(f) > r_+(f')$. By Def. 95, there exists $z_+ = z_+(f) \in \mathbf{I}$ with $r_+(f) = f(z_+)$. We now recall from part **a.** of Def. 97 that the condition

$$\sup\{f'(z') : z' \geq z_+\} \geq f(z_+) = r_+(f) \tag{1154}$$

is necessary for $f \sqsubseteq f'$. However,

$$\begin{aligned}
&\sup\{f'(z') : z' \geq z_+\} \\
&\leq \sup\{f'(z) : z \in \mathbf{I}\} \\
&= \max(f'(z_+(f')), \sup\{f'(z) : z \neq z_+(f')\}) && \text{by splitting sup-expression} \\
&= \max(r_+(f'), \sup\{f'(z) : z \neq z_+(f')\}) && \text{by Def. 95.a} \\
&= r_+(f') && \text{by Def. 95.b because } r_+(f') \geq \tfrac{1}{2} \\
&< r_+(f)\,. && \text{by assumption}
\end{aligned}$$

This demonstrates that condition (1154) is not valid, which is a necessary precondition of $f \sqsubseteq f'$.
Now we consider the remaining case that $r_+(f) > r_+(f')$. We recall from Def. 95 that there exists $z_+ = z_+(f')$ with $r_+(f') = f'(z_+)$. It is then immediate from Def. 97 that the following condition is necessary for $f \sqsubseteq f'$, viz

$$\sup\{f(z) : z \leq z_+\} \geq f'(z_+)\,, \tag{1155}$$

which is a specialization of Def. 97.**b**. As I will now show, this condition is not valid. This is because

$$\begin{aligned}
&\sup\{f(z) : z \leq z_+\} \\
&\leq \sup\{f(z) : z \in \mathbf{I}\} \\
&= \max(f(z_+(f)), \sup\{f(z) : z \neq z_+(f)\}) && \text{by splitting sup-expression} \\
&= \max(\underbrace{r_+(f)}_{\geq \frac{1}{2}}, \underbrace{\sup\{f(z) : z \neq z_+(f)\}}_{\leq 1 - r_+(f) \leq \frac{1}{2}}) && \text{by Def. 95} \\
&= r_+(f) \\
&< r_+(f')\,. && \text{by assumption}
\end{aligned}$$

Hence (1155) is indeed violated, which proves that $f \sqsubseteq f'$ is not possible.
To sum up, I have shown that $f \sqsubseteq f'$ does not hold if $r_+(f) \neq r_+(f')$. This proves that $r_+(f) = r_+(f')$ is entailed by $f \sqsubseteq f'$, as desired.

Let us now introduce another construction on $\mathbb{A}$. For all $A \in \mathbb{A}$, we define $A^\natural : \mathbf{I} \longrightarrow \mathcal{P}(\mathbf{I})$ by

$$A^\natural(z) = A(z) \cup \{\sup A(z)\}\,, \tag{1156}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 87 and $A \in \mathbb{A}$ that $A^\natural \in \mathbb{A}$ as well.

Lemma 189 *Consider $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined in terms of φ according to* (87)*. Then ψ is $^\natural$-invariant, i.e. $\psi(A^\natural) = \psi(A)$ for all $A \in \mathbb{A}$.*

Proof To see this, let $A \in \mathbb{A}$ be given. Now consider some $z \in \mathbf{I}$. Then

$$\begin{aligned} f_A(z) &= \sup A(z) && \text{by Def. 93} \\ &= \sup(A(z) \cup \{\sup A(z)\}) \\ &= \sup A^\natural(z) && \text{by (1156)} \\ &= f_{A^\natural}(z)\,. && \text{by Def. 93.} \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that

$$f_A = f_{A^\natural}\,. \tag{1157}$$

In turn

$$\begin{aligned} \psi(A^\natural) &= \varphi(f_{A^\natural}) && \text{by (87)} \\ &= \varphi(f_A) && \text{by (1157)} \\ &= \psi(A)\,. && \text{by (87)} \end{aligned}$$

In the following, we also need an additional construction on $\mathbb{X}$. For all $f \in \mathbb{X}$, we define $f^\ddagger : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$f^\ddagger(z) = \min(\sup\{f(z') : z' \leq z\}, \sup\{f(z') : z' \geq z\})\,, \tag{1158}$$

for all $z \in \mathbf{I}$.

Lemma 190 *Consider $f \in \mathbb{X}$ and suppose that $r_+ = f(z_+)$ for a given $z_+ \in \mathbf{I}$. Then for all $z \in \mathbf{I}$,*

a. if $z \leq z_+$, then $f^\ddagger(z) = \sup\{f(z') : z' \leq z\}$;

b. if $z \geq z_+$, then $f^\ddagger(z) = \sup\{f(z') : z' \geq z\}$.

Proof We recall from Def. 95 that

$$r_+ = f(z_+) \geq \tfrac{1}{2} \tag{1159}$$

and

$$f(z) \leq 1 - r_+ \leq r_+ \tag{1160}$$

for all $z \in \mathbf{I} \setminus \{z_+\}$.
Now consider a given $z \in \mathbf{I}$. In order to prove case **a.** of the lemma, we suppose that $z \leq z_+$. Then

$$\begin{aligned} &\sup\{f(z') : z' \geq z\} \\ &= \max(f(z_+), \sup\{f(z') : z' \geq z, z' \neq z_+\}) && \text{by splitting sup-expression} \\ &= \max(\underbrace{r_+}_{\geq \frac{1}{2}}, \underbrace{\sup\{f(z') : z' \geq z, z' \neq z_+\}}_{\leq \frac{1}{2}}) && \text{by (1159), (1160)} \\ &= r_+ \end{aligned}$$

and

$$\sup\{f(z') : z' \le z\} \le r_+$$

by (1159) and (1160). Hence

$$\begin{aligned} f^{\ddagger}(z) &= \min(\underbrace{\sup\{f(z') : z' \ge z\}}_{=r_+}, \underbrace{\sup\{f(z') : z' \le z\}}_{\le r_+}) && \text{by (1158)} \\ &= \sup\{f(z') : z' \le z\}\,. \end{aligned}$$

Case **b.** of the lemma is proven analogously. We then have $z \ge z_+$ and hence $\sup\{f(z') : z' \le z\} = r_+$ by (1159) and (1160). On the other hand, $\sup\{f(z') : z' \ge z\} \le r_+$. Therefore $f^{\ddagger}(z) = \min(\sup\{f(z') : z' \ge z\}, \sup\{f(z') : z' \le z\}) = \min(\sup\{f(z') : z' \ge z\}, r_+) = \sup\{f(z') : z' \ge z\}$, as desired.

Lemma 191 *Let $f \in \mathbb{X}$ be given and suppose that $A_f \in \mathbb{A}$ is defined in terms of f according to equation* (89)*. We further assume a choice of $z_+ \in \mathbf{I}$ with $f(z_+) \ge \frac{1}{2}$. Then for all $z_0 \in \mathbf{I}$,*

a. If $z_0 \ge z_+$, then ${A_f}^{\ddagger}(z_0) = \cup\{A_f(z') : z' \ge z_0\}$;

b. If $z \le z_+$, then ${A_f}^{\ddagger}(z_0) = \cup\{A_f(z') : z' \le z_0\}$.

Proof Let us recall from Def. 95 that $r_+ = f(z_+) \ge \frac{1}{2}$, and $f(z) \le 1 - r_+$ whenever $z \ne z_+$. Hence for all $z \in \mathbf{I} \setminus \{z_+\}$,

$$A_f(z) = [0, f(z)] \subseteq [0, r_+] \cup \{r_+\} = [0, f(z_+)] \cup \{f(z_+)\} = A_f(z_+)\,. \tag{1161}$$

In order to prove part **a.**, let us now suppose that $z_0 \ge z_+$. We then know from (1161) that

$$\cup\{A_f(z') : z' \le z_0\} = (\cup\{A_f(z') : z' \le z_0, z' \ne z_+\}) \cup A_f(z_+)\,, \quad \text{because } z_+ \le z_0$$

and hence

$$\cup\{A_f(z') : z' \le z_0\} = A_f(z_+) \tag{1162}$$

by (1161). On the other hand, the subsumption stated in (1161) entails that $A_f(z) \subseteq A_f(z_+)$ for all $z' \ge z_0$. Therefore

$$\cup\{A_f(z') : z' \ge z_0\} \subseteq A_f(z_+)\,. \tag{1163}$$

In turn,

$$\begin{aligned} {A_f}^{\ddagger}(z_0) &= (\cup\{A_f(z') : z' \ge z_0\}) \cap (\cup\{A_f(z') : z' \le z_0\}) && \text{by (1152)} \\ &= (\cup\{A_f(z') : z' \ge z_0\}) \cap A_f(z_+) && \text{by (1162)} \\ &= \cup\{A_f(z') \cap A_f(z_+) : z' \ge z_0\} && \text{by distributivity} \\ &= \cup\{A_f(z') : z' \ge z_0\}\,, && \text{by (1163)} \end{aligned}$$

as desired. In the remaining case **b.** that $z_0 \leq z_+$, we can proceed analogously. By similar reasoning as above, one then shows that

$$\cup\{A_f(z') : z' \geq z_0\} = A_f(z_+)$$

and

$$\cup\{A_f(z') : z' \leq z_0\} \subseteq A_f(z_+)\,.$$

Based on these results, it is then easy to finish the proof of $A_f{}^{\ddagger}(z_0) = \cup\{A_f(z') : z' \leq z_0\}$, again utilizing the distributivity of set operations and the law of absorption.

Lemma 192 *For all $f \in \mathbb{X}$,*

$$A_{(f^{\ddagger})} = (A_f)^{\ddagger\natural}\,.$$

Proof Let $z_0 \in \mathbf{I}$ be given.

a.: $A_{(f^{\ddagger})}(z_0) \subseteq (A_f)^{\ddagger\natural}(z_0)$**.**
To see this, consider $r_0 \in A_{f^{\ddagger}}(z_0)$. We abbreviate $r_+ = r_+(f)$. It is now convenient to discern the following two cases.

In the first case that $r_0 = r_+$ and $r_+ > \frac{1}{2}$, we know from Def. 95 that $r_+ = f(z_+)$ for a unique choice of $z_+ \in \mathbf{I}$ and that $f(z) \leq 1-r_+ < r_+$ for all $z \neq z_+$. We observe that for $z > z_+$,

$$\begin{aligned}
&f^{\ddagger}(z) \\
&= \sup\{f(z') : z' \geq z\} && \text{by L-190} \\
&\leq 1 - r_+
\end{aligned}$$

by Def. 95, because $z_+ < z' \leq z$. For similar reasons, we obtain that $f^{\ddagger}(z) \leq 1 - r_+$ in the case that $z < z_+$. Hence $f^{\ddagger}(z) = r_+$ is possible only if $z = z_+$. In fact, we then have:

$$\begin{aligned}
f^{\ddagger}(z_+) &= \sup\{f(z') : z' \geq z_+\} && \text{by L-190} \\
&= \max(\underbrace{f(z_+)}_{\geq\frac{1}{2}}, \underbrace{\sup\{f(z') : z' > z_+\}}_{\leq\frac{1}{2}}) && \text{by Def. 95} \\
&= f(z_+)\,.
\end{aligned}$$

We can hence conclude from $r_0 = r_+ > \frac{1}{2}$ that $z_0 = z_+$. Hence $A_f(z_0) = A_f(z_+) = [0, 1 - r_+] \cup \{r_+\}$ by (89). It is then obvious from $f(z) \leq 1 - r_+$ for all $z \neq z_+$ and from (1152) that $A_f{}^{\ddagger}(z_0) = [0, 1 - r_+] \cup \{r_+\}$ as well. It is apparent from our assumption $r_+ > \frac{1}{2}$ that $r_+ = \sup([0, 1 - r_+] \cup \{r_+\})$. Therefore

$$A_f{}^{\ddagger\natural}(z_+) = [0, 1 - r_+] \cup \{r_+\}\,, \tag{1164}$$

and in particular, $r_0 = r_+ \in A_f{}^{\ddagger\natural}(z_+) = A_f{}^{\ddagger\natural}(z_0)$, as desired.

