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Ingo Glöckner

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Abteilung Informationstechnik, Postfach 10 01 31, 33501 Bielefeld

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Ingo Glöckner

Universität Bielefeld, Technische Fakultät, AG Technische Informatik,
Postfach 10 01 31, 33501 Bielefeld,
mailto:ingo@techfak.uni-bielefeld.de

Contents

Abstract

DFS theory offers an axiomatic approach to fuzzy quantification. It attempts to characterise 'reasonable' models of fuzzy quantifiers in terms of formal conditions imposed on a fuzzification mapping. In this way, a large number of linguistically motivated adequacy criteria can be guaranteed to hold for all models of DFS theory, the so-called called determiner fuzzification schemes or DFSes for short. The report improves upon existing work on DFS theory by developing an alternative axiom system of only six basic axioms. The new axiom set is shown to be equivalent to the original one. A subclass of DFSes called $\mathcal{M}_{\mathcal{B}}$-DFSes is then introduced which can be defined in terms of a three-valued cutting mechanism. Based on an investigation of this mechanism, it becomes possible to prove that the new DFS axioms form an independent axiom system.
In addition, a number of novel properties of fuzzification mechanisms are discussed and examples of $\mathcal{M}_{\mathcal{B}}$-DFSes are investigated from this perspective. A particularly well-behaved DFS is being presented and its special properties as well as their uniqueness are established. This model can be shown to generalize the Sugeno integral (and hence the FG-count approach to fuzzy quantification) to the case of nonmonotonic and arbitrary multiplace quantifiers.
The report is intended as a technical reference and its purpose is to provide the proofs for a number of results related to DFS theory. The style of presentation is therefore rather technical and presentation order is guided by the natural order of the proofs.

1 DFS Theory: A Review of Concepts

DFS theory [9] is an axiomatic theory of fuzzy natural language (NL) quantification. It aims at providing linguistically adequate models for approximate quantifiers like *almost all*, as well as for quantification involving fuzzy arguments as in *all tall swedes are rich*. Unlike existing approaches to fuzzy quantification [27, 14, 24, 25], DFS theory does not rest on Zadeh's proposal of representing fuzzy quantifiers as fuzzy subsets of the non-negative reals (absolute kind like "about 10") or as fuzzy subsets of the unit interval (proportional kind like "more than 30 percent"). Consequently, DFS theory cannot rely on Zadeh's proposal of evaluating quantifying statements "Q X's are A's" by a fuzzy comparison of cardinalities "card A is Q". This departure from other work on fuzzy quantification is a consequence of negative results concerning the linguistic adequacy of existing approaches, which became apparent when an evaluation of these approaches based on criteria from linguistics was carried out [10]. In particular, none of these approaches provides a satisfying model of *two-place quantification*, as in *many tall people are lucky*, where both the restriction *tall people* and the scope *lucky* can be fuzzy. In order to overcome this problem of existing approaches and to guarantee a proper treatment of fuzzy multiplace quantification, DFS theory does not share their representation of fuzzy quantifiers by fuzzy numbers and abandons the search for an appropriate measure for the cardinality of fuzzy sets, because cardinality information is not sufficient to evaluate all quantifying expressions of interest.[1] Instead, we will assume the framework provided by the current *linguistic* theory of NL quantification, the theory of generalized quantifiers (TGQ [1, 2]), which has been developed independently of fuzzy set theory, and hence provides a conceptually rather different view of natural language quantification.

We shall introduce two-valued quantifiers in concordance with TGQ:

Definition 1 (Two-valued generalized quantifiers)
An n-ary generalized quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2} = \{0, 1\}$, *where* $\mathcal{P}(E)$ *is the powerset (set of subsets) of* E.

A two-valued quantifier hence assigns to each n-tuple of crisp subsets $X_1, \dots, X_n \in \mathcal{P}(E)$ a two-valued quantification result $Q(X_1, \dots, X_n) \in \mathbf{2}$. Well-known examples are

$$\begin{aligned}
&\forall_E(X) = 1 \Leftrightarrow X = E\\
&\exists_E(X) = 1 \Leftrightarrow X \neq \varnothing\\
&\textbf{all}_E(X_1,\, X_2) = 1 \Leftrightarrow X_1 \subseteq X_2\\
&\textbf{some}_E(X_1,\, X_2) = 1 \Leftrightarrow X_1 \cap X_2 \neq \varnothing\\
&\textbf{at least k}_E(X_1,\, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq k\,.
\end{aligned}$$

Whenever the base set is clear from the context, we drop the subscript E; $|\bullet|$ denotes cardinalityand $\cap$ denotes intersection $X \cap Y = \{e \in E : e \in X \text{ and } e \in Y\}$. For finite E, we can define proportional quantifiers like

$$\begin{aligned}
&[\textbf{rate} \geq r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq r\,|X_1|\\
&[\textbf{rate} > r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| > r\,|X_1|
\end{aligned}$$

for $r \in \mathbf{I}$, $X_1, X_2 \in \mathcal{P}(E)$. For example, "at least 30 percent of the X's are Y's" can be expressed as $[\textbf{rate} \geq 0.3](X, Y)$, while $[\textbf{rate} > 0.4]$ is suited to model "more than 40 percent". By the *scope*

[1]As we shall see later, it is possible to recover the cardinality-based approach to fuzzy quantification in the case of quantitative one-place quantifiers, see Th-129.

of an NL quantifier we denote the argument occupied by the verbal phrase (e.g. "sleep" in "all men sleep"); by convention, the scope is the last argument of a quantifier. The first argument of a two-place quantifier is its *restriction*. The two-place use of a two-place quantifier, like in "most X's are Y's" is called its *restricted use,* while its one-place use (relative to the whole domain E, like in "most elements of the domain are Y") is its *unrestricted use*. For example, the unrestricted use of **all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is modelled by $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$, which has $\forall(X) = \textbf{all}(E, X)$.
TGQ has classified the wealth of quantificational phenomena in natural languages in order to unveil universal properties shared by quantifiers in all natural languages, or to single out classes of quantifiers with specific properties (we shall describe some of these properties below). However, an extension to the continuous-valued case, in order to better capture the meaning of approximate quantifiers like "many" or "about ten", has not been an issue for TGQ. In addition, TGQ has ignored the problem of providing a convincing interpretation for quantifying statements in the presence of fuzziness, i.e. in the frequent case that the arguments of the quantifier are occupied by concepts like "tall" or "cloudy" which do not possess sharply defined boundaries.

Hence let us introduce the fuzzy framework. Suppose E is a given set. A fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ of E assigns to each $e \in E$ a membership grade $\mu_X(e) \in \mathbf{I} = [0, 1]$; we denote by $\widetilde{\mathcal{P}}(E)$ the set of all fuzzy subsets (fuzzy powerset) of E. Apparently $\widetilde{\mathcal{P}}(E) \cong \mathbf{I}^E$, where $\mathbf{I}^E$ denotes the set of membership functions $\mu_X : E \longrightarrow \mathbf{I}$. Some authors identify fuzzy subsets and membership functions, i.e. stipulate $\widetilde{\mathcal{P}}(E) = \mathbf{I}^E$. In the present sequel, I will not enforce this identification. It will be convenient to assume that every crisp subset $X \in \mathcal{P}(E)$ can be considered a special case of fuzzy subset of E (when identifying fuzzy subsets and membership functions, we would have to view the characteristic function of X as a special case of fuzzy subset).

Definition 2 (Fuzzy generalized quantifiers)
An n-ary fuzzy quantifier $\widetilde{Q}$ *on a base set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of fuzzy subsets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ assigns a gradual result $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$.*[2]

An example is $\widetilde{\textbf{some}}(X_1, X_2) = \sup\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}$, for all $X \in \widetilde{\mathcal{P}}(E)$. How can we justify that this operator is a good model of the NL quantifier "some"? How can we describe characteristics of fuzzy quantifiers and how can we locate a fuzzy quantifier based on a description of desired properties? Fuzzy quantifiers are possibly too rich a set of operators to investigate this question directly. Few intuitions apply to the behaviour of quantifiers in the case that the arguments are fuzzy, and the familiar concept of *cardinality* of crisp sets, which makes it easy to define quantifiers on crisp arguments, is no longer available.

We therefore have to introduce some kind of *simplified description* of the essential aspects of a fuzzy quantifier. In order to comply with linguistic theory, this representation should be rich enough to embed all two-valued quantifiers of TGQ.

Definition 3 (Semi-fuzzy quantifiers)
An n-ary semi-fuzzy quantifier *on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of* crisp *subsets $X_1, \ldots, X_n \in \mathcal{P}(E)$ assigns a gradual result $Q(X_1, \ldots, X_n) \in \mathbf{I}$.*[3]

[2] This definition closely resembles Zadeh's [28, pp.756] alternative view of fuzzy quantifiers as fuzzy second-order predicates, but models these as mappings in order to simplify notation. In addition, we permit for arbitrary $n \in \mathbb{N}$. The above definition of n-ary fuzzy quantifiers, originally dubbed *fuzzy determiners*, has first been used in in [9, p. 6].

[3] The concept of semi-fuzzy quantifiers (originally dubbed "fuzzy pre-determiners") has been introduced in [9, p. 7].

Semi-fuzzy quantifiers are half-way between two-valued quantifiers and fuzzy quantifiers because they have crisp input and fuzzy (gradual) output. In particular, every two-valued quantifier of TGQ is a semi-fuzzy quantifier by definition. To provide an example, a possible definition of the semi-fuzzy quantifier **almost all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is

$$\textbf{almost all}(X_1, X_2) = \begin{cases} f_{\text{almost all}}(\frac{|X_1 \cap X_2|}{|X_1|}) & : \quad X_1 \neq \varnothing \\ 1 & : \quad \text{else} \end{cases} \tag{1}$$

where $f_{\text{almost all}}(z) = S(z, 0.7, 0.9)$, using Zadeh's S-function (see Fig. 1). Unlike the represen-

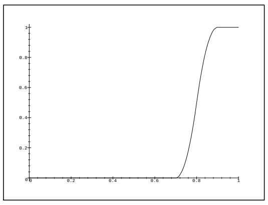

Figure 1: A possible definition of $f_{\text{almost all}}$

tations chosen by existing approaches to fuzzy quantification, semi-fuzzy quantifiers can express genuine multiplace quantification (arbitrary n); they are not restricted to the absolute and proportional types; they are not necessarily quantitative (in the sense of automorphism-invariance); and there is no a priori restriction to finite domains. Compared to fuzzy quantifiers, the main benefit of introducing semi-fuzzy quantifiers is conceptual simplicity due to the restriction to crisp argument sets, which usually makes it easy to understand the input-output behavior of a semi-fuzzy quantifier. Most importantly, we have the familiar concept of crisp cardinality available, which is of invaluable help in defining the quantifiers of interest. However, being half-way between two-valued generalized quantifiers and fuzzy quantifiers, semi-fuzzy quantifiers do not accept fuzzy input, and we have to make use of a fuzzification mechanism which transports these to fuzzy quantifiers.

Definition 4 (Quantifier Fuzzification Mechanism)
A quantifier fuzzification mechanism[4](QFM) $\mathcal{F}$ *assigns to every semi-fuzzy quantifier*

$$Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$$

a corresponding fuzzy quantifier

$$\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$$

of the same arity n and on the same base set E.

By modelling approaches to fuzzy quantification as instances of quantifier fuzzification mechanisms, we can express linguistic adequacy conditions on "intended" approaches in terms of preservation and homomorphism properties of the corresponding fuzzification mappings [9, 10].

To this end, we first need to introduce several concepts related to (semi-) fuzzy quantifiers.

[4]originally called "determiner fuzzification mechanism" in [9, p. 9].

Definition 5 (Underlying semi-fuzzy quantifier)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is a fuzzy quantifier. By* $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the* underlying semi-fuzzy quantifier, *viz.*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n) \tag{2}$$

for all crisp *subsets* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Every reasonable QFM $\mathcal{F}$ should correctly generalise the semi-fuzzy quantifiers to which it is applied, i.e. for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, we should have

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n) \tag{3}$$

for all *crisp* arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, or equivalently: $\mathcal{U}(\mathcal{F}(Q)) = Q$.
Let us now consider a special case of quantifiers, called projection quantifiers. Suppose E is a set of persons and John $\in E$. We can then express the membership assessment "Is John contained in Y?", where Y is a crisp subset $Y \in \mathcal{P}(E)$, by computing χ_Y(John), where $\chi_Y : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is the *characteristic function*

$$\chi_Y(e) = \begin{cases} 1 & : \quad e \in Y \\ 0 & : \quad e \notin Y\,. \end{cases} \tag{4}$$

for all $Y \in \mathcal{P}(E)$, $e \in E$. Similarly, we can evaluate the fuzzy membership assessment "To which grade is John contained in X?", where $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E, by computing μ_X(John). Abstracting from argument sets, we obtain the following definitions of projection quantifiers:

Definition 6 (Projection quantifiers)
Suppose $E \neq \varnothing$ *is given and* $e \in E$*. The* projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ *is defined by* $\pi_e(Y) = \chi_Y(e)$*, for all* $Y \in \mathcal{P}(E)$.

In the case of fuzzy projection quantifiers, we replace the characteristic function with the membership function of the fuzzy argument set:

Definition 7 (Fuzzy projection quantifiers)
For all base sets $E \neq \varnothing$ *and all* $e \in E$*, the* fuzzy projection quantifier $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is defined by* $\widetilde{\pi}_e(X) = \mu_X(e)$*, for all* $X \in \widetilde{\mathcal{P}}(E)$.

It is apparent from the relationship of these quantifiers with crisp / fuzzy membership assessments that $\widetilde{\pi}_e$ is the proper fuzzy counterpart of π_e, and we should have $\mathcal{F}(\pi_e) = \widetilde{\pi}_e$ in every reasonable QFM. Hence for the crisp subset **married** $\in \mathcal{P}(E)$,

$$\pi_{\text{John}}(\textbf{married}) = \begin{cases} 1 & : \quad \text{John} \in \textbf{married} \\ 0 & : \quad \text{else} \end{cases}$$

and we should also have that $\mathcal{F}(\pi_{\text{John}})(\textbf{lucky}) = \widetilde{\pi}_{\text{John}}(\textbf{lucky}) = \mu_{\textbf{lucky}}(\text{John})$, where **lucky** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of lucky people.
We expect that our framework not only provides an interpretation for quantifiers, but also for the propositional part of the logic. We therefore need to associate a suitable choice of fuzzy conjunction, fuzzy disjunction etc. with a given QFM $\mathcal{F}$. By a canonical construction, which we describe now, $\mathcal{F}$ induces a unique fuzzy operator for each of the propositional connectives (see [9]

for more details). Let $\{*\}$ be an arbitrary singleton set. We denote by π_* the projection quantifier $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{2}$ defined by Def. 6, i.e. $\pi_*(Y) = \chi_Y(*)$, where χ_Y is the characteristic function of $Y \in \mathcal{P}(\{*\})$ (see above). Similarly, we denote by $\widetilde{\pi}_*$ the fuzzy projection quantifier $\widetilde{\pi}_* : \widetilde{\mathcal{P}}(\{*\}) \longrightarrow \mathbf{I}$ defined by Def. 7, i.e. the bijection $\widetilde{\pi}_*(X) = \mu_X(*)$, where μ_X is the membership function of $X \in \widetilde{\mathcal{P}}(\{*\})$.

Definition 8 (Induced truth functions)
Suppose f is a semi-fuzzy truth function, i.e. a mapping $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ (e.g., two-valued conjunction $f = \wedge$ or two-valued disjunction $f = \vee$). We can view f as a semi-fuzzy quantifier $f^* : \mathcal{P}(\{*\})^n \longrightarrow \mathbf{I}$ by defining

$$f^*(X_1, \ldots, X_n) = f(\pi_*(X_1),\ \ldots,\ \pi_*(X_n))\,.$$

By applying $\mathcal{F}$, f^* is generalized to a fuzzy quantifier $\mathcal{F}(f^*) : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \mathbf{I}$, from which we obtain a fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$,

$$\widetilde{\mathcal{F}}(f)(x_1,\ \ldots x_n) = \mathcal{F}(f^*)(\widetilde{\pi}_*^{-1}(x_1),\ \ldots,\ \widetilde{\pi}_*^{-1}(x_n))$$

for all $x_1,\ \ldots,\ x_n \in \mathbf{I}$.

Whenever $\mathcal{F}$ is understood from context, we shall abbreviate $\widetilde{\mathcal{F}}(f)$ as $\widetilde{f}$. By pointwise application of the induced negation $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$, conjunction $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$, and disjunction $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$, $\mathcal{F}$ also induces a unique choice of fuzzy complement $\widetilde{\neg}$, fuzzy intersection $\widetilde{\cap}$, and fuzzy union $\widetilde{\cup}$:

Definition 9 (Induced operations on fuzzy sets)
Suppose $\mathcal{F}$ is a QFM, and E is some set. $\mathcal{F}$ induces fuzzy set operators

$$\begin{aligned}
\widetilde{\neg} &= \widetilde{\mathcal{F}}(\neg) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)\\
\widetilde{\cap} &= \widetilde{\mathcal{F}}(\cap) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)\\
\widetilde{\cup} &= \widetilde{\mathcal{F}}(\cup) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)
\end{aligned}$$

which are pointwise defined by

$$\begin{aligned}
\mu_{\widetilde{\neg} X}(e) &= \widetilde{\neg}\, \mu_X(e)\\
\mu_{X_1 \widetilde{\cap} X_2}(e) &= \mu_{X_1}(v) \mathbin{\widetilde{\wedge}} \mu_{X_2}(e)\\
\mu_{X_1 \widetilde{\cup} X_2}(e) &= \mu_{X_1}(v) \mathbin{\widetilde{\vee}} \mu_{X_2}(e)
\end{aligned}$$

for all $X,\ X_1,\ X_2 \in \widetilde{\mathcal{P}}(E)$ and $e \in E$.[5]

In the following, we will assume that an arbitrary but fixed choice of these connectives and fuzzy set operations is given.

In analogy to the external negation $\neg Q$ of a two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ in TGQ [8, p.236], we shall now introduce the external negation of (semi-)fuzzy quantifiers.

[5] The ambiguous use of $\widetilde{\neg}$ both as designating fuzzy negation and fuzzy complementation should not create confusion because the fuzzy negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ applies to gradual truth values $x \in \mathbf{I}$, while the fuzzy complement $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ applies to fuzzy subsets $X \in \widetilde{\mathcal{P}}(E)$ of a given set E. (Likewise for our notation of two-valued negation $\neg : \mathbf{2} \longrightarrow \mathbf{2}$ and crisp complementation $\neg : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$).

Definition 10 (External negation)
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, *the* external negation $\widetilde{\neg}\, Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$(\widetilde{\neg}\, Q)(X_1, \ldots, X_n) = \widetilde{\neg}(Q(X_1, \ldots, X_n))$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$. *In the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, *the external negation* $\widetilde{\neg}\, \widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined analogously, based on* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

In addition to the external negation $\neg Q$ of a two-valued quantifier, TGQ discerns another type of negation, which corresponds to the *antonym* or *internal negation* $Q\neg$ of a two-valued quantifier [8, p. 237]. Here we prefer the term *internal complementation* because the construction involves to complementation of one of the argument sets.

Definition 11 (Internal complementation)
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$, *the* antonym $Q\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$Q\neg(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{n-1}, \neg X_n)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$, *where* $\neg X_n$ *denotes complementation. The definition of the antonym* $\widetilde{Q}\widetilde{\neg}$ *of a fuzzy quantifier* $\widetilde{Q}$ *is analogous, but the fuzzy complement* $\widetilde{\neg}$ *must be used, and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

TGQ also knows the concept of the *dual* of a two-valued quantifier (written as $Q\Box$ in my notation), which is the antonym of the negation of a quantifier (or equivalently, the negation of the antonym) [8, p. 238].

Definition 12 (Dualisation)
For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$, *the* dual $Q\widetilde{\Box} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by* $Q\widetilde{\Box} = \widetilde{\neg}\, Q\neg$. *In the case of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, *we use an analogous definition, i.e.* $\widetilde{Q}\widetilde{\Box} = \widetilde{\neg}\, \widetilde{Q}\widetilde{\neg}$.

For example, **less than n** is the negation of **at least n**, **no** is the antonym of **all**, and **some** is the dual of **all**. It is straightforward to require that a QFM be compatible with these constructions, i.e. we desire that $\mathcal{F}$(**less than n**) be the negation of $\mathcal{F}$(**at least n**), $\mathcal{F}$(**no**) be the antonym of $\mathcal{F}$(**all**), and $\mathcal{F}$(**some**) be the dual of $\mathcal{F}$(**all**). We hence say that $\mathcal{F}$ is compatible with negation, antonyms, and dualisation, if $\mathcal{F}(\neg Q) = \neg \mathcal{F}(Q)$, $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\neg$ and $\mathcal{F}(Q\widetilde{\Box}) = \mathcal{F}(Q)\widetilde{\Box}$, respectively.
For example, compatibility with antonyms means that $\mathcal{F}$(**all**)(**rich**, $\widetilde{\neg}$ **lucky**) $=$ $\mathcal{F}$(**no**)(**rich**, **lucky**). Similarly, preservation of external negation ensures that

$$\mathcal{F}(\textbf{at most 10})(\textbf{young}, \textbf{rich}) = \widetilde{\neg}\, \mathcal{F}(\textbf{more than 10})(\textbf{young}, \textbf{rich})\,,$$

as desired.
For all $n \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, we define the *transposition* $\tau_i : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ by

$$\tau_i(k) = \begin{cases} n & : \quad k = i \\ i & : \quad k = n \\ k & : \quad \text{else} \end{cases}$$

for all $k \in \{1, \ldots, n\}$. It is apparent that each of these τ_i is self-inverse, i.e.

$$\tau_i = \tau_i^{-1}, \tag{5}$$

and that each permutation, i.e. bijection $\beta : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, can be expressed by a sequence of such transpositions. We utilize these transpositions to effect permutations of the arguments positions:

Definition 13 (Argument transpositions)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$ and $i \in \{0, \ldots, n\}$. By $Q\tau_i : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the semi-fuzzy quantifier defined by

$$Q\tau_i(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{i-1}, X_n, X_{i+1}, \ldots, X_{n-1}, X_i),$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$. In the case of fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we define $\widetilde{Q}\tau_i : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ analogously.

We require that every "reasonable" QFM $\mathcal{F}$ be compatible with argument transpositions, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $i \in \{1, \ldots, n\}$ are given, then $\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i$. Because every permutation can be expressed as a sequence of transpositions, this also ensures that $\mathcal{F}$ commutes with arbitrary permutations of the arguments of a quantifier. In particular, symmetry properties of a quantifier Q carry over to its fuzzified analogon $\mathcal{F}(Q)$. Hence $\mathcal{F}(\textbf{some})(\textbf{rich}, \textbf{young}) = \mathcal{F}(\textbf{some})(\textbf{young}, \textbf{rich})$, i.e. the meaning of "some rich people are young" and "some young people are rich" coincide.

Quantifying natural language expressions can involve composite quantifiers like "many married X's are Y", a construction known as *adjectival restriction* [8, p. 247].[6] The construction corresponds to an intersection with the extension of the adjective, thus $\textbf{many}(X \cap \textbf{married}, Y)$. Adjectival restriction can hence be decomposed into the constructions of intersecting argument sets and the insertion of constant arguments like **married**. Let us first consider intersections of argument sets.

Definition 14 (Internal meets)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$. The semi-fuzzy quantifier $Q\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by

$$Q\cap(X_1, \ldots, X_{n+1}) = Q(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1}),$$

for all $(X_1, \ldots, X_{n+1}) \in \mathcal{P}(E)^{n+1}$. In the case of a fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\widetilde{Q}\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously.

In order to allow for a compositional interpretation of composite quantifiers like "all X's are Y's and $Z's$", we require that a QFM $\mathcal{F}$ be compatible with intersections of the argument sets, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$, then $\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\cap}$. For example, the semi-fuzzy quantifier $\textbf{all}\cap$, which has $\textbf{all}\cap(X, Y, Z) = \textbf{all}(X, Y \cap Z)$ for crisp $X, Y, Z \in \mathcal{P}(E)$, should be mapped to $\mathcal{F}(\textbf{all})\widetilde{\cap}$, i.e. for all fuzzy subsets $X, Y, Z \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(\textbf{all}\cup)(X, Y, Z) = \mathcal{F}(\textbf{all})(X, Y \widetilde{\cap} Z)$. Similarly, we desire that $\mathcal{F}(\textbf{some}) = \mathcal{F}(\exists)\widetilde{\cap}$, because the two-place quantifier **some** can be expressed as $\textbf{some} = \exists\cap$.

[6] in this case, restriction of the first argument by the crisp adjective "married".

Definition 15 (Argument insertion)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier,* $n > 0$*, and* $A \in \mathcal{P}(E)$*. By* $Q \triangleleft A : \mathcal{P}(E)^{n-1} \longrightarrow \mathbf{I}$ *we denote the semi-fuzzy quantifier defined by*

$$Q \triangleleft A(X_1, \ldots, X_{n-1}) = Q(X_1, \ldots, X_{n-1}, A)$$

for all $X_1, \ldots, X_{n-1} \in \mathcal{P}(E)$*. (Analogous definition of* $\widetilde{Q} \triangleleft A$ *for fuzzy quantifiers).*

The main application of argument insertion is that of modelling adjectival restriction by a crisp adjective. For example, if **married** $\in \mathcal{P}(E)$ is extension of the crisp adjective "married", then $Q' = \textbf{many}\tau_1 \cap \triangleleft \textbf{married}\tau_1$, i.e. $Q'(X, Y) = \textbf{many}(\textbf{married} \cap X, Y)$, models the composite quantifier "many married X's are Y's". If a QFM $\mathcal{F}$ is compatible with argument insertion, then $\mathcal{F}(Q')(\textbf{rich}, \textbf{lucky}) = \mathcal{F}(\textbf{many})(\textbf{married} \widetilde{\cap} \textbf{rich}, \textbf{lucky})$, as desired. Let us remark that adjectival restriction with a *fuzzy* adjective cannot be modelled directly. This is because we cannot insert a fuzzy argument A into a semi-fuzzy quantifier.

Definition 16 (Monotonicity in arguments)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is said to be* nonincreasing *in its* i*-th argument (*$i \in \{1, \ldots, n\}, n > 0$*) iff for all* $X_1, \ldots, X_n, X'_i \in \mathcal{P}(E)$ *such that* $X_i \subseteq X'_i$*,*

$$Q(X_1, \ldots, X_n) \geq Q(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n)\,.$$

Nondecreasing monotonicity of Q *is defined by changing '*$\geq$*' to '*$\leq$*' in the above inequation. On fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*, we use an analog definition, where* $X_1, \ldots, X_n, X'_i \in \widetilde{\mathcal{P}}(E)$*, and "*$\subseteq$*" is the fuzzy inclusion relation, i.e.* $X \subseteq X'$ *iff* $\mu_X(e) \leq \mu_{X'}(e)$ *for all* $e \in E$*.*

For example, **all** is nonincreasing in the first and nondecreasing in the second argument. **most** is nondecreasing in its second argument, etc. It is natural to require that monotonicity properties of a quantifier in its arguments be preserved when applying a QFM $\mathcal{F}$. For example, we expect that $\mathcal{F}(\textbf{all})$ is nonincreasing in the first and nondecreasing in the second argument.

Definition 17
Every mapping $f : E \longrightarrow E'$ *uniquely determines a* powerset function $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$*, which is defined by* $\widehat{f}(X) = \{f(e) : e \in X\}$*, for all* $X \in \mathcal{P}(E)$*.*

The underlying mechanism which transports f to $\widehat{f}$ can be generalized to the case of fuzzy sets and is then called an *extension principle*.

Definition 18
An extension principle $\mathcal{E}$*,*

$$(f : E \longrightarrow E') \mapsto (\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')),$$

assigns to each mapping $f : E \longrightarrow E'$ *(where* $E, E' \neq \varnothing$*), a corresponding mapping* $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$*.*

Notes

- The case that $E = \varnothing$ or $E' = \varnothing$ has been excluded because it is irrelevant in this context. However, every extension principle (in the sense of the above definition) can be extended to a "full" extension principle which is also defined in the case $E = \varnothing$ or $E' = \varnothing$ in the obvious way, cf. [9, p. 20].

- Similarly, a generalisation to extension principles for n-ary mappings is obvious but irrelevant to our purposes.

Definition 19 (Induced extension principle)
Suppose $\mathcal{F}$ is a QFM. $\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E \longrightarrow E'$ (where $E,\, E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ defined by

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$.

The most prominent definition of an extension principle has been proposed by Zadeh [26]:

Definition 20
The standard extension principle $(\hat{\hat{\bullet}})$ *is defined by*

$$\mu_{\hat{\hat{f}}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}\,,$$

for all $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $e' \in E'$.

We can use the extension principle to construct quantifiers.

Definition 21 (Functional application)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $f_1, \dots, f_n : E' \longrightarrow E$ are given ($E' \neq \varnothing$). We define the semi-fuzzy quantifier $Q' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by

$$Q' = Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}, \qquad \textit{i.e.}$$
$$Q'(Y_1, \dots, Y_n) = Q(\widehat{f_1}(Y_1), \dots, \widehat{f_n}(Y_n)),$$

for $Y_1, \dots, Y_n \in \mathcal{P}(E')$. In the case of a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we define the fuzzy quantifier $\widetilde{Q}' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by

$$\widetilde{Q}' = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i), \qquad \textit{i.e.}$$
$$\widetilde{Q}'(X_1, \dots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f_1)(Y_1), \dots, \widehat{\mathcal{F}}(f_n)(X_n)),$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

It is natural to require that a QFM $\mathcal{F}$ be compatible with its induced extension principle, i.e. $\mathcal{F}(Q') = \widetilde{Q}'$, or equivalently

$$\mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i)\,. \tag{6}$$

Equation (6) hence establishes a relation between powerset functions and the induced extension principle $\widehat{\mathcal{F}}$. It is of particular importance to DFS theory because it is the only axiom which relates the behaviour of $\mathcal{F}$ on different domains $E,\, E'$.
The following definition of the DFS axioms summarises our above considerations on reasonable QFMs.[7]

[7] The DFS axioms have first been presented in [9, p. 22], which also provides ample motivation for the axioms.

Definition 22 (DFS: Determiner Fuzzification Scheme)
A QFM $\mathcal{F}$ is called a determiner fuzzification scheme (DFS) *iff the following axioms are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Preservation of constants	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n = 0$	(DFS 1)
Compatibility with π_*	$\mathcal{F}(\pi_*) = \widetilde{\pi}_*$	(DFS 2)
External negation	$\mathcal{F}(\widetilde{\neg}\, Q) = \widetilde{\neg}\, \mathcal{F}(Q)$	(DFS 3)
Argument transposition	$\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i$ *if* $i \in \{1, \ldots, n\}$	(DFS 4)
Internal complementation	$\mathcal{F}(Q\neg) = \mathcal{F}(Q)\widetilde{\neg}$ *if* $n > 0$	(DFS 5)
Internal meets	$\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\cap}$ *if* $n > 0$	(DFS 6)
Argument insertion	$\mathcal{F}(Q\triangleleft A) = \mathcal{F}(Q)\triangleleft A$ *if* $n > 0$, A *crisp*	(DFS 7)
Preservation of monotonicity	Q *noninc. in* n*-th arg* $\Rightarrow \mathcal{F}(Q)$ *noninc. in* n*-th arg,* $n > 0$	(DFS 8)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(DFS 9)

2 Some properties of DFSes

Here we review some properties of DFS which are crucial for the proofs to follow. Almost all propositions are cited from [9], and the appropriate reference to their proofs will be given. The only new results presented here are concerned with the interpretation of existential and universal quantifiers in DFSes. We will profit from work of Thiele [20] and give a precise description of the corresponding fuzzy quantifiers.

2.1 Correct generalisation

Let us firstly establish that $\mathcal{F}(Q)$ coincides with the original semi-fuzzy quantifier Q when all arguments are crisp sets, i.e. that $\mathcal{F}(Q)$ consistently extends Q.

Theorem 1 (Correct generalisation)
Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier. Then $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. for all crisp *subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,*

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)\,.$$

(Proof: see [9, Th-1, p. 27])

For example, if E is a set of persons, and $\textbf{women}, \textbf{married} \in \mathcal{P}(E)$ are the crisp sets of "women" and "married persons" in E, then

$$\mathcal{F}(\textbf{some})(\textbf{women}, \textbf{married}) = \textbf{some}(\textbf{women}, \textbf{married})\,,$$

i.e. the "fuzzy some" obtained by applying $\mathcal{F}$ coincides with the (original) "crisp some" whenever the latter is defined, which is of course highly desirable.

2.2 Properties of the induced truth functions

Let us now turn to the fuzzy truth functions induced by a DFS. As for negation, the standard choice in fuzzy logic is certainly $\neg : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\neg x = 1 - x$ for all $x \in \mathbf{I}$.
The essential properties of this and other reasonable negation operators are captured by the following definition.

Definition 23 (Strong negation)
$\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *iff it satisfies*

a. $\widetilde{\neg}\, 0 = 1$ *(boundary condition)*

b. $\widetilde{\neg}\, x_1 \geq \widetilde{\neg}\, x_2$ *for all* $x_1, x_2 \in \mathbf{I}$ *such that* $x_1 < x_2$ *(i.e.* $\widetilde{\neg}$ *is monotonically decreasing)*

c. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e.* $\widetilde{\neg}$ *is involutive).*

Note. Whenever the standard negation $\neg x = 1 - x$ is being assumed, we shall drop the 'tilde'-notation. Hence the standard fuzzy complement is denoted $\neg X$, where $\mu_{\neg X}(e) = 1 - \mu_X(e)$. Similarly, the external negation of a (semi-) fuzzy quantifier with respect to the standard negation is written $\neg Q$, and the antonym of a fuzzy quantifier with respect to the standard fuzzy complement is written as $\widetilde{Q}\neg$.
With conjunction, there are several common choices in fuzzy logic (although the standard is certainly $\wedge = \min$). All of these belong to the class of t-norms, which seems to capture what one would expect of a reasonable conjunction operator.

Definition 24 (t-norms)
A mapping $\widetilde{\wedge} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* t*-norm iff it satisfies the following conditions.*

a. $x \mathbin{\widetilde{\wedge}} 0 = 0$*, for all* $x \in \mathbf{I}$

b. $x \mathbin{\widetilde{\wedge}} 1 = x$*, for all* $x \in \mathbf{I}$ *(identity)*

c. $x_1 \mathbin{\widetilde{\wedge}} x_2 = x_2 \mathbin{\widetilde{\wedge}} x_1$ *for all* $x_1,\ x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \leq x'_1$*, then* $x_1 \mathbin{\widetilde{\wedge}} x_2 \leq x'_1 \mathbin{\widetilde{\wedge}} x_2$*, for all* $x_1,\ x'_1,\ x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\wedge}$ *is nondecreasing)*

e. $(x_1 \mathbin{\widetilde{\wedge}} x_2) \mathbin{\widetilde{\wedge}} x_3 = x_1 \mathbin{\widetilde{\wedge}} (x_2 \mathbin{\widetilde{\wedge}} x_3)$*, for all* $x_1,\ x_2,\ x_3 \in \mathbf{I}$ *(associativity).*

The dual concept of t-norm is that of an s-norm, which expresses the essential properties of fuzzy disjunction operators (cf. Schweizer & Sklar [16]).

Definition 25 (s-norms)
$\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is called an* s*-norm iff it satisfies the following conditions.*

a. $x \mathbin{\widetilde{\vee}} 1 = 1$*, for all* $x \in \mathbf{I}$

b. $x \mathbin{\widetilde{\vee}} 0 = x$*, for all* $x \in \mathbf{I}$

c. $x_1 \mathbin{\widetilde{\vee}} x_2 = x_2 \mathbin{\widetilde{\vee}} x_1$ *for all* $x_1,\ x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \leq x'_1$*, then* $x_1 \mathbin{\widetilde{\vee}} x_2 \leq x'_1 \mathbin{\widetilde{\vee}} x_2$*, for all* $x_1,\ x'_1,\ x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\vee}$ *is monotonically increasing)*

e. $(x_1 \mathbin{\widetilde{\vee}} x_2) \mathbin{\widetilde{\vee}} x_3 = x_1 \mathbin{\widetilde{\vee}} (x_2 \mathbin{\widetilde{\vee}} x_3)$*, for all* $x_1,\ x_2,\ x_3 \in \mathbf{I}$ *(associativity).*

The induced truth functions of DFSes can be shown to belong to these classes of "reasonable" choices:

Theorem 2
In every DFS $\mathcal{F}$*,*

a. $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$ *is the identity truth function;*

b. $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator;*

c. $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ *is a* t*-norm;*

d. $x_1 \mathbin{\widetilde{\vee}} x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, x_2)$*, i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$*,*

e. $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2$*, where* $\widetilde{\rightarrow} = \widetilde{\mathcal{F}}\rightarrow$*.*

(Proof: A.1, p.81+)

The fuzzy disjunction induced by $\mathcal{F}$ is therefore definable in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$, and the fuzzy implication induced by $\mathcal{F}$ is definable in terms of $\widetilde{\vee}$ and $\widetilde{\neg}$ (and hence also in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$).[8] A similar point can be made about all other two-place logical connectives except for antivalence xor and equivalence $\leftrightarrow$ (see remarks on p. 29 and [9, p. 52]).

[8]In particular, if $\neg x = 1 - x$ is the standard negation and $\vee = \max$ is the standard fuzzy disjunction, we obtain the Kleene-Dienes implication $x \rightarrow y = \max(1 - x, y)$.

2.3 Preservation of argument structure

We shall now discuss homomorphism properties of DFSes with respect to operations on the argument sets.

We already know from (DFS 3) and (DFS 5) that every DFS is compatible with external negation and formation of antonyms. Because of its compatibility with argument transpositions (DFS 4), we conclude that every DFS is compatible with complementation in arbitrary argument positions. Let us also remark that every DFS is compatible with dualisation, which is immediate from (DFS 3) and (DFS 5):

Theorem 3

Every DFS $\mathcal{F}$ is compatible with dualisation, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$, then

$$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}\,.$$

Hence $\mathcal{F}(\textsf{all})(\textsf{lucky}, \textsf{tall}) = \widetilde{\neg}\,\mathcal{F}(\textsf{some})(\textsf{lucky}, \widetilde{\neg}\,\textsf{tall})$, i.e. the meanings of "all lucky persons are tall" and "it is not the case that some lucky person is not tall" coincide.

Definition 26 (Internal joins)

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$. The semi-fuzzy quantifier $Q\cup : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by

$$Q\cup(X_1, \ \ldots, X_{n+1}) = Q(X_1, \ \ldots, X_{n-1}, X_n \cup X_{n+1})\,,$$

for all $X_1, \ \ldots, X_{n+1} \in \mathcal{P}(E)$. In the case of fuzzy quantifiers, $\widetilde{Q}\widetilde{\cup} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously.

In order to allow for a compositional interpretation of composite quantifiers like "all X's are Y's or $Z's$", we desire that a QFM $\mathcal{F}$ be compatible with unions of the argument sets:

Theorem 4 (Internal joins)

Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier ($n > 0$). Then

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}\,.$$

(Proof: see [9, Th-8, p. 32])

For example, the semi-fuzzy quantifier $\textsf{all}\cup$, which has $\textsf{all}\cup(X, Y, Z) = \textsf{all}(X, Y \cup Z)$ for crisp $X, Y, Z \in \mathcal{P}(E)$, should be mapped to $\mathcal{F}(\textsf{all})\widetilde{\cup}$, i.e. for all fuzzy subsets $X, Y, Z \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(\textsf{all}\cup)(X, Y, Z) = \mathcal{F}(\textsf{all})(X, Y \,\widetilde{\cup}\, Z)$.

Let us also remark that by (DFS 4), this property generalises to unions in arbitrary argument positions.

We already know from (DFS 5) that every DFS is compatible with argument-wise complementation. Let us now establish that $\mathcal{F}$ respects even more fine-grained application of the negation operator.

Definition 27
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier* ($n > 0$) *and* $A \in \mathcal{P}(E)$ *a crisp subset of* E*. By* $Q\triangle A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the semi-fuzzy quantifier defined by*

$$Q\triangle A(X_1, \ldots, X_n) = Q(X_1,\ \ldots,\ X_{n-1},\ X_n\triangle A)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$*, where* $\triangle$ *denotes the symmetrical set difference. For fuzzy quantifiers* $\widetilde{Q}$*, we define* $\widetilde{Q}\widetilde{\triangle}A$ *analogously, where the fuzzy symmetrical difference* $X_1 \mathbin{\widetilde{\triangle}} X_2 \in \widetilde{\mathcal{P}}(E)$ *is defined by* $\mu_{X_1 \widetilde{\triangle} X_2}(e) = \mu_{X_1}(e) \mathbin{\widetilde{\mathrm{xor}}} \mu_{X_2}(e)$ *for all* $e \in E$*.*

Theorem 5
Suppose $\mathcal{F}$ *is a DFS. Then for every semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ ($n > 0$) *and every crisp subset* $A \in \mathcal{P}(E)$*,* $\mathcal{F}(Q\triangle A) = \mathcal{F}(Q) \mathbin{\widetilde{\triangle}} A$*.*
(Proof: see [9, Th-9, p. 35])

2.4 Monotonicity properties

Theorem 6 (Monotonicity in i-th argument)
Suppose $\mathcal{F}$ *is a DFS and* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*. Then* Q *is monotonically nondecreasing (nonincreasing) in its* i*-th argument* ($i \leq n$) *if and only if* $\mathcal{F}(Q)$ *is monotonically nondecreasing (nonincreasing) in its* i*-th argument.*
(Proof: see [9, Th-4, p. 28])

For example, **some** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is monotonically nondecreasing in both arguments. By the theorem, then, $\mathcal{F}(\textbf{some}) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is nondecreasing in both arguments also. In particular,

$$\mathcal{F}(\textbf{some})(\textbf{young\_men}, \textbf{very\_tall}) \leq \mathcal{F}(\textbf{some})(\textbf{men}, \textbf{tall}),$$

i.e. “some young men are very tall” entails “some men are tall”, if **young_men** $\subseteq$ **men** and **very_tall** $\subseteq$ **tall**.

Let us now state that every DFS preserves monotonicity properties of semi-fuzzy quantifiers even if these hold only locally.

Definition 28 (Local monotonicity)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $U, V \in \mathcal{P}(E)^n$ *are given. We say that* Q is locally nondecreasing in the range (U, V) *iff for all* $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \mathcal{P}(E)$ *such that* $U_i \subseteq X_i \subseteq X'_i \subseteq V_i$ ($i = 1,\ \ldots,\ n$)*, we have* $Q(X_1, \ldots, X_n) \leq Q(X'_1,\ \ldots,\ X'_n)$*. We will say that* Q *is locally nonincreasing in the range* (U, V) *if under the same conditions,* $Q(X_1, \ldots, X_n) \leq Q(X'_1,\ \ldots,\ X'_n)$*. On fuzzy quantifiers, local monotonicity is defined analogously, but* $X_1, \ldots, X_n, X'_1, \ldots, X'_n$ *are taken from* $\widetilde{\mathcal{P}}(E)$*, and ‘*$\subseteq$*’ is the fuzzy inclusion relation.*

Theorem 7 (Preservation of local monotonicity)
Suppose $\mathcal{F}$ *is a DFS,* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a semi-fuzzy quantifier and* $U, V \in \mathcal{P}(E)^n$*. Then* Q *is locally nondecreasing (nonincreasing) in the range* (U, V) *iff* $\mathcal{F}(Q)$ *is locally nondecreasing (nonincreasing) in the range* (U, V)*.*
(Proof: see [9, Th-12, p. 36])

To present an example, consider the proportional quantifier

$$\textbf{more than 10 percent} = [\textbf{rate} > 0.1] : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}\,.$$

The quantifier is neither nonincreasing nor nondecreasing in its first argument. Nevertheless, some characteristics of the quantifier express themselves in its local monotonicity properties. For example, suppose $A, B \in \mathcal{P}(E)$ are subsets of E and $A \neq \varnothing$. Then **more than 10 percent** is locally nonincreasing in the range $((A, B), (A \cup \neg B, B))$, and it is locally nondecreasing in the range $((A, B), (A \cup B, B))$. The theorem ensures that such characteristics of a quantifier which become visible through its local monotonicity properties be preserved when applying a DFS.

DFSes can also be shown to be *monotonic* in the sense of preserving inequations between quantifiers. Let us firstly define a partial order $\leq$ on (semi-)fuzzy quantifiers.

Definition 29
Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers. Let us write $Q \leq Q'$ iff for all $X_1, \ldots, X_n \in \mathcal{P}(E)$, $Q(X_1, \ldots, X_n) \leq Q'(X_1, \ldots, X_n)$. On fuzzy quantifiers, we define $\leq$ analogously, where $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

For example, $[\textbf{rate} > 0.5] \leq [\textbf{rate} > 0.2]$, which reflects our intuition that "more than 50 percent of the X's are Y" is a stronger condition than "more than 20 percent of the X's are Y".

Theorem 8
Suppose $\mathcal{F}$ is a DFS and $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers. Then $Q \leq Q'$ if and only if $\mathcal{F}(Q) \leq \mathcal{F}(Q')$.
(Proof: see [9, Th-13, p. 36])

The theorem ensures that inequations between quantifiers carry over to the corresponding fuzzy quantifiers. Hence

$$\mathcal{F}(\textbf{more than 50 percent})(\textbf{blonde}, \textbf{tall}) \leq \mathcal{F}(\textbf{more than 20 percent})(\textbf{blonde}, \textbf{tall})\,,$$

as desired.

Definition 30
Suppose $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers and $U, V \in \mathcal{P}(E)^n$. We say that Q_1 is (not necessarily strictly) smaller than Q_2 in the range (U, V), in symbols: $Q_1 \leq_{(U,V)} Q_2$, iff for all $X_1, \ldots, X_n \in \mathcal{P}(E)$ such that $U_1 \subseteq X_1 \subseteq V_1, \ldots, U_n \subseteq X_n \subseteq V_n$,

$$Q_1(X_1, \ldots, X_n) \leq Q_2(X_1, \ldots, X_n)\,.$$

On fuzzy quantifiers, we define $\widetilde{Q}_1 \leq_{(U,V)} \widetilde{Q}_2$ analogously, but $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and '$\subseteq$' denotes the fuzzy inclusion relation.

Every DFS preserves inequations between quantifiers even if these hold only locally.

Theorem 9
Suppose $\mathcal{F}$ is a DFS, $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $U, V \in \mathcal{P}(E)^n$. Then

$$Q_1 \leq_{(U,V)} Q_2 \Leftrightarrow \mathcal{F}(Q_1) \leq_{(U,V)} \mathcal{F}(Q_2)\,.$$

(Proof: see [9, Th-14, p. 37])

For example, the two-place quantifier **all** is smaller than **some** whenever the first argument is nonempty, i.e.

$$\textbf{all} \leq_{((\{e\},E\}),\,(E,E))} \textbf{some}\,,$$

for all $e \in E$. The theorem ensures that such local inequations are preserved when applying a DFS. In particular, if $\textbf{tall}, \textbf{lucky} \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E and **tall** has nonempty support, then

$$\textbf{all}(\textbf{tall}, \textbf{lucky}) \leq \textbf{some}(\textbf{tall}, \textbf{lucky})\,,$$

as desired.

2.5 Miscellaneous Properties

Theorem 10 (Projection quantifiers)
Let $\mathcal{F}$ be a DFS, $E \neq \varnothing$ a base set and $n \in \mathbb{N}$. Then for all $e \in E$,

$$\mathcal{F}(\pi_e) = \widetilde{\pi}_e\,.$$

(Proof: see [9, Th-15, p. 37])

Hence in the example domain $E = \{\mathrm{John}, \mathrm{Lucas}, \mathrm{Mary}\}$, the crisp projection quantifier **john** $= \pi_{\mathrm{John}} : \mathcal{P}(E) \longrightarrow \mathbf{2}$ would be mapped to $\widetilde{\pi}_{\mathrm{John}} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, as desired.
The following theorem simplifies proofs related to the induced negation operator of a DFS.

Theorem 11
Suppose a QFM $\mathcal{F}$ satisfies (DFS 2)*, i.e. $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$, and $\neg' : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping such that for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,*

$$\mathcal{F}(\neg' Q) = \neg' \mathcal{F}(Q)\,.$$

Then $\neg' = \widetilde{\mathcal{F}}(\neg)$, i.e. $\neg'$ is is the negation operator induced by $\mathcal{F}$.
(Proof: see [9, Th-10, p. 35])

A similar point can be made about conjunction:

Theorem 12
Suppose a QFM $\mathcal{F}$ satisfies (DFS 2)*, i.e. $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$, and $\wedge' : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping such that for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,*

$$\mathcal{F}(Q\cap) = \mathcal{F}(Q)\cap'\,,$$

where $\cap'$ is the fuzzy intersection corresponding to $\wedge'$. Then $\wedge' = \widetilde{\mathcal{F}}(\wedge)$, i.e. $\wedge'$ is the conjunction operator induced by $\mathcal{F}$.
(Proof: see [9, Th-11, p. 35])

2.6 Properties of the induced extension principle

Theorem 13 ($\widehat{\mathcal{F}}$ is functorial)
Suppose $\mathcal{F}$ is a DFS and $\widehat{\mathcal{F}}$ the extension principle induced by $\mathcal{F}$. Then for all $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ (where $E \neq \varnothing$, $E' \neq \varnothing$, $E'' \neq \varnothing$),

a. $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$

b. $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$

(Proof: see [9, Th-16, p. 38])

The induced extension principles of all DFSes coincide on injective mappings.

Theorem 14 (Extension of injections)
Suppose $\mathcal{F}$ is a DFS and $f : E \longrightarrow E'$ is an injection. Then for all $X \in \widetilde{\mathcal{P}}(E)$, $v \in E'$,

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(v) = \begin{cases} \mu_X(f^{-1}(v)) & : \quad v \in \mathrm{Im}\, f \\ 0 & : \quad v \notin \mathrm{Im}\, f \end{cases}$$

(Proof: see [9, Th-17, p. 38])

Definition 31 (Quantitative semi-fuzzy quantifier)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is called quantitative *iff for all automorphisms*[9] *$\beta : E \longrightarrow E$ and all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,*

$$Q(Y_1, \ldots, Y_n) = Q(\widehat{\beta}(Y_1), \ \ldots, \ \widehat{\beta}(Y_n)) \, .$$

For example, **at least k** is quantitative, while the projection quantifier **john** $= \pi_{\mathrm{John}}$ is not (whenever E is not a singleton).

Definition 32 (Quantitative fuzzy quantifier)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is said to be quantitative *iff for all automorphisms $\beta : E \longrightarrow E$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\widetilde{Q}(X_1, \ldots, X_n) = \widetilde{Q}(\hat{\hat{\beta}}(X_1), \ \ldots, \ \hat{\hat{\beta}}(X_n)) \, ,$$

where $\hat{\hat{\beta}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ is obtained by applying the standard extension principle.

By Th-14, the induced extension principles of all DFSes coincide on injective mappings. Therefore, the explicit mention of the standard extension principle in the above definition does *not* limit its applicability to any particular choice of extension principle.

Theorem 15 (Preservation of quantitativity)
Suppose $\mathcal{F}$ is a DFS. For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, Q is quantitative if and only if $\mathcal{F}(Q)$ is quantitative.
(Proof: see [9, Th-18, p. 39])

[9] i.e. bijections of E into itself

For example, the quantitative quantifiers **all**, **some** and **at least k** are mapped to quantitative semi-fuzzy quantifiers $\mathcal{F}($**all**$)$, $\mathcal{F}($**some**$)$ and $\mathcal{F}($**at least k**$)$, respectively. On the other hand, the non-quantitative projection quantifier **john** $= \pi_{\mathrm{John}}$ is mapped to the fuzzy projection quantifier $\mathcal{F}($**john**$) = \widetilde{\pi}_{\mathrm{John}}$, which is also non-quantitative.
We will now establish that a DFS is compatible with exactly one extension principle.

Theorem 16
Suppose $\mathcal{F}$ is a DFS and $\mathcal{E}$ an extension principle such that for every semi-fuzzy quantifier $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ and all $f_1 : E \longrightarrow E', \ \ldots, \ f_n : E \longrightarrow E'$, $\mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i) = \mathcal{F}(Q) \circ \overset{n}{\underset{i=1}{\times}} \mathcal{E}(f_i)$. Then $\widehat{\mathcal{F}} = \mathcal{E}$.
(Proof: see [9, Th-19, p. 39])

The extension principle $\widehat{\mathcal{F}}$ of a DFS $\mathcal{F}$ is uniquely determined by the fuzzy existential quantifiers $\mathcal{F}(\exists) = \mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ induced by $\mathcal{F}$. The converse can also be shown: the fuzzy existential quantifiers obtained from a DFS $\mathcal{F}$ are uniquely determined by its extension principle $\widehat{\mathcal{F}}$.

Theorem 17
Suppose $\mathcal{F}$ is a DFS.

a. For every mapping $f : E \longrightarrow E'$ and all $e' \in E'$, $\mu_{\widehat{\mathcal{F}}(f)(\bullet)}(e') = \mathcal{F}(\exists)\widetilde{\cap}\triangleleft f^{-1}(e')$.

b. If $E \neq \varnothing$ and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$, then $\mathcal{F}(\exists) = \widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!)$, where $\{*\}$ is an arbitrary singleton set and $! : E \longrightarrow \{*\}$ is the mapping defined by $!(x) = *$ for all $x \in E$.

(Proof: A.2, p.81+)

A notion closely related to extension principles is that of *fuzzy inverse images*. Let us first recall the concept of inverse images in the crisp case:

Definition 33 (Inverse Images)
Suppose $f : E \longrightarrow E'$ is a mapping. The inverse image mapping $f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$ is defined by $f^{-1}(V) = \{e \in E : f(e) \in V\}$, for all $V \in \mathcal{P}(E')$.

Unlike in the case of extension principles, there is only one reasonable definition of fuzzy inverse images:

Definition 34 (Standard Fuzzy Inverse Images)
Suppose $f : E \longrightarrow E'$ is a mapping. The fuzzy inverse image mapping $\hat{\hat{f}}^{-1} : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ assigns to each $Y \in \widetilde{\mathcal{P}}(E')$ the fuzzy subset $\hat{\hat{f}}^{-1}(Y) \in \widetilde{\mathcal{P}}(E)$ defined by

$$\mu_{\hat{\hat{f}}^{-1}(Y)}(x) = \mu_Y(f(x))$$

for all $x \in E$.

Now, every quantifier fuzzification mechanism $\mathcal{F}$ induces fuzzy inverse images by means of the following construction.

Definition 35
Suppose $\mathcal{F}$ is a quantifier fuzzification mechanism and $f : E \longrightarrow E'$ is some mapping. $\mathcal{F}$ induces a fuzzy inverse image mapping $\widehat{\mathcal{F}}^{-1}(f) : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ *which to each $Y \in \widetilde{\mathcal{P}}(E')$ assigns the fuzzy subset $\widehat{\mathcal{F}}^{-1}(f)$ defined by*

$$\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mathcal{F}(\chi_{f^{-1}(\bullet)}(e))(Y)\,.$$

It is now easily shown that if $\mathcal{F}$ is a DFS, then its induced fuzzy inverse images coincide with the above "reasonable" definition.

Theorem 18 (Induced fuzzy inverse images)
Suppose $\mathcal{F}$ is a DFS, $f : E \longrightarrow E'$ is a mapping and $Y \in \widetilde{\mathcal{P}}(E')$. Then for all $e \in E$, $\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mu_Y(f(e))$.
(Proof: see [9, Th-23, p. 41])

2.7 Properties with respect to the standard quantifiers

Some first results on the interpretation of the standard quantifiers $\forall$ and $\exists$ in DFSes have been proven in [9, p. 41+]:

Theorem 19
Let $\mathcal{F}$ be a DFS, E a non-empty set and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

a. $\mathcal{F}(\exists)(\varnothing) = 0, \quad \mathcal{F}(\exists)(E) = 1$

b. $\mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} \{e\}) = \mu_X(e)$
for all $e \in E$

c. $\mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} A) = \mathop{\widetilde{\vee}}\limits_{i=1}^{m} \mu_X(a_i)$
for all finite $A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E)$ ($a_i \neq a_j$ if $i \neq j$)

d. $\mathcal{F}(\exists)(X) \geq \sup\left\{ \mathop{\widetilde{\vee}}\limits_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$

In particular, if E is finite, i.e. $E = \{e_1, \ldots, e_m\}$ where the e_i are pairwise distinct, then

$$\mathcal{F}(\exists)(X) = \mathop{\widetilde{\vee}}\limits_{i=1}^{m} \mu_X(e_i)\,.$$

(Proof: see [9, Th-24, p. 41])

A similar point can be made about the dual case of the universal quantifier.

Theorem 20
Let F be a DFS, E a non-empty set and $\forall = \forall_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

a. $\mathcal{F}(\forall)(\varnothing) = 0, \quad \mathcal{F}(\forall)(E) = 1$

b. $\mathcal{F}(\forall)(X \,\widetilde{\cup}\, \widetilde{\neg}\, \{e\}) = \mu_X(e)$
for all $e \in E$

c. $\mathcal{F}(\forall)(X \,\widetilde{\cup}\, \widetilde{\neg}\, A) = \mathop{\widetilde{\bigwedge}}_{i=1}^{m} \mu_X(a_i)$
for all finite $A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E)$ *(*$a_i \neq a_j$ *if* $i \neq j$*)*

d. $\mathcal{F}(\forall)(X) \leq \inf \left\{ \mathop{\widetilde{\bigwedge}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$

In particular, if E is finite, i.e. $E = \{e_1, \ldots, e_m\}$ where the e_i are pairwise distinct, then

$$\mathcal{F}(\forall)(X) = \mathop{\widetilde{\bigwedge}}_{i=1}^{m} \mu_X(e_i)\,.$$

(Proof: see [9, Th-26, p. 42])

In the following, we wish to improve upon this analysis by showing that in Th-19.c and Th-20.c, the inequations can actually be replaced with an equality. To this end, we will utilize some results due to Thiele [18, 19, 20]. Let us first introduce the concept of T-quantifiers and S-quantifiers (adapted to our notation):

Definition 36 (T-quantifier)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is called a T-quantifier *iff $\widetilde{Q}$ satisfies the following axioms:*

a. For all $X \in \widetilde{\mathcal{P}}(E)$ and $e \in E$,

$$\widetilde{Q}(X \cup \neg\{e\}) = \mu_X(e)$$

b. For all $X \in \widetilde{\mathcal{P}}(E)$ and $e \in E$,

$$\widetilde{Q}(X \cap \neg\{e\}) = 0$$

c. $\widetilde{Q}$ is nondecreasing, i.e. for all $X, X' \in \widetilde{\mathcal{P}}(E)$ such that $X \subseteq X'$,

$$\widetilde{Q}(X) \leq \widetilde{Q}(X')$$

d. $\widetilde{Q}$ is quantitative, i.e. for every automorphism (permutation) $\beta : E \longrightarrow E$,

$$\widetilde{Q} \circ \hat{\hat{\beta}} = \widetilde{Q}\,.$$

Note. In the definition, $\cap$ is the standard fuzzy intersection based on $\min$, and $\cup$ is the standard fuzzy union based on $\max$. However, all fuzzy intersections based on t-norms and all fuzzy unions based on s-norms will give the same results, because one of the arguments is a crisp subset of E.

A dual definition is introduced for S-quantifiers:

Definition 37 (S-quantifier)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is called an* S-quantifier *iff* $\widetilde{Q}$ *satisfies the following axioms:*

a. *For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$,

$$\widetilde{Q}(X \cup \{e\}) = 1$$

b. *For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $e \in E$,

$$\widetilde{Q}(X \cap \{e\}) = \mu_X(e)$$

c. $\widetilde{Q}$ *is nondecreasing, i.e. for all* $X, X' \in \widetilde{\mathcal{P}}(E)$ *such that* $X \subseteq X'$,

$$\widetilde{Q}(X) \leq \widetilde{Q}(X')$$

d. $\widetilde{Q}$ *is quantitative, i.e. for every automorphism (permutation)* $\beta : E \longrightarrow E$,

$$\widetilde{Q} \circ \hat{\hat{\beta}} = \widetilde{Q} \,.$$

Note. An analogous point about $\cap$ and $\cup$ can be made in this case.
It is apparent from the following theorem that in every DFS, the fuzzy universal quantifier is a T-quantifier, and the fuzzy existential quantifier is an S-quantifier:

Theorem 21 (T- and S-quantifiers in DFSes)
In every DFS $\mathcal{F}$*, the fuzzy universal quantifier* $\mathcal{F}(\forall_E)$ *is a T-quantifier, and the fuzzy existential quantifier* $\mathcal{F}(\exists_E)$ *is an S-quantifier, for all non-empty base sets* E.
(Proof: A.3, p.81+)

There is a close relationship between T-quantifiers and t-norms (and analogously between S-quantifiers and s-norms).

Definition 38 ($\widetilde{\wedge}_{\widetilde{Q}}$)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is a T-quantifier and* $|E| > 1$. $\widetilde{\wedge}_{\widetilde{Q}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$x_1 \,\widetilde{\wedge}_{\widetilde{Q}}\, x_2 = \widetilde{Q}(X)$$

for all $x_1, x_2 \in \mathbf{I}$*, where* $X \in \widetilde{\mathcal{P}}(E)$ *is defined by*

$$\mu_X(e) = \begin{cases} x_1 & : \quad e = e_1 \\ x_2 & : \quad e = e_2 \\ 1 & : \quad \textit{else} \end{cases} \tag{7}$$

where $e_1 \neq e_2$*,* $e_1, e_2 \in E$ *are two arbitrary distinct elements of* E.

Note. It is evident by the quantitativity of T-quantifiers that $x_1 \,\widetilde{\wedge}_{\widetilde{Q}}\, x_2$ does not depend on the choice of $e_1, e_2 \in E$.
A dual definition will be used for S-quantifiers:

Definition 39 ($\widetilde{\vee}_{\widetilde{Q}}$)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is an S-quantifier and $|E| > 1$. $\widetilde{\vee}_{\widetilde{Q}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is defined by

$$x_1 \,\widetilde{\vee}_{\widetilde{Q}}\, x_2 = \widetilde{Q}(X)$$

for all $x_1, x_2 \in \mathbf{I}$, where $X \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_X(e) = \begin{cases} x_1 & : & e = e_1 \\ x_2 & : & e = e_2 \\ 0 & : & \textit{else} \end{cases} \tag{8}$$

and $e_1 \neq e_2$, $e_1, e_2 \in E$ are two arbitrary distinct elements of E.

Note. Again, the independence of $\widetilde{\vee}_{\widetilde{Q}}$ on the chosen elements $e_1, e_2 \in E$ is apparent from the quantitativity of S-quantifiers.
We can now benefit from the following theorem of Thiele (in our notation):

Theorem 22 (T-quantifier)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is a T-quantifier where $|E| > 1$. Then $\widetilde{\wedge}_{\widetilde{Q}}$ is a t-norm, and

$$\widetilde{Q}(X) = \inf \left\{ \mathop{\widetilde{\wedge}_{\widetilde{Q}}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ \ldots, \ a_m\} \in \mathcal{P}(E) \textit{ finite}, a_i \neq a_j \textit{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(See Thiele [18, Th-8.1, p.47])

A dual theorem holds for S-quantifiers:

Theorem 23 (S-quantifiers)
Suppose $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is a S-quantifier where $|E| > 1$. Then $\widetilde{\vee}_{\widetilde{Q}}$ is an s-norm, and

$$\widetilde{Q}(X) = \sup \left\{ \mathop{\widetilde{\vee}_{\widetilde{Q}}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ \ldots, \ a_m\} \in \mathcal{P}(E) \textit{ finite}, a_i \neq a_j \textit{ if } i \neq j \right\}$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(See Thiele [18, Th-8.2, p.48])

Note. Some properties of s-norm aggregation of infinite collections in the form expressed by the theorem (and hence, as expressed by S-quantifiers) have been studied by Rovatti and Fantuzzi [15] who view S-quantifiers as a special type of non-additive functionals.
Connecting theorems Th-23 and Th-23 with our earlier results on the interpretation of the standard quantifiers in DFSes, we immediately obtain

Theorem 24 (Universal quantifiers in DFSes)
Suppose $\mathcal{F}$ is a DFS and $E \neq \varnothing$ is some base set. Then

$$\mathcal{F}(\forall)(X) = \inf \left\{ \mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ \ldots, \ a_m\} \in \mathcal{P}(E) \textit{ finite}, a_i \neq a_j \textit{ if } i \neq j \right\}$$

(Proof: A.4, p.82+)

Theorem 25 (Existential quantifiers in DFSes)
Suppose $\mathcal{F}$ is a DFS and E is a non-empty base set. Then

$$\mathcal{F}(\exists)(X) = \sup\left\{ \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$$

(Proof: A.5, p.82+)

This improves upon our analysis in Th-19 and Th-20 where we could only show that the fuzzy existential and universal quantifiers are bounded by the above expressions. In particular, the theorem shows that in every DFS, the fuzzy existential and fuzzy quantifiers are uniquely determined by the induced fuzzy disjunction and conjunction.

Theorem 26
Suppose $\mathcal{F}$ is a DFS, $\widehat{\mathcal{F}}$ its induced extension principle and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$.

a. $\widehat{\mathcal{F}}$ is uniquely determined by $\widetilde{\vee}$;

b. $\widetilde{\vee}$ is uniquely determined by $\widehat{\mathcal{F}}$, viz. $x_1 \,\widetilde{\vee}\, x_2 = (\widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!))(X)$ for all $x_1, x_2 \in \mathbf{I}$, where $X \in \widetilde{\mathcal{P}}(\{1, 2\})$ is defined by $\mu_X(1) = x_1$ and $\mu_X(2) = x_2$, $\{*\}$ is an arbitrary singleton set, and $!$ is the unique mapping $! : \{1, 2\} \longrightarrow \{*\}$.

(Proof: A.6, p.82+)

In particular, if $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$ is the standard extension principle, then $\widetilde{\vee} = \max$. Because I did not want QFMs in which $\widetilde{\vee} \neq \max$ to be a priori excluded from consideration, axiom (DFS 9) has been stated in terms of the extension principle $\widehat{\mathcal{F}}$ induced by $\mathcal{F}$, rather than requiring the compatibility of $\mathcal{F}$ to the standard extension principle.

2.8 Special subclasses of DFSes

We will now turn to subclasses of DFS models which satisfy some additional requirements.

Definition 40
Let $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ be a strong negation operator. A DFS $\mathcal{F}$ is called a $\widetilde{\neg}$-DFS if its induced negation coincides with $\widetilde{\neg}$, i.e. $\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}$. In particular, we will call $\mathcal{F}$ a $\neg$-DFS if it induces the standard negation $\neg x = 1 - x$.

Definition 41
Suppose $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ a bijection. For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we define

$$\mathcal{F}^\sigma(Q)(X_1, \ldots, X_n) = \sigma^{-1}\mathcal{F}(\sigma Q)(\sigma X_1, \ldots, \sigma X_n),$$

where σQ abbreviates $\sigma \circ Q$, and $\sigma X_i \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset with $\mu_{\sigma X_i} = \sigma \circ \mu_{X_i}$.

Theorem 27
If $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ an increasing bijection, then $\mathcal{F}^\sigma$ is a DFS.
(Proof: see [9, Th-27, p. 43])

It is well-known [12, Th-3.7] that for every strong negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ there is a monotonically increasing bijection $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ such that $\widetilde{\neg}\, x = \sigma^{-1}(1 - \sigma(x))$ for all $x \in \mathbf{I}$. The mapping σ is called the *generator* of $\widetilde{\neg}$.

Theorem 28
Suppose $\mathcal{F}$ is a $\widetilde{\neg}$-DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ the generator of $\widetilde{\neg}$. Then $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$ is a $\neg$-DFS and $\mathcal{F} = \mathcal{F}'^{\sigma}$.
(Proof: see [9, Th-28, p. 44])

This means that we can freely move from an arbitrary $\widetilde{\neg}$-DFS to a corresponding $\neg$-DFS and vice versa: in the following, we shall hence restrict attention to $\neg$-DFSes.

In [9, Def. 32] the notion of $\mathcal{E}$-DFS has been introduced in order to classify DFSes according to their induced extension principle, and a number of theorems have been proven for $\mathcal{E}$-DFSes.

Definition 42 ($\mathcal{E}$-DFS)
A $\neg$-DFS $\mathcal{F}$ which induces the extension principle $\mathcal{E} = \widehat{\mathcal{F}}$ will be called an $\mathcal{E}$-DFS.

We can now utilize theorem Th-17 which establishes that the induced extension principle of a DFS is uniquely determined by it induced disjunction, and that conversely the induced disjunction is uniquely determined by the induced extension principle. In other words, if two DFSes induce the same extension principle, then they also induce the same disjunction and vice versa. Because of this equivalence, the following definition of $\widetilde{\vee}$-DFSes gives rise to the same classes of DFSes as the definition of $\mathcal{E}$-DFSes.

Definition 43
A $\neg$-DFS $\mathcal{F}$ which induces a fuzzy disjunction $\widetilde{\vee}$ is called a $\widetilde{\vee}$-DFS.

In the following, we prefer to talk about $\widetilde{\vee}$-DFSes rather than $\mathcal{E}$-DFSes for reasons of simplicity. By our above reasoning, all results established for $\mathcal{E}$-DFSes in [9, p. 44+] carry over to $\widetilde{\vee}$-DFSes.

Theorem 29
Suppose $\mathcal{J}$ is a non-empty index set and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes. Further suppose that $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ satisfies the following conditions:

a. If $f \in \mathbf{I}^{\mathcal{J}}$ is constant, i.e. if there is a $c \in \mathbf{I}$ such that $f(j) = c$ for all $j \in \mathcal{J}$, then $\Psi(f) = c$.

b. $\Psi(1 - f) = 1 - \Psi(f)$, where $1 - f \in \mathbf{I}^{\mathcal{J}}$ is point-wise defined by $(1 - f)(j) = 1 - f(j)$, for all $j \in \mathcal{J}$.

c. Ψ is monotonically nondecreasing, i.e. if $f(j) \leq g(j)$ for all $j \in \mathcal{J}$, then $\Psi(f) \leq \Psi(g)$.

If we define $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ by

$$\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](Q)(X_1, \dots, X_n) = \Psi((\mathcal{F}_j(Q)(X_1, \dots, X_n))_{j\in\mathcal{J}})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, then $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ is a $\widetilde{\vee}$-DFS.
(Proof: see [9, Th-29, p. 44])

In particular, convex combinations (e.g., arithmetic mean) and stable symmetric sums [17] of $\widetilde{\vee}$-DFSes are again $\widetilde{\vee}$-DFSes.

The $\neg$-DFSes can be partially ordered by "specificity" or "cautiousness", in the sense of closeness to $\frac{1}{2}$. We shall define a partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ by

$$x \preceq_c y \Leftrightarrow y \leq x \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} \leq x \leq y\,, \tag{9}$$

for all $x, y \in \mathbf{I}$.[10]

Definition 44
Suppose $\mathcal{F}$, $\mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is consistently less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, iff for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}'(Q)(X_1, \ldots, X_n)\,.$$

We now wish to establish the existence of consistently least specific $\widetilde{\vee}$-DFSes. In order to be able to state the theorem, we firstly need to introduce the *fuzzy median* $\mathrm{m}_{\frac{1}{2}}$.

Definition 45
The fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}}(u_1,\, u_2) = \begin{cases} \min(u_1,\, u_2) & : \ \min(u_1,\, u_2) > \frac{1}{2} \\ \max(u_1,\, u_2) & : \ \max(u_1,\, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

$\mathrm{m}_{\frac{1}{2}}$ is an associative mean operator [4] and the only stable (i.e. idempotent) associative symmetric sum [17]. The fuzzy median can be generalised to an operator $\mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.

Definition 46
The generalised fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ is defined by

$$\mathrm{m}_{\frac{1}{2}}\, X = \mathrm{m}_{\frac{1}{2}}(\inf X,\, \sup X)\,,$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Note. The generalised fuzzy median is obviously related to median quantifiers, studied by Thiele [21].