Next we consider the remaining case that $r_0 \neq r_+$. It is then apparent from Def. 95 and (89) that $r_0 \leq 1 - r_+$. In the special case that $z_0 = z_+$, we hence already know from (1164) that indeed $r_0 \in A_f{}^{\ddagger\natural}(z_0)$. Hence suppose that $z_0 \neq z_+$. It is then apparent from L-190 and $f(z') \leq 1 - r_+$ for all $z' \neq z_+$, as ensured by Def. 95, that in fact

$$f^{\ddagger}(z_0) \leq 1 - r_+ \,. \tag{1165}$$

Hence by (89),

$$A_{f^{\ddagger}}(z_0) = [0, f^{\ddagger}(z_0)] \,,$$

and in particular

$$r_0 \leq f^{\ddagger}(z_0) \,.$$

We now observe from (89) that $0 \in A_f(z)$, regardless of $z \in \mathbf{I}$. In particular $0 \in A_f(z_0)$. This proves that $r_0 \in A_f(z_0)$ in the special case that $r_0 = 0$. Hence suppose that $r_0 > 0$. It is now convenient to discern two more cases.
In the case that $z_0 > z_+$, we know from L-190 that

$$f^{\ddagger}(z_0) = \sup\{f(z') : z' \geq z_0\} \,. \tag{1166}$$

We can then consider some $r' \in [0, r_0)$. In particular, $r' < f^{\ddagger}(z_0)$ and by (1166), $f^{\ddagger}(z_0) = \sup\{f(z') : z' \geq z_0\} > 0$. Hence there exists $z' \geq z_0$ with

$$f(z') > r' \,. \tag{1167}$$

Because $z' \geq z_0 > z_+$, we also know that $f(z') \leq 1 - r_+ \leq \frac{1}{2}$. Therefore $A_f(z') = [0, f(z')]$. In turn, we conclude from $z' \geq z_0$ and (1167) that

$$[0, r'] \subseteq [0, f(z')] = A_f(z') \subseteq \cup\{A_f(z) : z \geq z_0\} \,. \tag{1168}$$

Because $r' \leq 1 - r_+$, we also know from (89) and $z_+ < z_0$ that

$$[0, r'] \subseteq [0, 1 - r_+] \subseteq A_f(z_+) \subseteq \cup\{A_f(z) : z \leq z_0\} \,. \tag{1169}$$

Recalling (1152), this proves that

$$[0, r'] \subseteq (\cup\{A_f(z) : z \geq z_0\}) \cap (\cup\{A_f(z) : z \leq z_0\}) = A_f{}^{\ddagger}(z_0) \,.$$

Because $r' < r_0$ was arbitrarily chosen, we conclude that

$$[0, r_0) \subseteq A_f{}^{\ddagger}(z_0) \,.$$

In particular, $s = \sup A_f{}^{\ddagger}(z_0) \geq r_0$. Let us now recall from (89) that every $A_f(z)$, $z \neq z_+$, is a closed interval of the form $[0, a]$. Because $z_0 \neq z_+$, we hence obtain from L-191 that $A_f{}^{\ddagger}(z_0)$ is a union of closed intervals of the above form. Hence

${A_f}^{\ddagger}(z_0)$ has one of the following forms, ${A_f}^{\ddagger}(z_0) = [0, s)$ or ${A_f}^{\ddagger}(z_0) = [0, s]$, where $s = \sup {A_f}^{\ddagger}(z_0)$ as above. In any case, ${A_f}^{\ddagger}(z_0) \cup \{s\} = [0, s]$. Recalling (1156), this proves that ${A_f}^{\ddagger\natural}(z_0) = [0, s]$, where $s \geq r_0$. In particular $r_0 \in {A_f}^{\ddagger\natural}(z_0)$, as desired.
The proof of the remaining case that $z_0 < z_+$ is completely analogous to the above prove of the case $z_0 > z_+$. We again utilize L-190, which in this case states that $f^{\ddagger}(z_0) = \sup\{f(z') : z' \leq z_0\}$. Apart from reversal of the inequations (considering $z' \leq z_0$ instead of $z' \geq z_0$), the present case can hence be proven in the same way as the case $z_0 > z_+$.

b.: $(A_f)^{\ddagger\natural}(z_0) \subseteq A_{(f^{\ddagger})}(z_0)$.
To see this, consider $r_0 \in {A_f}^{\ddagger\natural}(z_0)$.
Let us first suppose that $z_0 = z_+$. We notice that $A_f(z) \subseteq [0, 1 - r_+]$ for all $z \neq z_+$ and $A_f(z_+) = [0, 1 - r_+] \cup \{r_+\}$ by Def. 95 and (89). Therefore ${A_f}^{\ddagger}(z_+) = [0, 1 - r_+] \cup \{r_+\}$ and

$$ {A_f}^{\ddagger\natural}(z_+) = [0, 1 - r_+] \cup \{r_+\} \tag{1170} $$

as well, see (1152) and (1156).
We further deduce from $f(z_+) = r_+$ and (1158) that $f^{\ddagger}(z_+) = r_+$ as well. By (89), then, we obtain that

$$ A_{f^{\ddagger}}(z_+) = [0, 1 - r_+] \cup \{r_+\} $$

because $r_+ \geq \frac{1}{2}$. Combining this with (1170) results in $A_{f^{\ddagger}}(z_+) = {A_f}^{\ddagger\natural}(z_+)$. In particular $r_0 \in A_{f^{\ddagger}}(z_0)$ and $z_0 = z_+$ entail that $r_0 \in {A_f}^{\ddagger\natural}(z_0)$.
Let us now treat the remaining case that $z_0 \neq z_+$. Let us recall from (89) and Def. 95 that $A_f(z) = [0, f(z)]$ for all $z \neq z_+$. Hence for $z_0 > z_+$,

$$
\begin{aligned}
&{A_f}^{\ddagger}(z_0) \\
&= \cup\{A_f(z') : z' \geq z_0\} && \text{by L-191} \\
&= \cup\{[0, f(z')] : z' \geq z_0\} && \text{by (89) and Def. 95 because } z_+ < z_0 \\
&\subseteq [0, \sup\{f(z') : z' \geq z_0\}] \\
&= [0, f^{\ddagger}(z_0)]\,, && \text{by L-190}
\end{aligned}
$$

and for $z_0 < z_+$,

$$
\begin{aligned}
&{A_f}^{\ddagger}(z_0) \\
&= \cup\{A_f(z') : z' \leq z_0\} && \text{by L-191} \\
&= \cup\{[0, f(z')] : z' \leq z_0\} && \text{by (89) and Def. 95 because } z_+ > z_0 \\
&\subseteq [0, \sup\{f(z') : z' \leq z_0\}] \\
&= [0, f^{\ddagger}(z_0)]\,. && \text{by L-190}
\end{aligned}
$$

To sum up, I have shown that for all $z_0 \neq z_+$,

$$ {A_f}^{\ddagger}(z_0) \subseteq [0, f^{\ddagger}(z_0)]\,. $$

In particular, $\sup A_{f}{}^{\ddagger}(z_0) \leq f^{\ddagger}(z_0)$, i.e. $\sup A_{f}{}^{\ddagger}(z_0) \in [0, f^{\ddagger}(z_0)]$. Recalling (1156), this proves that $A_{f}{}^{\ddagger\natural}(z_0) \subseteq [0, f^{\ddagger}(z_0)]$. Let us further notice that

$$\begin{aligned} f^{\ddagger}(z_0) &\leq \sup\{f(z') : z' \neq z\} && \text{by L-190 and } z \neq z_+ \\ &\leq 1 - r_+ && \text{by Def. 95} \\ &\leq \tfrac{1}{2}\,. \end{aligned}$$

Hence by (89), $A_{f^{\ddagger}}(z_0) = [0, f^{\ddagger}(z_0)]$, i.e. $A_{f}{}^{\ddagger\natural}(z_0) \subseteq [0, f^{\ddagger}(z_0)] = A_{f^{\ddagger}}(z_0)$. In particular, $r_0 \in A_{f}{}^{\ddagger\natural}(z_0)$ entails that $r_0 \in A_{f^{\ddagger}}(z_0)$, which completes the proof of part **b.** of the lemma.

Lemma 193 *Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by* (87)*. If ψ satisfies* (ψ-4)*, then φ is ‡-invariant, i.e. $\varphi(f^{\ddagger}) = \varphi(f)$ for all $f \in \mathbb{X}$.*

Proof To see this, let $f \in \mathbb{X}$. Then

$$\begin{aligned} \varphi(f^{\ddagger}) &= \psi(A_{f^{\ddagger}}) && \text{by L-185} \\ &= \psi(A_{f}{}^{\ddagger\natural}) && \text{by L-192} \\ &= \psi(A_f) && \text{by L-189, L-187} \\ &= \varphi(f)\,. && \text{by L-185} \end{aligned}$$

In the following, we introduce a preorder $\trianglelefteq$ on $\mathbb{X}$. For all $f, f' \in \mathbb{X}$, let us write $f \trianglelefteq f'$ if and only if the following two conditions are satisfied.

a.: for all $z \in \mathbf{I}$, there exists $z' \geq z$ with $f'(z') \geq f(z)$; (1171)

b.: for all $z' \in \mathbf{I}$, there exists $z \leq z'$ with $f(z) \geq f'(z')$. (1172)

Lemma 194 *Let $f, f' \in \mathbb{X}$ be given and suppose that $f \sqsubseteq f'$. Then $f^{\ddagger} \trianglelefteq f'^{\ddagger}$.*

Proof In order to prove this, we first observe that $r_+(f^{\ddagger}) = r_+(f)$ and $r_+(f'^{\ddagger}) = r_+(f')$, which is apparent from (1158). In addition, we know from L-188 that $f \sqsubseteq f'$ entails that $r_+(f) = r_+(f')$. To sum up, there exists $r_+ \in \mathbf{I}$ with

$$r_+ = r_+(f) = r_+(f^{\ddagger}) = r_+(f'^{\ddagger}) = r_+(f')\,. \tag{1173}$$

It is also worth noticing that

$$f^{\ddagger}(z_+) = f(z_+) = r_+ \tag{1174}$$

and

$$f'^{\ddagger}(z_+{}') = f'(z_+{}') = r_+\,, \tag{1175}$$

where $z_+ = z_+(f)$ and $z_+{}' = z_+(f')$, which is apparent from (1158) and the fact that the maximum of f is achieved at z_+.

Let us now consider the preconditions (1171) and (1172) of $f^\ddagger \trianglelefteq f'^\ddagger$ in turn.
To see that (1171) holds, suppose that $z \in \mathbf{I}$. In the case that $z \leq z_+' = z_+(f')$, we proceed as follows.

$$\begin{aligned} f^\ddagger(z) &\leq r_+(f^\ddagger) && \text{by Def. 95} \\ &= r_+(f') && \text{by (1173)} \\ &= f'(z_+') && \text{see Def. 95} \\ &= f'^\ddagger(z_+')\,. && \text{by (1175)} \end{aligned}$$

Hence $z' = z_+'$ is a legal choice of $z' \geq z$ with $f^\ddagger(z) \leq f'^\ddagger(z')$.
In the remaining case that $z > z_+(f')$,

$$\begin{aligned} f'^\ddagger(z) &= \sup\{f'(z') : z' \geq z\} && \text{by L-190} \\ &= \sup\{\sup\{f(z'') : z'' \geq z'\} : z' \geq z\} \\ &\geq \sup\{f(z') : z' \geq z\} && \text{by Def. 97} \\ &\geq f^\ddagger(z)\,, && \text{by (1158)} \end{aligned}$$

i.e. $z' = z$ is a suitable choice of $z' \geq z$ with $f'^\ddagger(z') \geq f^\ddagger(z)$, as desired.
Now we focus on (1172). Hence consider $z' \in \mathbf{I}$. In the case that $z_+ = z_+(f) \leq z'$,

$$\begin{aligned} f^\ddagger(z_+) &= r_+ && \text{by (1174)} \\ &= r_+(f'^\ddagger) && \text{by (1173)} \\ &\geq f'^\ddagger(z')\,. && \text{see Def. 95} \end{aligned}$$

Hence $z = z_+$ is a suitable choice of $z \leq z'$ with $f^\ddagger(z) \geq f'^\ddagger(z')$.
Finally if $z_+ = z_+(f) > z'$, then

$$\begin{aligned} f^\ddagger(z') &= \sup\{f(z) : z \leq z'\} && \text{by L-190} \\ &= \sup\{\sup\{f(z'') : z'' \leq z\} : z \leq z'\} \\ &\geq \sup\{f'(z) : z \leq z'\} && \text{by Def. 97} \\ &\geq f'^\ddagger(z')\,. && \text{by (1158)} \end{aligned}$$

This proves that $z = z'$ is a legal choice of $z \leq z'$ with $f^\ddagger(z) \geq f'^\ddagger(z')$. Because both preconditions (1171) and (1172) are satisfied, we conclude the desired $f^\ddagger \trianglelefteq f'^\ddagger$.

Lemma 195 *Let $f, f' \in \mathbb{X}$ be given and suppose that $f \trianglelefteq f'$. Then $f \sqsubseteq f'$ as well.*

Proof Hence let $f, f' \in \mathbb{X}$ with $f \trianglelefteq f'$ be given. I first show that condition **a.** of Def. 97 is satisfied. Hence let $z \in \mathbf{I}$ be given. We then know from (1171) that there exists $z_0 \geq z$ with $f'(z_0) \geq f(z)$. In particular

$$\begin{aligned} \sup\{f'(z') : z' \geq z\} &\geq f'(z_0) && \text{because } z_0 \geq z \\ &\geq f(z)\,. && \text{by (1171)} \end{aligned}$$

Now let us discuss condition **b.** of Def. 97. Hence consider $z' \in \mathbf{I}$. By (1172), then, there exists $z_1 \leq z'$ with $f(z_1) \geq f'(z')$. Therefore

$$\begin{aligned} \sup\{f(z) : z \leq z'\} &\leq f(z_1) && \text{because } z_1 \leq z' \\ &\geq f'(z') \,. && \text{by (1172)} \end{aligned}$$

This completes the proof that both preconditions of Def. 97 are valid; hence indeed $f \sqsubseteq f'$.

Lemma 196 *Let* $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ *be given and suppose that* $\psi : \mathbb{A} \longrightarrow \mathbf{I}$, *defined by* (87), *satisfies* (ψ-4). *Then for all* $f, f' \in \mathbb{X}$,

$$\varphi(f) \leq \varphi(f') \tag{1176}$$

provided that $f \trianglelefteq f'$.

Proof Hence let $f, f' \in \mathbb{X}$ with $f \trianglelefteq f'$ be given. We then know from L-195 that $f \sqsubseteq f'$. Hence L-188 is applicable, and

$$r_+ = r_+(f) = r_+(f') \,. \tag{1177}$$

I will now show that $A_f \sqsubseteq A_{f'}$, by proving that conditions **a.** and **b.** of Def. 89 are fulfilled by $A_f, A_{f'}$.
To see that condition **a.** holds, consider $z \in \mathbf{I}$. If $f(z) \geq \frac{1}{2}$, i.e. $f(z) = r_+(f)$, then $A_f(z) = [0, 1 - r_+] \cup \{r_+\}$ by (89) and (1177). We then conclude from (1171) that there exists $z' \geq z$ with $f'(z') \geq f(z)$. Because $r_+(f') = r_+$ by (1177), this is only possible if $f'(z') = r_+$. Hence by (89), $A_{f'}(z') = [0, 1 - r_+] \cup \{r_+\} = A_f(z)$. In particular, $r \in A_f(z)$ entails that $r \in A_{f'}(z')$ for the given $z' \geq z$, as desired.
In the remaining case that $f(z) < \frac{1}{2}$, we know from (89) that $A_f(z) = [0, f(z)]$. From (1171), we obtain that there exists $z' \geq z$ with $f'(z') \geq f(z)$. Hence $[0, f(z)] \subseteq [0, f'(z')] \subseteq A_{f'}(z')$. In particular $r \in A_f(z) = [0, f(z)]$ entails that $r \in A_{f'}(z')$ for the given $z' \geq z$. This completes the proof that condition **a.** of Def. 89 holds.

We now discuss condition **b.** of Def. 89. Hence let $z' \in \mathbf{I}$ be given. If $f'(z') \geq \frac{1}{2}$, i.e. $f'(z') = r_+(f')$, then then $A_{f'}(z') = [0, 1 - r_+] \cup \{r_+\}$ by (89) and (1177). We obtain from (1172) that there exists $z \leq z'$ with $f(z) \geq f'(z')$. Because $r_+(f) = r_+$ by (1177), this is only possible if $f(z) = r_+$ as well. Hence by (89), $A_f(z) = [0, 1 - r_+] \cup \{r_+\} = A_{f'}(z')$. In particular, $r \in A_{f'}(z')$ entails that $r \in A_f(z)$ for the given $z \leq z'$.
Finally in the case that $f'(z') < \frac{1}{2}$, (89) results in $A_{f'}(z') = [0, f'(z')]$. From (1172), we obtain that there exists $z \leq z'$ with $f(z) \geq f'(z')$. Hence $[0, f'(z')] \subseteq [0, f(z)] \subseteq A_f(z)$. In particular $r \in A_{f'}(z') = [0, f'(z')]$ entails that $r \in A_f(z)$ for the given $z \leq z'$. Hence condition **b.** of Def. 89 holds as well. This completes the proof that indeed

$$A_f \sqsubseteq A_{f'} \,. \tag{1178}$$

Consequently

$$\begin{aligned}
\varphi(f) &= \psi(A_f) && \text{by L-185, (88)}\\
&\leq \psi(A_{f'}) && \text{by } (\psi\text{-4}) \text{ and (1178)}\\
&= \varphi(f')\,, && \text{by L-185, (88)}
\end{aligned}$$

i.e. $f \trianglelefteq f'$ entails that $\varphi(f) \leq \varphi(f')$, as claimed by the lemma.

Proof of Theorem 126

Consider $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ and suppose that $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ is defined by (87). We shall prove the equivalences stated in the theorem in due turn. For convenience, every equivalence will be split into two implications, which are proven separately.

a.1: If φ satisfies (φ-1) then ψ satisfies (ψ-1).
To see this, suppose that φ satisfies (φ-1) and consider $A \in \mathbb{A}$ with $D(A) = \{1\}$, i.e. $A(z_+) = \{1\}$ and $A(z) = \varnothing$ for all $z \neq z_+$. It is then apparent from Def. 93 that $f_A(z_+) = \sup A(z_+) = \sup\{1\} = 1$ and $f_A(z) = \sup A(z) = \sup \varnothing = 0$ for all $z \neq z_+$. Hence

$$f_A(z) = \begin{cases} 1 & : \quad z = z_+ \\ 0 & : \quad z \neq z_+ \end{cases}$$

for all $z \in \mathbf{I}$. In particular $f^{-1}((0,1]) = \{z_+\}$ and $f(z_+) = 1$. Therefore

$$\begin{aligned}
\psi(A) &= \varphi(f_A) && \text{by (87)}\\
&= z_+\,. && \text{by } (\varphi\text{-1})
\end{aligned}$$

a.2: If ψ satisfies (ψ-1) then φ satisfies (φ-1).
Hence suppose that ψ satisfies (ψ-1) and let $f \in \mathbb{X}$ be given such that $f^{-1}((0,1]) = \{z_+\}$ and $f(z_+) = 1$. We now define $A \in \mathbb{A}$ by

$$A(z) = \begin{cases} \{1\} & : \quad z = z_+ \\ \varnothing & : \quad \text{else} \end{cases}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 93 that

$$f_A = f\,. \tag{1179}$$

We further notice that $z_+(A)$ coincides with the given z_+, and $D(A) = \{1\}$, i.e. (ψ-1) is applicable. Therefore

$$\begin{aligned}
\varphi(f) &= \varphi(f_A) && \text{by (1179)}\\
&= \psi(A) && \text{by (87)}\\
&= z_+\,. && \text{by } (\psi\text{-1})
\end{aligned}$$

b.1: If φ satisfies (φ-2) then ψ satisfies (ψ-2).
Hence suppose that φ satisfies (φ-2) and let $A, A' \in \mathbb{A}$ be given such that

$$A'(z) = A(1-z) \tag{1180}$$

for all $z \in \mathbf{I}$.
Now we consider $f_A{}'(z)$ vs. $f_A(z)$ for a given $z \in \mathbf{I}$. Apparently

$$\begin{aligned} f_A{}'(z) &= \sup A'(z) && \text{by Def. 93} \\ &= \sup A(1-z) && \text{by (1180)} \\ &= f_A(1-z)\,. && \text{by Def. 93} \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $f_A{}'(z) = f_A(1-z)$ for all $z \in \mathbf{I}$, and (φ-2) is applicable. Consequently

$$\begin{aligned} \psi(A') &= \varphi(f_{A'}) && \text{by (87)} \\ &= 1 - \varphi(f_A) && \text{by (φ-2)} \\ &= 1 - \psi(A)\,. && \text{by (87)} \end{aligned}$$

b.2: If ψ satisfies (ψ-2) then φ satisfies (φ-2).
To see this, assume that ψ satisfies (ψ-2) and consider $f, f' \in \mathbb{X}$ with

$$f'(z) = f(1-z) \tag{1181}$$

for all $z \in \mathbf{I}$. As I will now prove, this entails that

$$A_{f'}(z) = A_f(1-z) \tag{1182}$$

for all $z \in \mathbf{I}$. Hence let $z \in \mathbf{I}$. In the case that $f'(z) > \frac{1}{2}$, then $f(1-z) = f'(z) > \frac{1}{2}$ as well. Therefore

$$\begin{aligned} A_{f'}(z) &= [0, 1 - f'(z)] \cup \{f'(z)\} && \text{by (89)} \\ &= [0, 1 - f(1-z)] \cup \{f(1-z)\} && \text{by (1181)} \\ &= A_f(1-z)\,. && \text{by (89)} \end{aligned}$$

In the remaining case that $f'(z) \le \frac{1}{2}$, we conclude from (1181) that $f(1-z) = f'(z) \le \frac{1}{2}$ as well. Hence

$$\begin{aligned} A_{f'}(z) &= [0, f'(z)] && \text{by (89)} \\ &= [0, f(1-z)] && \text{by (1181)} \\ &= A_f(1-z)\,. && \text{by (89)} \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrary, this proves that $A_{f'}(z) = A_f(1-z)$ for all $z \in \mathbf{I}$, i.e. (ψ-2) is applicable. Hence

$$\begin{aligned} \varphi(f') &= \psi(A_{f'}) && \text{by L-185, (88)} \\ &= 1 - \psi(A_f) && \text{by (ψ-2)} \\ &= 1 - \varphi(f)\,. && \text{by L-185 and (88)} \end{aligned}$$

c.1: If φ satisfies (φ-3) then ψ satisfies (ψ-3).
Hence suppose that φ satisfies (φ-3) and consider $A \in \mathbb{A}$ with $\mathrm{NV}(A) \subseteq \{0,1\}$ and $r_+ \in A(1)$. Hence $A(z) = \varnothing$ for all $z \in (0,1)$, see (78). Recalling Def. 93, this proves that $f_A(z) = \sup A(z) = \sup \varnothing = 0$ for all $z \in (0,1)$. In other words, ${f_A}^{-1}((0,1]) \subseteq \{0,1\}$. Because $r_+ \in A(1)$ and $r_+ \geq \frac{1}{2}$, we further obtain from Def. 93 that $f_A(1) = \sup A(1) \geq r_+ \geq \frac{1}{2}$. Summarizing these results, we have shown that (φ-3) is applicable to f_A, and hence

$$\varphi(f_A) = 1 - f_A(0) = 1 - \sup A(0) \tag{1183}$$

by Def. 93. In turn

$$\begin{aligned} \psi(A) &= \varphi(f_A) && \text{by (87)} \\ &= 1 - \sup A(0)\,, && \text{by (1183)} \end{aligned}$$

i.e. (ψ-3) holds, as desired.