Theorem 30
Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j \in \mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$. Then there exists a greatest lower specificity bound on $(\mathcal{F}_j)_{j \in \mathcal{J}}$, i.e. a $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{glb}}$ such that $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}_j$ for all $j \in \mathcal{J}$ (i.e. $\mathcal{F}_{\mathrm{glb}}$ is a lower specificity bound), and for all other lower specificity bounds $\mathcal{F}'$,

[10] $\preceq_c$ is Mukaidono's ambiguity relation, see [13].

$\mathcal{F}' \preceq_c \mathcal{F}_{\text{glb}}$.
$\mathcal{F}_{\text{glb}}$ is defined by

$$\mathcal{F}_{\text{glb}}(Q)(X_1,\ldots,X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_j(Q)(X_1,\ldots,X_n) : j \in \mathcal{J}\},$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$.
(Proof: A.7, p.82+)

In particular, the theorem asserts the existence of least specific $\widetilde{\vee}$-DFSes, i.e. whenever $\widetilde{\vee}$ is an s-norm such that $\widetilde{\vee}$-DFSes exist, then there exists a least specific $\widetilde{\vee}$-DFS (just apply the above theorem to the collection of all $\widetilde{\vee}$-DFSes).

Definition 47
A DFS $\mathcal{F}$ such that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is called a $(\widetilde{\neg}, \widetilde{\vee})$-DFS.

In the theory of generalized quantifiers there are constructions $Q \wedge Q'$ and $Q \vee Q'$ of forming the conjunction (disjunction) of two-valued quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$. We shall generalize these constructions to (semi)-fuzzy quantifiers.

Definition 48
Suppose $\widetilde{\wedge}, \widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ are given. For all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the conjunction $Q \mathbin{\widetilde{\wedge}} Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the disjunction $Q \mathbin{\widetilde{\vee}} Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of Q and Q' are defined by

$$(Q \mathbin{\widetilde{\wedge}} Q')(X_1,\ldots,X_n) = Q(X_1,\ldots,X_n) \mathbin{\widetilde{\wedge}} Q'(X_1,\ldots,X_n)$$
$$(Q \mathbin{\widetilde{\vee}} Q')(X_1,\ldots,X_n) = Q(X_1,\ldots,X_n) \mathbin{\widetilde{\vee}} Q'(X_1,\ldots,X_n)$$

for all $X_1,\ldots,X_n \in \mathcal{P}(E)$. For fuzzy quantifiers, $\widetilde{Q} \mathbin{\widetilde{\wedge}} \widetilde{Q}'$ and $\widetilde{Q} \mathbin{\widetilde{\vee}} \widetilde{Q}'$ are defined analogously.

In the following, we shall be concerned with $(\widetilde{\neg}, \max)$-DFSes, i.e. DFSes which induce the standard disjunction $\widetilde{\mathcal{F}}(\vee) = \vee = \max$. In the case of $(\widetilde{\neg}, \max)$-DFSes, we can establish a theorem on conjunctions and disjunctions of (semi-)fuzzy quantifiers.

Theorem 31
Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

a. $\mathcal{F}(Q \wedge Q') \leq \mathcal{F}(Q) \wedge \mathcal{F}(Q')$

b. $\mathcal{F}(Q \vee Q') \geq \mathcal{F}(Q) \vee \mathcal{F}(Q')$.

(Proof: see [9, Th-32, p. 48])

Note. Because the theorem refers to the standard fuzzy conjunction and disjunction, the constructions on quantifiers have been written $Q \wedge Q'$ and $Q \vee Q'$, omiting the 'tilde' notation for fuzzy connectives. Similarly, the standard fuzzy intersection and standard fuzzy union will be written $X \cap Y$ and $X \cup Y$, resp., where $\mu_{X\cap Y}(e) = \min(\mu_X(e), \mu_Y(e))$ and $\mu_{X\cup Y}(e) = \max(\mu_X(e), \mu_Y(e))$. The same conventions are stipulated for intersections $\widetilde{Q}\cap$ and unions $\widetilde{Q}\cup$ of the arguments of a fuzzy quantifier, as well as for duals $Q\square$ of semi-fuzzy quantifiers or $\widetilde{Q}\square$ of fuzzy quantifiers, based on the standard negation.
We have so far not made any claims about the interpretation of $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow)$ and $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor})$ in a given DFS $\mathcal{F}$.

Theorem 32
Suppose $\mathcal{F}$ is a $(\tilde{\neg}, \max)$-DFS. Then for all $x_1,\ x_2 \in \mathbf{I}$,

a. $x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 = (x_1 \wedge x_2) \vee (\tilde{\neg}\, x_1 \wedge \tilde{\neg}\, x_2)$

b. $x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 = (x_1 \wedge \tilde{\neg}\, x_2) \vee (\tilde{\neg}\, x_1 \wedge x_2)\,.$

(Proof: see [9, Th-33, p. 48])

Note. In the case of a $\neg$-DFS, we obtain the equivalence $x_1 \leftrightarrow x_2 = \max(\min(x_1, x_2), \min(1 - x_1, 1 - x_2))$ and the antivalence $x_1 \mathrm{\ xor\ } x_2 = \max(\min(x_1, 1 - x_2), \min(1 - x_1, x_2))$. The corresponding fuzzy set difference is denoted $X \triangle Y$, where $\mu_{X\triangle Y}(e) = \mu_X(e) \mathrm{\ xor\ } \mu_Y(e)$, and the fuzzy quantifier which uses this particular fuzzy difference in its last argument is denoted $\widetilde{Q}\triangle A$.

Definition 49 (Standard DFS)
By a standard DFS *we denote a $(\neg, \max)$-DFS.*

Standard DFS induce the standard connectives of fuzzy logic and by Th-26, they also induce the standard extension principle.[11] It has been remarked in [9, p.49] that the propositional part of a standard DFS coincides with the well-known K-standard sequence logic of Dienes [6]. In particular, the three-valued fragment is Kleene's three-valued logic. Standard DFSes represent a "boundary" case of DFSes because they induce the smallest fuzzy existential quantifiers, the smallest extension principle, and the largest fuzzy universal quantifiers.

[11] Originally, standard DFSes have been introduced as the class of $\neg$-DFSes which induce the standard extension principle [9, p. 49], because it was not apparent at that point that this class of DFSes coincides with that of $(\neg, \max)$-DFSes. Utilizing the new results presented in this report, we can now apply theorem Th-26 to conclude that every $(\neg, \max)$-DFS is guaranteed to be a standard DFS (in the sense of inducing the standard extension principle), i.e. the two classes indeed coincide.

3 An Alternative Characterisation of DFSes

3.1 An Alternative Construction of Induced Connectives

In Def. 8, we have presented a canonical construction for obtaining induced fuzzy truth functions $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ for given semi-fuzzy truth functions $f : \mathbf{2}^n \longrightarrow \mathbf{I}$. Basically, this construction utilizes the bijection between the set of two-valued truth values $\mathbf{2} = \{0, 1\}$ and the powerset $\mathcal{P}(\{*\}) = \{\varnothing, \{*\}\}$ of an arbitrary singleton set $\{*\}$. The semi-fuzzy truth function $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ was then transformed into a semi-fuzzy quantifier $f^* : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \mathbf{I}$, to which the QFM $\mathcal{F}$ can be applied. The resulting fuzzy quantifier was then transformed into a fuzzy truth function by utilizing the bijection between the set of continuous truth values, i.e. the unit interval $\mathbf{I}$, and the fuzzy powerset $\widetilde{\mathcal{P}}(\{*\})$ of a singleton set $\{*\}$.

Although this construction is perfectly reasonable and gives the intended results, it has the disadvantage that the semi-fuzzy quantifier f^* associated with f is still an n-place quantifier. It would be beneficial to use a construction which only involves monadic (one-place) quantifiers for two reasons.

1. Existing approaches to fuzzy quantification like the FE-count approach [14] are often limited to one-place or special forms of two-place quantification. It probably becomes easier to discuss these approaches within our framework if the construction of induced connectives involves one-place quantifiers only.

2. Because the proposed construction of induced connectives assigns n-place fuzzy truth functions to n-place fuzzy quantifiers, we need those axioms which involve multiplace quantification, viz argument transposition (DFS 4), internal meets (DFS 6), argument insertion (DFS 7) and functional application (DFS 9) to establish the desired properties of the basic two-place truth functions like conjunction or disjunction. If we had a construction of induced connectives which maps conjunction and disjunction to one-place quantifiers, we might eliminate some of the axioms related to multi-place quantification, which only serve to ensure a reasonable interpretation of the induced conjunction and disjunction.

These considerations have led us to introduce an alternative construction for induced fuzzy truth functions. Both constructions will of course coincide in every DFS, but they might produce different results if the QFM under consideration fails on some of the DFS axioms.

As the starting point for the alternative construction of induced connectives, let us observe that $\mathbf{2}^n$ and $\mathcal{P}(\{1, \ldots, n\})$ are isomorphic. This is apparent if we recall that for every set A, $\mathbf{2}^A$ (the set of characteristic functions of A) and $\mathcal{P}(A)$ (the set of subsets of A) are isomorphic, and if we utilize the set-theoretic construction of natural numbers as $0 = \varnothing$, $1 = \{0\}$, $\mathbf{2} = \{0, 1\}$, $n = \{0, \ldots, n-1\}$. We then have $\mathbf{2}^n \cong \mathcal{P}(n) = \mathcal{P}(\{0, \ldots, n-1\})$. For convenience, as we have numbered the arguments of n-ary quantifiers from 1 to n, rather than 0 to $n-1$, we shall replace $\{0, \ldots, n-1\}$ by the isomorphic $\{1, \ldots, n\}$, to obtain $\mathbf{2}^n \cong \mathcal{P}(\{1, \ldots, n\})$, using the following bijection.

Definition 50
For all $n \in \mathbb{N}$, the bijection $\eta : \mathbf{2}^n \longrightarrow \mathcal{P}(\{1, \ldots, n\})$ is defined by

$$\eta(x_1, \ldots, x_n) = \{k \in \{1, \ldots, n\} : x_k = 1\},$$

for all $x_1, \ldots, x_n \in \mathbf{2}$.

An analogous construction is possible in the fuzzy case, where we have $\mathbf{I}^n \cong \widetilde{\mathcal{P}}(\{1, \ldots, n\})$:

Definition 51
For all $n \in \mathbb{N}$, the bijection $\widetilde{\eta} : \mathbf{I}^n \longrightarrow \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ is defined by

$$\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(k) = x_k$$

for all $x_1, \ldots, x_n \in \mathbf{I}$ and $k \in \{1, \ldots, n\}$.

(It is apparent that both η and $\widetilde{\eta}$ are indeed bijections). By making use of η and $\widetilde{\eta}$, we can transform each semi-fuzzy truth function $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ into a corresponding *one-place* semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$, apply $\mathcal{F}$ to obtain $\mathcal{F}(Q_f) : \widetilde{\mathcal{P}}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$, and translate this into a fuzzy truth function $\widetilde{\widetilde{\mathcal{F}}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$:

Definition 52 (Alternative Construction of Induced Fuzzy Truth Functions)
Suppose $\mathcal{F}$ is a QFM and f is a semi-fuzzy truth function (i.e. a mapping $f : \mathbf{2}^n \longrightarrow \mathbf{I}$) of arity $n > 0$. The semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ is defined by

$$Q_f(X) = f(\eta^{-1}(X))$$

for all $X \in \mathcal{P}(\{1, \ldots, n\})$. The induced fuzzy truth function $\widetilde{\widetilde{\mathcal{F}}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{\widetilde{\mathcal{F}}}(f)(x_1, \ldots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n))\,,$$

for all $x_1, \ldots, x_n \in \mathbf{I}$. If $f : \mathbf{I}^0 \longrightarrow \mathbf{I}$ is a nullary semi-fuzzy truth function (i.e., a constant), we shall define $\widetilde{\widetilde{\mathcal{F}}}(f) : 2^0 \longrightarrow \mathbf{I}$ by $\widetilde{\widetilde{\mathcal{F}}}(f) = \widetilde{\mathcal{F}}(f)$, i.e. using the original construction of induced connectives.

Notes

- The special treatment of nullary semi-fuzzy truth functions is necessary because in this case, we would have $Q_f : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which does not conform to our definition of semi-fuzzy quantifiers and fuzzy quantifiers based on nonempty base-sets. Rather than adapting the definition of semi-fuzzy and fuzzy quantifiers such as to allow for empty base sets, and hence cover Q_f for nullary f, too, we prefer to treat the case of nullary truth functions separately, i.e. by the original construction $\widetilde{\mathcal{F}}$.[12]

- Whenever $\mathcal{F}$ is understood from context, we shall abbreviate $\widetilde{\widetilde{\mathcal{F}}}(f)$ as $\widetilde{\widetilde{f}}$; in particular, we will write $\widetilde{\widetilde{\neg}}$, $\widetilde{\widetilde{\wedge}}$, $\widetilde{\widetilde{\vee}}$, $\widetilde{\widetilde{\rightarrow}}$ and $\widetilde{\widetilde{\leftrightarrow}}$ for the negation, conjunction, disjunction, implication and equivalence, respectively, as obtained from $\widetilde{\widetilde{\mathcal{F}}}$.

[12]Empty base sets are generally avoided in logic because for the empty domain $E = \varnothing$, the tautology $\forall x\varphi \rightarrow \exists x\varphi$ fails.

3.2 Equivalence of both Constructions with respect to DFSes

Having two constructions of induced fuzzy truth functions, which are both straightforward, we would of course like to have that the results of both constructions coincide, at least if the QFM $\mathcal{F}$ of interest is sufficiently well-behaved. As it turns out, the key tool for the proof that the original and the new construction of induced connectives coincide in every DFS is the *reduction* of an n-place quantifier to a monadic quantifier. We will see that n-place quantification can be reduced to one-place quantification on a suitable base set.

Why is it always possible to reduce n-place quantification to one-place quantification? To see this, let us recall that for arbitrary sets A, B, C, it always holds that

$$A^{B\times C} \cong (A^B)^C \,,$$

a relationship known as "currying". We then have

$$\mathcal{P}(E)^n \cong (\mathbf{2}^E)^n \cong \mathbf{2}^{E\times n} \cong \mathcal{P}(E\times n) \cong \mathcal{P}(E\times\{1,\ldots,n\})\,,$$

and similarly

$$\widetilde{\mathcal{P}}(E)^n \cong (\mathbf{I}^E)^n \cong \mathbf{I}^{E\times n} \cong \widetilde{\mathcal{P}}(E\times n) \cong \widetilde{\mathcal{P}}(E\times\{1,\ldots,n\})\,.$$

For convenience, we have replaced $E\times n$ by $E\times\{1,\ldots,n\}$, which better suits our habit of numbering the arguments of an n-place quantifier from 1 to n (rather than from 0 to $n-1$).

This suggests that by exploiting the bijection $\mathcal{P}(E)^n \cong \mathcal{P}(E\times\{1,\ldots,n\})$, n-place quantification as expressed by some $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ can be replaced by one-place quantification using a one-place quantifier $\langle Q\rangle : \mathcal{P}(E\times\{1,\ldots,n\}) \longrightarrow \mathbf{I}$, and that conversely $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ can always be recovered from $\mathcal{F}(\langle Q\rangle) : \widetilde{\mathcal{P}}(E\times\{1,\ldots,n\}) \longrightarrow \mathbf{I}$. To this end, let us introduce some formal machinery.

Definition 53 (E_n)
If E is some set and $n\in\mathbb{N}$, we will abbreviate

$$E_n = E\times\{1,\ldots,n\}\,.$$

This is only to provide a short notation for the base sets of the resulting monadic quantifiers. For $n=0$, we obtain the empty product $E_0=\varnothing$.

Definition 54 ($\iota_i^{n,E}$)
Let E be a given set, $n\in\mathbb{N}\setminus\{0\}$ and $i\in\{1,\ldots,n\}$. By $\iota_i^{n,E} : E \longrightarrow E_n$ we denote the inclusion defined by

$$\iota_i^{n,E}(e) = (e,i)\,,$$

for all $e\in E$.

Notes

- Being an inclusion, $\iota_i^{n,E}$ is of course injective (one-to-one), a fact which we will use repeatedly. It is also apparent that

$$\operatorname{Im}\iota_i^{n,E} = \{(e,i) : e\in E\}\,. \tag{10}$$

- the crisp extension (powerset mapping, see Def. 17) of $\iota_i^{n,E} : E \longrightarrow E_n$ will be denoted by $\widehat{\iota}_i^{n,E} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E_n)$. The inverse image mapping of $\iota_i^{E,n}$ will be denoted $(\iota_i^{E,n})^{-1} : \mathcal{P}(E_n) \longrightarrow \mathcal{P}(E)$, cf. Def. 33.

- Let us also remark that for all $n, n' \in \mathbb{N}$ such that $i \leq n$, $i \leq n'$,

$$\iota_i^{n,E}(e) = \iota_i^{n',E}(e) \tag{11}$$

for all $e \in E$. We have only used the superscript n in $\iota_i^{n,E}$ because we wanted the notation to disambiguate the codomain (or range) of $\iota_i^{n,E}$, viz E_n.

We can use these injections to define the monadic quantifier $\langle Q \rangle$ of interest.

Definition 55 ($\langle Q \rangle$)
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-place semi-fuzzy quantifier, where $n > 0$. The semi-fuzzy quantifier $\langle Q \rangle : \mathcal{P}(E_n) \longrightarrow \mathbf{I}$ is defined by

$$\langle Q \rangle(X) = Q((\iota_1^{n,E})^{-1}(X), \ldots, (\iota_n^{n,E})^{-1}(X))$$

for all $X \in \mathcal{P}(E_n)$.

For fuzzy quantifiers, $\langle \widetilde{Q} \rangle$ is defined similarly, using the fuzzy inverse image mapping $(\hat{\hat{\iota}}_i^{n,E})^{-1} : \widetilde{\mathcal{P}}(E_n) \longrightarrow \widetilde{\mathcal{P}}(E)$ of $\iota_i^{n,E}$:

Definition 56 ($\langle \widetilde{Q} \rangle$)
Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ a fuzzy quantifier, $n > 0$. The fuzzy quantifier $\langle \widetilde{Q} \rangle : \widetilde{\mathcal{P}}(E_n) \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{Q}(X) = \widetilde{Q}((\hat{\hat{\iota}}_1^{n,E})^{-1}(X), \ldots, (\hat{\hat{\iota}}_n^{n,E})^{-1}(X)),$$

for all $X \in \widetilde{\mathcal{P}}(E_n)$.

We now wish to establish the relationship between $\langle Q \rangle$ and Q (semi-fuzzy case) and $\langle \widetilde{Q} \rangle$ and $\widetilde{Q}$ (fuzzy case). In the fuzzy case, this will involve the use of the induced fuzzy disjunction (or union). Let us first introduce a concise notation for iterated unions of a quantifier's arguments:

Definition 57 ($Q\cup^k$ and $\widetilde{Q}\widetilde{\widetilde{\cup}}^k$)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$ and $k \in \mathbb{N} \setminus \{0\}$. The semi-fuzzy quantifier $Q\cup^k : \mathcal{P}(E)^{n+k-1} \longrightarrow \mathbf{I}$ is inductively defined as follows:

a. $Q\cup^1 = Q$;

b. $Q\cup^k = Q\cup^{k-1}\cup$ *if* $k > 1$.

For fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\widetilde{Q}\widetilde{\widetilde{\cup}}^k : \widetilde{\mathcal{P}}(E)^{n+k-1} \longrightarrow \mathbf{I}$ is defined analogously.

Theorem 33 (Reduction to monadic semi-fuzzy quantifiers)
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ where $n > 0$,

$$Q = \langle Q \rangle \cup^n \circ \bigtimes_{i=1}^{n} \widehat{\iota}_i^{n,E}.$$

(Proof: B.1, p.83+)

Note. This demonstrates that n-place crisp (or "semi-fuzzy") quantification, where $n > 0$, can always be reduced to one-place quantification. Would we allow for empty base sets, this would also go through for $n = 0$; we only had to exclude this case because $E_0 = \varnothing$, i.e. in this case we have $\langle Q \rangle : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which we reject as a semi-fuzzy quantifier because the base set is empty.

In the following, we will need a notation for multiple fuzzy disjunctions and fuzzy unions with unambiguous bracketing order (we will not presuppose associativity).

Definition 58 **(**$[\widetilde{\widetilde{\vee}}]_{i=1}^{n} x_i$**)**
Let $\widetilde{\widetilde{\vee}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *be a given mapping,* $n \in \mathbb{N} \setminus \{0\}$ *and* $x_1, \ldots, x_n \in \mathbf{I}$*.* $[\widetilde{\widetilde{\vee}}]_{i=1}^{n} x_i \in \mathbf{I}$ *is inductively defined as follows.*

a. $[\widetilde{\widetilde{\vee}}]_{i=1}^{1} x_i = x_1$.

b. $[\widetilde{\widetilde{\vee}}]_{i=1}^{n} x_i = x_1 \,\widetilde{\widetilde{\vee}}\, [\widetilde{\widetilde{\vee}}]_{i=1}^{n-1} x_{i+1}$ *if* $n > 1$.

And similarly for a fuzzy union $\widetilde{\widetilde{\cup}}$:

Definition 59
Assume $\widetilde{\widetilde{\vee}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is some mapping and* $\widetilde{\widetilde{\cup}}$ *is the fuzzy set operator element-wise defined in terms of* $\widetilde{\widetilde{\vee}}$*. Further suppose* E *is some set and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *are fuzzy subsets of* E*.* $[\widetilde{\widetilde{\cup}}]_{i=1}^{n} X_i \in \widetilde{\mathcal{P}}(E)$ *is inductively defined by*

a. $[\widetilde{\widetilde{\cup}}]_{i=1}^{1} X_i = X_1$

b. $[\widetilde{\widetilde{\cup}}]_{i=1}^{n} X_i = x_1 \,\widetilde{\widetilde{\cup}}\, [\widetilde{\widetilde{\cup}}]_{i=1}^{n-1} X_{i+1}$ *if* $n > 1$.

Notes

- $[\widetilde{\widetilde{\cup}}]_{i=1}^{n}$ permits us to express multiple fuzzy unions with a uniquely defined bracketing order. This is important when knowledge about the associativity of $\widetilde{\widetilde{\vee}}$ (and hence $\widetilde{\widetilde{\cup}}$) is lacking. The particular bracketing order chosen ensures that for every fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, $n \in \mathbb{N} \setminus \{0\}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,

$$\widetilde{Q}\widetilde{\widetilde{\cup}}^{n}(X_1, \ldots, X_n) = \widetilde{Q}([\widetilde{\widetilde{\cup}}]_{i=1}^{n} X_i)\,. \tag{12}$$

- Because of the element-wise definition of $\widetilde{\widetilde{\cup}}$ in terms of $\widetilde{\widetilde{\vee}}$ and because of the parallel definitions of $[\widetilde{\widetilde{\vee}}]_{i=1}^{n}$ and $[\widetilde{\widetilde{\cup}}]_{i=1}^{n}$, it is apparent that

$$\mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^{n} X_i}(e) = [\widetilde{\widetilde{\vee}}]_{i=1}^{n} \mu_{X_i}(e)\,. \tag{13}$$

  for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $e \in E$.

A theorem analogous to Th-33 can be proven in the fuzzy case:

Theorem 34 (Reduction to monadic fuzzy quantifiers)
Suppose $\widetilde{\widetilde{\vee}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *has* $x \,\widetilde{\widetilde{\vee}}\, 0 = 0 \,\widetilde{\widetilde{\vee}}\, x = x$ *for all* $x \in \mathbf{I}$*, and* $\widetilde{\widetilde{\cup}}$ *is the fuzzy union element-wise defined in terms of* $\widetilde{\widetilde{\vee}}$*. For every fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*,* $n > 0$*,*

$$\widetilde{Q} = \langle \widetilde{Q} \rangle \widetilde{\widetilde{\cup}}^n \circ \bigtimes_{i=1}^{n} \hat{\hat{\imath}}_i^{n,E} .$$

In other words,

$$\widetilde{Q}(X_1, \ldots, X_n) = \langle \widetilde{Q} \rangle ([\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\imath}}_i^{n,E}(X_i))$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*.*
(Proof: B.2, p.84+)

Note. The theorem shows that it is possible to reduce n-place quantification to one-place quantification in the fuzzy case as well. Nullary quantifiers ($n = 0$) had to be excluded for same reason than with Th-33.
The central fact which we need for the subsequent proofs is the following:

Theorem 35
Suppose $\mathcal{F}$ *is a QFM with the following properties:*

a. $x \,\widetilde{\vee}\, 0 = 0 \,\widetilde{\vee}\, x = x$ *for all* $x \in \mathbf{I}$*;*

b. for all semi-fuzzy quantifiers $Q : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *where* $n > 0$*,*

$$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\widetilde{\cup}};$$

c. $\mathcal{F}$ *satisfies* (DFS 9) *(functional application);*

d. If E, E' *are nonempty sets and* $f : E \longrightarrow E'$ *is an injective mapping, then* $\widehat{\mathcal{F}}(f) = \hat{\hat{f}}$*, i.e.* $\mathcal{F}$ *coincides with the standard extension principle on injections.*

Then for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$*,*

$$\mathcal{F}(\langle Q \rangle) = \langle \mathcal{F}(Q) \rangle .$$

(Proof: B.3, p.86+)

Based on these results, it is now easy to prove the desired result that both constructions of induced truth functions coincide in every DFS.

Theorem 36 (Equivalence of induced truth function constructions)
In every DFS $\mathcal{F}$*,*

$$\widetilde{\mathcal{F}} = \widetilde{\widetilde{\mathcal{F}}}.$$

(Proof: B.4, p.87+)

In the same way than with $\widetilde{\mathcal{F}}$, we can define fuzzy set operations $\widetilde{\widetilde{\cap}}$ (induced fuzzy intersection), $\widetilde{\widetilde{\cup}}$ (induced fuzzy union) and $\widetilde{\widetilde{\neg}}$ (induced fuzzy complement) by element-wise applying $\widetilde{\widetilde{\wedge}}$, $\widetilde{\widetilde{\vee}}$ and $\widetilde{\widetilde{\neg}}$ (the fuzzy negation), respectively. In turn, these can be used to define $\widetilde{\widetilde{\neg}}\,Q$, $Q\widetilde{\widetilde{\neg}}$ and $Q\widetilde{\widetilde{\cap}}$ in the straightforward way, i.e. simply by replacing $\widetilde{\neg}$ and $\widetilde{\cap}$ in Def. 10, Def. 11 and Def. 14 with $\widetilde{\widetilde{\neg}}$ and $\widetilde{\widetilde{\cap}}$. Other constructions like dualisation $Q\widetilde{\widetilde{\square}}$ and $\widetilde{Q}\widetilde{\widetilde{\square}}$, intersections in arguments $Q\widetilde{\widetilde{\cup}}$, symmetrical difference $\widetilde{Q}\widetilde{\widetilde{\triangle}}A$ based on $\widetilde{\widetilde{\mathrm{xor}}}$, conjunction $Q\widetilde{\widetilde{\wedge}}Q'$ and disjunction $Q\widetilde{\widetilde{\vee}}Q'$ are obtained analogously. We can then ask if by adapting the definitions of $\widetilde{\neg}\,Q$, $Q\widetilde{\neg}$ and $Q\widetilde{\cap}$ to the new construction of induced truth functions, we still obtain the same class of DFS models, or if the adapted axiom system has different models as the original axiom set (based on $\widetilde{\mathcal{F}}$). Happily, nothing is altered, as the next theorem states:

Theorem 37 (Adapting the DFS Axioms to the New Induced Connectives)
Let us denote by DFS' the modified set of DFS axioms where $\widetilde{\neg}$ is replaced by $\widetilde{\widetilde{\neg}}$, and $\widetilde{\cap}$ is replaced by $\widetilde{\widetilde{\cap}}$. DFS and DFS' are equivalent, i.e. they possess the same models.
(Proof: B.5, p.89+)

3.3 The Revised Set of DFS Axioms

We have introduced an alternative construction of induced connectives of a QFM and shown that it coincides with our original construction in every DFS. In addition, it has been shown that the definition of DFSes is not altered if the DFS axioms are changed so that they are based on the new constructions. The rationale was that the new construction $\widetilde{\widetilde{\mathcal{F}}}$ depends on the behaviour of $\mathcal{F}$ for quantifiers of arities $n \leq 1$ only. As we shall see, this renders it possible to prove the desired properties of conjunction and disjunction without having to use some of the axioms which express properties of multi-place quantifiers, viz. “argument insertion” (DFS 7) and “argument transposition” (DFS 4). Based on the known properties of disjunction and conjunction, it can then be shown that these axioms are dependent on the remaining axioms and can hence be eliminated, to justify the reduced set of independent axioms. This reduction seems not to be possible with the original definition of induced connectives. Let us now state the revised set of DFS axioms.

Definition 60 (Alternative Characterisation of DFSes)
The revised set of DFS axioms comprises the following conditions (Z-1) to (Z-6). For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(Z-2)
Dualisation	$\mathcal{F}(Q\widetilde{\widetilde{\square}}) = \mathcal{F}(Q)\widetilde{\widetilde{\square}}$ $n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\widetilde{\cup}}$ $n > 0$	(Z-4)
Preservation of monotonicity	Q *noninc. in* n*-th arg* $\Rightarrow \mathcal{F}(Q)$ *noninc. in* n*-th arg,* $n > 0$	(Z-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(Z-6)

Compared to the original set of DFS axioms, DFS-1 (Preservation of constants) has been modified to require proper generalisation in the case $n = 1$ as well. DFS-2 (Compatibility with π_*) has been generalised to arbitrary projection quantifiers π_e. DFS-3 (External negation) and DFS-5 (Internal Complementation) have been combined into the single axiom Z-3, which requires the compatibility of $\mathcal{F}$ with dualisation. DFS-4 (Argument transposition) has been omitted. DFS-6 (Internal meets) has been replaced with Z-4 (Internal joins), which states the compatibility of $\mathcal{F}$ with the dual construction. DFS-7 (Argument insertion) has been omitted. DFS-8 (Preservation of monotonicity) and DFS-9 (functional application) have not been altered and become the new axioms Z-5 and Z-6. In addition, the "old" construction $\widetilde{\mathcal{F}}$ of induced fuzzy truth functions has consistently been replaced with the "new" construction $\widetilde{\widetilde{\mathcal{F}}}$.

3.4 Equivalence of Old and New Characterisation of DFSes

It is this revised axiom set (Z-1) to (Z-6) of which we intend to prove independency. However, before doing this, we have to establish the equivalence of the "new" and "old" characterisation of DFSes.

Theorem 38 (Equivalence of Characterisations)
For every QFM $\mathcal{F}$, the following are equivalent:

a. $\mathcal{F}$ satisfies (DFS 1) to (DFS 9)

b. $\mathcal{F}$ satisfies (Z-1) to (Z-6)

(Proof: B.6, p.90+)

4 Characterisation of the class of $\mathcal{M}_{\mathcal{B}}$-DFSes

Having shown the equivalence of the new axiom set with the original set of DFS axioms, we are now ready to investigate the independence of the revised axiom set. We are interested in establishing the independence of the axiom set because an independent axiom set provides a *minimal* characterisation of DFSes, and hence simplifies proofs whether a given QFM is a DFS.

In order to show that the new axiom set is independent, we have to prove that none of the axioms (Z-1) to (Z-6) is entailed by the remaining axioms. This can be done by showing that for each (Z-i), there exists a QFM $\mathcal{F}$ which satisfies all axioms (Z-j), where $j \neq i$, except for (Z-i).

In order to find suitable QFM candidates, we introduce the class of $\mathcal{M}_{\mathcal{B}}$-QFMs, i.e. the class of QFMs definable in terms of three-valued cuts of argument sets and subsequent aggregation of quantification results by applying the fuzzy median. We hence generalise the construction successfully used in [9] to define DFSes. We then investigate necessary and sufficient conditions under which the resulting QFMs satisfy each of (Z-1) to (Z-6). Based on these known properties of $\mathcal{M}_{\mathcal{B}}$-QFMs, it will then be easy to construct the counterexamples needed for the independence proof. As a by-product of this investigation, we obtain a characterisation of the class of $\mathcal{M}_{\mathcal{B}}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping $\mathcal{B}$.

In order to define the unrestricted class of $\mathcal{M}_{\mathcal{B}}$-QFMs, let us recall some concepts introduced in [9]. We have decided to somewhat change notation in this report in order to clarify the distinction between three-valued cuts and three-valued cut ranges (see below).

We shall first introduce some concepts related to three-valued subsets. We will model three-valued subsets of a given base set E in a way analogous to fuzzy subsets, i.e. we shall assume that each three-valued subset X of E is uniquely characterised by its membership function $\nu_X : E \longrightarrow \{0, \frac{1}{2}, 1\}$ (we use the symbol ν_X rather than μ_X in order make unambiguous that the membership function is three-valued). The collection of three-valued subsets of a given E will be denoted $\breve{\mathcal{P}}(E)$; we shall assume that $\breve{\mathcal{P}}(E)$ is a set, and clearly we have $\breve{\mathcal{P}}(E) \cong \{0, \frac{1}{2}, 1\}^E$. As in the case of fuzzy subsets, it might be convenient to identify three-valued subsets and their membership functions, i.e. to stipulate $\breve{\mathcal{P}}(E) = \{0, \frac{1}{2}, 1\}^E$. However, we will again not enforce this identification.

We shall assume that each crisp subset $X \in \mathcal{P}(E)$ can be viewed as a three-valued subset of E, and that each three-valued subset X of E can be viewed as a fuzzy subset of E. E.g., we will at times use the same symbol X to denote a particular crisp subset of E, as well as the corresponding three-valued and fuzzy subsets. If one chooses to identify membership functions and three-valued/fuzzy subsets, then the crisp subset X is distinct from its representation as a three-valued or fuzzy subset, which corresponds to its characteristic function χ_X. In this case, it is understood that the appropriate transformations (i.e., using characteristic function) are performed whenever X is substituted for a three-valued or fuzzy subset.

Our construction of DFSes in [9] relies heavily on the fact that each three-valued subset $X \in \breve{\mathcal{P}}(E)$ can be represented by a closed range of crisp subsets of E as follows.

Definition 61 (Crisp range of a three-valued set)
Suppose E is some set and $X \in \breve{\mathcal{P}}(E)$ is a three-valued subset of E. We associate with X crisp subsets $X^{\min}, X^{\max} \in \mathcal{P}(E)$, defined by

$$X^{\min} = \{e \in E : \nu_X(e) = 1\}$$
$$X^{\max} = \{e \in E : \nu_X(e) \geq \tfrac{1}{2}\}.$$

Based on $X^{\min}$ and $X^{\max}$, we associate with X a range of crisp sets $\mathcal{T}(X) \subseteq \mathcal{P}(E)$ defined by

$$\mathcal{T}(X) = \{Y \in \mathcal{P}(E) : X^{\min} \subseteq Y \subseteq X^{\max}\} .$$

Now that we have the required concepts on three-valued sets available, we can turn to three-valued cuts.

Definition 62 ($t_\gamma(x)$)
For every $x \in \mathbf{I}$ and $\gamma \in \mathbf{I}$, the three-valued cut of x at γ is defined by

$$t_\gamma(x) = \begin{cases} 1 & : \quad x \geq \frac{1}{2} + \frac{1}{2}\gamma \\ \frac{1}{2} & : \quad \frac{1}{2} - \frac{1}{2}\gamma < x < \frac{1}{2} + \frac{1}{2}\gamma \\ 0 & : \quad x \leq \frac{1}{2} - \frac{1}{2}\gamma \end{cases}$$

if $\gamma > 0$, and

$$t_0(x) = \begin{cases} 1 & : \quad x > \frac{1}{2} \\ \frac{1}{2} & : \quad x = \frac{1}{2} \\ 0 & : \quad x < \frac{1}{2} \end{cases}$$

in the case that $\gamma = 0$.

The cutting parameter γ can be conceived of as a degree of "cautiousness" because the larger γ becomes, the closer $t_\gamma(x)$ will approach the "undecided" result of $\frac{1}{2}$. Hence $t_{\gamma'}(x) \preceq_c t_\gamma(x)$ whenever $\gamma \leq \gamma'$. The three-valued cut mechanism can be extended to three-valued cuts of fuzzy subsets by applying it element-wise to the membership functions:

Definition 63 ($\mathrm{T}_\gamma(X)$)
Suppose E is some set and $X \in \widetilde{\mathcal{P}}(E)$ a subset of E. The three-valued cut of X at $\gamma \in \mathbf{I}$ is the three-valued subset $\mathrm{T}_\gamma(X) \in \breve{\mathcal{P}}(E)$ defined by

$$\nu_{\mathrm{T}_\gamma(X)}(e) = t_\gamma(\mu_X(e)) ,$$

for all $e \in E$.

We will most often not need the three-valued cut directly, but rather the crisp range corresponding to the cut, which can be represented by a pair of α-cuts. The definition of α-cuts and strict α-cuts is a usual:

Definition 64 (Alpha cuts)
Let E be a given set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\alpha \in \mathbf{I}$. By $(X)_{\geq\alpha} \in \mathcal{P}(E)$ we denote the α-cut

$$(X)_{\geq\alpha} = \{x \in E : \mu_X(x) \geq \alpha\} .$$

Definition 65 (Strict alpha cuts)
Let $X \in \widetilde{\mathcal{P}}(E)$ be given and $\alpha \in \mathbf{I}$. By $(X)_{>\alpha} \in \mathcal{P}(E)$ the strict α-cut

$$(X)_{>\alpha} = \{x \in E : \mu_X(x) > \alpha\} .$$

The α-cuts are related to the the crisp range of a three-valued cut as follows.

Definition 66 ($(X)^{\min}_\gamma$, $(X)^{\max}_\gamma$, $\mathcal{T}_\gamma(X)$)
Let E some set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\gamma \in \mathbf{I}$. $(X)^{\min}_\gamma, (X)^{\max}_\gamma \in \mathcal{P}(E)$ and $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ are defined by

$$\begin{aligned}
(X)^{\min}_\gamma &= (\mathrm{T}_\gamma(X))^{\min} \\
(X)^{\max}_\gamma &= (\mathrm{T}_\gamma(X))^{\max} \\
\mathcal{T}_\gamma(X) &= \mathcal{T}(\mathrm{T}_\gamma(X)) = \{Y : (X)^{\min}_\gamma \subseteq Y \subseteq (X)^{\max}_\gamma\}\,.
\end{aligned}$$

Hence if $\gamma > 0$, then

$$\begin{aligned}
(X)^{\min}_\gamma &= (X)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \\
(X)^{\max}_\gamma &= (X)_{> \frac{1}{2}-\frac{1}{2}\gamma}\,,
\end{aligned}$$

and in the case that $\gamma = 0$,

$$\begin{aligned}
(X)^{\min}_0 &= (X)_{> \frac{1}{2}} \\
(X)^{\max}_0 &= (X)_{\geq \frac{1}{2}}\,.
\end{aligned}$$

The basic idea is to view the crisp ranges corresponding to three-valued cuts of a fuzzy subset as providing a number of alternatives to be checked. For example, in order to evaluate a semi-fuzzy quantifier Q at a certain cut level γ, we have to consider all choices of $Q(Y_1, \ldots, Y_n)$, where $Y_i \in \mathcal{T}_\gamma(X_i)$. The set of results obtained in this way must then be aggregated to a single result in the unit interval. The generalised fuzzy median introduced on p. 27 is particularly suited to carry out this aggregation because the resulting fuzzification mechanisms are consistent with Kleene's three-valued logic. This is beneficial because the three-valued portion of each Standard-DFS corresponds to Kleene's three-valued logic, cf. [9, p.49].

We can use the crisp ranges which correspond to the three-valued cuts of a quantifier's argument sets to define a family of QFMs $(\bullet)_\gamma$, indexed by the cautiousness parameter $\gamma \in \mathbf{I}$:

Definition 67 ($(\bullet)_\gamma$)
For every $\gamma \in \mathbf{I}$, we denote by $(\bullet)_\gamma$ the QFM defined by

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Note. From Def. 46, it is apparent that for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$,

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}(Q^{\min}_\gamma(X_1, \ldots, X_n), Q^{\max}_\gamma(X_1, \ldots, X_n)) \tag{14}$$

where $Q^{\min}_\gamma, Q^{\max}_\gamma$ are defined by

$$Q^{\min}_\gamma(X_1, \ldots, X_n) = \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} \tag{15}$$

$$Q^{\max}_\gamma(X_1, \ldots, X_n) = \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}\,. \tag{16}$$

This reformulation will be useful in some of the proofs to follow.

None of the QFMs $(\bullet)_\gamma$ is a DFS, the fuzzy median suppresses too much structure. Nevertheless, these QFMs prove useful in defining DFSes. The basic idea is that in order to interpret $\mathcal{F}(Q)(X_1, \ldots, X_n)$, we consider the results obtained at all levels of cautiousness γ, i.e. the γ-indexed family $(Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}}$. We can then apply various aggregation operators on these γ-indexed results to obtain new QFMs, which might be DFSes. We will use the symbol $\mathcal{B}$ for such aggregation operators, and the resulting QFM will be denoted $\mathcal{M}_{\mathcal{B}}$. In order to determine the proper domain for the aggregation operator, we need to recall some monotonicity properties of $Q_\gamma(X_1, \ldots, X_n)$.

Theorem 39 (Monotonicity properties of $(\bullet)_\gamma$)
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. If $Q_0(X_1, \ldots, X_n) > \frac{1}{2}$, then $Q_\gamma(X_1, \ldots, X_n)$ is monotonically nonincreasing in γ and $Q_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

b. If $Q_0(X_1, \ldots, X_n) = \frac{1}{2}$, then $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. If $Q_0(X_1, \ldots, X_n) < \frac{1}{2}$, then $Q_\gamma(X_1, \ldots, X_n)$ is monotonically nondecreasing in γ and $Q_\gamma(X_1, \ldots, X_n) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

(Proof: see [9, Th-42, p. 61])

Definition 68 ($\mathbb{B}$)
$\mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^-$ and $\mathbb{B} \subseteq \mathbf{I}^{\mathbf{I}}$ are defined by

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) > \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [\tfrac{1}{2}, 1] \text{ and } f \text{ nonincreasing } \}\\
\mathbb{B}^{\frac{1}{2}} &= \{f \in \mathbf{I}^{\mathbf{I}} : f(x) = \tfrac{1}{2} \text{ for all } x \in \mathbf{I} \}\\
\mathbb{B}^- &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) < \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [0, \tfrac{1}{2}] \text{ and } f \text{ nondecreasing } \}\\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^- .
\end{aligned}$$

Note. In the following, we shall denote by $\mathrm{c}_a : \mathbf{I} \longrightarrow \mathbf{I}$ the constant mapping

$$\mathrm{c}_a(x) = a\,, \tag{17}$$

for all $a, x \in \mathbf{I}$. Using this notation, apparently $\mathbb{B}^{\frac{1}{2}} = \{\mathrm{c}_{\frac{1}{2}}\}$.

In terms of these abbreviations, we then have

Theorem 40
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. If $Q_0(X_1, \ldots, X_n) > \frac{1}{2}$, then $(Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}} \in \mathbb{B}^+$;

b. If $Q_0(X_1, \ldots, X_n) = \frac{1}{2}$, then $(Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}} \in \mathbb{B}^{\frac{1}{2}}$ (i.e. constantly $\frac{1}{2}$);

c. If $Q_0(X_1, \ldots, X_n) < \frac{1}{2}$, then $(Q_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}} \in \mathbb{B}^-$.

Proof: Apparent from Th-39.

In particular, we know that regardless of $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B} .$$

In addition, the set $\mathbb{B}$ is exhausted by $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$ in the sense that for each $f \in \mathbb{B}$ there exists choices of Q and $X_1, \dots, X_n$ such that $f = (Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$, as we shall now state:

Theorem 41
Suppose f is some mapping $f \in \mathbb{B}$.

a. *Define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by*

$$Q(Y) = f(\inf Y) \qquad \text{(Th-41.a.i)}$$

for all $Y \in \mathcal{P}(\mathbf{I})$ and let $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ the fuzzy subset with membership function

$$\mu_X(z) = \tfrac{1}{2} + \tfrac{1}{2} z \qquad \text{(Th-41.a.ii)}$$

for all $z \in \mathbf{I}$. Then

$$Q_\gamma(X) = f(\gamma)$$

for all $\gamma \in \mathbf{I}$.

b. *Define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by*

$$Q(Y) = f(\sup Y) \qquad \text{(Th-41.b.i)}$$

for all $Y \in \mathcal{P}(\mathbf{I})$ and let $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ the fuzzy subset with membership function

$$\mu_X(z) = \tfrac{1}{2} - \tfrac{1}{2} z \qquad \text{(Th-41.b.ii)}$$

for all $z \in \mathbf{I}$. Then

$$Q_\gamma(X) = f(\gamma)$$

for all $\gamma \in \mathbf{I}$.

(Proof: C.1, p.103+)

Let us now return to the original idea of aggregating over the results of $Q_\gamma(X_1, \dots, X_n)$ for all choices of the cautiousness parameter. By the above theorems, we know that a corresponding aggregation operator must be defined on $\mathbb{B}$, because $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}$, and no smaller set $A \subseteq \mathbf{I}^{\mathbf{I}}$ will suffice. Relative to an aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$, we define the QFM $\mathcal{M}_{\mathcal{B}}$ which corresponds to $\mathcal{B}$ as follows.

Definition 69 ($\mathcal{M}_{\mathcal{B}}$)
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. The QFM $\mathcal{M}_{\mathcal{B}}$ is defined by

$$\mathcal{M}_{\mathcal{B}}(Q)(X_1, \dots, X_n) = \mathcal{B}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) ,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1 \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{M}_\mathcal{B}$-QFMs we mean the class of all QFMs $\mathcal{M}_\mathcal{B}$ defined in this way. It is apparent that if we do not impose restrictions on admissible choices of $\mathcal{B}$, the resulting QFMs will often fail to be DFSes. Let us now give an example of an $\mathcal{M}_\mathcal{B}$-QFM which is a DFS.

Definition 70 ($\mathcal{M}$)
The $\mathcal{M}_\mathcal{B}$-QFM $\mathcal{M}$ is defined by

$$\mathcal{M}(Q)(X_1, \ldots, X_n) = \int_0^1 Q_\gamma(X_1, \ldots, X_n)\, d\gamma\,,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Theorem 42
$\mathcal{M}$ is a standard DFS.
(Proof: see [9, Th-44, p. 63])

We are now interested in stating necessary and sufficient conditions on $\mathcal{B}$ for $\mathcal{M}_\mathcal{B}$ to be a DFS. In order to do so, we first need to introduce some constructions on $\mathbb{B}$.

Definition 71
Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotonic mapping (i.e., either nondecreasing or nonincreasing). The mappings $f^\flat,\ f^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$ are defined by:

$$f^\sharp = \begin{cases} \lim_{y \to x^+} f(y) & : \quad x < 1 \\ f(1) & : \quad x = 1 \end{cases}$$
$$f^\flat = \begin{cases} \lim_{y \to x^-} f(y) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases}$$

for all $x \in \mathbf{I}$.

Notes

- It is apparent that if $f \in \mathbb{B}$, then $f^\sharp \in \mathbb{B}$ and $f^\flat \in \mathbb{B}$, which can be easily checked for the cases $f \in \mathbb{B}^+$, $f \in \mathbb{B}^-$ and $f \in \mathbb{B}^{\frac{1}{2}}$.

Let us remark that $f^\sharp$ and $f^\flat$ are well-defined, i.e. the limites in the above expressions exist, regardless of f. This is apparent from the following observation.

Theorem 43
Suppose $f : [a, b] \longrightarrow [c, d]$ is a mapping, where $a \leq b$, $c \leq d$.

a. *If f is nondecreasing, then $\lim\limits_{x \to z^-} f(x)$ exists for all $z \in (a, b]$, and*

$$\lim_{x \to z^-} f(x) = \sup\{f(x) : x < z\}\,.$$

b. *If f is nondecreasing, then $\lim\limits_{x \to z^+} f(x)$ exists for all $z \in [a, b)$, and*

$$\lim_{x \to z^+} f(x) = \inf\{f(x) : x > z\}\,.$$

c. If f is nonincreasing, then $\lim_{x \to z^-} f(x)$ exists for all $z \in (a, b]$, and

$$\lim_{x \to z^-} f(x) = \inf\{f(x) : x < z\} .$$

d. If f is nonincreasing, then $\lim_{x \to z^+} f(x)$ exists for all $z \in [a, b)$, and

$$\lim_{x \to z^+} f(x) = \sup\{f(x) : x > z\} .$$

(Proof: C.2, p.106+)

Let us also introduce a number of coefficients which describe certain aspects of a mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Definition 72
Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotone mapping (i.e., either nondecreasing or nonincreasing). We define

$$f_0^* = \lim_{\gamma \to 0^+} f(\gamma) \tag{18}$$
$$f_*^0 = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{19}$$
$$f_*^{\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \tag{20}$$
$$f_1^* = \lim_{\gamma \to 1^-} f(\gamma) \tag{21}$$
$$f_*^1 = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} . \tag{22}$$

As usual, we stipulate that $\sup \varnothing = 0$ *and* $\inf \varnothing = 1$.

Note. All limites in the definition of these coefficients are known to exist by Th-43.

Based on these concepts, we can now state a number of axioms governing the behaviour of reasonable choices of $\mathcal{B}$.

Definition 73 (Axioms B-1 to B-5)
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathcal{B}$, we define the following conditions on $\mathcal{B}$:

$$\mathcal{B}(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{B-1}$$
$$\mathcal{B}(1 - f) = 1 - \mathcal{B}(f) \tag{B-2}$$

If $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$, then (B-3)

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases}$$

$$\mathcal{B}(f^\sharp) = \mathcal{B}(f^\flat) \tag{B-4}$$

If $f \le g$, then $\mathcal{B}(f) \le \mathcal{B}(g)$ (B-5)

Let us briefly comment on the meaning of these conditions. (B-1) states that $\mathcal{B}$ preserves constants. In particular, all $\mathcal{M}_{\mathcal{B}}$-QFMs such that $\mathcal{B}$ satisfies (B-1) coincide on three-valued argument sets. (B-2) expresses that $\mathcal{B}$ is compatible to the standard negation $1 - x$. (B-3) ensures that all conforming $\mathcal{M}_{\mathcal{B}}$-QFMs coincide on three-valued quantifiers. (B-4) expresses some kind of insensitivity property of $\mathcal{B}$, which turns out to be crucial with respect to $\mathcal{M}_{\mathcal{B}}$ satisfying *functional application* (Z-6). Finally, (B-5) expresses that $\mathcal{B}$ is monotonic, i.e. application of $\mathcal{B}$ preserves inequations. Let us now investigate how these conditions on $\mathcal{B}$ are related to the DFS axioms. As we shall see, (B-1) to (B-5) are necessary and sufficient for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.

Theorem 44
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-1)*, then* $\mathcal{M}_{\mathcal{B}}(Q)(X_1, \dots, X_n) = \mathcal{M}(Q)(X_1, \dots, X_n)$ *for all three-valued argument sets* $X_1, \dots, X_n \in \breve{\mathcal{P}}(E)$.
(Proof: C.3, p.108+)

As a corollary, we obtain the following.

Theorem 45
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-1)*, then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-1).
(Proof: C.4, p.109+)

Theorem 46
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-3)*, then* $\mathcal{M}_{\mathcal{B}}(Q) = \mathcal{M}(Q)$ *for all three-valued semi-fuzzy quantifiers* $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$.
(Proof: C.5, p.109+)

As a corollary, we then have

Theorem 47
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-3)*, then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-2).
(Proof: C.6, p.110+)

In addition, we know that if $\mathcal{B}$ satisfies (B-3), then $\widetilde{\widetilde{\mathcal{M}}}_{\mathcal{B}}(\neg) = \widetilde{\widetilde{\mathcal{M}}}(\neg) = \widetilde{\mathcal{M}}(\neg) = 1 - x$ and similarly $\widetilde{\widetilde{\mathcal{M}}}_{\mathcal{B}}(\wedge) = \min$ and $\widetilde{\widetilde{\mathcal{M}}}_{\mathcal{B}}(\vee) = \max$. Because the induced extension principle $\widehat{\mathcal{M}}_{\mathcal{B}}$ is obtained from applying $\mathcal{M}_{\mathcal{B}}$ to a two-valued quantifier, we also know that in this case, $\widehat{\mathcal{M}}_{\mathcal{B}}$ is the standard extension principle.

Theorem 48
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-2) *and* (B-3)*, then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-3).
(Proof: C.7, p.110+)

Theorem 49
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-3)*, then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-4).
(Proof: C.8, p.112+)

Theorem 50
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-5)*, then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-5).
(Proof: C.9, p.115+)

Theorem 51
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-3), (B-4) *and* (B-5), *then* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-6).
(Proof: C.10, p.116+)

We can summarise these results as follows:

Theorem 52
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-1) *to* (B-5), *then* $\mathcal{M}_{\mathcal{B}}$ *is a Standard DFS.*
(Proof: C.11, p.121+)

We hence know that the conditions (B-1) to (B-5) are sufficient for $\mathcal{M}_{\mathcal{B}}$ to be a DFS. Let us now turn to the converse problem of showing that these conditions are also necessary for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.
Let us first observe the following. If $\mathcal{B}$ satisfies (B-2), then apparently $\mathcal{B}$ is completely determined by its behaviour on $\mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$: if $f \in \mathbb{B}^-$, then $1 - f \in \mathbb{B}^+$ and hence $\mathcal{B}(f) = 1 - \mathcal{B}(1 - f)$. In particular, every $\mathcal{B}$ which satisfies (B-2) apparently has $\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = 1 - \mathcal{B}(1 - \mathrm{c}_{\frac{1}{2}}) = 1 - \mathcal{B}(\mathrm{c}_{\frac{1}{2}})$, i.e. $\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$.

Definition 74 (BB)
Let us denote by BB *the set of those* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *with the following properties:*

a. $\mathcal{B}$ *satisfies* (B-2)

b. $\mathcal{B}(f) \geq \frac{1}{2}$ *for all* $f \in \mathbb{B}^+$.

It is apparent that all $\mathcal{B}$ such that $\mathcal{M}_{\mathcal{B}}$ is a DFS are contained in BB, i.e. we can restrict attention to BB without losing any of the models of interest. We shall formally establish this later (see p.50). For those $\mathcal{B}$ contained in BB, we can give a more concise description.

Definition 75 ($\mathbb{H}$)
By $\mathbb{H} \subseteq \mathbf{I}^{\mathbf{I}}$ *we denote the set of nonincreasing mappings* $f : \mathbf{I} \longrightarrow \mathbf{I}$, $f \neq \mathrm{c}_0$, *i.e.*

$$\mathbb{H} = \{ f \in \mathbf{I}^{\mathbf{I}} : f \text{ nonincreasing and } f(0) > 0 \} .$$

We can associate with each $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ a $\mathcal{B} \in$ BB as follows:

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f - 1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1 - 2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{23}$$

It is apparent that each $\mathcal{B}$ constructed from some $\mathcal{B}'$ satisfies (B-2).
On the other hand, we can associate with each $\mathcal{B} \in$ BB the following $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$:

$$\mathcal{B}'(f) = 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 \tag{24}$$

for all $f \in \mathbb{H}$. These constructions are obviously inverse of each others, i.e. there is a one-to-one correspondence between $\mathcal{B}'$ and the set of those $\mathcal{B} : \mathbb{H} \longrightarrow \mathbf{I}$ which satisfy (B-2) and have $\mathcal{B}(f) \geq \frac{1}{2}$ for all $f \in \mathbb{B}^+$.

In the case of the DFS $\mathcal{M} = \mathcal{M}_{\mathcal{B}_\int}$, for example, we have

$$\mathcal{B}_\int(f) = \int_0^1 f(\gamma)\, d\gamma$$

for all $f \in \mathbb{B}$, and similarly

$$\mathcal{B}'_\int(g) = \int_0^1 g(\gamma)\, d\gamma\,,$$

for all $g \in \mathbb{H}$, which is apparent from Def. 70. If we restrict attention to the above subclass BB of $\mathcal{B}$'s, we can focus on properties of the corresponding $\mathcal{B}'$'s in $\mathbb{H}$:

Definition 76
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathbb{H}$, we define the following conditions on $\mathcal{B}'$:

$$\mathcal{B}'(f) = f(0) \quad \textit{if } f \textit{ is constant, i.e. } f(x) = f(0) \textit{ for all } x \in \mathbf{I} \tag{C-1}$$
$$\textit{If } f(\mathbf{I}) \subseteq \{0,1\}, \textit{ then } \mathcal{B}'(f) = f_*^0, \tag{C-2}$$
$$\mathcal{B}'(f) = 0 \quad \textit{if } \widehat{f}((0,1]) = \{0\} \tag{C-3.a}$$
$$\mathcal{B}'(f^\sharp) = \mathcal{B}'(f^\flat) \quad \textit{if } \widehat{f}((0,1]) \neq \{0\} \tag{C-3.b}$$
$$\textit{If } f \leq g, \textit{ then } \mathcal{B}'(f) \leq \mathcal{B}'(g) \tag{C-4}$$

Theorem 53
If $\mathcal{B} \in$ BB and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is the corresponding mapping as defined by (24)*, then the following conditions are equivalent:*

a. (B-1) *and* (C-1)*;*

b. (B-3) *and* (C-2)*;*

c. (B-4) *and the conjunction of* (C-3.a) *and* (C-3.b)*;*

d. (B-5) *and* (C-4).

(Proof: C.12, p.121+)

Our introducing of $\mathcal{B}'$ is mainly a matter of convenience. We can now succintly define some examples of $\mathcal{M}_{\mathcal{B}}$-QFMs.

Definition 77 ($\mathcal{M}^*$)
By $\mathcal{M}^*$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}^{*\prime}(f) = f_*^0 \cdot f_0^*\,,$$

for all $f \in \mathbb{H}$, where the coefficients f_*^0 and f_0^* are defined by (19) *and* (18)*, resp.*

Theorem 54
$\mathcal{M}^*$ is a standard DFS.
(Proof: see [9, Th-45, p. 63])

Definition 78 ($\mathcal{M}_*$)
By $\mathcal{M}_*$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'_*(f) = \sup\{x \cdot f(x) : x \in \mathbf{I}\} ,$$

for all $f \in \mathbb{H}$.

Theorem 55
$\mathcal{M}_*$ is a standard DFS.
(Proof: see [9, Th-47, p. 64])

Let us now address the issue of whether B-1 to B-5 are necessary conditions for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.

Theorem 56
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-1)*, then $\mathcal{M}_{\mathcal{B}}$ does not satisfy* (Z-1).
(Proof: C.13, p.125+)

Theorem 57
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-3)*, then $\mathcal{M}_{\mathcal{B}}$ is not a DFS.*
(Proof: C.14, p.125+)

As a corollary, we obtain that

Theorem 58
Every $\mathcal{M}_{\mathcal{B}}$-DFS is a standard DFS.
(Proof: C.15, p.137+)

Theorem 59
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3) *but does not satisfy* (B-2)*, then $\mathcal{M}_{\mathcal{B}}$ does not satisfy* (Z-3).
(Proof: C.16, p.137+)

Theorem 60
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3) *but does not satisfy* (B-4)*, then $\mathcal{M}_{\mathcal{B}}$ does not satisfy* (Z-6).
(Proof: C.17, p.138+)

Theorem 61
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-5)*, then $\mathcal{M}_{\mathcal{B}}$ does not satisfy* (Z-5)
(Proof: C.18, p.138+)

Summarising these results, we obtain

Theorem 62
The conditions (B-1) to (B-5) are necessary and sufficient for $\mathcal{M}_{\mathcal{B}}$ to be a DFS.
(Proof: C.19, p.139+)

Based on these results, it is now easy to show that when restricting attention to BB, and hence considering the simplified representation $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ of a $\mathcal{M}_{\mathcal{B}}$-QFM, no QFMs of interest are lost:

Theorem 63
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. If $\mathcal{B} \notin \mathrm{BB}$, then $\mathcal{M}_\mathcal{B}$ is not a DFS.
(Proof: C.20, p.139+)

Let us also observe that condition (C-3.a) is dependent on the other conditions.

Theorem 64
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given and $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equation (23) *and Def. 69. Then $\mathcal{M}_\mathcal{B}$ is a DFS if and only if $\mathcal{B}'$ satisfies* (C-1), (C-2), (C-3.b), *and* (C-4).
(Proof: C.21, p.140+)

5 Independence Proof for the Revised Axiom Set

We have introduced the class of $\mathcal{M}_{\mathcal{B}}$-QFMs and we have provided necessary and sufficient conditions for $\mathcal{M}_{\mathcal{B}}$ to be a DFS. More specifically, we have taken a closer look on the behaviour of $\mathcal{M}_{\mathcal{B}}$ in case only some of the conditions (B-1) to (B-5) are satisfied, which will prove useful for establishing the independence of the revised DFS axiom set. In order to prove that these axioms are independent, we have to provide for each (Z-i) a QFM $\mathcal{F}_i$ which satisfies all DFS axioms except for (Z-i). Table 1 summarises our chances of using $\mathcal{M}_{\mathcal{B}}$-QFMs for this task, based on the above theorems.

$\mathcal{B}$		$\mathcal{M}_{\mathcal{B}}$	
satisfies	fails on	guaranteed	fails on
(B-2), (B-3), (B-4), (B-5)	(B-1)	all except (Z-1) (Correct generalisation)	(Z-1)
(B-1), (B-3), (B-4), (B-5)	(B-2)	all except (Z-3) (Dualisation)	(Z-3)
(B-1), (B-2), (B-3), (B-4)	(B-5)	all except (Z-5) (Preservation of monotonicity)	(Z-5)
(B-1), (B-2), (B-3), (B-5)	(B-4)	all except (Z-6) (Functional application)	(Z-6)

Table 1: Dependencies between (B-1) to (B-5) and revised DFS axioms

As indicated by the results in the table, there is a chance of using $\mathcal{M}_{\mathcal{B}}$-QFMs for proving the independence of (Z-1), (Z-3), (Z-5) and (Z-6). We only need to find choices of $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ which violate one of (B-1), (B-2), (B-4) and (B-5) and satisfy all remaining 'B-conditions'. In order to prove the independence of (Z-2), an $\mathcal{M}_{\mathcal{B}}$-QFM will be used which disvalidates (B-3); in this case, I cannot rely on general results (as in the table above) and hence must prove for this *particular* choice of $\mathcal{B}$ that the remaining axioms are valid. Finally, the independence of (Z-4) from the other axioms can be proven by piecewise combining two different $\mathcal{M}_{\mathcal{B}}$-QFMs, one for quantifiers of arity $n \leq 1$, and the other for quantifiers of arity $n \geq 2$.

Hence let us firstly consider (Z-1), (Z-3), (Z-5) and (Z-6) (the cases covered by the above table). Actually, what we shall prove is that the conditions (B-1) to (B-5) on $\mathcal{B}$ are independent. It will then be a corollary that (Z-1), (Z-2), (Z-3), (Z-5) and (Z-6) are independent, too.

Theorem 65
(B-1) *is independent of* (B-2), (B-3), (B-4) *and* (B-5).
(Proof: D.1, p.140+)

Theorem 66
(B-2) *is independent of* (B-1), (B-3), (B-4) *and* (B-5).
(Proof: D.2, p.143+)

Theorem 67
(B-4) *is independent of* (B-1), (B-2), (B-3) *and* (B-5).
(Proof: D.3, p.146+)

Theorem 68
(B-5) *is independent of* (B-1), (B-2), (B-3) *and* (B-4).
(Proof: D.4, p.147+)

This immediately gives us the following:

Theorem 69
Each of the following axioms of (Z-1) to (Z-6) is independent of the remaining axioms:

a. (Z-1),

b. (Z-3),

c. (Z-5),

d. (Z-6).

(Proof: D.5, p.150+)

Theorem 70
(B-3) *is independent of* (B-1), (B-2), (B-4) *and* (B-5).
(Proof: D.6, p.151+)

Hence the 'B-conditions' (B-1) to (B-5) provide a characterisation of $\mathcal{M}_{\mathcal{B}}$-DFSes through an independent axiom set. A similar point can be made about the reduced set of 'C-conditions':

Theorem 71
The conditions (C-1), (C-2), (C-3.b) *and* (C-4) *are independent.*
(Proof: D.7, p.153+)

Returning to the independence proof for (Z-1) to (Z-6), we can now utilize Th-70 to show that

Theorem 72
(Z-2) *is independent of* (Z-1) *and* (Z-3) *to* (Z-6).
(Proof: D.8, p.153+)

Theorem 73
(Z-4) *is independent of* (Z-1) *to* (Z-3) *and* (Z-5), (Z-6).
(Proof: D.9, p.167+)

This finishes the independence proof of the revised DFS axioms (Z-1) to (Z-6).

6 Further Properties of DFSes and Principled Adequacy Bounds

In order to be able to discuss certain aspects of $\mathcal{M}_{\mathcal{B}}$-DFSes, we shall now introduce a number of novel concepts which describe specific properties of natural language quantifiers and further adequacy conditions on QFMs. New results on properties of DFSes will also be presented, which are now easy to prove because the improved, smaller set of DFS axioms (Z-1) to (Z-6) is available. In addition, it will be shown that there are principled adequacy bounds on QFMs. For example, no reasonable QFM can preserve general convexity properties of quantifiers.

6.1 Existence of Upper Specificity Bounds

In section 2.8, we have introduced the partial specificity order $\preceq_c$. In addition, a result concerning the existence of greatest lower bounds $\mathcal{F}_{\mathrm{glb}}$ of collections of $\widetilde{\vee}$-DFSes has been established. We shall now address the converse issue of most specific DFSes, i.e. least upper bounds with respect to $\preceq_c$.

Definition 79
Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes $\mathcal{F}_j$, $j \in \mathcal{J}$ where $\mathcal{J} \neq \varnothing$. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is called specificity consistent *iff for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, either $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ or $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$, where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.*

Theorem 74
Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$.

a. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ has upper specificity bounds exactly if $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent.

b. If $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent, then its least upper specificity bound is the $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{lub}}$ defined by

$$\mathcal{F}_{\mathrm{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.

(Proof: E.1, p.170+)

In the following, we shall discuss a number of additional adequacy criteria for approaches to fuzzy quantification.

6.2 Continuity Conditions

Definition 80
We say that a QFM $\mathcal{F}$ is arg-continuous *if and only if $\mathcal{F}$ maps all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to continuous fuzzy quantifiers $\mathcal{F}(Q)$, i.e. for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\mathcal{F}(Q)(X_1, \ldots, X_n), \mathcal{F}(Q)(X'_1, \ldots, X'_n)) < \varepsilon$ for all $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ with $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$; where*

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) = \max_{i=1}^{n} \sup\{|\mu_{X_i}(e) - \mu_{X'_i}(e)| : e \in E\} \,. \qquad (25)$$

Definition 81
We say that a QFM $\mathcal{F}$ is Q-continuous *if and only if for each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mathcal{F}(Q), \mathcal{F}(Q')) < \varepsilon$ whenever $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ satisfies $d(Q, Q') < \delta$; where*

$$d(Q, Q') = \sup\{|Q(Y_1, \ldots, Y_n) - Q'(Y_1, \ldots, Y_n)| : Y_1, \ldots, Y_n \in \mathcal{P}(E)\}$$
$$d(\mathcal{F}(Q), \mathcal{F}(Q')) = \sup\{|\mathcal{F}(Q)(X_1, \ldots, X_n) - \mathcal{F}(Q')(X_1, \ldots, X_n)| : X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\}\,.$$

Arg-continuity means that a small change in the membership grades $\mu_{X_i}(e)$ of the argument sets does not change $\mathcal{F}(Q)(X_1, \ldots, X_n)$ drastically; it hence expresses an important robustness condition with respect to noise. Q-continuity captures an important aspect of robustness with respect to imperfect knowledge about the precise definition of a quantifier; i.e. slightly different definitions of Q will produce similar quantification results. Both condition are crucial to the utility of a DFS and should be possessed by every practical model. They are not part of the DFS axioms because we wanted to have DFSes for general t-norms (including the discontinuous variety).

6.3 Propagation of Fuzziness

Finally, let us recall the specificity order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ defined by equation (9). We can extend $\preceq_c$ to fuzzy sets $X \in \widetilde{\mathcal{P}}(E)$, semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ as follows:

$$\begin{aligned} X \preceq_c X' &\iff \mu_X(e) \preceq_c \mu_{X'}(e) && \text{for all } e \in E;\\ Q \preceq_c Q' &\iff Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n) && \text{for all } Y_1, \ldots, Y_n \in \mathcal{P}(E);\\ \widetilde{Q} \preceq_c \widetilde{Q}' &\iff \widetilde{Q}(X_1, \ldots, X_n) \preceq_c \widetilde{Q}'(X_1, \ldots, X_n) && \text{for all } X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\,. \end{aligned}$$

Intuitively, we should expect that the quantification results become less specific when the quantifier or the argument sets become less specific. In other words: the fuzzier the input, the fuzzier the output.

Definition 82
We say that a QFM $\mathcal{F}$ propagates fuzziness in arguments *if and only if the following property is satisfied for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n, X'_1, \ldots, X'_n$: If $X_i \preceq_c X'_i$ for all $i = 1, \ldots, n$, then $\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}(Q)(X'_1, \ldots, X'_n)$.*
We say that $\mathcal{F}$ propagates fuzziness in quantifiers *if and only if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.*

6.4 Contextuality

Let us now introduce another very fundamental adequacy condition on QFMs. Suppose $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset. The *support* $\text{spp}(X) \in \mathcal{P}(E)$ and the *core*, $\text{core}(X) \in \mathcal{P}(E)$ are defined by

$$\text{spp}(X) = \{e \in E : \mu_X(e) > 0\} \tag{26}$$
$$\text{core}(X) = \{e \in E : \mu_X(e) = 1\}\,. \tag{27}$$

$\text{spp}(X)$ contains all elements which potentially belong to X and $\text{core}(X)$ contains all elements which fully belong to X. The interpretation of a fuzzy subset X is hence ambiguous only with respect to crisp subsets Y in the context range

$$\text{cxt}(X) = \mathcal{T}_1(X) = \{Y \in \mathcal{P}(E) : \text{core}(X) \subseteq Y \subseteq \text{spp}(Y)\}\,. \tag{28}$$

For example, let $E = \{a, b, c\}$ and suppose $X \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset

$$\mu_X(e) = \begin{cases} 1 & : \quad x = a \text{ or } x = b \\ \frac{1}{2} & : \quad x = c \end{cases} \tag{29}$$

The corresponding context range is

$$\text{cxt}(X) = \{Y : \{a, b\} \subseteq Y \subseteq \{a, b, c\}\} = \{\{a, b\}, \{a, b, c\}\}\,.$$

Now let us consider $\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Because $\exists(\{a, b\}) = \exists(\{a, b, c\}) = 1$, $\exists(Y) = 1$ for all crisp subsets in the context of X. We hence expect that $\mathcal{F}(\exists)(X) = 1$: regardless of whether we assume that $c \in X$ or $c \notin X$, the quantification result is always equal to one.

Definition 83 (Contextually equal)
Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We say that Q and Q' are contextually equal *relative to $(X_1, \ldots, X_n)$, in symbols: $Q \sim_{(X_1,\ldots,X_n)} Q'$, if and only if*

$$Q|_{\text{cxt}(X_1)\times\cdots\times\text{cxt}(X_n)} = Q'|_{\text{cxt}(X_1)\times\cdots\times\text{cxt}(X_n)}\,,$$

i.e.

$$Q(Y_1, \ldots, Y_n) = Q'(Y_1, \ldots, Y_n)$$

for all $Y_1 \in \text{cxt}(X_1), \ldots, Y_n \in \text{cxt}(X_n)$.

Note. It is apparent that for each $E \neq \varnothing$, $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $\sim_{(X_1,\ldots,X_n)}$ is an equivalence relation on the set of all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Definition 84
A QFM $\mathcal{F}$ is said to be contextual *iff for all $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and every choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$:*

$$Q \sim_{(X_1,\ldots,X_n)} Q' \quad \Rightarrow \quad \mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(X_1, \ldots, X_n)\,.$$

As illustrated by our motivating example, it is highly desirable that a QFM satisfies this very elementary and fundamental adequacy condition. And indeed, every DFS can be shown to fulfill this condition.