c.2: If ψ satisfies (ψ-3) then φ satisfies (φ-3).
Suppose that ψ satisfies (ψ-3) and consider $f \in \mathbb{X}$ with $f^{-1}((0,1]) \subseteq \{0,1\}$ and $f(1) \geq \frac{1}{2}$. We define $A \in \mathbb{A}$ by

$$A(z) = \begin{cases} \{f(0)\} & : \quad z = 0 \\ \varnothing & : \quad z \in (0,1) \\ [0, 1 - f(1)] \cup \{f(1)\} & : \quad z = 1 \end{cases} \tag{1184}$$

for all $z \in \mathbf{I}$. It is then apparent from Def. 93 that

$$f = f_A\,. \tag{1185}$$

In addition, $\mathrm{NV}(A) = \{0,1\}$ by (78) and $r_+(A) = f(1) \in A(1)$. Hence (ψ-3) is applicable, and

$$\psi(A) = 1 - \sup A(0) = 1 - f(0) \tag{1186}$$

by (1184). This proves that

$$\begin{aligned} \varphi(f) &= \varphi(f_A) && \text{by (1185)} \\ &= \psi(A) && \text{by (87)} \\ &= 1 - f(0)\,, && \text{by (1186)} \end{aligned}$$

i.e. (φ-3) is indeed valid.

d.1: If φ satisfies (φ-4) then ψ satisfies (ψ-4).
Let us assume that φ fulfills (φ-4) and suppose that $A, A' \in \mathbb{A}$ satisfy $A \sqsubseteq A'$. I now show that

$$f_A \sqsubseteq f_{A'}\,. \tag{1187}$$

To see that condition **a.** of Def. 97 is satisfied, consider $z \in \mathbf{I}$. If $A(z) = \varnothing$, then $f_A(z) = \sup A(z) = \sup \varnothing = 0$ and hence trivially $\sup\{f_{A'}(z') : z' \geq z\} \geq 0 = f_A(z)$. In the remaining case that $A(z) \neq \varnothing$, let $\varepsilon > 0$. Because $f_A(z) = \sup A(z)$, there exists $r \in A(z)$ with $r > f_A(z) - \varepsilon$. Recalling Def. 89, we conclude from $A \sqsubseteq A'$ that there exists $z'_0 \geq z$ with $r \in A(z'_0)$. Hence $f_{A'}(z'_0) = \sup A'(z'_0) \geq r > f_A - \varepsilon$ for this choice of $z'_0 \geq z$, and in turn

$$\sup\{f_{A'}(z') : z' \geq z\} \geq f_{A'}(z'_0) > f_A(z) - \varepsilon$$

because $z'_0 \geq z$. $\varepsilon \to 0$ then yields the desired

$$\sup\{f_{A'}(z') : z' \geq z\} \geq f_A(z)\,,$$

i.e. condition **a.** of Def. 97 is satisfied. Let us now consider the second condition of Def. 97. Hence let $z' \in \mathbf{I}$. If $A'(z') = \varnothing$, then ${f_A}'(z') = \sup A'(z') = \sup \varnothing = 0$, see Def. 93. It then holds trivially that $\sup\{f_A(z) : z \leq z'\} \geq 0 = f_{A'}(z')$. In the remaining case that $A'(z') \neq \varnothing$, let $\varepsilon > 0$. Then there exists $r \in A'(z')$ with $r > \sup A'(z') - \varepsilon = f_{A'}(z') - \varepsilon$, see Def. 93. Recalling condition **b.** of Def. 89, we conclude from $A \sqsubseteq A'$ that there exist $z_0 \leq z'$ with $r \in A(z_0)$. In particular $\sup A(z_0) \geq r$ and hence $f_A(z_0) \geq r > f_{A'}(z') - \varepsilon$, cf. Def. 93. Because $z_0 \leq z'$,

$$\sup\{f_A(z) : z \leq z'\} \geq f_A(z_0) > f_{A'}(z) - \varepsilon\,.$$

Noticing that $\varepsilon > 0$ was chosen arbitrarily, we hence deduce that $\sup f_A(z) : z \leq z'\} \geq f_{A'}(z)$. This proves that the second condition of Def. 97 is also satisfied; which permits us to deduce that (1187) is indeed valid. Therefore

$$\begin{aligned}
\psi(A) &= \varphi(f_A) && \text{by (87)}\\
&\leq \varphi(f_{A'}) && \text{by (φ-4) and (1187)}\\
&= \psi(A')\,, && \text{by (87)}
\end{aligned}$$

as desired.

d.2: If ψ satisfies (ψ-4) then φ satisfies (φ-4).
Hence suppose that (ψ-4) is valid for the given ψ. In order to prove that φ satisfies (φ-4), we consider $f, f' \in \mathbb{X}$ with $f \sqsubseteq f'$. We then know from L-194 that

$$f^{\ddagger} \trianglelefteq {f'}^{\ddagger}\,. \tag{1188}$$

Therefore

$$\begin{aligned}
\varphi(f) &= \varphi(f^{\ddagger}) && \text{by L-193}\\
&\leq \varphi({f'}^{\ddagger}) && \text{by L-196 and (1188)}\\
&= \varphi(f')\,, && \text{by L-193}
\end{aligned}$$

i.e. φ indeed satisfies (φ-4).

e.1: If φ satisfies (φ-5) then ψ satisfies (ψ-5).
Suppose that φ satisfies (φ-5) and consider some $A \in \mathbb{A}$. Then for all $z \in \mathbf{I}$,

$$f_A(z) = \sup A(z) \tag{1189}$$

by Def. 93, and

$$\begin{aligned} f_{\boxplus A}(z) &= \sup[0, \widehat{\boxplus} A(z)] && \text{by Def. 93 and Def. 91} \\ &= \widehat{\boxplus} A(z) \\ &= \min(\sup A(z), \tfrac{1}{2}) && \text{by (73)} \\ &= \min(f_A(z), \tfrac{1}{2})\,. && \text{by (1189)} \end{aligned}$$

Hence (φ-5) is applicable, and

$$\varphi(f_A) = \varphi(f_{\boxplus A})\,. \tag{1190}$$

This in turn proves the desired

$$\begin{aligned} \psi(A) &= \varphi(f_A) && \text{by (87)} \\ &= \varphi(f_{\boxplus A}) && \text{by (1190)} \\ &= \psi(\boxplus A)\,, && \text{by (87)} \end{aligned}$$

i.e. ψ indeed satisfies (ψ-5).

e.2: If ψ satisfies (ψ-5) then φ satisfies (φ-5).
To see this, let us assume that ψ satisfies (ψ-5). Now consider $f \in \mathbb{X}$ and define $f' \in \mathbb{X}$ by

$$f'(z) = \min(f(z), \tfrac{1}{2}) \tag{1191}$$

for all $z \in \mathbf{I}$. Recalling (89), f and f' apparently result in

$$A_f(z) = \begin{cases} [0, f(z)] & : \quad f(z) \leq \frac{1}{2} \\ [0, 1 - f(z)] \cup \{f(z)\} & : \quad f(z) > \frac{1}{2} \end{cases} \tag{1192}$$

and

$$A_{f'}(z) = [0, f'(z)] \tag{1193}$$

for all $z \in \mathbf{I}$. Now expanding $f'(z)$ by (1191), we obtain

$$\begin{aligned} A_{f'}(z) &= [0, \min(f(z), \tfrac{1}{2})] && \text{by (1191), (1193)} \\ &= [0, \min(\sup A_f(z), \tfrac{1}{2})] && \text{by (1192)} \\ &= \boxplus A_f(z)\,. && \text{by Def. 91} \end{aligned}$$

Because $z \in \mathbf{I}$ was arbitrarily chosen, this proves that

$$A_{f'} = \boxplus A_f\,. \tag{1194}$$

Therefore

$$\begin{aligned}
\varphi(f) &= \psi(A_f) && \text{by L-185 and (88)} \\
&= \psi(\boxplus A_f) && \text{by } (\psi\text{-5}) \\
&= \psi(A_{f'}) && \text{by (1194)} \\
&= \varphi(f')\,, && \text{by L-185 and (88)}
\end{aligned}$$

which completes the proof that (φ-5) holds provided that (ψ-5) is valid.

B.36 Proof of Theorem 127

Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given and suppose that φ satisfies (φ-1)–(φ-5). We then know from Th-126 that the mapping $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ defined by (87) satisfies (ψ-1)–(ψ-5). In turn, we conclude from Th-107 that $\mathcal{F}_\psi$ is a standard DFS. But $\mathcal{F}_\varphi = \mathcal{F}_\psi$ by Th-124, which proves that $\mathcal{F}_\varphi$ is indeed a standard DFS.

B.37 Proof of Theorem 128

Let $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ be given and suppose that $\mathcal{F}_\varphi$ is a DFS. Let us define $\psi : \mathbb{A} \longrightarrow \mathbf{I}$ in terms of φ according to (87). We then know that $\mathcal{F}_\psi = \mathcal{F}_\varphi$, i.e. $\mathcal{F}_\psi$ is a DFS. Hence Th-116 states that ψ satisfies (ψ-1)–(ψ-5). In turn, we obtain from Th-126 that φ satisfies (φ-1)–(φ-5), as desired.

B.38 Proof of Theorem 129

The independence of most of the 'φ-conditions' of the remaining conditions is apparent from Th-126 and Th-121. In fact the only condition the independence of which needs to be verified separately is (φ-5). To see that (φ-5) is independent of (φ-1), (φ-2), (φ-3) and (φ-4), we introduce a mapping $\varphi_* : \mathbb{X} \longrightarrow \mathbf{I}$ which satisfies all of these conditions except for (φ-5). Hence let us define a number of coefficients based on a given f, viz

$$\ell(f) = \begin{cases} \max(\inf f^{-1}((0,1]), 1 - \sup \widehat{f}([0,1))) & : \quad f(1) = r_+ \\ \max(\inf f^{-1}((0,1]), f(1)) & : \quad f(1) \neq r_+ \end{cases} \tag{1195}$$

$$u(f) = \begin{cases} \min(\sup f^{-1}((0,1]), \sup \widehat{f}((0,1])) & : \quad f(0) = r_+ \\ \min(\sup f^{-1}((0,1]), 1 - f(0)) & : \quad f(0) \neq r_+ \end{cases} \tag{1196}$$

$$\alpha = \alpha(f) = 2 \cdot r_+(f) - 1 \tag{1197}$$

for all $f \in \mathbb{X}$, where $r_+ = r_+(f)$. In terms of these coefficients, we then define $\varphi_* : \mathbb{X} \longrightarrow \mathbf{I}$ by

$$\varphi_*(f) = \begin{cases} \alpha \cdot \ell(f) + (1-\alpha)u(f) & : \quad \ell(f) > \frac{1}{2} \\ \alpha \cdot u(f) + (1-\alpha)\ell(f) & : \quad u(f) < \frac{1}{2} \\ \frac{1}{2} & : \quad \ell(f) \leq \frac{1}{2} \leq u(f) \end{cases} \tag{1198}$$

for all $f \in \mathbb{X}$. In the following, I will first prove that φ_* is well-defined, by showing that $\ell(f) \leq u(f)$ for all $f \in \mathbb{X}$, as required by (1198). To this end, we need some observations on how the computation of $\ell(f)$ and $u(f)$ can be simplified.

Lemma 197 *Let $f \in \mathbb{X}$ be given. Then*

a. if $f(0) \neq 0$ and $f(1) \neq r_+$, then $\ell(f) = f(1)$;

b. if $f(0) \neq 0$ and $f(1) = r_+$, then $\ell(f) = 1 - \sup \widehat{f}([0,1))$;

c. if $f(1) = 0$, then $\ell(f) = \inf f^{-1}((0,1])$.

Proof

a.: $f(0) \neq 0$ **and** $f(1) \neq r_+$**.**
Then $0 \in f^{-1}((0,1])$ and hence

$$\inf f^{-1}((0,1]) = 0\,. \tag{1199}$$

Therefore

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), f(1)) && \text{by (1195)}\\
&= \max(0, f(1)) && \text{by (1199)}\\
&= f(1)\,.
\end{aligned}$$

b.: $f(0) \neq 0$ **and** $f(1) = r_+$**.** Then again $0 \in f^{-1}((0,1])$ and hence

$$\inf f^{-1}((0,1]) = 0\,. \tag{1200}$$

In turn

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), 1 - \sup \widehat{f}([0,1))) && \text{by (1195)}\\
&= \max(0, 1 - \sup \widehat{f}([0,1))) && \text{by (1200)}\\
&= 1 - \sup \widehat{f}([0,1))\,.
\end{aligned}$$

c.: $f(1) = 0$
Then clearly $f(1) \neq r_+$ because $r_+ \geq \frac{1}{2}$. Therefore

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), f(1)) && \text{by (1195)}\\
&= \max(\inf f^{-1}((0,1]), 0) && \text{by assumption that } f(1) = 0\\
&= \inf f^{-1}((0,1])\,.
\end{aligned}$$

In the case of $u(f)$, we obtain similar results.