Theorem 75 (DFSes are contextual)
Every DFS $\mathcal{F}$ is contextual.
(Proof: E.2, p.173+)

6.5 Convexity

Definition 85 *Convex quantifiers*
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier such that $n > 0$. Q is said to be convex in its i-th argument, *where $i \in \{1, \ldots, n\}$, if and only if*

$$Q(X_1, \ldots, X_n) \geq \min(Q(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n), Q(X_1, \ldots, X_{i-1}, X''_i, X_{i+1}, \ldots, X_n))$$

whenever $X_1, \ldots, X_n, X'_i, X''_i \in \mathcal{P}(E)$ *and* $X'_i \subseteq X_i \subseteq X''_i$.
Similarly, a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *is called* convex in its i-th argument, *where* $i \in \{1, \ldots, n\}$*, if and only if*

$$\widetilde{Q}(X_1, \ldots, X_n) \geq \min(\widetilde{Q}(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n), \widetilde{Q}(X_1, \ldots, X_{i-1}, X''_i, X_{i+1}, \ldots, X_n))$$

whenever $X_1, \ldots, X_n, X'_i, X''_i \in \widetilde{\mathcal{P}}(E)$ *and* $X'_i \subseteq X_i \subseteq X''_i$*, where '*$\subseteq$*' is the fuzzy inclusion relation.*

Note. In the literature on TGQ, those quantifiers which I call 'convex' are usually dubbed 'continuous', see e.g. [8, Def. 16, p. 250]. I have decided to change terminology because of the possible ambiguity of 'continuous', which could also mean 'smooth'.

Some well-known properties of convex quantifiers (in the sense of TGQ) carry over to convex semi-fuzzy and convex fuzzy quantifier.

Theorem 76 (Conjunctions of convex semi-fuzzy quantifiers)
Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers of arity* $n > 0$ *which are convex in the* i*-th argument, where* $i \in \{1, \ldots, n\}$*. Then the semi-fuzzy quantifier* $Q \wedge Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*, defined by*

$$(Q \wedge Q')(X_1, \ldots, X_n) = \min(Q(X_1, \ldots, X_n), Q'(X_1, \ldots, X_n))$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$*, is also convex in the* i*-th argument.*
(Proof: E.3, p.174+)

Note. The theorem states that conjunctions of convex semi-fuzzy quantifiers are convex (provided the standard fuzzy conjunction $\wedge = \min$ is chosen). A similar point can be made about fuzzy quantifiers.

Theorem 77 (Conjunctions of convex fuzzy quantifiers)
Suppose $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *are fuzzy quantifiers of arity* $n > 0$ *which are convex in the* i*-th argument, where* $i \in \{1, \ldots, n\}$*. Then the fuzzy quantifier* $\widetilde{Q} \wedge \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*, defined by*

$$(\widetilde{Q} \wedge \widetilde{Q}')(X_1, \ldots, X_n) = \min(\widetilde{Q}(X_1, \ldots, X_n), \widetilde{Q}'(X_1, \ldots, X_n))$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*, is also convex in the* i*-th argument.*
(Proof: E.4, p.175+)

Let us also state that every convex semi-fuzzy quantifier can be decomposed into a conjunction of a nonincreasing and a nondecreasing semi-fuzzy quantifier:

Theorem 78 (Decomposition of convex semi-fuzzy quantifiers)
A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is convex in its* i*-th argument,* $i \in \{1, \ldots, n\}$*, if and only if* Q *is the conjunction of a nondecreasing and a nonincreasing semi-fuzzy quantifier, i.e. if there exist* $Q^+, Q^- : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *such that* Q^+ *is nondecreasing in its* i*-th argument;* Q^- *is nonincreasing in its* i*-th argument, and* $Q = Q^+ \wedge Q^-$.
(Proof: E.5, p.176+)

Again, a similar point can be made about fuzzy quantifiers:

Theorem 79 (Decomposition of convex fuzzy quantifiers)
A fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is convex in its i-th argument, $i \in \{1, \ldots, n\}$, if and only if $\widetilde{Q}$ is the conjunction of a nondecreasing and a nonincreasing fuzzy quantifier, i.e. if there exist $\widetilde{Q}^+, \widetilde{Q}^- : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ such that $\widetilde{Q}^+$ is nondecreasing in its i-th argument; $\widetilde{Q}^-$ is nonincreasing in its i-th argument, and $\widetilde{Q} = \widetilde{Q}^+ \wedge \widetilde{Q}^-$.
(Proof: E.6, p.178+)

We say that a QFM $\mathcal{F}$ preserves convexity iff convexity of a quantifier in its arguments is preserved when applying $\mathcal{F}$.

Definition 86
Suppose $n \in \mathbb{N} \setminus \{0\}$. A QFM $\mathcal{F}$ is said to preserve convexity of n-ary quantifiers *if and only if every n-ary semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which is convex in its i-th argument is mapped to a fuzzy quantifier $\mathcal{F}(Q)$ which is also convex in its i-th argument.*
$\mathcal{F}$ is said to preserve convexity *if $\mathcal{F}$ preserves the convexity of n-ary quantifiers for all $n > 0$.*

As we shall now see, preservation of convexity is an adequacy property which in this strong form conflicts with other desirable properties.
Let us now provide a first hint that contextuality of a QFM excludes preservation of convexity.

Theorem 80
Suppose $\mathcal{F}$ is a contextual QFM with the following properties: for every base set $E \neq \varnothing$,

a. the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;

b. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;

c. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists $e \in E$ such that $\mu_X(e) < 1$, then $\mathcal{F}(\sim\forall)(X) > 0$, where $\sim\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by

$$(\sim\forall)(Y) = \begin{cases} 1 & : \quad X \neq E \\ 0 & : \quad X = E \end{cases}$$

Then $\mathcal{F}$ does not preserve convexity of one-place quantifiers $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on finite base sets $E \neq \varnothing$.
(Proof: E.7, p.178+)

This means that even if we restrict to the simple case of one-place quantifiers, and even if we restrict to the simple case of finite base sets, there is still no QFM $\mathcal{F}$ which satisfies the very important adequacy conditions imposed by the theorem and at the same time preserves convexity under these simplifying assumptions. In particular, there is no QFM which both satisfies these fundamental adequacy conditions and also preserves convexity.
As a corollary, we obtain

Theorem 81 (No DFS preserves convexity)
Suppose $\mathcal{F}$ is a DFS. Then $\mathcal{F}$ does not preserve convexity of one-place quantifiers on finite domains. In particular, $\mathcal{F}$ does not preserve convexity.
(Proof: E.8, p.179+)

Because contextuality is a rather fundamental condition, it seems to be advantageous to weaken our requirements on the preservation of convexity properties, rather than compromising contextuality or any of the other very elementary conditions.

To this end, we will restrict attention to a limited class of convex quantifiers, namely quantitative convex quantifiers (see Def. 31 and Def. 32). Many natural language quantifiers are quantitative, i.e. the limited class still covers most important quantifiers. Unfortunately, it turns out that restricting the condition of preserving convexity to quantitative quantifiers is still insufficient.

In order to state a corresponding theorem, let us first consider another important adequacy condition.

Definition 87

Suppose $\mathcal{F}$ is some QFM. We say that $\mathcal{F}$ is compatible with cylindrical extensions *iff the following condition holds for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Whenever $n' \in \mathbb{N}$, $n' \geq n$; $i_1, \ldots, i_n \in \{1, \ldots, n'\}$ such that $1 \leq i_1 < i_2 < \cdots < i_n \leq n'$, and $Q' : \mathcal{P}(E)^{n'} \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1, \ldots, Y_{n'}) = Q(Y_{i_1}, \ldots, Y_{i_n})$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, then

$$\mathcal{F}(Q')(X_1, \ldots, X_{n'}) = \mathcal{F}(Q)(X_{i_1}, \ldots, X_{i_n}),$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Note. This property of being compatible with cylindrical extensions is very fundamental. It simply states that vacuous argument positions of a quantifier can be eliminated. For example, if $Q' : \mathcal{P}(E)^4 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and if there exists a semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ such that $Q'(Y_1, Y_2, Y_3, Y_4) = Q(Y_3)$ for all $Y_1, \ldots, Y_4 \in \mathcal{P}(E)$, then we know that Q' does not really depend on all arguments; it is apparent that the choice of Y_1, Y_2 and Y_4 has no effect on the quantification result. It is hence straightforward to require that $\mathcal{F}(Q')(X_1, X_2, X_3, X_4) = \mathcal{F}(Q)(X_3)$ for all $X_1, \ldots, X_4 \in \widetilde{\mathcal{P}}(E)$, i.e. $\mathcal{F}(Q')$ is also independent of X_1, X_2, X_4, and it can be computed from $\mathcal{F}(Q)$. Let us remark that every DFS fulfills this property:

Theorem 82

Every DFS $\mathcal{F}$ is compatible with cylindrical extensions.
(Proof: E.9, p.180+)

Theorem 83

Suppose $\mathcal{F}$ is a contextual QFM which is compatible with cylindrical extensions and satisfies the following properties: for all base sets $E \neq \varnothing$,

a. *the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;*

b. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;*

c. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e') = 0$ for all $e' \in E \setminus \{e\}$ and $\mu_X(e) < 1$, then $\mathcal{F}(\sim\exists)(X) > 0$, where $\sim\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by*

$$(\sim\exists)(Y) = \begin{cases} 1 & : \quad X = \varnothing \\ 0 & : \quad X \neq \varnothing \end{cases}$$

Then $\mathcal{F}$ does not preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$ even on finite base sets.
(Proof: E.10, p.180+)

Theorem 84
No DFS preserves the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$ in their arguments.
(Proof: E.11, p.182+)

This leaves open the possibility that certain DFSes will preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n = 1$. For simplicity, we shall investigate this preservation property for the case of finite domains only.

Definition 88
A QFM $\mathcal{F}$ is said to weakly preserve convexity *iff $\mathcal{F}$ preserves the convexity of quantitative one-place quantifiers on finite domains, i.e. whenever $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a convex quantitative semi-fuzzy quantifier such that $E \neq \varnothing$ is finite, then the resulting fuzzy quantifier $\mathcal{F}(Q)$ is convex as well.*

In this case, we get positive results on the existence of DFSes that weakly preserve convexity. For example, the $\mathcal{M}_{\mathcal{B}}$-DFS $\mathcal{M}_{CX}$, to be introduced on page 62, can be shown to fulfill this adequacy criterion.
In general, we have negative results concerning the preservation of convexity of quantitative two-place quantifiers (see Th-84). However, as I will now show, it is possible for a DFS to preserve convexity properties of two-place quantifiers in a special case of interest to natural language interpretation.

Theorem 85
Suppose $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is an absolute quantifier on a finite base set, i.e. there exists a quantitative one-place quantifier $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ such that $Q = Q'\cap$. If a DFS $\mathcal{F}$ has the property of weakly preserving convexity and Q is convex in its arguments, then $\mathcal{F}(Q)$ is also convex in its arguments.
(Proof: E.12, p.182+)

6.6 Fuzzy Argument Insertion

In our comments on argument insertion (see page 10) we have remarked that adjectival restriction with fuzzy adjectives cannot be modelled directly: if $A \in \widetilde{\mathcal{P}}(E)$ is a *fuzzy* subset of E, then only $\mathcal{F}(Q) \triangleleft A$ is defined, but not $Q \triangleleft A$. However, one can ask if $\mathcal{F}(Q) \triangleleft A$ can be represented by a semi-fuzzy quantifier Q', i.e. if there is a Q' such that

$$\mathcal{F}(Q) \triangleleft A = \mathcal{F}(Q') \,. \tag{30}$$

The obvious choice for Q' is the following.

Definition 89 (Fuzzy Argument Insertion)
Suppose $\mathcal{F}$ is a QFM, $Q : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $A \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E. The semi-fuzzy quantifier $Q \,\widetilde{\triangleleft}\, A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$Q \,\widetilde{\triangleleft}\, A = \mathcal{U}(\mathcal{F}(Q) \triangleleft A) \,,$$

i.e.

$$Q \mathbin{\widetilde{\triangleleft}} A(Y_1, \ldots, Y_n) = \mathcal{F}(Q)(Y_1, \ldots, Y_n, A)\,,$$

for all crisp subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Notes

- $Q \mathbin{\widetilde{\triangleleft}} A$ is written with the "tilde" notation $\widetilde{\triangleleft}$ in order to emphasise that it depends on the chosen QFM $\mathcal{F}$.
- as already noted in [9, p. 54], $Q' = Q \mathbin{\widetilde{\triangleleft}} A$ is the only choice of Q' which possibly satisfies (30), because any Q' which satisfies $\mathcal{F}(Q') = \mathcal{F}(Q) \triangleleft A$ also satisfies

  $$Q' = \mathcal{U}(\mathcal{F}(Q')) = \mathcal{U}(\mathcal{F}(Q) \triangleleft A) = Q \mathbin{\widetilde{\triangleleft}} A\,,$$

  which is apparent from Th-1.

Unfortunately, $Q \mathbin{\widetilde{\triangleleft}} A$ is not necessarily a good model of the semi-fuzzy quantifier obtained from Q by inserting a fuzzy argument set. This is because $Q \mathbin{\widetilde{\triangleleft}} A$ is not guaranteed to fulfill (30) in a particular QFM (not even in a DFS). Let us hence turn this equation into an adequacy condition which ensures that $Q \mathbin{\widetilde{\triangleleft}} A$ conveys the intended meaning in a given QFM $\mathcal{F}$:

Definition 90 (Compatibility with Fuzzy Argument Insertion)
Suppose $\mathcal{F}$ is a QFM. We say that $\mathcal{F}$ is compatible with fuzzy argument insertion *iff for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and every $A \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}(Q \mathbin{\widetilde{\triangleleft}} A) = \mathcal{F}(Q) \triangleleft A\,.$$

The main application of this property in natural language is that of adjectival restriction of a quantifier by means of a fuzzy adjective. For example, suppose E is a set of people, and **lucky** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of those people in E who are lucky. Further suppose **almost all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier which models "almost all". Finally, suppose the DFS $\mathcal{F}$ is chosen as the model of fuzzy quantification. We can then construct the semi-fuzzy quantifier $Q' =$ **almost all**$\cap \widetilde{\triangleleft}$ **lucky**. If $\mathcal{F}$ is compatible with fuzzy argument insertion, then the semi-fuzzy quantifier Q' is guaranteed to adequately model the composite expression "almost all X's are lucky Y's", because

$$\mathcal{F}(Q')(X_1, X_2) = \mathcal{F}(Q)(X, Y \mathbin{\widetilde{\cap}} \textbf{lucky})$$

for all fuzzy arguments $X, Y \in \widetilde{\mathcal{P}}(E)$, which (relative to $\mathcal{F}$) is the proper expression for interpreting "almost all X's are lucky Y's" in the fuzzy case. Compatibility with fuzzy argument insertion is a very restrictive adequacy condition. We shall present the unique standard DFS which fulfills this condition on page 62 below.

7 Properties of $\mathcal{M}_\mathcal{B}$-DFSes

Let us now take a closer look at $\mathcal{M}_\mathcal{B}$-DFSes. We will present more examples of $\mathcal{M}_\mathcal{B}$-DFSes, discuss a number of properties shared by all $\mathcal{M}_\mathcal{B}$-DFSes, and we will identify certain $\mathcal{M}_\mathcal{B}$-DFSes with special properties. For example, it will be proven which $\mathcal{M}_\mathcal{B}$-DFSes are the least and the most specific. In addition, an $\mathcal{M}_\mathcal{B}$-DFS called $\mathcal{M}_{CX}$ will be presented which can shown to exhibit unique adequacy properties.

Let us first state a theorem which simplifies the comparison of $\mathcal{M}_\mathcal{B}$-DFSes with respect to specificity:

Theorem 86
Suppose $\mathcal{B}'_1, \mathcal{B}'_2 : \mathbb{H} \longrightarrow \mathbf{I}$ are given. Further suppose that $\mathcal{B}_1, \mathcal{B}_2 \in$ BB are the mappings associated with $\mathcal{B}'_1$ and $\mathcal{B}'_2$, respectively, according to equation (23), and $\mathcal{M}_{\mathcal{B}_1}$, $\mathcal{M}_{\mathcal{B}_2}$ are the corresponding QFMs defined by Def. 69. Then $\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2}$ iff $\mathcal{B}'_1 \leq \mathcal{B}'_2$.
(Proof: F.1, p.183+)

Next we shall investigate extreme cases of $\mathcal{M}_\mathcal{B}$-DFSes with respect to the specificity order.

Definition 91 ($\mathcal{M}_U$)
By $\mathcal{M}_U$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by

$$\mathcal{B}'_U(f) = \max(f_*^1, f_1^*)$$

for all $f \in \mathbb{H}$, where the coefficients f_1^* and f_*^1 are defined by equations (21) *and* (22), *resp.*

Theorem 87
Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^1 \oplus f_1^* \,, \qquad \text{(Th-87.a)}$$

for all $f \in \mathbb{H}$ (see (22), (21) *for the definition of the coefficients f_*^1 and f_1^*, resp.). Further suppose that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'$ according to equation* (23)*, and that $\mathcal{M}_\mathcal{B}$ is the QFM defined in terms of $\mathcal{B}$ according to Def. 69. The QFM $\mathcal{M}_\mathcal{B}$ is a standard DFS.*
(Proof: F.2, p.184+)

Note. In particular, $\mathcal{M}_U$ is a standard DFS.
$\mathcal{M}_U$ is an extreme case of $\mathcal{M}_\mathcal{B}$-DFS in terms of specificity.

Theorem 88
$\mathcal{M}_U$ is the least specific $\mathcal{M}_\mathcal{B}$-DFS.
(Proof: F.3, p.190+)

Let us now consider the question of the existence of most specific $\mathcal{M}_\mathcal{B}$-DFSes. We first observe that

Theorem 89
All $\mathcal{M}_\mathcal{B}$-DFSes are specificity consistent standard DFSes.
(Proof: F.4, p.192+)

It is then immediate from Th-74 that there exists a least upper specificity bound $\mathcal{F}_{\text{lub}}$ on the collection of all $\mathcal{M}_\mathcal{B}$-DFSes. As we will now show, $\mathcal{F}_{\text{lub}}$ is in fact an $\mathcal{M}_\mathcal{B}$-DFS.

Theorem 90
Let $(\mathcal{M}_{\mathcal{B}_j})_{j\in\mathcal{J}}$ a $\mathcal{J}$-indexed collection of $\mathcal{M}_{\mathcal{B}}$-DFSes where $\mathcal{J} \neq \varnothing$. The least upper specificity bound $\mathcal{F}_{\mathrm{lub}}$ of $(\mathcal{M}_{\mathcal{B}_j})_{j\in\mathcal{J}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS.
(Proof: F.5, p.192+)

Definition 92 ($\mathcal{M}_S$)
By $\mathcal{M}_S$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'_S(f) = \min(f_*^0, f_0^*)$$

for all $f \in \mathbb{H}$, using the same abbreviations than with $\mathcal{M}^*$.

Theorem 91
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the the QFM $\mathcal{M}_{\mathcal{B}}$ is defined in terms of $\mathcal{B}'$ according to (23) *and Def. 69. Then $\mathcal{M}_{\mathcal{B}}$ is a standard DFS.*
(Proof: F.6, p.195+)

Note. In particular, $\mathcal{M}_S$ is a standard DFS.
$\mathcal{M}_S$ also represents an extreme case of $\mathcal{M}_{\mathcal{B}}$-DFS in terms of specificity, as we shall now state:

Theorem 92
$\mathcal{M}_S$ is the most specific $\mathcal{M}_{\mathcal{B}}$-DFS.
(Proof: F.7, p.196+)

Definition 93 ($\mathcal{M}_{CX}$)
By $\mathcal{M}_{CX}$ we denote the $\mathcal{M}_{\mathcal{B}}$-QFM defined by

$$\mathcal{B}'_{CX}(f) = \sup\{\min(x, f(x)) : x \in \mathbf{I}\}$$

for all $f \in \mathbb{H}$.

Theorem 93
Suppose $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a continuous t-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} \qquad \text{(Th-93.a)}$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'$ according to equation 23. The QFM $\mathcal{M}_{\mathcal{B}}$, defined by Def. 69, is a standard DFS.
(Proof: F.8, p.198+)

Note. In particular, $\mathcal{M}_{CX}$ is a standard DFS.
As we shall see below, $\mathcal{M}_{CX}$ is a DFS with unique properties. In order to carry out these proofs, we need to observe that there are various ways of computing $\mathcal{B}'_{CX}$:

Theorem 94
For all $f \in \mathbb{H}$,

$$\begin{aligned}&\sup\{\min(x, f(x)) : x \in \mathbf{I}\}\\ =&\inf\{\max(x, f(x)) : x \in \mathbf{I}\}\\ =&\sup\{x \in \mathbf{I} : f(x) > x\}\\ =&\inf\{x \in \mathbf{I} : f(x) < x\}\\ =&\text{the unique } x \text{ s.th. } f(y) > y \text{ for all } y < x \text{ and } f(y) < y \text{ for all } y > x\,.\end{aligned}$$

(Proof: F.9, p.202+)

In addition, we shall need some abbreviations. Given a base set $E \neq \varnothing$, a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$, let us stipulate

$$|X|_\gamma^{\min} = |(X)_\gamma^{\min}| \tag{31}$$
$$|X|_\gamma^{\max} = |(X)_\gamma^{\max}| \tag{32}$$

for all $\gamma \in \mathbf{I}$.
In order to establish these results, we need some more observations on quantitative quantifiers and on quantitative convex quantifiers.

Theorem 95
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a one-place semi-fuzzy quantifier on a finite base set $E \neq \varnothing$. Then Q is quantitative if and only if there exists a mapping $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|)\,,$$

for all $Y \in \mathcal{P}(E)$. q is defined by

$$q(j) = Q(Y_j) \tag{33}$$

for $j \in \{0, \ldots, |E|\}$, where $Y_j \in \mathcal{P}(E)$ is an arbitrary subset of cardinality $|Y_j| = j$.
(Proof: F.10, p.205+)

Theorem 96
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative semi-fuzzy quantifier on a finite base set and $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is defined by (33)*. Then Q is convex if and only if q has the following property: whenever $j', j, j'' \in \{0, \ldots, |E|\}$ and $j' \le j \le j''$, then*

$$q(j) \ge \min(q(j'), q(j''))\,.$$

(Proof: F.11, p.206+)

Theorem 97
Suppose $m \in \mathbb{N} \setminus \{0\}$ and $q : \{0, \ldots, m\}$ has the following property: whenever $j', j, j'' \in \{0, \ldots, |E|\}$ and $j' \le j \le j''$, then $q(j) \ge \min(q(j'), q(j''))$. Then there exists $j_{\mathrm{pk}} \in \{0, \ldots, m\}$ such that

$$q(j) \le q(j')$$

for all $j \leq j'' \leq j_{\mathrm{pk}}$*, and*

$$q(j) \geq q(j')$$

for all $j_{\mathrm{pk}} \leq j \leq j'$*.*
(Proof: F.12, p.207+)

We need some more notation in order to be able to simplify the computation of $Q_\gamma(X)$ in the case of quantitative quantifiers.

Definition 94
Suppose $m \in \mathbb{N}$ *and* $q : \{0, \ldots, m\} \longrightarrow \mathbf{I}$*. For all* $\ell, u \in \{0, \ldots, m\}$*, we define*

$$q^{\min}(\ell, u) = \min\{q(j) : \ell \leq j \leq u\}$$
$$q^{\max}(\ell, u) = \max\{q(j) : \ell \leq j \leq u\}\,.$$

Theorem 98
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is a one-place quantitative semi-fuzzy quantifier on a finite base set, and assume* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is the mapping defined by* (33)*. For all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $\gamma \in \mathbf{I}$*,*

$$Q_\gamma^{\min}(X) = q^{\min}(\ell, u)$$
$$Q_\gamma^{\max}(X) = q^{\max}(\ell, u)$$

where $\ell = |X|_\gamma^{\min}\gamma$ *and* $u = |X|_\gamma^{\max}\gamma$*. Hence*

$$Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(q^{\min}(\ell, u), q^{\max}(\ell, u)) = \mathrm{m}_{\frac{1}{2}}\{q(j) : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}\,.$$

(Proof: F.13, p.209+)

Theorem 99
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is a quantitative convex quantifier on a finite base set. Further suppose* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is the mapping defined by* (33) *and* $j_{\mathrm{pk}} \in \{0, \ldots, |E|\}$ *is chosen as in Th-97. Then*

$$q^{\min}(\ell, u) = \min(q(\ell), q(u))$$
$$q^{\max}(\ell, u) = \begin{cases} q(\ell) & : \quad \ell > j_{\mathrm{pk}} \\ q(u) & : \quad u < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \quad \ell \leq j_{\mathrm{pk}} \leq u \end{cases}$$

i.e.

$$Q_\gamma^{\min}(X) = \min(q(|X|_\gamma^{\min}), q(|X|_\gamma^{\max}))$$
$$Q_\gamma^{\max}(X) = \begin{cases} q(|X|_\gamma^{\min}) & : \quad |X|_\gamma^{\min} > j_{\mathrm{pk}} \\ q(|X|_\gamma^{\max}) & : \quad |X|_\gamma^{\max} < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \quad |X|_\gamma^{\min} \leq j_{\mathrm{pk}} \leq |X|_\gamma^{\max} \end{cases}$$

for all $X \in \widetilde{\mathcal{P}}(E)$ *and* $\gamma \in \mathbf{I}$*.*
(Proof: F.14, p.210+)

Based on these theorems, it is now easy to show the following.

Theorem 100
The DFS $\mathcal{M}_{CX}$ weakly preserves convexity.
(Proof: F.15, p.212+)

Theorem 101
Suppose $\mathcal{M}_{\mathcal{B}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS. If $\mathcal{M}_{\mathcal{B}}$ weakly preserves convexity, then $\mathcal{M}_{CX} \preceq_c \mathcal{M}_{\mathcal{B}}$.
(Proof: F.16, p.227+)

In addition to weakly preserving convexity, $\mathcal{M}_{CX}$ can also be shown to be compatible with fuzzy argument insertion.

Theorem 102
The DFS $\mathcal{M}_{CX}$ is compatible with fuzzy argument insertion.
(Proof: F.17, p.231+)

As we shall see in the next section, $\mathcal{M}_{CX}$ is in fact the only standard DFS which fulfills this adequacy condition.
Now we shall discuss Q-continuity and arg-continuity of $\mathcal{M}_{\mathcal{B}}$-DFSes. Let us first make some general observations how the continuity conditions are related to $(\bullet)_\gamma$.

Theorem 103
Let $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given. Then $d(Q_\gamma, Q'_\gamma) \leq d(Q, Q')$ for all $\gamma \in \mathbf{I}$.
(Proof: F.18, p.252+)

We can use this inequation to formulate a condition on $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ which is necessary and sufficient for $\mathcal{M}_{\mathcal{B}}$ to be Q-continuous. To this end, we first define a metric $d : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbf{I}$ by

$$d(f, g) = \sup\{|f(\gamma) - g(\gamma)| : \gamma \in \mathbf{I}\} \tag{34}$$

for all $f, g \in \mathbb{H}$.

Theorem 104
Suppose $\mathcal{M}_{\mathcal{B}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS and $\mathcal{B}'$ is the corresponding mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$.
$\mathcal{M}_{\mathcal{B}}$ is Q-continuous iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < \delta$.
(Proof: F.19, p.255+)

In the case of continuity in arguments, we need a different metric on $\mathbb{H}$. Hence let us define $d' : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbf{I}$ by

$$d'(f, g) = \sup\{\inf\{\gamma' - \gamma : \gamma' \in \mathbf{I}, \max(f(\gamma'), g(\gamma')) \leq \min(f(\gamma), g(\gamma))\} : \gamma \in \mathbf{I}\}, \tag{35}$$

for all $f, g \in \mathbb{H}$.
We can now characterise the arg-continuous $\mathcal{M}_{\mathcal{B}}$-DFSes in terms of the mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ as follows.

Theorem 105
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies (C-2), (C-3.b) *and* (C-4). *Further suppose that $\mathcal{M}_{\mathcal{B}}$ is defined in terms of $\mathcal{B}'$ according to* (23) *and Def. 69. Then the following conditions are equivalent:*

a. *$\mathcal{M}_\mathcal{B}$ is arg-continuous.*

b. *for all $f \in \mathbb{H}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $d'(f, g) < \delta$.*

(Proof: F.20, p.260+)

Sometimes the following sufficient condition is simpler to check.

Theorem 106
Let $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (C-2), (C-3.b) *and* (C-4). *If for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon$ whenever $f \leq g$ and $d'(f, g) < \delta$, then $\mathcal{M}_\mathcal{B}$ is arg-continuous.*
(Proof: F.21, p.272+)

We shall now apply these theorems to establish or reject Q-continuity and arg-continuity of our examples of $\mathcal{M}_\mathcal{B}$-DFSes.

Theorem 107
Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by (Th-87.a). *Further suppose that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'$ according to equation* (23)*, and that $\mathcal{M}_\mathcal{B}$ is the QFM defined in terms of $\mathcal{B}$ according to Def. 69. The QFM $\mathcal{M}_\mathcal{B}$ is neither Q-continuous nor arg-continuous.*
(Proof: F.22, p.273+)

In particular, $\mathcal{M}_U$ fails on both continuity conditions.

Theorem 108
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the the QFM $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (23) *and Def. 69. Then $\mathcal{M}_\mathcal{B}$ is neither Q-continuous nor arg-continuous.*
(Proof: F.23, p.274+)

In particular, $\mathcal{M}_S$ and $\mathcal{M}^*$ fail on both continuity conditions. These results illustrate that $\mathcal{M}_U$ and $\mathcal{M}_S$ are only of theoretical interest, because they represent extreme cases in terms of specificity. Due to their discontinuity, these models are not suited for applications. The DFS $\mathcal{M}^*$ is also impractical.
We shall now discuss practical models. In order to establish that $\mathcal{M}$ is arg-continuous, we first observe how the metrics d and d' are related. To this end, we introduce the mapping $(\bullet)^\Diamond : \mathbb{H} \longrightarrow \mathbb{H}$ defined by

$$f^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : f(\gamma) < v\} \tag{36}$$

for all $f \in \mathbb{H}$ and $v \in \mathbf{I}$. It is easily checked that indeed $f^\Diamond \in \mathbb{H}$ whenever $f \in \mathbb{H}$. Let us now utilize this definition to unveil the relationship between the metrics d and d'.

Theorem 109
For all $f, g \in \mathbb{H}$,

$$d'(f, g) = d(f^\Diamond, g^\Diamond)\,.$$

(Proof: F.24, p.276+)

It is now easy to show that the DFS $\mathcal{M}$ satisfies both continuity conditions:

Theorem 110
$\mathcal{M}$ is both Q-continuous and arg-continuous.
(Proof: F.25, p.280+)

Theorem 111
Let $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ be a uniform continuous t-norm, i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_1 \odot y_1 - x_2 \odot y_2| < \varepsilon$ whenever $x_1, x_2, y_1, y_2 \in \mathbf{I}$ satisfy $\|(x_1, y_1) - (x_2, y_2)\| < \delta$. Further suppose that $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by equation (Th-93.a)*, and define the QFM $\mathcal{M}_{\mathcal{B}}$ in terms of $\mathcal{B}'$ according to* (23) *and Def. 69 as usual. Then $\mathcal{M}_{\mathcal{B}}$ is both Q-continuous and arg-continuous.*
(Proof: F.26, p.282+)

In particular, the DFS $\mathcal{M}_{CX}$ which exhibits the best theoretical properties, is indeed a good choice for applications, because it satisfies both continuity conditions. The theorem also shows that $\mathcal{M}_*$, presented on page 49, is a practical DFS.

Turning to $\mathcal{M}_{\mathcal{B}}$-DFSes in general, we shall now investigate the two types of propagation of fuzziness, see Def. 82. With respect to these criteria, $\mathcal{M}_{\mathcal{B}}$-DFSes represent a particularly well-behaved subclass of standard DFSes:

Theorem 112
Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in quantifiers.
(Proof: F.27, p.283+)

Theorem 113
Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in arguments.
(Proof: F.28, p.286+)

8 Upper and Lower Bounds on Quantification Results

In the following, we shall consider upper and lower bounds on the quantification results of DFSes. These will prove useful in order to establish some novel theorems; for example, it becomes possible to show that all standard DFSes coincide on two-valued quantifiers. In addition, this chapter will further enhance the unique properties of the DFS $\mathcal{M}_{CX}$.
Let us firstly define infima and suprema of families of (semi-)fuzzy quantifiers.

Definition 95
Suppose $E \neq \varnothing$ a nonempty base set and a nonnegative number $n \in \mathbb{N}$ are given. Further let $(Q^j)_{j\in\mathcal{J}}$ be a $\mathcal{J}$-indexed family of semi-fuzzy quantifiers $Q^j : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, where $\mathcal{J}$ is an arbitrary index set. We define semi-fuzzy quantifiers $\inf\{Q^j : j \in \mathcal{J}\}, \sup\{Q^j : j \in \mathcal{J}\} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$(\inf\{Q^j : j \in \mathcal{J}\})(Y_1, \ldots, Y_n) = \inf\{Q^j(Y_1, \ldots, Y_n) : j \in \mathcal{J}\}$$
$$(\sup\{Q^j : j \in \mathcal{J}\})(Y_1, \ldots, Y_n) = \sup\{Q^j(Y_1, \ldots, Y_n) : j \in \mathcal{J}\}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.
In the case of fuzzy quantifiers $\widetilde{Q}^j : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, the fuzzy quantifiers $\inf\{\widetilde{Q}^j : j \in \mathcal{J}\}, \sup\{\widetilde{Q}^j : j \in \mathcal{J}\} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ are defined analogously.

Theorem 114
Suppose a base set $E \neq \varnothing$ and an arity $n \in \mathbb{N}$ are given. Further assume that $\mathcal{J}$ is an arbitrary index set and $(Q^j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed family of semi-fuzzy quantifiers $Q^j : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $j \in \mathcal{J}$.

a. In every DFS $\mathcal{F}$,

$$\mathcal{F}(\sup\{Q^j : j \in \mathcal{J}\}) \geq \sup\{\mathcal{F}(Q^j) : j \in \mathcal{J}\}$$

b. In every DFS $\mathcal{F}$,

$$\mathcal{F}(\inf\{Q^j : j \in \mathcal{J}\}) \leq \inf\{\mathcal{F}(Q^j) : j \in \mathcal{J}\}\,.$$

(Proof: G.1, p.288+)

We shall now consider a particular choice of upper and lower bounds on the quantification results of a DFS $\mathcal{F}$. Let us first define the upper bound and lower bounds on a semi-fuzzy quantifier Q in a given argument range.

Definition 96
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given semi-fuzzy quantifier and $V, W \in \mathcal{P}(E)^n$. We define

$$U(Q, V, W) = \sup\{Q(Z_1, \ldots, Z_n) : V_1 \subseteq Z_1 \subseteq W_1, \ldots, V_n \subseteq Z_n \subseteq W_n\}$$
$$L(Q, V, W) = \inf\{Q(Z_1, \ldots, Z_n) : V_1 \subseteq Z_1 \subseteq W_1, \ldots, V_n \subseteq Z_n \subseteq W_n\}$$

Definition 97 ($Q^L_{V,W}$, $Q^U_{V,W}$)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $V, W \in \mathcal{P}(E)^n$. We shall define semi-fuzzy quantifiers $Q^U_{V,W}, Q^L_{V,W} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$Q^U_{V,W}(Y_1, \ldots, Y_n) = \begin{cases} U(Q, V, W) & : \quad V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n \\ 1 & : \quad \text{else} \end{cases}$$

$$Q^L_{V,W}(Y_1, \ldots, Y_n) = \begin{cases} L(Q, V, W) & : \quad V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Definition 98
Let $\mathcal{F}$ be a given QFM. For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $V, W \in \mathcal{P}(E)^n$, we define fuzzy quantifiers $\widetilde{Q}^L_{V,W}, \widetilde{Q}^U_{V,W} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$\widetilde{Q}^L_{V,W} = \mathcal{F}(Q^L_{V,W})$$
$$\widetilde{Q}^U_{V,W} = \mathcal{F}(Q^U_{V,W})\,.$$

Definition 99 ($\widetilde{Q}^L$, $\widetilde{Q}^U$)
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $\mathcal{F}$ is a QFM. The fuzzy quantifiers $\widetilde{Q}^U, \widetilde{Q}^L : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ are defined by

$$\widetilde{Q}^U = \inf\{\widetilde{Q}^U_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}$$
$$\widetilde{Q}^L = \sup\{\widetilde{Q}^L_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}$$

for every $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Theorem 115
In every DFS $\mathcal{F}$ and for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\widetilde{Q}^L \le \mathcal{F}(Q) \le \widetilde{Q}^U\,.$$

(Proof: G.2, p.289+)

Next, let us take a closer look at the fuzzy quantifiers $\widetilde{Q}^U_{V,W}$ and $\widetilde{Q}^L_{V,W}$. As it turns out, these can be expressed in a rather simple form.
To this end, let us introduce some more notation.

Definition 100
For all $a \in \mathbf{I}$, we define semi-fuzzy truth functions $b_a, p_a : \mathbf{2} \longrightarrow \mathbf{I}$ by

$$b_a(x) = \begin{cases} 0 & : \quad x = 0 \\ a & : \quad x = 1 \end{cases}$$
$$p_a(x) = \begin{cases} 1 & : \quad x = 0 \\ a & : \quad x = 1 \end{cases}$$

If $\mathcal{F}$ is some given QFM, we shall abbreviate by $\widetilde{\widetilde{b}}_a, \widetilde{\widetilde{p}}_a : \mathbf{I} \longrightarrow \mathbf{I}$ the induced truth functions $\widetilde{\widetilde{b}}_a = \widetilde{\widetilde{\mathcal{F}}}(b_a)$ and $\widetilde{\widetilde{p}}_a = \widetilde{\widetilde{\mathcal{F}}}(p_a)$.

Definition 101
Let a nonempty base set E be given and $V, W \in \mathcal{P}(E)$. The two-valued quantifier $\Xi_{V,W} : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by

$$\Xi_{V,W}(Y) = \begin{cases} 1 & : \quad V \subseteq Y \subseteq W \\ 0 & : \quad \textit{else} \end{cases}$$

for all $Y \in \mathcal{P}(E)$. Similarly if $V, W \in \mathcal{P}(E)^n$ for arbitrary $n \in \mathbb{N}$, we define $\Xi_{V,W} : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ by

$$\Xi_{V,W}(Y_1, \ldots, Y_n) = \bigwedge_{i=1}^{n} \Xi_{V_i,W_i}(Y_i)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *Given a QFM* $\mathcal{F}$, *we shall abbreviate*

$$\widetilde{\widetilde{\Xi}}_{V,W} = \mathcal{F}(\Xi_{V,W}) \,.$$

Let us first see how $\widetilde{\widetilde{\Xi}}_{V,W}$ looks like when $\mathcal{F}$ is a DFS.

Theorem 116

Suppose $\mathcal{F}$ *is a DFS,* $E \neq \varnothing$ *is a base set and* $V, W \in \mathcal{P}(E)^n$, $n \in \mathbb{N}$, *such that* $V_i \subseteq W_i$ *for all* $i = 1, \ldots, n$. *Then for all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,

$$\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n) = \mathop{\widetilde{\bigwedge}}_{i=1}^{n} \mathcal{F}(\forall)(Z_i)$$

where

$$Z_i = (X_i \mathbin{\widetilde{\widetilde{\cup}}} (W_i \setminus V_i)) \mathbin{\widetilde{\widetilde{\triangle}}} \neg W_i$$

i.e. the $Z_i \in \widetilde{\mathcal{P}}(E)$ *are defined by*

$$\mu_{Z_i}(e) = \begin{cases} \mu_{X_i}(e) & : \quad e \in V_i \\ 1 & : \quad e \in W_i \setminus V_i \\ \widetilde{\widetilde{\neg}}\, \mu_{X_i}(e) & : \quad e \notin W_i \end{cases}$$

for all $e \in E$.
(Proof: G.3, p.292+)

The relevance of these concepts becomes apparent from the next theorem.

Theorem 117

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier,* $V, W \in \mathcal{P}(E)^n$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. *Then in every DFS* $\mathcal{F}$,

$$\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n) = \widetilde{\widetilde{p}}_{U(Q,V,W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n))$$

and

$$\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) = \widetilde{\widetilde{b}}_{L(Q,V,W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n)) \,.$$

(Proof: G.4, p.296+)

Next, we shall take a closer look at the interpretation of $\widetilde{\widetilde{b}}_a$ and $\widetilde{\widetilde{p}}_a$ in certain DFSes $\mathcal{F}$.

Theorem 118

In every DFS $\mathcal{F}$,

$$\widetilde{\widetilde{b}}_a(x) \leq \min(a, x)$$
$$\widetilde{\widetilde{p}}_a(x) \geq \max(a, \widetilde{\widetilde{\neg}}\, x)$$

for all $a, x \in \mathbf{I}$.
(Proof: G.5, p.302+)

This is sufficient to establish upper and lower bounds on the interpretation of certain quantifiers of interest. For example, we are interested how absolute quantifiers like “at least k” are interpreted in DFSes. First we need some more notation.

Definition 102
Suppose $E \neq \varnothing$ is a finite base set of cardinality $|E| = m$. For a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$, let us denote by $\mu_{[j]}(X) \in \mathbf{I}$, $j = 1, \ldots, m$, the j-th largest membership value of X (including duplicates).[13] We shall also stipulate that $\mu_{[0]}(X) = 1$ and $\mu_{[j]}(X) = 0$ whenever $j > m$.

Now let us turn attention to the following class of quantifiers.

Definition 103
Suppose $E \neq \varnothing$ is a base set and $k \in \mathbb{N}$. The quantifier $[\geq k] : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by

$$[\geq k](Y) = \begin{cases} 1 & : \quad |Y| \geq k \\ 0 & : \quad \textit{else} \end{cases}$$

for all $Y \in \mathcal{P}(E)$.

Theorem 119
Let $E \neq \varnothing$ be a given finite base set and $k \in \mathbb{N}$. Then in every DFS $\mathcal{F}$,

$$\mu_{[1]}(X) \mathbin{\widetilde{\widetilde{\wedge}}} \ldots \mathbin{\widetilde{\widetilde{\wedge}}} \mu_{[k]}(X) \leq \mathcal{F}([\geq k])(X) \leq \mu_{[k]}(X) \mathbin{\widetilde{\widetilde{\vee}}} \ldots \mathbin{\widetilde{\widetilde{\vee}}} \mu_{[|E|]}(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$.
(Proof: G.6, p.302+)

In the case of a standard DFS, we obtain

Theorem 120
Suppose that $\mathcal{F}$ is a standard DFS, $E \neq \varnothing$ is a nonempty base set and $k \in \mathbb{N}$. Then

$$\mathcal{F}([\geq k])(X) = \sup\{\alpha \in \mathbf{I} : |(X)_{\geq \alpha}| \geq k\}\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$. In particular, if E is finite, then

$$\mathcal{F}([\geq k])(X) = \mu_k(X)\,.$$

(Proof: G.7, p.306+)

Theorem 121
Let a QFM $\mathcal{F}$ with the following properties be given:

a. $0 \mathbin{\widetilde{\widetilde{\wedge}}} z = 0$ *for all $z \in \mathbf{I}$;*

b. $0 \mathbin{\widetilde{\widetilde{\wedge}}} z = z$ *for all $z \in \mathbf{I}$;*

[13] More formally, we can order the elements of E such that $E = \{e_1, \ldots, e_m\}$ and $\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)$ and then define $\mu_{[j]}(X) = \mu_X(e_j)$.

c. $\mathcal{F}$ *satisfies* (DFS 6)*;*

d. $\mathcal{F}(\pi_1) = \widetilde{\pi}_1$, *where* $\pi_1 : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$*;*

e. $\widetilde{\widetilde{\neg}}$ *is a strong negation operator;*

f. $x_1 \,\widetilde{\widetilde{\vee}}\, x_2 = \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\wedge}}\, \widetilde{\widetilde{\neg}}\, x_2)$ *for all* $x_1, x_2 \in \mathbf{I}$ *(DeMorgan's Law).*

If $\mathcal{F}$ *is compatible with fuzzy argument insertion, then*

$$\widetilde{\widetilde{b}}_a(x) = x \,\widetilde{\widetilde{\wedge}}\, a$$
$$\widetilde{\widetilde{p}}_a(x) = \widetilde{\widetilde{\neg}}\, x \,\widetilde{\widetilde{\vee}}\, a$$

for all $a, x \in \mathbf{I}$.
(Proof: G.8, p.307+)

Our main result on $\mathcal{M}_{CX}$ is then the following:

Theorem 122
For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\widetilde{Q}^L = \mathcal{M}_{CX}(Q) = \widetilde{Q}^U .$$

(Proof: G.9, p.308+)

In particular, this means that $\mathcal{M}_{CX}$ is a concrete implementation of a so-called "substitution approach" to fuzzy quantification [22], i.e. the fuzzy quantifier is modelled by constructing an equivalent logical formula.[14] This is apparent is we simply expand $\widetilde{Q}^U$ and $\widetilde{Q}^L$ in the above theorem:

Theorem 123
For every $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}&\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)\\ &= \sup\{\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\\ &= \inf\{\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\end{aligned}$$

where

$$\begin{aligned}&\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n)\\ &= \min(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \textit{all } i\})\\ &\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n)\\ &= \max(1 - \widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), \sup\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \textit{all } i\})\\ &\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n)\\ &= \min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) .\end{aligned}$$

(Proof: G.10, p.317+)

[14] In the finite case, inf and sup reduce to logical connectives $\wedge = \max$ and $\vee = \min$ as usual. We need to allow for occurrences of constants $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$ in the resulting formula because the fuzzification mechanism is applied to semi-fuzzy quantifiers, not only to two-valued quantifiers.

Combining Th-122 with theorems Th-115 and Th-121, we can also conclude that

Theorem 124
$\mathcal{M}_{CX}$ is the only standard DFS which is compatible with fuzzy argument insertion.
(Proof: G.11, p.318+)

Another direct consequence of the above results is the following:

Theorem 125
All standard DFSes coincide on two-valued quantifiers, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued quantifier, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ is a choice of fuzzy arguments and $\mathcal{F}, \mathcal{F}'$ are given standard DFSes, then

$$\mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}'(Q)(X_1, \ldots, X_n) .$$

(Proof: G.12, p.319+)

Now let us establish some further results on $\mathcal{M}_{CX}$.

Definition 104
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing semi-fuzzy quantifier and $X \in \widetilde{\mathcal{P}}(E)$. The Sugeno integral *$(S) \int X \, dQ$ is defined by*

$$(S) \int X \, dQ = \sup\{\min(\alpha, Q((X)_{\geq\alpha})) : \alpha \in \mathbf{I}\} .$$

Let us now state that $\mathcal{M}_{CX}$ properly generalises the Sugeno integral:

Theorem 126
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nondecreasing. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

$$(S) \int X \, dQ = \mathcal{M}_{CX}(Q)(X) .$$

(Proof: G.13, p.320+)

Hence $\mathcal{M}_{CX}$ coincides with the Sugeno integral whenever the latter is defined.
Another nice property of $\mathcal{M}_{CX}$ is that it does not 'invent' any new truth-values. $\mathcal{M}_{CX}$ hence combines well with ordinal scales of truth values provided these are closed under negation:

Theorem 127
Let $\Omega \subset \mathbf{I}$ be a given set with the following properties:

- *Ω is finite;*
- *if $\omega \in \Omega$, then $1 - \omega \in \Omega$;*
- *$\{0, 1\} \subseteq \Omega$.*

Further suppose that $Q : \mathcal{P}(E)^n \longrightarrow \Omega$ is a semi-fuzzy quantifier with quantification results in Ω and that $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are Ω-valued fuzzy subsets of E, i.e. $\mu_{X_i}(e) \in \Omega$ for all $i = 1, \ldots, n$ and all $e \in E$. Then $\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) \in \Omega$.
(Proof: G.14, p.321+)

In practise, there is seldom perfect knowledge concerning the precise choice of numeric membership degrees. As stated by the next theorem, $\mathcal{M}_{CX}$ is robust with respect to changes in the numeric membership degrees as long as these changes are symmetrical with respect to negation. Recalling Def. 41, we can express this as follows.

Theorem 128
Suppose $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ *is a mapping with the following properties:*

- σ *is a bijection;*
- σ *is nondecreasing;*
- σ *is symmetrical with respect to negation, i.e.* $\sigma(1-x) = 1 - \sigma(x)$ *for all* $x \in \mathbf{I}$.

Then ${\mathcal{M}_{CX}}^{\sigma} = \mathcal{M}_{CX}$.
(Proof: G.15, p.322+)

Let us now show that in the case of $\mathcal{M}_{CX}$, we can use the following *fuzzy interval cardinality* to evaluate quantitative one-place quantifiers.

Definition 105
For every fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$*, the* fuzzy interval cardinality $\|X\|_{\text{iv}} \in \widetilde{\mathcal{P}}(\mathbb{N} \times \mathbb{N})$ *is defined by*

$$\mu_{\|X\|_{\text{iv}}}(\ell, u) = \begin{cases} \min(\mu_{[\ell]}(X), 1 - \mu_{[u+1]}(X)) & : \ \ell \leq u \\ 0 & : \ \textit{else} \end{cases} \qquad \textit{for all } \ell, u \in \mathbb{N}. \tag{37}$$

Intuitively, $\mu_{\|X\|_{\text{iv}}}(\ell, u)$ expresses the degree to which X has between ℓ and u elements.
Recalling our abbreviations $q^{\min}$ and $q^{\max}$ of definition Def. 94, we may express the result on quantitative quantifiers as follows.

Theorem 129
For every quantitative one-place quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *on a finite base set and all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$\begin{aligned} \mathcal{M}_{CX}(Q)(X) &= \max\{\min(\mu_{\|X\|_{\text{iv}}}(\ell, u), q^{\min}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\} \\ &= \min\{\max(1 - \mu_{\|X\|_{\text{iv}}}(\ell, u), q^{\max}(\ell, u)) : 0 \leq \ell \leq u \leq |E|\}\,. \end{aligned}$$

(Proof: G.16, p.324+)

Theorem 130
A quantitative one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *on a finite base set is nondecreasing (nonincreasing) if and only if the mapping* q *defined by* (33) *is nondecreasing (nonincreasing).*
(Proof: G.17, p.326+)

Theorem 131
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is a quantitative one-place quantifier on a finite base set and* q *is the mapping defined by* (33)*. Then*

$$\begin{array}{lll} q^{\min}(\ell, u) = q(\ell) & q^{\max}(\ell, u) = q(u) & \textit{if } Q \textit{ nondecreasing} \\ q^{\min}(\ell, u) = q(u) & q^{\max}(\ell, u) = q(\ell) & \textit{if } Q \textit{ nonincreasing} \end{array}$$

(Proof: G.18, p.326+)

Based on these concepts, it is now easy to establish the following theorem:

Theorem 132
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a nondecreasing quantitative one-place quantifier on a finite base set, and* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is the mapping defined by equation* (33)*. Then for all* $X \in \widetilde{\mathcal{P}}(E)$*,*

$$\mathcal{M}_{CX}(Q)(X) = \max\{\min(q(j), \mu_{[j]}(X)) : 0 \le j \le |E|\} ,$$

i.e. $\mathcal{M}_{CX}$ *consistently generalises the FG-count approach of [27, 23].*
(Proof: G.19, p.326+)

Note. Actually, this is a corollary of Th-126, if we recall the known relationship between the Sugeno integral and the FG-count approach, see [5].

9 Conclusion

In the report, we have reviewed the basic concepts of DFS theory by introducing two-valued quantifiers, fuzzy quantifiers, semi-fuzzy quantifiers and quantifier fuzzification mechanisms. We have also addressed the question of well-behavedness of a quantifier fuzzification mechanism. Our approach is basically an algebraic one: rather than making any claims on ‘the meaning’ of a natural language quantifier, and its corresponding model as a fuzzy quantifier, we assume that most (if not all) important aspects of the meaning of a quantifier express in terms of its observable behaviour.
We have therefore considered several properties of semi-fuzzy quantifiers and fuzzy quantifiers each of which captures some aspects of the behaviour of a quantifier. The properties presented are sometimes borrowed from logic, but our prime source of these properties is the logico-linguistic Theory of Generalized Quantifiers (TGQ), which has focused on those criteria which are essential from a linguistic perspective. A ‘reasonable’ choice of QFM is expected to preserve the important properties of all quantifiers (e.g. monotonicity properties).
In addition to preserving linguistic properties of quantifiers, we are interested in obtaining a system which also preserves important relationships between quantifiers. The prime example are functional relationships between quantifiers which are established by certain constructions (like dualisation or negation). Compatibility with such constructions corresponds to the well-known mathematical concept of a homomorphism (structure-preserving mapping). A slight pecularity of these constructions is that some of these depend on the quantifier fuzzification mechanism chosen. Our goal is to obtain a ‘self-consistent’ system in which such ‘induced’ constructions are interpreted in such a way that the quantifier fuzzification mechanism is compatible to its own induced constructions.
When discussing fuzzy quantifiers, it should be kept in mind that supporting these only *extends*, but does not replace, the propositional part of a fuzzy logic (i.e. fuzzy truth functions). In addition, there is an obvious relationship between fuzzy quantifiers and extension principles; each QFM gives rise to an induced set of fuzzy truth functions and an induced extension principle in a canonical way. We can therefore judge the adequacy of a quantifier fuzzification mechanism not only by considering its behaviour on fuzzy quantifiers; apart from the quantifier perspective, we can also investigate the properties of the induced truth functions and the induced extension principle.
When compiling the ‘DFS axioms’ from all these adequacy conditions, which single out the specific subclass of ‘reasonable’ QFMs which qualify as DFSes (determiner fuzzification schemes), I have tried to select a *minimal* set of conditions which entails all other important adequacy properties. The DFS axioms (DFS-1) to (DFS-9), as well as the revised set (Z-1) to (Z-6), might therefore appear rather abstract and compressed at first sight. However, as illustrated by the findings on properties of DFSes in chapter two, these axioms fulfill their purpose of capturing a larger number of desired adequacy properties in a condensed form. The reader interested in the motivation of these axioms and in the proofs of the properties cited is invited to consult the original presentation of DFS theory in [9]. The only novel material in chapter two is concerned with the interpretation of existential and universal quantifiers in DFSes. By utilizing the results of H. Thiele, it was possible to precisely characterise how these quantifiers are modelled in a DFS.

After portraying the most important properties of DFSes, we turn to the foundational question of defining DFSes through an independent axiom set. The original axiom set (DFS-1) to (DFS-9) still contains some subtle interdependencies some of which are caused with the use of multiplace quantifiers in the construction of induced truth functions. It was therefore decided to replace the original construction of induced fuzzy truth functions by an alternative definition which does not depend on quantifiers of arities greater than one. The alternative definition is justified by the fact that it coincides with the original construction in all DFSes, as has been shown in the third chapter. We can hence replace the old construction with the new one without altering the target class of QFMs that qualify as DFSes. By exploiting the benefits of the new construction for induced connectives, it was possible to replace the original DFS axioms with an equivalent axiom set (Z-1) to (Z-6). In order to prove that the new axiom set is independent, we have introduced a special class of QFMs definable in terms of a three-valued cutting mechanism $(\bullet)_\gamma$. These provide the required candidate models which validate all of the axioms (Z-1) to (Z-6) except for one particular axiom, thus proving its independence. To facilitate the independence proof, the dependencies between certain conditions on the aggregation mappings $\mathcal{B}$ and the 'Z-conditions' have been studied in detail. In particular, we have provided a characterisation of the class of $\mathcal{M}_\mathcal{B}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping. Based on these preparations, the independence proof of the revised axiom system is straightforward.

After enhancing the foundations of DFS theory, a number of additional adequacy conditions have been formalized and investigated. Some novel results on properties of DFSes have also been presented. For example, we have introduced the notion of a specificity consistent $\widetilde{\vee}$-DFSes and shown that a collection of such DFSes has an upper specificity bound exactly if it is specificity consistent. This is important because one is often interested in obtaining as specific results as possible. A formula which reveals the exact form of the least upper bound has also been presented.

Next, two continuity conditions on QFMs have been introduced, which capture distinct smoothness or robustness considerations on QFMs: A QFM is Q-continuous if similar semi-fuzzy quantifiers are mapped to similar fuzzy quantifiers, while it is arg-continuous if the resulting fuzzy quantifiers are smooth (i.e. similar fuzzy argument sets map to similar results). These continuity conditions have not been incorporated into the DFS axioms in order not to exclude certain boundary cases of DFSes (e.g. DFSes which induce the drastic product t-norm, and are hence discontinuous in their arguments). However, every practical model should possess these properties.

Apart from these continuity conditions, two additional properties have been formalized which are concerned with the propagation of fuzziness. It seems reasonable to assume that whenever either the quantifier or the argument sets become fuzzier, the computed quantification results should become fuzzier, too. As we have seen, all $\mathcal{M}_\mathcal{B}$-DFSes fulfill these properties, but other DFSes may fall short of it.

We have also considered an insensitivity property called contextuality, and all DFSes have been shown to satisfy this very elementary adequacy criterion. Unfortunately, contextuality conflicts with another desirable property, viz. the preservation of convexity. It has been argued that under conditions considerably weaker than the DFS axioms, it is not possible to preserve convexity properties of a quantifier, not even of the quantitative variety. Only if one restricts attention to quantitative one-place quantifiers on finite base sets, convexity properties can be preserved by some DFSes (e.g. $\mathcal{M}_{CX}$).

Finally, we have discussed the construction of fuzzy argument insertion: A QFM compliant with fuzzy argument insertion permits for a representation of intermediate quantifiers which result from inserting a fuzzy argument into one of the argument positions through suitable semi-fuzzy quantifiers. This property is important because it allows for a compositional interpretation of quantifiers restricted by fuzzy adjectives; however, it is very restrictive. For example, we have shown that among the class of standard DFSes, only a single model exists that fulfills this property, again the DFS $\mathcal{M}_{CX}$.

Turning to $\mathcal{M}_\mathcal{B}$-DFSes, we have shown these can be compared for specificity and used this to spell out the most specific and least specific $\mathcal{M}_\mathcal{B}$-DFSes. We have then introduced the particularly well-behaved DFS $\mathcal{M}_{CX}$, and shown that it weakly preserves convexity and even complies with fuzzy argument insertion. Next we have presented necessary and sufficient conditions for an $\mathcal{M}_{\mathcal{B}}$-DFS to be Q-continuous or arg-continuos. Based on these conditions, it was then easily shown that a number of DFSes (including the model $\mathcal{M}_{CX}$) satisfy both continuity conditions, while other $\mathcal{M}_{\mathcal{B}}$-DFSes (in particular the most specific and least specific cases) fail to be continuous and are hence not practical models.

In order to gain more knowledge on DFSes, and in particular the special role of $\mathcal{M}_{CX}$, an apparent way of obtaining upper and lower bounds on the quantification results of DFSes has been formalized and investigated. In some cases, these bounds are specific enough to uniquely determine the interpretation of a given quantifier (for example, this is the case with quantifiers of the type *at least k*, but only when restricting attention to standard DFSes). In the case of a DFS which fulfills the condition on fuzzy argument insertion, the upper and lower bounds reduce to a particularly simple form. This can be used to show that among all standard DFSes, only $\mathcal{M}_{CX}$ is compatible with fuzzy argument insertion. In addition, this uncovers that $\mathcal{M}_{CX}$ can be evaluated through (possibly infinitary) logical formulas, and hence implements the so-called substitution approach to fuzzy quantification.

Some other theorems are also apparent from this representation of $\mathcal{M}_{CX}$. For example, it is easily shown that $\mathcal{M}_{CX}$ generalizes the Sugeno integral, and hence the FG-count approach to fuzzy quantification. The representation of $\mathcal{M}_{CX}$ can be further simplified in the case of quantitative one-place quantifiers. Using a suitable fuzzy interval cardinality measure, the quantification results of $\mathcal{M}_{CX}$ can be computed directly from interval cardinality information. Some further insensitivity properties of $\mathcal{M}_{CX}$ can also be proven from this representation.

Finally, we were able to use these results to prove that all standard DFSes agree on two-valued quantifiers, and in this case, they coincide with the well-known fuzzification mechanism proposed by B.R. Gaines as a foundation for fuzzy reasoning [7].[15]

[15] This is apparent from [9, Th-59, p.77].

Appendix

I have adopted the convention of calling any proposition which occurs in the main text a *theorem*, and of calling any proposition which only occurs in the proofs a *lemma*. Theorems are referred to as Th-n, where n is the number of the theorem, and lemmata are referred to as L-n, where n is the number of the lemma. Equations which are embedded in proofs are referred to as (n), where n is the number of the equation.

A Proofs of Theorems in Chapter 2

A.1 Proof of Theorem 2

The claim a. that $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$ has been shown in [9, Th-2, p.27]. The claim b. that $\widetilde{\neg}$ is a strong negation operator has been proven in [9, Th-3, p.28]. Finally, it has been shown in [9, Th-6, p.30] and [9, Th-7, p.31] that $\widetilde{\wedge}$ is a t-norm, $\widetilde{\vee}$ is an s-norm and $\widetilde{\rightarrow}$ can be expressed as $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2$.

A.2 Proof of Theorem 17

The claim of part a. has been shown in [9, Th-21, p.40]. Part b. corresponds to [9, Th-22, p.40].

A.3 Proof of Theorem 21

Suppose $E \neq \varnothing$ is an arbitrary nonempty base set.

1. $\mathcal{F}(\forall_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is a T-quantifier. Let us consider the defining properties **a.** to **d.** of Def. 36.

a. This property is ensured by Th-20.b;

b. This property is apparent from the fact that $\forall_E : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is constantly zero in the range $(\varnothing, E \cap \neg\{e\})$. By Th-7, $\mathcal{F}(\forall_E)$ is locally constant in that range, too, and Th-1 tells us that it is constantly zero in that range;

c. This property is ensured by Th-6;

d. This property is ensured by Th-15, noting that the crisp universal quantifier $\forall_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is quantitative.

2. $\mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is an S-quantifier. Here we consider the defining properties **a.** to **d.** of Def. 37.

a. This property is apparent from the fact that $\exists_E : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is constantly one in the range $(\{e\}, E)$. By Th-7, $\mathcal{F}(\exists_E)$ is locally constant in that range, too, and Th-1 tells us that it is constantly one in that range;

b. This property is ensured by Th-19.b;

c. This property is ensured by Th-6;

d. This property is ensured by Th-15, noting that the crisp existential quantifier $\exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is quantitative.

A.4 Proof of Theorem 24

By Th-21, $\mathcal{F}(\forall_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is a T-quantifier. By Th-22, $\mathcal{F}(\forall_E)$ can be decomposed in the way claimed by the theorem. The only claim which remains to be shown is that $\widetilde{\wedge}_{\mathcal{F}(\forall_E)} = \widetilde{\wedge}$. But this is apparent from Th-20.c and Def. 38.

A.5 Proof of Theorem 25

By Th-21, $\mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is an S-quantifier. By Th-23, $\mathcal{F}(\exists_E)$ can be decomposed in the form claimed by the theorem. Again, the only claim which remains to be shown is that $\widetilde{\vee}_{\mathcal{F}(\exists_E)} = \widetilde{\vee}$. This is obvious from Th-19 and Def. 39.

A.6 Proof of Theorem 26

Suppose $\mathcal{F}$ is a DFS.

a. By Th-17, the induced extension principle $\widehat{\mathcal{F}}$ is uniquely determined by the interpretation of existential quantifiers in $\mathcal{F}$. Hence By Th-25, the interpretation of existential quantifiers in $\mathcal{F}$ is uniquely determined by its induced disjunction $\widetilde{\vee}$. Hence $\widehat{\mathcal{F}}$ is uniquely determined by the induced disjunction $\widetilde{\vee}$.

b. See [9, Th-25, p. 41].

A.7 Proof of Theorem 30

It is apparent from Th-29 that $\mathcal{F}_{\text{glb}}$ is a $\widetilde{\vee}$-DFS. It is also apparent from $\mathrm{m}_{\frac{1}{2}} X \preceq_c x$ for all $x \in X$ that $\mathcal{F}_{\text{glb}} \preceq_c \mathcal{F}_j$ for all $j \in \mathcal{J}$, i.e. $\mathcal{F}_{\text{glb}}$ is a lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$. It remains to be shown that $\mathcal{F}_{\text{glb}}$ is a *greatest* lower bound. Hence let $\mathcal{F}$ a lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$; we shall assume that $\mathcal{F} \preceq_c \mathcal{F}_{\text{glb}}$ fails. Then there exists $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ such that $\mathcal{F}(Q)(X_1, \ldots, X_n) \not\preceq_c \mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n)$. We may assume without loss of generality that $\mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n) \geq \frac{1}{2}$ (otherwise, we might use $\neg Q$ rather than Q). Then $\mathcal{F}(Q)(X_1, \ldots, X_n) \not\preceq_c \mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n)$ means that either

a. $\mathcal{F}(Q)(X_1, \ldots, X_n) > \mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n)$, or

b. $\mathcal{F}(Q)(X_1, \ldots, X_n) < \frac{1}{2}$.

We shall consider these cases in turn.

Case a.: $\mathcal{F}(Q)(X_1, \ldots, X_n) > \mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n)$**.**
Because $\mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n) \geq \frac{1}{2}$, we know from Def. 46, Def. 45 that

$$\begin{aligned}\mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} \\ &= \max(\tfrac{1}{2}, \inf\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}\,.\end{aligned}$$

Hence for all $\varepsilon > 0$, there exists $j' \in \mathcal{J}$ such that

$$\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) < \mathcal{F}_{\text{glb}}(Q)(X_1, \ldots, X_n) + \varepsilon\,.$$

In particular, substituting $\varepsilon = \mathcal{F}(Q)(X_1, \ldots, X_n) - \mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) > 0$, we conclude that there exists $j' \in \mathcal{J}$ such that

$$\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) < \mathcal{F}(Q)(X_1, \ldots, X_n)\,.$$

Because $\mathcal{F}(Q)(X_1, \ldots, X_n) > \frac{1}{2}$, this means that $\mathcal{F}(Q)(X_1, \ldots, X_n) \not\preceq_c \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n)$, cf. (9). Hence by Def. 44, $\mathcal{F} \not\preceq_c \mathcal{F}_{j'}$, which contradicts the assumption that $\mathcal{F}$ be a lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$.

Case b.: $\mathcal{F}(Q)(X_1, \ldots, X_n) < \frac{1}{2}$**.**
In this case, we conclude from $\mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) \geq \frac{1}{2}$ and Def. 45, Def. 46 that

$$\sup\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} \geq \tfrac{1}{2}\,.$$

Hence for all $\varepsilon > 0$, there exists $j' \in \mathcal{J}$ such that

$$\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \tfrac{1}{2} - \varepsilon\,.$$

In particular, substituting $\varepsilon = \frac{1}{2} - \mathcal{F}(Q)(X_1, \ldots, X_n) > 0$, we conclude that there exists $j \in \mathcal{J}$ such that

$$\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \mathcal{F}(Q)(X_1, \ldots, X_n)\,.$$

Because $\mathcal{F}(Q)(X_1, \ldots, X_n) < \frac{1}{2}$, this means by (9) that

$$\mathcal{F}(Q)(X_1, \ldots, X_n) \not\preceq_c \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n)$$

and in turn, $\mathcal{F} \not\preceq_c \mathcal{F}_{j'}$ by Def. 44. Hence $\mathcal{F}$ is not a lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$.

B Proofs of Theorems in Chapter 3

B.1 Proof of Theorem 33

Let us first observe that by (10), we have

$$\mathrm{Im}\,\iota_i^{n,E} \cap \mathrm{Im}\,\iota_j^{n,E} = \varnothing \tag{38}$$

if $i \neq j$, and hence

$$\widehat{\iota}_i^{n,E}(X_i) \cap \mathrm{Im}\,\iota_j^{n,E} = \begin{cases} \varnothing & : \quad i \neq j \\ \widehat{\iota}_j^{n,E}(X_j) & : \quad i = j \end{cases} \tag{39}$$

for all $i, j \in \{1, \ldots, n\}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Therefore

$$\begin{aligned}
&(\iota_j^{n,E})^{-1}(\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i)) \\
&= (\iota_j^{n,E})^{-1}((\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i)) \cap \mathrm{Im}\,\iota_j^{n,E}) && \text{because } f^{-1}(A) = f^{-1}(A \cap \mathrm{Im}\,f) \\
&= (\iota_j^{n,E})^{-1}(\bigcup_{i=1}^{n} (\widehat{\iota}_i^{n,E}(X_i) \cap \mathrm{Im}\,\iota_j^{n,E})) && \text{by distributivity of } \cap \text{ and } \cup \\
&= (\iota_j^{n,E})^{-1}(\widehat{\iota}_j^{n,E}(X_j)) && \text{by (39)} \\
&= X_j\,,
\end{aligned}$$

i.e.

$$(\iota_j^{n,E})^{-1}(\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i)) = X_j \,. \tag{40}$$

It is then apparent that

$$\begin{aligned}
&\langle Q\rangle(\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i)) \\
&= Q((\iota_1^{n,E})^{-1}(\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i)), \ldots, (\iota_n^{n,E})^{-1}(\bigcup_{i=1}^{n} \widehat{\iota}_i^{n,E}(X_i))) && \text{by Def. 55} \\
&= Q(X_1, \ldots, X_n) && \text{by (40).}
\end{aligned}$$

B.2 Proof of Theorem 34

Let us first state some lemmata which will facilitate the proof of the theorem.

Lemma 1
Let $\widetilde{\widetilde{\vee}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *a mapping such that* $0 \,\widetilde{\widetilde{\vee}}\, x = x \,\widetilde{\widetilde{\vee}}\, 0 = x$ *for all* $x \in \mathbf{I}$. *Further let* $n \in \mathbb{N} \setminus \{0\}$, $j \in \{1, \ldots, n\}$ *and* $x_j \in \mathbf{I}$. *If* $x_i = 0$ *for all* $i \neq j$, *then*

$$\mathop{[\widetilde{\widetilde{\vee}}]}_{i=1}^{n} x_i = x_j \,.$$

Proof By induction on n.

a. $n = 1$.
Then $j = 1$ and hence

$$\begin{aligned}
\mathop{[\widetilde{\widetilde{\vee}}]}_{i=1}^{1} x_j &= x_1 && \text{by Def. 58.a} \\
&= x_j \,.
\end{aligned}$$

b. $n > 1$. We will distinguish two cases.

1. $j = 1$. Then

$$\begin{aligned}
\mathop{[\widetilde{\widetilde{\vee}}]}_{i=1}^{n} x_i &= x_1 \,\widetilde{\widetilde{\vee}} \mathop{[\widetilde{\widetilde{\vee}}]}_{i=1}^{n-1} x_{i+1} && \text{by Def. 58.b} \\
&= x_1 \,\widetilde{\widetilde{\vee}} \mathop{[\widetilde{\widetilde{\vee}}]}_{i=1}^{n-1} 0 && \text{by condition on } x_i,\ j = 1 \\
&= x_1 \,\widetilde{\widetilde{\vee}}\, 0 && \text{by induction hypothesis for } n-1 \\
&= x_1 && \text{by condition on } \widetilde{\widetilde{\vee}}.
\end{aligned}$$

2. $j > 1$. Then

$$\begin{aligned}
\underset{i=1}{\overset{n}{[\widetilde{\widetilde{\vee}}]}}\, x_i &= x_1 \,\widetilde{\widetilde{\vee}}\, \underset{i=1}{\overset{n-1}{[\widetilde{\widetilde{\vee}}]}}\, x_{i+1} && \text{by Def. 58.b}\\
&= 0 \,\widetilde{\widetilde{\vee}}\, \underset{i=1}{\overset{n-1}{[\widetilde{\widetilde{\vee}}]}}\, x_{i+1} && \text{by assumption on } x_i \text{ and } j > 1\\
&= \underset{i=1}{\overset{n-1}{[\widetilde{\widetilde{\vee}}]}}\, x_{i+1} && \text{by condition on } \widetilde{\widetilde{\vee}}\\
&= x_j && \text{by induction hypothesis for } n-1.
\end{aligned}$$

Lemma 2
Suppose $\widetilde{\widetilde{\vee}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *satisfies* $x \,\widetilde{\widetilde{\vee}}\, 0 = 0 \,\widetilde{\widetilde{\vee}}\, x = x$ *for all* $x \in \mathbf{I}$*. Further suppose* E *is a non-empty set and* $n \in \mathbb{N} \setminus \{0\}$*. For all* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*,* $[\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i) \in \widetilde{\mathcal{P}}(E_n)$ *is the fuzzy set with membership degrees*

$$\mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j) = \mu_{X_j}(e)\,,$$

for all $(e, j) \in E_n$*.*

Proof Let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $(e, k) \in E_n$ be given. By the definition of the standard extension principle,

$$\mu_{\hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j) = \begin{cases} \mu_{X_i}(e) & : \quad j = i \\ 0 & : \quad j \neq i \end{cases} \tag{41}$$

for all $i \in \{1, \ldots, n\}$, because $\iota_i^{n,E}$ is injective and $\operatorname{Im} \iota_i^{n,E} = \{(e, i) : e \in E\}$.
In particular, $x_i = \mu_{\hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j)$ satisfies the condition of L-1, which shows that

$$\underset{i=1}{\overset{n}{[\widetilde{\widetilde{\vee}}]}}\, \mu_{\hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j) = \mu_{\hat{\hat{\iota}}_j^{n,E}(X_j)}(e, j) = \mu_{X_j}(e)\,. \tag{42}$$

for all $(e, j) \in E_n$. Hence

$$\begin{aligned}
&\mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j)\\
&= \underset{i=i}{\overset{n}{[\widetilde{\widetilde{\vee}}]}}\, \mu_{\hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j) && \text{by (13)}\\
&= \mu_{X_j}(e) && \text{by (42),}
\end{aligned}$$

which finishes the proof.

Proof of Theorem 34

Let $X \in \widetilde{\mathcal{P}}(E_n)$ and $j \in \{1, \dots, n\}$. Consider some $e \in E$.

$$
\begin{aligned}
&\mu_{(\hat{\iota}_j^{n,E})^{-1}([\widetilde{\widetilde{\cup}}]_{i=1}^n \hat{\iota}_i^{n,E}(X_i))}(e) \\
&= \mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^n \hat{\iota}_i^{n,E}(X_i)}(\iota_j^{n,E}(e)) && \text{by Def. 34} \\
&= \mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^n \hat{\iota}_i^{n,E}(X_i)}(e, j) && \text{by Def. 54} \\
&= \mu_{X_j}(e) && \text{by L-2,}
\end{aligned}
$$

i.e.

$$
(\hat{\iota}_j^{n,E})^{-1}([\widetilde{\widetilde{\bigcup}}]_{i=1}^{n} \hat{\iota}_i^{n,E}(X_i)) = X_j \tag{43}
$$

From this we obtain

$$
\begin{aligned}
&\langle \widetilde{Q} \rangle ([\widetilde{\widetilde{\bigcup}}]_{i=1}^{n} \hat{\iota}_i^{n,E}(X_i)) \\
&= \widetilde{Q}((\hat{\iota}_1^{n,E})^{-1}(X), \dots, (\hat{\iota}_n^{n,E})^{-1}(X)) && \text{by Def. 56} \\
&= \widetilde{Q}(X_1, \dots, X_n) && \text{by (43).}
\end{aligned}
$$

B.3 Proof of Theorem 35

An immediate consequence of Th-33 is the following lemma:

Lemma 3
Suppose $\mathcal{F}$ is a QFM with the following properties:

a. *for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$,*

$$
\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\widetilde{\cup}}\,; \tag{44}
$$

b. *$\mathcal{F}$ satisfies* (DFS 9) *(functional application).*

Then for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$,

$$
\mathcal{F}(Q) = \mathcal{F}(\langle Q \rangle)\widetilde{\widetilde{\cup}}^{n} \circ \bigtimes_{i=1}^{n} \widehat{\mathcal{F}}(\iota_i^{n,E})\,.
$$

Notes

- The conditions of the lemma are of course satisfied by every DFS $\mathcal{F}$.
- The lemma has been stated in somewhat more general form (rather than for DFSes only) in order to allow for the case where we have limited knowledge about the properties of $\mathcal{F}$ beyond those mentioned in the theorem.

Proof Suppose $\mathcal{F}$ satisfies the requirements of the lemma, and let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an arbitrary semi-fuzzy quantifier ($n > 0$). Then for all $X_1, \ldots, X_n$,

$$\begin{aligned}
&\mathcal{F}(Q)(X_1, \ldots, X_n) \\
&= \mathcal{F}(\langle Q \rangle \cup^n \circ \mathop{\times}_{i=1}^{n} \widehat{\iota}_i^{n,E})(X_1, \ldots, X_n) && \text{by Th-33} \\
&= \mathcal{F}(\langle Q \rangle) \widetilde{\cup}^n \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(\iota_i^{n,E})(X_1, \ldots, X_n) && \text{by (44), (DFS 9).}
\end{aligned}$$

Proof of Theorem 35

Consider $X \in \widetilde{\mathcal{P}}(E_n)$. With $X_1 = (\hat{\hat{\iota}}_1^{n,E})^{-1}(X), \ldots, X_n = (\hat{\hat{\iota}}_n^{n,E})^{-1}(X)$ we have

$$X = [\widetilde{\widetilde{\bigcup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)\,. \tag{45}$$

Hence

$$\begin{aligned}
&\mathcal{F}(\langle Q \rangle)(X) \\
&= \mathcal{F}(\langle Q \rangle)([\widetilde{\widetilde{\bigcup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)) && \text{by (45)} \\
&= \mathcal{F}(Q)(X_1, \ldots, X_n) && \text{by L-3, (12)} \\
&= \langle \mathcal{F}(Q) \rangle([\widetilde{\widetilde{\bigcup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)) && \text{by Th-34} \\
&= \langle \mathcal{F}(Q) \rangle(X) && \text{by (45).}
\end{aligned}$$

B.4 Proof of Theorem 36

Lemma 4
For all $n \in \mathbb{N} \setminus \{0\}$, $i \in \{1, \ldots, n\}$ *and* $Z \in \mathcal{P}(\{)1, \ldots, n\}$,

$$(\eta^{-1}(Z))_i = \pi_i(Z)\,.$$

Proof By the definition of η (see Def. 50), it is apparent that

$$\eta^{-1}(Z) = (\chi_Z(1), \ldots, \chi_Z(n))\,,$$

for all $Z \in \mathcal{P}(\{1, \ldots, n\})$. The claim of the lemma is obtained recalling that

$$\pi_i(Z) = \chi_Z(i)$$

by Def. 6.

Lemma 5
Suppose $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy truth function with arity* $n > 0$. *Then*

$$\langle f^* \rangle = Q_f \circ \widehat{\varrho}\,,$$

where $\varrho : \{*\}_n \longrightarrow \{1, \ldots, n\}$ *is defined by*

$$\varrho(*, i) = i \tag{46}$$

for all $(*, i) \in \{*\}_n$.

Proof Easy noting that for all $W \in \mathcal{P}(\{*\}_n)$, $i \in \{1, \ldots, n\}$,

$$\begin{aligned}
&\pi_i(\widehat{\varrho}(W)) && && (47)\\
&= \chi_{\widehat{\varrho}(W)}(i) && \text{by Def. 6} && (48)\\
&= \chi_W(*, i) && \text{by Def. 17, } \varrho \text{ bijection, } \varrho^{-1}(i) = (*, i). && (49)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\pi_*((\iota_i^{n,\{*\}})^{-1}(W)) && && (50)\\
&= \chi_{(\iota_i^{n,\{*\}})^{-1}(W)}(*) && \text{by Def. 6} && (51)\\
&= \chi_W(*, i) && \text{by Def. 54, Def. 33, } \varrho \text{ bijection.} && (52)
\end{aligned}$$

Therefore

$$\begin{aligned}
&(Q_f \circ \widehat{\varrho})(W) &&\\
&= Q_f(\widehat{\varrho}(W)) &&\\
&= f(\eta^{-1}(\widehat{\varrho}(W))) && \text{by Def. 52}\\
&= f(\pi_1(\widehat{\varrho}(W)), \ldots, \pi_n(\widehat{\varrho}(W))) && \text{by L-4}\\
&= f(\chi_W(*, 1), \ldots, \chi_W(*, n)) && \text{by (49)}\\
&= f(\pi_*((\iota_1^{n,\{*\}})^{-1}(W)), \ldots, \pi_*((\iota_n^{n,\{*\}})^{-1}(W))) && \text{by (52)}\\
&= \langle f \rangle(W) && \text{by Def. 55.}
\end{aligned}$$

Lemma 6

For all $n \in \mathbb{N} \setminus \{0\}$ *and* $x_1, \ldots, x_n \in \mathbf{I}$,

$$\widetilde{\eta}(x_1, \ldots, x_n) = \hat{\hat{\varrho}}([\widetilde{\bigcup}]_{i=1}^{n} \hat{\iota}_i^{n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i))) \,,$$

where $\varrho : \{*\}_n \longrightarrow \{1, \ldots, n\}$ *is defined as in L-5.*

Proof To this end, let us observe that

$$\begin{aligned}
&\mu_{\hat{\hat{\varrho}}([\widetilde{\cup}]_{i=1}^{n} \hat{\iota}_i^{n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)))}(j) &&\\
&= \mu_{[\widetilde{\cup}]_{i=1}^{n} \hat{\iota}_i^{n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i))}(*, j) && \text{because } \varrho \text{ bijection, } \varrho^{-1}(j) = (*, j)\\
&= \mu_{\widetilde{\pi}_*^{-1}(x_j)}(*) && \text{by L-2}\\
&= x_j && \text{by Def. 6}\\
&= \mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(j) && \text{by Def. 51.}
\end{aligned}$$

Hence

$$\mu_{\hat{\hat{\varrho}}([\widetilde{\cup}]_{i=1}^{n} \hat{\iota}_i^{n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)))}(j) = \mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(j)$$

for all $n \in \mathbb{N} \setminus \{0\}$, $j \in \{1, \ldots, n\}$ and $x_1, \ldots, x_n \in \mathbf{I}$, i.e.

$$\hat{\hat{\varrho}}([\widetilde{\bigcup}]_{i=1}^{n} \hat{\iota}_i^{n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i))) = \widetilde{\eta}(x_1, \ldots, x_n) \,.$$

Proof of Theorem 36

Suppose $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ is a semi-fuzzy truth function and $x_1, \ldots, x_n \in \mathbf{I}$. Then

$$
\begin{aligned}
&\widetilde{\mathcal{F}}(f)(x_1, \ldots, x_n) \\
&= \mathcal{F}(f^*)(\widetilde{\pi}_*^{-1}(x_1), \ldots, \widetilde{\pi}_*^{-1}(x_n)) && \text{by Def. 8} \\
&= \langle \mathcal{F}(f^*)\rangle \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(\iota_i^{n,\{*\}})(\widetilde{\pi}_*^{-1}(x_i)) && \text{by Th-34} \\
&= \langle \mathcal{F}(f^*)\rangle \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{\,n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)) && \text{by Th-14} \\
&= \mathcal{F}(\langle f^*\rangle) \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{\,n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)) && \text{by Th-35} \\
&= \mathcal{F}(Q_f \circ \widehat{\varrho}) \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{\,n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)) && \text{by L-5} \\
&= \mathcal{F}(Q_f) \circ \hat{\hat{\varrho}} \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{\,n,\{*\}}(\widetilde{\pi}_*^{-1}(x_i)) && \text{by (DFS 9), Th-14 (ϱ injective)} \\
&= \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n)) && \text{by L-6} \\
&= \widetilde{\widetilde{\mathcal{F}}}(f)(x_1, \ldots, x_n) && \text{by Def. 52.}
\end{aligned}
$$

B.5 Proof of Theorem 37

a. Suppose $\mathcal{F}$ is a DFS (i.e. a model of the axiom set DFS). Then by Th-36,

$$\widetilde{\mathcal{F}}(\neg) = \widetilde{\widetilde{\mathcal{F}}}(\neg)$$
$$\widetilde{\mathcal{F}}(\wedge) = \widetilde{\widetilde{\mathcal{F}}}(\wedge)\,,$$

i.e. DFS' (where $\widetilde{\neg}$ is replaced by $\widetilde{\widetilde{\neg}}$, and $\widetilde{\wedge}$ is replaced by $\widetilde{\widetilde{\wedge}}$) coincides with DFS, of which $\mathcal{F}$ is a model by assumption.

b. Suppose $\mathcal{F}$ is a model of DFS'. Then in particular, $\mathcal{F}$ satisfies (DFS 2) (which coincides with (DFS 2)', because it does not refer to negation or conjunction). In addition, $\mathcal{F}$ satisfies (DFS 3)', i.e.

$$\mathcal{F}(\widetilde{\widetilde{\neg}} Q) = \widetilde{\widetilde{\neg}} \mathcal{F}(Q)$$

for arbitrary semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. The premises of Th-11 are hence satisfied, according to which $\widetilde{\neg} = \widetilde{\widetilde{\neg}}$, as desired, i.e. (DFS 3) holds for $\mathcal{F}$. Because $\widetilde{\neg} = \widetilde{\widetilde{\neg}}$, (DFS 5) and (DFS 5)' become identical, too, i.e. (DFS 5) is satisfied by $\mathcal{F}$.
By similar reasoning, we observe that $\mathcal{F}$ satisfies (DFS 6)':

$$\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\widetilde{\cap}}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$; in this case, the premises of Th-12 are satisfied, which tells us that $\widetilde{\widetilde{\wedge}} = \widetilde{\wedge}$ and hence (DFS 6) holds.
All other axioms (DFS i)' do not refer to the induced negation and induced conjunction and are hence identical to the corresponding axiom (DFS i), which is then known to hold because $\mathcal{F}$ satisfies all (DFS i)' by assumption.
This finishes the proof that $\mathcal{F}$ is a model of the original DFS axioms.

B.6 Proof of Theorem 38

Let us first show that every QFM which satisfies the DFS axioms also satisfies Z-1 to Z-6.

Lemma 7
Every DFS $\mathcal{F}$ (i.e. model of the (DFS 1) to (DFS 9)) satisfies Z-1 to Z-6.

Proof

a. (Z-1) (correct generalisation) is apparent from Th-1.

b. (Z-2) (projection quantifiers) is immediate from Th-10.

c. (Z-3) (dualisation) is apparent from Th-3 and Th-36;

d. (Z-4) (internal joins) is known to hold from Th-4 and Th-36;

e. (Z-5) coincides with (DFS 8).

f. (Z-6) coincides with (DFS 9).

This finishes the proof that every DFS satisfies Z-1 to Z-6.
Let us now turn to the converse problem of showing that every model of Z-1 to Z-6 also satisfies the original DFS axioms. We shall exploit that by Th-37, we can use DFS' where the induced negation and conjunction according to the "new" construction of induced connectives are used.
It is then apparent that the following DFS axioms hold:

Lemma 8
Suppose $\mathcal{F}$ is a QFM. If $\mathcal{F}$ satisfies (Z-1) to (Z-6), then $\mathcal{F}$ satisfies (DFS 1), (DFS 2), (DFS 8) *and* (DFS 9).

Proof Trivial:

a. (DFS 1) is a subcase of (Z-1);

b. (DFS 2) is a subcase of (Z-2);

c. (DFS 8) coincides with (Z-5);

d. (DFS 9) coincides with (Z-6).

It remains to be shown that every model of Z-1 to Z-6 also satisfies (DFS 3), (DFS 4), (DFS 5), (DFS 6) and (DFS 7).

Lemma 9
Suppose $f : E \longrightarrow E'$ is some mapping, where E and E' are nonempty, and further suppose $\mathcal{F}$ is a QFM which satisfies (Z-2). *If $e' \in E'$ is such that*

$$\chi_{\widehat{f}(\bullet)}(e') = \pi_e \tag{53}$$

for some $e \in E$, then

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \widetilde{\pi}_e(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Proof Trivial:

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(e') &= \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X) && \text{by Def. 19} \\
&= \mathcal{F}(\pi_e)(X) && \text{by (53)} \\
&= \widetilde{\pi}_e(X) && \text{by (Z-2).}
\end{aligned}$$

Lemma 10
Suppose $f : E \longrightarrow E'$ is some mapping, where E, E' are nonempty, and suppose $\mathcal{F}$ is a QFM which satisfies (Z-1) *and* (Z-5)*. Then for all $e' \in E' \setminus \operatorname{Im} f$,*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = 0\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Proof Trivial:
If $e' \in E' \setminus \operatorname{Im} f$, then

$$\chi_{\widehat{f}(X)}(e') = 0 \tag{54}$$

for all $X \in \mathcal{P}(E)$, i.e. $\chi_{\widehat{f}(\bullet)}(e') : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is constantly zero. In particular, it is nonincreasing, i.e. $\mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))$ is nonincreasing by (Z-5). We hence obtain:

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(e') &= \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X) && \text{by Def. 19} \\
&\leq \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(\varnothing) && \text{nonincreasing quantifier (see above)} \\
&= \chi_{\widehat{f}(\varnothing)}(e') && \text{by (Z-1)} \\
&= 0 && \text{by (54).}
\end{aligned}$$

Because $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') \in \mathbf{I}$, i.e. $\mu_{\widehat{\mathcal{F}}(f)(X)}(e') \geq 0$, this proves the claim.
From this, we obtain that the induced extension principle of a QFM which satisfies (Z-1), (Z-2) and (Z-5) is guaranteed to behave as expected for injective mappings:

Lemma 11
Suppose $\mathcal{F}$ is a QFM which satisfies (Z-1), (Z-2) *and* (Z-5)*. Further let nonempty sets E, E' be given. If $f : E \longrightarrow E'$ is injective, then $\widehat{\mathcal{F}}(f) = \hat{\hat{f}}$, i.e.*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \begin{cases} \mu_X(f^{-1}(e')) & : \quad e' \in \operatorname{Im} f \\ 0 & : \quad e' \notin \operatorname{Im} f \end{cases}$$

for all $X \in E$, $e' \in E'$.

Proof

a. $e' \in \operatorname{Im} f$. Because f is injective, $e = f^{-1}(e')$ is the only element of E such that $f(e) = e'$. Therefore

$$e' \in \widehat{f}(X) \Leftrightarrow e \in X$$

for all $X \in \mathcal{P}(E)$; in other words,

$$\chi_{\widehat{f}(X)}(e') = \pi_e(X)$$

for all $X \in \mathcal{P}(E)$. The preconditions of L-9 are satisfied, and we obtain

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \widetilde{\pi}_e(X)$$

for all $X \in \widetilde{\mathcal{P}}(E)$, as desired.

b. $e' \notin \operatorname{Im} f$.
Then by L-10,

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = 0\,.$$

Lemma 12 (Induced negation)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then $\widetilde{\widetilde{\neg}} = \widetilde{\widetilde{\mathcal{F}}}(\neg) : \mathbf{I} \longrightarrow \mathbf{I}$ is a strong negation operator.

Proof

a. $\widetilde{\widetilde{\neg}}\, 0 = 1$ and $\widetilde{\widetilde{\neg}}\, 1 = 0$ by (Z-1).

b. $\widetilde{\widetilde{\neg}}$ is monotonically nonincreasing by (Z-5) because $\neg$ is nonincreasing.

c. $\widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, x = x$ for all $x \in \mathbf{I}$, i.e. $\widetilde{\widetilde{\neg}}$ is involutive.
To see this, choose some singleton set $\{*\}$. We then have

$$\pi_* \widetilde{\widetilde{\Box}} = \pi_*\,, \qquad (55)$$

because we already know by **a.** that $\widetilde{\widetilde{\neg}} \neg x = x$ for $x \in \{0, 1\}$. Hence

$$\begin{aligned}
\widetilde{\pi}_* &= \mathcal{F}(\pi_*) && \text{by (Z-2)}\\
&= \mathcal{F}(\pi_* \widetilde{\widetilde{\Box}}) && \text{by (55)}\\
&= \mathcal{F}(\pi_*)\widetilde{\widetilde{\Box}} && \text{by (Z-3)}\\
&= \widetilde{\pi}_* \widetilde{\widetilde{\Box}} && \text{by (Z-2)}\,,
\end{aligned}$$

i.e.

$$\widetilde{\pi}_* = \widetilde{\pi}_* \widetilde{\widetilde{\square}} \,. \tag{56}$$

Now let $x \in \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\{*\})$, defined by $\mu_X(*) = x$. Then

$$\begin{aligned}
x &= \mu_X(*) && \text{by definition of } X \\
&= \widetilde{\pi}_* X && \text{by definition of } \widetilde{\pi}_* \\
&= \widetilde{\pi}_* \widetilde{\widetilde{\square}} X && \text{by (56)} \\
&= \widetilde{\widetilde{\neg}}\, \widetilde{\pi}_*(\widetilde{\widetilde{\neg}}\, X) && \text{by definition of dualisation} \\
&= \widetilde{\widetilde{\neg}}\, \widetilde{\widetilde{\neg}}\, x \,,
\end{aligned}$$

where the last step is apparent because $\widetilde{\widetilde{\neg}}\, X \in \widetilde{\mathcal{P}}(\{*\})$ has $\mu_{\widetilde{\widetilde{\neg}} X}(*) = \widetilde{\widetilde{\neg}}\, \mu_X(*) = \widetilde{\widetilde{\neg}}\, x$ and hence $\widetilde{\pi}_*(\widetilde{\widetilde{\neg}}\, X) = \mu_{\widetilde{\widetilde{\neg}} X}(*) = \widetilde{\widetilde{\neg}}\, x$, i.e. $\widetilde{\widetilde{\neg}}\, \widetilde{\pi}_*(\widetilde{\widetilde{\neg}}\, X) = \widetilde{\widetilde{\neg}}\, \widetilde{\widetilde{\neg}}\, x$. This finishes the proof that $\widetilde{\widetilde{\neg}}$ is a strong negation operator.

Lemma 13

$$Q_\vee = \exists_{\{1,2\}} \,,$$

where $\exists_{\{1,2\}} : \mathcal{P}(\{1,2\}) \longrightarrow \mathbf{2}$ *is defined by*

$$\exists_{\{1,2\}}(X) = \begin{cases} 1 & : \quad X \neq \varnothing \\ 0 & : \quad X = \varnothing \end{cases} \tag{57}$$

for all $X \in \mathcal{P}(\{1,2\})$.

Proof Trivial, by checking all four choices of $X \in \mathcal{P}(\{1,2\})$.

Lemma 14 (Induced disjunction)
If a QFM $\mathcal{F}$ *satisfies Z-1 to Z-6, then* $\widetilde{\widetilde{\vee}} = \widetilde{\widetilde{\mathcal{F}}}(\vee) : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *has the following properties:*

a. $x \,\widetilde{\widetilde{\vee}}\, 0 = x$, *for all* $x \in \mathbf{I}$

b. $x_1 \,\widetilde{\widetilde{\vee}}\, x_2 = x_2 \,\widetilde{\widetilde{\vee}}\, x_1$ *for all* $x_1,\, x_2 \in \mathbf{I}$ *(commutativity)*

Note. This means that at this point, we have shown that $\widetilde{\widetilde{\vee}}$ has certain important properties of an s-norm. The remaining properties of s-norms ($x \,\widetilde{\widetilde{\vee}}\, 1 = 1$, nondecreasing monotonicity, and associativity) also hold but we have dropped the proofs in the sake of brevity because these properties will not be needed in the following.

Proof

a. $x \mathbin{\widetilde{\widetilde{\vee}}} 0 = x$, for all $x \in \mathbf{I}$
To this end, let us consider $c_1 : \{1\} \longrightarrow \{1, 2\}$, defined by $c_1(1) = 1$. Observing that

$$\pi_1 = \exists_{\{1,2\}} \circ \widehat{c_1} \,, \tag{58}$$

we obtain by (Z-6) and (Z-2) that

$$\widetilde{\pi}_1 = \mathcal{F}(\exists_{\{1,2\}}) \circ \widehat{\mathcal{F}}(c_1) \,. \tag{59}$$

Now let us consider $\widehat{\mathcal{F}}(c_1) \in \widetilde{\mathcal{P}}(\{1, 2\})$.
Clearly $\chi_{\widehat{c_1}(X)}(1) = \chi_X(1) = \pi_1(X)$ for all $X \in \mathcal{P}(\{1\})$, i.e.

$$\chi_{\widehat{c_1}(\bullet)}(1) = \pi_1 \,. \tag{60}$$

By L-9, we then have

$$\mu_{\widehat{\mathcal{F}}(c_1)(X)}(1) = \widetilde{\pi}_1(X) \tag{61}$$

for all $X \in \widetilde{\mathcal{P}}(\{1\})$.
Furthermore $2 \notin \operatorname{Im} c_1$ and hence by L-10,

$$\mu_{\widehat{\mathcal{F}}(c_1)(X)}(2) = 0 \tag{62}$$

for all $X \in \widetilde{\mathcal{P}}(\{1\})$.
Combining this, we obtain for all $x \in \mathbf{I}$,

$$\begin{aligned} x &= \widetilde{\pi}_1(\widetilde{\eta}(x)) && \text{by Def. 7, Def. 51} \\ &= \mathcal{F}(\exists_{\{1,2\}})(\widehat{\mathcal{F}}(c_1)(\widetilde{\eta}(x))) && \text{by (59)} \\ &= \mathcal{F}(\exists_{\{1,2\}})(\widetilde{\eta}(x, 0)) && \text{by (61), (62), Def. 51} \\ &= x \mathbin{\widetilde{\widetilde{\vee}}} 0 && \text{by Def. 52, L-13.} \end{aligned}$$

b. $x_1 \mathbin{\widetilde{\widetilde{\vee}}} x_2 = x_2 \mathbin{\widetilde{\widetilde{\vee}}} x_1$ for all $x_1,\, x_2 \in \mathbf{I}$, i.e. $\widetilde{\widetilde{\vee}}$ is commutative.
To see this, let us first consider $\beta : \{1, 2\} \longrightarrow \{1, 2\}$ defined by $\beta(1) = 2$, $\beta(2) = 1$. Then

$$\chi_{\widehat{\beta}(Y)}(1) = \begin{cases} 1 & : \quad 2 \in Y \\ 0 & : \quad 2 \notin Y \end{cases}$$

for all $Y \in \mathcal{P}(\{1, 2\})$, i.e.

$$\chi_{\widehat{\beta}(\bullet)}(1) = \pi_2 \,. \tag{63}$$

By analogous reasoning,

$$\chi_{\widehat{\beta}(\bullet)}(2) = \pi_1 \,. \tag{64}$$

Recalling the definition of the induced extension principle, we then have

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(\beta)(X)}(1) &= \mathcal{F}(\chi_{\widehat{\beta}(\bullet)}(1))(X) && \text{by Def. 19} \\
&= \mathcal{F}(\pi_2)(X) && \text{by (63)} \\
&= \widetilde{\pi}_2(X) && \text{by (Z-2)} \\
&= \mu_X(2) && \text{by Def. 7}
\end{aligned}$$

for all $X \in \widetilde{\mathcal{P}}(\{1,2\})$, i.e.

$$\mu_{\widehat{\mathcal{F}}(\beta)(X)}(1) = \mu_X(2)\,. \tag{65}$$

By analogous reasoning using (64),

$$\mu_{\widehat{\mathcal{F}}(\beta)(X)}(2) = \mu_X(1)\,. \tag{66}$$

Let is now recall the alternative construction of induced connectives. For all $x_1, x_2 \in \mathbf{I}$,

$$\begin{aligned}
x_1 \mathbin{\widetilde{\widetilde{\vee}}} x_2 &= \mathcal{F}(Q_\vee)(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52} \\
&= \mathcal{F}(\exists_{\{1,2\}})(\widetilde{\eta}(x_1, x_2)) && \text{by L-13} \\
&= \mathcal{F}(\exists_{\{1,2\}} \circ \widehat{\beta})(\widetilde{\eta}(x_1, x_2)) && \text{because } \exists_{\{1,2\}} \text{ quantitative, } \beta \text{ automorphism} \\
&= \mathcal{F}(\exists_{\{1,2\}})(\widehat{\mathcal{F}}(\beta)(\widetilde{\eta}(x_1, x_2))) && \text{by (Z-6)} \\
&= \mathcal{F}(\exists_{\{1,2\}})(\widetilde{\eta}(x_2, x_1)) && \text{by (65),(66) and Def. 51} \\
&= \mathcal{F}(Q_\vee)(\widetilde{\eta}(x_2, x_1)) && \text{by L-13} \\
&= x_2 \mathbin{\widetilde{\widetilde{\vee}}} x_1 && \text{by Def. 52.}
\end{aligned}$$

In order to prove that the new axiom system still entails (DFS 7), we will first state some lemmata.

Lemma 15
Suppose $\mathcal{F}$ is a QFM which satisfies Z-1 to Z-6 and $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a monadic semi-fuzzy quantifier. Then for all $A \in \mathcal{P}(E)$,

$$\mathcal{F}(Q \triangleleft A) = \mathcal{F}(Q) \triangleleft A\,.$$

Proof Both $\mathcal{F}(Q \triangleleft A) : \widetilde{\mathcal{P}}(E)^0 \longrightarrow \mathbf{I}$ and $\mathcal{F}(Q) \triangleleft A : \widetilde{\mathcal{P}}(E)^0 \longrightarrow \mathbf{I}$ are nullary fuzzy quantifiers and hence defined on the empty tuple $\varnothing$ only. So let us check their behaviour on the empty argument:

$$\begin{aligned}
\mathcal{F}(Q) \triangleleft A(\varnothing) &= \mathcal{F}(Q)(A) && \text{by Def. 15} \\
&= Q(A) && \text{by (Z-1)} \\
&= (Q \triangleleft A)(\varnothing) && \text{by Def. 15} \\
&= \mathcal{F}(Q \triangleleft A)(\varnothing) && \text{by (Z-1).}
\end{aligned}$$

Lemma 16

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier of arity* $n > 1$ *and* $A \in \mathcal{P}(E)$. *Then*

$$Q \triangleleft A = \langle Q \rangle \circ \widetilde{\widetilde{\widehat{h}\Box}} \circ \widetilde{\widetilde{\widehat{k}\Box}} \cup^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\iota}_i^{\,n-1,E} \tag{67}$$

where

$$E'' = E_{n-1} \cup \{(e, n) : e \in A\}\,, \tag{68}$$

$k : E_{n-1} \longrightarrow E''$ *is the inclusion*

$$k(e, i) = (e, i) \tag{69}$$

for all $(e, i) \in E_{n-1}$, *and* $h : E'' \longrightarrow E_n$ *is the inclusion*

$$h(e, i) = (e, i) \tag{70}$$

for all $(e, i) \in E''$.

Proof Let us abbreviate

$$Z = \bigcup_{i=1}^{n-1} \widehat{\iota}_i^{\,n-1,E}(X_i)\,. \tag{71}$$

We shall first take a closer look at some sets which will occur as subexpressions in the main proof of the lemma. Clearly

$$\widehat{k}(\neg Z) = \widehat{k}(E_{n-1} \setminus Z) = E_{n-1} \setminus Z \tag{72}$$

by (69). Because $A \setminus (A \setminus B) = A \cap B$, we have

$$E_{n-1} \setminus (E_{n-1} \setminus Z) = E_{n-1} \cap Z \tag{73}$$

$$= Z \qquad \text{because } Z \subseteq E_{n-1}. \tag{74}$$

Abbreviating

$$A' = \{(e, n) : e \in A\}\,, \tag{75}$$

and noting that $E_{n-1} \cap A' = \varnothing$, in particular $(E_{n-1} \setminus Z) \cap A' = \varnothing$, we further have

$$A' \setminus (E_{n-1} \setminus Z) = A'\,. \tag{76}$$

Combining this, we obtain that $\neg\widehat{k}(\neg Z) \in \mathcal{P}(E'')$ is the set

$$\begin{aligned}
\neg\widehat{k}(\neg Z) &= E'' \setminus \widehat{k}(\neg Z) \\
&= (E_{n-1} \cup A') \setminus (E_{n-1} \setminus Z) && \text{by (68), (72), (75)} \\
&= (E_{n-1} \setminus (E_{n-1} \setminus Z)) \cup (A' \setminus (E_{n-1} \setminus Z)) && \text{by } A \setminus B = A \cap \neg B \text{ and distributivity} \\
&= Z \cup A' && \text{by (74), (76)}
\end{aligned}$$

From this we obtain by (70),

$$\widehat{h}(\neg\widehat{k}(\neg Z)) = Z \cup A' \,. \tag{77}$$

Noting that

$$A' = \widehat{\iota}_i^{\,n,E}(A)\,,$$

we may substitute $X_n = A$ and utilize (11) to obtain

$$\widehat{h}(\neg\widehat{k}(\neg Z)) = \bigcup_{i=1}^{n} \widehat{\iota}_i^{\,n,E}(X_i)\,. \tag{78}$$

Based on this equation, the proof of the lemma is now easy:

$$
\begin{aligned}
&(\langle Q\rangle \circ \widehat{h}\widetilde{\widetilde{\Box}} \circ \widehat{k}\widetilde{\widetilde{\Box}}\cup^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\iota}_i^{\,n-1,E})(X_1, \ldots, X_{n-1}) && \\
&= (\langle Q\rangle \circ \widehat{h}\widetilde{\widetilde{\Box}} \circ \widehat{k}\widetilde{\widetilde{\Box}})(Z) && \text{by (71)} \\
&= \widetilde{\widetilde{\neg}}(\langle Q\rangle \circ \widehat{h}\widetilde{\widetilde{\Box}} \circ \widehat{k})(\neg Z) && \text{by Def. 12} \\
&= \widetilde{\widetilde{\neg}}(\langle Q\rangle \circ \widehat{h}\widetilde{\widetilde{\Box}}(\widehat{k}(\neg Z)) && \text{by Def. 21} \\
&= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}(\langle Q\rangle \circ \widehat{h})(\neg\widehat{k}(\neg Z)) && \text{by Def. 12} \\
&= (\langle Q\rangle \circ \widehat{h})(\neg\widehat{k}(\neg Z)) && \text{by L-12, } \widetilde{\widetilde{\neg}} \text{ involutive} \\
&= \langle Q\rangle(\widehat{h}(\neg\widehat{k}(\neg Z))) && \text{by Def. 21} \\
&= \langle Q\rangle(\bigcup_{i=1}^{n} \widehat{\iota}_i^{\,n,E}(X_i)) && \text{by (78)} \\
&= Q(X_1, \ldots, X_{n-1}, A) && \text{by Th-33, } X_n = A.
\end{aligned}
$$

Lemma 17

Suppose $\mathcal{F}$ is a QFM which satisfies Z-1 to Z-6, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 1$, and $A \in \mathcal{P}(E)$. Then (using the same abbreviations as in the previous lemma),

$$\langle \mathcal{F}(Q)\rangle \circ \hat{\hat{h}}\widetilde{\widetilde{\Box}} \circ \hat{\hat{k}}\widetilde{\widetilde{\Box}}\widetilde{\widetilde{\cup}}^{\,n-1} \circ \mathop{\times}_{i=1}^{n-1} \hat{\hat{\iota}}_i^{\,n-1,E} = \langle \mathcal{F}(Q)\rangle\widetilde{\widetilde{\cup}}^{\,n} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{\,n,E} \triangleleft A\,.$$

Proof Let $X_1, \ldots, X_{n-1} \in \widetilde{\mathcal{P}}(E)$ and let us abbreviate

$$X = [\widetilde{\widetilde{\bigcup}}]_{i=1}^{n-1} \hat{\hat{\iota}}_i^{\,n-1,E}(X_i)\,.$$

Then by L-2 and Def. 20,

$$\mu_{\hat{\hat{k}}(\widetilde{\widetilde{\neg}}X)}(e, j) = \begin{cases} \widetilde{\widetilde{\neg}}\,\mu_{X_j}(e) & : \quad j \neq n \\ 0 & : \quad j = n \end{cases} \tag{79}$$

for all $(e, j) \in E''$. Hence by L-12 ($\widetilde{\widetilde{\neg}}$ involutive, $\widetilde{\widetilde{\neg}}\,0 = 1$),

$$\mu_{\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}}X)}(e, j) = \begin{cases} \mu_{X_j}(e) & : \quad j \neq n \\ 1 & : \quad j = n \end{cases} \tag{80}$$

for all $(e, j) \in E''$. Therefore

$$\mu_{\hat{\hat{h}}(\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}}X))}(e, j) = \begin{cases} \mu_{\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}}X)}(e, j) & : \quad (e, j) \in E'' \\ 0 & : \quad \text{else} \end{cases} \tag{81}$$

$$= \begin{cases} \mu_{X_j}(e) & : \quad j \neq n \\ 1 & : \quad j = n \text{ and } e \in A \\ 0 & : \quad j = n \text{ and } e \notin A \end{cases} \tag{82}$$

$$= \begin{cases} \mu_{X_j}(e) & : \quad j \neq n \\ \chi_A(e) & : \quad j = n \end{cases} \tag{83}$$

$$= \mu_{[\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)}(e, j) \tag{84}$$

for all $(e, j) \in E_n$, where the last step holds by L-2 provided we substitute $X_n = A$.

$$
\begin{aligned}
&(\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}}\widetilde{\widetilde{\square}} \circ \hat{\hat{k}}\widetilde{\widetilde{\square}}\widetilde{\widetilde{\cup}}^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \hat{\hat{\iota}}_i^{n-1,E})(X_1, \ldots, X_{n-1}) && \\
&= (\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}}\widetilde{\widetilde{\square}} \circ \hat{\hat{k}}\widetilde{\widetilde{\square}})(X) && \text{by (79), (12)} \\
&= \widetilde{\widetilde{\neg}}(\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}}\widetilde{\widetilde{\square}} \circ \hat{\hat{k}})(\widetilde{\widetilde{\neg}} X) && \text{by Def. 12} \\
&= \widetilde{\widetilde{\neg}}(\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}}\widetilde{\widetilde{\square}})(\hat{\hat{k}}(\widetilde{\widetilde{\neg}} X)) && \text{by Def. 21} \\
&= \widetilde{\widetilde{\neg}}\widetilde{\widetilde{\neg}}(\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}})(\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}} X)) && \text{by Def. 12} \\
&= (\langle \mathcal{F}(Q) \rangle \circ \hat{\hat{h}})(\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}} X)) && \text{by L-12, } \widetilde{\widetilde{\neg}} \text{ involutive} \\
&= \langle \mathcal{F}(Q) \rangle(\hat{\hat{h}}(\widetilde{\widetilde{\neg}}\hat{\hat{k}}(\widetilde{\widetilde{\neg}} X))) && \text{by Def. 21} \\
&= \langle \mathcal{F}(Q) \rangle([\widetilde{\widetilde{\cup}}]_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E}(X_i)) && \text{by (84), } X_n = A \\
&= (\langle \mathcal{F}(Q) \rangle \widetilde{\widetilde{\cup}}^{n} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E})(X_1, \ldots, X_{n-1}, A) && \text{by (12), } X_n = A \\
&= (\langle \mathcal{F}(Q) \rangle \widetilde{\widetilde{\cup}}^{n} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E} \triangleleft A)(X_1, \ldots, X_{n-1}) && \text{by Def. 15.}
\end{aligned}
$$

Lemma 18 (Argument insertion)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then it also satisfies (DFS 7).

Proof The case $n = 1$ is covered by L-15. Hence let us assume that $n > 1$. Then, using the abbreviations of L-16,

$$\begin{aligned}
&= \mathcal{F}(Q \triangleleft A) \\
&= \mathcal{F}(\langle Q \rangle \circ \widehat{h}\widetilde{\widetilde{\Box}} \circ \widehat{k}\widetilde{\widetilde{\Box}}\cup^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\iota}_i^{n-1,E}) && \text{by L-16} \\
&= \mathcal{F}(\langle Q \rangle) \circ \widehat{\mathcal{F}}(h)\widetilde{\widetilde{\Box}} \circ \widehat{\mathcal{F}}(k)\widetilde{\widetilde{\Box}}\widetilde{\widetilde{\cup}}^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\mathcal{F}}(\iota_i^{n-1,E}) && \text{by (Z-3), (Z-6)} \\
&= \mathcal{F}(\langle Q \rangle \circ \widehat{h}\widetilde{\widetilde{\Box}} \circ \widehat{k}\widetilde{\widetilde{\Box}}\cup^{n-1} \circ \mathop{\times}_{i=1}^{n-1} \widehat{\iota}_i^{n-1,E}) && \text{by L-11} \\
&= \langle \mathcal{F}(Q) \rangle \widetilde{\widetilde{\cup}}^{n} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{\iota}}_i^{n,E} \triangleleft A && \text{by L-17} \\
&= \mathcal{F}(Q) \triangleleft A && \text{by Th-34.}
\end{aligned}$$

Lemma 19 (External negation)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then it also satisfies (DFS 3)*'.*

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier, $n > 0$. Then apparently

$$\widetilde{\widetilde{\neg}} Q = Q \cup \widetilde{\widetilde{\Box}} \triangleleft E\,. \tag{85}$$

From this, we obtain

$$\begin{aligned}
\mathcal{F}(\widetilde{\widetilde{\neg}} Q) &= \mathcal{F}(Q \cup \widetilde{\widetilde{\Box}} \triangleleft E) && \text{by (85)} \\
&= \mathcal{F}(Q) \widetilde{\cup} \widetilde{\widetilde{\Box}} \triangleleft E && \text{by L-18, (Z-3) and (Z-4)} \\
&= \widetilde{\widetilde{\neg}} \mathcal{F}(Q)\,,
\end{aligned}$$

where the last step hold because $\widetilde{\widetilde{\neg}} 1 = 0$ by L-12, i.e. $\widetilde{\widetilde{\neg}} E = \varnothing$, and because $x \,\widetilde{\widetilde{\vee}}\, 0 = x$ by L-14, i.e. $X \,\widetilde{\widetilde{\cup}}\, \varnothing = X$ for all $X \in \widetilde{\mathcal{P}}(E)$.

Lemma 20 (Internal complementation)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then it also satisfies (DFS 5)*'.*

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier, $n > 0$. Because $\widetilde{\widetilde{\neg}}$ is involutive by L-12, it holds that

$$Q\neg = \widetilde{\widetilde{\neg}} Q \widetilde{\widetilde{\Box}}\,. \tag{86}$$

For the same reason,

$$\mathcal{F}(Q)\widetilde{\widetilde{\neg}} = \widetilde{\widetilde{\neg}} \mathcal{F}(Q)\widetilde{\widetilde{\Box}}\,. \tag{87}$$

Hence

$$\begin{aligned}
\mathcal{F}(Q\neg) &= \mathcal{F}(\widetilde{\widetilde{\neg}} Q \widetilde{\widetilde{\Box}}) && \text{by (86)} \\
&= \widetilde{\widetilde{\neg}} \mathcal{F}(Q)\widetilde{\widetilde{\Box}} && \text{by L-19, (Z-3)} \\
&= \mathcal{F}(Q)\widetilde{\widetilde{\neg}} && \text{by (87).}
\end{aligned}$$

Lemma 21
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier of arity* $n > 0$ *and* $i \in \{1, \ldots, n\}$. *Then*

$$\langle Q\tau_i \rangle = \langle Q \rangle \circ \widehat{\beta}\,,$$

where $\beta : E_n \longrightarrow E_n$ *is defined by*

$$\beta(e, j) = (e, \tau_i(j))\,, \tag{88}$$

for all $(e, j) \in E_n$.

Proof Let us first observe that

$$
\begin{aligned}
&(\iota_k^{n,E})^{-1}(\widehat{\beta}(X)) \\
&= \{e \in E : (e, k) \in \widehat{\beta}(X)\} && \text{by Def. 33, Def. 54} \\
&= \{e \in E : \beta^{-1}(e, k) \in X\} && \text{by Def. 17, } \beta \text{ bijective} \\
&= \{e \in E : (e, \tau_i^{-1}(k)) \in X\} && \text{by (88)} \\
&= \{e \in E : (e, \tau_i(k) \in X\} && \text{by (5)} \\
&= (\iota_{\tau_i(k)}^{n,E})^{-1}(X)\,.
\end{aligned}
$$

i.e.

$$(\iota_k^{n,E})^{-1}(\widehat{\beta}(X)) = (\iota_{\tau_i(k)}^{n,E})^{-1}(X) \tag{89}$$

Therefore

$$
\begin{aligned}
&(\langle Q \rangle \circ \widehat{\beta})(X) \\
&= Q((\iota_1^{n,E})^{-1}(\widehat{\beta}(X)), \ldots, (\iota_n^{n,E})^{-1}(\widehat{\beta}(X))) && \text{by Def. 55} \\
&= Q((\iota_{\tau_i(1)}^{n,E})^{-1}(X), \ldots, (\iota_{\tau_i(n)}^{n,E})^{-1}(X)) && \text{by (89)} \\
&= (Q\tau_i)((\iota_1^{n,E})^{-1}(X), \ldots, (\iota_n^{n,E})^{-1}(X)) && \text{by Def. 13} \\
&= \langle Q\tau_i \rangle(X) && \text{by Def. 55.}
\end{aligned}
$$

Lemma 22
Suppose $\widetilde{\widetilde{\vee}} : \mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$ *satisfies* $x \widetilde{\widetilde{\vee}} 0 = 0 \widetilde{\widetilde{\vee}} x = x$ *for all* $x \in \mathbf{I}$, *and* $\widetilde{\widetilde{\cup}}$ *is the fuzzy union elementwise defined in terms of* $\widetilde{\widetilde{\vee}}$. *Then for all base sets* $E \neq \varnothing$ *and all* $n \in \mathbb{N} \setminus \{0\}$, $i \in \{1, \ldots, n\}$,

$$\hat{\hat{\beta}}([\overset{n}{\underset{j=1}{\widetilde{\widetilde{\cup}}}}]\, \hat{\hat{\iota}}_j^{\,n,E}(X_j)) = [\overset{n}{\underset{j=1}{\widetilde{\widetilde{\cup}}}}]\, \hat{\hat{\iota}}_{\tau_i(j)}^{\,n,E}(X_j)\,,$$

where β *is defined as in L-21.*

Proof For all $(e,k) \in E_n$,

$$\begin{aligned}
&\mu_{\hat{\hat{\beta}}([\widetilde{\widetilde{\cup}}]_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E}(X_j))}(e,k) \\
&= \mu_{[\widetilde{\widetilde{\cup}}]_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E}(X_j)}(\beta^{-1}(e,k)) && \text{by Def. 20, } \beta \text{ bijective} \\
&= \mu_{[\widetilde{\widetilde{\cup}}]_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E}(X_j)}(e,\tau_i^{-1}(k)) && \text{by (88)} \\
&= \mu_{X_{\tau_i^{-1}(k)}}(e) && \text{by L-2} \\
&= \mu_{X_{\tau_i(k)}}(e) && \text{by (5)} \\
&= \mu_{[\widetilde{\widetilde{\cup}}]_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E}(X_{\tau_i(j)})}(e,k) && \text{by L-2.}
\end{aligned}$$

Lemma 23 (Argument transposition)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then it also satisfies (DFS 4).

Proof

$$\begin{aligned}
&\mathcal{F}(Q\tau_i) \\
&= \mathcal{F}(\langle Q\tau_i\rangle \cup^n \circ \mathop{\times}_{j=1}^{n} \widehat{\iota}_j^{n,E}) && \text{by Th-33} \\
&= \mathcal{F}(\langle Q\tau_i\rangle)\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \widehat{\mathcal{F}}(\iota_j^{n,E}) && \text{by (Z-4), (Z-6)} \\
&= \mathcal{F}(\langle Q\tau_i\rangle)\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} && \text{by L-11} \\
&= \mathcal{F}(\langle Q\rangle \circ \widehat{\beta})\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} && \text{by L-21} \\
&= \mathcal{F}(\langle Q\rangle) \circ \widehat{\mathcal{F}}(\beta)\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} && \text{by (Z-6)} \\
&= \mathcal{F}(\langle Q\rangle) \circ \hat{\hat{\beta}}\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} && \text{by L-11, } \beta \text{ injection} \\
&= \langle \mathcal{F}(Q)\rangle \circ \hat{\hat{\beta}}\widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} && \text{by Th-35} \\
&= \langle \mathcal{F}(Q)\rangle \widetilde{\widetilde{\cup}}^n \circ \mathop{\times}_{j=1}^{n} \hat{\hat{\iota}}_j^{n,E} \tau_i && \text{by L-22} \\
&= \mathcal{F}(Q)\tau_i && \text{by Def. 13.}
\end{aligned}$$

Lemma 24
For all $x_1, x_2 \in \mathbf{I}$ and fuzzy complement operators defined by some $\widetilde{\widetilde{\neg}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$,

$$\widetilde{\eta}(\widetilde{\widetilde{\neg}}\, x_1, \widetilde{\widetilde{\neg}}\, x_2) = \widetilde{\widetilde{\neg}}\, \widetilde{\eta}(x_1, x_2)\,.$$

Proof $\widetilde{\eta}(\widetilde{\widetilde{\neg}}\, x_1, \widetilde{\widetilde{\neg}}\, x_2), \widetilde{\widetilde{\neg}}\, \widetilde{\eta}(x_1, x_2) \in \widetilde{\mathcal{P}}(\{1, 2\})$, i.e. we can show element-wise for $e \in \{1, 2\}$ that both fuzzy subsets coincide. If $e = 1$,

$$\begin{aligned}
&\mu_{\widetilde{\widetilde{\neg}}\, \widetilde{\eta}(x_1,x_2)}(1) && \\
&= \widetilde{\widetilde{\neg}}\, \mu_{\widetilde{\eta}(x_1,x_2)}(1) && \text{by Def. 9} \\
&= \widetilde{\widetilde{\neg}}\, x_1 && \text{by Def. 51} \\
&= \mu_{\widetilde{\eta}(\widetilde{\widetilde{\neg}}\, x_1, \widetilde{\widetilde{\neg}}\, x_2)}(1) && \text{by Def. 51.}
\end{aligned}$$

The case $e = 2$ is treated analogously.

Lemma 25 (Duality of $\widetilde{\widetilde{\wedge}}$ and $\widetilde{\widetilde{\vee}}$)
Suppose $\mathcal{F}$ is a QFM which satisfies Z-1 to Z-6. Then for all $x_1, x_2 \in \mathbf{I}$,

$$\begin{aligned}
x_1 \,\widetilde{\widetilde{\vee}}\, x_2 &= \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\wedge}}\, \widetilde{\widetilde{\neg}}\, x_1) \\
x_1 \,\widetilde{\widetilde{\wedge}}\, x_2 &= \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\vee}}\, \widetilde{\widetilde{\neg}}\, x_2)\,.
\end{aligned}$$

Proof Let us first observe that by similar reasoning as in L-13, we have

$$Q_\wedge = \forall_{\{1,2\}}\,, \tag{90}$$

where $\forall_{\{1,2\}} : \mathcal{P}(\{1, 2\}) \longrightarrow \mathbf{2}$ is defined by

$$\forall_{\{1,2\}}(X) = \begin{cases} 1 & : \quad X = \{1, 2\} \\ 0 & : \quad \text{else} \end{cases}$$

for all $X \in \mathcal{P}(\{1, 2\})$.
Let $x_1, x_2 \in \mathbf{I}$. Then

$$\begin{aligned}
x_1 \,\widetilde{\widetilde{\vee}}\, x_2 &= \mathcal{F}(Q_\vee)(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52} \\
&= \mathcal{F}(\exists_{\{1,2\}})(\widetilde{\eta}(x_1, x_2)) && \text{by L-13} \\
&= \mathcal{F}(\forall_{\{1,2\}}\widetilde{\widetilde{\Box}})(\widetilde{\eta}(x_1, x_2)) && \text{by duality of } \exists \text{ and } \forall\text{, and } \widetilde{\widetilde{\neg}}\, 0 = 1, \\
& && \widetilde{\widetilde{\neg}}\, 1 = 0 \text{ (see L-12)} \\
&= \mathcal{F}(\forall_{\{1,2\}})\widetilde{\widetilde{\Box}}(\widetilde{\eta}(x_1, x_2)) && \text{by (Z-3)} \\
&= \widetilde{\widetilde{\neg}}\, \mathcal{F}(\forall_{\{1,2\}})\widetilde{\widetilde{\neg}}(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 12} \\
&= \widetilde{\widetilde{\neg}}\, \mathcal{F}(\forall_{\{1,2\}})(\widetilde{\widetilde{\neg}}\, \widetilde{\eta}(x_1, x_2)) && \text{by Def. 10, Def. 11} \\
&= \widetilde{\widetilde{\neg}}\, \mathcal{F}(\forall_{\{1,2\}})(\widetilde{\eta}(\widetilde{\widetilde{\neg}}\, x_1, \widetilde{\widetilde{\neg}}\, x_2)) && \text{by L-24} \\
&= \widetilde{\widetilde{\neg}}\, \mathcal{F}(Q_\wedge)(\widetilde{\eta}(\widetilde{\widetilde{\neg}}\, x_1, \widetilde{\widetilde{\neg}}\, x_2)) && \text{by (90)} \\
&= \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\wedge}}\, \widetilde{\widetilde{\neg}}\, x_2) && \text{by Def. 52.}
\end{aligned}$$

This proves the first equation. From this we obtain:

$$\begin{aligned}
x_1 \,\widetilde{\widetilde{\wedge}}\, x_2 &= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\wedge}}\, \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, x_2) && \text{by L-12, } \widetilde{\widetilde{\neg}} \text{ is involutive} \\
&= \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, x_1 \,\widetilde{\widetilde{\vee}}\, \widetilde{\widetilde{\neg}}\, x_2) && \text{by first equation of lemma.}
\end{aligned}$$

Lemma 26 (Internal meets)
If a QFM $\mathcal{F}$ satisfies Z-1 to Z-6, then it also satisfies (DFS 6)*'.*

Proof This is simple noting that

$$Q\cap = Q\neg\cup\neg\tau_n\neg\tau_n \tag{91}$$

by De Morgan's law.
Therefore

$$\begin{aligned}
\mathcal{F}(Q\cap) &= \mathcal{F}(Q\neg\cup\neg\tau_n\neg\tau_n) && \text{by (91)}\\
&= \mathcal{F}(Q)\widetilde{\widetilde{\neg}}\widetilde{\widetilde{\cup}}\widetilde{\widetilde{\neg}}\tau_n\widetilde{\widetilde{\neg}}\tau_n && \text{by L-23, L-20, (Z-4)}\\
&= \mathcal{F}(Q)\widetilde{\widetilde{\cap}} && \text{by L-25.}
\end{aligned}$$

Proof of Theorem 38

L-7 shows that every model of the DFS axioms also satisfies Z-1 to Z-6.
As concerns the converse direction, a QFM $\mathcal{F}$ which satisfies Z-1 to Z-6 is known to fulfill

Axiom	by Lemma
(DFS 1)	L-8
(DFS 2)	L-8
(DFS 3)'	L-19
(DFS 4)	L-23
(DFS 5)'	L-20
(DFS 6)'	L-26
(DFS 7)	L-18
(DFS 8)	L-8
(DFS 9)	L-8

Recalling that (DFS i) coincides with (DFS i)' for $i \in \{1,2,4,7,8,9\}$, this means that $\mathcal{F}$ is a model of DFS'. From Th-37, we obtain that $\mathcal{F}$ is a model of the DFS axioms.

C Proofs of Theorems in Chapter 4

C.1 Proof of Theorem 41

a. Suppose $f \in \mathbb{B}$ is some mapping and $Q : \mathcal{P}(I) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ are defined by (Th-41.a.i) and (Th-41.a.ii), resp.
By the definition of X, $\mu_X(z) \geq \frac{1}{2}$ for all $z \in \mathbf{I}$ and hence

$$(X)^{\max}_{\gamma} = \mathbf{I} \qquad \text{by Def. 66} \tag{92}$$

i.e.

$$\inf\,(X)^{\max}_{\gamma} = 0 \tag{93}$$

for all $\gamma \in \mathbf{I}$. Considering $(X)^{\min}_{\gamma}$, we firstly have

$$\begin{aligned}
(X)^{\min}_0 &= (X)_{>\frac{1}{2}} && \text{by Def. 66}\\
&= \{z \in \mathbf{I} : \mu_X(z) > \tfrac{1}{2}\} && \text{by Def. 65}\\
&= \{z \in \mathbf{I} : z > 0\} && \text{by (Th-41.a.ii)}
\end{aligned}$$

i.e.

$$(X)^{\min}_0 = (0,1]\,. \tag{94}$$

In the case that $\gamma > 0$, we obtain

$$\begin{aligned}
(X)^{\min}_\gamma &= (X)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} && \text{by Def. 66}\\
&= \{z \in \mathbf{I} : \mu_X(z) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\} && \text{by Def. 64}\\
&= \{z \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}z \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\} && \text{by (Th-41.a.ii)}\\
&= \{z \in \mathbf{I} : z \geq \gamma\}\,,
\end{aligned}$$

i.e.

$$(X)^{\min}_\gamma = [\gamma, 1]\,. \tag{95}$$

Summarising (94) and (95), we have

$$\inf\,(X)^{\min}_\gamma = \gamma\,. \tag{96}$$

We shall now discern the cases that $f \in \mathbb{B}^+$, $f \in \mathbb{B}^{\frac{1}{2}}$ and $f \in \mathbb{B}^-$.

1. $f \in \mathbb{B}^+$. Then

$$\begin{aligned}
Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma),\, Q((X)^{\max}_\gamma)) && \text{by monotonicity of } Q \text{ and (14)}\\
&= \mathrm{m}_{\frac{1}{2}}(f(\inf\,(X)^{\min}_\gamma),\, f(\inf\,(X)^{\max}_\gamma)) && \text{by (Th-41.a.i)}\\
&= \mathrm{m}_{\frac{1}{2}}(f(\gamma), f(0)) && \text{by (93), (96)}\\
&= f(\gamma)
\end{aligned}$$

   where the last step holds by the definition of $\mathrm{m}_{\frac{1}{2}}$ because $\frac{1}{2} \leq f(\gamma) \leq f(0)$ by the nonincreasing monotonicity of $f \in \mathbb{B}^+$ (see Def. 45 and Def. 68).

2. $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$. Then

$$Q(Y) = \mathrm{c}_{\frac{1}{2}}(\inf Y) = \tfrac{1}{2} \tag{97}$$

   for all $Y \in \mathcal{P}(\mathbf{I})$ and hence

$$\begin{aligned}
Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{\tfrac{1}{2} : Y \in \mathcal{T}_\gamma(X)\} && \text{by (97), } \mathcal{T}_\gamma(X) \neq \varnothing\\
&= \mathrm{m}_{\frac{1}{2}}\{\tfrac{1}{2}\}\\
&= \tfrac{1}{2} && \text{by Def. 46}\\
&= \mathrm{c}_{\frac{1}{2}}(\gamma)
\end{aligned}$$

   i.e. $Q_\gamma(X) = f(\gamma)$, as desired.

3. $f \in \mathbb{B}^-$. Then

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)_\gamma^{\min}),\, Q((X)_\gamma^{\max})) && \text{by monotonicity of } Q \text{ and (14)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(\inf\,(X)_)^{\min}\gamma,\, f(\inf\,(X)_\gamma^{\max})) && \text{by (Th-41.a.i)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(\gamma), f(0)) && \text{by (93), (96)} \\ &= f(\gamma) \end{aligned}$$

where the last step holds by the definition of $\mathrm{m}_{\frac{1}{2}}$ because $f(0) \leq f(\gamma) \leq \frac{1}{2}$ by the nondecreasing monotonicity of $f \in \mathbb{B}^+$ (see Def. 45 and Def. 68).

b. Suppose $f \in \mathbb{B}$ is some mapping and $Q : \mathcal{P}(I) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ are defined by (Th-41.b.i) and (Th-41.b.ii), resp.
By the definition of X, $\mu_X(z) \leq \frac{1}{2}$ for all $z \in \mathbf{I}$ and hence

$$(X)_\gamma^{\min} = \varnothing \qquad \text{by Def. 66} \tag{98}$$

i.e.

$$\sup\,(X)_\gamma^{\min} = \sup \varnothing = 0 \tag{99}$$

for all $\gamma \in \mathbf{I}$. Considering $(X)_\gamma^{\max}$, we firstly have

$$\begin{aligned} (X)_0^{\max} &= (X)_{\geq \frac{1}{2}} && \text{by Def. 66} \\ &= \{z \in \mathbf{I} : \mu_X(z) \geq \tfrac{1}{2}\} && \text{by Def. 64} \\ &= \{0\} && \text{by (Th-41.b.ii)} \end{aligned}$$

i.e.

$$(X)_0^{\max} = \{0\}\,. \tag{100}$$

In the case that $\gamma > 0$, we obtain

$$\begin{aligned} (X)_\gamma^{\max} &= (X)_{> \frac{1}{2} - \frac{1}{2}\gamma} && \text{by Def. 66} \\ &= \{z \in \mathbf{I} : \mu_X(z) > \tfrac{1}{2} - \tfrac{1}{2}\gamma\} && \text{by Def. 65} \\ &= \{z \in \mathbf{I} : \tfrac{1}{2} - \tfrac{1}{2}z > \tfrac{1}{2} - \tfrac{1}{2}\gamma\} && \text{by (Th-41.a.ii)} \\ &= \{z \in \mathbf{I} : z < \gamma\}\,, \end{aligned}$$

i.e.

$$(X)_\gamma^{\max} = [0, \gamma)\,. \tag{101}$$

Summarising (100) and (101), we have

$$\inf\,(X)_\gamma^{\max} = \gamma\,. \tag{102}$$

We shall now discern the cases that $f \in \mathbb{B}^+$, $f \in \mathbb{B}^{\frac{1}{2}}$ and $f \in \mathbb{B}^-$.

1. $f \in \mathbb{B}^+$. Then

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma),\ Q((X)^{\max}_\gamma)) && \text{by monotonicity of } Q \text{ and (14)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(\sup{(X)^{\min}_\gamma}),\ f(\sup{(X)^{\max}_\gamma})) && \text{by (Th-41.b.i)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(0), f(\gamma)) && \text{by (99), (102)} \\ &= f(\gamma) \end{aligned}$$

where the last step holds by the definition of $\mathrm{m}_{\frac{1}{2}}$ because $\frac{1}{2} \leq f(\gamma) \leq f(0)$ by the nonincreasing monotonicity of $f \in \mathbb{B}^+$ (see Def. 45 and Def. 68).

2. $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$. Then

$$Q(Y) = \mathrm{c}_{\frac{1}{2}}(\sup Y) = \tfrac{1}{2} \tag{103}$$

for all $Y \in \mathcal{P}(\mathbf{I})$ and hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 67} \\ &= \mathrm{m}_{\frac{1}{2}}\{\tfrac{1}{2} : Y \in \mathcal{T}_\gamma(X)\} && \text{by (103), } \mathcal{T}_\gamma(X) \neq \varnothing \\ &= \mathrm{m}_{\frac{1}{2}}\{\tfrac{1}{2}\} \\ &= \tfrac{1}{2} && \text{by Def. 46} \\ &= \mathrm{c}_{\frac{1}{2}}(\gamma) \end{aligned}$$

i.e. $Q_\gamma(X) = f(\gamma)$, as desired.

3. $f \in \mathbb{B}^-$. Then

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma),\ Q((X)^{\max}_\gamma)) && \text{by monotonicity of } Q \text{ and (14)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(\sup{(X)^{\min}_)}\gamma,\ f(\sup{(X)^{\max}_\gamma})) && \text{by (Th-41.b.i)} \\ &= \mathrm{m}_{\frac{1}{2}}(f(0), f(\gamma)) && \text{by (99), (102)} \\ &= f(\gamma) \end{aligned}$$

where the last step holds by the definition of $\mathrm{m}_{\frac{1}{2}}$ because $f(0) \leq f(\gamma) \leq \frac{1}{2}$ by the nondecreasing monotonicity of $f \in \mathbb{B}^+$ (see Def. 45 and Def. 68).

C.2 Proof of Theorem 43

Assume $f : [a,b] \longrightarrow [c,d]$ is a given mapping, where $a \leq b$ and $c \leq d$.

a. f nondecreasing, $z \in (a,b]$.
Let us recall that a sequence $(a_n)_{n\in\mathbb{N}}$ ($a_n \in \mathbb{R}$ for all $n \in \mathbb{N}$) converges towards $a \in \mathbb{R}$ (in symbols, $\lim_{n\to\infty} a_n = a$) iff for every $\varepsilon > 0$, there exists some $N(\varepsilon) \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N(\varepsilon)$.
Let us also recall the definition of (lefthand-side) limes, i.e. we write $\lim_{x\to z^-} f(x) = y$ for some $y \in [c,d]$ iff for every sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in [a,z)$ and $\lim_{n\to\infty} x_n = z$, we have $\lim_{n\to\infty} f(x_n) = y$.

Hence let $(x_n)_{n\in\mathbb{N}}$ a sequence such that $x_n \in [a, z)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = z$. We have to show that $\lim_{n\to\infty} f(x_n) = s$, where I have abbreviated $s = \sup\{f(x) : x < z\}$.
To see this, let $\varepsilon > 0$. Because $z \in (a, b]$, we know that $\{f(x) : x < z\} = \{f(x) : x \in [a, z)\} \neq \varnothing$, and by the definition of sup, there exists $x' \in [a, z)$ such that

$$|s - f(x')| = s - f(x') < \varepsilon\,.$$

Let $\varepsilon' = z - x' > 0$.
Because $\lim_{n\to\infty} x_n = z$, there is an $N(\varepsilon')$ such that

$$z - x_n = |z - x_n| < \varepsilon'$$

for all $n \geq N(\varepsilon')$, i.e.

$$x_n > z - \varepsilon' = z - (z - x') = x'$$

for all $n \geq N(\varepsilon')$. Therefore

$$f(x_n) \geq f(x')$$

for all $n \geq N(\varepsilon')$, i.e.

$$|s - f(x_n)| = s - f(x_n) \leq s - f(x') < \varepsilon\,.$$

We conclude that for all $\varepsilon > 0$, there exists $N(\varepsilon) = N(\varepsilon') \in \mathbb{N}$ such that $|s - f(x_n)| < \varepsilon$ for all $n \geq N(\varepsilon)$, i.e. $\lim_{n\to\infty} f(x_n)$ exists and $\lim_{n\to\infty} f(x_n) = s = \sup\{f(x) : x < z\}$.

c. f nonincreasing, $z \in (a, b]$.
Then $-f : [a, b] \longrightarrow [-d, -c]$ is nondecreasing and

$$\begin{aligned}
\lim_{x\to z^-} f(x) &= -\lim_{x\to z^-} -f(x) \\
&= -\sup\{-f(x) : x < z\} && \text{by part a. of theorem} \\
&= \inf\{-(-f(x)) : x < z\} \\
&= \inf\{f(x) : x < z\}\,.
\end{aligned}$$

b. f nondecreasing, $z \in [a, b)$.
Then $g : [-b, -a] \longrightarrow [c, d]$, defined by $g(x) = f(-x)$ for all $x \in [-b, -a]$, is nonincreasing, and

$$\begin{aligned}
\lim_{x\to z^+} f(x) &= \lim_{x\to(-z)^-} f(-x) \\
&= \lim_{x\to(-z)^-} g(x) \\
&= \inf\{g(x) : x < -z\} && \text{by part c. of the theorem} \\
&= \inf\{f(-x) : x < -z\} \\
&= \inf\{f(x) : x > z\}\,.
\end{aligned}$$

d. f nonincreasing, $z \in [a, b)$.
Then $-f : [a, b] \longrightarrow [-d, -c]$ is nondecreasing, and

$$\begin{aligned}\lim_{x\to z^+} f(x) &= -\lim_{x\to z^+} -f(x) \\ &= -\inf\{-f(x) : x > z\} && \text{by part b. of the theorem} \\ &= \sup\{-(-f(x)) : x > z\} \\ &= \sup\{f(x) : x > z\}\,.\end{aligned}$$

C.3 Proof of Theorem 44

Lemma 27
Suppose $E \neq \varnothing$ is some base set and $X \in \breve{\mathcal{P}}(E)$ is a three-valued subset of E. Then

$$\mathrm{T}_\gamma(X) = X$$

for all $\gamma \in \mathbf{I}$.

Proof

a. $\gamma = 0$. Then

$$\begin{aligned}\nu_{\mathrm{T}_0(X)}(e) &= t_0(\nu_X(e)) && \text{by Def. 63, } X \text{ three-valued} \\ &= \begin{cases} 1 & : \quad \nu_X(e) > \frac{1}{2} \\ \frac{1}{2} & : \quad \nu_X(e) = \frac{1}{2} \\ 0 & : \quad \nu_X(e) < \frac{1}{2} \end{cases} && \text{by Def. 62} \\ &= \begin{cases} 1 & : \quad \nu_X(e) = 1 \\ \frac{1}{2} & : \quad \nu_X(e) = \frac{1}{2} \\ 0 & : \quad \nu_X(e) = 0 \end{cases} && \text{because } X \text{ is three-valued} \\ &= \nu_X(e)\,,\end{aligned}$$

for all $e \in E$, i.e. $\mathrm{T}_\gamma(X) = X$.

b. $\gamma > 0$. In this case

$$\begin{aligned}\nu_{\mathrm{T}_\gamma(X)}(e) &= t_\gamma(\nu_X(e)) && \text{by Def. 63, } X \text{ three-valued} \\ &= \begin{cases} 1 & : \quad \nu_X(e) \geq \frac{1}{2} + \frac{1}{2}\gamma \\ \frac{1}{2} & : \quad \frac{1}{2} - \frac{1}{2}\gamma < \nu_X(e) < \frac{1}{2} + \frac{1}{2}\gamma \\ 0 & : \quad \nu_X(e) \leq \frac{1}{2} - \frac{1}{2}\gamma \end{cases} && \text{by Def. 62} \\ &= \begin{cases} 1 & : \quad \nu_X(e) = 1 \\ \frac{1}{2} & : \quad \nu_X(e) = \frac{1}{2} \\ 0 & : \quad \nu_X(e) = 0 \end{cases} && \text{because } \nu_X(e) \text{ is three-valued} \\ &= \nu_X(e)\,,\end{aligned}$$

for all $e \in E$, i.e. $\mathrm{T}_\gamma(X) = X$.

Proof of Theorem 44

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \dots, X_n \in \breve{\mathcal{P}}(E)$ are three-valued argument sets. Noting that

$$Q_\gamma(X_1, \dots, X_n) = Q_0(X_1, \dots, X_n) \tag{104}$$

because for three-valued argument sets, $\mathcal{T}_\gamma(X_i) = \mathcal{T}_0(X_i)$ for all $\gamma \in \mathbf{I}$ by L-27 and Def. 66, we have

$$\begin{aligned} \mathcal{M}_\mathcal{B}(Q)(X_1, \dots, X_n) &= \mathcal{B}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\ &= Q_0(X_1, \dots, X_n) && \text{by (B-1), (104)} \\ &= \mathcal{M}(Q)(X_1, \dots, X_n)\,, \end{aligned}$$

where the last step holds by [9, Th-48, p. 65].

C.4 Proof of Theorem 45

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-1) and let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given where $n \in \{0, 1\}$.

a. $n = 0$. Then $\mathcal{U}(\mathcal{M}_\mathcal{B}(Q)) = Q$ iff $Q(\varnothing) = \mathcal{M}_\mathcal{B}(Q)(\varnothing)$, where $\varnothing$ is the only choice of arguments (empty or null-tuple of crisp subsets). In particular, $\varnothing$ is also an empty tuple of three-valued subsets. This gives us

$$\begin{aligned} Q(\varnothing) &= \mathcal{M}(Q)(\varnothing) && \text{by Th-42 and Th-1} \\ &= \mathcal{M}_\mathcal{B}(Q)(\varnothing) && \text{by Th-44.} \end{aligned}$$

b. $n = 1$. In this case, the condition $\mathcal{U}(\mathcal{M}_\mathcal{B}(Q)) = Q$ is equivalent to

$$\mathcal{M}_\mathcal{B}(Q)(X) = Q(X)$$

for all *crisp* subsets $X \in \mathcal{P}(E)$. Noting that every crisp subset $X \in \mathcal{P}(E)$ is also a three-valued subset of E, we immediately have

$$\begin{aligned} \mathcal{M}_\mathcal{B}(Q)(X) &= \mathcal{M}(Q)(X) && \text{by Th-44, } X \text{ three-valued} \\ &= Q(X) && \text{by Th-42 and Th-1} \end{aligned}$$

which finishes the proof.