Lemma 198 *Let $f \in \mathbb{X}$ be given. Then*

a. if $f(1) \neq 0$ and $f(0) \neq r_+$, then $u(f) = 1 - f(0)$;

b. if $f(1) \neq 0$ and $f(0) = r_+$, then $u(f) = \sup \widehat{f}((0,1])$;

c. if $f(0) = 0$, then $u(f) = \sup f^{-1}((0,1])$.

Proof

a.: $f(1) \neq 0$ **and** $f(0) \neq r_+$**.**
Then $1 \in f^{-1}((0,1])$ and hence

$$\sup f^{-1}((0,1]) = 1\,. \tag{1201}$$

Therefore

$$\begin{aligned} u(f) &= \min(\sup f^{-1}((0,1]), 1 - f(0)) && \text{by (1196)} \\ &= \min(1, 1 - f(0)) && \text{by (1201)} \\ &= 1 - f(0)\,. \end{aligned}$$

b.: $f(1) \neq 0$ **and** $f(0) = r_+$**.** Then again $1 \in f^{-1}((0,1])$ and hence

$$\sup f^{-1}((0,1]) = 1\,. \tag{1202}$$

In turn

$$\begin{aligned} u(f) &= \min(\sup f^{-1}((0,1]), \sup \widehat{f}((0,1])) && \text{by (1196)} \\ &= \min(1, \sup \widehat{f}((0,1])) && \text{by (1202)} \\ &= \sup \widehat{f}((0,1])\,. \end{aligned}$$

c.: $f(0) = 0$
Then in particular $f(0) \neq r_+$ because $r_+ \geq \frac{1}{2}$. Therefore

$$\begin{aligned} u(f) &= \min(\sup f^{-1}((0,1]), 1 - f(0)) && \text{by (1196)} \\ &= \min(\sup f^{-1}((0,1]), 1) && \text{by assumption that } f(0) = 0 \\ &= \sup f^{-1}((0,1])\,. \end{aligned}$$

Lemma 199 *φ_* is well-defined, i.e. for all $f \in \mathbb{X}$, $\ell(f) \leq u(f)$.*

Proof To see this, consider $f \in \mathbb{X}$. It is convenient to discern the following cases.

a.: $f(0) = r_+$ **and** $f(1) = 0$.
Then

$$\begin{aligned} \ell(f) &= f(1) && \text{by L-197.}\textbf{a} \\ &= 0 && \text{by assumption} \\ &\le u(f)\,. \end{aligned}$$

b.: $f(0) = r_+$**,** $f(1) \neq r_+$ **and** $f(1) \neq 0$.
Then

$$\begin{aligned} \ell(f) &= f(1) && \text{by L-197.}\textbf{a} \\ &\le \sup \widehat{f}((0,1]) && \text{because } 1 \in (0,1] \text{ and } f(1) \in \widehat{f}((0,1]) \\ &= u(f)\,. && \text{by L-198.}\textbf{b} \end{aligned}$$

c.: $f(0) = r_+$ **and** $f(1) = r_+$.
We then know from Def. 95 that $r_+ = \frac{1}{2}$. Therefore

$$\begin{aligned} \ell(f) &= 1 - \sup \widehat{f}([0,1)) && \text{by L-197.}\textbf{b} \\ &= 1 - r_+ && \text{see Def. 95} \\ &= r_+ && \text{because } r_+ = \tfrac{1}{2} \\ &= \sup \widehat{f}((0,1]) && \text{see Def. 95} \\ &= u(f)\,. && \text{by L-198.}\textbf{b} \end{aligned}$$

d.: $f(0) = 0$ **and** $f(1) = r_+$.
Then $1 \in f^{-1}((0,1])$ and hence

$$\sup f^{-1}((0,1]) = 1\,. \tag{1203}$$

In particular

$$\begin{aligned} \ell(f) &\le 1 \\ &= \sup f^{-1}((0,1]) && \text{by (1203)} \\ &= u(f)\,. && \text{by L-198.}\textbf{c} \end{aligned}$$

e.: $f(0) \neq 0$**,** $f(0) \neq r_+$ **and** $f(1) = r_+$.
Then

$$\begin{aligned} \ell(f) &= 1 - \sup \widehat{f}([0,1)) && \text{by L-197.}\textbf{b} \\ &\le 1 - f(0) && \text{because } 0 \in [0,1) \text{ and hence } f(0) \in \widehat{f}([0,1)) \\ &= u(f)\,. && \text{by L-198.}\textbf{a} \end{aligned}$$

f.: $f(0) = 0$ **and** $f(1) = 0$.
Then

$$\begin{aligned} \ell(f) &= \inf f^{-1}((0,1]) && \text{by L-197.}\mathbf{c} \\ &\leq \sup f^{-1}((0,1]) && \text{because } f^{-1}((0,1]) \neq \varnothing \\ &= u(f)\,. && \text{by L-198.}\mathbf{c} \end{aligned}$$

g.: $f(0) = 0$ **and** $f(1) \neq 0$**,** $f(1) \neq r_+$.
Then $1 \in f^{-1}((0,1])$ and hence

$$\sup f^{-1}((0,1]) = 1\,. \tag{1204}$$

Therefore

$$\begin{aligned} \ell(f) &\leq 1 \\ &= \sup f^{-1}((0,1]) && \text{by (1204)} \\ &= u(f)\,. && \text{by L-198.}\mathbf{c} \end{aligned}$$

h.: $f(0) \neq 0$**,** $f(0) \neq r_+$ **and** $f(1) = 0$.
Then

$$\begin{aligned} \ell(f) &= f(1) && \text{by L-197.}\mathbf{a} \\ &= 0 && \text{by assumption on } f \\ &\leq u(f)\,. \end{aligned}$$

i.: $f(0) \neq 0$**,** $f(0) \neq r_+$**,** $f(1) \neq 0$ **and** $f(1) \neq r_+$.
Then

$$\begin{aligned} \ell(f) &= f(1) && \text{by L-197.}\mathbf{a} \\ &\leq 1 - r_+ && \text{by Def. 95 because } f(1) \neq r_+ \\ &\leq r_+ && \text{because } r_+ \geq \tfrac{1}{2} \\ &\leq 1 - f(0) && \text{by Def. 95 because } f(0) \neq r_+ \\ &= u(f)\,. && \text{by L-198.}\mathbf{a} \end{aligned}$$

Lemma 200 *Let $f \in \mathbb{X}$ be given and suppose that $\ell(f) = u(f)$. Then $\varphi_*(f) = \ell(f)$.*

Proof It is useful to discern three cases. Firstly if $\ell(f) > \frac{1}{2}$, then

$$\begin{aligned} \varphi_*(f) &= \alpha\,\ell(f) + (1-\alpha)u(f) && \text{by (1198)} \\ &= \alpha\,\ell(f) + (1-\alpha)\ell(f) && \text{by assumption} \\ &= \alpha\,\ell(f) + \ell(f) - \alpha\ell(f) \\ &= \ell(f)\,. \end{aligned}$$

In the case that $\ell(f) = \frac{1}{2}$, we directly obtain from (1198) that $\varphi_*(f) = \frac{1}{2} = \ell(f)$. Finally if $\ell(f) < \frac{1}{2}$, then in particular $u(f) = \ell(f) < \frac{1}{2}$ and hence

$$\begin{aligned}
\varphi_*(f) &= \alpha\, u(f) + (1-\alpha)\ell(f) && \text{by (1198)} \\
&= \alpha\, \ell(f) + (1-\alpha)\ell(f) && \text{by assumption} \\
&= \alpha\, \ell(f) + \ell(f) - \alpha\ell(f) \\
&= \ell(f)\,,
\end{aligned}$$

as desired.

Lemma 201 *Let $f \in \mathbb{X}$ be given and suppose that $f' \in \mathbb{X}$ is defined by*

$$f'(z) = f(1-z) \tag{1205}$$

for all $z \in \mathbf{I}$. Then

a. $\ell(f') = 1 - u(f)$;

b. $u(f') = 1 - \ell(f)$.

Proof To see this, we first notice that

$$f'(1) = f(0)\,, \tag{1206}$$
$$f'(0) = f(1)\,. \tag{1207}$$

This is apparent from (1205). In addition, we observe that

$$\begin{aligned}
{f'}^{-1}((0,1]) &= \{z : f'(z) > 0\} && \text{by definition of inverse images} \\
&= \{z : f(1-z) > 0\} && \text{by (1205)} \\
&= \{1 - z' : f(z') > 0\} && \text{by substitution } z' = 1 - z \\
&= \{1 - z : z \in \{z' : f(z') > 0\}\}\,,
\end{aligned}$$

and hence

$${f'}^{-1}((0,1]) = \{1 - z : z \in f^{-1}((0,1])\}\,. \tag{1208}$$

This proves that

$$\inf {f'}^{-1}((0,1]) = \inf\{1 - z : z \in f^{-1}((0,1])\} = 1 - \sup f^{-1}((0,1]) \tag{1209}$$

and

$$\sup {f'}^{-1}((0,1]) = \sup\{1 - z : z \in f^{-1}((0,1])\} = 1 - \inf f^{-1}((0,1]) \tag{1210}$$

Finally

$$\begin{aligned}
\sup \widehat{f'}([0,1)) &= \sup\{f'(z) : z \in [0,1)\} && \text{by Def. 15} \\
&= \sup\{f(z) : z \in (0,1]\}\,, && \text{by (1205)}
\end{aligned}$$

i.e.

$$\sup \widehat{f'}([0,1)) \sup \widehat{f}((0,1]) , \tag{1211}$$

and similarly

$$\begin{aligned} \sup \widehat{f'}((0,1]) &= \sup\{f'(z) : z \in (0,1]\} && \text{by Def. 15} \\ &= \sup\{f(z) : z \in [0,1)\} , && \text{by (1205)} \end{aligned}$$

which shows that

$$\sup \widehat{f'}((0,1]) = \sup \widehat{f}([0,1)) . \tag{1212}$$

We can now put the pieces together, and prove that part **a.** of the lemma is valid. Firstly if $f'(1) = r_+$, then we know from (1206) that $f(0) = f'(1) = r_+$ as well, where I have abbreviated $r_+ = r_+ f$, noticing that $r_+(f) = r_+(f')$. Therefore

$$\begin{aligned} \ell(f') &= \max(\inf f'^{-1}((0,1]), 1 - \sup \widehat{f'}([0,1))) && \text{by (1195)} \\ &= \max(1 - \sup f^{-1}((0,1]), 1 - \sup \widehat{f}((0,1])) && \text{by (1209), (1211)} \\ &= 1 - \min(\sup f^{-1}((0,1]), \sup \widehat{f}((0,1])) && \text{by De Morgan's law} \\ &= 1 - u(f) . && \text{by (1196)} \end{aligned}$$

In the remaining case that $f'(1) \neq r_+$, we again conclude from $f(0) = f'(1)$ that $f(0) \neq r_+$ as well. Therefore

$$\begin{aligned} \ell(f') &= \max(\inf f'^{-1}((0,1]), f'(1)) && \text{by (1195)} \\ &= \max(1 - \sup f^{-1}((0,1]), f(0)) && \text{by (1206) and (1209)} \\ &= \max(1 - \sup f^{-1}((0,1]), 1 - (1 - f(0))) && \text{because } 1 - x \text{ involution} \\ &= 1 - \min(\sup f^{-1}((0,1]), 1 - f(0)) && \text{by De Morgan's law} \\ &= 1 - u(f) . && \text{by (1196)} \end{aligned}$$

This completes the proof of part **a.** of the lemma. As concerns part **b.**, we proceed as follows.

$$\begin{aligned} u(f') &= 1 - (1 - u(f')) && \text{because } 1 - x \text{ involution} \\ &= 1 - \ell(f) \end{aligned}$$

by part **a.** of the lemma, utilizing that $f(z) = f(1-(1-z)) = f'(1-z)$ for all $z \in \mathbf{I}$, which is apparent from (1205).

Lemma 202 *For all $f \in \mathbb{X}$,*

a. $\ell(f^{\ddagger}) = \ell(f)$*;*

b. $u(f^{\ddagger}) = u(f)$*;*

c. $\alpha(f^{\ddagger}) = \alpha(f)$*; and in particular*

d. $\varphi_*(f^{\ddagger}) = \varphi_*(f)$.

Proof Consider some $f \in \mathbb{X}$.

a.: $\ell(f^\ddagger) = \ell(f)$**.**
We first notice from (1158) that $r_+(f^\ddagger) = r_+(f)$; in the following, I will hence abbreviate both coefficients as r_+. Recalling (1195), is then sufficient to show that $\inf {f^\ddagger}^{-1}((0,1]) = \inf f^{-1}((0,1])$, $\sup \widehat{f^\ddagger}([0,1)) = \sup \widehat{f}([0,1))$, and $f^\ddagger(1) = f(1)$.
As concerns $\inf {f^\ddagger}^{-1}((0,1])$, we first recall that $f^\ddagger \geq f$ and hence ${f^\ddagger}^{-1}((0,1]) \supseteq f^{-1}((0,1])$. Hence

$$\inf {f^\ddagger}^{-1}((0,1]) \leq \inf f^{-1}((0,1]) \,. \tag{1213}$$

Now let us consider the converse inequation. Let $\varepsilon > 0$. Because $f^\ddagger(z_+) = r_+ > 0$, we know that ${f^\ddagger}^{-1}((0,1]) \neq \varnothing$. Hence there exists $z' \in \mathbf{I}$ with

$$f^\ddagger(z') > 0 \tag{1214}$$

and

$$z' < \inf {f^\ddagger}^{-1}((0,1]) + \varepsilon \,. \tag{1215}$$

It is apparent from (1158) that

$$\sup\{f(z) : z \leq z'\} \geq f^\ddagger(z') \,. \tag{1216}$$

Recalling (1214), we can choose some $r' \in (0, f^\ddagger(z'))$. By (1216), then, there exists $z'' \in \mathbf{I}$ with $z'' \leq z'$, i.e.

$$z'' < \inf {f^\ddagger}^{-1}((0,1]) + \varepsilon \tag{1217}$$

by (1215); and with the additional property that $f(z'') > r'$, in particular

$$f(z'') > 0 \,.$$

We hence know that $z'' \in f^{-1}((0,1])$. In turn by (1217),

$$\inf f^{-1}((0,1]) \leq z'' < \inf {f^\ddagger}^{-1}((0,1]) + \varepsilon \,. \tag{1218}$$

$\varepsilon \to 0$ then proves that $\inf f^{-1}((0,1]) \leq z'' \leq \inf {f^\ddagger}^{-1}((0,1])$. Recalling (1213), we have hence shown that indeed

$$\inf f^{-1}((0,1]) = \inf {f^\ddagger}^{-1}((0,1]) \,.$$

Now let us turn attention to $\sup \widehat{f^\ddagger}([0,1))$ vs. $\sup \widehat{f}([0,1))$. We first observe that

$$\sup \widehat{f^\ddagger}([0,1)) \geq \sup \widehat{f}([0,1)) \tag{1219}$$

because $f^\ddagger \geq f$. Now let us show that the converse inequation is also valid. Hence let $\varepsilon > 0$. Then there exists $z_0 \in [0,1)$ with

$$f^\ddagger(z_0) > \sup \widehat{f^\ddagger}([0,1)) - \tfrac{\varepsilon}{2} \,. \tag{1220}$$

By (1158), then, we know that

$$\sup\{f(z) : z \leq z_0\} \geq f^{\ddagger}(z_0)\,.$$

In particular, there exists $z_1 \in [0, z_0]$ with

$$f(z_1) > \sup\{f(z) : z \leq z_0\} - \tfrac{\varepsilon}{2}\,,$$

i.e.

$$f(z_1) > \sup \widehat{f^{\ddagger}}([0,1)) - \varepsilon$$

by (1220). Because $z_1 \in [0, z_0] \subseteq [0,1)$, we conclude that

$$\sup \widehat{f}([0,1)) = \sup\{f(z) : z \in [0,1)\} \geq f(z_1) > \sup \widehat{f^{\ddagger}}([0,1)) - \varepsilon\,.$$

Noticing that $\varepsilon > 0$ was arbitrarily chosen, this proves that in fact

$$\sup \widehat{f}([0,1)) \geq \sup \widehat{f^{\ddagger}}([0,1))\,.$$

Combining this with (1219), we hence obtain that $\sup \widehat{f}([0,1)) = \sup \widehat{f^{\ddagger}}([0,1))$.
Now let us discuss $f^{\ddagger}(1)$ vs. $f(1)$. In this case, we simply observe that

$$\begin{aligned} f^{\ddagger}(1) &= \sup\{f(z) : z \geq 1\} && \text{by L-190} && (1221)\\ &= \sup\{f(1)\} && \text{because } z \leq 1 && (1222)\\ &= f(1)\,, && && (1223) \end{aligned}$$

as desired. This finishes the proof that $\ell(f^{\ddagger}) = \ell(f)$ because $\ell(\bullet)$ only depends on the coefficients discussed above, and these coefficients have been shown to be ‡-invariant.

b.: $u(f^{\ddagger}) = u(f)$**.**
To see this, suppose that $f', f'' \in \mathbb{X}$ are defined by

$$f'(z) = f(1-z) \qquad (1224)$$

$$f''(z) = f^{\ddagger}(1-z) \qquad (1225)$$

for all $z \in \mathbf{I}$. Then

$$\begin{aligned} f'^{\ddagger}(z) &= \min(\sup\{f'(z') : z' \leq z\}, \\ &\qquad \sup\{f'(z') : z' \geq z\}) && \text{by (1158)}\\ &= \min(\sup\{f(1-z') : z' \leq z\}, \\ &\qquad \sup\{f(1-z') : z' \geq z\}) && \text{by (1224)}\\ &= \min(\sup\{f(z'') : z'' \geq 1-z\}, \\ &\qquad \sup\{f(z'') : z'' \leq 1-z\}) && \text{substituting } z'' = 1-z'\\ &= f^{\ddagger}(1-z)\,, && \text{by (1158)} \end{aligned}$$

and hence

$$f'^{\ddagger}(z) = f''(z) \tag{1226}$$

by (1225). Therefore

$$\begin{aligned} u(f^{\ddagger}) &= 1 - \ell(f'') && \text{by L-201 and (1225)} \\ &= 1 - \ell(f'^{\ddagger}) && \text{by (1226)} \\ &= 1 - \ell(f') && \text{by part } \textbf{a.} \text{ of the lemma} \\ &= u(f)\,. && \text{by L-201 and (1224)} \end{aligned}$$

c.: $\alpha(f^{\ddagger}) = \alpha(f)$**.**
This is apparent from (1197) and $r_+(f^{\ddagger}) = r_+(f)$. Hence $\alpha(f^{\ddagger}) = 2r_+(f^{\ddagger}) - 1 = 2r_+(f) - 1 = \alpha(f)$, as desired.

d.: $\varphi_*(f^{\ddagger}) = \varphi_*(f)$**.**
We already know from part **a.**–**c.** of the lemma that $\ell(f)$, $u(f)$ and $\alpha(f)$ are ‡-invariant. Now φ_* is a function of these coefficients, which is apparent from (1198). Hence φ_* is ‡-invariant as well, and indeed $\varphi_*(f^{\ddagger}) = \varphi_*(f)$.

Lemma 203 *Let $f, f' \in \mathbb{X}$ be given and suppose that $f \trianglelefteq f'$. Then*

a. $\ell(f) \leq \ell(f')$*;*

b. $u(f) \leq u(f')$*;*

c. $\alpha(f) = \alpha(f')$*;*

d. $\varphi_*(f) \leq \varphi_*(f')$*.*

Proof

a.: $\ell(f) \leq \ell(f')$**.**
I first show that

$$\inf f^{-1}((0,1]) \leq \inf f'^{-1}((0,1])\,. \tag{1227}$$

Hence let $\varepsilon > 0$. Because $f'^{-1}((0,1]) \neq \varnothing$, there exists $z' \in \mathbf{I}$ with $z' \in f'^{-1}((0,1])$, i.e.

$$f'(z') > 0 \tag{1228}$$

and

$$z' < \inf f'^{-1}((0,1]) + \varepsilon\,. \tag{1229}$$

By (1172), there exists $z \leq z'$ with $f(z) \geq f'(z')$. We then conclude from (1228) that $f(z) > 0$ and hence $z \in f^{-1}((0,1])$. In addition, we conclude from (1229) that

$$z \leq z' < \inf f'^{-1}((0,1]) + \varepsilon$$

and hence

$$\inf f^{-1}((0,1]) \leq z < \inf f'^{-1}((0,1]) + \varepsilon\,,$$

because $z \in f^{-1}((0,1])$. $\varepsilon \to 0$ then proves the target inequation (1227).
I next prove that

$$\sup \widehat{f}([0,1)) \geq \sup \widehat{f'}([0,1))\,. \tag{1230}$$

Hence let $\varepsilon > 0$. Because $\widehat{f'}([0,1)) \neq \varnothing$, there exists $r \in \widehat{f'}([0,1))$ with $r > \sup \widehat{f'}([0,1)) - \varepsilon$. In turn, we obtain from Def. 15 that there exists $z' \in [0,1)$ with

$$f'(z') = r > \sup \widehat{f'}([0,1)) - \varepsilon\,. \tag{1231}$$

By (1172), then, there exists $z \leq z'$ with $f(z) \geq f'(z')$. In particular $z \in [0,1)$ and hence $f(z) \in \widehat{f}([0,1))$. This proves that

$$\sup \widehat{f}([0,1)) \geq f(z) \geq f'(z') > \sup \widehat{f'}([0,1)) - \varepsilon\,, \tag{1232}$$

see (1231). The desired inequation (1230) is then obtained for $\varepsilon \to 0$.
Let us further observe that

$$f(1) \leq f'(1) \tag{1233}$$

This is apparent from (1171), which states the existence of $z' \geq 1$ with $f(1) \leq f'(z')$. But z' is restricted to the unit range $z' \in [0,1]$. Hence in fact $z' = 1$ and consequently $f(1) \leq f'(1)$, i.e. (1233) is satisfied.
Finally we recall from L-195 that $f \sqsubseteq f'$. Hence L-188 is applicable, and

$$r_+ = r_+(f) = r_+(f')\,. \tag{1234}$$

Taking into account (1195), (1233) and (1234), it is now sufficient to consider the following cases.
Firstly if $f(1) < r_+$ and $f'(1) < r_+$,

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), f(1)) && \text{by (1195)}\\
&\leq \max(\inf f'^{-1}((0,1]), f'(1)) && \text{by (1227) and (1233)}\\
&= \ell(f')\,. && \text{by (1195)}
\end{aligned}$$

Secondly if $f(1) < r_+$ and $f'(1) = r_+$, then

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), f(1)) && \text{by (1195)}\\
&\leq \max(\inf f^{-1}((0,1]), \tfrac{1}{2}) && \text{by Def. 95 because } f(1) \neq r_+\\
&\leq \max(\inf f'^{-1}((0,1]), \tfrac{1}{2}) && \text{by (1227)}\\
&\leq \max(\inf f'^{-1}((0,1]), 1 - \sup \widehat{f'}([0,1)))\,,
\end{aligned}$$

where the last step is apparent from Def. 95, because $f'(1) = r_+$ and hence $f'(z) \leq 1 - r_+$ for all $z \neq 1$. From this, we obtain that $\ell(f) \leq \ell(f')$, which is immediate from $f'(1) = r_+$ and (1195).
Finally if $f(1) = r_+$ and $f'(1) = r_+$, then

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), 1 - \sup \widehat{f}([0,1))) && \text{by (1195)} \\
&\leq \max(\inf {f'}^{-1}((0,1]), 1 - \sup \widehat{f}'([0,1))) && \text{by (1227) and (1230)} \\
&= \ell(f')\,. && \text{by (1195)}
\end{aligned}$$

This completes the proof of part **a.**, i.e. $f \trianglelefteq f'$ entails that $\ell(f) \leq \ell(f')$, as desired.