C.5 Proof of Theorem 46

We first need some lemma.

Lemma 28 ($\mathcal{M}$ on three-valued quantifiers)
Let $Q : \mathcal{P}(E)^n \longrightarrow \{0, \tfrac{1}{2}, 1\}$ a three-valued semi-fuzzy quantifier and $(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\mathcal{M}(Q)(X_1, \dots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f(0) > \frac{1}{2} \\ \frac{1}{2} & : \quad f(0) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f(0) < \frac{1}{2} \end{cases}$$

where we have abbreviated

$$f(\gamma) = Q_\gamma(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$.
(See [9, L-33, p. 139])

Proof of Theorem 46

Let us consider some three-valued semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ and a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Noting that in this case, $Q_\gamma(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$ for all $\gamma \in \mathbf{I}$, the conditions of (B-3) are fulfilled and hence

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) &= \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 69}\\
&= \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \\ \frac{1}{2} & : \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \end{cases} && \text{by (B-3), abbreviating } f(\gamma) = Q_\gamma(X_1, \ldots, X_n)\\
&= \mathcal{M}(Q)(X_1, \ldots, X_n) && \text{by L-28}
\end{aligned}$$

C.6 Proof of Theorem 47

Let $E \neq \varnothing$ be given and let $e \in E$. Then

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(\pi_e) &= \mathcal{M}(\pi_e) && \text{by Th-46, } \pi_e \text{ two-valued}\\
&= \widetilde{\pi}_e && \text{by Th-42}
\end{aligned}$$

C.7 Proof of Theorem 48

Let us recall some lemmata:

Lemma 29
If $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, then*

$$(\neg Q)_\gamma(X_1, \ldots, X_n) = \neg(Q_\gamma(X_1, \ldots, X_n))$$

for all $\gamma \in \mathbf{I}$*, where* $\neg : \mathbf{I} \longrightarrow \mathbf{I}$ *is the standard negation* $\neg x = 1 - x$*.*
(See [9, L-20])

Lemma 30
Suppose $E \neq \varnothing$ *is a given base set and* $X \in \widetilde{\mathcal{P}}(E)$*. Then*

$$\mathcal{T}_0(\neg X) = \neg \mathcal{T}_0(X)\,,$$

where

$$\neg \mathcal{T}_0(X) = \{\neg Y : Y \in \mathcal{T}_0(X)\}\,.$$

Proof By Def. 66,

$$\mathcal{T}_0(X) = \{Y \in \mathcal{P}(E) : (X)_0^{\min} \subseteq Y \subseteq (X)_0^{\max}\}$$

where

$$(X)_0^{\min} = (X)_{>\frac{1}{2}}$$
$$(X)_0^{\max} = (X)_{\geq\frac{1}{2}}\,.$$

Similarly,

$$\mathcal{T}_0(\neg X) = \{Y \in \mathcal{P}(E) : (\neg X)_0^{\min} \subseteq Y \subseteq (\neg X)_0^{\max}\},$$

where

$$(\neg X)_0^{\min} = (\neg X)_{>\frac{1}{2}} = \neg((X)_{\geq 1-\frac{1}{2}}) = \neg((X)_{\geq\frac{1}{2}}) = \neg(X)_0^{\max} \tag{105}$$
$$(\neg X)_0^{\max} = (\neg X)_{\geq\frac{1}{2}} = \neg((X)_{>1-\frac{1}{2}}) = \neg((X)_{>\frac{1}{2}}) = \neg(X)_0^{\min}\,. \tag{106}$$

Because $A \subseteq B \subseteq C$ is equivalent to $\neg C \subseteq \neg B \subseteq \neg A$ for arbitrary $A, B, C \in \mathcal{P}(E)$, it is apparent that

$$\begin{aligned}\neg\mathcal{T}_0(X) &= \{\neg Y : Y \in \mathcal{T}_0(X)\}\\ &= \{\neg Y : (X)_{>\frac{1}{2}} \subseteq Y \subseteq (X)_{\geq\frac{1}{2}}\}\\ &= \{\neg Y : \neg((X)_{\geq\frac{1}{2}}) \subseteq \neg Y \subseteq \neg((X)_{>\frac{1}{2}})\}\\ &= \{Z : (\neg X)_0^{\min} \subseteq Z \subseteq (\neg X)_0^{\max}\} && \text{by (105), (106), substitution } Z = \neg Y\\ &= \mathcal{T}_0(\neg X)\,.\end{aligned}$$

Lemma 31

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. *Then*

$$(Q\neg)_\gamma(X_1, \ldots, X_n) = Q_\gamma(X_1, \ldots, X_{n-1}, \neg X_n)$$

for all $\gamma \in \mathbf{I}$.

Proof The case that $\gamma \in (0, 1]$ is covered by [9, L-22, p. 127]. It remains to be shown that the equation holds if $\gamma = 0$.
Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier where $n > 0$, and suppose a choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ is given. Then

$$\begin{aligned}&(Q\neg)_0(X_1, \ldots, X_n)\\ &= \mathrm{m}_{\frac{1}{2}}\{Q\neg(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_0(X_i)\} && \text{by Def. 67}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, \neg Y_n) : Y_i \in \mathcal{T}_0(X_i)\} && \text{by Def. 11}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, Z)\\ &\qquad : Y_1 \in \mathcal{T}_0(X_1), \ldots, Y_{n-1} \in \mathcal{T}_0(X_{n-1}), Z \in \neg\mathcal{T}_0(X_n)\} && \text{by substitution } Z = \neg Y_n\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, Z)\\ &\qquad : Y_1 \in \mathcal{T}_0(X_1), \ldots, \mathcal{T}_0(X_{n-1}), Z \in \mathcal{T}_0(\neg X_n)\} && \text{by L-30}\\ &= Q_0(X_1, \ldots, X_{n-1}, \neg X_n)\,, && \text{by Def. 67}\end{aligned}$$

as desired.

Proof of Theorem 48

Let us first recall that by Th-46 and Th-42, $\mathcal{M}_{\mathcal{B}}$ induces the standard negation. Now let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier ($n > 0$) and suppose $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given. Then, abbreviating $\neg x = 1 - x$,

$$(Q\widetilde{\widetilde{\Box}})_\gamma(X_1, \ldots, X_n) = \neg Q_\gamma(X_1, \ldots, X_{n-1}, \neg X_n) \tag{107}$$

by L-29 and L-31. Hence

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(Q\widetilde{\widetilde{\Box}})(X_1, \ldots, X_n) &= \mathcal{B}((Q\widetilde{\widetilde{\Box}}_\gamma(X_1, \ldots, X_n))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&= \mathcal{B}(\neg(Q_\gamma(X_1, \ldots, X_{n-1}, \neg X_n))_{\gamma\in\mathbf{I}}) && \text{by (107)}\\
&= \neg\mathcal{B}((Q_\gamma(X_1, \ldots, X_{n-1}, \neg X_n))_{\gamma\in\mathbf{I}}) && \text{by (B-2)}\\
&= \neg\mathcal{M}_{\mathcal{B}}(Q)(X_1, \ldots, X_{n-1}, \neg X_n) && \text{by Def. 69}\\
&= \mathcal{M}_{\mathcal{B}}(Q)\widetilde{\widetilde{\Box}}(X_1, \ldots, X_n) && \text{by Def. 12}
\end{aligned}$$

which finishes the proof.

C.8 Proof of Theorem 49

We need some lemmata.

Lemma 32
Suppose $E \neq \varnothing$ is some base set and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Then

a. $\mathcal{T}_0(X_1 \cup X_2) = \mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2)$.

b. $\mathcal{T}_0(X_1 \cap X_2) = \mathcal{T}_0(X_1) \cap \mathcal{T}_0(X_2)$.

Proof

a. Suppose $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are given.

$$\begin{aligned}
\mathcal{T}_0(X_1 \cup X_2) &= \{Z : (X_1 \cup X_2)_0^{\min} \subseteq Z \subseteq (X_1 \cup X_2)_0^{\max}\} && \text{by Def. 66}\\
&= \{Z : (X_1 \cup X_2)_{>\frac{1}{2}} \subseteq Z \subseteq (X_1 \cup X_2)_{\geq\frac{1}{2}} && \text{by Def. 66}\\
&= \{Z : (X_1)_{>\frac{1}{2}} \cup (X_2)_{>\frac{1}{2}} \subseteq Z \subseteq (X_1)_{\geq\frac{1}{2}} \cup (X_2)_{\geq\frac{1}{2}}\} && \text{by properties of } \alpha\text{-cuts}\\
&= \{Z : (X_1)_0^{\min} \cup (X_2)_0^{\min} \subseteq Z \subseteq (X_1)_0^{\max} \cup (X_2)_0^{\max}\}\,. && \text{by Def. 66}
\end{aligned}$$

On the other hand,

$$\mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2) = \{Z_1 \cup Z_2 : (X_1)_0^{\min} \subseteq Z_1 \subseteq (X_1)_0^{\max}, \quad (X_2)_0^{\min} \subseteq Z_2 \subseteq (X_2)_0^{\max}\}\,.$$

We will now show that $\mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2) = \mathcal{T}_0(X_1 \cup X_2)$ by proving both inclusions $\mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2) \subseteq \mathcal{T}_0(X_1 \cup X_2)$ and $\mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2) \supseteq \mathcal{T}_0(X_1 \cup X_2)$.

To see that the first inclusion holds, let $Y_1 \in \mathcal{T}_0(X_1)$ and $Y_2 \in \mathcal{T}_0(X_2)$. Then $Y_1 \supseteq (X_1)_0^{\min}$, $Y_2 \supseteq (X_2)_0^{\min}$ and by the monotonicity $\cup$,

$$Y_1 \cup Y_2 \supseteq (X_1)_0^{\min} \cup (X_2)_0^{\min} = (X_1 \cup X_2)_0^{\min} . \tag{108}$$

(The last equation holds because $(X_1)_0^{\min} \cup (X_2)_0^{\min} = (X_1)_{>\frac{1}{2}} \cup (X_2)_{>\frac{1}{2}} = (X_1 \cup X_2)_{>\frac{1}{2}} = (X_1 \cup X_2)_0^{\min}$).
Again because $Y_1 \in \mathcal{T}_0(X_1)$ and $Y_2 \in \mathcal{T}_0(X_2)$, we have $Y_1 \subseteq (X_1)_0^{\max}$ and $Y_2 \subseteq (X_2)_0^{\max}$ and by the monotonicity of $\cup$,

$$Y_1 \cup Y_2 \subseteq (X_1)_0^{\max} \cup (X_2)_0^{\max} = (X_1 \cup X_2)_0^{\max} . \tag{109}$$

(Again, the last equation holds because α-cuts are homomorphic with respect to the "union" operation $\cup$). From (108), (109) we conclude by Def. 66 that $Y_1 \cup Y_2 \in \mathcal{T}_0(X_1 \cup X_2)$; this proves the first inclusion.
Let us now prove the second inclusion $\mathcal{T}_0(X_1 \cup X_2) \subseteq \mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2)$. Hence let $Z \in \mathcal{T}_0(X_1 \cup X_2)$, i.e.

$$(X_1 \cup X_2)_{\subseteq}^{\min} Z \subseteq (X_1 \cup X_2)_0^{\max} .$$

Let us define

$$\begin{aligned} Y_1 &= Z \cap (X_1)_0^{\max} \\ Y_2 &= Z \cap (X_2)_0^{\max} . \end{aligned}$$

Then

$$\begin{aligned} Y_1 \cup Y_2 &= (Z \cap (X_1)_0^{\max}) \cup (Z \cap (X_2)_0^{\max}) \\ &= (Z \cup Z) \cap (Z \cup (X_2)_0^{\max}) \\ &\quad \cap ((X_1)_0^{\max} \cup Z) \cap ((X_1)_0^{\max} \cup (X_2)_0^{\max}) && \text{by distributivity of } \cup, \cap \\ &= Z \cap (Z \cup (X_2)_0^{\max}) \cap ((X_1)_0^{\max} \cup Z) \cap ((X_1)_0^{\max} \cup (X_2)_0^{\max}) && \text{because } \cup \text{ is idempotent} \\ &= Z \cap ((X_1)_0^{\max} \cup X_2 0) && \text{by absorption} \\ &= Z , \end{aligned}$$

where the last step holds because $Z \subseteq ((X_1)_0^{\max} \cup (X_2)_0^{\max})$.
It remains to be shown that $Y_1 \in \mathcal{T}_0(X_1)$ and $Y_2 \in \mathcal{T}_0(Y_2)$. Clearly $Z \supseteq (X_1 \cup X_2)_0^{\min} = (X_1)_0^{\min} \cup (X_2)_0^{\min}$ and hence

$$\begin{aligned} Y_1 &= Z \cap (X_1)_0^{\max} \\ &\supseteq ((X_1)_0^{\min} \cup (X_2)_0^{\min}) \cap (X_1)_0^{\max} && \text{by monotonicity of } \cap \\ &= ((X_1)_0^{\min} \cap (X_1)_0^{\max}) \cup ((X_2)_0^{\min} \cap (X_1)_0^{\max}) && \text{by distributivity} \\ &= (X_1)_0^{\min} \cup ((X_2)_0^{\min} \cap (X_1)_0^{\max}) && \text{because } (X_1)_0^{\min} \subseteq (X_1)_0^{\max} \\ &\supseteq (X_1)_0^{\min} . \end{aligned}$$

In addition,

$$Y_1 = Z \cap (X_1)_0^{\max} \subseteq (X_1)_0^{\max} .$$

Hence $Y_1 \in \mathcal{T}_0(X_1)$. By analogous reasoning, $Y_2 \in \mathcal{T}_0(X_2)$. This proves that $\mathcal{T}_0(X_1 \cup X_2) \subseteq \mathcal{T}_0(X_1) \cup \mathcal{T}_0(X_2)$.

b. Suppose $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are given. Then

$$\begin{aligned}
\mathcal{T}_0(X_1 \cap X_2) &= \mathcal{T}_0(\neg(\neg X_1 \cup \neg X_2)) && \text{by De Morgan's Law}\\
&= \neg\mathcal{T}_0(\neg X_1 \cup \neg X_2) && \text{by L-30}\\
&= \neg(\mathcal{T}_0(\neg X_1) \cup \mathcal{T}_0(\neg X_2)) && \text{by part a. of this lemma}\\
&= \neg(\neg\mathcal{T}_0(X_1) \cup \neg\mathcal{T}_0(X_2)) && \text{by L-30}\\
&= \mathcal{T}_0(X_1) \cap \mathcal{T}_0(X_2)\,. && \text{by De Morgan's Law}
\end{aligned}$$

Lemma 33
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-ary semi-fuzzy quantifier and $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in \mathbf{I}$,

$$(Q\cap)_\gamma(X_1, \ldots, X_{n+1}) = Q_\gamma(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})\,.$$

Proof
The case $\gamma \neq 0$ is covered by [9, L-23].
Let us now show that the above property also holds in the case that $\gamma = 0$. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier such that $n > 0$ and suppose $X_1, \dot{;} X_{n+1} \in \widetilde{\mathcal{P}}(E)$ are given. Then

$$\begin{aligned}
&(Q\cap)_0(X_1, \ldots, X_{n+1})\\
&= \mathrm{m}_{\frac{1}{2}}\{Q\cap(Y_1, \ldots, Y_{n+1}) : Y_i \in \mathcal{T}_0(X_i)\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1}) : Y_i \in \mathcal{T}_0(X_i)\} && \text{by Def. 14}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, Z) : Y_1 \in \mathcal{T}_0(X_1), \ldots, Y_{n-1} \in \mathcal{T}_0(X_{n-1}),\\
&\qquad Z \in \mathcal{T}_0(X_n) \cap \mathcal{T}_0(X_{n+1})\} && \text{substitution } Z = Y_n \cap Y_{n+1}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_{n-1}, Z) : Y_1 \in \mathcal{T}_0(X_1), \ldots, Y_{n-1} \in \mathcal{T}_0(X_{n-1}),\\
&\qquad Z \in \mathcal{T}_0(X_n \cap X_{n+1})\} && \text{by L-32}\\
&= Q_0(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})\,, && \text{by Def. 67}
\end{aligned}$$

as desired.

Lemma 34
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $1 \leq k \leq n$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$Q_\gamma(X_1, \ldots, X_{k-1}, X_n, X_{k+1}, \ldots, X_{n-1}, X_k) = (Q\tau_k)_\gamma(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$.
(See [9, L-21, p. 126])

Lemma 35
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-ary semi-fuzzy quantifier and $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in \mathbf{I}$,

$$(Q\cup)_\gamma(X_1, \ldots, X_{n+1}) = Q_\gamma(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})\,.$$

Proof This is immediate if we observe that

$$\begin{aligned}
&(Q_\gamma\cup)(X_1,\dots,X_{n+1})\\
&= (Q\neg\cap\neg\tau_n\neg\tau_n)_\gamma(X_1,\dots,X_{n+1}) && \text{by Def. 11, Def. 13, Def. 26 and DeMorgan's law}\\
&= (Q_\gamma\neg\cap\neg\tau_n\neg\tau_n)(X_1,\dots,X_{n+1}) && \text{by L-31, L-33, L-34}\\
&= Q_\gamma(X_1,\dots,\neg(\neg X_n\cap\neg X_{n+1})) && \text{by Def. 11, Def. 14, Def. 13}\\
&= Q_\gamma(X_1,\dots,X_{n-1},X_n\cup X_{n+1}) && \text{by DeMorgan's law.}
\end{aligned}$$

Proof of Theorem 49

If $\mathcal{B}:\mathbb{B}\longrightarrow\mathbf{I}$ satisfies (B-3), then

$$\begin{aligned}
\widetilde{\widetilde{\mathcal{M}_\mathcal{B}}}(\vee) &= \widetilde{\widetilde{\mathcal{M}}}(\vee) && \text{by Th-46}\\
&= \widetilde{\mathcal{M}}(\vee) && \text{by Th-42, Th-36}\\
&= \vee
\end{aligned}$$

because $\mathcal{M}$ is Standard-DFS by Th-42 ($\vee:\mathbf{I}\times\mathbf{I}\longrightarrow\mathbf{I}$ denotes the standard disjunction $a\vee b = \max(a,b)$). Hence $\mathcal{M}_\mathcal{B}$ induces the standard union $\cup$ of fuzzy sets (based on $\max$), and

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q\cup)(X_1,\dots,X_{n+1}) &= \mathcal{B}(((Q\cup)_\gamma(X_1,\dots,X_{n+1}))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&= \mathcal{B}((Q_\gamma(X_1,\dots,X_{n-1},X_n\cup X_{n+1}))_{\gamma\in\mathbf{I}}) && \text{by L-35}\\
&= \mathcal{M}_\mathcal{B}(Q)(X_1,\dots,X_{n-1},X_n\cup X_{n+1}) && \text{by Def. 69}\\
&= \mathcal{M}_\mathcal{B}(Q)\widetilde{\widetilde{\Box}}(X_1,\dots,X_{n+1}) && \text{by Def. 26}
\end{aligned}$$

as desired.

C.9 Proof of Theorem 50

Lemma 36

Assume $Q:\mathcal{P}(E)^n\longrightarrow\mathbf{I}$ is nonincreasing in its n-th argument. Let $X_1,\ \dots,\ X_n,\ X'_n\in\widetilde{\mathcal{P}}(E)$ where $X_n\subseteq X'_n$. Then for all $\gamma\in\mathbf{I}$,

$$Q_\gamma(X_1,\dots,X_n)\geq Q_\gamma(X_1,\dots,X_{n-1},X'_n)\,.$$

(See [9, L-25])

Proof of Theorem 50

Suppose $\mathcal{B}:\mathbb{B}\longrightarrow\mathbf{I}$ satisfies (B-5), and $Q:\mathcal{P}(E)^n\longrightarrow\mathbf{I}$ is a semi-fuzzy quantifier ($n>0$) which is nonincreasing in its n-th argument. Further let $X_1,\dots,X_n,X'_n\in\widetilde{\mathcal{P}}(E)$, $X_n\subseteq X'_n$. Then for all $\gamma\in\mathbf{I}$,

$$Q_\gamma(X_1,\dots,X_n)\geq Q_\gamma(X_1,\dots,X_{n-1},X'_n) \tag{110}$$

by L-36, i.e.

$$(Q_\gamma(X_1,\dots,X_n))_{\gamma\in\mathbf{I}}\geq (Q_\gamma(X_1,\dots,X_{n-1},X'_n))_{\gamma\in\mathbf{I}}\,. \tag{111}$$

Hence

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(Q)(X_1,\ldots,X_n) &= \mathcal{B}((Q_\gamma(X_1,\ldots,X_n))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&\geq \mathcal{B}((Q_\gamma(X_1,\ldots,X_{n-1},X'_n))_{\gamma\in\mathbf{I}}) && \text{by (111)}\\
&= \mathcal{M}_{\mathcal{B}}(Q)(X_1,\ldots,X_{n-1},X'_n) && \text{by Def. 69.}
\end{aligned}$$

C.10 Proof of Theorem 51

In order to establish the relationship between (B-4) and (Z-6), we will utilize an alternative definition of three-valued cuts (for rationale, see [9, p.133]). These are defined as follows.

Definition 106 (Alternative definition of three-valued cuts)

$$\begin{aligned}
Q^\blacktriangledown_\gamma(X_1,\ldots,X_n) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}^\blacktriangledown_\gamma(X_1),\ldots,Y_n \in \mathcal{T}^\blacktriangledown_\gamma(X_n)\}\\
\mathcal{T}^\blacktriangledown_\gamma(X_i) &= \{Y : (X_i)^{\blacktriangledown\min}_\gamma \subseteq Y \subseteq (X_i)^{\blacktriangledown\max}_\gamma\}\\
(X_i)^{\blacktriangledown\min}_\gamma &= (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma}\\
(X_i)^{\blacktriangledown\max}_\gamma &= \begin{cases} (X_i)_{\geq\frac{1}{2}} & : \gamma = 0\\ (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma > 0\end{cases}
\end{aligned}$$

This modified definition of three-valued cuts has the following property:

Lemma 37
Let $Q : \mathcal{P}(E)'^m \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier $f_1,\ldots,f_n : E \longrightarrow E'$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in \mathbf{I}\setminus\{0\}$,

$$(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})^\blacktriangledown_\gamma(X_1,\ldots,X_n) = Q^\blacktriangledown_\gamma(\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n))\,,$$

i.e. $(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})^\blacktriangledown_\gamma = Q^\blacktriangledown_\gamma \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i$.
(See [9, L-27,p.133])

Let us also recall that the modified cuts are very closely related to our original definition of three-valued cuts:

Lemma 38
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$.

a. If $Q_0(X_1,\ldots,X_n) \geq \frac{1}{2}$, then $Q_\gamma(X_1,\ldots,X_n)$ and $Q^\blacktriangledown_\gamma(X_1,\ldots,X_n)$ are nonincreasing in γ and for all $\gamma \in \mathbf{I}$,

$$Q_\gamma(X_1,\ldots,X_n) \geq Q^\blacktriangledown_\gamma(X_1,\ldots,X_n) \tag{a.i}$$

$$Q_{\gamma'}(X_1,\ldots,X_n) \leq Q^\blacktriangledown_\gamma(X_1,\ldots,X_n) \quad \textit{for all } \gamma' > \gamma. \tag{a.ii}$$

b. If $Q_0(X_1,\ldots,X_n) \leq \frac{1}{2}$, then $Q_\gamma(X_1,\ldots,X_n)$ and $Q^\blacktriangledown_\gamma(X_1,\ldots,X_n)$ are nondecreasing in γ and for all $\gamma \in \mathbf{I}$,

$$Q_\gamma(X_1,\ldots,X_n) \leq Q^\blacktriangledown_\gamma(X_1,\ldots,X_n) \tag{b.i}$$

$$Q_{\gamma'}(X_1,\ldots,X_n) \geq Q^\blacktriangledown_\gamma(X_1,\ldots,X_n) \quad \textit{for all } \gamma' > \gamma. \tag{b.ii}$$

(See [9, L-31,p.137])

Lemma 39

a. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is nonincreasing, then*

$$f^\sharp \le f \le f^\flat .$$

b. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a constant mapping, then* $f^\sharp = f^\flat = f$*.*

c. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is nondecreasing, then*

$$f^\flat \le f \le f^\sharp .$$

Proof

a. Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nonincreasing. We will first show that $f^\sharp \le f$, i.e. $f^\sharp(x) \le f(x)$ for each $x \in \mathbf{I}$. If $x = 1$, this holds trivially because by Def. 71, $f^\sharp(1) = f(1)$. If $x \in [0, 1)$, then

$$f(x) \ge f(y) \tag{112}$$

for all $y > x$ because f is assumed to be nonincreasing. In particular,

$$\begin{aligned} f(x) &\ge \sup\{f(y) : y \in \mathbf{I}, \quad y > x\} && \text{by (112)} \\ &= \lim_{y \to x^+} f(y) && \text{by Th-43} \\ &= f^\sharp . && \text{by Def. 71} \end{aligned}$$

To see that $f \le f^\flat$, consider some $x \in \mathbf{I}$. In the case that $x = 0$, $f(0) = f^\flat(0)$ holds by Def. 71. If $x \in (0, 1]$, we can utilize that

$$f(x) \le f(y) \tag{113}$$

for all $y < x$, which holds because $f \in \mathbb{B}^+$ is nonincreasing. Hence

$$\begin{aligned} f(x) &\le \inf\{f(y) : y \in \mathbf{I}, \quad y < x\} && \text{by (113)} \\ &= \lim_{y \to x^-} f(y) && \text{by Th-43} \\ &= f^\flat(x) && \text{by Def. 71.} \end{aligned}$$

b. Obvious from Def. 71.

c. In the case that $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing, the proof is analogous to that of **a.** However, because of the converse monotonicity property, all inequations must be reversed compared to **a.**.

Lemma 40
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-4) *and* (B-5)*, then*

$$\mathcal{B}(f^\sharp) = \mathcal{B}(f) = \mathcal{B}(f^\flat)$$

for all $f \in \mathbb{B}$*.*

Proof Suppose $f \in \mathbb{B}$ is given.

a. If $f \in \mathbb{B}^+$, then

$$\begin{aligned} \mathcal{B}(f^\sharp) &\leq \mathcal{B}(f) && \text{by L-39, (B-5)} \\ &\leq \mathcal{B}(f^\flat) && \text{by L-39, (B-5)} \\ &= \mathcal{B}(f^\sharp) && \text{by (B-4).} \end{aligned}$$

b. If $f \in \mathbb{B}^-$, then similarly

$$\begin{aligned} \mathcal{B}(f^\flat) &\leq \mathcal{B}(f) && \text{by L-39, (B-5)} \\ &\leq \mathcal{B}(f^\sharp) && \text{by L-39, (B-5)} \\ &= \mathcal{B}(f^\flat) && \text{by (B-4).} \end{aligned}$$

c. The case $f = \mathrm{c}_{\frac{1}{2}}$ is trivial by L-39.b.

Lemma 41
If $f, g \in \mathbb{B}$ satisfy

$$f|_{(0,1)} = g|_{(0,1)}$$

i.e. $f(x) = g(x)$ for all $x \in \mathbf{I} \setminus \{0, 1\}$, then

$$f^{\flat\sharp} = g^{\flat\sharp} .$$

Proof If $x \in (0, 1]$, then

$$\begin{aligned} f^\flat(x) &= \lim_{y \to x^-} f(y) && \text{by Def. 71} \\ &= \lim_{y \to x^-} g(y) && \text{because } f|_{(0,1)} = g|_{(0,1)} \\ &= g^\flat(x) && \text{by Def. 71,} \end{aligned}$$

i.e.

$$f^\flat|_{(0,1)} = g^\flat|_{(0,1)} . \tag{114}$$

From this we obtain for all $x \in (0, 1)$

$$\begin{aligned} f^{\flat\sharp} &= \lim_{y \to x^+} f^\flat(x) && \text{by Def. 71} \\ &= \lim_{y \to x^+} g^\flat(x) && \text{because of (114)} \\ &= g^{\flat\sharp} && \text{by Def. 71.} \end{aligned}$$

In the case that $x = 1$,

$$\begin{aligned} f^{\flat\sharp}(1) &= f^\flat(1) && \text{by Def. 71} \\ &= g^\flat(1) && \text{by (114)} \\ &= g^{\flat\sharp}(1) && \text{by Def. 71.} \end{aligned}$$

In the remaining case that $x = 0$,

$$\begin{aligned} f^{\flat\sharp}(0) &= \lim_{y \to 0^+} f^\flat(y) && \text{by Def. 71} \\ &= \lim_{y \to 0^+} g^\flat(y) && \text{because of (114)} \\ &= g^{\flat\sharp}(0) && \text{by Def. 71.} \end{aligned}$$

Lemma 42
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-4) *and* (B-5) *and $f, g \in \mathbb{B}$ satisfy*

$$f|_{(0,1)} = g|_{(0,1)} \,,$$

then

$$\mathcal{B}(f) = \mathcal{B}(g) \,.$$

Proof This is now trivial:

$$\begin{aligned} \mathcal{B}(f) &= \mathcal{B}(f^{\flat\sharp}) && \text{by L-40} \\ &= \mathcal{B}(g^{\flat\sharp}) && \text{by L-41} \\ &= \mathcal{B}(g) && \text{by L-40.} \end{aligned}$$

Lemma 43
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-4) *and* (B-5)*, then*

$$\mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) = \mathcal{B}((Q_\gamma^\blacktriangledown(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof We shall discern three cases.

a. $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$. Then

$$Q_\gamma(X_1, \ldots, X_n) \geq Q_\gamma^\blacktriangledown(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$ by L-38.a.i and hence

$$\mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) \geq \mathcal{B}((Q_\gamma^\blacktriangledown(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) \tag{115}$$

by (B-5).
On the other hand,

$$\begin{aligned} {Q^\sharp}_\gamma(X_1, \ldots, X_n) &= \lim_{\gamma' \to \gamma^+} Q_{\gamma'}(X_1, \ldots, X_n) && \text{by Def. 71} \\ &= \sup\{Q_{\gamma'}(X_1, \ldots, X_n) : \gamma' > \gamma\} && \text{by Th-43} \\ &\leq Q_\gamma^\blacktriangledown(X_1, \ldots, X_n) && \text{by L-38.a.ii} \end{aligned}$$

i.e.

$$Q^{\sharp}{}_{\gamma}(X_1, \ldots, X_n) \leq Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \qquad (116)$$

for all $\gamma \in [0, 1)$. Introducing $f : \mathbf{I} \longrightarrow \mathbf{I}$, defined by

$$f(\gamma) = \begin{cases} Q^{\sharp}{}_{\gamma}(X_1, \ldots, X_n) & : \quad \gamma < 1 \\ \frac{1}{2} & : \quad \gamma = 1 \end{cases} \qquad (117)$$

we clearly have

$$f(\gamma) \leq Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \qquad (118)$$

for all $\gamma \in \mathbf{I}$. This is obvious from (116) in the case $\gamma < 1$, and from $Q^{\blacktriangledown}_1(X_1, \ldots, X_n) \geq \frac{1}{2} = f(1)$. Hence

$$\begin{aligned} \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) &= \mathcal{B}((Q^{\sharp}{}_{\gamma}(X_1, \ldots, X_n)) && \text{by (B-4)} \\ &= \mathcal{B}(f) && \text{by L-42} \\ &\leq \mathcal{B}((Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by (118), (B-5)} \end{aligned}$$

i.e.

$$\mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) \leq \mathcal{B}((Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}})\,. \qquad (119)$$

Combining the inequations (115) and (119), we get the desired result.

b. $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$. Then

$$Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) = Q_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}$$

for all $\gamma \in \mathbf{I}$, i.e. $\mathcal{B}((Q^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}})$ holds trivially.

b. $Q_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$. The proof of this case is entirely analogous to that of **a.**, using L-38.b.i and L-38.b.ii rather than L-38.a.i and L-38.a.ii.

Proof of Theorem 51

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3), (B-4) and (B-5). Further suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $f_1, \ldots, f_n : E' \longrightarrow E$ are mappings and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$. Then

$$\begin{aligned} &\mathcal{M}_{\mathcal{B}}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})(X_1, \ldots, X_n) \\ &= \mathcal{B}(((Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\ &= \mathcal{B}(((Q \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by L-43} \\ &= \mathcal{B}((Q^{\blacktriangledown}_{\gamma}(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)))_{\gamma \in \mathbf{I}}) && \text{by L-37, L-42} \\ &= \mathcal{B}((Q_\gamma(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)))_{\gamma \in \mathbf{I}}) && \text{by L-43} \\ &= \mathcal{M}_{\mathcal{B}}(Q)(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)) && \text{by Def. 69} \end{aligned}$$

i.e. (Z-6) holds, as desired.

C.11 Proof of Theorem 52

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-1) to (B-5). Then (Z-1) to (Z-6) hold by Th-45, Th-47, Th-48, Th-49, Th-50 and Th-51. By Th-38, this proves that $\mathcal{M}_{\mathcal{B}}$ is a DFS. From Th-46 and the fact that $\mathcal{M}$ is a standard DFS by Th-42 it follows that $\mathcal{M}_{\mathcal{B}}$ is a standard DFS, too.

C.12 Proof of Theorem 53

Suppose $\mathcal{B} \in \mathrm{BB}$, and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is related to $\mathcal{B}$ according to (24).

a.: (B-1) and (C-1) are equivalent. Suppose $\mathcal{B}$ satisfies (B-1) and let $f \in \mathbb{H}$ a constant, i.e. $f = \mathrm{c}_a$ for some $a \in (0, 1]$. Then

$$\tfrac{1}{2} + \tfrac{1}{2}f = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{c}_a = \mathrm{c}_{\frac{1}{2}+\frac{1}{2}a} \tag{120}$$

(see (17)) and hence

$$\begin{aligned}
\mathcal{B}'(f) &= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 && \text{by (24)}\\
&= 2\mathcal{B}(\mathrm{c}_{\frac{1}{2}+\frac{1}{2}a}) - 1 && \text{by (120)}\\
&= 2(\tfrac{1}{2} + \tfrac{1}{2}a) - 1 && \text{by (B-1)}\\
&= a\\
&= f(0)\,, && \text{because } f = \mathrm{c}_a
\end{aligned}$$

i.e. (C-1) holds.
To see that (C-1) entails (B-1), suppose $\mathcal{B}'$ satisfies (C-1) and let $f \in \mathbb{B}$ a constant, i.e. $f = \mathrm{c}_a$ for some $a \in \mathbf{I}$.
i. $a > \frac{1}{2}$. Then $f \in \mathbb{B}^+$ and

$$\begin{aligned}
\mathcal{B}(f) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f - 1) && \text{by (23), } f \in \mathbb{B}^+\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2\mathrm{c}_a - 1) && \text{because } f = \mathrm{c}_a\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(\mathrm{c}_{2a-1}) && \text{by (17)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}(2a - 1) && \text{by (C-1)}\\
&= a\\
&= f(0) && \text{by assumption, } f = \mathrm{c}_a
\end{aligned}$$

This proves that $\mathcal{B}$ satisfies (B-1) in the case that $f = \mathrm{c}_a$ where $a > \frac{1}{2}$. The cases **ii.** ($a = \frac{1}{2}$) and **iii.** ($a < \frac{1}{2}$) are treated similarly.

b.: (B-3) and (C-2) are equivalent Suppose $\mathcal{B}$ satisfies (B-3) and let $f \in \mathbb{H}$ a mapping such that $f(\mathbf{I}) \subseteq \{0, 1\}$, and define $g \in \mathbb{B}^+$ by

$$g(\gamma) = \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) \tag{121}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
f_*^0 &= \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{by (19)}\\
&= \inf\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) = \tfrac{1}{2}\}\\
&= \inf\{\gamma \in \mathbf{I} : g(\gamma) = \tfrac{1}{2}\} && \text{by (121)}\\
&= g_*^{\frac{1}{2}}\,, && \text{by (20)}
\end{aligned}$$

i.e.

$$f_*^0 = g_*^{\frac{1}{2}} . \tag{122}$$

In addition, it is apparent from (121) and $f(\mathbf{I}) \subseteq \{0, 1\}$ that

$$g(\mathbf{I}) \subseteq \{\tfrac{1}{2}, 1\} . \tag{123}$$

Therefore

$$\begin{aligned}
\mathcal{B}'(f) &= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 && \text{by (24)} \\
&= 2\mathcal{B}(g) - 1 && \text{by (121)} \\
&= 2(\tfrac{1}{2} + \tfrac{1}{2}g_*^{\frac{1}{2}}) - 1 && \text{by (B-3), (123)} \\
&= g_*^{\frac{1}{2}} \\
&= f_*^0 , && \text{by (122)}
\end{aligned}$$

i.e. (C-2) holds.
To see that the converse direction of the equivalence holds, let us assume that $\mathcal{B}'$ satisfies (C-2). Furthermore, let $f \in \mathbb{B}$ a mapping such that $f(\mathbf{I}) \subseteq \{0,\ \frac{1}{2},\ 1\}$.
i. $f \in \mathbb{B}^+$, i.e. $f(\mathbf{I}) \subseteq \{\frac{1}{2}, 1\}$ by Def. 68. Let us define $h \in \mathbb{H}$ by $h = 2f - 1$. Then

$$h(\mathbf{I}) \subseteq \{0, 1\} \tag{124}$$

and

$$\begin{aligned}
h_*^0 &= \inf\{\gamma \in \mathbf{I} : h(\gamma) = 0\} && \text{by (19)} \\
&= \inf\{\gamma \in \mathbf{I} : 2f(\gamma) - 1 = 0\} && \text{by definition of } h \\
&= \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \\
&= f_*^{\frac{1}{2}} , && \text{by (20)}
\end{aligned}$$

i.e.

$$h_*^0 = f_*^{\frac{1}{2}} . \tag{125}$$

Therefore

$$\begin{aligned}
\mathcal{B}(f) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f - 1) && \text{by (23), } f \in \mathbb{B}^+ \\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(h) && \text{by definition of } h \\
&= \tfrac{1}{2} + \tfrac{1}{2}h_*^0 && \text{by (C-2)} \\
&= \tfrac{1}{2} + \tfrac{1}{2}f_*^{\frac{1}{2}} , && \text{by (125)}
\end{aligned}$$

i.e. (B-3) holds if $f \in \mathbb{B}^+$. The remaining cases **ii.** ($f \in \mathbb{B}^{\frac{1}{2}}$) and **iii.** ($f \in \mathbb{B}^-$) are treated analogously.

c.: (B-4) and the conjunction of (C-3.a) and (C-3.b) are equivalent Suppose $\mathcal{B}$ satisfies (B-4) and let $f \in \mathbb{H}$.

If $\widehat{f}((0,1] = \{0\}$, then $f(\gamma) = 0$ for all $\gamma > 0$. It is easily seen from Def. 71 that in this case,

$$(\tfrac{1}{2} + \tfrac{1}{2}f)^{\flat} = \tfrac{1}{2} + \tfrac{1}{2}f \tag{126}$$

and

$$(\tfrac{1}{2} + \tfrac{1}{2}f)^{\sharp} = \mathrm{c}_{\frac{1}{2}}\,. \tag{127}$$

We may then conclude from (B-4) and (23) that that

$$\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) = \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2}\,. \tag{128}$$

Therefore

$$\begin{aligned}
\mathcal{B}'(f) &= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 && \text{by (24)}\\
&= 2 \cdot \tfrac{1}{2} - 1 && \text{by (128)}\\
&= 0\,,
\end{aligned}$$

i.e. (C-3.a) holds.
In the remaining case that $\widehat{f}((0,1] \neq \{0\}$,

$$\begin{aligned}
\mathcal{B}'(f^{\sharp}) &= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f^{\sharp}) - 1 && \text{by (24)}\\
&= 2\mathcal{B}((\tfrac{1}{2} + \tfrac{1}{2}f)^{\sharp}) - 1 && \text{apparent from Def. 71}\\
&= 2\mathcal{B}((\tfrac{1}{2} + \tfrac{1}{2}f)^{\flat}) - 1 && \text{by (B-4)}\\
&= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f^{\flat}) - 1 && \text{apparent from Def. 71}\\
&= \mathcal{B}'(f^{\flat})\,. && \text{by (24)}
\end{aligned}$$

Now let us consider the reverse direction of the equivalence. Suppose $\mathcal{B}'$ satisfies (C-3.a) and (C-3.b), and let $f \in \mathbb{B}$ be given.
i. $f \in \mathbb{B}^+$. Let us first consider the case that $\widehat{f}((0,1]) = \{\frac{1}{2}\}$. Then $f^{\sharp} = \mathrm{c}_{\frac{1}{2}}$ and $f^{\flat} = f$ (this is apparent from Def. 71). We compute:

$$\begin{aligned}
\mathcal{B}(f^{\sharp}) &= \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) && \text{because } f^{\sharp} = \mathrm{c}_{\frac{1}{2}}\\
&= \tfrac{1}{2} && \text{by (23)}\\
&= \tfrac{1}{2} + \tfrac{1}{2} \cdot 0\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f - 1) && \text{by (C-3.a)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f^{\flat} - 1) && \text{because } f^{\flat} = f\text{, see above}\\
&= \mathcal{B}(f^{\flat})\,.
\end{aligned}$$

Equation (C-3.a) is applicable because $\widehat{f}((0,1]) = \{\frac{1}{2}\}$ entails that $(\widehat{2f-1})((0,1]) = \{0\}$. If $\widehat{f}((0,1]) \neq \{\frac{1}{2}\}$, then $(\widehat{2f-1})((0,1]) \neq \{0\}$, i.e. (C-3.b) is applicable. Therefore

$$\begin{aligned}
\mathcal{B}(f^\sharp) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f^\sharp - 1) && \text{by (23)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'((2f-1)^\sharp) && \text{apparent from Def. 71}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'((2f-1)^\flat) && \text{by (C-3.b)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2f^\flat - 1) && \text{apparent from Def. 71}\\
&= \mathcal{B}(f^\flat)\,. && \text{by (23)}
\end{aligned}$$

ii., iii.: The remaining cases $f \in \mathbb{B}^{\frac{1}{2}}$ and $f \in \mathbb{B}^-$ are treated analogously.

d.: (B-5) and (C-4) are equivalent Suppose $\mathcal{B}$ satisfies (B-5) and let $f, g \in \mathbb{H}$ such that $f \leq g$. Then

$$\begin{aligned}
\mathcal{B}'(f) &= 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 && \text{by (24)}\\
&\leq 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}g) - 1 && \text{by (B-5), } f \leq g\\
&= \mathcal{B}'(g)\,, && \text{by (24)}
\end{aligned}$$

i.e. (C-4) holds.
To see that the reverse direction of the equivalence holds, let us assume that $\mathcal{B}'$ satisfies (C-4). Further suppose that $f, g \in \mathbb{B}$ such that $f \leq g$. Let us first observe that

$$\begin{aligned}
h \in \mathbb{B}^+ &\implies \mathcal{B}(h) \geq \tfrac{1}{2}\\
h \in \mathbb{B}^{\frac{1}{2}} &\implies \mathcal{B}(h) = \tfrac{1}{2}\\
h \in \mathbb{B}^- &\implies \mathcal{B}(h) \leq \tfrac{1}{2}
\end{aligned}$$

for $h \in \{f, g\}$, which is apparent from (23). Because of these inequations, the only nontrivial cases to be shown are

i. $f,\, g \in \mathbb{B}^+$

ii. $f,\, g \in \mathbb{B}^-$.

We shall only prove **ii.** here because the proof of **i.** is analogous.
Hence let us assume that $f,\, g \in \mathbb{B}^-$, $f \leq g$. Then

$$1 - 2f \geq 1 - 2g\,. \tag{129}$$

Therefore

$$\begin{aligned}
\mathcal{B}(f) &= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}'(1 - 2f) && \text{by (23), } f \in \mathbb{B}^-\\
&\leq \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}'(1 - 2g) && \text{by (C-4), (129)}\\
&= \mathcal{B}(g)\,. && \text{by (23), } g \in \mathbb{B}^-
\end{aligned}$$

C.13 Proof of Theorem 56

Lemma 44

For every nullary semi-fuzzy quantifier $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ and all $\gamma \in \mathbf{I}$,

$$Q_\gamma(\varnothing) = Q(\varnothing)\,.$$

(See [9, L-18, p. 124])

Proof of Theorem 56

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-1), i.e. there is some constant $f \in \mathbb{B}$ such that

$$\mathcal{B}(f) \neq f(0)\,. \tag{130}$$

Let $E \neq \varnothing$ some set and $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ the constant semi-fuzzy quantifier defined by

$$Q(\varnothing) = f(0)\,. \tag{131}$$

Then

$$\begin{aligned} Q_\gamma(\varnothing) &= Q(\varnothing) && \text{by L-44} \\ &= f(0) && \text{by (131)} \\ &= f(\gamma) && \text{because } f \text{ constant} \qquad (+) \end{aligned}$$

for all $\gamma \in \mathbf{I}$ and hence

$$\begin{aligned} Q(\varnothing) &= f(0) \\ &\neq \mathcal{B}(f) && \text{by (130)} \\ &= \mathcal{B}((Q_\gamma(\varnothing))_{\gamma\in\mathbf{I}}) && \text{by (+)} \\ &= \mathcal{M}_\mathcal{B}(Q)(\varnothing) && \text{by Def. 69} \end{aligned}$$

i.e. (Z-1) fails on Q.

C.14 Proof of Theorem 57

In order to prove that (B-3) is necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS, we first need some lemmata which highlight the role of $f^\flat$ and $f^\sharp$.

Lemma 45

a. If $f \in \mathbb{B}^+$, then

$$f^\sharp(x) \leq f^\flat(x)$$

for all $x \in \mathbf{I}$ and

$$f^\sharp(x) \geq f^\flat(y)$$

for all $x \in \mathbf{I}$ and $y > x$.

b. If $f \in \mathbb{B}^-$, then

$$f^\sharp(x) \geq f^\flat(x)$$

for all $x \in \mathbf{I}$ and

$$f^\sharp(x) \leq f^\flat(y)$$

for all $x \in \mathbf{I}$, $y > x$.

Note. The lemma can be summarized as stating that for all $f \in \mathbb{B}$ and $x \in \mathbf{I}$, $f^\sharp(x) \preceq_c f^\flat(x)$, but $f^\flat(y) \preceq_c f^\sharp(x)$ for all $y > x$.

Proof

a. Firstly

$$\begin{aligned}
f^\sharp(0) &= \lim_{x\to 0^+} f(x) && \text{by Def. 71}\\
&= \sup\{f(x) : x > 0\} && \text{by Th-43}\\
&\leq \sup\{f(0) : x > 0\} && \text{because } f \in \mathbb{B}^+ \text{ nonincreasing}\\
&= f(0)\\
&= f^\flat(0)\,. && \text{by Def. 71}
\end{aligned}$$

If $0 < x < 1$, then similarly

$$\begin{aligned}
f^\sharp(x) &= \lim_{z\to x^+} f(z) && \text{by Def. 71}\\
&= \sup\{f(z) : z > x\} && \text{by Th-43}\\
&\leq \sup\{f(x) : z > x\} && \text{because } f \in \mathbb{B}^+ \text{ nonincreasing}\\
&= f(x)\\
&= \inf\{f(x) : z < x\}\\
&\leq \inf\{f(z) : z < x\} && \text{because } f \in \mathbb{B}^+ \text{ nonincreasing}\\
&= \lim_{z\to x^-} f(z) && \text{by Th-43}\\
&= f^\flat(x)\,. && \text{by Def. 71}
\end{aligned}$$

Finally if $x = 1$,

$$\begin{aligned}
f^\sharp(1) &= f(1) && \text{by Def. 71}\\
&= \inf\{f(1) : z < 1\}\\
&\leq \inf\{f(z) : z < 1\} && \text{because } f \in \mathbb{B}^+ \text{ nonincreasing}\\
&= \lim_{z\to 1^-} f(z) && \text{by Th-43}\\
&= f^\flat(1)\,. && \text{by Def. 71}
\end{aligned}$$

This finishes the proof that $f^\sharp(x) \leq f^\flat(x)$.

To see that the second inequation holds, let $x, y \in \mathbf{I}$, $y > x$. Then $x < 1$ by our assumption $x < y \leq 1$, and $y > 0$ by our assumption $y > x$. Choosing some $w \in (x, y)$, we hence have

$$\begin{aligned}
f^\sharp(x) &= \lim_{z \to x^+} f(z) && \text{by Def. 71} \\
&= \sup\{f(z) : z > x\} && \text{by Th-43} \\
&= \sup\{f(z) : x < z < y\} && \text{because } f \text{ nonincreasing, } y > 0 \\
&\geq \sup\{f(z) : w < z < y\} && \text{by monotonicity of } \sup \\
&\geq \inf\{f(z) : w < z < y\} && \text{because } (w, y) \neq \varnothing \\
&= \inf\{f(z) : z < y\} && \text{because } f \text{ nonincreasing, } w < y \\
&= \lim_{z \to y^-} f(z) && \text{by Th-43} \\
&= f^\flat(y)\,. && \text{by Def. 71}
\end{aligned}$$

b. The proof of the case $f \in \mathbb{B}^-$ is completely analogous to that of **a.** It is also easily obtained from **a.** by observing that $(1 - f)^\sharp = 1 - f^\sharp$ and $(1 - f)^\flat = 1 - f^\flat$.
The previous lemma becomes useful for our purposes when combined with the following lemma on properties of the fuzzy median.

Lemma 46
For all $x, y \in \mathbf{I}$, if $x \preceq_c y$, then $\mathrm{m}_{\frac{1}{2}}(x, y) = x$.

Proof If $x \preceq_c y$, then either $\frac{1}{2} \leq x \leq y$ or $\frac{1}{2} \geq x \geq y$. In the first case, $\min(x, y) \geq \frac{1}{2}$ and hence $\mathrm{m}_{\frac{1}{2}}(x, y) = \min(x, y) = x$ by Def. 45. In the second case, $\max(x, y) \leq \frac{1}{2}$ and hence $\mathrm{m}_{\frac{1}{2}}(x, y) = \max(x, y) = x$ by Def. 45.
In addition, we shall need the following.

Lemma 47
For all $f \in \mathbb{B}$ and $x \in \mathbf{I}$,

$$f(y) \preceq_c f(x)$$

for all $y \geq x$.

Proof Suppose f is a mapping $f \in \mathbb{B}$ and let $x, y \in \mathbf{I}$ such that $x \leq y$.

- If $f \in \mathbb{B}^+$, then f is nonincreasing and $f \geq \frac{1}{2}$ by Def. 68. Hence $f(x) \geq f(y) \geq \frac{1}{2}$, i.e. $f(y) \preceq_c f(x)$ by Def. 44.
- If $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$, then $f(x) = f(y) = \frac{1}{2}$ and hence $f(y) \preceq_c f(x)$ by Def. 44.
- If $f \in \mathbb{B}^-$, then f is nondecreasing and $f \leq \frac{1}{2}$ by Def. 68. Hence $f(x) \leq f(y) \leq \frac{1}{2}$, i.e. $f(y) \preceq_c f(x)$ by Def. 44.

Lemma 48
Suppose f is some mapping $f \in \mathbb{B}$, and let us define $f_1 \in \mathbb{B}$ by

$$f_1(x) = \begin{cases} f^\sharp(x) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases} \tag{L-48.1}$$

for all $x \in \mathbf{I}$.

a. There exist $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ *such that*

$$(Q \circ \widehat{g})_\gamma(X) = f_1(\gamma)$$
$$Q_\gamma(\hat{g}(X)) = f^\flat(\gamma)$$

for all $\gamma \in \mathbf{I}$.

b. There exist $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ *such that*

$$(Q \circ \widehat{g})_\gamma(X) = f_1(\gamma)$$
$$Q_\gamma(\hat{g}(X)) = f^\sharp(\gamma)$$

for all $\gamma \in \mathbf{I}$.

Proof We will use the same mapping $g : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ and the same fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ for the proofs of **a.** and **b**.
We shall define the mapping $g : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(z_1, z_2) = z_1 \tag{132}$$

for all $z_1, z_2 \in \mathbf{I}$, i.e. g is the projection of (z_1, z_2) to its first argument.
Furthermore, we shall define the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ by

$$\mu_X(z_1, z_2) = \begin{cases} \frac{1}{2} + \frac{1}{2} z_2 & : \quad z_2 < z_1 \\ \frac{1}{2} z_2 & : \quad z_1 = 0 \text{ and } z_2 < 1 \\ 0 & : \quad \text{else} \end{cases} \tag{133}$$

for all $z_1, z_2 \in \mathbf{I}$.
Abbreviating $V = \hat{g}(X) \in \widetilde{\mathcal{P}}(\mathbf{I})$, we have

$$\begin{aligned} \mu_V(z_1) &= \sup\{\mu_X(z_1, z_2) : z_2 \in \mathbf{I}\} && \text{by Def. 20, (132)} \\ &= \begin{cases} \sup\{\frac{1}{2} + \frac{1}{2} z_2 : z_2 < z_1\} & : \quad z_1 \neq 0 \\ \sup\{\frac{1}{2} z_2 : z_2 < 1\} & : \quad z_1 = 0 \end{cases} && \text{by (133)} \\ &= \begin{cases} \frac{1}{2} + \frac{1}{2} z_1 & : \quad z_1 \neq 0 \\ \frac{1}{2} & : \quad z_1 = 0 \end{cases} \end{aligned}$$

for all $z_1 \in \mathbf{I}$. From this it is apparent by Def. 66 that

$$(V)_0^{\min} = (V)_{>\frac{1}{2}} = (0, 1] \tag{134}$$
$$(V)_0^{\max} = (V)_{\geq\frac{1}{2}} = [0, 1] \tag{135}$$

and if $\gamma > 0$,

$$(V)_\gamma^{\min} = \{z_1 \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2} z_1 \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\} = [\gamma, 1] \tag{136}$$
$$(V)_\gamma^{\max} = (Z)_{>\frac{1}{2} - \frac{1}{2}\gamma} = [0, 1]\,. \tag{137}$$

Investigating the three-valued cut ranges of X itself, we have (by Def. 66 and (133)):

$$(X)_0^{\min} = (X)_{>\frac{1}{2}} = \{(z_1, z_2) : 0 < z_2 < z_1\}$$
$$(X)_0^{\max} = (X)_{\geq\frac{1}{2}} = \{(z_1, z_2) : z_2 < z_1\}$$

If $\gamma > 0$,

$$\begin{aligned}(X)_\gamma^{\min} &= (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} && \text{by Def. 66}\\ &= \{(z_1, z_2) : z_2 < z_1 \text{ and } \tfrac{1}{2} + \tfrac{1}{2}z_2 \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\} && \text{by (133)}\\ &= \{(z_1, z_2) : \gamma \leq z_2 < z_1\}\end{aligned}$$

and similarly

$$\begin{aligned}(X)_\gamma^{\max} &= (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} && \text{by Def. 66}\\ &= \{(z_1, z_2) : z_2 < z_1\} \cup \{(0, z_2) : \tfrac{1}{2}z_2 > \tfrac{1}{2} - \tfrac{1}{2}\gamma \text{ and } z_2 < 1\} && \text{by (133)}\\ &= \{(z_1, z_2) : z_2 < z_1\} \cup \{(0, z_2) : 1 - \gamma < z_2 < 1\}\,.\end{aligned}$$

From this characterisation of $(X)_\gamma^{\min}$ and $(X)_\gamma^{\max}$, it is easily seen that by Def. 17,

$$\widehat{g}((X)_0^{\min}) = (0, 1] \tag{138}$$
$$\widehat{g}((X)_0^{\max}) = (0, 1] \tag{139}$$

and if $\gamma > 0$,

$$\widehat{g}((X)_\gamma^{\min}) = (\gamma, 1] \tag{140}$$
$$\widehat{g}((X)_\gamma^{\max}) = [0, 1] \tag{141}$$

We are now ready to turn to part a. and b. of the lemma.

a. To prove this part of the lemma, let us define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q(Y) = \begin{cases} f^\flat(y) & : \quad y \in Y \text{ and } y \neq 0 \\ f^\sharp(y) & : \quad y \notin Y \text{ and } y \neq 0 \\ f(0) & : \quad y = 0 \end{cases} \tag{142}$$

where we have abbreviated $y = \inf Y$. We will assume that g and X are defined as above.

By applying L-45, it is easily observed that Q is monotonically nondecreasing (if $f \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$) or nonincreasing (if $f \in \mathbb{B}^-$) in its argument. We can utilize the monotonicity of Q in some of the following computations.

$$\begin{aligned}Q_0(\hat{g}(X)) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (\hat{g}(X))_0^{\min} \subseteq Y \subseteq (\hat{g}(X))_0^{\max}\} && \text{by Def. 67}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (0, 1] \subseteq Y \subseteq [0, 1]\} && \text{by (134), (135)}\\ &= \mathrm{m}_{\frac{1}{2}}\{f(0)\} && \text{by (142)}\\ &= f(0) && \text{by Def. 46}\\ &= f^\flat(0)\,. && \text{by Def. 71}\end{aligned}$$

$$
\begin{aligned}
(Q \circ \widehat{g})_0(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(\widehat{g}(Y)) : (X)_0^{\min} \subseteq Y \subseteq (X)_0^{\max}\} && \text{by Def. 67} \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : \widehat{g}((X)_0^{\min}) \subseteq Z \subseteq \widehat{g}((X)_0^{\max})\} \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : (0,1] \subseteq Z \subseteq (0,1]\} && \text{by (138), (139)} \\
&= \mathrm{m}_{\frac{1}{2}}\{f(0)\} && \text{by (142)} \\
&= f(0) && \text{by Def. 46} \\
&= f_1(0)\,. && \text{by (L-48.1)}
\end{aligned}
$$

If $\gamma > 0$, then

$$
\begin{aligned}
Q_\gamma(\hat{\hat{g}}(X)) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (\hat{\hat{g}}(X))_\gamma^{\min} \subseteq Y \subseteq (\hat{\hat{g}}(X))_\gamma^{\max}\} && \text{by Def. 67} \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\} && \text{by (136), (137)} \\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\}, \\
&\qquad \sup\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\}) && \text{by Def. 46} \\
&= \mathrm{m}_{\frac{1}{2}}(Q([\gamma,1]),\, Q([0,1])) && \text{by monotonicity of } Q \\
&= \mathrm{m}_{\frac{1}{2}}(f^\flat(\gamma), f(0)) && \text{by (142)} \\
&= f^\flat(\gamma)\,. && \text{by L-46 and L-47, noting that } f^\flat \in \mathbb{B}
\end{aligned}
$$

and

$$
\begin{aligned}
(Q \circ \widehat{g})_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(\widehat{g}(Y)) : (X)_\gamma^{\min} \subseteq Y \subseteq (X)_\gamma^{\max}\} && \text{by Def. 71} \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : \widehat{g}((X)_\gamma^{\min}) \subseteq Z \subseteq \widehat{g}((X)_\gamma^{\max})\} \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\} && \text{by (140), (141)} \\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\}, \\
&\qquad \sup\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\}) && \text{by Def. 46} \\
&= \mathrm{m}_{\frac{1}{2}}(Q((\gamma,1]),\, Q([0,1])) && \text{by monotonicity of } Q \\
&= \mathrm{m}_{\frac{1}{2}}(f^\sharp(\gamma),\, f(0)) && \text{by (142)} \\
&= f^\sharp(\gamma)\,. && \text{by L-46, L-45, } f(0) = f^\flat(0)
\end{aligned}
$$

This finishes the proof of part a. of the lemma.

b. In this case, we shall define $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ as follows:

$$
Q(Y) = \begin{cases} f^\sharp(y) & : \quad y > 0 \\ f(0) & : \quad y = 0 \text{ and } 0 \notin Y \\ f^\sharp(0) & : \quad y = 0 \text{ and } 0 \in Y \end{cases} \tag{143}
$$

where we again abbreviate $y = \inf Y$. In addition, we will assume that g and X are defined as above. Then

$$\begin{aligned}
Q_0(\hat{\hat{g}}(X)) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (\hat{\hat{g}}(X))_0^{\min} \subseteq Y \subseteq (\hat{\hat{g}}(X))_0^{\max}\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (0,1] \subseteq Y \subseteq [0,1]\} && \text{by (134), (135)}\\
&= \mathrm{m}_{\frac{1}{2}}\{f(0),\ f^\sharp(0)\} && \text{by (143)}\\
&= f^\sharp(0) && \text{by L-46, L-45, } f(0) = f^\flat(0)
\end{aligned}$$

and

$$\begin{aligned}
(Q \circ \widehat{g})_0(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(\widehat{g}(Y)) : (X)_0^{\min} \subseteq Y \subseteq (X)_0^{\max}\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : \widehat{g}((X)_0^{\min}) \subseteq Z \subseteq \widehat{g}((X)_0^{\max})\}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : (0,1] \subseteq Z \subseteq (0,1]\} && \text{by (138), (139)}\\
&= \mathrm{m}_{\frac{1}{2}}\{f(0)\} && \text{by (143)}\\
&= f(0) && \text{by Def. 46}\\
&= f_1(0) && \text{by (L-48.1)}
\end{aligned}$$

And if $\gamma > 0$,

$$\begin{aligned}
Q_\gamma(\hat{\hat{g}}(X)) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : (\hat{\hat{g}}(X))_\gamma^{\min} \subseteq Y \subseteq (\hat{\hat{g}}(X))_\gamma^{\max}\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\} && \text{by (136), (137)}\\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\},\\
&\qquad \sup\{Q(Y) : [\gamma,1] \subseteq Y \subseteq [0,1]\}) && \text{by Def. 46}\\
&= \mathrm{m}_{\frac{1}{2}}(Q([\gamma,1]),\ Q([0,1])) && \text{by L-45, monotonicity of } f^\sharp \in \mathbb{B}\\
&= \mathrm{m}_{\frac{1}{2}}\{f^\sharp(\gamma), f(0)\} && \text{by (143)}\\
&= f^\sharp(\gamma) && \text{by L-46, L-39,L-47}
\end{aligned}$$

and finally

$$\begin{aligned}
(Q \circ \widehat{g})_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(\widehat{g}(Y)) : (X)_\gamma^{\min} \subseteq Y \subseteq (X)_\gamma^{\max}\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : \widehat{g}((X)_\gamma^{\min}) \subseteq Z \subseteq \widehat{g}((X)_\gamma^{\max})\}\\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\} && \text{by (140), (141)}\\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\},\\
&\qquad \sup\{Q(Z) : (\gamma,1] \subseteq Z \subseteq [0,1]\}) && \text{by Def. 46}\\
&= \mathrm{m}_{\frac{1}{2}}(Q((\gamma,1]), Q([0,1])) && \text{by L-45, monotonicity of } f^\sharp \in \mathbb{B}\\
&= \mathrm{m}_{\frac{1}{2}}(f^\sharp(\gamma), f(0)) && \text{by (143)}\\
&= f^\sharp(\gamma) && \text{by L-46, L-39,L-47}
\end{aligned}$$

which completes the proof of part **b.**

Lemma 49

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given. If* $\widehat{\mathcal{M}_{\mathcal{B}}} = (\hat{\hat{\bullet}})$ *and* $\mathcal{M}_{\mathcal{B}}$ *satisfies* (Z-6)*, then* $\mathcal{B}$ *satisfies* (B-4).

Proof Suppose f is some mapping $f \in \mathbb{B}$. By L-48.a, there exist $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ such that

$$(Q \circ \widehat{g})_\gamma(X) = f_1(\gamma) \tag{144}$$

$$Q_\gamma(\hat{\hat{g}}(X)) = f^\flat(\gamma) \tag{145}$$

for all $\gamma \in \mathbf{I}$, where $f_1 \in \mathbb{B}$ is defined by (L-48.1). Therefore

$$\begin{aligned}
\mathcal{B}(f_1) &= \mathcal{B}(((Q \circ \widehat{g})_\gamma(X))_{\gamma \in \mathbf{I}}) && \text{by (144)} \\
&= \mathcal{M}_\mathcal{B}(Q \circ \widehat{g})(X) && \text{by Def. 69} \\
&= \mathcal{M}_\mathcal{B}(Q)(\widehat{\mathcal{M}_\mathcal{B}}(g)(X)) && \text{by assumption of (Z-6)} \\
&= \mathcal{M}_\mathcal{B}(Q)(\hat{\hat{g}}(X)) && \text{by assumption that } \widehat{\mathcal{M}_\mathcal{B}} = (\hat{\hat{\bullet}}) \\
&= \mathcal{B}((Q_\gamma(\hat{\hat{g}}(X)))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\
&= \mathcal{B}(f^\flat) && \text{by (145)}
\end{aligned}$$

i.e.

$$\mathcal{B}(f_1) = \mathcal{B}(f^\flat)\,. \tag{146}$$

On the other hand, we know that by L-48.b, there exist $Q' : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, $g' : \mathbf{I} \times \mathbf{I}$ and $X' \in \widetilde{\mathcal{P}}(\mathbf{I} \times \mathbf{I})$ such that

$$(Q' \circ \widehat{g'})_\gamma(X) = f_1(\gamma) \tag{147}$$

$$Q'_\gamma(\hat{\hat{g}}'(X)) = f^\sharp(\gamma)\,. \tag{148}$$

$$\tag{149}$$

Therefore

$$\begin{aligned}
\mathcal{B}(f_1) &= \mathcal{B}(((Q' \circ \widehat{g'})_\gamma(X))_{\gamma \in \mathbf{I}}) && \text{by (147)} \\
&= \mathcal{M}_\mathcal{B}(Q' \circ \widehat{g'})(X) && \text{by Def. 69} \\
&= \mathcal{M}_\mathcal{B}(Q')(\widehat{\mathcal{M}_\mathcal{B}}(g')(X)) && \text{by assumption of (Z-6)} \\
&= \mathcal{M}_\mathcal{B}(Q')(\hat{\hat{g}}'(X)) && \text{by assumption that } \widehat{\mathcal{M}_\mathcal{B}} = (\hat{\hat{\bullet}}) \\
&= \mathcal{B}((Q'_\gamma(\hat{\hat{g}}'(X)))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\
&= \mathcal{B}(f^\sharp)\,, && \text{by (148)}
\end{aligned}$$

i.e.

$$\mathcal{B}(f_1) = \mathcal{B}(f^\sharp)\,. \tag{150}$$

Combining (146) and (150), we obtain the desired $\mathcal{B}(f^\flat) = \mathcal{B}(f^\sharp)$, i.e. (B-4) holds.
We shall now introduce a weakened form of (B-3) and show that it is necessary for $\mathcal{M}_\mathcal{B}$ to satisfy (Z-2).

The condition we impose on $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is the following. If $f \in \mathbb{B}$, $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$ and $f(f_*^{\frac{1}{2}}) = f(0)$, then

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases} \qquad \text{(B-3')}$$

where we have abbreviated

$$f_*^{\frac{1}{2}} = \inf\{x \in \mathbf{I} : f(x) = \tfrac{1}{2}\}\,.$$

Lemma 50
If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *does not satisfy* (B-3'), *then* $\mathcal{M}_{\mathcal{B}}$ *does not satisfy* (Z-2).

Proof If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-3'), then there exists some $f \in \mathbb{B}$ such that $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$, $f(f_*^{\frac{1}{2}}) = f(0)$ and

$$\mathcal{B}(f) \neq \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases} \qquad (151)$$

a. $f \in \mathbb{B}^+$. In this case, let $E = \{*\}$ and define $X \in \widetilde{\mathcal{P}}(E)$ by

$$\mu_X(*) = \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}}\,. \qquad (152)$$

Then

$$f(\gamma) = \pi_{*\gamma}(X) \qquad (153)$$

for all $\gamma \in \mathbf{I}$ and hence

$$\begin{aligned} \mathcal{M}_{\mathcal{B}}(\pi_*)(X) &= \mathcal{B}((\pi_{*\gamma}(X))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\ &= \mathcal{B}(f) && \text{by (153)} \\ &\neq \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}} && \text{by assumption (151)} \\ &= \mu_X(*) && \text{by (152)} \\ &= \widetilde{\pi}_*(X)\,, && \text{by Def. 6} \end{aligned}$$

which shows that $\mathcal{M}_{\mathcal{B}}$ does not satisfy (Z-2).

b. $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$, and (151) becomes

$$\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) \neq \tfrac{1}{2}\,. \qquad (154)$$

Again choosing $E = \{*\}$, and defining $X \in \widetilde{\mathcal{P}}(E)$ by

$$\mu_X(*) = \tfrac{1}{2}\,, \tag{155}$$

we observe that $\pi_*\gamma(X) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ and hence

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(\pi_*)(X) &= \mathcal{B}((\pi_{*\gamma}(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&= \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) && \text{by definition of } X\\
&\neq \tfrac{1}{2} && \text{by (154)}\\
&= \mu_X(*) && \text{by (155)}\\
&= \widetilde{\pi}_*(X) && \text{by Def. 6}
\end{aligned}$$

which again proves that $\mathcal{M}_{\mathcal{B}}$ does not satisfy (Z-2).

c. $f \in \mathbb{B}^-$. The proof is analogous to that of case **a.** and **b.**

Lemma 51
Let $x_1, x_2 \in \mathbf{I}$ and let us abbreviate $z = \max(x_1, x_2)$.

a. If $z > \frac{1}{2}$, then

$$Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \begin{cases} 1 & : \quad \gamma \leq 2z - 1 \\ \frac{1}{2} & : \quad \gamma > 2z - 1 \end{cases}$$

b. If $z = \frac{1}{2}$, then $Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. If $z < \frac{1}{2}$, then

$$Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2z \\ \frac{1}{2} & : \quad \gamma > 1 - 2z \end{cases}$$

Proof Suppose $x_1, x_2 \in \mathbf{I}$ are given. Without loss of generality, we shall assume that $x_1 \geq x_2$, i.e.

$$x_1 = z = \max(x_1, x_2)\,. \tag{156}$$

In the following, we will abbreviate

$$X = \widetilde{\eta}(x_1, x_2)\,, \tag{157}$$

i.e. $\mu_X(1) = x_1$ and $\mu_X(2) = x_2$. Let us further observe that by Def. 52,

$$Q_{\vee}(V) = \begin{cases} 1 & : \quad V \neq \varnothing \\ 0 & : \quad V = \varnothing \end{cases} \tag{158}$$

for all $V \in \mathcal{P}(\{1, 2\})$.
We shall prove separately the cases a., b. and c.:

a. $x_1 > \frac{1}{2}$. Then by Def. 66 and (157),

$$(X)_\gamma^{\max} \supseteq \{1\} \neq \varnothing$$

forall $\gamma \in \mathbf{I}$. Furthermore

$$(X)_\gamma^{\min} \supseteq \{1\} \neq \varnothing$$

if $\gamma \leq 2x_1 - 1$, and

$$(X)_\gamma^{\min} = \varnothing$$

if $\gamma > 2x_1 - 1$. Hence by Def. 67 and (158),

$$Q_{\vee\gamma}(X) = \begin{cases} 1 & : \quad \gamma \leq 2x_1 - 1 \\ \frac{1}{2} & : \quad \gamma > 2x_1 - 1 \end{cases}$$

b. $x_1 = \frac{1}{2}$. Then $(X)_\gamma^{\max} \supseteq \{1\} \neq \varnothing$ for all $\gamma \in \mathbf{I}$ and $(X)_\gamma^{\min} = \varnothing$ for all $\gamma \in \mathbf{I}$. Hence by Def. 67 and (158),

$$Q_{\vee\gamma}(X) = \tfrac{1}{2}$$

for all $\gamma \in \mathbf{I}$.

c. $x_1 < \frac{1}{2}$. In this case, $(X)_\gamma^{\min} = \varnothing$ for all $\gamma \in \mathbf{I}$ and $(X)_\gamma^{\max} = \varnothing$ for all $\gamma \leq 1 - 2x_1$, $(X)_\gamma^{\max} \supseteq \{1\} \neq \varnothing$ for all $\gamma > 1 - 2x_1$. Hence by Def. 67 and (158),

$$Q_{\vee\gamma}(X) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2x_1 \\ \frac{1}{2} & : \quad \gamma > 1 - 2x_1 \end{cases}$$

Lemma 52

If $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3'), *then $\mathcal{M}_\mathcal{B}$ induces the standard disjunction $\widetilde{\widetilde{\mathcal{M}_\mathcal{B}}}(\vee) = \vee$, i.e. $x \vee y = \max(x, y)$ for all $x, y \in \mathbf{I}$.*

Proof Suppose $x_1, x_2 \in \mathbf{I}$ are given. We will abbreviate

$$z = \max(x_1, x_2)\,.$$

We shall discern three cases.

i. $z > \frac{1}{2}$. Then by L-51.a,

$$Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \begin{cases} 1 & : \quad \gamma \leq 2z - 1 \\ \frac{1}{2} & : \quad \gamma > 2z - 1 \end{cases}$$

By Def. 52 and our assumption (B-3'), this proves that

$$\widetilde{\widetilde{\mathcal{M}_\mathcal{B}}}(\vee)(x_1, x_2) = \tfrac{1}{2} + \tfrac{1}{2}(2z - 1) = z = \max(x_1, x_2)\,,$$

noting that $(Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)))_*^{\frac{1}{2}} = 2z - 1$.

ii. $z = \frac{1}{2}$. Then by L-51.b,

$$Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \tfrac{1}{2}$$

for all $\gamma \in \mathbf{I}$. By the same reasoning as in case a.,

$$\widetilde{\widetilde{\mathcal{M}_{\mathcal{B}}}}(\vee)(x_1, x_2) = \tfrac{1}{2} = \max(x_1, x_2)\,.$$

iii. $z < \frac{1}{2}$. In this case,

$$Q_{\vee\gamma}(\widetilde{\eta}(x_1, x_2)) = \begin{cases} 0 & : \quad \gamma \le 1 - 2z \\ \frac{1}{2} & : \quad \gamma > 1 - 2z \end{cases}$$

by L-51.c. For the same reasons as in case a. and b., we obtain

$$\widetilde{\widetilde{\mathcal{M}_{\mathcal{B}}}}(\vee)(x_1, x_2) = \tfrac{1}{2} - \tfrac{1}{2}(1 - 2z) = z = \max(x_1, x_2)\,.$$

Lemma 53
Suppose $\mathcal{B}$ is some mapping $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$. If $\mathcal{M}_{\mathcal{B}}$ is a DFS, then $\widehat{\mathcal{M}_{\mathcal{B}}} = (\hat{\hat{\bullet}})$, i.e. the induced extension principle of $\mathcal{M}_{\mathcal{B}}$ is the standard extension principle.