b.: $u(f) \leq u(f')$**.**
To see this, define $f_1, f_1' \in \mathbb{X}$ by $f_1(z) = f(1-z)$ and $f_1'(z) = f'(1-z)$ for all $z \in \mathbf{I}$. It is then apparent from $f \trianglelefteq f'$ that

$$f_1' \trianglelefteq f_1\,. \tag{1235}$$

Therefore

$$\begin{aligned}
u(f) &= 1 - \ell(f_1) && \text{by L-201} \\
&\leq 1 - \ell(f_1') && \text{by part } \textbf{a.} \text{ of the lemma} \\
&= u(f')\,, && \text{by L-201}
\end{aligned}$$

as desired.

c.: $\alpha(f) = \alpha(f')$**.**
To this end, we simply notice that

$$\begin{aligned}
\alpha(f) &= 2r_+(f) - 1 && \text{by (1197)} \\
&= 2r_+(f') - 1 && \text{by (1234)} \\
&= \alpha(f')\,. && \text{by (1197)}
\end{aligned}$$

d.: $\varphi_*(f) \leq \varphi_*(f')$**.**
We know from the previous parts of the lemma and L-199 that $\ell(f) \leq \ell(f') \leq u(f')$ and $\ell(f) \leq u(f) \leq u(f')$. In addition, we know that $\alpha(f) = \alpha(f')$; this coefficient will hence be abbreviated as α. Due to the above inequations, it is sufficient to discern the following cases.
If $\ell(f) > \frac{1}{2}$, then $\ell(f') > \frac{1}{2}$ as well and hence

$$\begin{aligned}
\varphi_*(f) &= \alpha\,\ell(f) + (1-\alpha)u(f) && \text{by (1198)} \\
&\leq \alpha\,\ell(f') + (1-\alpha)u(f') && \text{by parts } \textbf{a.}, \textbf{b.} \text{ of the lemma} \\
&= \varphi_*(f')\,. && \text{by (1198)}
\end{aligned}$$

If $u(f') \geq \frac{1}{2}$ and $\ell(f) \leq \frac{1}{2}$, then $\varphi_*(f) \leq \frac{1}{2} \leq \varphi_*(f')$, which is clear from (1198). Finally if $u(f') < \frac{1}{2}$, then $u(f) < \frac{1}{2}$ as well. Hence

$$\begin{aligned} \varphi_*(f) &= \alpha\, u(f) + (1-\alpha)\ell(f) && \text{by (1198)} \\ &\leq \alpha\, u(f') + (1-\alpha)\ell(f') && \text{by parts } \textbf{a.}, \textbf{b.} \text{ of the lemma} \\ &= \varphi_*(f')\,. && \text{by (1198)} \end{aligned}$$

Lemma 204 *Suppose that $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ is ‡-invariant and monotonic with respect to $\trianglelefteq$, i.e.*

$$\varphi(f^{\ddagger}) = \varphi(f) \tag{1236}$$

for all $f \in \mathbb{X}$, and

$$\varphi(f) \leq \varphi(f') \tag{1237}$$

whenever $f \trianglelefteq f'$. Then φ satisfies (φ-4)*.*

Proof Hence let $f, f' \in \mathbb{X}$ with $f \sqsubseteq f'$ be given. We then know from L-194 that

$$f^{\ddagger} \trianglelefteq f'^{\ddagger}\,. \tag{1238}$$

This proves the desired

$$\begin{aligned} \varphi(f) &= \varphi(f^{\ddagger}) && \text{by (1236)} \\ &\leq \varphi(f'^{\ddagger}) && \text{by (1238) and (1237)} \\ &= \varphi(f')\,. && \text{by (1236)} \end{aligned}$$

Lemma 205 *The condition* (φ-5) *is independent of* (φ-1)–(φ-4)*.*

Proof In order to prove this, I show that $\varphi_* : \mathbb{X} \longrightarrow \mathbf{I}$ as defined by (1198) satisfies (φ-1)–(φ-4) and fails on (φ-5).

a.: φ_* satisfies (φ-1).
Hence consider $f \in \mathbb{X}$ with $f^{-1}((0,1]) = \{z_+\}$ and $f(z_+) = 1$. Then in particular

$$\inf f^{-1}((0,1]) = \sup f^{-1}((0,1]) = z_+\,. \tag{1239}$$

In the case that $z_+ = 0$, we know from L-197.**a** that

$$\ell(f) = f(1) = 0\,. \tag{1240}$$

In addition, $f(z) = 0$ for all $z > 0$ entails that $\sup f^{-1}((0,1]) = \sup \varnothing = 0$ and $\sup \widehat{f}((0,1]) = \sup\{0\} = 0$. Therefore

$$u(f) = \min(\sup f^{-1}((0,1]), \sup \widehat{f}((0,1])) = \min(0,0) = 0 \tag{1241}$$

by (1196). In turn

$$\begin{aligned} \varphi_*(f) &= \alpha\, u(f) + (1-\alpha)\ell(f) && \text{by (1198)} \\ &= \alpha \cdot 0 + (1-\alpha) \cdot 0 && \text{by (1240), (1241)} \qquad = 0 \\ &= z_+ \,. \end{aligned}$$

In the case that $z_+ = 1$, we know from Def. 95 and $r_+ = 1$ that $f(z) = 0$ for all $z \in [0,1)$. Therefore $f^{-1}((0,1]) = \{1\}$ and

$$\inf f^{-1}((0,1]) = 1\,.$$

In addition

$$\sup \widehat{f}[0,1) = \sup\{0\} = 0\,.$$

We conclude that

$$\ell(f) = \max(\inf f^{-1}((0,1]), 1 - \sup \widehat{f}[0,1)) = \max(1,1) = 1 \tag{1242}$$

by (1195). As concerns $u(f)$, we obtain from L-198.**a** that

$$u(f) = 1 - f(0) = 1 - 0 = 1\,. \tag{1243}$$

Hence in this case,

$$\begin{aligned} \varphi_*(f) &= \alpha\, \ell(f) + (1-\alpha)u(f) && \text{by (1198)} \\ &= \alpha \cdot 1 + (1-\alpha) \cdot 1 && \text{by (1242), (1243)} \\ &= 1 \\ &= z_+ \,. \end{aligned}$$

In the remaining case that $z_+ \in (0,1)$, we conclude from Def. 95 and $r_+ = 1$ that $f(0) = 0$ and $f(1) = 0$. Hence by L-197.**c** and (1239),

$$\ell(f) = \inf f^{-1}((0,1]) = z_+\,.$$

We further obtain from L-198.**c** and (1239) that

$$\ell(f) = \sup f^{-1}((0,1]) = z_+\,.$$

We can then apply lemma L-200 and conclude that $\varphi_*(f) = \ell(f) = z_+$.

b.: φ_* **satisfies** (φ-2)**.**
Hence let $f \in \mathbb{X}$ and suppose that $f' \in \mathbb{X}$ is defined by $f'(z) = f(1-z)$ for all $z \in \mathbf{I}$. Then apparently $r_+(f') = r_+(f)$. In particular, $\alpha(f') = \alpha(f)$, see (1197). In the following, I will hence abbreviate $\alpha = \alpha(f) = \alpha(f')$.
Now suppose that $\ell(f') > \frac{1}{2}$. We then know from L-201.**b** that $u(f) = 1 - \ell(f') < \frac{1}{2}$. Therefore

$$\begin{aligned}
\varphi_*(f') &= \alpha\,\ell(f') + (1-\alpha)u(f') && \text{by (1198)}\\
&= \alpha(1-u(f)) + (1-\alpha)(1-\ell(f)) && \text{by L-201}\\
&= \alpha - \alpha u(f) + (1-\alpha) - (1-\alpha)\ell(f)\\
&= 1 - (\alpha u(f) + (1-\alpha)\ell(f))\\
&= 1 - \varphi_*(f)\,. && \text{by (1198)}
\end{aligned}$$

Now suppose that $u(f') < \frac{1}{2}$. We then know from L-201.**a** that $\ell(f) = 1 - u(f') > \frac{1}{2}$. Therefore

$$\begin{aligned}
\varphi_*(f') &= \alpha\,u(f') + (1-\alpha)\ell(f') && \text{by (1198)}\\
&= \alpha(1-\ell(f)) + (1-\alpha)(1-u(f)) && \text{by L-201}\\
&= \alpha - \alpha\ell(f) + (1-\alpha) - (1-\alpha)u(f)\\
&= 1 - (\alpha\ell(f) + (1-\alpha)u(f))\\
&= 1 - \varphi_*(f)\,. && \text{by (1198)}
\end{aligned}$$

In the remaining case that $\ell(f') \le \frac{1}{2} \le u(f')$, we know from L-201 that $\ell(f) = 1 - u(f') \le \frac{1}{2} \le 1 - \ell(f') = u(f)$ as well. Hence $\varphi_*(f') = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \varphi_*(f)$ by (1198).

c.: φ_* **satisfies** (φ-3)**.**
To see this, consider $f \in \mathbb{X}$ with $f^{-1}((0,1]) \subseteq \{0,1\}$ and $f(1) \ge \frac{1}{2}$. Hence $f(1) = r_+$ and $1 \in f^{-1}((0,1])$. In the following I discern two main cases.
If $f(0) = 0$, then $f^{-1}((0,1]) = \{1\}$. In particular

$$\inf f^{-1}((0,1]) = \sup f^{-1}((0,1]) = 1\,. \tag{1244}$$

In addition, $f^{-1}((0,1]) = \{1\}$ means that $f(z) = 0$ for all $z < 1$. Hence

$$\sup \widehat{f}([0,1)) = \sup\{0\} = 0\,. \tag{1245}$$

It therefore holds that

$$\begin{aligned}
\ell(f) &= \max(\inf f^{-1}((0,1]), 1 - \sup \widehat{f}([0,1))) && \text{by (1195)}\\
&= \max(1, 1-0) && \text{by (1244), (1245)}\\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
u(f) &= \sup f^{-1}((0,1]) && \text{by L-198.}\mathbf{c}\\
&= 1\,. && \text{by (1244)}
\end{aligned}$$

By applying L-200, we obtain the desired $\varphi_*(f) = 1 = 1 - 0 = 1 - f(0)$.
Next we consider the case that $f(0) > 0$. Then $f^{-1}((0,1]) = \{0,1\}$, i.e.

$$\inf f^{-1}((0,1]) = 0 \tag{1246}$$

$$\sup f^{-1}((0,1]) = 1\,. \tag{1247}$$

Recalling that $f(z) = 0$ for all $z \in (0,1)$ by assumption, it is further apparent that

$$\sup \widehat{f}([0,1)) = \sup\{f(0), 0\} = f(0)\,. \tag{1248}$$

Therefore

$$\begin{aligned} \ell(f) &= 1 - \sup \widehat{f}([0,1)) && \text{by L-197.}\mathbf{b} \\ &= 1 - f(0) && \text{by (1248)} \end{aligned}$$

As concerns $u(f)$, we observe that $u(f) = 1 - f(0)$ by L-198.**a**, provided that $f(0) \neq r_+$. If on the other and $f(0) = r_+$, then we know from Def. 95 and the assumption that $f(1) = r_+$ that indeed $r_+ = \frac{1}{2}$. Recalling that $f(z) = 0$ for all $z \in (0,1)$, we then obtain that

$$\sup \widehat{f}((0,1]) = \sup\{f(1), 0\} = f(1) = r_+\,. \tag{1249}$$

Therefore

$$\begin{aligned} u(f) &= \sup \widehat{f}((0,1]) && \text{by L-198.}\mathbf{b} \\ &= r_+ && \text{by (1249)} \\ &= 1 - r_+ && \text{because } r_+ = \tfrac{1}{2} \\ &= 1 - f(0)\,. \end{aligned}$$

This completes the proof that $u(f) = 1 - f(0)$ regardless of $f(0)$. Recalling the above result that $\ell(f) = 1 - f(0)$ as well, we can hence apply L-200 and conclude that $\varphi_*(f) = 1 - f(0)$. This proves that φ_* indeed satisfies (φ-3).

d.: φ_* satisfies (φ-4).
This is apparent from the above lemmata: L-202 states that φ_* is $\bullet^{\ddagger}$-invariant, and L-203 states that φ_* is monotonic with respect to $\trianglelefteq$. Hence lemma L-204 is applicable, and we conclude that φ_* satisfies (φ-4).