Proof By L-50, we know that from $\mathcal{M}_{\mathcal{B}}$ is a DFS, we may conclude that $\mathcal{B}$ satisfies (B-3'). Abbreviating $\widetilde{\vee} = \widetilde{\mathcal{M}_{\mathcal{B}}}(\vee)$ and $\widetilde{\widetilde{\vee}} = \widetilde{\widetilde{\mathcal{M}_{\mathcal{B}}}}(\vee)$, we know by L-52 that $\widetilde{\widetilde{\vee}}$ is the standard disjunction (i.e., $\max$). From Th-36, we then conclude that $\widetilde{\vee} = \widetilde{\widetilde{\vee}} = \max$.
Now let $f : E \longrightarrow E'$ some mapping, $X \in \widetilde{\mathcal{P}}(E)$ and $z \in E'$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{M}_{\mathcal{B}}}(f)(X)}(z) &= \mathcal{M}_{\mathcal{B}}(\exists)(X \cap f^{-1}(z)) && \text{by Th-17} \\
&= \sup \Big\{ \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_{X \cap f^{-1}(z)}(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite}, \\
&\qquad a_i \neq a_j \text{ if } i \neq j \Big\} && \text{by Th-25, } \widetilde{\vee} = \vee \\
&= \sup\{\mu_{X \cap f^{-1}(z)}(e) : e \in E\} \\
&= \sup\{\mu_X(e) : e \in f^{-1}(z)\} \\
&= \mu_{\hat{\hat{f}}(X)}(e)\,.
\end{aligned}$$

We are now prepared to prove the main theorem.

Proof of Theorem 57

Suppose $\mathcal{M}_{\mathcal{B}}$ is a DFS and $f \in \mathbb{B}$ such that $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$.

a. $f \in \mathbb{B}^+$. If $f(f_*^{\frac{1}{2}}) = f(0)$, then $\mathcal{B}(f) = \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}}$ because (B-3') holds in every DFS by L-50.
If $f(f_*^{\frac{1}{2}}) \neq f(0)$, then apparently $f(f_*^{\frac{1}{2}}) = \frac{1}{2}$. This is because $f(\mathbf{I}) \subseteq \{\frac{1}{2}, 1\}$ in the case

$f \in \mathbb{B}^+$ (see Def. 68). We further know that $f(0) = 1$ (otherwise we had $f(x) = \frac{1}{2}$ for all $x \in \mathbf{I}$, i.e. $f = c_{\frac{1}{2}}$, which contradicts our assumption $f \in \mathbb{B}^+$). Therefore, f has the form

$$f(x) = \begin{cases} 1 & : \quad x < f_*^{\frac{1}{2}} \\ \frac{1}{2} & : \quad x \geq f_*^{\frac{1}{2}} \end{cases}$$

It is then apparent from Def. 71 that

$$f^\sharp = f \tag{159}$$

and

$$f^\flat(x) = \begin{cases} 1 & : \quad x \leq f_*^{\frac{1}{2}} \\ \frac{1}{2} & : \quad x > f_*^{\frac{1}{2}} \end{cases} \tag{160}$$

Hence

$$\begin{aligned} \mathcal{B}(f) &= \mathcal{B}(f^\sharp) && \text{by (159)} \\ &= \mathcal{B}(f^\flat) && \text{by (B-4) (holds by L-53, L-49)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}} && \text{by (B-3') (holds by L-50)} \end{aligned}$$

b. $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = c_{\frac{1}{2}}$. This case is already covered by L-50.

The remaining case

c. $f \in \mathbb{B}^-$

can be treated in analogy to case **a.**

C.15 Proof of Theorem 58

Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be given. If $\mathcal{M}_\mathcal{B}$ is a DFS, then $\mathcal{B}$ satisfies (C-2) by Th-57. From Th-46 and Th-42, we conclude that then $\widetilde{\widetilde{\mathcal{M}}}_\mathcal{B}(\neg) = \widetilde{\widetilde{\mathcal{M}}}(\neg) = \neg$ is the standard negation and $\widetilde{\widetilde{\mathcal{M}}}_\mathcal{B}(\vee) = \widetilde{\widetilde{\mathcal{M}}} = \max$ is the standard disjunction. Hence $\mathcal{M}_\mathcal{B}$ is a standard DFS by Def. 49.

C.16 Proof of Theorem 59

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3), but fails on (B-2).

By Th-46, we know that $\widetilde{\mathcal{M}}_\mathcal{B}(\neg) = \widetilde{\mathcal{M}}(\neg) = \neg$ and also $\widetilde{\widetilde{\mathcal{M}}}_\mathcal{B}(\neg) = \widetilde{\widetilde{\mathcal{M}}}(\neg) = \neg$, i.e. the induced negation of $\mathcal{M}_\mathcal{B}$ is the standard negation $\neg x = 1 - x$.

Because $\mathcal{B}$ does not satisfy (B-2), there exists some $f \in \mathbb{B}$ such that

$$\mathcal{B}(1 - f) \neq 1 - \mathcal{B}(f)\,. \tag{161}$$

By Th-41, there exists $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ such that

$$Q_\gamma(X) = f(\gamma) \tag{162}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(Q\widetilde{\widetilde{\Box}})(\neg X) &= \mathcal{B}(((Q\widetilde{\widetilde{\Box}})_\gamma(\neg X))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&= \mathcal{B}(((\widetilde{\widetilde{\neg}} Q\neg)_\gamma(\neg X))_{\gamma\in\mathbf{I}}) && \text{by Def. 12}\\
&= \mathcal{B}(((\neg Q\neg)_\gamma(\neg X))_{\gamma\in\mathbf{I}}) && \text{because } \widetilde{\widetilde{\neg}} = \neg\\
&= \mathcal{B}(1 - (Q_\gamma(\neg\neg X))_{\gamma\in\mathbf{I}}) && \text{by L-31, L-29}\\
&= \mathcal{B}(1 - (Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{because } \neg = 1 - x \text{ is involutory}\\
&= \mathcal{B}(1 - f) && \text{by (162)}\\
&\neq 1 - \mathcal{B}(f) && \text{by (161)}\\
&= 1 - \mathcal{B}((Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{by (162)}\\
&= 1 - \mathcal{M}_{\mathcal{B}}(Q)(X) && \text{by Def. 69}\\
&= \neg\mathcal{M}_{\mathcal{B}}(Q)(\neg\neg X) && \text{because } \neg = 1 - x \text{ is involutory}\\
&= \mathcal{M}_{\mathcal{B}}(Q)\widetilde{\widetilde{\Box}}(\neg X)\,, && \text{by Def. 12, } \widetilde{\widetilde{\neg}} = \neg
\end{aligned}$$

i.e. (Z-3) fails.

C.17 Proof of Theorem 60

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3), but fails to satisfy (B-4).

Because $\mathcal{B}$ satisfies (B-3), we know by Def. 19, Th-46 and Th-42 that $\widehat{\mathcal{M}_{\mathcal{B}}} = \widehat{\mathcal{M}} = (\hat{\hat{\bullet}})$, i.e. $\mathcal{M}_{\mathcal{B}}$ induces the standard extension principle. The conditions of L-49 are hence fulfilled, which tells us that (B-4) is a necessary condition for (Z-6). Because (B-4) is violated, we conclude that (Z-6) does not hold, too.

C.18 Proof of Theorem 61

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ does not satisfy (B-5). Then there are $f, g \in \mathbb{B}$ such that $f \leq g$ but

$$\mathcal{B}(f) > \mathcal{B}(g)\,. \tag{163}$$

By Th-41.a, there are semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, viz. $Q(Y) = f(\inf Y)$ and $Q'(Y) = g(\inf Y)$ for all $Y \in \mathcal{P}(\mathbf{I})$, and a fuzzy argument set $X \in \widetilde{\mathcal{P}}(\mathbf{I})$, $\mu_X(z) = \frac{1}{2} + \frac{1}{2}z$, such that

$$Q_\gamma(X) = f(\gamma) \tag{164}$$

and

$$Q'_\gamma(X) = g(\gamma) \tag{165}$$

for all $\gamma \in \mathbf{I}$. Let us now define $Q'' : \mathcal{P}(\mathbf{I}) \times \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q''(Y_1, Y_2) = \begin{cases} Q'(Y_1) & : \quad Y_2 = \varnothing \\ Q(Y_1) & : \quad Y_2 \neq \varnothing \end{cases} \tag{166}$$

Because $Q'(Y_1) = g(\inf Y_1) \geq f(\inf Y_1) = Q(Y_1)$, for all $Y_1 \in \mathcal{P}(\mathbf{I})$, it is apparent that Q'' is nonincreasing in its second argument. $\mathcal{M}_\mathcal{B}$, however, does not preserve the monotonicity of Q'':

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q'')(X, \varnothing) &= \mathcal{B}((Q''_\gamma(X, \varnothing))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\
&= \mathcal{B}((Q'_\gamma(X))_{\gamma \in \mathbf{I}}) && \text{by (166)} \\
&= \mathcal{B}(g) && \text{by (165)} \\
&< \mathcal{B}(f) && \text{by (163)} \\
&= \mathcal{B}((Q_\gamma(X))_{\gamma \in \mathbf{I}}) && \text{by (164)} \\
&= \mathcal{B}((Q''_\gamma(X, \mathbf{I}))_{\gamma \in \mathbf{I}}) && \text{by (166)} \\
&= \mathcal{M}_\mathcal{B}(Q'')(X, \mathbf{I}) \,, && \text{by Def. 69}
\end{aligned}$$

i.e. $\mathcal{M}_\mathcal{B}(Q'')(X, \varnothing) < \mathcal{M}(Q'')(X, \mathbf{I})$, which proves that the nonincreasing monotonicity of Q'' in its second argument is not preserved by $\mathcal{M}_\mathcal{B}$.

C.19 Proof of Theorem 62

By Th-56, Th-57 and Th-61, we already know that (B-1) (B-3) and (B-5), resp., are necessary conditions for $\mathcal{M}_\mathcal{B}$ to be a DFS.

- If (B-3) does not hold, then $\mathcal{M}_\mathcal{B}$ is not a DFS by Th-57. In particular, if (B-2) or (B-4) does not hold, then $\mathcal{M}_\mathcal{B}$ is not a DFS (because it is not a DFS anyway). This proves that (B-2) and (B-4) are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS in the case that (B-3) does not hold.
- If (B-3) holds, then Th-59 and Th-60 apply, which tell us that (B-2) and (B-4) are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS provided that (B-3) holds.

We may summarize these results as stating that (B-2) and (B-4) are necessary conditions as well, i.e. all conditions (B-1) to (B-5) are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.
The converse claim that (B-1) to (B-5) are also sufficient for $\mathcal{M}_\mathcal{B}$ to be a DFS has already been proven in Th-52.

C.20 Proof of Theorem 63

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is not contained in BB. Then $\mathcal{B}$ fails one or both of the defining conditions of BB (see Def. 74). If $\mathcal{B}$ does not satisfy (B-2), then $\mathcal{M}_\mathcal{B}$ fails to be a DFS by Th-59. In the remaining case that $\mathcal{B}$ satisfies (B-2), but fails Def. 74.b, there is some $f \in \mathbb{B}^+$ such that

$$\mathcal{B}(f) < \tfrac{1}{2}. \tag{167}$$

By (B-2), we have

$$\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \mathcal{B}(1 - (1 - \mathrm{c}_{\frac{1}{2}})) = 1 - \mathcal{B}(1 - \mathrm{c}_{\frac{1}{2}}) = 1 - \mathcal{B}(\mathrm{c}_{\frac{1}{2}})$$

i.e.

$$\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2} \,. \tag{168}$$

We hence have $f \geq \mathrm{c}_{\frac{1}{2}}$ (because $f \in \mathbb{B}^+$) but $\mathcal{B}(f) < \frac{1}{2} = \mathcal{B}(\mathrm{c}_{\frac{1}{2}})$, i.e. $\mathcal{B}$ fails (B-5). By Th-61, $\mathcal{M}_\mathcal{B}$ is not a DFS.

C.21 Proof of Theorem 64

It is apparent from Th-62 and Th-53 that the conditions (C-1), (C-2), (C-3.a), (C-3.b) and (C-4) on $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ are necessary and sufficient for $\mathcal{M}_\mathcal{B}$ to be a DFS. It remains to be shown that (C-3.a) is entailed by the remaining axioms.
Hence suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies (C-1), (C-2), (C-3.b) and (C-4). Further let $f \in \mathbb{H}$ be given such that $\widehat{f}((0,1]) = \{0\}$. Clearly

$$f \leq g \tag{169}$$

where $g \in \mathbb{H}$ is defined by

$$g(\gamma) = \begin{cases} 1 & : \quad \gamma = 0 \\ 0 & : \quad \text{else} \end{cases} \tag{170}$$

Apparently $\widehat{g}(\mathbf{I}) \subseteq \{0,1\}$; hence by (C-2)

$$\mathcal{B}'(g) = g_*^0 = 0\,, \tag{171}$$

see (19) and (170). Because $\mathcal{B}'$ satisfies (C-4), we conclude from (169) and (171) that $\mathcal{B}'(f) \leq \mathcal{B}'(g) = 0$. But $\mathcal{B}'(f) \geq 0$; hence $\mathcal{B}'(f) = 0$. This proves that (C-3.a) holds, as desired.

D Proof of Theorems in Chapter 5

D.1 Proof of Theorem 65

Lemma 54
Suppose $f \in \mathbb{H}$ is some mapping.

a. *If $\widehat{f}((0,1]) = \{0\}$, then $f_*^0 = 0$.*

b. *If $\widehat{f}((0,1]) \neq \{0\}$, then $(f^\sharp)_*^0 = (f^\flat)_*^0$.*

Proof

a. Suppose $f \in \mathbb{H}$ and $\widehat{f}((0,1]) = \{0\}$, i.e.

$$f(\gamma) = 0 \tag{172}$$

for all $\gamma \in (0,1]$.
We also know by Def. 75 that

$$f(0) \neq 0\,. \tag{173}$$

Hence

$$\begin{aligned} f_*^0 &= \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{by (19)} \\ &= \inf(0,1] && \text{by (172), (173)} \\ &= 0\,. \end{aligned}$$

b. Suppose $f \in \mathbb{H}$ and $\widehat{f}((0,1]) \neq \{0\}$.
In order to prove that $(f^\flat)^0_* = (f^\sharp)^0_*$, we will show that $f^0_* = (f^\sharp)^0_*$ and that $f^0_* = (f^\flat)^0_*$.
To prove the first equation, let us introduce abbreviations

$$F = \{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{174}$$
$$S = \{\gamma \in \mathbf{I} : \sup\{f(\gamma') : \gamma' > \gamma\} = 0\} \tag{175}$$

Apparently

$$f^0_* = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} = \inf F \tag{176}$$

by (19) and (174), and

$$\begin{aligned}
(f^\sharp)^0_* &= \inf\{\gamma \in \mathbf{I} : f^\sharp = 0\} && \text{by (19)}\\
&= \inf\{\gamma \in [0,1) : \lim_{\gamma' \to \gamma^+} f(\gamma') = 0\} \cup \{1 : f(1) = 0\} &&\\
&= \inf\{\gamma \in [0,1) : \lim_{\gamma' \to \gamma^+} f(\gamma') = 0\} && \text{because } \inf \varnothing = 1\\
&= \inf\{\gamma \in [0,1) : \sup\{f(\gamma') : \gamma' > \gamma\} = 0\} && \text{by Th-43}\\
&= \inf\{\gamma \in \mathbf{I} : \sup\{f(\gamma') : \gamma' > \gamma\} = 0\} && \text{because } \inf X = \inf(X \cup \{1\})\\
& && \text{for all } X \in \mathcal{P}(\mathbf{I})\\
&= \inf S && \text{by (175)}\,,
\end{aligned}$$

i.e.

$$(f^\sharp)^0_* = \inf S\,. \tag{177}$$

Let us consider some $\gamma > f^0_*$. Then by (19), $f(\gamma) = 0$ and hence

$$\sup\{f(\gamma') : \gamma' > \gamma\} = \sup\{0\} = 0 \tag{178}$$

because f is nonincreasing and nonnegative. This proves that $(f^0_*, 1] \subseteq S$ and hence

$$\begin{aligned}
(f^\sharp)^0_* &= \inf S && \text{by (177)}\\
&\leq \inf (f^0_*, 1] && \text{because } (f^0_*, 1] \subseteq S\\
&= f^0_*\,.
\end{aligned}$$

It remains to be shown that $(f^\sharp)^0_* \geq f^0_*$.
Let us assume to the contrary that

$$(f^\sharp)^0_* < f^0_*\,. \tag{179}$$

Then in particular $(f^\sharp)^0_* < 1$, i.e. $\inf S < 1$ and $S \neq \varnothing$ by (177). Hence there exists some $\gamma_0 \in S$ such that

$$\gamma_0 < f^0_* \tag{180}$$

and (because $\gamma_0 \in S$):

$$\sup\{f(\gamma') : \gamma' > \gamma_0\} = 0\,. \tag{181}$$

This means that $f(\gamma') = 0$ for all $\gamma' > \gamma_0$, i.e. $(\gamma_0, 1] \subseteq F$ by (174). Therefore

$$\begin{aligned} f_*^0 &= \inf F && \text{by (176)} \\ &\le \inf(\gamma_0, 1] && \text{because } (\gamma_0, 1] \subseteq F \\ &= \gamma_0 \,. \end{aligned}$$

This contradicts (180), which states that $\gamma_0 < f_*^0$. We conclude that our assumption (179) is false, i.e. $(f^\sharp)_*^0 \ge f_*^0$.

This finishes the proof that $f_*^0 = (f^\sharp)_*^0$. The second equation, $f_*^0 = (f^\flat)_*^0$, can be shown analogously.

Lemma 55

Suppose $f, g \in \mathbb{H}$. *If* $f \le g$, *then* $f_*^0 \le g_*^0$.

Proof

Assume f, g are mappings in $\mathbb{H}$ and suppose that

$$f(\gamma) \le g(\gamma)$$

for all $\gamma \in \mathbf{I}$. Let us abbreviate

$$F = \{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{182}$$
$$G = \{\gamma \in \mathbf{I} : g(\gamma) = 0\} \,. \tag{183}$$

It is then apparent that

$$\begin{aligned} F &= \{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{by (182)} \\ &= \{\gamma \in \mathbf{I} : f(\gamma) \le 0\} && \text{because } f(\gamma) \ge 0 \text{ for all } \gamma \in \mathbf{I} \\ &\supseteq \{\gamma \in \mathbf{I} : g(\gamma) \le 0\} && \text{because } f(\gamma) \le g(\gamma) \text{ for all } \gamma \in \mathbf{I} \\ &= \{\gamma \in \mathbf{I} : g(\gamma) = 0\} && \text{because } g(\gamma) \ge 0 \text{ for all } \gamma \in \mathbf{I} \\ &= G \,, && \text{by (183)} \end{aligned}$$

i.e. $F \supseteq G$ and

$$\begin{aligned} f_*^0 &= \inf F && \text{by (19), (182)} \\ &\le \inf G && \text{because } F \supseteq G \\ &= g_*^0 \,. && \text{by (19), (183)} \end{aligned}$$

To show that (B-1) is independent of the other conditions, let us consider $\mathcal{B}'_{(\mathrm{B-1})} : \mathbb{H} \longrightarrow \mathbf{I}$, defined by

$$\mathcal{B}'_{(\mathrm{B-1})}(f) = f_*^0 \,, \tag{184}$$

for all $f \in \mathbb{H}$.

Lemma 56

$\mathcal{B}'_{(\mathrm{B-1})}$ *satisfies* (C-2), (C-3.a), (C-3.b) *and* (C-4), *but violates* (C-1).

Proof Let us first show that (C-1) fails. To this end, let $\mathrm{c}_{\frac{1}{2}} \in \mathbb{H}$ the constant $\mathrm{c}_{\frac{1}{2}}(x) = \frac{1}{2}$ for all $x \in \mathbf{I}$. Then

$$\begin{aligned}
(\mathrm{c}_{\frac{1}{2}})^0_* &= \inf\{x \in \mathbf{I} : \mathrm{c}_{\frac{1}{2}}(x) = 0\} && \text{by (19)} \\
&= \inf\{x \in \mathbf{I} : \tfrac{1}{2} = 0\} && \text{by def. of } \mathrm{c}_{\frac{1}{2}} \\
&= \inf \varnothing \\
&= 1 \\
&\neq \tfrac{1}{2} \\
&= \mathrm{c}_{\frac{1}{2}}(0)\,.
\end{aligned}$$

This proves that $\mathcal{B}'_{(\mathrm{B}-1)}$ does not satisfy (C-1). We shall now discuss the other properties:
Concerning (C-2), let $f \in \mathbb{H}$ some mapping such that $f(\mathbf{I}) \subseteq \{0,1\}$. Then $\mathcal{B}'_{(\mathrm{B}-1)}(f) = f^0_*$ by (184), i.e. (C-2) holds.
As to (C-3.a) and (C-3.b), the result has already been proven in lemma L-54.
Finally $\mathcal{B}'_{(\mathrm{B}-1)}$ is known to satisfy (C-4) by L-55.

Proof of Theorem 65

We shall assume that $\mathcal{B}_{(\mathrm{B}-1)} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'_{(\mathrm{B}-1)}$ in the usual way, i.e. as described by equation (23). The claim of the theorem is then apparent from L-56 and Th-53, noting that (B-2) holds because $\mathcal{B}_{(\mathrm{B}-1)}$ is related to $\mathcal{B}'_{(\mathrm{B}-1)}$ by (23) and hence $\mathcal{B}_{(\mathrm{B}-1)} \in \mathrm{BB}$.

D.2 Proof of Theorem 66

Recalling Def. 70 and Def. 77, we shall define $\mathcal{B}_{(\mathrm{B}-2)} : \mathbb{B} \longrightarrow \mathbf{I}$ by

$$\mathcal{B}_{(\mathrm{B}-2)}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'_{\int}(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}^{*\prime}(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{185}$$

for all $f \in \mathbb{B}$, i.e.

$$\mathcal{B}_{(\mathrm{B}-2)}(f) = \begin{cases} \mathcal{B}_{\int}(f) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \mathcal{B}^*(f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{186}$$

We already know by Th-42 and Th-54 that $\mathcal{M} = \mathcal{M}_{\mathcal{B}_{\int}}$ and $\mathcal{M}^* = \mathcal{M}_{\mathcal{B}^*}$ are DFSes. By Th-62, we know that both $\mathcal{B}_{\int}$ and $\mathcal{B}^*$ satisfy conditions (B-1) to (B-5). Let us now investigate the interesting properties of $\mathcal{B}_{(\mathrm{B}-2)}$.
In order to prove that $\mathcal{B}_{(\mathrm{B}-2)}$ satisfies (B-1), suppose that $f \in \mathbb{B}$ is given such that $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$ (i.e. f is constant). If $f \in \mathbb{B}^+$, then

$$\begin{aligned}
\mathcal{B}_{(\mathrm{B}-2)}(f) &= \mathcal{B}_{\int}(f) && \text{by (186)} \\
&= f(0)\,. && \text{because } \mathcal{B}_{\int} \text{ satisfies (B-1)}
\end{aligned}$$

If $f \in \mathbb{B}^{\frac{1}{2}}$, then $f = \mathrm{c}_{\frac{1}{2}}$, i.e. $f(\gamma) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ and hence

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(f) &= \tfrac{1}{2} && \text{by (186)}\\ &= f(0)\,.\end{aligned}$$

Finally if $f \in \mathbb{B}^{-}$, then

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(f) &= \mathcal{B}^{*}(f) && \text{by (186)}\\ &= f(0)\,. && \text{because } \mathcal{B}^{*} \text{ satisfies (B-1)}\end{aligned}$$

Considering (B-2), suppose $f \in \mathbb{B}$ is defined by

$$f(\gamma) = 1 - \tfrac{1}{2}\gamma \qquad (187)$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(f) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{\int}(2f-1) && \text{by (185)}\\ &= \tfrac{1}{2} + \tfrac{1}{2}\int_0^1 (2(1-\tfrac{1}{2}\gamma)-1)\,d\gamma && \text{by Def. 70}\\ &= \tfrac{1}{2} + \tfrac{1}{2}\int_0^1 (1-\gamma)\,d\gamma\\ &= \tfrac{1}{2} + \tfrac{1}{2}(1-\tfrac{1}{2})\\ &= \tfrac{3}{4}\,.\end{aligned}$$

Hence

$$1 - \mathcal{B}_{(\mathrm{B}-2)}(f) = 1 - \tfrac{3}{4} = \tfrac{1}{4}\,. \qquad (188)$$

On the other hand, $1 - f \in \mathbb{B}^{-}$ and hence

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(1-f) &= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}^{*\prime}(1-2(1-f)) && \text{by (185)}\\ &= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}^{*\prime}(1-2(1-(1-\tfrac{1}{2}\gamma))) && \text{by (187)}\\ &= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}^{*\prime}(1-\gamma)\\ &= \tfrac{1}{2} - \tfrac{1}{2}((1-\gamma)_{*}^{0}\cdot(1-\gamma)_{0}^{*}) && \text{by Def. 77}\\ &= \tfrac{1}{2} - \tfrac{1}{2}(1\cdot 1) && \text{by (19), (18)}\\ &= 0\,.\end{aligned}$$

Hence $\mathcal{B}_{(\mathrm{B}-2)}(1-f) = 0 \neq \frac{1}{4} = 1 - \mathcal{B}_{(\mathrm{B}-2)}(f)$ (see (188)), i.e. (B-2) fails, as desired.
In order to show that $\mathcal{B}_{(\mathrm{B}-2)}$ satisfies (B-3), suppose $f \in \mathbb{B}$ such that $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$. If $f \in \mathbb{B}^{+}$, then

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(f) &= \mathcal{B}_{\int}(f) && \text{by (186)}\\ &= \tfrac{1}{2} + \tfrac{1}{2}f_{*}^{\frac{1}{2}}\,,\end{aligned}$$

because $\mathcal{B}_{\int}$ is known to satisfy (B-3) (see above). Similarly if $f \in \mathbb{B}^{-}$,

$$\begin{aligned}\mathcal{B}_{(\mathrm{B}-2)}(f) &= \mathcal{B}^{*}(f) && \text{by (186)}\\ &= \tfrac{1}{2} - \tfrac{1}{2}f_{*}^{\frac{1}{2}}\,,\end{aligned}$$

because $\mathcal{B}^*$ is known to satisfy (B-3). In the remaining case that $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$, (185) directly yields the desired $\mathcal{B}_{(\mathrm{B-2})}(f) = \frac{1}{2}$.
Let us now consider (B-4).

- If $f \in \mathbb{B}^+$, then $f^\flat \in \mathbb{B}^+$ and $f^\sharp \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$. In the case that $f^\sharp \in \mathbb{B}^+$, we have

$$\begin{aligned} \mathcal{B}_{(\mathrm{B-2})}(f^\flat) &= \mathcal{B}_\int(f^\flat) && \text{by (186), } f^\flat \in \mathbb{B}^+ \\ &= \mathcal{B}_\int(f^\sharp) && \text{because } \mathcal{B}_\int \text{ satisfies (B-4)} \\ &= \mathcal{B}_{(\mathrm{B-2})}(f^\sharp)\,. && \text{by (186), } f^\sharp \in \mathbb{B}^+ \end{aligned}$$

  In the case that $f^\sharp \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f^\sharp = \mathrm{c}_{\frac{1}{2}}$,

$$\begin{aligned} \mathcal{B}_{(\mathrm{B-2})}(f^\flat) &= \mathcal{B}_\int(f^\flat) && \text{by (186), } f^\flat \in \mathbb{B}^+ \\ &= \mathcal{B}_\int(f^\sharp) && \text{because } \mathcal{B}_\int \text{ satisfies (B-4)} \\ &= \tfrac{1}{2} && \text{because } f^\sharp = \mathrm{c}_{\frac{1}{2}} \text{ and } \mathcal{B}_\int \text{ satisfies (B-3)} \\ &= \mathcal{B}_{(\mathrm{B-2})}(f^\sharp)\,. && \text{by (185), } f^\sharp \in \mathbb{B}^{\frac{1}{2}} \end{aligned}$$

- If $f \in \mathbb{B}^{\frac{1}{2}}$, then $f = f^\sharp = f^\flat = \mathrm{c}_{\frac{1}{2}}$ and hence trivially $\mathcal{B}_{(\mathrm{B-2})}(f^\sharp) = \mathcal{B}_{(\mathrm{B-2})}(f^\flat)$.
- The case that $f \in \mathbb{B}^-$ is analogous to $f \in \mathbb{B}^+$, using $\mathcal{B}^*$ rather than $\mathcal{B}_\int$.

Finally let us show that $\mathcal{B}_{(\mathrm{B-2})}$ satisfies (B-5). To this end, let us observe that for all $f \in \mathbb{B}$,

- if $f \in \mathbb{B}^+$, then $\mathcal{B}_{(\mathrm{B-2})}(f) = \mathcal{B}_\int(f) = \frac{1}{2} + \frac{1}{2}\mathcal{B}'_\int(2f-1) \geq \frac{1}{2}$ by (23), $\mathcal{B}_\int \in \mathrm{BB}$;
- if $f \in \mathbb{B}^{\frac{1}{2}}$, then $\mathcal{B}_{(\mathrm{B-2})}(f) = \frac{1}{2}$ by (185);
- if $f \in \mathbb{B}^-$, then $\mathcal{B}_{(\mathrm{B-2})}(f) = \mathcal{B}^*(f) = \frac{1}{2} - \frac{1}{2}\mathcal{B}^{*\prime}(1-2f) \leq \frac{1}{2}$ by (23), $\mathcal{B}^* \in \mathrm{BB}$.

The only critical cases with respect to (B-5) are hence (a) $f, g \in \mathbb{B}^+$ and (b) $f, g \in \mathbb{B}^-$.

(a) Suppose $f, g \in \mathbb{B}^+$ and $f \leq g$. Then

$$\begin{aligned} \mathcal{B}_{(\mathrm{B-2})}(f) &= \mathcal{B}_\int(f) && \text{by (186), } f \in \mathbb{B}^+ \\ &\leq \mathcal{B}_\int(g) && \text{because } \mathcal{B}_\int \text{ satisfies (B-5)} \\ &= \mathcal{B}_{(\mathrm{B-2})}(g)\,. && \text{by (186), } g \in \mathbb{B}^+ \end{aligned}$$

(b) Suppose $f, g \in \mathbb{B}^-$ and $f \leq g$. Then similarly

$$\begin{aligned} \mathcal{B}_{(\mathrm{B-2})}(f) &= \mathcal{B}^*(f) && \text{by (186), } f \in \mathbb{B}^- \\ &\leq \mathcal{B}^*(g) && \text{because } \mathcal{B}^* \text{ satisfies (B-5)} \\ &= \mathcal{B}_{(\mathrm{B-2})}(g)\,. && \text{by (186), } g \in \mathbb{B}^- \end{aligned}$$

D.3 Proof of Theorem 67

Let us define $\mathcal{B}'_{(\mathrm{B}-4)} : \mathbb{H} \longrightarrow \mathbf{I}$ by

$$\mathcal{B}'_{(\mathrm{B}-4)}(f) = f(0) \cdot f^0_* \tag{189}$$

for all $f \in \mathbb{H}$.

Lemma 57
$\mathcal{B}'_{(\mathrm{B}-4)}$ *satisfies* (C-1), (C-2) *and* (C-4), *but fails on* (C-3.b).

Proof Considering the conditions on $\mathcal{B}'_{(\mathrm{B}-4)}$, let us firstly show that (C-1) is satisfied. Suppose $f \in \mathbb{H}$ is constant, i.e. $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$. By Def. 75, we know that $f(0) > 0$ and hence $f(\gamma) = f(0) > 0$ for all $\gamma \in \mathbf{I}$. Recalling (19), this proves that $f^0_* = 1$, and hence

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B}-4)}(f) &= f(0) f^0_* && \text{by (189)} \\ &= f(0)\,, && \text{by } f^0_* = 1 \end{aligned}$$

as desired.
Let us now show that (C-2) holds. To this end, let $f \in \mathbb{H}$ such that $f(\mathbf{I}) \subseteq \{0, 1\}$. Then $f(0) = 1$, because $f \in \mathbb{H}$ has $f(0) > 0$ by Def. 75, and $f(0) = 1$ is the only alternative result because $f(0) \in f(\mathbf{I}) \subseteq \{0, 1\}$. Therefore

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B}-4)}(f) &= f(0) f^0_* && \text{by (189)} \\ &= f^0_*\,. && \text{because } f(0) = 1 \end{aligned}$$

In order to show that (C-4) holds, suppose $f, g \in \mathbb{H}$ are mappings such that

$$f(\gamma) \le g(\gamma) \tag{190}$$

for all $\gamma \in \mathbf{I}$. In particular, $f(0) \le g(0)$. In addition, we know from L-55 that $f^0_* \le g^0_*$. By the monotonicity of multiplication,

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B}-4)}(f) &= f(0) f^0_* && \text{by (189)} \\ &\le g(0) g^0_* && \text{because } g(0) \le f(0),\ f^0_* \le g^0_* \\ &= \mathcal{B}'_{(\mathrm{B}-4)} g\,. && \text{by (189)} \end{aligned}$$

Finally, let us show that $\mathcal{B}'_{(\mathrm{B}-4)}$ does not satisfy (C-3.b). Define $f \in \mathbb{H}$ by

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma = 0 \\ \frac{1}{2} & : \quad \gamma > 0 \end{cases} \tag{191}$$

Then by Def. 71, $f^\sharp = \mathrm{c}_{\frac{1}{2}}$ and $f^\flat = f$. By (19),

$$(f^\sharp)^0_* = f^0_* = 1 \tag{192}$$

and hence

$$\begin{aligned}
\mathcal{B}'_{(\mathrm{B-4})}(f^\flat) &= f^\flat(0)\cdot(f^\flat)^0_* && \text{by (189)}\\
&= f(0)\cdot f^0_* && \text{because } f^\flat = f\\
&= 1\cdot 1 && \text{by (191), (192)}\\
&= 1\\
&\neq \tfrac{1}{2}\\
&= \tfrac{1}{2}\cdot 1\\
&= f^\sharp(0)\cdot(f^\sharp)^0_* && \text{by (192) and } f^\sharp = \mathrm{c}_{\frac{1}{2}}\\
&= \mathcal{B}'_{(\mathrm{B-4})}(f^\sharp) && \text{by (189)}\,,
\end{aligned}$$

i.e. $\mathcal{B}'_{(\mathrm{B-4})}$ does not satisfy (C-3.b).

Proof of Theorem 67

We shall assume that $\mathcal{B}_{(\mathrm{B-4})} \in \mathrm{BB}$ is defined in terms of $\mathcal{B}'_{(\mathrm{B-4})}$ according to equation (23). We already know from L-57 that $\mathcal{B}'_{(\mathrm{B-4})}$ satisfies (C-1), (C-2) and (C-4) but fails on (C-3.b). By Th-53, this is entails that $\mathcal{B}_{(\mathrm{B-4})}$ satisfies (B-5), (B-3) and (B-5) but fails on (B-4). Recalling that $\mathcal{B}_{(\mathrm{B-4})} \in \mathrm{BB}$ satisfies (B-2) by Def. 74, $\mathcal{B}_{(\mathrm{B-4})}$ satisfies all conditions except (B-4), and is hence a proper example for establishing the independence of (B-4) from the other conditions.

D.4 Proof of Theorem 68

Lemma 58
Suppose $f \in \mathbb{H}$ *and* $\widehat{f}((0,1]) \neq \{0\}$. *Then*

$$\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} = \sup\{f(\gamma) : \gamma > 0\}\,.$$

Proof

a. $\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} \geq \sup\{f(\gamma) : \gamma > 0\}$.
To see this, let $z \in (0,1]$ and $\gamma \in (0,z)$. Then

$$\begin{aligned}
\inf\{f(\gamma') : \gamma' < \gamma\} &\geq \inf\{f(z) : \gamma' < \gamma\} && \text{because } f \in \mathbb{H} \text{ nonincreasing, } \gamma' < z\\
&= f(z)
\end{aligned}$$

Hence

$$\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} \geq f(z)$$

for all $z \in (0,1]$.
It follows that

$$\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} \geq \sup\{f(z) : z > 0\}\,,$$

as desired.

b. $\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} \leq \sup\{f(\gamma) : \gamma > 0\}$.
Let us assume to the contrary that

$$\sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} > \sup\{f(\gamma) : \gamma > 0\}\,. \tag{193}$$

Then there exists some $\gamma_0 > 0$ such that

$$\inf\{f(\gamma') : \gamma' < \gamma_0\} > \sup\{f(\gamma) : \gamma > 0\}\,. \tag{194}$$

Now let $\omega \in (0, \gamma_0)$. By the (nonincreasing) monotonicity of $f \in \mathbb{H}$,

$$f(\omega) \geq \inf\{f(\gamma') : \gamma' < \gamma_0\}\,.$$

On the other hand, $\omega > 0$, i.e.

$$f(\omega) \in \{f(\gamma) : \gamma > 0\}\,,$$

and hence

$$\sup\{f(\gamma) : \gamma > 0\} \geq f(\omega) \geq \inf\{f(\gamma') : \gamma' < \gamma_0\}\,,$$

which contradicts (194).

Lemma 59
Suppose $f \in \mathbb{H}$. Then

a. if $\widehat{f}((0,1]) = \{0\}$, then $f_0^* = 0$.

b. if $\widehat{f}((0,1]) \neq \{0\}$, then $(f^\sharp)_0^* = (f^\flat)_0^*$.

Proof

a. In this case, $f(\gamma) = 0$ for all $\gamma > 0$ and hence apparently

$$f_0^* = \lim_{\gamma \to 0^+} f(\gamma) = 0\,.$$

b. Suppose $f \in \mathbb{H}$ and $\widehat{f}((0,1]) \neq \{0\}$. Then

$$\begin{aligned}
(f^\sharp)_0^* &= \lim_{\gamma \to 0^+} \gamma^\sharp && \text{by (18)} \\
&= \sup\{f^\sharp(\gamma) : \gamma > 0\} && \text{because } f^\sharp \in \mathbb{H} \text{ nonincreasing} \\
&= \sup\{\sup\{f(\gamma') : \gamma' > \gamma\} : \gamma > 0\} && \text{by Def. 71} \\
&= \sup\{f(\gamma') : \gamma' > 0\} && \text{(apparent)} \\
&= \lim_{\gamma \to 0^+} f(\gamma) && \text{because } f \in \mathbb{H} \text{ nonincreasing} \\
&= f_0^*\,. && \text{by (18)}
\end{aligned}$$

Similarly

$$
\begin{aligned}
(f^\flat)^*_0 &= \lim_{\gamma \to 0^+} f^\flat(\gamma) && \text{by (18)} \\
&= \sup\{f^\flat(\gamma) : \gamma > 0\} && \text{because } f^\flat \in \mathbb{H} \text{ nonincreasing} \\
&= \sup\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma > 0\} && \text{by Def. 71} \\
&= \sup\{f(\gamma) : \gamma > 0\} && \text{by L-58} \\
&= \lim_{\gamma \to 0^+} f(\gamma) && \text{because } f \in \mathbb{H} \text{ nonincreasing} \\
&= f^*_0 \,, && \text{by (18)}
\end{aligned}
$$

which finishes the proof of part **b.** of the lemma.
Now let us define the mapping $\mathcal{B}'_{(\mathrm{B}-5)} : \mathbb{H} \longrightarrow \mathbf{I}$ by

$$
\mathcal{B}'_{(\mathrm{B}-5)}(f) = \begin{cases} f^0_* & : \quad f^*_0 = 1 \\ f^*_0 & : \quad f^*_0 < 1 \end{cases} \tag{195}
$$

Lemma 60
The mapping $\mathcal{B}'_{(\mathrm{B}-5)}$ *satisfies* (C-1), (C-2), (C-3.a) *and* (C-3.b), *but violates* (C-4).

Proof To see that $\mathcal{B}'_{(\mathrm{B}-5)}$ satisfies (C-1), let $f \in \mathbb{H}$ a constant function, i.e. $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$. In particular,

$$
f^*_0 = f(0) \tag{196}
$$

by (18). Recalling Def. 75, we further know that $f(0) > 0$ and hence $f(\gamma) > 0$ for all $\gamma \in \mathbf{I}$, i.e.

$$
f^0_* = 1 \,. \tag{197}
$$

by (19).

- if $f(0) = 1$, then $f^*_0 = f(0) = 1$ by (196), $f^0_* = 1$ by (197) and hence $\mathcal{B}'_{(\mathrm{B}-5)}(f) = f^0_* = 1 = f(0)$.
- if $f(0) < 1$, then $f^*_0 = f(0) < 1$ by (196). Hence by (195), $\mathcal{B}'_{(\mathrm{B}-5)}(f) = f^*_0 = f(0)$.

In order to prove that (C-2) holds, let $f \in \mathbb{H}$ such that $f(\mathbf{I}) \subseteq \{0, 1\}$.

- if $f^*_0 = 1$, then diectly $\mathcal{B}'_{(\mathrm{B}-5)} f = f^0_*$ by (195);
- if $f^*_0 = 0$, then $f(0) = 1$ (because $f(0) > 0$, $f(0) \in \{0, 1\}$) and $f(\gamma) = 0$ for all $\gamma > 0$ by (18). Recalling (19), it is apparent that $f^0_* = 0$ as well, i.e.

  $$
  \mathcal{B}'_{(\mathrm{B}-5)}(f) = f^*_0 = 0 = f^0_* \,,
  $$

  as desired.

Considering (C-3.a), suppose $f \in \mathbb{H}$ has $\widehat{f}((0,1]) = \{0\}$, i.e. $f(\gamma) = 0$ for all $\gamma > 0$. Applying L-59.a, we obtain that $f_0^* = 0$. Hence by (195), $\mathcal{B}'_{(\mathrm{B-5})}(f) = f_0^* = 0$.

In order to show that (C-3.b) holds, let $f \in \mathbb{H}$ such that $\widehat{f}((0,1]) \neq \{0\}$. Then

$$\begin{aligned}
\mathcal{B}'_{(\mathrm{B-5})}(f^\sharp) &= \begin{cases} (f^\sharp)_*^0 & : \quad (f^\sharp)_0^* = 1 \\ (f^\sharp)_0^* & : \quad (f^\sharp)_0^* < 1 \end{cases} && \text{by (195)} \\
&= \begin{cases} (f^\flat)_*^0 & : \quad (f^\flat)_0^* = 1 \\ (f^\flat)_0^* & : \quad (f^\flat)_0^* < 1 \end{cases} && \text{by L-54.b, L-59.b} \\
&= \mathcal{B}'_{(\mathrm{B-5})}(f^\flat)\,.
\end{aligned}$$

Finally, let us show that (C-4) is violated. To this end, let us define $f, g \in \mathbb{H}$ by

$$f(\gamma) = \begin{cases} \frac{2}{3} & : \quad x \leq \frac{1}{3} \\ 0 & : \quad x > \frac{1}{3} \end{cases}$$

and

$$g(\gamma) = \begin{cases} 1 & : \quad x \leq \frac{1}{3} \\ 0 & : \quad x > \frac{1}{3} \end{cases}$$

for all $\gamma \in \mathbf{I}$. It is apparent from these definitions and (19), (18) that

$$\begin{aligned}
f_*^0 &= \tfrac{1}{3} \\
g_*^0 &= \tfrac{1}{3} \\
f_0^* &= \tfrac{2}{3} \\
g_0^* &= 1\,.
\end{aligned}$$

Hence

$$\mathcal{B}'_{(\mathrm{B-5})}(f) = f_0^* = \tfrac{2}{3} > \tfrac{1}{3} = g_*^0 = \mathcal{B}'_{(\mathrm{B-5})}(g)\,,$$

although clearly $f \leq g$.

Proof of Theorem 68

We shall define $\mathcal{B}_{(\mathrm{B-5})} \in \mathrm{BB}$ according to equation (23). Noting that $\mathcal{B}_{(\mathrm{B-5})} \in \mathrm{BB}$ satisfies (B-2) and recalling Th-53, we can prove the independence of (B-5) from (B-1) to (B-4) by showing that $\mathcal{B}'_{(\mathrm{B-5})}$ satisfies (C-1), (C-2) and (C-3.a)/(C-3.b), but fails on (C-4). This has been done in lemma L-60.

D.5 Proof of Theorem 69

a. By Th-65, there exists a choice of $\mathcal{B}_{(\mathrm{B-1})} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies (B-2), (B-3), (B-4) and (B-5), but violates (B-1). Abbreviating $\mathcal{M}_{(\mathrm{B-1})} = \mathcal{M}_{\mathcal{B}_{(\mathrm{B-1})}}$, the sufficiency theorems Th-47, Th-48, Th-49, Th-50 and Th-51 tell us that $\mathcal{M}_{(\mathrm{B-1})}$ satisfies (Z-2), (Z-3), (Z-4), (Z-5) and (Z-6), respectively. We may then conclude from the "necessity theorem" Th-56 that $\mathcal{M}_{(\mathrm{B-1})}$ violates (Z-1), i.e. (Z-1) is independent of the remaining conditions (Z-2) to (Z-6).

b. Analogous: By Th-66, there exists a choice of $\mathcal{B}_{(\mathrm{B}-2)} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies (B-1), (B-3), (B-4) and (B-5), but violates (B-2). Abbreviating $\mathcal{M}_{(\mathrm{B}-2)} = \mathcal{M}_{\mathcal{B}_{(\mathrm{B}-2)}}$, the sufficiency theorems Th-45, Th-47, Th-49, Th-50 and Th-51 tell us that $\mathcal{M}_{(\mathrm{B}-2)}$ satisfies (Z-1), (Z-2), (Z-4), (Z-5) and (Z-6), respectively. We may then conclude from the "necessity theorem" Th-59 that $\mathcal{M}_{(\mathrm{B}-2)}$ violates (Z-3), i.e. (Z-3) is independent of the remaining conditions in (Z-1) to (Z-6).

c. Again analogous: By Th-68, there exists a choice of $\mathcal{B}_{(\mathrm{B}-5)} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies (B-1), (B-2), (B-3) and (B-4), but violates (B-5). Abbreviating $\mathcal{M}_{(\mathrm{B}-5)} = \mathcal{M}_{\mathcal{B}_{(\mathrm{B}-5)}}$, the sufficiency theorems Th-45, Th-47, Th-48, Th-49 and Th-51 tell us that $\mathcal{M}_{(\mathrm{B}-5)}$ satisfies (Z-1), (Z-2), (Z-3), (Z-4) and (Z-6), respectively. We may then conclude from the "necessity theorem" Th-61 that $\mathcal{M}_{(\mathrm{B}-5)}$ violates (Z-5), i.e. (Z-5) is independent of the remaining conditions in (Z-1) to (Z-6).

d. Same argumentation: By Th-67, there exists a choice of $\mathcal{B}_{(\mathrm{B}-4)} : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies (B-1), (B-2), (B-3) and (B-5), but violates (B-4). Abbreviating $\mathcal{M}_{(\mathrm{B}-4)} = \mathcal{M}_{\mathcal{B}_{(\mathrm{B}-4)}}$, the sufficiency theorems Th-45, Th-47, Th-48, Th-49 and Th-50 tell us that $\mathcal{M}_{(\mathrm{B}-4)}$ satisfies (Z-1), (Z-2), (Z-3), (Z-4) and (Z-5), respectively. We may then conclude from the "necessity theorem" Th-60 that $\mathcal{M}_{(\mathrm{B}-4)}$ violates (Z-6), i.e. (Z-6) is independent of the remaining conditions (Z-1) to (Z-5).

D.6 Proof of Theorem 70

In order to show that (B-3) is independent of the remaining conditions on $\mathcal{B}$, let us define $\mathcal{B}'_{(\mathrm{B}-3)} : \mathbb{H} \longrightarrow \mathbf{I}$ as follows:

$$\mathcal{B}'_{(\mathrm{B}-3)}(f) = f_0^* \tag{198}$$

for all $f \in \mathbb{H}$, where $f_0^* = \lim_{x \to 0^+} f(x)$, as in (18).

Lemma 61
$\mathcal{B}'_{(\mathrm{B}-3)}$ *satisfies* (C-1), (C-3.a), (C-3.b) *and* (C-4), *but violates* (C-2).

Proof
Let us consider the conditions on $\mathcal{B}'_{(\mathrm{B}-3)}$ in turn.

$\mathcal{B}'_{(\mathrm{B}-3)}$ **satisfies** (C-1) If $f \in \mathbb{H}$ is constant, then $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$ and hence

$$\lim_{\gamma \to 0^+} f(\gamma) = f(0)\,. \tag{199}$$

Therefore

$$\begin{aligned}
\mathcal{B}'_{(\mathrm{B}-3)}(f) &= f_0^* && \text{by (198)}\\
&= \lim_{\gamma \to 0^+} f(\gamma) && \text{by (18)}\\
&= f(0)\,, && \text{by (199)}
\end{aligned}$$

i.e. (C-1) holds.

$\mathcal{B}'_{(\mathrm{B-3})}$ **satisfies** (C-3.a) Suppose $f \in \mathbb{H}$ is a mapping such that $\widehat{f}((0,1]) = \{0\}$. Then

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B-3})}(f) &= f_0^* && \text{by (198)} \\ &= 0\,. && \text{by L-59.a} \end{aligned}$$

$\mathcal{B}'_{(\mathrm{B-3})}$ **satisfies** (C-3.b) Let $f \in \mathbb{H}$ a mapping such that $\widehat{f}((0,1]) \neq \{0\}$. Then

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B-3})}(f^\sharp) &= (f^\sharp)_0^* && \text{by (198)} \\ &= (f^\flat)_0^* && \text{by L-59.b} \\ &= \mathcal{B}'_{(\mathrm{B-3})}(f^\flat)\,. && \text{by (198)} \end{aligned}$$

$\mathcal{B}'_{(\mathrm{B-3})}$ **satisfies** (C-4) Suppose $f, g \in \mathbb{H}$ such that $f \leq g$. Then

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B-3})}(f) &= f_0^* && \text{by (198)} \\ &= \lim_{x\to 0^+} f(x) && \text{by (18)} \\ &= \sup\{f(x) : x > 0\} && \text{by Th-43, } f \text{ nonincreasing} \\ &\leq \sup\{g(x) : x > 0\} && \text{because } f \leq g \\ &= \lim_{x\to 0^+} g(x) && \text{by Th-43, } g \text{ nonincreasing} \\ &= g_0^* && \text{by (18)} \\ &= \mathcal{B}'_{(\mathrm{B-3})}(g)\,. && \text{by (198)} \end{aligned}$$

$\mathcal{B}'_{(\mathrm{B-3})}$ **violates** (C-2) Let $f \in \mathbb{H}$ the mapping defined by

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma < \frac{1}{2} \\ 0 & : \quad \gamma \geq \frac{1}{2} \end{cases}$$

Then by (19),

$$f_*^0 = \tfrac{1}{2} \tag{200}$$

and by (18),

$$f_0^* = 1\,. \tag{201}$$

Therefore

$$\begin{aligned} \mathcal{B}'_{(\mathrm{B-3})}(f) &= f_0^* && \text{by (198)} \\ &= 1 && \text{by (201)} \\ &\neq \tfrac{1}{2} \\ &= f_*^0\,, && \text{by (200)} \end{aligned}$$

i.e. (C-2) does not hold.

Proof of Theorem 70

We shall assume that $\mathcal{B}_{(\mathrm{B}-3)} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'_{(\mathrm{B}-3)}$ according to equation (23). In particular, we hence know that $\mathcal{B}_{(\mathrm{B}-3)} \in$ BB, i.e. $\mathcal{B}_{(\mathrm{B}-3)}$ satisfies (B-2) by Def. 74. In addition, we know from L-61 that $\mathcal{B}'_{(\mathrm{B}-3)}$ satisfies (C-1), (C-3.a), (C-3.b) and (C-4), but violates (C-2). By Th-53, $\mathcal{B}_{(\mathrm{B}-3)}$ satisfies (B-1), (B-4) and (B-5), but violates (B-3).

D.7 Proof of Theorem 71

The proof is immediate from previous lemmata (see table below).

Condition	Independence proven in
(C-1)	L-56
(C-2)	L-61
(C-3.b)	L-57
(C-4)	L-60

D.8 Proof of Theorem 72

We will need a series of lemmata.

Lemma 62
Let $x \in \mathbf{I}$.

a. If $x > \frac{1}{2}$, then

$$Q_{\neg\gamma}(\widetilde{\eta}(x)) = \begin{cases} 0 & : \quad \gamma \leq 2x - 1 \\ \frac{1}{2} & : \quad \gamma > 2x - 1 \end{cases}$$

b. If $x = \frac{1}{2}$, then $Q_{\neg\gamma}(\widetilde{\eta}(x)) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. If $x < \frac{1}{2}$, then

$$Q_{\neg\gamma}(\widetilde{\eta}(x)) = \begin{cases} 1 & : \quad \gamma \leq 1 - 2x \\ \frac{1}{2} & : \quad \gamma > 1 - 2x \end{cases}$$

Proof
Let us first observe that by Def. 52, $Q_\neg : \mathcal{P}(\{1\}) \longrightarrow \mathbf{I}$ is the semi-fuzzy quantifier defined by

$$Q_\neg(V) = \begin{cases} 0 & : \quad X = \{1\} \\ 1 & : \quad X = \varnothing \end{cases} \tag{202}$$

Now suppose $x \in \mathbf{I}$. We will abbreviate $X = \widetilde{\eta}(x) \in \widetilde{\mathcal{P}}(\{1\})$, i.e. X is defined by $\mu_X(1) = x$. In the following, we shall treat separately the three cases of the lemma.

a. $x > \frac{1}{2}$. Then $1 \in (X)^{\max}_\gamma$ for all $\gamma \in \mathbf{I}$. Furthermore, $1 \in (X)^{\min}_\gamma$ if $\gamma \leq 2x - 1$, and $1 \notin (X)^{\min}_\gamma$ if $\gamma > 2x - 1$, as is easily seen from Def. 66. Hence by Def. 67 and (202),

$$Q_{\neg\gamma}(X) = \begin{cases} 0 & : \quad \gamma \leq 2x - 1 \\ \frac{1}{2} & : \quad \gamma > 2x - 1 \end{cases}$$

as desired.

b. Then $1 \in (X)^{\max}_{\gamma}$ and $1 \notin (X)^{\min}_{\gamma}$ for all $\gamma \in \mathbf{I}$ by Def. 66, i.e. $(X)^{\min}_{\gamma} = \varnothing$ and $(X)^{\max}_{\gamma} = \{1\}$ for all γ. By Def. 67 and (202), $Q_{\neg\gamma}(X) = \mathrm{m}_{\frac{1}{2}}\{Q_{\neg}(\varnothing), Q_{\neg}(\{1\}\} = \mathrm{m}_{\frac{1}{2}}(0,1) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. Then $1 \notin (X)^{\min}_{\gamma}$, i.e. $(X)^{\min}_{\gamma} = \varnothing$, for all $\gamma \in \mathbf{I}$. Furthermore $1 \notin (X)^{\max}_{\gamma}$ for all $\gamma \leq 1-2x$ and $1 \in (X)^{\max}_{\gamma}$ if $\gamma > 1-2x$, which is obvious from Def. 66. This means that $(X)^{\max}_{\gamma} = \varnothing$ if $\gamma \leq 1-2x$, and $(X)^{\max}_{\gamma} = \{1\}$ if $\gamma > 1-2x$. By Def. 67 and (202),

$$Q_{\neg\gamma}(X) = \begin{cases} 1 & : \quad \gamma \leq 1-2x \\ \frac{1}{2} & : \quad \gamma > 1-2x \end{cases}$$

which finishes the proof of the lemma.

Lemma 63
Suppose E is some nonempty set, $e \in E$ and $X \in \widetilde{\mathcal{P}}(E)$. We will abbreviate $z = \mu_X(e)$. Then

a. If $z > \frac{1}{2}$, then

$$\pi_{e\gamma}(X) = \begin{cases} 0 & : \quad \gamma \leq 2z-1 \\ \frac{1}{2} & : \quad \gamma > 2z-1 \end{cases}$$

b. If $z = \frac{1}{2}$, then $\pi_{e\gamma}(X) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. If $z < \frac{1}{2}$, then

$$\pi_{e\gamma}(X) = \begin{cases} 1 & : \quad \gamma \leq 1-2z \\ \frac{1}{2} & : \quad \gamma > 1-2z \end{cases}$$

Proof
Suppose E is some nonempty set, $e \in E$, $X \in \widetilde{\mathcal{P}}(E)$ and $z = \mu_X(e)$. Let us recall that by Def. 6,

$$\pi_e(V) = \begin{cases} 1 & : \quad e \in V \\ 0 & : \quad e \notin V \end{cases} \tag{203}$$

for all $V \in \mathcal{P}(E)$. We will prove separately the three cases of the lemma.

a. $z > \frac{1}{2}$. Then $e \in (X)^{\max}_{\gamma}$ for all $\gamma \in \mathbf{I}$. Furthermore $e \in (X)^{\min}_{\gamma}$ if $\gamma \leq 2z-1$, and $e \notin (X)^{\min}_{\gamma}$ if $\gamma > 2z-1$. This is apparent from Def. 66. Hence $\pi_e((X)^{\max}_{\gamma}) = 1$ for all $\gamma \in \mathbf{I}$, $\pi_e((X)^{\min}_{\gamma}) = 1$ for all $\gamma \leq 2z-1$, and $\pi_e((X)^{\min}_{\gamma}) = 0$ for all $\gamma > 2z-1$; see (203). Therefore by Def. 67,

$$\pi_{e\gamma}(X) = \begin{cases} 1 & : \quad \gamma \leq 2z-1 \\ \frac{1}{2} & : \quad \gamma > 2z-1 \end{cases}$$

b. $z = \frac{1}{2}$. Then $e \in (X)^{\max}_{\gamma}$ and $e \notin (X)^{\min}_{\gamma}$ for all $\gamma \in \mathbf{I}$, see Def. 66. By (203), $\pi_e((X)^{\min}_{\gamma}) = 0$ and $\pi_e((X)^{\max}_{\gamma}) = 1$ for all $\gamma \in \mathbf{I}$. From Def. 67, it is then apparent that $\pi_{e\gamma}(X) = \mathrm{m}_{\frac{1}{2}}(0,1) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

c. $z < \frac{1}{2}$. In this case, $e \notin (X)^{\min}_\gamma$ for all $\gamma \in \mathbf{I}$. In addition, $e \notin (X)^{\max}_\gamma$ if $\gamma \leq 1 - 2z$, and $e \in (X)^{\max}_\gamma$ if $\gamma > 1 - 2z$. This is apparent from Def. 66. Therefore $\pi_e((X)^{\min}_\gamma) = 0$ for all $\gamma \in \mathbf{I}$, $\pi_e((X)^{\max}_\gamma) = 0$ if $\gamma \leq 1 - 2z$, and $\pi_e((X)^{\max}_\gamma) = 1$ if $\gamma > 1 - 2z$, as is easily seen from (203). Hence by Def. 67,

$$\pi_{e\gamma}(X) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2z \\ \frac{1}{2} & : \quad \gamma > 1 - 2z \end{cases}$$

which we intended to show.

Lemma 64

Suppose $f : E \longrightarrow E'$ is some mapping where E, E' are nonempty sets; further suppose that $X \in \widetilde{\mathcal{P}}(E)$ and $z \in E'$ are given. We shall abbreviate

$$\begin{aligned} V &= \{\mu_X(e) : e \in f^{-1}(z)\} \\ s &= \sup V \\ Q_\gamma(X) &= \chi_{\widehat{f}(X)}(z)\,. \end{aligned}$$

1. *If $s > \frac{1}{2}$ and $s \in V$, then*

$$Q_\gamma(X) = \begin{cases} 1 & : \quad \gamma \leq 2s - 1 \\ \frac{1}{2} & : \quad \gamma > 2s - 1 \end{cases}$$

2. *If $s > \frac{1}{2}$ and $s \notin V$, then*

$$Q_\gamma(X) = \begin{cases} 1 & : \quad \gamma < 2s - 1 \\ \frac{1}{2} & : \quad \gamma \geq 2s - 1 \end{cases}$$

3. *If $s = \frac{1}{2}$ and $s \in V$, then $Q_\gamma(X) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$;*

4. *If $s = \frac{1}{2}$ and $s \notin V$, then*

$$Q_\gamma(X) = \begin{cases} 0 & : \quad \gamma = 0 \\ \frac{1}{2} & : \quad \gamma > 0 \end{cases}$$

5. *If $s < \frac{1}{2}$ and $s \in V$, then*

$$Q_\gamma(X) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2s \\ \frac{1}{2} & : \quad \gamma > 1 - 2s \end{cases}$$

6. *If $s < \frac{1}{2}$ and $s \notin V$, then*

$$Q_\gamma(X) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2s \\ \frac{1}{2} & : \quad \gamma > 1 - 2s \end{cases}$$

Proof
Let us first observe that

$$Q(Y) = \begin{cases} 1 & : \quad f^{-1}(z) \cap Y \neq \varnothing \\ 0 & : \quad f^{-1}(z) \cap Y = \varnothing \end{cases} \tag{204}$$

for all $Y \in \mathcal{P}(E)$. Apparently

$$Q(Y) = \exists(Y \cap f^{-1}(z)), \tag{205}$$

in particular, Q is a nondecreasing quantifier. Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 67} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\}, \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\}) && \text{by Def. 46} \\ &= \mathrm{m}_{\frac{1}{2}}(Q((X)_\gamma^{\min}), Q((X)_\gamma^{\max})) && \text{because } Q \text{ nondecreasing} \end{aligned}$$

i.e.

$$Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(Q((X)_\gamma^{\min}), Q((X)_\gamma^{\max})). \tag{206}$$

We are now prepared to prove the claims of the lemma.

1. If $s > \frac{1}{2}$ and $s \in V$, then there is some $e \in f^{-1}(z)$ such that $\mu_X(e) = s > \frac{1}{2}$ and

$$\mu_X(e') \leq \mu_X(e) \tag{207}$$

for all $e' \in f^{-1}(z)$; this is apparent from the definition of $s = \sup V$ and V. Because $\mu_X(e) > \frac{1}{2}$, we know by Def. 66 that $e \in (X)_\gamma^{\max}$ for all γ, i.e.

$$\begin{aligned} Q((X)_\gamma^{\max}) &\geq Q((X)_\gamma^{\max} \cap \{e\}) && \text{because } Q \text{ nondecreasing} \\ &= Q(\{e\}) && \text{because } e \in (X)_\gamma^{\max}\} \\ &= 1, && \text{by (204)} \end{aligned}$$

i.e.

$$Q((X)_\gamma^{\max}) = 1 \tag{208}$$

for all $\gamma \in \mathbf{I}$. By (207) and $\mu_X(e) = s$, we further know that $e \in (X)_\gamma^{\min}$ for all $\gamma \leq 2s - 1$, and

$$(X)_\gamma^{\min} \cap f^{-1}(z) = \varnothing \tag{209}$$

for all $\gamma > 2s - 1$; this is apparent from Def. 66. Hence in the case that $\gamma \leq 2s - 1$,

$$\begin{aligned} Q((X)_\gamma^{\min}) &\geq Q((X)_\gamma^{\min} \cap \{e\}) && \text{because } Q \text{ nondecreasing} \\ &= Q(\{e\}) && \text{because } e \in (X)_\gamma^{\min}\} \\ &= 1, && \text{by (204)} \end{aligned}$$

i.e.

$$Q((X)_\gamma^{\min}) = 1. \tag{210}$$

In the case that $\gamma > 2s - 1$, however,

$$\begin{aligned} Q((X)^{\min}_\gamma) &= \exists(X \cap f^{-1}(z)) && \text{by (205)} \\ &= \exists(\varnothing) && \text{by (209)} \\ &= 0\,, \end{aligned}$$

i.e.

$$Q((X)^{\min}_\gamma) = 0 \tag{211}$$

if $\gamma > 2s - 1$. Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \begin{cases} \mathrm{m}_{\frac{1}{2}}(1,1) & : \quad \gamma \le 2s-1 \\ \mathrm{m}_{\frac{1}{2}}(0,1) & : \quad \gamma > 2s-1 \end{cases} && \text{by (208), (210) and (211)} \\ &= \begin{cases} 1 & : \quad \gamma \le 2s-1 \\ \frac{1}{2} & : \quad \gamma > 2s-1 \end{cases} && \text{by Def. 45.} \end{aligned}$$

2. $s > \frac{1}{2}$ and $s \notin V$. Suppose $\gamma < 2s - 1$, i.e. $\frac{1}{2} + \frac{1}{2}\gamma < s$. Because $s = \sup V > 0$ and $s > \frac{1}{2} + \frac{1}{2}\gamma$, there exists $v \in V$ such that $v > \frac{1}{2} + \frac{1}{2}\gamma$. By the definition of V, there exists some $e \in f^{-1}(z)$ such that $\mu_X(e) = v > \frac{1}{2} + \frac{1}{2}\gamma$. By Def. 66, it is apparent that $e \in (X)^{\max}_\gamma$ and $e \in (X)^{\min}_\gamma$, i.e. $Q((X)^{\max}_\gamma) = Q((X)^{\min}_\gamma) = 1$ (by similar reasoning as in case 1.). Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(1,1) \\ &= 1\,, \end{aligned}$$

if $\gamma < 2s - 1$.
Let us now consider the case that $\gamma \ge 2s - 1$. Because $s = \sup V \notin V$, we know that $\mu_X(e) < s$ for all $e \in f^{-1}(z)$ Hence $(X)^{\min}_\gamma \cap f^{-1}(z) = \varnothing$ for all $\gamma \ge 2s - 1$ and by (205), $Q((X)^{\min}_\gamma) = 0$. As in the previous case where $\gamma' < 2s-1$, we still have $(X)^{\max}_\gamma \cap f^{-1} \neq \varnothing$, i.e. $Q((X)^{\max}_\gamma) = 1$. Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0,1) && \text{(see above)} \\ &= \tfrac{1}{2}\,, \end{aligned}$$

if $\gamma \ge 2s - 1$.

3. $s = \frac{1}{2}$ and $s \in V$. Then there exists some $e \in f^{-1}(z)$ such that $\mu_X(e) = \frac{1}{2}$. By Def. 66, $e \in (X)^{\max}_\gamma$ for all $\gamma \in \mathbf{I}$. Because $\mu_X(e') \le \mu_X(e)$ for all $e' \in f^{-1}(z)$, which is apparent from the $\mu_X(e) = \sup V$ and $\mu_X(e') \in V$ for all $e' \in f^{-1}(z)$, we also know that $(X)^{\min}_\gamma \cap f^{-1}(z) = \varnothing$ for all $\gamma \in \mathbf{I}$. By similar reasoning as above, we conclude that $Q((X)^{\min}_\gamma) = 0$ and $Q((X)^{\max}_\gamma) = 1$ for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0,1) \\ &= \tfrac{1}{2}\,. \end{aligned}$$

for all $\gamma \in \mathbf{I}$.

4. $s = \frac{1}{2}$ and $s \notin V$. By the definition of V and $s = \sup V > 0$, it is apparent that
$$\mu_X(e') < \tfrac{1}{2} \tag{212}$$
for all $e' \in f^{-1}(z)$, and that for each $s' < \frac{1}{2}$, there exists some $e \in f^{-1}(z)$ such that
$$\mu_X(e) > s' \,. \tag{213}$$
Let us firstly consider the case that $\gamma = 0$. Then $(X)^{\min}_0 = (X)_{>\frac{1}{2}}$ and $(X)^{\max}_0 = (X)_{\geq\frac{1}{2}}$, see Def. 66. From (212), we conclude that $(X)^{\min}_0 \cap f^{-1}(z) = (X)^{\max}_0 \cap f^{-1}(z) = \varnothing$ and hence by (205), $Q((X)^{\min}_0) = Q((0)^{\max}_{)} = 0$, i.e.
$$\begin{aligned} Q_0(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_0), Q((X)^{\max}_0)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0,0) \\ &= 0\,. \end{aligned}$$
Let us now consider the case that $\gamma > 0$. Firstly $(X)^{\min}_\gamma \cap f^{-1}(z) = \varnothing$ for all $\gamma > 0$, which is apparent Def. 66 and (212). By (213) and Def. 66, $(X)^{\max}_\gamma \cap f^{-1}(z) = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} \cap f^{-1}(z) \neq \varnothing$, i.e $Q((X)^{\max}_\gamma) = 1$ by (205) for all $\gamma > 0$. Hence
$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0,1) \\ &= \tfrac{1}{2} \end{aligned}$$
for all $\gamma > 0$.

5. If $s < \frac{1}{2}$ and $s \in V$, then there is some $e \in f^{-1}(z)$ such that $\mu_X(e) = v = \sup V$, and
$$\mu_X(e') \leq \mu_X(e) \tag{214}$$
for all $e' \in f^{-1}(z)$. Because of (213) and observing that $\mu_X(e) = v < \frac{1}{2}$, it is apparent from Def. 66 that $(X)^{\min}_\gamma \cap f^{-1}(z) = \varnothing$ for all $\gamma \in \mathbf{I}$ and hence
$$Q((X)^{\min}_\gamma) = 0 \tag{215}$$
for all $\gamma \in \mathbf{I}$.
In the case that $\gamma \leq 1 - 2s$,
$$\tfrac{1}{2} - \tfrac{1}{2}\gamma \geq \tfrac{1}{2} - \tfrac{1}{2}(1-2s) = s = \mu_X(e) \geq \mu_X(e')$$
for all $e' \in f^{-1}(z)$ by (214). Hence by Def. 66, $(X)^{\max}_\gamma \cap f^{-1}(z) = \varnothing$ and hence $Q((X)^{\max}_\gamma) = 0$ for all $\gamma \leq s$, i.e.
$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_\gamma), Q((X)^{\max}_\gamma)) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0,0) \\ &= 0\,. \end{aligned}$$

Considering the case that $\gamma > 1 - 2s$, i.e. $\frac{1}{2} - \frac{1}{2}\gamma < s$, we conclude from Def. 66 that $(X)^{\max}_{\gamma} = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} \ni \{e\}$ because $\mu_X(e) = s > \frac{1}{2} - \frac{1}{2}\gamma$, and hence $Q((X)^{\max}_{\gamma}) = 1$ by (206). Combining this with (215), we obtain

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q((X)^{\min}_{\gamma}), Q((X)^{\max}_{\gamma})) && \text{by (206)} \\ &= \mathrm{m}_{\frac{1}{2}}(0, 1) \\ &= \tfrac{1}{2}\,, \end{aligned}$$

for all $\gamma > 1 - 2s$.