e.: φ_* violates (φ-5).
To see this, consider $f \in \mathbb{X}$ defined by

$$f(z) = \begin{cases} \frac{3}{4} & : \quad z = \frac{4}{5} \\ \frac{1}{4} & : \quad z \in \{\frac{3}{5}, 1\} \\ 0 & : \quad \text{else} \end{cases} \tag{1250}$$

for all $z \in \mathbf{I}$. Then clearly

$$f^{-1}((0,1]) = \{\tfrac{3}{5}, \tfrac{4}{5}, 1\}\,. \tag{1251}$$

Therefore

$$\ell(f) = \inf f^{-1}((0,1]) = \tfrac{3}{5} \tag{1252}$$

by L-197.**c** and (1251). In addition

$$u(f) = \sup f^{-1}((0,1]) = 1 \tag{1253}$$

by L-198.**c** and (1251). Let us further notice from (1250) that $r_+ = r_+(f) = \frac{3}{4}$ and hence $\alpha = \alpha(f) = \frac{1}{2}$ by (1197). Because $\ell(f) = \frac{3}{5} > \frac{1}{2}$, we then obtain from (1198) that

$$\varphi_*(f) = \alpha\,\ell(f) + (1-\alpha)u(f) = \tfrac{1}{2}\cdot\tfrac{3}{5} + \tfrac{1}{2}\cdot 1 = \tfrac{3}{1}0 + \tfrac{1}{2} = \tfrac{4}{5}\,. \tag{1254}$$

Now consider $f' \in \mathbb{X}$ defined by

$$f'(z) = \min(f(z), \tfrac{1}{2})$$

for all $z \in \mathbf{I}$, i.e.

$$f'(z) = \begin{cases} \frac{1}{2} & : \quad z = \frac{4}{5} \\ \frac{1}{4} & : \quad z \in \{\frac{3}{5}, 1\} \\ 0 & : \quad \text{else} \end{cases} \tag{1255}$$

by (1250). We observe that again

$$f'^{-1}((0,1]) = \{\tfrac{3}{5}, \tfrac{4}{5}, 1\}\,.$$

Because $f'(1) = 0$, we hence obtain from L-197.**c** that

$$\ell(f') = \inf f'^{-1}((0,1]) = \tfrac{3}{5}\,. \tag{1256}$$

We further obtain from L-198 tha

$$u(f') = \sup f'^{-1}((0,1]) = 1\,. \tag{1257}$$

Let us now notice from (1255) that $r_+(f') = \frac{1}{2}$. Hence $\alpha' = \alpha(f') = 0$ by (1197), and consequently

$$\varphi_*(f') = \alpha'\,\ell(f') + (1-\alpha')u(f') = u(f') = 1 \tag{1258}$$

by (1198), (1256), (1257) and recalling that $\alpha' = 0$. To sum up, equations (1254) and (1258) prove that $\varphi_*(f') = 1 \neq \frac{4}{5} = \varphi_*(f)$ although $f'(z) = \min(f(z), \frac{1}{2})$ for all $z \in \mathbf{I}$. Hence condition (φ-5) is not valid in the case of φ_*. Because all other 'φ-conditions' are satisfied by φ_*, this completes the proof that condition (φ-5) is indeed independent of the remaining conditions.

Proof of Theorem 129

We know from Th-121 that (ψ-1), (ψ-2), (ψ-3) and (ψ-4) are independent of the remaining 'ψ-conditions' of (ψ-1)–(ψ-5). Hence there exist $\psi_j : \mathbb{A} \longrightarrow \mathbf{I}$, $j \in \{1, 2, 3, 4\}$, which violate condition (ψ-j) and satisfy all conditions (ψ-k), where $k \in \{1, 2, 3, 4, 5\} \setminus \{j\}$. In particular, every ψ_j satisfies (ψ-5). Let us now define $\varphi_j : \mathbb{X} \longrightarrow \mathbf{I}$ relative to ψ_j according to (88). We then know from L-185 and Th-126 that every φ_j, $j \in \{1, 2, 3, 4\}$, violates condition (φ-j) and satisfies all other conditions (φ-k), where $k \in \{1, 2, 3, 4, 5\} \setminus \{j\}$. This proves that ($\varphi$-1), ($\varphi$-2), ($\varphi$-3), and ($\varphi$-4) are each independent of the other conditions in (φ-1)–(φ-5).
It remains to be shown that (φ-5) is independent of (φ-1)–(φ-4). This condition has been treated separately, and I simply refer to L-205.

B.39 Proof of Theorem 130

Lemma 206 *Let $E \neq \varnothing$ be given, $X \in \widetilde{\mathcal{P}}(E)$ and $Y \in \mathcal{P}(E)$. Then for all $e \in E$,*

$$\theta(\mu_X(e),\, \chi_Y(e)) = \min(2\,\delta_{X,Y}(e), 1)\,.$$

Proof Consider $x \in \mathbf{I}$ and $y \in \{0, 1\}$.

- If $y = 1$ and $x \leq \frac{1}{2}$, then $2x \leq 1$ and hence $\min(2x, 1) = 2x = \theta(x,\, y)$;
- If $y = 1$ and $x > \frac{1}{2}$, then $2x > 1$ and hence $\min(2x, 1) = 1 = \theta(x,\, y)$;
- If $y = 0$ and $x < \frac{1}{2}$, then $2 - 2x > 1$ and hence $\min(2 - 2x, 1) = 1 = \theta(x,\, y)$;
- If $y = 0$ and $x \geq \frac{1}{2}$, then $2 - 2x \leq 1$ and hence $\min(2 - 2x, 1) = 2 - 2x = \theta(x,\, y)$.

Combining these cases, this proves that

$$\theta(x,\, y) = \begin{cases} \min(2x, 1) & : \quad y = 1 \\ \min(2 - 2x, 1) & : \quad y = 0 \end{cases} \tag{1259}$$

for all $x \in \mathbf{I}$ and $y \in \{0, 1\}$. Now consider a choice of $E \neq \varnothing$, $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$ and $e \in E$. Then

$$\begin{aligned} &\theta(\mu_X(e),\, \chi_Y(e)) \\ &= \begin{cases} \min(2\,\mu_X(e), 1) & : \quad \chi_Y(e) = 1 \\ \min(2 - 2\,\mu_X(e), 1) & : \quad \chi_Y(e) = 0 \end{cases} && \text{by (1259)} \\ &= \begin{cases} \min(2\,\mu_X(e), 1) & : \quad e \in Y \\ \min(2(1 - \mu_X(e)), 1) & : \quad e \notin Y \end{cases} && \text{by (1)} \\ &= \min(2\,\delta_{X,Y}(e), 1)\,, && \text{by (60)} \end{aligned}$$

as desired.

Lemma 207 *Let $E \neq \varnothing$, $X \in \widetilde{\mathcal{P}}(E)$ and $Y \in \mathcal{P}(E)$ be given. Then $\Theta(X,\, Y) = \min(2\,\Xi_X(Y), 1)$.*

Proof Straightforward:

$$\begin{aligned}\Theta(X, Y) &= \inf\{\theta(\mu_X(e), \chi_Y(e)) : e \in E\} && \text{by Def. 100}\\ &= \inf\{\min(2\,\delta_{X,Y}(e), 1) : e \in E\} && \text{by L-206}\\ &= \min(2\inf\{\delta_{X,Y}(e) : e \in E\}, 1) && \text{(apparent)}\\ &= \min(2\Xi_X(Y), 1)\,. && \text{by Def. 83}\end{aligned}$$

Lemma 208 *Let $E \neq \varnothing$ be given, $n \in \mathbb{N}$ and consider a choice of $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Then*

$$\min_{i=1}^{n} \Theta(X_i, Y_i) = \min(2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n), 1)\,.$$

Proof Trivial consequence of the previous lemma.

$$\begin{aligned}&\min_{i=1}^{n} \Theta(X_i, Y_i)\\ &= \min_{i=1}^{n} \min(2\,\Xi_{Y_i}(X_i), 1) && \text{by L-207}\\ &= \min(2 \cdot \min_{i=1}^{n} \Xi_{Y_i}(X_i), 1) && \text{(apparent)}\\ &= \min(2\,\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n), 1)\,. && \text{by Def. 83}\end{aligned}$$

Lemma 209 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments. Then for all $z \in \mathbf{I}$,*

$$\widetilde{Q}_z(X_1, \dots, X_n) = \min(2 f_{Q,X_1,\dots,X_n}(z), 1)\,.$$

Proof Apparent from a simple computation.

$$\begin{aligned}&\widetilde{Q}_z(X_1, \dots, X_n)\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \dots, Y_n) \in Q^{-1}(z)\} && \text{by Def. 101}\\ &= \sup\{\min(2\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n), 1) :\\ &\qquad (Y_1, \dots, Y_n) \in Q^{-1}(z)\} && \text{by L-208}\\ &= \min(2\sup\{\Xi_{Y_1,\dots,Y_n}(X_1, \dots, X_n) :\\ &\qquad (Y_1, \dots, Y_n) \in Q^{-1}(z)\}, 1) && \text{(obvious)}\\ &= \min(2\, f_{Q,X_1,\dots,X_n}(z), 1)\,. && \text{by Def. 94}\end{aligned}$$

Proof of Theorem 130

Let a QFM $\mathcal{F}$ be given and suppose that condition **a.** is satisfied. Hence there exists $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ with $\mathcal{F} = \mathcal{F}_\varphi$, which satisfies ($\varphi$-5). For a given $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, let us abbreviate

$$f'(z) = \min(f_{Q,X_1,\ldots,X_n}(z), \tfrac{1}{2}) \tag{1260}$$

for all $z \in \mathbf{I}$. We then know from (φ-5) and Def. 96 that $\mathcal{F}_\varphi(Q)(X_1, \ldots, X_n) = \varphi(f_{Q,X_1,\ldots,X_n}) = \varphi(f')$. Noticing that

$$f'(z) = \tfrac{1}{2}\min(2\, f_{Q,X_1,\ldots,X_n}(z), 1) = \tfrac{1}{2}\widetilde{Q}_z(X_1, \ldots, X_n)\,,$$

this completes the proof that $\mathcal{F}_\psi$ can be defined in terms of $\widetilde{Q}_z$.
Now consider the converse situation that $\mathcal{F}$ can be defined in terms of $\widetilde{Q}_z$. Hence there exists a mapping $G : \mathbf{I}^\mathbf{I} \longrightarrow \mathbf{I}$ such that

$$\mathcal{F}(Q)(X_1, \ldots, X_n) = G((\widetilde{Q}_z(X_1, \ldots, X_n))_{z\in\mathbf{I}}) \tag{1261}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and choices of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. In order to see that $\mathcal{F}$ is an $\mathcal{F}_\varphi$-QFM based on a choice of φ which satisfies (φ-5), we now define $\varphi : \mathbb{X} \longrightarrow \mathbf{I}$ by

$$\varphi(f) = G(\min(2f, 1))\,, \tag{1262}$$

for all $f \in \mathbb{X}$. Now consider a choice of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{F}_\varphi(Q)(X_1, \ldots, X_n) &= \varphi(f_{Q,X_1,\ldots,X_n}) && \text{by Def. 96}\\
&= G(\min(2f_{Q,X_1,\ldots,X_n}, 1)) && \text{by (1262)}\\
&= G((\widetilde{Q}_z(X_1, \ldots, X_n))_{z\in\mathbf{I}}) && \text{by L-209}\\
&= \mathcal{F}(Q)(X_1, \ldots, X_n)\,. && \text{by (1261)}
\end{aligned}$$

Hence $\mathcal{F}$ is indeed an $\mathcal{F}_\varphi$-QFM. It remains to be shown that (φ-5) is satisfied. Hence let $f \in \mathbb{X}$ be given. Then

$$\begin{aligned}
\varphi(\min(f, \tfrac{1}{2})) &= G(\min(2(\min(f, \tfrac{1}{2}), 1))) && \text{by (1262)}\\
&= G(2\min(f, \tfrac{1}{2})) && \text{because } \min(f, \tfrac{1}{2}) \le \tfrac{1}{2}\\
&= G(\min(2f, 1)) && \text{(apparent)}\\
&= \varphi(f)\,. && \text{by (1262)}
\end{aligned}$$

Hence φ indeed satisfies (φ-5), as desired.

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