6. $s < \frac{1}{2}$ and $s \notin V$. Then

$$\mu_X(e') < s \tag{216}$$

for all $e' \in f^{-1}(z)$, and for all $s' < s$ there exists $e \in f^{-1}(z)$ such that

$$\mu_X(e) > s'\,. \tag{217}$$

We will firstly discuss the case that $\gamma \le 1 - 2s$. By (216),

$$\tfrac{1}{2} - \tfrac{1}{2}\gamma \ge \tfrac{1}{2} - \tfrac{1}{2}(1 - 2s) = s > \mu_X(e')$$

for all $e' \in f^{-1}(z)$ and hence $(X)^{\max}_{\gamma} \cap f^{-1}(z) = (X)^{\min}_{\gamma} \cap f^{-1}(z) = \varnothing$ by Def. 66. By (205), $Q((X)^{\min}_{\gamma}) = Q((X)^{\max}_{\gamma}) = 0$ and hence by (206), $Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(0, 0) = 0$ for all $\gamma \le 1 - 2s$.

Finally, let us consider the case that $\gamma > 1 - 2s$. By (217), there exists some $e \in f^{-1}(z)$ such that $\mu_X(e) > \frac{1}{2} - \frac{1}{2}\gamma$, i.e. $(X)^{\max}_{\gamma} \cap f^{-1}(z) \neq \varnothing$ and by (205), $Q((X)^{\max}_{\gamma}) = 1$. On the other hand, $(X)^{\min}_{\gamma} \cap f^{-1}(z) = \varnothing$ (as above) and hence $Q((X)^{\min}_{\gamma}) = 0$. By (206), $Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(0, 1) = \frac{1}{2}$.

We will now introduce an $\mathcal{M}_{\mathcal{B}}$-QFM which will serve as an example of the independence of (Z-2) of the other axioms.

Definition 107

The $\mathcal{M}_{\mathcal{B}}$-QFM $\mathcal{M}_{(\mathrm{Z-2})}$ is defined by $\mathcal{M}_{(\mathrm{Z-2})} = \mathcal{M}_{\mathcal{B}_{(\mathrm{Z-2})}}$, where $\mathcal{B}'_{(\mathrm{Z-2})} : \mathbb{H} \longrightarrow \mathbf{I}$ is

$$\mathcal{B}'_{(\mathrm{Z-2})}(f) = f(1)$$

for all $f \in \mathbb{H}$ and $\mathcal{B}_{(\mathrm{Z-2})} : \mathbb{B} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B}'_{(\mathrm{Z-2})}$ according to equation (23)*, i.e.*

$$\mathcal{B}_{(\mathrm{Z-2})}(f) = f(1)$$

for all $f \in \mathbb{B}$.

Let us now investigate the induced connectives of $\mathcal{M}_{(\mathrm{Z-2})}$.

Lemma 65

The induced negation $\widetilde{\widetilde{\neg}} = \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)$ of $\mathcal{M}_{(\mathrm{Z-2})}$ is

$$\widetilde{\widetilde{\neg}}\, x = t_1(\neg x)$$

for all $x \in \mathbf{I}$, where $\neg : \mathbf{I} \longrightarrow \mathbf{I}$ is the standard negation $\neg x = 1 - x$ and $t_1(\bullet)$ is the three-valued cut at cut-level 1*.*

In particular, the induced fuzzy complement of $\mathcal{M}_{(\mathrm{Z-2})}$ is

$$\widetilde{\widetilde{\neg}}\, X = \mathrm{T}_1(\neg X)$$

Proof Let us first recall that for all $x \in (0,1)$,

$$t_1(x) = \tfrac{1}{2}$$

by Def. 62. Observing that if $x \in (0,1)$, then $\neg x = 1 - x \in (0,1)$, too, we obtain that

$$t_1(\neg x) = \tfrac{1}{2} \tag{218}$$

for all $x \in (0,1)$.
To simplify the proof of the lemma, we shall discern five cases:

a. $x = 1$. Then

$$\begin{aligned} \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)(x) &= (Q_\neg)_1(\widetilde{\eta}(x)) && \text{by Def. 107, Def. 52} \\ &= 0 && \text{by L-62.a, } x = 1 = \gamma \\ &= t_1(0) && \text{by Def. 62} \\ &= t_1(\neg x)\,. && \text{because } \neg x = \neg 1 = 1 - 1 = 0 \end{aligned}$$

b. $\frac{1}{2} < x < 1$. Then

$$\begin{aligned} \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)(x) &= (Q_\neg)_1(\widetilde{\eta}(x)) && \text{by Def. 107, Def. 52} \\ &= \tfrac{1}{2} && \text{by L-62.a, } x < 1 = \gamma \\ &= t_1(\neg x)\,. && \text{by (218)} \end{aligned}$$

c. $x = \frac{1}{2}$. Then

$$\begin{aligned} \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)(x) &= (Q_\neg)_1(\widetilde{\eta}(x)) && \text{by Def. 107, Def. 52} \\ &= \tfrac{1}{2} && \text{by L-62.b} \\ &= t_1(\neg x)\,. && \text{by (218)} \end{aligned}$$

d. $0 < x < \frac{1}{2}$. In this case,

$$\begin{aligned} \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)(x) &= (Q_\neg)_1(\widetilde{\eta}(x)) && \text{by Def. 107, Def. 52} \\ &= \tfrac{1}{2} && \text{by L-62.c, } \gamma = 1 > 1 - 2x \\ &= t_1(\neg x)\,. && \text{by (218)} \end{aligned}$$

e. $x = 0$. Then

$$\begin{aligned} \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-2})}(\neg)(x) &= (Q_\neg)_1(\widetilde{\eta}(x)) && \text{by Def. 107, Def. 52} \\ &= 1 && \text{by L-62.b, } \gamma = 1 \le 1 - 2x \\ &= t_1(1) && \text{by Def. 62} \\ &= t_1(\neg x)\,. && \text{because } \neg x = \neg 0 = 1 - 0 = 1 \end{aligned}$$

Lemma 66

The induced disjunction $\widetilde{\widetilde{\vee}} = \widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)$ *of* $\mathcal{M}_{(\mathrm{Z}-2)}$ *is*

$$x_1 \mathbin{\widetilde{\widetilde{\vee}}} x_2 = t_1(x_1 \vee x_2)$$

for all $x_1, x_2 \in \mathbf{I}$*, where* $\vee : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is the standard disjunction* $x_1 \vee x_2 = \max(x_1, x_2)$ *and* $t_1(\bullet)$ *is the three-valued cut at cut-level* 1*.*
In particular, the induced fuzzy union of $\mathcal{M}_{(\mathrm{Z}-2)}$ *is*

$$X_1 \mathbin{\widetilde{\widetilde{\cup}}} X_2 = \mathrm{T}_1(X_1 \cup X_2)\,.$$

Proof Suppose $x_1, x_2 \in \mathbf{I}$. Let us abbreviate $z = \max(x_1, x_2)$. We shall again discern five cases.

a. $z = 1$. Then

$$\begin{aligned}\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)(x_1, x_2) &= (Q_\vee)_1(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52, Def. 107}\\ &= 1 && \text{by L-51.a, } \gamma = 1 \leq 2z - 1 = 1\\ &= t_1(z)\,. && \text{by Def. 62, } z = 1\end{aligned}$$

b. $\frac{1}{2} < z < 1$. Then

$$\begin{aligned}\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)(x_1, x_2) &= (Q_\vee)_1(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52, Def. 107}\\ &= \tfrac{1}{2} && \text{by L-51.a, } \gamma = 1 > 2z - 1\\ &= t_1(z)\,. && \text{by Def. 62, } z \in (0, 1)\end{aligned}$$

c. $z = \frac{1}{2}$. Then

$$\begin{aligned}\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)(x_1, x_2) &= (Q_\vee)_1(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52, Def. 107}\\ &= \tfrac{1}{2} && \text{by L-51.b}\\ &= t_1(z)\,. && \text{by Def. 62, } z \in (0, 1)\end{aligned}$$

d. $0 < z < \frac{1}{2}$. Then

$$\begin{aligned}\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)(x_1, x_2) &= (Q_\vee)_1(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52, Def. 107}\\ &= \tfrac{1}{2} && \text{by L-51.c, } \gamma = 1 > 1 - 2z\\ &= t_1(z)\,. && \text{by Def. 62, } z \in (0, 1)\end{aligned}$$

e. $z = 0$. Then

$$\begin{aligned}\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-2)}(\vee)(x_1, x_2) &= (Q_\vee)_1(\widetilde{\eta}(x_1, x_2)) && \text{by Def. 52, Def. 107}\\ &= 0 && \text{by L-51.c, } \gamma = 1 \leq 1 - 2z\\ &= t_1(z)\,. && \text{by Def. 62, } z = 0\end{aligned}$$

Lemma 67

Suppose $f : E \longrightarrow E'$ *is some mapping where* E, E' *are nonempty sets; further suppose that* $X \in \widetilde{\mathcal{P}}(E)$ *and* $z \in E'$ *are given. Then*

$$\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f)(X)}(z) = \begin{cases} 1 & : \quad \mu_X(e) = 1 \text{ for some } e \in f^{-1}(z)\\ 0 & : \quad \mu_X(e) = 0 \text{ for all } e \in f^{-1}(z)\\ \frac{1}{2} & : \quad \text{else}\end{cases}$$

Proof Let us define $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ by $Q(Y) = \chi_{\widehat{f}(Y)}(z)$. Let us further abbreviate $V = \{\mu_X(e) : e \in f^{-1}(z)$ and $s = \sup V$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) &= \mathcal{M}_{(\mathrm{Z-2})}(Q)(X) && \text{by Def. 19}\\
&= Q_1(X) && \text{by Def. 107}\\
&= \begin{cases}
1 & : \quad s = 1 \text{ and } s \in V \\
\frac{1}{2} & : \quad s = 1 \text{ and } s \notin V \\
\frac{1}{2} & : \quad \frac{1}{2} < s < 1 \text{ and } s \in V \\
\frac{1}{2} & : \quad \frac{1}{2} < s < 1 \text{ and } s \notin V \\
\frac{1}{2} & : \quad s = \frac{1}{2} \text{ and } s \in V \\
\frac{1}{2} & : \quad s = \frac{1}{2} \text{ and } s \notin V \\
\frac{1}{2} & : \quad 0 < s < \frac{1}{2} \text{ and } s \in V \\
\frac{1}{2} & : \quad 0 < s < \frac{1}{2} \text{ and } s \notin V \\
0 & : \quad s = 0 \text{ and } s \in V \\
0 & : \quad s = 0 \text{ and } s \notin V
\end{cases}
&& \begin{array}{l}
\text{by L-64.1, } 1 = \gamma \le 2s - 1 = 1 \\
\text{by L-64.2, } 1 = \gamma \ge 2s - 1 = 1 \\
\text{by L-64.1, } 1 = \gamma > 2s - 1 \\
\text{by L-64.2, } 1 = \gamma \ge 2s - 1 \\
\text{by L-64.3} \\
\text{by L-64.4, } \gamma = 1 \\
\text{by L-64.5, } 1 = \gamma > 1 - 2s \\
\text{by L-64.6, } 1 = \gamma > 1 - 2s \\
\text{by L-64.5, } 1 = \gamma \le 1 - 2s = 1 \\
\text{by L-64.6, } 1 = \gamma \le 1 - 2s = 1
\end{array}
\end{aligned}$$

i.e.

$$\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) = \begin{cases} 1 & : \quad s = 1 \text{ and } s \in V \\ 0 & : \quad s = 0 \\ \frac{1}{2} & : \quad \text{else} \end{cases} \tag{219}$$

It is apparent from (219) that $\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) = 1$ iff $s = 1$ and $s \in V$. By the definition of $V = \{\mu_X(e) : e \in f^{-1}(z)\}$ and $s = \sup V$, $s \in V$ means that there exists some $e \in f^{-1}(z)$ such that $1 = s = \mu_X(e)$.
By (219), $\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) = 0$ is equivalent to $s = 0$. Recalling that $s = \sup V = \sup\{\mu_X(e) : e \in f^{-1}(z)\}$, the criterion $s = 0$ is in turn equivalent to the condition that $\mu_X(e) = 0$ for all $e \in f^{-1}(z)$.
Hence

$$\mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) = \begin{cases} 1 & : \quad \mu_X(e) = 1 \text{ for some } e \in f^{-1}(z) \\ 0 & : \quad \mu_X(e) = 0 \text{ for all } e \in f^{-1}(z) \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

as desired.

Lemma 68
Suppose $f : E \longrightarrow E'$ is a mapping where E, E' are nonempty. Further suppose that $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E. Then

$$\mathcal{T}_1(\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)) = \widehat{f}(\mathcal{T}_1(X))\,,$$

where we have abbreviated

$$\widehat{f}(\mathcal{T}_1(X)) = \{\widehat{f}(Y) : Y \in \mathcal{T}_1(X)\}\,.$$

Proof By [9, L-26, p.133], it is sufficient to show the following.

a. $\widehat{f}((X)_1^{\min}) = (\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X))_1^{\min}$;

b. $\widehat{f}((X)_1^{\max}) = (\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X))_1^{\max}$.

a. To prove the first claim, let us recall that by Def. 66, $(X)_1^{\min} = (X)_{\geq 1}$, i.e. the above equation becomes

$$\widehat{f}((X)_{\geq 1}) = \left(\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)\right)_{\geq 1}.$$

To see that this equation holds, let $z \in E'$. Then

$$\begin{aligned} z \in \left(\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)\right)_{\geq 1} &\Leftrightarrow \mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) = 1 && \text{by Def. 64} \\ &\Leftrightarrow \text{there is some } e \in f^{-1}(z) \text{ s.th. } \mu_X(e) = 1 && \text{by L-67} \\ &\Leftrightarrow \text{there is some } e \in f^{-1}(z) \text{ s.th. } e \in (X)_{\geq 1} && \text{by Def. 64} \\ &\Leftrightarrow z \in \widehat{f}((X)_{\geq 1}). && \text{by Def. 17} \end{aligned}$$

b. We will show in an analogous way that equation b. holds. Recalling that by Def. 66, $(X)_1^{\max} = (X)_{>0}$, equation b. becomes

$$\widehat{f}((X)_{>0}) = \left(\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)\right)_{>0}.$$

To see that this equation holds, let $z \in E'$. Then

$$\begin{aligned} z \in \left(\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)\right)_{>0} &\Leftrightarrow \mu_{\widehat{\mathcal{M}}_{(\mathrm{Z-2})}(f)(X)}(z) > 0 && \text{by Def. 65} \\ &\Leftrightarrow \text{there is some } e \in f^{-1}(z) \text{ s.th. } \mu_X(e) > 0 && \text{by L-67} \\ &\Leftrightarrow \text{there is some } e \in f^{-1}(z) \text{ s.th. } e \in (X)_{>0} && \text{by Def. 65} \\ &\Leftrightarrow z \in \widehat{f}((X)_{>0}). && \text{by Def. 17} \end{aligned}$$

Lemma 69

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy argument sets and $i \in \{1, \ldots, n\}$. Then

$$\mathcal{M}_{(\mathrm{Z-2})}(Q)(X_1, \ldots, X_n) = \mathcal{M}_{(\mathrm{Z-2})}(Q)(X_1, \ldots, X_{i-1}, \mathrm{T}_1(X_i), X_{i+1}, X_n),$$

where $\mathrm{T}_1(X_i)$ is the three-valued cut of X_i at cut-level 1; see Def. 63.

Proof
Let us first observe that

$$\begin{aligned}\mathcal{T}_1(X_i) &= \mathcal{T}(\mathrm{T}_1(X_i)) && \text{by Def. 66}\\ &= \mathcal{T}(\mathrm{T}_1(\mathrm{T}_1(X_i))) && \text{by L-27, } \mathrm{T}_1(X_i) \text{ three-valued}\\ &= \mathcal{T}_1(\mathrm{T}_1(X_i)) && \text{by Def. 66}\,,\end{aligned}$$

i.e.

$$\mathcal{T}_1(X_i) = \mathcal{T}_1(\mathrm{T}_1(X_i))\,. \tag{220}$$

Hence

$$\begin{aligned}&\mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1, \dots, X_n)\\ &= \mathcal{B}_{(\mathrm{Z}-2)}((Q_\gamma(X_1, \dots, X_n))_{\gamma\in\mathbf{I}}) && \text{by Def. 107, Def. 69}\\ &= Q_1(X_1, \dots, X_n) && \text{by Def. 107}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_1(X_1), \dots, Y_n \in \mathcal{T}_1(X_n)\} && \text{by Def. 67}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_1(X_1), \dots, Y_{i-1} \in \mathcal{T}_1(X_{i-1}),\\ &\qquad Y_i \in \mathcal{T}_1(\mathrm{T}_1(X_i)), Y_{i+1} \in \mathcal{T}_1(X_{i+1}), \dots, Y_n \in \mathcal{T}_1(X_n)\} && \text{by L-27, Def. 66}\\ &= Q_1(X_1, \dots, X_{i-1}, \mathrm{T}_1(X_i), X_{i+1}, \dots, X_n) && \text{by Def. 67}\\ &= \mathcal{B}_{(\mathrm{Z}-2)}((Q_\gamma(X_1, \dots, X_{i-1}, \mathrm{T}_1(X_i), X_{i+1}, \dots, X_n))_{\gamma\in\mathbf{I}}) && \text{by Def. 107}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1, \dots, X_{i-1}, \mathrm{T}_1(X_i), X_{i+1}, \dots, X_n)\,. && \text{by Def. 107, Def. 69}\end{aligned}$$

Lemma 70
Suppose $X \in \mathcal{P}(\mathbf{I})$ is an arbitrary subset of $\mathbf{I}$. Then

$$\mathrm{m}_{\frac{1}{2}}\{t_1(x) : x \in X\} = t_1(\mathrm{m}_{\frac{1}{2}} X)\,.$$

Proof From the definition of fuzzy median Def. 45, it is apparent that

$$\mathrm{m}_{\frac{1}{2}}(x, y) = 1 \Leftrightarrow x = y = 1$$

and

$$\mathrm{m}_{\frac{1}{2}}(x, y) = 0 \Leftrightarrow x = y = 0\,.$$

It is then obvious from the definition of the generalized fuzzy median Def. 46 that

$$\mathrm{m}_{\frac{1}{2}} Z = 1 \Leftrightarrow Z = \{1\} \tag{221}$$

and

$$\mathrm{m}_{\frac{1}{2}} Z = 0 \Leftrightarrow Z = \{0\} \tag{222}$$

for all $Z \in \mathcal{P}(\mathbf{I})$.

Now suppose $X \in \mathcal{P}(\mathbf{I})$ is given. We shall abbreviate

$$L = \mathrm{m}_{\frac{1}{2}}\{t_1(x) : x \in X\} \tag{223}$$

and

$$R = t_1(\mathrm{m}_{\frac{1}{2}} X) \tag{224}$$

It is easily seen from Def. 62 and Def. 45 that $R \in \{0, \frac{1}{2}, 1\}$ and $L \in \{0, \frac{1}{2}, 1\}$. We can hence show the equality of R and L by proving the following:

a. $L = 1$ iff $R = 1$;

b. $L = 0$ iff $R = 0$.

a. This is apparent from the following chain of equivalences:

$$\begin{aligned} R = 1 &\Leftrightarrow t_1(\mathrm{m}_{\frac{1}{2}} X) = 1 && \text{by (224)} \\ &\Leftrightarrow \mathrm{m}_{\frac{1}{2}} X = 1 && \text{by Def. 62} \\ &\Leftrightarrow X = \{1\} && \text{by (221)} \\ &\Leftrightarrow \{t_1(x) : x \in X\} = \{1\} && \text{by Def. 62} \\ &\Leftrightarrow \mathrm{m}_{\frac{1}{2}}\{t_1(x) : x \in X\} = 1 && \text{by (221)} \\ &\Leftrightarrow L = 1\,. && \text{by (223)} \end{aligned}$$

b. The proof of this case is analogous to that of **a.**:

$$\begin{aligned} R = 0 &\Leftrightarrow t_1(\mathrm{m}_{\frac{1}{2}} X) = 0 && \text{by (224)} \\ &\Leftrightarrow \mathrm{m}_{\frac{1}{2}} X = 0 && \text{by Def. 62} \\ &\Leftrightarrow X = \{0\} && \text{by (222)} \\ &\Leftrightarrow \{t_1(x) : x \in X\} = \{0\} && \text{by Def. 62} \\ &\Leftrightarrow \mathrm{m}_{\frac{1}{2}}\{t_1(x) : x \in X\} = 0 && \text{by (222)} \\ &\Leftrightarrow L = 0\,, && \text{by (223)} \end{aligned}$$

as desired.

Let us recall one more lemma on properties of $(\bullet)_\gamma$:

Lemma 71

Suppose $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ *is a constant semi-fuzzy quantifier. Then* $\mathcal{U}(Q_\gamma) = Q$ *for all* $\gamma \in \mathbf{I}$.

Proof Apparent from [9, L-18, p. 124] and Def. 5.

With the help of these lemmata, it is now easy to show that $\mathcal{M}_{(\mathrm{Z}-2)}$ satisfies all “Z-axioms” except (Z-2).

Proof of Theorem 72

$\mathcal{M}_{(\mathrm{Z-2})}$ **satisfies** (Z-1) Let us first consider the case of nullary (i.e. constant) semi-fuzzy quantifiers. To this end, let $Q : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant semi-fuzzy quantifier. Then

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-2})}(Q)(\varnothing) &= Q_1(\varnothing) && \text{by Def. 107} \\ &= Q(\varnothing)\,. && \text{by L-71} \end{aligned}$$

The remaining case is that of unary quantifiers. Hence suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ and a *crisp* subset $X \in \mathcal{P}(E)$ are given. Then

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-2})}(Q)(X) &= Q_1(X) && \text{by Def. 107} \\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \mathcal{T}_1(X)\} && \text{by Def. 67} \\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \{X\}\} && \text{by Def. 66, } X \text{ crisp} \\ &= \mathrm{m}_{\frac{1}{2}}\{Q(X)\} \\ &= Q(X)\,. && \text{by Def. 46} \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z-2})}$ **does not satisfy** (Z-2) To see that (Z-2) is violated, let $E = \{e\}$ some singleton base set and let $x \in (\frac{1}{2}, 1)$, e.g. $x = \frac{3}{4}$. Further suppose that $X \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset defined by $\mu_X(e) = x = \frac{3}{4}$. Then

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-2})}(\pi_e)(X) &= \mathcal{B}_{(\mathrm{Z-2})}((\pi_{e\gamma}(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 107, Def. 69} \\ &= \pi_{e1}(X) && \text{by Def. 107} \\ &= \tfrac{1}{2} && \text{by L-63} \\ &\neq \tfrac{3}{4} \\ &= \mu_X(e) && \text{by definition of } X \\ &= \widetilde{\pi}_e X\,. && \text{by Def. 7} \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z-2})}$ **satisfies** (Z-3)

$$\begin{aligned} &\mathcal{M}_{(\mathrm{Z-2})}(Q\widetilde{\widetilde{\Box}})(X_1, \dots, X_n) \\ &= (Q\widetilde{\widetilde{\Box}})_1(X_1, \dots, X_n) && \text{by Def. 107} \\ &= \mathrm{m}_{\frac{1}{2}}\{Q\widetilde{\widetilde{\Box}}(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_1(X_i)\} && \text{by Def. 67} \\ &= \mathrm{m}_{\frac{1}{2}}\{\widetilde{\widetilde{\neg}}\, Q(Y_1, \dots, Y_{n-1}, \widetilde{\widetilde{\neg}}\, Y_n) : Y_i \in \mathcal{T}_1(X_i)\} && \text{by Def. 12} \\ &= \mathrm{m}_{\frac{1}{2}}\{t_1(\neg Q(Y_1, \dots, Y_n)) : Y_1 \in \mathcal{T}_1(X_1), \dots, Y_{n-1} \in \mathcal{T}_1(X_{n-1}), Y_n \in \mathcal{T}_1(\neg X_n)\} && \text{by L-65, L-31} \\ &= t_1(\neg\, \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_1(X_1), \dots, Y_{n-1} \in \mathcal{T}_1(X_{n-1}), Y_n \in \mathcal{T}_1(\neg X_n)\}) && \text{by L-70, L-29} \\ &= \widetilde{\widetilde{\neg}}\, Q_1(X_1, \dots, X_{n-1}, \neg X_n) && \text{by L-65, Def. 67} \\ &= \widetilde{\widetilde{\neg}}\, \mathcal{M}_{(\mathrm{Z-2})}(Q)(X_1, \dots, X_{n-1}, \neg X_n) && \text{by Def. 107} \\ &= \widetilde{\widetilde{\neg}}\, \mathcal{M}_{(\mathrm{Z-2})}(Q)(X_1, \dots, X_{n-1}, \mathrm{T}_1(\neg X_n)) && \text{by L-69} \\ &= \widetilde{\widetilde{\neg}}\, \mathcal{M}_{(\mathrm{Z-2})}(Q)(X_1, \dots, X_{n-1}, \widetilde{\widetilde{\neg}}\, X_n) && \text{by L-65} \\ &= \mathcal{M}_{(\mathrm{Z-2})}(Q)\widetilde{\widetilde{\Box}}(X_1, \dots, X_n)\,. && \text{by Def. 12} \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z}-2)}$ **satisfies** (Z-4)

$$\begin{aligned}\mathcal{M}_{(\mathrm{Z}-2)}(Q\cup)(X_1,\dots,X_{n+1}) &= (Q\cup)_1(X_1,\dots,X_{n+1}) && \text{by Def. 107}\\ &= Q_1(X_1,\dots,X_{n-1},X_n\cup X_{n+1}) && \text{by L-35}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1,\dots,X_{n-1},X_n\cup X_{n+1}) && \text{by Def. 107}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1,\dots,X_{n-1},\mathrm{T}_1(X_n\cup X_{n+1})) && \text{by L-69}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1,\dots,X_{n-1},X_n\,\widetilde{\widetilde{\cup}}\,X_{n+1}) && \text{by L-66}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)\widetilde{\widetilde{\cup}}(X_1,\dots,X_{n+1})\,. && \text{by Def. 26}\end{aligned}$$

$\mathcal{M}_{(\mathrm{Z}-2)}$ **satisfies** (Z-5) Suppose $Q:\mathcal{P}(E)^n\longrightarrow\mathbf{I}$ is nonincreasing in its n-th argument. Further suppose that $X_1,\dots,X_n,X_n'\in\widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E such that $X_n\subseteq X_n'$. Then

$$\begin{aligned}\mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1,\dots,X_n) &= Q_1(X_1,\dots,X_n) && \text{by Def. 107}\\ &\geq Q_1(X_1,\dots,X_{n-1},X_n') && \text{by L-36}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(X_1,\dots,X_{n-1},X_n')\,. && \text{by Def. 107}\end{aligned}$$

$\mathcal{M}_{(\mathrm{Z}-2)}$ **satisfies** (Z-6) Suppose $Q:\mathcal{P}(E)'^n\longrightarrow\mathbf{I}$ is a semi-fuzzy quantifier, $f_1,\dots,f_n:E\longrightarrow E'$ are mappings ($E\neq\varnothing$) and $X_1,\dots,X_n\in\widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}&\mathcal{M}_{(\mathrm{Z}-2)}(Q\circ\mathop{\times}_{i=1}^{n}\widehat{f}_i)(X_1,\dots,X_n)\\ &= (Q\circ\mathop{\times}_{i=1}^{n}\widehat{f}_i)_1(X_1,\dots,X_n) && \text{by Def. 107}\\ &= \mathrm{m}_{\frac{1}{2}}\{(Q\circ\mathop{\times}_{i=1}^{n}\widehat{f}_i)(Y_1,\dots,Y_n):Y_i\in\mathcal{T}_1(X_i)\} && \text{by Def. 67}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(\widehat{f}_1(Y_1),\dots,\widehat{f}_n(Y_n)):Y_i\in\mathcal{T}_1(X_i)\} && \text{by Def. 21}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Z_1,\dots,Z_n):Z_i\in\widehat{f}_i(\mathcal{T}_1(X_i))\} && \text{using the abbr. of L-68}\\ &= \mathrm{m}_{\frac{1}{2}}\{Q(Z_1,\dots,Z_n):Z_i\in\mathcal{T}_1(\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f_i)(X_i))\} && \text{by L-68}\\ &= Q_1(\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f_1)(X_1),\dots,\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f_n)(X_n)) && \text{by Def. 67}\\ &= \mathcal{M}_{(\mathrm{Z}-2)}(Q)(\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f_1)(X_1),\dots,\widehat{\mathcal{M}}_{(\mathrm{Z}-2)}(f_n)(X_n))\,. && \text{by Def. 107}\end{aligned}$$

This finishes the proof that $\mathcal{M}_{(\mathrm{Z}-2)}$ satisfies all properties except for (Z-2), i.e. (Z-2) is independent of the remaining axioms (Z-1) and (Z-3) to (Z-6).

D.9 Proof of Theorem 73

We shall define the QFM $\mathcal{M}_{(\mathrm{Z}-4)}$ by

$$\mathcal{M}_{(\mathrm{Z}-4)}(Q)=\begin{cases}\mathcal{M}(Q) & : \quad n\leq 1\\ \mathcal{M}^*(Q) & : \quad n>1\end{cases} \tag{225}$$

for all semi-fuzzy quantifiers $Q:\mathcal{P}(E)^n\longrightarrow\mathbf{I}$.

Because $\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-4)}$ and $\widehat{\mathcal{M}}_{(\mathrm{Z}-4)}$ are defined in terms of nullary or one-place semi-fuzzy quantifiers (see Def. 52 and Def. 19, resp.), it is apparent that

$$\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z}-4)}=\widetilde{\widetilde{\mathcal{M}}} \tag{226}$$

and

$$\widehat{\mathcal{M}}_{(\mathrm{Z-4})} = \widehat{\mathcal{M}} = (\hat{\hat{\bullet}}) . \tag{227}$$

The last equation is apparent from the fact that $\mathcal{M}$ is a standard DFS by Th-42.
Another consequence of $\mathcal{M}$ being a standard DFS and (226) is that $\mathcal{M}_{(\mathrm{Z-4})}$ induces the standard fuzzy negation

$$\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-4})}(\neg)(x) = 1 - x \tag{228}$$

for all $x \in \mathbf{I}$, and the standard fuzzy disjunction

$$\widetilde{\widetilde{\mathcal{M}}}_{(\mathrm{Z-4})}(\vee)(x_1, x_2) = \max(x_1, x_2) , \tag{229}$$

for all $x_1, x_2 \in \mathbf{I}$.
Let us now show that $\mathcal{M}_{(\mathrm{Z-4})}$ satisfies all "Z-axioms" except for (Z-4), i.e. that (Z-4) is independent of the remaining conditions.

$\mathcal{M}_{(\mathrm{Z-4})}$ **satisfies** (Z-1) Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier where $n \leq 1$. Then

$$\begin{aligned} \mathcal{U}(\mathcal{M}_{(\mathrm{Z-4})}(Q)) &= \mathcal{U}(\mathcal{M}(Q)) && \text{by (225), } n \leq 1 \\ &= Q . && \text{because } \mathcal{M} \text{ DFS by Th-42} \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z-4})}$ **satisfies** (Z-2) Suppose $E \neq \varnothing$ is some nonempty base set, $e \in E$ an arbitrary element of E.

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-4})}(\pi_e) &= \mathcal{M}(\pi_e) && \text{by (225), } n = 1 \\ &= \widetilde{\pi}_e . && \text{because } \mathcal{M} \text{ DFS by Th-42} \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z-4})}$ **satisfies** (Z-3) Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier where $n > 0$. Recalling that $\mathcal{M}_{(\mathrm{Z-4})}$ and $\mathcal{M}$ both induce the standard negation (by (228) and Th-42, respectively) we obtain in the case that $n = 1$,

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-4})}(Q\square) &= \mathcal{M}(Q\square) && \text{by (225), } n = 1 \\ &= \mathcal{M}(Q)\square && \text{because } \mathcal{M} \text{ standard DFS by Th-42} \\ &= \mathcal{M}_{(\mathrm{Z-4})}(Q)\square . && \text{by (225), } n = 1 \end{aligned}$$

In the case that $n > 1$, we recall that $\mathcal{M}^*$ also induces the standard fuzzy negation by Th-54. Therefore

$$\begin{aligned} \mathcal{M}_{(\mathrm{Z-4})}(Q\square) &= \mathcal{M}^*(Q\square) && \text{by (225), } n > 1 \\ &= \mathcal{M}^*(Q)\square && \text{because } \mathcal{M}^* \text{ standard DFS by Th-54} \\ &= \mathcal{M}_{(\mathrm{Z-4})}(Q)\square . && \text{by (225), } n > 1 \end{aligned}$$

$\mathcal{M}_{(\mathrm{Z-4})}$ **violates** (Z-4) Suppose $f \in \mathbb{B}$ is the mapping defined by

$$f(\gamma) = 1 - \tfrac{1}{4}\gamma \tag{230}$$

for all $\gamma \in \mathbf{I}$. By Th-41, there exists a monadic semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ such that

$$Q_\gamma(X) = f(\gamma) \tag{231}$$

for all $\gamma \in \mathbf{I}$. We compute:

$$\begin{aligned}
\widetilde{\widetilde{\cup}}\mathcal{M}_{(\mathrm{Z-4})}(Q)(X, \varnothing) &= \mathcal{M}_{(\mathrm{Z-4})}(Q)(X \cup \varnothing) && \text{by Def. 26, (229)}\\
&= \mathcal{M}_{(\mathrm{Z-4})}(Q)(X)\\
&= \mathcal{M}(Q)(X) && \text{by (225), } n = 1\\
&= \int_0^1 Q_\gamma(X)\, d\gamma && \text{by Def. 70}\\
&= \int_0^1 1 - \tfrac{1}{4}\gamma\, d\gamma \text{by (230), (231)}\\
&= (\gamma - \tfrac{1}{8}\gamma^2)\big|_0^1\\
&= 1 - \tfrac{1}{8}\\
&= \tfrac{7}{8}\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{M}_{(\mathrm{Z-4})}(\widetilde{\widetilde{\cup}}Q)(X, \varnothing) &= \mathcal{M}_{(\mathrm{Z-4})}(Q\cup)(X, \varnothing) && \text{by (229)}\\
&= \mathcal{M}^*(Q\cup)(X, \varnothing) && \text{by (225), } Q\cup \text{ two-place}\\
&= \mathcal{M}^*(Q)\cup(X, \varnothing) && \text{by Th-54, } \mathcal{M}^* \text{ is standard DFS}\\
&= \mathcal{M}^*(Q)(X \cup \varnothing) && \text{by Def. 26}\\
&= \mathcal{M}^*(Q)(X)\\
&= \mathcal{B}^*((Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 77, Def. 69}\\
&= \mathcal{B}^*(f) && \text{by (231)}\\
&= 1\,,
\end{aligned}$$

which is apparent from Def. 70 and (230), observing that $f_0^* = 1$ and $f_*^{\frac{1}{2}} = 1$. Comparing $\mathcal{M}_{(\mathrm{Z-4})}(Q)\widetilde{\widetilde{\square}}(X, \varnothing)$ and $\mathcal{M}_{(\mathrm{Z-4})}Q\widetilde{\widetilde{\square}}(X, \varnothing)$, it is apparent that (Z-4) fails.

$\mathcal{M}_{(\mathrm{Z-4})}$ **satisfies** (Z-5) We shall discern monadic quantifiers and quantifiers of arities $n > 1$.

a. $n = 1$. Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nonincreasing in its argument. Let $X, X' \in \widetilde{\mathcal{P}}(E)$ such that $X \subseteq X'$. Then

$$\begin{aligned}
\mathcal{M}_{(\mathrm{Z-4})}(Q)(X) &= \mathcal{M}(Q)(X) && \text{by (225), } n = 1\\
&\geq \mathcal{M}(Q)(X') && \text{by Th-42, } \mathcal{M} \text{ satisfies (Z-5)}\\
&= \mathcal{M}_{(\mathrm{Z-4})}(Q)(X')\,. && \text{by (225), } n = 1
\end{aligned}$$

b. $n > 1$. Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is nonincreasing in its n-th argument ($n > 1$). By (225), we know that $\mathcal{M}_{(\mathrm{Z-4})}(Q) = \mathcal{M}^*(Q)$. By Th-54, we know that $\mathcal{M}^*$ is a DFS and hence fulfills (Z-5), i.e. $\mathcal{M}_{(\mathrm{Z-4})}(Q) = \mathcal{M}^*(Q)$ is nondecreasing in its n-th argument, as desired.

$\mathcal{M}_{(\text{Z}-4)}$ **satisfies** (Z-6) We shall again treat separately semi-fuzzy quantifiers of arity $n \leq 1$ and those of arity $n > 1$.

a. Suppose $Q : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier where $n \leq 1$. Further suppose that $f_1, \ldots, f_n : E \longrightarrow E'$ are mappings, where $E \neq \varnothing$. Then

$$\begin{aligned}
\mathcal{M}_{(\text{Z}-4)}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) &= \mathcal{M}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) && \text{by (225), } n \leq 1 \\
&= \mathcal{M}(Q) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i && \text{by Th-42, } \mathcal{M} \text{ is standard DFS} \\
&= \mathcal{M}_{(\text{Z}-4)}(Q) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i && \text{by (225), } n \leq 1 \\
&= \mathcal{M}_{(\text{Z}-4)}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{M}}_{(\text{Z}-4)}(f_i)\,. && \text{by (227)}
\end{aligned}$$

b. Assume $Q : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier where $n > 1$. Further suppose that $f_1, \ldots, f_n : E \longrightarrow E'$ are mappings, where $E \neq \varnothing$. Then

$$\begin{aligned}
\mathcal{M}_{(\text{Z}-4)}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) &= \mathcal{M}^*(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) && \text{by (225), } n > 1 \\
&= \mathcal{M}^*(Q) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i && \text{by Th-54, } \mathcal{M}^* \text{ is standard DFS} \\
&= \mathcal{M}_{(\text{Z}-4)}(Q) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i && \text{by (225), } n > 1 \\
&= \mathcal{M}_{(\text{Z}-4)}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{M}}_{(\text{Z}-4)}(f_i)\,, && \text{by (227)}
\end{aligned}$$

i.e. (Z-6) holds.

E Proofs of Theorems in Chapter 6

E.1 Proof of Theorem 74

Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$.

a. Let us first prove that $(\mathcal{F}_j)_{j\in\mathcal{J}}$ does not have upper specificity bounds if $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is not specificity consistent.

The proof is by contradiction. Hence let us assume that $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is not specificity consistent and that $\mathcal{F}^*$ is an upper bound of $(\mathcal{F}_j)_{j\in\mathcal{J}}$. Because $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is not specificity consistent, we conclude from Def. 79 that there exists $j, j' \in \mathcal{J}$, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ such that

$$\begin{aligned}
&\mathcal{F}_j(Q)(X_1, \ldots, X_n) < \tfrac{1}{2} \\
&\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \tfrac{1}{2}\,.
\end{aligned}$$

Then $\mathcal{F}_j \preceq_c \mathcal{F}^*$ entails that

$$\mathcal{F}^*(Q)(X_1, \ldots, X_n) \leq \mathcal{F}_j(Q)(X_1, \ldots, X_n) < \tfrac{1}{2}$$

and $\mathcal{F}_{j'} \preceq_c \mathcal{F}^*$ entails that

$$\mathcal{F}^*(Q)(X_1, \ldots, X_n) \geq \mathcal{F}_j(Q)(X_1, \ldots, X_n) > \tfrac{1}{2}\,,$$

i.e. $\mathcal{F}^*(Q)(X_1, \ldots, X_n) < \frac{1}{2}$ and $\mathcal{F}^*(Q)(X_1, \ldots, X_n) > \frac{1}{2}$, a contradiction. We conclude that there is no such upper bound $\mathcal{F}^*$. It remains to be shown that $(\mathcal{F}_j)_{j\in\mathcal{J}}$ has upper specificity bounds whenever it is specificity consistent. This case is covered by part b. of the theorem, which we prove below.

b. Suppose that $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent. Let us define the QFM $\mathcal{F}_{\text{lub}}$ by

$$\mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases} \tag{232}$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$. We have to show that

1. $\mathcal{F}_{\text{lub}}$ is a $\widetilde{\vee}$-DFS.
2. $\mathcal{F}_{\text{lub}}$ is an upper specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$.
3. $\mathcal{F}_{\text{lub}}$ is the *least* upper specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$.

ad 1. To prove that $\mathcal{F}_{\text{lub}}$ is a $\widetilde{\vee}$-DFS, we first observe that $\mathcal{F}_{\text{lub}}$ can be defined in terms of an aggregation mapping $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$, viz

$$\Psi(f) = \begin{cases} \sup \widehat{f}(\mathcal{J}) & : \quad \widehat{f}(\mathcal{J}) \subseteq [\frac{1}{2}, 1] \\ \inf \widehat{f}(\mathcal{J}) & : \quad \widehat{f}(\mathcal{J}) \subseteq [0, \frac{1}{2}] \\ \frac{1}{2} & : \quad \text{else} \end{cases} \tag{233}$$

for all $f : \mathcal{J} \longrightarrow \mathbf{I}$; this is apparent because the "else" case will never occur since $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent, and hence $\mathcal{F}_{\text{lub}} = \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$, using the abbrevation of Th-29. Let us now show that Ψ satisfies the conditions of this theorem.
Hence suppose that $f : \mathcal{J} \longrightarrow \mathbf{I}$ is a constant mapping, i.e. there exists $c \in \mathbf{I}$ such that $f(j) = c$ for all $j \in \mathcal{J}$. If $c \geq \frac{1}{2}$, then $\Psi(f) = \sup \widehat{f}(\mathcal{J}) = \sup\{c\} = c$ by (233). Similarly if $c \leq \frac{1}{2}$, then $\Psi(f) = \inf \widehat{f}(\mathcal{J}) = \inf\{c\} = c$. Hence Ψ satisfies condition a. of Th-29.
Now suppose that $f : \mathcal{J} \longrightarrow \mathbf{I}$ and $g(j) = 1 - f(j)$ for all $j \in \mathcal{J}$. If $\widehat{f}(\mathcal{J}) \subseteq [\frac{1}{2}, 1]$, then $\widehat{g}(\mathcal{J}) = \{1 - f(j) : j \in \mathcal{J}\} \subseteq [0, \frac{1}{2}]$ and hence

$$\begin{aligned} \Psi(g) &= \inf\{1 - f(j) : j \in \mathcal{J}\} && \text{by (233)} \\ &= 1 - \sup\{f(j) : j \in \mathcal{J}\} \\ &= 1 - \Psi(f)\,. && \text{by (233)} \end{aligned}$$

Similarly if $\widehat{f}(\mathcal{J}) \subseteq [0, \frac{1}{2}]$, then $\widehat{g}(\mathcal{J}) \subseteq [\frac{1}{2}, 1]$ and hence

$$\begin{aligned} \Psi(g) &= \sup\{1 - f(j) : j \in \mathcal{J}\} && \text{by (233)} \\ &= 1 - \inf\{f(j) : j \in \mathcal{J}\} \\ &= 1 - \Psi(f)\,. && \text{by (233)} \end{aligned}$$

Finally if $\widehat{f}(\mathcal{J}) \not\subseteq [\frac{1}{2}, 1]$ and $\widehat{f}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$, then $\widehat{g}(\mathcal{J}) \not\subseteq [\frac{1}{2}, 1]$ and $\widehat{g}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$ as well, and hence $\Psi(g) = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \Psi(f)$. Summarising, Ψ satisfies condition b. of Th-29.
Let us now assume that $f, g : \mathcal{J} \longrightarrow \mathbf{I}$ satisfy $f \leq g$. We shall discern the following cases to prove that condition c. of Th-29 is satisfied, too.

i. $\widehat{f}(\mathcal{J}) \subseteq [0, \frac{1}{2}]$ and $\widehat{g}(\mathcal{J}) \subseteq [0, \frac{1}{2}]$.
Then $\Psi(f) = \inf \widehat{f}(\mathcal{J}) \le \inf \widehat{g}(\mathcal{J}) = \Psi(g)$.

ii. $\widehat{f}(\mathcal{J}) \subseteq [0, \frac{1}{2}]$ and $\widehat{g}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$.
Then $\Psi(f) = \inf \widehat{f}(\mathcal{J}) \le \frac{1}{2} \le \Psi(g)$.

iii. $\widehat{f}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$, $\widehat{f}(\mathcal{J}) \not\subseteq [\frac{1}{2}, 1]$, $\widehat{g}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$, $\widehat{g}(\mathcal{J}) \not\subseteq [\frac{1}{2}, 1]$.
Then $\Psi(f) = \frac{1}{2} = \Psi(g)$.

iv. $\widehat{f}(\mathcal{J}) \not\subseteq [0, \frac{1}{2}]$, $\widehat{f}(\mathcal{J}) \not\subseteq [\frac{1}{2}, 1]$, $\widehat{g}(\mathcal{J}) \subseteq [\frac{1}{2}, 1]$.
Then $\Psi(f) = \frac{1}{2} \le \sup \widehat{g}(\mathcal{J}) = \Psi(g)$.

v. $\widehat{f}(\mathcal{J}) \subseteq [\frac{1}{2}, 1]$, $\widehat{g}(\mathcal{J}) \subseteq [\frac{1}{2}, 1]$.
Then $\Psi(f) = \sup \widehat{f}(\mathcal{J}) \le \sup \widehat{g}(\mathcal{J}) = \Psi(g)$.

We can hence apply Th-29 and conclude that $\mathcal{F}_{\text{lub}}$ is a $\widetilde{\vee}$-DFS.

ad 2. Let us now show that $\mathcal{F}_{\text{lub}}$ is an upper specificity bound on all $\mathcal{F}_j$, $j \in \mathcal{J}$. Hence let $j' \in \mathcal{J}$, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. If $\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \frac{1}{2}$, then $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$ because $(\mathcal{F}_j)_{j \in \mathcal{J}}$ is assumed to be specificity consistent. Hence

$$\begin{aligned} \mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) &= \sup\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} && \text{by definition of } \mathcal{F}_{\text{lub}} \\ &\ge \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) && \text{because } j' \in \mathcal{J} \\ &> \tfrac{1}{2}\,, \end{aligned}$$

i.e. $\mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) \succeq_c \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n)$ by Def. 44. Similarly if $\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) < \frac{1}{2}$, then $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ because $(\mathcal{F}_j)_{j \in \mathcal{J}}$ is assumed to be specificity consistent. Therefore

$$\begin{aligned} \mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) &= \inf\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} && \text{by definition of } \mathcal{F}_{\text{lub}} \\ &\le \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) && \text{because } j' \in \mathcal{J} \\ &< \tfrac{1}{2}\,, \end{aligned}$$

i.e. $\mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) \succeq_c \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n)$ by Def. 44. Finally if $\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) = \frac{1}{2}$, then trivially $\mathcal{F}_{\text{lub}}(Q)(X_1, \ldots, X_n) \succeq_c \frac{1}{2} = \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n)$ by Def. 44. Summarising these results, $\mathcal{F}_{\text{lub}}$ is an upper specificity bound for $(\mathcal{F}_j)_{j \in \mathcal{J}}$.

ad 3. It remains to be shown that $\mathcal{F}_{\text{lub}}$ is the *least* upper specificity bound.
Hence suppose that the $\widetilde{\vee}$-DFS $\mathcal{F}^*$ is an upper specificity bound for $(\mathcal{F}_j)_{j \in \mathcal{J}}$. We will show that $\mathcal{F}_{\text{lub}} \preceq_c \mathcal{F}^*$. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Suppose there is some $j' \in \mathcal{J}$ such that $\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \frac{1}{2}$. Then by our assumption that $\mathcal{F}^*$ be an upper specificity bound,

$$\mathcal{F}^*(Q)(X_1, \ldots, X_n) \ge \mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \tfrac{1}{2}\,, \tag{234}$$

cf. Def. 44. We further conclude from $\mathcal{F}_{j'}(Q)(X_1, \ldots, X_n) > \frac{1}{2}$ and the fact that all $\mathcal{F}_j$ are specificity consistent that $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$, i.e. $\mathcal{F}(Q)(X_1, \ldots, X_n) \ge \frac{1}{2}$ for all $j \in \mathcal{J}$. Now let j an arbitrary choice of $j \in \mathcal{J}$. Because $\mathcal{F}^*$ is an upper specificity bound, we know that

$\mathcal{F}^*(Q)(X_1,\ldots,X_n) \succeq_c \mathcal{F}_j(Q)(X_1,\ldots,X_n)$.
Because $\mathcal{F}_j(Q)(X_1,\ldots,X_n) \geq \frac{1}{2}$ and $\mathcal{F}^*(Q)(X_1,\ldots,X_n) > \frac{1}{2}$ by (234), this means that

$$\mathcal{F}^*(Q)(X_1,\ldots,X_n) \geq \mathcal{F}_j(Q)(X_1,\ldots,X_n)\,,$$

see (9). Because $j \in \mathcal{J}$ was chosen arbitrarily, we conclude that

$$\begin{aligned} \mathcal{F}^*(Q)(X_1,\ldots,X_n) &\geq \sup\{\mathcal{F}_j(Q)(X_1,\ldots,X_n) : j \in \mathcal{J}\} \\ &= \sup R_{Q,X_1,\ldots,X_n} && \text{by definition of } R_{Q,X_1,\ldots,X_n}\text{, see Def. 79} \\ &= \mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n) && \text{by (232)} \end{aligned}$$

i.e. by (9),

$$\mathcal{F}^*(Q)(X_1,\ldots,X_n) \succeq_c \mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n)\,.$$

The case that $\mathcal{F}_{j'}(Q)(X_1,\ldots,X_n) < \frac{1}{2}$ for some $j' \in \mathcal{J}$ is proven analogously. Finally if $\mathcal{F}_j(Q)(X_1,\ldots,X_n) = \frac{1}{2}$ for all $j \in \mathcal{J}$, then $\mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n) = \frac{1}{2}$ and trivially

$$\mathcal{F}^*(Q)(X_1,\ldots,X_n) \succeq_c \tfrac{1}{2} = \mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n)\,,$$

see (9). Summarising, $\mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n) \preceq_c \mathcal{F}^*(Q)(X_1,\ldots,X_n)$ for all Q and $X_1,\ldots,X_n$. Hence by Def. 44. $\mathcal{F}_{\text{lub}} \preceq_c \mathcal{F}^*$, as desired.

E.2 Proof of Theorem 75

Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$ are chosen such that $Q \sim_{(X_1,\ldots,X_n)} Q'$. Recalling that by Def. 66,

$$\mathcal{T}_1(X_i) = \{Y \in \mathcal{P}(E) : (X_i)_{\geq 1} \subseteq Y \subseteq (X_i)_{>0}\} \tag{235}$$

for $i = 1,\ldots,n$, it is apparent that

$$(Y \cup (X_i)_{\geq 1}) \cap (X_i)_{>0} = Y$$

for all $Y \in \mathcal{T}_1(X_i)$, and

$$(Y \cup (X_i)_{\geq 1}) \cap (X_i)_{>0} \in \mathcal{T}_1(X_i)$$

for all $Y \in \mathcal{P}(E)$. Hence by Def. 83, $Q \sim_{(X_1,\ldots,X_n)} Q'$ is equivalent to

$$\begin{aligned} & Q((Y_1 \cup (X_1)_{\geq 1}) \cap (X_1)_{>0}, \\ & \qquad \ldots, \\ & \qquad (Y_n \cup (X_n)_{\geq 1}) \cap (X_n)_{>0}) \\ = \ & Q'((Y_1 \cup (X_1)_{\geq 1}) \cap (X_1)_{>0}, \\ & \qquad \ldots \\ & \qquad (Y_n \cup (X_n)_{\geq 1}) \cap (X_n)_{>0}) \end{aligned} \tag{236}$$

for all $Y_1,\ldots,Y_n \in \mathcal{P}(E)$. Let us now define $Q'' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$Q''(Y_1,\ldots,Y_n) = Q((Y_1 \cup (X_1)_{\geq 1}) \cap (X_1)_{>0},\ldots,(Y_n \cup (X_n)_{\geq 1}) \cap (X_n)_{>0}) \tag{237}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. It is then apparent from (236) that

$$Q''(Y_1, \ldots, Y_n) = Q'((Y_1 \cup (X_1)_{\geq 1}) \cap (X_1)_{>0}, \ldots, (Y_n \cup (X_n)_{\geq 1}) \cap (X_n)_{>0}) \tag{238}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.
Let us further observe that

$$(X_i)_{\geq 1} \subseteq X_i \subseteq (X_i)_{>0}$$

and hence

$$(X_i \mathbin{\widetilde{\cup}} (X_i)_{\geq 1}) \mathbin{\widetilde{\cap}} (X_i)_{>0} = X_i \tag{239}$$

for $i = 1, \ldots, n$. Therefore

$$\begin{aligned}
&\mathcal{F}(Q)(X_1, \ldots, X_n) \\
&= \mathcal{F}(Q)((X_1 \mathbin{\widetilde{\cup}} (X_1)_{\geq 1}) \mathbin{\widetilde{\cap}} (X_1)_{>0}, \\
&\qquad \ldots \\
&\qquad (X_n \mathbin{\widetilde{\cup}} (X_n)_{\geq 1}) \mathbin{\widetilde{\cap}} (X_n)_{>0}) && \text{by (239)} \\
&= \mathcal{F}(Q'')(X_1, \ldots, X_n) && \text{by (DFS 4), (DFS 6), Th-4 and (237)} \\
&= \mathcal{F}(Q)'((X_1 \mathbin{\widetilde{\cup}} (X_1)_{\geq 1}) \mathbin{\widetilde{\cap}} (X_1)_{>0}, \\
&\qquad \ldots \\
&\qquad (X_n \mathbin{\widetilde{\cup}} (X_n)_{\geq 1}) \mathbin{\widetilde{\cap}} (X_n)_{>0}) && \text{by (DFS 4), (DFS 6), Th-4 and (238)} \\
&= \mathcal{F}(Q)'(X_1, \ldots, X_n)\,, && \text{by (239)}
\end{aligned}$$

as desired.

E.3 Proof of Theorem 76

Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are convex in the i-th argument, $i \in \{1, \ldots, n\}$. Then by Def. 85,

$$Q(X_1, \ldots, X_n) \geq \min(Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) \tag{240}$$

$$Q'(X_1, \ldots, X_n) \geq \min(Q'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), Q'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))\,, \tag{241}$$

for all $X_1, \ldots, X_n, X_i', X_i'' \in \mathcal{P}(E)$ such that $X_i' \subseteq X_i \subseteq X_i''$. Hence

$$\begin{aligned}
&(Q \wedge Q')(X_1, \ldots, X_n) \\
&= \min(Q(X_1, \ldots, X_n), \\
&\quad Q'(X_1, \ldots, X_n)) && \text{by definition of } Q \wedge Q' \\
&\geq \min(\min(Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)), \\
&\quad \min(Q'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad Q'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))) && \text{by (240), (241)} \\
&= \min(\min(Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad Q'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)), \\
&\quad \min(Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n), \\
&\qquad Q'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))) \\
&= \min((Q \wedge Q')(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\quad (Q \wedge Q')(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)), && \text{by definition of } Q \wedge Q'
\end{aligned}$$

i.e. $Q \wedge Q'$ is convex in its i-th argument.

E.4 Proof of Theorem 77

Suppose $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ are convex in the i-th argument, $i \in \{1, \ldots, n\}$. Then by Def. 85,

$$\widetilde{Q}(X_1, \ldots, X_n) \geq \min(\widetilde{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \widetilde{Q}(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) \tag{242}$$

$$\widetilde{Q}'(X_1, \ldots, X_n) \geq \min(\widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)), \tag{243}$$

for all $X_1, \ldots, X_n, X_i', X_i'' \in \widetilde{\mathcal{P}}(E)$ such that $X_i' \subseteq X_i \subseteq X_i''$. Hence

$$\begin{aligned}
&(\widetilde{Q} \wedge \widetilde{Q}')(X_1, \ldots, X_n) \\
&= \min(\widetilde{Q}(X_1, \ldots, X_n), \\
&\quad \widetilde{Q}'(X_1, \ldots, X_n)) && \text{by definition of } \widetilde{Q} \wedge \widetilde{Q}' \\
&\geq \min(\min(\widetilde{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad \widetilde{Q}(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)), \\
&\quad \min(\widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad \widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))) && \text{by (242), (243)} \\
&= \min(\min(\widetilde{Q}(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\qquad \widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)), \\
&\quad \min(\widetilde{Q}(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n), \\
&\qquad \widetilde{Q}'(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))) \\
&= \min((\widetilde{Q} \wedge \widetilde{Q}')(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\
&\quad (\widetilde{Q} \wedge \widetilde{Q}')(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)), && \text{by definition of } \widetilde{Q} \wedge \widetilde{Q}'
\end{aligned}$$

i.e. $\widetilde{Q} \wedge \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is convex in its i-th argument.

E.5 Proof of Theorem 78

a. Suppose $Q^+, Q^- : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are n-ary semi-fuzzy quantifiers and $i \in \{1, \ldots, n\}$. Further suppose that Q^+ is nondecreasing in its i-th argument and that Q^- is nonincreasing in its i-th argument. Define the semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by $Q = Q^+ \wedge Q^-$. We have to show that Q is convex in the i-th argument. Hence let $X_1, \ldots, X_n, X_i', X_i'' \in \mathcal{P}(E)$ such that

$$X_i' \subseteq X_i \subseteq X_i'' . \tag{244}$$

Because Q^+ is nondecreasing in the i-th argument, we conclude from (244) that

$$\begin{aligned}&Q^+(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)\\ &\leq Q^+(X_1, \ldots, X_n)\\ &\leq Q^+(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n) ,\end{aligned}$$

i.e.

$$Q^+(X_1, \ldots, X_n) \;\geq\; \min(\begin{array}{l} Q^+(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\ Q^+(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) . \end{array} \tag{245}$$

Similarly, we conclude from (244) and the fact that Q^- is nonincreasing that

$$\begin{aligned}&Q^-(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)\\ &\geq Q^-(X_1, \ldots, X_n)\\ &\geq Q^-(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n) ,\end{aligned}$$

i.e.

$$Q^-(X_1, \ldots, X_n) \;\geq\; \min(\begin{array}{l} Q^-(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\ Q^-(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) . \end{array} \tag{246}$$

Therefore

$$\begin{aligned}
&Q(X_1, \ldots, X_n)\\
&= \min(Q^+(X_1, \ldots, X_n), Q^-(X_1, \ldots, X_n)) && \text{by definition of } Q = Q^+ \wedge Q^-\\
&\geq \min(\min(Q^+(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),\\
&\qquad\qquad Q^+(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)),\\
&\qquad \min(Q^-(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),\\
&\qquad\qquad Q^-(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))) && \text{by (245), (246)}\\
&= \min(\min(Q^+(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),\\
&\qquad\qquad Q^-(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n)),\\
&\qquad \min(Q^+(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n),\\
&\qquad\qquad Q^-(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)))\\
&= \min(Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n),\\
&\qquad Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n)) , && \text{because } Q = Q^+ \wedge Q^-
\end{aligned}$$

i.e. $Q = Q^+ \wedge Q^-$ is convex in the i-th argument, as desired.

b. To show that the reverse direction of the claimed equivalence also holds, let us suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier which is convex in the i-th argument, where $i \in \{1, \ldots, n\}$. Let us define $Q^+, Q^- : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$Q^+ = \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : Z \subseteq X_i\} \tag{247}$$

and

$$Q^- = \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : X_i \subseteq Z \subseteq E\} \tag{248}$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$.
To prove that Q^+ is nondecreasing in the i-th argument, let us consider a choice of arguments $X_1, \ldots, X_n, X'_i \in \mathcal{P}(E)$ such that $X_i \subseteq X'_i$. Then

$$\begin{aligned} Q^+(X_1, \ldots, X_n) &= \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : Z \subseteq X_i\} && \text{by (247)} \\ &\le \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : Z \subseteq X'_i\} && \text{because } X_i \subseteq X'_i \\ &= Q^+(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n) && \text{by (247)}\,. \end{aligned}$$

Similarly,

$$\begin{aligned} Q^-(X_1, \ldots, X_n) &= \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : X_i \subseteq Z \subseteq E\} && \text{by (248)} \\ &\ge \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : X'_i \subseteq Z \subseteq E\} && \text{because } X_i \subseteq X'_i \\ &= Q^-(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n) && \text{by (248)}\,. \end{aligned}$$

whenever $X_1, \ldots, X_n, X'_i \in \mathcal{P}(E)$ such that $X_i \subseteq X'_i$, i.e. Q^- is nonincreasing in the i-th argument. It remains to be shown that $Q = Q^+ \wedge Q^-$. Apparently

$$\begin{aligned} &Q^+(X_1, \ldots, X_n) \\ &= \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : Z \subseteq X_i\} && \text{by (247)} \\ &\ge Q(X_1, \ldots, X_n) && \text{because } X_i \in \{Z : Z \subseteq X_i\} \end{aligned}$$

and

$$\begin{aligned} &Q^-(X_1, \ldots, X_n) \\ &= \sup\{Q(X_1, \ldots, X_{i-1}, Z, X_{i+1}, X_n) : X_i \subseteq Z \subseteq E\} && \text{by (248)} \\ &\ge Q(X_1, \ldots, X_n)\,, && \text{because } X_i \in \{Z : X_i \subseteq Z \subseteq E\} \end{aligned}$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$. Therefore

$$Q \le Q^+ \wedge Q^-\,. \tag{249}$$

To see that $Q \ge Q^+ \wedge Q^-$, let $\varepsilon > 0$. By (247), there exists $X'_i \subseteq X_i$ such that

$$Q(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n) > Q^+(X_1, \ldots, X_n) - \varepsilon\,. \tag{250}$$

Similarly, we conclude from (248) that there exists $X''_i \supseteq X_i$ such that

$$Q(X_1, \ldots, X_{i-1}, X''_i, X_{i+1}, \ldots, X_n) > Q^-(X_1, \ldots, X_n) - \varepsilon\,. \tag{251}$$

Therefore

$$\begin{aligned}
&Q(X_1, \ldots, X_n) \\
&\geq \min(Q(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n), \\
&\qquad Q(X_1, \ldots, X_{i-1}, X'_i, X_{i+1}, \ldots, X_n)) && \text{because } Q \text{ convex in } i\text{-th argument} \\
&> \min(Q^+(X_1, \ldots, X_n) - \varepsilon, Q^-(X_1, \ldots, X_n) - \varepsilon) && \text{by (250), (251)} \\
&= \min(Q^+(X_1, \ldots, X_n), Q^-(X_1, \ldots, X_n)) - \varepsilon\,.
\end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitarily, this means that

$$Q(X_1, \ldots, X_n) \geq \min(Q^+(X_1, \ldots, X_n), Q^-(X_1, \ldots, X_n))\,,$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$, i.e.

$$Q \geq Q^+ \wedge Q^-\,. \tag{252}$$

Combining (249) and (252) yields the desired result $Q = Q^+ \wedge Q^-$.

E.6 Proof of Theorem 79

The proof of this theorem is entirely analogous to that of Th-79. In this case, the arguments X_i of the quantifiers are fuzzy subsets $X_i \in \widetilde{\mathcal{P}}(E)$, and '$\subseteq$' is the fuzzy inclusion relation.

E.7 Proof of Theorem 80

Let us assume that $E \neq \varnothing$ is a base set of cardinality $|E| \geq 3$. Then there exist pairwise distinct elements $a, b, c \in E$. We will define a semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ by

$$Q(Y) = \begin{cases} 1 & : \quad Y = \{a, b\} \vee Y = \{c\} \\ 0 & : \quad \text{else} \end{cases} \tag{253}$$

for all $Y \in \mathcal{P}(E)$. It is apparent from Th-78 that Q is convex, because Q can be decomposed into a conjunction $Q = Q^+ \wedge Q^-$, where $Q^+ : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nondecreasing and $Q^- : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nonincreasing. Q^+ is defined by

$$Q^+(Y) = \begin{cases} 1 & : \quad \{a, b\} \subseteq Y \vee \{c\} \subseteq Y \\ 0 & : \quad \text{else} \end{cases}$$

and Q^- is defined by

$$Q^-(Y) = \begin{cases} 1 & : \quad Y \subseteq \{a, b\} \vee Y \subseteq \{c\} \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y \in \mathcal{P}(E)$. It is obvious from these definitions that Q^+ is nondecreasing, Q^- is nonincreasing, and $Q = Q^+ \wedge Q^-$, i.e. Q is convex.

Let us now consider $X', X, X'' \in \widetilde{\mathcal{P}}(E)$, defined by

$$\mu_{X'}(e) = \begin{cases} \frac{1}{2} & : \quad e = c \\ 0 & : \quad \text{else} \end{cases}$$

$$\mu_{X}(e) = \begin{cases} \frac{1}{2} & : \quad e = c \\ 1 & : \quad e = a \\ 0 & : \quad \text{else} \end{cases}$$

$$\mu_{X''}(e) = \begin{cases} \frac{1}{2} & : \quad e = c \\ 1 & : \quad e = a \vee e = b \\ 0 & : \quad \text{else} \end{cases}$$

for all $e \in E$. Clearly

$$X' \subseteq X \subseteq X'' . \tag{254}$$

Considering $\mathcal{F}(Q)(X')$, let us observe that $\mathcal{T}_1(X') = \{\varnothing, \{c\}\}$ and $Q(\varnothing) = 0 = \exists(\varnothing)$, $Q(\{c\}) = 1 = \exists(\{c\})$. Therefore $Q \sim_{(X')} \exists$, and

$$\mathcal{F}(Q)(X') = \mathcal{F}(\exists)(X') > 0 \tag{255}$$

because $\mathcal{F}$ is required to be contextual and to satisfy assumption b. of the theorem.
Similarly, $\mathcal{T}_1(X) = \{\{a\}, \{a, c\}\}$, i.e. $Q(\{a\}) = 0 = \mathbb{O}(\{a\})$ and $Q(\{a, c\}) = 0 = \mathbb{O}(\{a, c\})$ and hence $Q \sim_{(X)} \mathbb{O}$. Therefore

$$\mathcal{F}(Q)(X) = \mathcal{F}(\mathbb{O})(X) = 0 \tag{256}$$

because $\mathcal{F}$ is required to be contextual and to satisfy assumption a. of the theorem.
Finally, $\mathcal{T}_1(X'') = \{\{a, b\}, \{a, b, c\}\}$, i.e. $Q(\{a, b\}) = 1 = (\sim\forall)(\{a, b\})$ and $Q(\{a, b, c\}) = 0 = (\sim\forall)(\{a, b, c\})$ and hence $Q \sim_{(X'')} \sim\forall$. We conclude that

$$\mathcal{F}(Q)(X'') = \mathcal{F}(\sim\forall)(X'') > 0 \tag{257}$$

because $\mathcal{F}$ is required to be contextual and to satisfy assumption c. of the theorem.
Summarizing, we obtain that

$$\begin{aligned} \mathcal{F}(Q)(X) &= 0 && \text{by (256)} \\ &< \min(\mathcal{F}(Q)(X'), \mathcal{F}(Q)(X'')) && \text{by (255), (257)} \end{aligned}$$

although $X' \subseteq X \subseteq X''$ by (254), i.e. $\mathcal{F}(Q)$ fails to be convex in its argument.

E.8 Proof of Theorem 81

Suppose $\mathcal{F}$ is a DFS. It is sufficient to show that the preconditions of Th-80 are satisfied. By Th-75, we already know that $\mathcal{F}$ is contextual. Now assume $E \neq \varnothing$ is some base set.
Let us consider the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$ defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, and let

$X \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
0 &= \mathbb{O}(\varnothing) && \text{by definition of } \mathbb{O} \\
&= \mathcal{F}(\mathbb{O})(\varnothing) && \text{by (Z-1)} \\
&\le \mathcal{F}(\mathbb{O})(X) && \text{because } \mathbb{O} \text{ nondecreasing, } \varnothing \subseteq X \\
&\le \mathcal{F}(\mathbb{O})(E) && \text{because } \mathbb{O} \text{ nondecreasing, } X \subseteq E \\
&= \mathbb{O}(E) && \text{by (Z-1)} \\
&= 0\,. && \text{by definition of } \mathbb{O}
\end{aligned}$$

Hence $\mathcal{F}(\mathbb{O})(X) = 0$, i.e. condition a. of Th-80 holds.
By Th-21, we know that $\mathcal{F}(\exists)$ is an S-quantifier; hence condition b. of Th-80 is fulfilled by Th-23.
Similarly, we know from Th-21 that $\mathcal{F}(\forall)$ is a T-quantifier. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists $e \in E$ such that $\mu_X(e) < 1$, then we conclude from Th-22 that $\mathcal{F}(\forall)(X) < 1$. Because $\widetilde{\neg}$ is a strong negation operator and $\mathcal{F}$ is compatible with external negation by (DFS 3), we conclude from $\widetilde{\neg}\,\forall = {\sim}\forall$ that $\mathcal{F}({\sim}\forall) = \mathcal{F}(\widetilde{\neg}\,\forall) = \widetilde{\neg}\,\mathcal{F}(\forall) > 0$, which is apparent because $\widetilde{\neg}$ is a strong negation, i.e. $x < 1$ implies that $\widetilde{\neg}\,x > 0$. This proves that condition c. of Th-80 is satisfied, too.
We can hence apply Th-80 to obtain the desired result.

E.9 Proof of Theorem 82

The claim of the theorem is apparent from repeated application of [9, Th-5,p.28] and (DFS 4).

E.10 Proof of Theorem 83

Let us assume that E is a base set of cardinality $|E| \ge 2$. We shall define $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ by

$$Q(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 0 \wedge |Y_2| = 0 \\ 1 & : \quad |Y_1| \ne 0 \wedge |Y_2| = 2 \\ 0 & : \quad \text{else} \end{cases} \tag{258}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. It is apparent from Def. 31 that Q is quantitative because $|\widehat{\beta}(Y_1)| = |Y_1|$ and $|\widehat{\beta}(Y_2)| = |Y_2|$ for every automorphism $\beta : E \longrightarrow E$ and every choice of $Y_1, Y_2 \in \mathcal{P}(E)$.
In addition, Q can be decomposed into a conjunction $Q = Q^+ \wedge Q^-$ of semi-fuzzy quantifiers $Q^+, Q^- : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$, where

$$Q^+(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 0 \\ 1 & : \quad |Y_1| \ne 0 \wedge |Y_2| \ge 2 \\ 0 & : \quad \text{else} \end{cases}$$

$$Q^-(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1| = 0 \wedge |Y_2| = 0 \\ 1 & : \quad |Y_1| \ne 0 \wedge |Y_2| \le 2 \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. It is obvious from these definitions that Q^+ is nondecreasing in the second argument and that Q^- is nonincreasing in the second argument. By Th-78, Q is convex in the second argument.
Because $|E| \ge 2$, we can choose $a, b \in E$ such that $a \ne b$. Let $X_1 \in \widetilde{\mathcal{P}}(E)$ the fuzzy subset defined by

$$\mu_{X_1}(e) = \begin{cases} \frac{1}{2} & : \quad e = a \\ 0 & : \quad \text{else} \end{cases}$$

for all $e \in E$ and let $X_2' = \varnothing$, $X_2 = \{a\}$, $X_2'' = \{a, b\}$. Clearly

$$X_2' \subseteq X_2 \subseteq X_2'' . \tag{259}$$

Considering X_1, X_2', we have $\mathcal{T}_1(X_1) = \{\varnothing, \{a\}\}$ and $\mathcal{T}_1(X_2') = \{\varnothing\}$. Observing that $Q(\varnothing, \varnothing) = 1 = (\sim\exists)(\varnothing)$ and $Q(\{a\}, \varnothing) = 0 = (\sim\exists)(\{a\})$, we conclude that $Q \sim_{(X_1, X_2')} Q'$, where $Q' : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is defined by

$$Q'(Y_1, Y_2) = (\sim\exists)(Y_1) ,$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. Therefore

$$\mathcal{F}(Q)(X_1, X_2') = \mathcal{F}(Q')(X_1, X_2') = \mathcal{F}(\sim\exists)(X_1) > 0 , \tag{260}$$

because $\mathcal{F}$ is contextual and compatible with cylindrical extensions, and because $\mathcal{F}$ satisfies assumption c. of the theorem.
Considering X_1, X_2, we have $\mathcal{T}_1(X_1) = \{\varnothing, \{a\}\}$ and $\mathcal{T}_1(X_2) = \{\{a\}\}$. In this case, $Q(\varnothing, \{a\}) = 0 = \mathbb{O}(\varnothing)$ and $Q(\{a\}, \{a\}) = 0 = \mathbb{O}(\{a\})$, i.e. $Q \sim_{(X_1, X_2)} Q''$, where $Q'' : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is defined by

$$Q'(Y_1, Y_2) = \mathbb{O}(Y_1) ,$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. Therefore

$$\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q'')(X_1, X_2) = \mathcal{F}(\mathbb{O})(X_1) = 0 \tag{261}$$

because $\mathcal{F}$ is contextual and compatible with cylindrical extensions, and because $\mathcal{F}$ satisfies assumption a. of the theorem.
Finally, let us consider X_1 and X_2''. Then $\mathcal{T}_1(X_1) = \{\varnothing, \{a\}\}$ and $\mathcal{T}_1(X_2'') = \{\{a, b\}\}$, i.e.

$$Q(\varnothing, \{a, b\}) = 0 = \exists(\varnothing) \qquad Q(\{a\}, \{a, b\}) = 1 = \exists(\{a\})$$

and hence $Q \sim_{(X_1, X_2'')} Q'''$, where $Q''' : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is defined by

$$Q'''(Y_1, Y_2) = \exists(Y_1) ,$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$. Therefore

$$\mathcal{F}(Q)(X_1, X_2'') = \mathcal{F}(Q'')(X_1, X_2'') = \mathcal{F}(\exists)(X_1) > 0 \tag{262}$$

because $\mathcal{F}$ is contextual and compatible with cylindrical extensions, and because $\mathcal{F}$ satisfies assumption b. of the theorem. Summarizing, we have

$$\begin{aligned} Q(X_1, X_2) &= 0 && \text{by (261)} \\ &< \min(Q(X_1, X_2'), Q(X_1, X_2'')) && \text{by (260) and (262)} \end{aligned}$$

although $X_2' \subseteq X_2 \subseteq X_2''$ by (259), i.e. $\mathcal{F}(Q)$ is not convex in its second argument.

E.11 Proof of Theorem 84

Suppose $\mathcal{F}$ is a DFS. Let us consider the preconditions of Th-83. We already know from Th-75 that $\mathcal{F}$ is contextual. We also know from Th-82 that $\mathcal{F}$ is compatible with cylindrical extensions. As concerns the conditions a. to c. of Th-83, we observe that Th-83.a and Th-83.b are identical to Th-80.a and Th-80.b, respectively. We have already shown in the proof of Th-81 that every DFS fulfills these conditions. Let us now show that $\mathcal{F}$ also fulfills the remaining condition c. of Th-83. Suppose $E \neq \varnothing$ is a base set and $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E with the property that there is some $e \in E$ such that $\mu_X(e') = 0$ for all $e' \in E \setminus \{e\}$ (in particular, $X \cap \{e\} = X$) and that $\mu_X(e) < 1$. By Th-21, we know that $\mathcal{F}(\exists)$ is an S-quantifier. By Def. 37.b, $\mathcal{F}(\exists)(X) = \mu_X(e) < 1$. Therefore

$$\begin{aligned}
\mathcal{F}(\sim\exists)(X) &= \mathcal{F}(\widetilde{\neg}\,\exists)(X) && \text{because } \widetilde{\neg} \text{ strong negation, } \sim\exists = \widetilde{\neg}\,\exists \\
&= \widetilde{\neg}\,\mathcal{F}(\exists)(X) && \text{by (DFS 3)} \\
&= \widetilde{\neg}\,\mu_X(e) && \text{by above reasoning} \\
&> 0\,, && \text{because } \widetilde{\neg} \text{ strong negation and } \mu_X(e) < 1
\end{aligned}$$

i.e. condition c. of Th-83 holds. We can hence apply Th-83 to obtain the desired result.

E.12 Proof of Theorem 85

Suppose $E \neq \varnothing$ is a finite base set, $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is quantitative, and $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is defined by $Q = Q'\cap$. Further suppose that $\mathcal{F}$ is a DFS which weakly preserves convexity. Suppose Q is convex in its second argument. Then Q' is convex in its argument as well. To see this, let $Y', Y, Y'' \in \mathcal{P}(E)$ such that $Y' \subseteq Y \subseteq Y''$. Then

$$\begin{aligned}
Q'(Y) &= Q'(E \cap Y) \\
&= Q(E, Y) && \text{because } Q = Q'\cap \\
&\geq \min(Q(E, Y'), Q(E, Y'')) && \text{by convexity of } Q \text{ in 2nd arg} \\
&= \min(Q'(E \cap Y'), Q'(E \cap Y'')) && \text{because } Q = Q'\cap \\
&= \min(Q'(Y'), Q'(Y''))\,,
\end{aligned}$$

i.e. Q' is convex in its argument.
Now let $X_1, X_2', X_2, X_2'' \in \widetilde{\mathcal{P}}(E)$ such that $X_2' \subseteq X_2 \subseteq X_2''$. Then

$$\begin{aligned}
&\mathcal{F}(Q)(X_1, X_2) \\
&= \mathcal{F}(Q'\cap)(X_1, X_2) && \text{because } Q = Q'\cap \\
&= \mathcal{F}(Q')(X_1 \,\widetilde{\cap}\, X_2) && \text{by (DFS 6)} \\
&\geq \min(\mathcal{F}(Q')(X_1 \,\widetilde{\cap}\, X_2'), \mathcal{F}(Q')(X_1 \,\widetilde{\cap}\, X_2'')) && \text{by convexity of } Q',\ \mathcal{F} \text{ weakly convex} \\
&= \min(\mathcal{F}(Q'\cap)(X_1, X_2'), \mathcal{F}(Q'\cap)(X_1, X_2'')) && \text{by (DFS 6)} \\
&= \min(\mathcal{F}(Q)(X_1, X_2'), \mathcal{F}(Q)(X_1, X_2''))\,, && \text{because } Q = Q'\cap
\end{aligned}$$

i.e. $\mathcal{F}(Q)$ is convex in the second argument, as desired.
Because of the symmetrical definition of Q (i.e. $Q = Q\tau_1$), the proof that convexity in the first argument is also preserved is completely analogous.

F Proofs of Theorems in Chapter 7

F.1 Proof of Theorem 86

Suppose $\mathcal{B}'_1, \mathcal{B}'_2 : \mathbb{H} \longrightarrow \mathbf{I}$ are given. Further suppose that $\mathcal{B}_1, \mathcal{B}_2 \in$ BB are the mappings associated with $\mathcal{B}'_1$ and $\mathcal{B}'_2$, respectively, according to equation (23), and $\mathcal{M}_{\mathcal{B}_1}$, $\mathcal{M}_{\mathcal{B}_2}$ are the corresponding QFMs defined by Def. 69. The theorem states that the condition

$$\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2} \tag{263}$$

is equivalent to

$$\mathcal{B}'_1 \leq \mathcal{B}'_2 \,. \tag{264}$$

a. (263) **entails** (264)**.**
Suppose $\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2}$. Further suppose that $f \in \mathbb{H}$ is some mapping; we have to show that $\mathcal{B}'_1(f) \leq \mathcal{B}'_2(f)$.
To this end, let us define $g : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(\gamma) = \tfrac{1}{2} + \tfrac{1}{2} f(\gamma) \,, \tag{265}$$

for all $\gamma \in \mathbf{I}$. By Th-41, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ such that

$$Q_\gamma(X) = g(\gamma) \tag{266}$$

for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned}
\mathcal{B}'_1(f) &= \mathcal{B}'_1(2(\tfrac{1}{2} + \tfrac{1}{2}f) - 1) \\
&= \mathcal{B}'_1(2g - 1) && \text{by (265)} \\
&= 2(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_1(2g-1)) - 1 \\
&= 2\mathcal{B}_1(g) - 1 && \text{by (23)} \\
&= 2\mathcal{B}_1((Q_\gamma(X))_{\gamma\in\mathbf{I}}) - 1 && \text{by (266)} \\
&= 2\mathcal{M}_{\mathcal{B}_1}(Q)(X) - 1 && \text{by Def. 69} \\
&\leq 2\mathcal{M}_{\mathcal{B}_2}(Q)(X) - 1 && \text{by assumption, (263) holds} \\
& && \mathcal{M}_{\mathcal{B}_1}(Q)(X) \geq \tfrac{1}{2}, \mathcal{M}_{\mathcal{B}_2}(Q)(X) \geq \tfrac{1}{2} \text{ because } g \in \mathbb{B}^+ \\
&= 2\mathcal{B}_2((Q_\gamma(X))_{\gamma\in\mathbf{I}}) - 1 && \text{by Def. 69} \\
&= 2\mathcal{B}_2(g) - 1 && \text{by (266)} \\
&= 2(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_2(2g-1)) - 1 && \text{by (23)} \\
&= \mathcal{B}'_2(2(\tfrac{1}{2} + \tfrac{1}{2}f) - 1)) && \text{by (265)} \\
&= \mathcal{B}'_2(f) \,.
\end{aligned}$$

b. (264) **entails** (263)**.**
Suppose $\mathcal{B}'_1 \leq \mathcal{B}'_2$. We have to show that $\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2}$. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ fuzzy argument sets. Let us abbreviate

$$f(\gamma) = Q_\gamma(X_1, \ldots, X_n) \tag{267}$$

for all $\gamma \in \mathbf{I}$.

i. If $f \in \mathbb{B}^+$, then

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) &= \mathcal{B}_1(f) && \text{by Def. 69, (267)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}_1'(2f-1) && \text{by (23), } f \in \mathbb{B}^+\\
&\le \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}_2'(2f-1) && \text{by assumption, } \mathcal{B}_1' \le \mathcal{B}_2'\\
&= \mathcal{B}_2(f) && \text{by (23), } f \in \mathbb{B}^+\\
&= \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n)\,. && \text{by Def. 69, (267)}
\end{aligned}$$

Hence $\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) \preceq_c \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n)$ by Def. 44 because both are in the range $[\frac{1}{2},1]$.

ii. $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$. Then

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) &= \mathcal{B}_1(\mathrm{c}_{\frac{1}{2}}) && \text{by Def. 69, (267)}\\
&= \tfrac{1}{2} && \text{by (23)}\\
&= \mathcal{B}_2(\mathrm{c}_{\frac{1}{2}}) && \text{by (23)}\\
&= \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n)\,. && \text{by Def. 69, (267)}
\end{aligned}$$

In particular, $\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) \preceq_c \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n)$ by Def. 44.

iii. $f \in \mathbb{B}^-$. Then

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) &= \mathcal{B}_1(f) && \text{by Def. 69, (267)}\\
&= \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}_1'(1-2f) && \text{by (23), } f \in \mathbb{B}^-\\
&\ge \tfrac{1}{2} - \tfrac{1}{2}\mathcal{B}_2'(1-2f) && \text{because } \mathcal{B}_1' \le \mathcal{B}_2' \text{ by assumption}\\
&= \mathcal{B}_2(f) && \text{by (23), } f \in \mathbb{B}^-\\
&= \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n) && \text{by Def. 69, (267)}
\end{aligned}$$

i.e. $\mathcal{M}_{\mathcal{B}_1}(Q)(X_1,\ldots,X_n) \preceq_c \mathcal{M}_{\mathcal{B}_2}(Q)(X_1,\ldots,X_n)$ by Def. 44 because both are in the range $[0,\frac{1}{2}]$.

F.2 Proof of Theorem 87

Lemma 72

Suppose $f \in \mathbb{H}$ is some mapping.

a. If $\widehat{f}((0,1]) = \{0\}$, then $f_*^1 = 0$.

b. If $\widehat{f}((0,1]) \neq \{0\}$, then $(f^\sharp)_*^1 = (f^\flat)_*^1$.

Proof

a. $\widehat{f}((0,1]) = \{0\}$. Then $f(\gamma) = 0$ for all $\gamma \in (0,1]$ and hence

$$\begin{aligned}
f_*^1 &= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (22)}\\
&\le \sup\{0\} && \text{because } \{\gamma : f(\gamma) = 1\} \subseteq \{0\}\\
&= 0\,,
\end{aligned}$$

i.e. $f_*^1 = 0$ (because it is nonnegative by definition).

b. $\widehat{f}((0,1]) \neq \{0\}$. In this case,

$$\begin{aligned}
&(f^\sharp)^1_* \\
&= \sup\{\gamma \in \mathbf{I} : f^\sharp(\gamma) = 1\} && \text{by (22)} \\
&= \sup\{\gamma \in [0,1) : f^\sharp(\gamma) = 1\} \cup \{1 : f^\sharp(1) = 1\} \\
&= \sup\{\gamma \in [0,1) : f^\sharp(\gamma) = 1\} && \text{because } f^\sharp(1) = 1 \text{ implies } f^\sharp(\gamma) = 1 \text{ for } \gamma \in \mathbf{I} \\
&= \sup\{\gamma \in [0,1) : \lim_{\gamma' \to \gamma^+} f(\gamma') = 1\} && \text{by Def. 71} \\
&= \sup\{\gamma \in [0,1) : \sup\{f(\gamma') : \gamma' > \gamma\} = 1\} && \text{by Th-43} \\
&= \sup\{\gamma \in [0,1] : \sup\{f(\gamma') : \gamma' > \gamma\} = 1\}
\end{aligned}$$

where the last equation holds because for $\gamma = 1$, we have $\sup\{f(\gamma') : \gamma' > 1\} = \sup \varnothing = 0$, and clearly $\sup X = \sup X \cup \{0\}$ if $X \in \mathcal{P}(\mathbf{I})$. We can summarize this as

$$(f^\sharp)^1_* = \sup L \tag{268}$$

where we have abbreviated

$$L = \{\gamma \in \mathbf{I} : \sup\{f(\gamma') : \gamma' > \gamma\} = 1\}\,. \tag{269}$$

Considering $(f^\flat)^1_*$,

$$\begin{aligned}
(f^\flat)^1_* &= \sup\{\gamma \in \mathbf{I} : f^\flat(\gamma) = 1\} && \text{by (22)} \\
&= \sup A\,,
\end{aligned}$$

abbreviating

$$A = \{\gamma \in \mathbf{I} : f^\flat(\gamma) = 1\}\,. \tag{270}$$

Observing that $f^\flat \in \mathbb{H}$ is nonincreasing (this is apparent from the fact that $f \in \mathbb{H}$ is nonincreasing by Def. 75, and by the definition of $f^\flat$ in terms of f, see Def. 71), we may conclude that $A \in \mathcal{P}(\mathbf{I})$ is an interval of one of the following forms:

i. $A = [0, s]$

ii. $A = [0, s)$

where $s = \sup A$, i.e. $s = (f^\flat)^1_*$. In both cases,

$$\sup A \setminus \{0\} = \sup A \tag{271}$$

and

$$\sup A \cup \{0\} = \sup A\,. \tag{272}$$

Therefore

$$\begin{aligned}
(f^\flat)^1_* &= \sup A && \text{by (270), see above} \\
&= \sup\{\gamma \in (0.1] : \lim_{\gamma' \to \gamma^-} f(\gamma') = 1\} && \text{by (271), Def. 71} \\
&= \sup\{\gamma \in (0,1] : \inf\{f(\gamma') : \gamma' < \gamma\} = 1\} && \text{by Th-43} \\
&= \sup\{\gamma \in \mathbf{I} : \inf\{f(\gamma') : \gamma' < \gamma\} = 1\} && \text{by (272)}\,,
\end{aligned}$$

i.e.

$$(f^\flat)^1_* = \sup R\,, \tag{273}$$

abbreviating

$$R = \{\gamma \in \mathbf{I} : \inf\{f(\gamma') : \gamma' < \gamma\} = 1\}\,. \tag{274}$$

By (268) and (273), it is hence sufficient to prove that $\sup L = \sup R$.
Let us first prove that $\sup L \leq \sup R$. We have to show that for each $\gamma_0 \in L$, there is some $\gamma_0' \in R$ such that $\gamma_0' \geq \gamma_0$.
Hence let $\gamma_0 \in L$. Then by (269),

$$\sup\{f(\gamma') : \gamma' > \gamma_0\} = 1\,.$$

In particular, $\{f(\gamma') : \gamma' > \gamma_0\} \neq \varnothing$, i.e. $\gamma_0 < 1$.
Let us now show that $f(\gamma_0) = 1$. Suppose $f(\gamma_0) = 1 - \varepsilon$, where $\varepsilon > 0$. Because $\sup\{f(\gamma') : \gamma' > \gamma_0\} = 1$, there exists some $\gamma' > \gamma$ such that $f(\gamma') > 1 - \varepsilon$. Observing that $f \in \mathbb{H}$ is nonincreasing by Def. 75, we conclude from $\gamma' > \gamma_0$ that $f(\gamma_0) \geq f(\gamma') > 1 - \varepsilon = f(\gamma_0)$, a contradiction. Therefore $f(\gamma_0 = 1$.
Again utilizing that f is nonincreasing, we obtain that $f(\gamma) = 1$ for all $\gamma \leq \gamma_0$. In particular,

$$\inf\{f(\gamma') : \gamma' < \gamma_0\} = \inf\{1 : \gamma' < \gamma_0\} = 1\,,$$

i.e. $\gamma_0 \in R$. This proves that $\sup L \leq \sup R$.
In order to show that $\sup L \geq \sup R$, let us abbreviate $s = \sup R$.

i. If $s = 0$, then clearly

$$\sup L \geq 0 = \sup R\,.$$

ii. If $s > 0$, then $[0, s)$ is nonempty and we may choose some $\gamma_0 \in [0, s)$. We know that $f(\gamma) = 1$ for all $\gamma \in [0, s)$ (this is apparent from (274) and the fact that f is nonincreasing). In particular,

$$f(\gamma) = 1 \tag{275}$$

for all γ in the nonempty open interval (γ_0, s). Therefore

$$\begin{aligned}\sup\{f(\gamma') : \gamma' > \gamma_0\} &\geq \sup\{f(\gamma') : \gamma' \in (\gamma_0, s)\} \\ &= \sup\{1\} && \text{by (275)} \\ &= 1\,,\end{aligned}$$

i.e. $\sup\{f(\gamma') : \gamma' > \gamma_0\} = 1$. Recalling (269), this means that $\gamma_0 \in L$. Because γ_0 was chosen arbitrarily, we conclude that $[0, s) \subseteq L$. Therefore

$$\begin{aligned}\sup L &\geq \sup[0, s) && \text{because } [0, s) \subseteq L \\ &= s \\ &= \sup R\,,\end{aligned}$$

as desired.

Lemma 73
Suppose $f \in \mathbb{H}$.

a. if $\widehat{f}((0, 1]) = \{0\}$, *then* $f^*_1 = 0$.

b. if $\widehat{f}((0, 1]) \neq \{0\}$, *then* $(f^\sharp)^*_1 = (f^\flat)^*_1$.

Proof

a. $\widehat{f}((0,1]) = \{0\}$. Then $f(\gamma) = 0$ for all $\gamma > 0$. Therefore

$$\begin{aligned} f_1^* &= \lim_{\gamma \to 1^-} f(\gamma) && \text{by (21)} \\ &= \lim_{\gamma \to 1^-} 0 \\ &= 0\,. \end{aligned}$$

b. $\widehat{f}((0,1]) \neq \{0\}$. Let us first perform some simplifications.

$$\begin{aligned} (f^\sharp)_1^* &= \lim_{\gamma \to 1^-} f^\sharp && \text{by (21)} \\ &= \lim_{\gamma \to 1^-} \lim_{\gamma' \to \gamma^+} f(\gamma') && \text{by Def. 71} \\ &= \lim_{\gamma \to 1^-} \sup\{f(\gamma') : \gamma' > \gamma\} && \text{by Th-43} \\ &= \inf\{\sup\{f(\gamma') : \gamma' > \gamma\} : \gamma < 1\}\,, && \text{by Th-43} \end{aligned}$$

i.e.

$$(f^\sharp)_1^* = \inf L\,, \tag{276}$$

abbreviating

$$L = \{\sup\{f(\gamma') : \gamma' > \gamma\} : \gamma < 1\} \tag{277}$$

Similarly

$$\begin{aligned} (f^\flat)_1^* &= \lim_{\gamma \to 1^-} f^\flat(\gamma) && \text{by (21)} \\ &= \inf\{f^\flat(\gamma) : \gamma < 1\} && \text{by Th-43} \\ &= \inf\{\lim_{\gamma' \to \gamma^-} f(\gamma') : \gamma \in (0,1)\} \cup \{f(0)\} && \text{by Def. 71} \\ &= \inf\{\lim_{\gamma' \to \gamma^-} f(\gamma') : \gamma \in (0,1)\} && \text{because } f \in \mathbb{H} \text{ is nonincreasing} \\ &= \inf\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\} && \text{by Th-43} \\ &= \inf\{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma < 1\}\,. && \text{because } \inf\{f(\gamma') : \gamma' < 0\} = \inf \varnothing = 1 \end{aligned}$$

Therefore

$$(f^\flat)_1^* = \inf R\,, \tag{278}$$

where

$$R = \{\inf\{f(\gamma') : \gamma' < \gamma\} : \gamma < 1\} \tag{279}$$

By (276) and (278), it is sufficient to show that $\inf L = \inf R$.
Let us first show that $\inf L \leq \inf R$. We will show that for each $x_0 \in R$, there exists $x_0' \in L$ such that $x_0' \leq x_0$.
Hence let $x_0 \in R$. By (279), there exists $\gamma_0 \in [0,1)$ such that

$$x_0 = \inf\{f(\gamma') : \gamma' < \gamma_0\}\,. \tag{280}$$

Because $\gamma_0 \in [0, 1)$, the open interval $(\gamma_0, 1)$ is nonempty. We may hence chose some $\gamma'_0 \in (\gamma_0, 1)$ and define

$$x'_0 = \sup\{f(\gamma') : \gamma' > \gamma'_0\} . \tag{281}$$

Clearly $x_0 \in L$, see (277). We compute:

$$\begin{aligned}
x'_0 &= \sup\{f(\gamma') : \gamma' > \gamma'_0\} && \text{by (281)} \\
&\le \sup\{f(\gamma'_0) : \gamma' > \gamma'_0\} && \text{because } f \in \mathbb{H} \text{ is nonincreasing, see Def. 75} \\
&= f(\gamma'_0) \\
&\le f(\gamma_0) && \text{because } f \text{ nonincreasing, } \gamma_0 < \gamma'_0 \\
&= \inf\{f(\gamma_0) : \gamma' < \gamma_0\} \\
&\le \inf\{f(\gamma') : \gamma' < \gamma_0\} && \text{because } f \text{ nonincreasing} \\
&= x_0 . && \text{by (280)}
\end{aligned}$$

In order to prove that $\inf L \ge \inf R$, let us consider some $x_0 \in L$, we will show that there is some $x'_0 \in R$ such that $x'_0 \le x_0$.
Because $x_0 \in L$, we know from (277) that there exists $\gamma_0 \in [0, 1)$ such that

$$x_0 = \sup\{f(\gamma') : \gamma' > \gamma_0\} . \tag{282}$$

Because $\gamma_0 \in [0, 1)$, the open interval $(\gamma_0, 1)$ is nonempty. We can hence choose $\hat{\gamma}, \gamma'_0 \in (\gamma_0, 1)$ such that $\gamma_0 < \hat{\gamma} < \gamma'_0$. Let us further define

$$x'_0 = \inf\{f(\gamma') : \gamma' < \gamma'_0\} . \tag{283}$$

It is apparent from (279) that $x'_0 \in R$. To see that $x'_0 \le x_0$, we compute

$$\begin{aligned}
x'_0 &= \inf\{f(\gamma') : \gamma' < \gamma'_0\} && \text{by (283)} \\
&= \inf\{f(\gamma') : \gamma' \in [\hat{\gamma}, \gamma'_0)\} && \text{because } f \text{ nonincreasing, } \hat{\gamma} < \gamma'_0 \\
&\le \inf\{f(\hat{\gamma}) : \gamma' \in [\hat{\gamma}, \gamma'_0)\} && \text{because } f \text{ nonincreasing} \\
&= f(\hat{\gamma}) \\
&\le \sup\{f(\gamma') : \gamma' > \gamma_0\} && \text{because } \hat{\gamma} > \gamma_0, \text{ i.e. } f(\hat{\gamma}) \in \{f(\gamma') : \gamma' > \gamma_0\} \\
&= x_0 . && \text{by (282)}
\end{aligned}$$

Proof of Theorem 87

Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined in terms of $\oplus$ according to (Th-87.a). By Th-64, we only need to show that $\mathcal{B}'$ satisfies (C-1), (C-2), (C-3.b) and (C-4) in order to prove that $\mathcal{M}_{\mathcal{B}}$ is a DFS.

$\mathcal{B}'$ satisfies (C-1).
Suppose $f \in \mathbb{H}$ is a constant, i.e. $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
f_*^1 &= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (22)} \\
&= \begin{cases} 1 & : \quad f(0) = 1 \\ 0 & : \quad f(0) < 1 \end{cases}
\end{aligned}$$

because f is constant. For the same reason,

$$\begin{aligned} f_1^* &= \lim_{\gamma\to 1^-} f(\gamma) && \text{by (21)} \\ &= \lim_{\gamma\to 1^-} f(0) && \text{because } f \text{ constant} \\ &= f(0)\,. \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{B}'(f) &= f_*^1 \oplus f_1^* && \text{by (Th-87.a)} \\ &= \begin{cases} 0 \oplus f(0) & : \quad f(0) < 1 \\ 1 \oplus f(0) & : \quad f(0) = 1 \end{cases} \\ &= \begin{cases} f(0) & : \quad f(0) < 1 \\ 1 & : \quad f(0) = 1 \end{cases} \\ &= f(0)\,, \end{aligned}$$

where the second-last equation holds because 0 is the identity of the s-norm $\oplus$, and because every s-norm has $1 \oplus x = 1$ for all $x \in \mathbf{I}$.

$\mathcal{B}'$ satisfies (C-2).
Suppose $f \in \mathbb{H}$ is a mapping such that $f(\mathbf{I}) \subseteq \{0, 1\}$. It is then apparent from the fact that f is nonincreasing (see Def. 75) that

$$\begin{aligned} f_*^1 &= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (22)} \\ &= \inf\{\gamma \in \mathbf{I} : f(\gamma) < 1\} && \text{because } f \text{ is nonincreasing} \\ &= \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{because } f \text{ is two-valued} \\ &= f_*^0\,, && \text{by (19)} \end{aligned}$$

i.e.

$$f_*^1 = f_*^0\,. \tag{284}$$

In addition,

$$\begin{aligned} f_1^* &= \lim_{\gamma\to 1^-} f(\gamma) && \text{by (21)} \\ &= \begin{cases} 1 & : \quad f(\gamma) = 1 \text{ for all } \gamma < 1 \\ 0 & : \quad \text{else} \end{cases} \end{aligned}$$

again because f is two-valued and nonincreasing. Therefore

- If $f(\gamma) = 1$ for all $\gamma \in [0, 1)$, then clearly

  $$f_*^1 = f_*^0 = 1 \tag{285}$$

  (see (22) and (284)) and hence

  $$\begin{aligned} \mathcal{B}'(f) &= f_*^1 \oplus f_1^* && \text{by (Th-87.a)} \\ &= 1 \oplus f_1^* && \text{by (285)} \\ &= 1 && \text{because } 1 \oplus x = 1 \text{ for every } s\text{-norm} \\ &= f_*^0\,. && \text{by (285)} \end{aligned}$$

- In the remaining case that there exists $\gamma \in [0,1)$ such that $f(\gamma) \neq 1$ (i.e. $f(\gamma) = 0$ because f is two-valued), we we know that $f_1^* = 0$ (see above). Hence

$$\begin{aligned} \mathcal{B}'(f) &= f_*^1 \oplus f_1^* && \text{by (Th-87.a)} \\ &= f_*^1 \oplus 0 && \text{because } f_1^* = 0 \\ &= f_*^1 && \text{because } x \oplus 0 = x \text{ for every } s\text{-norm} \\ &= f_*^0 \,. && \text{by (284)} \end{aligned}$$

$\mathcal{B}'$ **satisfies** (C-3.b)**.**
Suppose $f \in \mathbb{H}$ is a mapping such that $\widehat{f}((0,1]) \neq \{0\}$. Then

$$\begin{aligned} \mathcal{B}'(f^\sharp) &= (f^\sharp)_*^1 \oplus (f^\sharp)_1^* && \text{by (Th-87.a)} \\ &= (f^\flat)_*^1 \oplus (f^\flat)_1^* && \text{by L-72.b, L-73.b} \\ &= \mathcal{B}'(f^\flat) \,. && \text{by (Th-87.a)} \end{aligned}$$

$\mathcal{B}'$ **satisfies** (C-4)**.**
Suppose $f, g \in \mathbb{H}$ where $f \leq g$. Then clearly

$$\begin{aligned} f_*^1 &= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (22)} \\ &\leq \sup\{\gamma \in \mathbf{I} : g(\gamma) = 1\} && \text{because } f \leq g \leq 1 \\ &= g_*^1 && \text{by (22)} \end{aligned}$$

and

$$\begin{aligned} f_1^* &= \lim_{\gamma \to 1^-} f(\gamma) && \text{by (21)} \\ &\leq \lim_{\gamma \to 1^-} g(\gamma) && \text{by monotonicity of lim and } f \leq g \\ &= g_1^* \,. && \text{by (21)} \end{aligned}$$

Hence

$$\begin{aligned} \mathcal{B}'(f) &= f_*^1 \oplus f_1^* && \text{by (Th-87.a)} \\ &\leq g_*^1 \oplus g_1^* && \text{by } f_*^1 \leq g_*^1,\ f_1^* \leq g_1^* \text{ and monotonicity of } \oplus \\ &= \mathcal{B}'(g) \,. && \text{by (Th-87.a)} \end{aligned}$$

F.3 Proof of Theorem 88

We already know from Th-87 that $\mathcal{M}_U$ is a DFS. It remains to be shown that $\mathcal{M}_U$ is less specific that every other $\mathcal{M}_\mathcal{B}$-DFS.
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{M}_\mathcal{B}$, defined by Def. 69, is a DFS. Then by Th-62, $\mathcal{B}$ satisfies (B-1) to (B-5) and by Th-63, $\mathcal{B} \in$ BB. Hence there is some $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ such that $\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equation (23), and $\mathcal{B}'$ satisfies (C-1) to (C-4) by Th-53.
We shall utilize Th-86 to prove that $\mathcal{M}_U \preceq_c \mathcal{M}_\mathcal{B}$ by showing that $\mathcal{B}'_U \leq \mathcal{B}'$.
Hence let $f \in \mathbb{H}$.
Let us define $g : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(\gamma) = \begin{cases} 1 & : \quad f(\gamma) = 1 \\ 0 & : \quad \text{else} \end{cases} \tag{286}$$

for all $\gamma \in \mathbf{I}$.

- If $g(\gamma) = 0$ for all $\gamma \in \mathbf{I}$, then $f_*^1 = 0$ by (22) and clearly
$$\mathcal{B}'(f) \geq 0 = f_*^1\,. \tag{287}$$
- If $g \neq \mathrm{c}_0$, then $g \in \mathbb{H}$ (see Def. 68 and (286)) and:
$$g \leq f \tag{288}$$
(apparent from (286)),
$$g_*^0 = g_*^1 = f_*^1 \tag{289}$$
(apparent from (286), (19) and (22)). Hence
$$\begin{aligned} \mathcal{B}'(f) &\geq \mathcal{B}'(g) && \text{by (C-4) and } f \geq g \\ &= g_*^0 && \text{by (C-2)} \\ &= f_*^1\,, && \text{by (289)} \end{aligned}$$
i.e.
$$\mathcal{B}'(f) \geq f_*^1\,. \tag{290}$$

Now let us consider f_1^*.

- If $f_1^* = 0$, then clearly
$$\mathcal{B}'(f) \geq 0 = f_1^*\,. \tag{291}$$
- If $f_1^* > 0$, let us define $h \in \mathbb{H}$ by $h = \mathrm{c}_{f_1^*}$, i.e. h is the constant
$$h(\gamma) = f_1^* \tag{292}$$
for all $\gamma \in \mathbf{I}$. Because $f_1^* \neq \varnothing$, it is apparent from (21) that
$$\widehat{f}((0,1]) \neq \{0\}\,. \tag{293}$$
In addition, we may conclude from Def. 71 that
$$f^\flat \geq h\,. \tag{294}$$
Observing that $f^\flat \geq f \geq f^\sharp$, $\mathcal{B}'(f^\flat) \geq \mathcal{B}'(f) \geq \mathcal{B}'(f^\sharp)$. By (C-3.b) and (293),
$$\mathcal{B}'(f^\flat = \mathcal{B}'f = \mathcal{B}'f^\sharp\,. \tag{295}$$
Hence
$$\begin{aligned} \mathcal{B}'(f) &= \mathcal{B}'(f^\flat) && \text{by (295)} \\ &\geq \mathcal{B}'(h) && \text{by (C-4), (294)} \\ &= f_1^*\,, && \text{by (C-1), } h = \mathrm{c}_{f_1^*} \text{ by (292)} \end{aligned}$$
i.e.
$$\mathcal{B}'(f) \geq f_1^*\,. \tag{296}$$

Summarizing (287), (290), (291) and (296), we obtain $\mathcal{B}'(f) \geq \max(f_*^1, f_1^*) = \mathcal{B}'_U(f)$.

F.4 Proof of Theorem 89

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. In order to show that all $\mathcal{M}_\mathcal{B}$-DFSes are specificity consistent, we shall discern three cases.

a. There exists an $\mathcal{M}_\mathcal{B}$-DFS $\mathcal{M}_\mathcal{B}$ such that $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) > \frac{1}{2}$.
We know from Th-63 that $\mathcal{B} \in$ BB. We may hence define $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ by (24), i.e. $\mathcal{B}'(f) = 2\mathcal{B}(\frac{1}{2} + \frac{1}{2}f) - 1$ for all $f \in \mathbb{H}$, to obtain that

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1-2f) & : \quad f \in \mathbb{B}^- \end{cases}$$

by (23). Because of this relationship, we conclude from

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) > \tfrac{1}{2}$$

that

$$(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} \in \mathbb{B}^+ \,. \tag{297}$$

Now let $\mathcal{M}_{\mathcal{B}_*}$ another $\mathcal{M}_\mathcal{B}$-DFS. Again, we can apply Th-63 and conclude that $\mathcal{B}_* \in$ BB. Hence $\mathcal{B}_*$ is related to $\mathcal{B}'_* : \mathbb{H} \longrightarrow \mathbf{I}$ (again defined by (24)), according to equation (23). In particular, we conclude from (297) that

$$\mathcal{M}_{\mathcal{B}_*}(Q)(X_1, \ldots, X_n) = \mathcal{B}_*((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) = \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_*(2 \cdot (Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} - 1) \geq \tfrac{1}{2} \,.$$

Because $\mathcal{M}_{\mathcal{B}_*}$ was chosen arbitrarily, this means that $\mathcal{M}_{\mathcal{B}_*}(Q)(X_1, \ldots, X_n) \in [\frac{1}{2}, 1]$ for all $\mathcal{M}_\mathcal{B}$-DFSes $\mathcal{M}_{\mathcal{B}_*}$, as desired.

b. There exists an $\mathcal{M}_\mathcal{B}$-DFS $\mathcal{M}_\mathcal{B}$ such that $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) < \frac{1}{2}$.
This case can be reduced to **a.** noting that $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = 1 - \mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n)$ because all $\mathcal{M}_\mathcal{B}$-DFSes are standard DFSes by Th-52, Th-62. We then obtain from **a.** that $\mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n) \in [\frac{1}{2}, 1]$ for all $\mathcal{M}_\mathcal{B}$-DFSes. Hence $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) \in [0, \frac{1}{2}]$ in every $\mathcal{M}_\mathcal{B}$-DFS.

c. For all $\mathcal{M}_\mathcal{B}$-DFSes, $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = \frac{1}{2}$. In this case, clearly $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) \in [\frac{1}{2}, 1]$ for all $\mathcal{M}_\mathcal{B}$.
Summarising cases **a.** to **c.**, we have shown that for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, it either holds that $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $\mathcal{M}_\mathcal{B}$-DFSes, or it holds that $\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) \leq \frac{1}{2}$ for all $\mathcal{M}_\mathcal{B}$-DFSes. Hence the collection of $\mathcal{M}_\mathcal{B}$-DFSes is specificity consistent by Def. 79.

F.5 Proof of Theorem 90

Let $(\mathcal{M}_{\mathcal{B}_j})_{j \in \mathcal{J}}$ be a $\mathcal{J}$-indexed collection of $\mathcal{M}_\mathcal{B}$-DFSes $\mathcal{M}_{\mathcal{B}_j}$, $j \in \mathcal{J}$ where $\mathcal{J} \neq \varnothing$, and let $(\mathcal{B}'_j)_{j \in \mathcal{J}}$ the corresponding family of mappings $\mathcal{B}'_j : \mathbb{H} \longrightarrow \mathbf{I}$, $j \in \mathcal{J}$. Because all $\mathcal{M}_{\mathcal{B}_j}$ are

DFSes, we know from Th-64 that each $\mathcal{B}'_j$ satisfies (C-1), (C-2), (C-3.b), and (C-4).
Now let us define $\mathcal{B}'_{\text{lub}} : \mathbb{H} \longrightarrow \mathbf{I}$ by

$$\mathcal{B}'_{\text{lub}}(f) = \sup\{\mathcal{B}'_j(f) : j \in \mathcal{J}\}, \tag{298}$$

for all $f \in \mathbb{H}$. By Th-64, we can prove that $\mathcal{M}_{\mathcal{B}_{\text{lub}}}$ is a DFS by showing that $\mathcal{B}'_{\text{lub}}$ satisfies (C-1), (C-2), (C-3.b), and (C-4).
Hence let $a \in (0,1]$ and $\mathrm{c}_a \in \mathbb{H}$ the constant mapping $\mathrm{c}_a(x) = a$ for all $x \in \mathbf{I}$. Then

$$\begin{aligned}
\mathcal{B}'_{\text{lub}}(\mathrm{c}_a) &= \sup\{\mathcal{B}'_j(\mathrm{c}_a) : j \in \mathcal{J}\} && \text{by (298)}\\
&= \sup\{a : j \in \mathcal{J}\} && \text{by (C-1) for all } \mathcal{B}'_j\\
&= a\,,
\end{aligned}$$

i.e. $\mathcal{B}'_{\text{lub}}$ satisfies (C-1), as desired.
Concerning (C-2), let $f \in \mathbb{H}$ be given such that $\widehat{f}(\mathbf{I}) \subseteq \{0,1\}$. Then

$$\begin{aligned}
\mathcal{B}'_{\text{lub}}(f) &= \sup\{\mathcal{B}'_j(f) : j \in \mathcal{J}\} && \text{by (298)}\\
&= \sup\{f_*^0 : j \in \mathcal{J}\} && \text{by (C-2) for all } \mathcal{B}'_j\\
&= f_*^0\,,
\end{aligned}$$

i.e. (C-2) holds.
As to (C-3.b), let $f \in \mathbb{H}$ a mapping such that $\widehat{f}((0,1] \neq \{0\}$. Then

$$\begin{aligned}
\mathcal{B}'_{\text{lub}}(f^\sharp) &= \sup\{\mathcal{B}'_j(f^\sharp) : j \in \mathcal{J}\} && \text{by (298)}\\
&= \sup\{\mathcal{B}'_j(f^\flat) : j \in \mathcal{J}\} && \text{by (C-3.b) for all } \mathcal{B}'_j\\
&= \mathcal{B}'_{\text{lub}}(f^\flat)\,. && \text{by (298)}
\end{aligned}$$

Finally let us consider (C-4). Hence let $f, g \in \mathbb{H}$ such that $f \leq g$. Then

$$\begin{aligned}
\mathcal{B}'_{\text{lub}}(f) &= \sup\{\mathcal{B}'_j(f) : j \in \mathcal{J}\} && \text{by (298)}\\
&\leq \sup\{\mathcal{B}'_j(g) : j \in \mathcal{J}\} && \text{because } \mathcal{B}'_j(f) \leq \mathcal{B}'_j(g) \text{ by (C-4)}\\
&= \mathcal{B}'_{\text{lub}}(g)\,, && \text{by (298)}
\end{aligned}$$

as desired. Hence $\mathcal{M}_{\mathcal{B}_{\text{lub}}}$ is a DFS. Clearly $\mathcal{B}'_j \leq \mathcal{B}'_{\text{lub}}$ for all $j \in \mathcal{J}$ and hence $\mathcal{M}_{\mathcal{B}_j} \preceq_c \mathcal{M}_{\mathcal{B}_{\text{lub}}}$ by Th-86. This proves that $\mathcal{M}_{\mathcal{B}_{\text{lub}}}$ is an upper specificity bound of all $\mathcal{M}_{\mathcal{B}_j}$, $j \in \mathcal{J}$.
It remains to be shown that $\mathcal{M}_{\mathcal{B}_{\text{lub}}}$ is the *least* upper specificity bound $\mathcal{F}_{\text{lub}}$ of $(\mathcal{M}_{\mathcal{B}_j})_{j\in\mathcal{J}}$, i.e. that $\mathcal{M}_{\mathcal{B}_{\text{lub}}} = \mathcal{F}_{\text{lub}}$. By Th-74, $\mathcal{F}_{\text{lub}}$ is defined by

$$\mathcal{F}_{\text{lub}}(Q)(X_1,\ldots,X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2},1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0,\frac{1}{2}] \end{cases} \tag{299}$$

where

$$R_{Q,X_1,\ldots,X_n} = \{\mathcal{M}_{\mathcal{B}_j}(Q)(X_1,\ldots,X_n) : j \in \mathcal{J}\}, \tag{300}$$

for all semi-fuzzy quanitifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$. By (298) and (23),

$$\mathcal{B}_{\text{lub}}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'_{\text{lub}}(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'_{\text{lub}}(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{301}$$

for all $f \in \mathbb{B}$. Similarly

$$\mathcal{B}_j(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'_j(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'_j(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{302}$$

for all $j \in \mathcal{J}$. Hence

$$\begin{aligned}
\mathcal{B}_{\text{lub}}(f) &= \begin{cases} \frac{1}{2} + \frac{1}{2}\sup\{\mathcal{B}'_j(2f-1) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\sup\{\mathcal{B}'_j(1-2f) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^- \end{cases} && \text{by (298), (301)} \\
&= \begin{cases} \sup\{\frac{1}{2} + \frac{1}{2}\sup\{\mathcal{B}'_j(2f-1) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \inf\{\frac{1}{2} - \frac{1}{2}\mathcal{B}'_j(1-2f) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^- \end{cases} \\
&= \begin{cases} \sup\{\mathcal{B}_j(f) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \inf\{\mathcal{B}_j(f) & : \quad f \in \mathbb{B}^- \end{cases} && \text{by (302)}
\end{aligned}$$

Hence by Def. 69,

$$\mathcal{M}_{\mathcal{B}_{\text{lub}}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup\{\mathcal{M}_{\mathcal{B}_j}(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \inf\{\mathcal{M}_{\mathcal{B}_j}(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\} & : \quad f \in \mathbb{B}^- \end{cases}$$

where $f(\gamma) = Q_\gamma(X_1, \ldots, X_n)$ for all $\gamma \in \mathbf{I}$. If $f \in \mathbb{B}^+$, then $\mathcal{M}_{\mathcal{B}_j}(Q)(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $j \in \mathcal{J}$, i.e.

$$R_{Q,X_1,\ldots,X_n} \subseteq [\tfrac{1}{2}, 1]\,. \tag{303}$$

Similarly if $f \in \mathbb{B}^-$, then $\mathcal{M}_{\mathcal{B}_j}(Q)(X_1, \ldots, X_n) \leq \frac{1}{2}$ for all $j \in \mathcal{J}$ and hence by (300),

$$R_{Q,X_1,\ldots,X_n} \subseteq [0, \tfrac{1}{2}]\,. \tag{304}$$

If $f \in \mathbb{B}^{\frac{1}{2}}$, then $\mathcal{M}_{\mathcal{B}_j}(Q)(X_1, \ldots, X_n) = \frac{1}{2}$ for all $j \in \mathcal{J}$. Hence

$$\inf R_{Q,X_1,\ldots,X_n} = \inf\{\tfrac{1}{2}\} = \tfrac{1}{2} \qquad \sup R_{Q,X_1,\ldots,X_n} = \sup\{\tfrac{1}{2}\} = \tfrac{1}{2}\,. \tag{305}$$

Combining (303), (304) and (305), we may reformulate the last equation for $\mathcal{M}_{\mathcal{B}_{\text{lub}}}$ into

$$\mathcal{M}_{\mathcal{B}_{\text{lub}}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

i.e. $\mathcal{M}_{\mathcal{B}_{\text{lub}}} = \mathcal{F}_{\text{lub}}$ by (299), as desired.

F.6 Proof of Theorem 91

Suppose that $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}'(f) = f_*^0 \odot f_0^* \,, \tag{306}$$

for all $f \in \mathbb{H}$. By Th-64, it is sufficient to show that $\mathcal{B}'$ satisfies the conditions (C-1) to (C-4).

$\mathcal{B}'$ satisfies (C-1).
Suppose $f \in \mathbb{H}$ is a constant, i.e. $f(\gamma) = f(0)$ for all $\gamma \in \mathbf{I}$. Because $f \in \mathbb{H}$, $f(0) > 0$ by Def. 75, therefore $f(\gamma) = f(0) > 0$ for all $\gamma \in \mathbf{I}$. Hence

$$f_*^0 = \inf \varnothing = 1 \tag{307}$$

by (19), and

$$\begin{aligned}
\mathcal{B}'(f) &= f_*^0 \odot f_0^* && \text{by (306)}\\
&= 1 \odot f_0^* && \text{by (307)}\\
&= f_0^* && \text{because } 1 \text{ is identity of } t\text{-norm } \odot\\
&= \lim_{\gamma \to 0^+} f(\gamma) && \text{by (18)}\\
&= \lim_{\gamma \to 0^+} f(0) && \text{because } f \text{ is constant}\\
&= f(0)\,.
\end{aligned}$$

$\mathcal{B}'$ satisfies (C-2).
Suppose $f \in \mathbb{H}$ is a mapping such that $f(\mathbf{I}) \subseteq \{0, 1\}$. Because $f \in \mathbb{H}$, $f(0) > 0$ by Def. 75, i.e. $f(0) = 1$. We shall discern two cases.

i. $\widehat{f}((0, 1]) = \{0\}$. Then $f(\gamma) = 0$ for all $\gamma > 0$ and hence

$$\begin{aligned}
f_*^0 &= \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} && \text{by (19)}\\
&= \inf(0, 1]\\
&= 0\,,
\end{aligned}$$

i.e.

$$f_*^0 = 0 \tag{308}$$

Therefore

$$\begin{aligned}
\mathcal{B}'(f) &= f_*^0 \odot f_0^* && \text{by (306)}\\
&= 0 \odot f_0^* && \text{by (308)}\\
&= 0 && \text{because } \odot \text{ is } t\text{-norm}\\
&= f_*^0\,. && \text{by (308)}
\end{aligned}$$

ii. $\widehat{f}((0, 1]) \neq \{0\}$. Because $f \in \mathbb{H}$ is nonincreasing (by Def. 75) and two-valued (by assumption), we know that there is some $\gamma' > 0$ such that $f(\gamma) = 1$ for all $\gamma < \gamma'$, and $f(\gamma) > 0$ for all $\gamma > \gamma'$. Observing that $f(\gamma) = 1$ for all γ in the nonempty interval $[0, \gamma')$, it is apparent from (18) that

$$f_0^* = \lim_{\gamma \to 0^+} f(\gamma) = \lim_{\gamma \to 0^+} 1 = 1\,. \tag{309}$$

Hence

$$\begin{aligned} \mathcal{B}'(f) &= f_*^0 \odot f_0^* && \text{by (306)} \\ &= f_*^0 \odot 1 && \text{by (309)} \\ &= f_*^0\,, && \text{because } 1 \text{ is identity of } t\text{-norm } \odot \end{aligned}$$

as desired.

$\mathcal{B}'$ satisfies (C-3.b).
Suppose $f \in \mathbb{H}$ is a mapping such that $\widehat{f}((0,1]) \neq \{0\}$. Then

$$\begin{aligned} \mathcal{B}'(f^\flat) &= (f^\flat)_*^0 \odot (f^\flat)_0^* && \text{by (306)} \\ &= (f^\sharp)_*^0 \odot (f^\sharp)_0^* && \text{by L-54.b, L-59.b} \\ &= \mathcal{B}'(f^\sharp)\,. && \text{by (306)} \end{aligned}$$

$\mathcal{B}'$ satisfies (C-4).
Let $f, g \in \mathbb{H}$, $f \leq g$. Clearly

$$\begin{aligned} f_0^* &= \lim_{\gamma \to 0^+} f(\gamma) && \text{by (18)} \\ &\leq \lim_{\gamma \to 0^+} g(\gamma) && \text{by monotonicity of } \lim \text{ and } f \leq g \\ &= g_0^*\,, && \text{by (18)} \end{aligned}$$

i.e.

$$f_0^* \leq g_0^*\,. \tag{310}$$

We compute

$$\begin{aligned} \mathcal{B}'(f) &= f_*^0 \odot f_0^* && \text{by (306)} \\ &\leq g_*^0 \odot f_0^* && \text{by L-55 and monotonicity of } \odot \\ &\leq g_*^0 \odot g_0^* && \text{by (310) an monotonicity of } \odot \\ &= \mathcal{B}'(g)\,. && \text{by (306)} \end{aligned}$$

F.7 Proof of Theorem 92

We already know from Th-91 that $\mathcal{M}_S$ is a DFS. It remains to be shown that every other $\mathcal{M}_{\mathcal{B}}$-DFS is less specific than $\mathcal{M}_S$.
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{M}_{\mathcal{B}}$, defined by Def. 69, is a DFS. Then by Th-62, $\mathcal{B}$ satisfies (B-1) to (B-5) and by Th-63, $\mathcal{B} \in$ BB. Hence there is some $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ such that $\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equation (23), and $\mathcal{B}'$ satisfies (C-1) to (C-4) by Th-53.
We shall utilize Th-86 to prove that $\mathcal{M}_{\mathcal{B}} \preceq_c \mathcal{M}_S$ by showing that $\mathcal{B}' \leq \mathcal{B}'_S$.
Hence let $f \in \mathbb{H}$.

a. If $\widehat{f}((0,1]) = \{0\}$, then

$$\mathcal{B}'(f) = \mathcal{B}'_S(f) = 0$$

by (C-3.a), i.e. $\mathcal{B}'(f) \leq \mathcal{B}'_S(f)$.

b. $\widehat{f}((0,1]) \neq \{0\}$. In this case, let us define $g : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(x) = \begin{cases} f_0^* & : \quad x = 0 \\ f(x) & : \quad \text{else} \end{cases} \tag{311}$$

for all $x \in \mathbf{I}$. By Def. 68, apparently $f \in \mathbb{H}$.
Because $f_0^* = f^\sharp(0)$ by (18), Def. 71, we obtain that

$$f^\flat(0) \geq g(0) = f^\sharp(0)$$

by L-39. In the case that $\gamma > 0$,

$$f^\flat(\gamma) \geq f(\gamma) = g(\gamma) \geq f^\sharp(\gamma)\,,$$

again by L-39. Hence $f^\flat \geq g \geq f^\sharp$ and by (C-4), $\mathcal{B}'(f^\flat) \geq \mathcal{B}'(g) \geq \mathcal{B}'(f^\sharp)$. By applying L-39 and (C-4) directly, we also have $\mathcal{B}'(f^\flat) \geq \mathcal{B}'(f) \geq \mathcal{B}'(f^\sharp)$. Then by (C-3.b),

$$\mathcal{B}'(f^\flat) = \mathcal{B}'(f) = \mathcal{B}'(g) = \mathcal{B}'(f^\sharp) \tag{312}$$

On the other hand,

$$g(\gamma) \leq f_0^* \tag{313}$$

for all $\gamma \in \mathbf{I}$, which is apparent from (311), (18) and the fact that $f \in \mathbb{H}$ is nonincreasing. Hence (recalling (17)),

$$\begin{aligned} \mathcal{B}'(g) &\leq \mathcal{B}'(\mathrm{c}_{f_0^*}) && \text{by (C-4) and (313)} \\ &= f_0^*\,. && \text{by (C-1)} \end{aligned}$$

Combining this with (312) yields

$$\mathcal{B}'(f) \leq f_0^*\,. \tag{314}$$

Now let us define $h \in \mathbb{H}$ by

$$h(\gamma) = \begin{cases} 1 & : \quad f(\gamma) > 0 \\ 0 & : \quad f(\gamma) = 0 \end{cases} \tag{315}$$

Then clearly $f \leq h$. Furthermore, it is apparent from the definition of h that $h(\gamma) = 0$ exactly if $f(\gamma) = 0$. Therefore

$$h_*^0 = f_*^0 \tag{316}$$

by (19), and

$$\begin{aligned} \mathcal{B}'(f) &\leq \mathcal{B}'(h) && \text{by (C-4), } f \leq h \\ &= h_*^0 && \text{by (C-2)} \\ &= f_*^0\,, && \text{by (316)} \end{aligned}$$

i.e.

$$\mathcal{B}'(f) \leq f_*^0\,. \tag{317}$$

Combining (314) and (317) yields the desired $\mathcal{B}'(f) \leq \min(f_0^*, f_*^0) = \mathcal{B}'_S(f)$ in case **b.**.

F.8 Proof of Theorem 93

We shall treat separately two parts of the main proof.

Lemma 74
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by* (Th-93.a). *Then* $\mathcal{B}'$ *satisfies* (C-4).

Proof Suppose $f, g \in \mathbb{H}$, $f \leq g$. Then

$$\begin{aligned} \mathcal{B}'(f) &= \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)} \\ &\leq \sup\{\gamma \odot g(\gamma) : \gamma \in \mathbf{I}\} && \text{by monotonicity of } \odot \text{ and } \sup;\ f(\gamma) \leq g(\gamma) \\ &= \mathcal{B}'(g)\,. && \text{by (Th-93.a)} \end{aligned}$$

Lemma 75
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by* (Th-93.a). *Then* $\mathcal{B}'$ *satisfies* (C-3.b).

Proof Suppose $f \in \mathbb{H}$ where $\widehat{f}((0,1]) \neq \{0\}$. We know from L-45 that $f^\sharp \leq f^\flat$. Hence by L-74, $\mathcal{B}'(f^\sharp) \leq \mathcal{B}'(f^\flat)$. It remains to be shown that $\mathcal{B}'(f^\sharp) \geq \mathcal{B}'(f^\flat)$. Let us first perform some simplifications:

$$\begin{aligned} \mathcal{B}'(f^\sharp) &= \sup\{\gamma \odot f^\sharp(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)} \\ &= \max(\sup\{\gamma \odot \lim_{\gamma' \to \gamma^+} f(\gamma') : \gamma \in [0,1)\}, 1 \odot f(1)) && \text{by Def. 71} \\ &= \max(\sup\{\gamma \odot \sup\{f(\gamma') : \gamma' > \gamma\} : \gamma \in [0,1)\}, f(1)) && \text{by Th-43} \\ &= \max(\sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in [0,1)\}, f(1))\,, \end{aligned}$$

where the last step holds because $\odot$ is nondecreasing and continuous.
Now suppose that

$$\max(\sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in [0,1)\}, f(1)) = f(1)$$

i.e.

$$\sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in [0,1)\} \leq f(1)\,.$$

Clearly

$$\begin{aligned} &\sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma < 1\} \\ &\geq \sup\{\gamma \odot f(1) : \gamma < 1\} && \text{because } f \in \mathbb{H} \text{ is nonincreasing and } \odot \text{ nondecreasing} \\ &= \sup\{\gamma : \gamma < 1\} \odot f(1) && \text{because } \odot \text{ nondecreasing and continuous} \\ &= 1 \odot f(1) \\ &= f(1) && \text{because } 1 \text{ is identity of } t\text{-norm } \odot. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{B}'(f^\sharp) &= \sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma < 1\} \\ &= \max(\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in (0,1)\}, \\ &\qquad \sup\{0 \odot f(\gamma') : \gamma' > 0\}) \\ &= \max(\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in (0,1)\}, 0) \\ &= \sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in (0,1)\}\,, \end{aligned}$$

i.e.

$$\mathcal{B}'(f^\sharp) = \sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in (0,1)\}\,. \tag{318}$$

By similar reasoning,

$$\begin{aligned}
&\mathcal{B}'(f^\flat)\\
&= \sup\{\gamma \odot f^\flat(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)}\\
&= \max(0 \odot f^\flat(0),\\
&\qquad \sup\{\gamma \odot f^\flat(\gamma) : \gamma \in (0,1]\})\\
&= \max(0, \sup\{\gamma \odot f^\flat(\gamma) : \gamma \in (0,1]\}) && \text{by } \odot\ t\text{-norm}\\
&= \sup\{\gamma \odot f^\flat(\gamma) : \gamma \in (0,1]\} && \text{because } 0 \text{ is identity of } \max\\
&= \sup\{\gamma \odot \lim_{\gamma' \to \gamma^-} f(\gamma') : \gamma \in (0,1]\} && \text{by Def. 71}\\
&= \sup\{\gamma \odot \inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1]\} && \text{by Th-43}\\
&= \sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1]\} && \text{because } \odot \text{ nondec and continuous}\\
&= \max(\sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\},\\
&\qquad \inf\{1 \odot f(\gamma') : \gamma' < 1\})\\
&= \max(\sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\},\\
&\qquad \inf\{f(\gamma') : \gamma' < 1\}) && \text{because } 1 \text{ identity of } t\text{-norm } \odot.
\end{aligned}$$

Now suppose that

$$\max(\sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\}, \inf\{f(\gamma') : \gamma' < 1\}) = \inf\{f(\gamma') : \gamma' < 1\}\,,$$

i.e.

$$\sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\} \le \inf\{f(\gamma') : \gamma' < 1\}\,. \tag{319}$$

We will show that for each $\varepsilon > 0$,

$$\sup\{\inf\{\gamma \odot f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\} \ge \inf\{f(\gamma') : \gamma' < 1\} - \varepsilon\,.$$

Hence let $\varepsilon > 0$. By the continuity of $\odot$, there is some $\gamma_0 \in (0,1)$ such that

$$\gamma_0 \odot \inf\{f(\gamma') : \gamma' < 1\} \ge 1 \odot \inf\{f(\gamma') : \gamma' < 1\} - \varepsilon\,,$$

i.e.

$$\gamma_0 \odot \inf\{f(\gamma') : \gamma' < 1\} \ge \inf\{f(\gamma') : \gamma' < 1\} - \varepsilon\,. \tag{320}$$

Hence

$$\begin{aligned}
&\sup\{\gamma \odot \inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\}\\
&\ge \gamma_0 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\}\\
&\ge \gamma_0 \odot \inf\{f(\gamma') : \gamma < 1\}\\
&\ge \inf\{f(\gamma') : \gamma' < 1\} - \varepsilon\,. && \text{by (320)}
\end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily, this proves that in the case (319),

$$\sup\{\gamma \odot \inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\} = \inf\{f(\gamma') : \gamma' < 1\}\,.$$

We can hence simplify $\mathcal{B}'(f^\flat)$ as follows.

$$\mathcal{B}'(f^\flat) = \sup\{\gamma \odot \inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\}\,. \tag{321}$$

Making use of (318) and (321), let us now prove that

$$\mathcal{B}'(f^\sharp) \geq \mathcal{B}'(f^\flat)\,.$$

We will prove the above inequation by showing that $\mathcal{B}'(f^\sharp) > \mathcal{B}'(f^\flat) - \varepsilon$ for each $\varepsilon > 0$. Hence let $\varepsilon > 0$. By the (321), there is some

$$y_0 \in \{\gamma \odot \inf\{f(\gamma') : \gamma' < \gamma\} : \gamma \in (0,1)\}$$

such that

$$\mathcal{B}'(f^\flat) - y_0 < \tfrac{\varepsilon}{2}\,. \tag{322}$$

Let us denote by γ_0 a choice of $\gamma_0 \in (0,1)$ such that

$$y_0 = \gamma_0 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\} \tag{323}$$

Because $\odot$ is assumed to be continuous, there is some $\gamma_1 \in (0, \gamma_0)$ such that

$$\gamma_1 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\} > \gamma_0 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\} - \tfrac{\varepsilon}{2}\,. \tag{324}$$

Let us choose some $\gamma_* \in (\gamma_1, \gamma_0)$. Then

$$\begin{aligned}
\mathcal{B}'(f^\sharp) &= \sup\{\sup\{\gamma \odot f(\gamma') : \gamma' > \gamma\} : \gamma \in (0,1)\} && \text{by (318)}\\
&\geq \sup\{\gamma_1 \odot f(\gamma') : \gamma' > \gamma_1\} \\
&\geq \gamma_1 \odot f(\gamma_*) && \text{because } \gamma_* > \gamma_1\\
&\geq \gamma_1 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\} && \text{because } \gamma_* < \gamma_0\\
&> \gamma_0 \odot \inf\{f(\gamma') : \gamma' < \gamma_0\} - \tfrac{\varepsilon}{2} && \text{by (324)}\\
&= y_0 - \tfrac{\varepsilon}{2} && \text{by (323)}\\
&> (\mathcal{B}'(f^\flat) - \tfrac{\varepsilon}{2}) - \tfrac{\varepsilon}{2} && \text{by (322)}\\
&= \mathcal{B}'(f^\flat) - \varepsilon\,.
\end{aligned}$$

We conclude that $\mathcal{B}'(f^\sharp) \geq \mathcal{B}'(f^\flat)$, because $\mathcal{B}'(f^\sharp) > \mathcal{B}'(f^\flat) - \varepsilon$ for all $\varepsilon > 0$.

Proof of Theorem 93

Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by (Th-93.a). By Th-64, we only need to prove that $\mathbb{H}$ satisfies (C-1) to (C-4) in order to show that $\mathcal{M}_\mathcal{B}$ is a standard DFS.

$\mathcal{B}'$ **satisfies** (C-1)**.**
Suppose $f \in \mathbb{H}$ is a constant, i.e.

$$f(\gamma) = f(0) \tag{325}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
\mathcal{B}'(f) &= \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)}\\
&= \sup\{\gamma \odot f(0) : \gamma \in \mathbf{I}\} && \text{by (325)}\\
&= 1 \odot f(0) && \text{by monotonicity of } \odot\\
&= f(0) && \text{because } 1 \text{ is identity of } \odot.
\end{aligned}$$

$\mathcal{B}'$ **satisfies** (C-2)**.**
Suppose $f \in \mathbb{H}$ is two-valued, i.e. $f(\mathbf{I}) \subseteq \{0, 1\}$. We shall discern two cases.

a. $f(f_*^0) = 1$. Then

$$\begin{aligned}
\mathcal{B}'(f) &= \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)}\\
&= \max(\sup\{\gamma \odot f(\gamma) : \gamma \in [0, f_*^0]\},\\
&\qquad \sup\{\gamma \odot f(\gamma) : \gamma \in (f_*^0, 1]\})\\
&= \max(\sup\{\gamma \odot 1 : \gamma \in [0, f_*^0]\},\\
&\qquad \sup\{\gamma \odot 0 : \gamma \in (f_*^0, 1]\}) && \text{because } f \text{ two-valued}\\
& && \text{and } f \text{ nonincreasing by Def. 75}\\
&= \max(\sup\{\gamma : \gamma \in [0, f_*^0]\}, \sup\{0 : \gamma \in (f_*^0, 1]\}) && \text{because } \odot \text{ is } t\text{-norm}\\
&= \max(\sup[0, f_*^0]\}, 0)\\
&= f_*^0 .
\end{aligned}$$

b. $f(f_*^0) = 0$. Then by similar reasoning,

$$\begin{aligned}
\mathcal{B}'(f) &= \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} && \text{by (Th-93.a)}\\
&= \max(\sup\{\gamma \odot f(\gamma) : \gamma \in [0, f_*^0)\},\\
&\qquad \sup\{\gamma \odot f(\gamma) : \gamma \in [f_*^0, 1]\})\\
&= \max(\sup\{\gamma \odot 1 : \gamma \in [0, f_*^0)\},\\
&\quad \sup\{\gamma \odot 0 : \gamma \in [f_*^0, 1]\}) && \text{because } f \text{ two-valued and nonincreasing}\\
&= \max(\sup[0, f_*^0), 0)\\
&= f_*^0 .
\end{aligned}$$

$\mathcal{B}'$ **satisfies** (C-3.b)**.**
See L-75.

$\mathcal{B}'$ **satisfies** (C-4)**.**
See L-74.

F.9 Proof of Theorem 94

Lemma 76
For all $f \in \mathbb{H}$, there is at most one $x' \in \mathbf{I}$ such that

a. $f(y) > y$ *for all* $y < x'$*;*

b. $f(y) < y$ *for all* $y > x'$*.*

Proof Suppose x', $x'' \in \mathbf{I}$ both satisfy the conditions of the lemma. Without loss of generality, we may assume that $x' \leq x''$. If $x' < x''$, then there is some $z \in (x', x'')$.
By condition b. and $z > x'$, we obtain

$$f(z) < z\,.$$

By condition a. and $z < x''$, we have

$$f(z) > z\,,$$

a contradiction. This proves that $x' = x''$.

Lemma 77
For all $f \in \mathbb{H}$,

$$x' = \sup\{z : f(z) > z\}$$

satisfies the conditions a. and b. of L-76.

Note. Combined with L-76, this guarantees the existence of a unique $x' \in \mathbf{I}$ with the desired properties for each choice of $f \in \mathbb{H}$.

Proof Suppose $f \in \mathbb{H}$ is given. Let us abbreviate

$$Z = \{z \in \mathbf{I} : f(z) > z\}\,.$$

By Def. 75, we know that $f(0) > 0$ and hence $0 \in Z$, in particular $Z \neq \varnothing$.
Now let $x' = \sup Z$ and let $y \in [0, x')$. Because $Z \neq \varnothing$, we know that there exists some $z \in [y, x']$ such that $f(z) > z$. Hence

$$\begin{aligned} f(y) &\geq f(z) && \text{by } f \in \mathbb{H} \text{ nonincreasing, } y \leq z \\ &> z && \text{by choice of } z \\ &\geq y\,, \end{aligned}$$

i.e. condition a. of L-76 holds for all $y < x'$.
To see that condition b. of L-76 also holds, let us observe that by definition of x',

$$f(z) \leq z \tag{326}$$

for all $z > x'$.
Now suppose $y > x'$ and $z \in (x', y)$. Then

$$\begin{aligned} f(y) &=\leq f(z) && \text{by } f \in \mathbb{H} \text{ nonincreasing, } z < y \\ &\leq z && \text{by (326)} \\ &< y && \text{by choice of } z \end{aligned}$$

i.e. condition b. holds.

Lemma 78
For all $f \in \mathbb{H}$ *and* $x' = \sup\{z : f(z) > z\}$,

$$\sup\{f(z) : z > x'\} \leq x' .$$

Proof Let us abbreviate

$$s = \sup\{f(z) : z > x'\} . \tag{327}$$

Suppose $s > x'$. By (327), there exists $z > x'$ such that

$$f(z) > x' . \tag{328}$$

We hence have

$$\begin{aligned} z &> f(z) && \text{by L-77.b and } z > x' \\ &> x' && \text{by (328).} \end{aligned}$$

Because $f(z) < z$ and $f \in \mathbb{H}$ nonincreasing, we may conclude that

$$f(f(z)) \geq f(z)$$

which contradicts L-76.b because $f(z) > x'$. It follows that $s \leq x'$.

Lemma 79
For each $f \in \mathbb{H}$,

$$\sup\{\min(z, f(z)) : z \in \mathbf{I}\} = \sup\{z \in \mathbf{I} : f(z) > z\} .$$

Proof Abbreviating

$$x' = \sup\{z \in \mathbf{I} : f(z) > z\} ,$$

we obtain

$$\begin{aligned} \sup\{\min(z, f(z)) : z \in \mathbf{I}\} &= \max\{\sup\{\min(z, f(z)) : z < x'\}, \\ &\quad \min(x', f(x')), \\ &\quad \sup\{\min(z, f(z)) : z > x'\}\} \\ &= \max\{\sup\{z : z < x'\}, \\ &\quad \min(x', f(x')), \\ &\quad \sup\{f(z) : z > x'\}\} && \text{by L-77} \\ &= \max\{x', \\ &\quad \min(x', f(x')), \\ &\quad \sup\{f(z) : z > x'\}\} \\ &= \max\{x', \\ &\quad \sup\{f(z) : z > x'\}\} && \text{because } x' \geq \min(x', f(x')) \\ &= x' . && \text{by L-78} \end{aligned}$$

Lemma 80
For all $f \in \mathbb{H}$ *and* $x' = \sup\{z : f(z) > z\}$,

$$\inf\{f(z) : z < x'\} \geq x' .$$

Proof Suppose $f \in \mathbb{H}$, $x' = \sup\{z : f(z) > z\}$ and let us abbreviate

$$i = \inf\{f(z) : z < x'\}\,. \tag{329}$$

Assume $i < x'$. Then by (329), we know that there is some $z < x'$ such that

$$f(z) < x'\,. \tag{330}$$

By L-77.a, we also have

$$f(z) > z\,. \tag{331}$$

Because $f \in \mathbb{H}$ is nonincreasing,

$$f(f(z)) \leq f(z) \qquad \text{by (331), } f \text{ noninc}$$

which contradicts

$$f(f(z)) > f(z)\,. \qquad \text{by (330), L-77.a}$$

Hence $i \geq x$, as desired.

Lemma 81
For all $f \in \mathbb{H}$,

$$\inf\{\max(z, f(z)) : z \in \mathbf{I}\} = \sup\{z \in \mathbf{I} : f(z) > z\}\,.$$

Proof Suppose $f \in \mathbb{H}$ is given and let us abbreviate $x' = \sup\{z \in \mathbf{I} : f(z) > z\}$. Then

$$\begin{aligned}
\inf\{\max(z, f(z)) : z \in \mathbf{I}\} &= \min\{\inf\{\max(z, f(z)) : z < x'\}, \\
&\qquad \max(x', f(x')), \\
&\qquad \inf\{\max(z, f(z)) : z > x'\}\} \\
&= \min\{\inf\{f(z) : z < x'\}, \\
&\qquad \max(x', f(x')), \\
&\qquad \inf\{z : z > x'\}\} && \text{by L-77} \\
&= \min\{\inf\{f(z) : z < x'\}, \\
&\qquad \max(x', f(x')), \\
&\qquad x'\} \\
&= x'
\end{aligned}$$

where the last equation holds because $x' \leq \max(x', f(x'))$ and because $x' \leq \inf\{f(z) : z < x'\}$ by L-80.

Based on these lemmata, the proof of Th-94 is now trivial:

Proof of Theorem 94

Let $f \in \mathbb{H}$ be given. We have to show that

$$\begin{aligned} &\sup\{\min(x, f(x)) : x \in \mathbf{I}\} \\ =&\inf\{\max(x, f(x)) : x \in \mathbf{I}\} \\ =&\text{the unique } x' \text{ s.th. } f(y) > y \text{ for all } y < x' \text{ and } f(y) < y \text{ for all } y > x' \,. \end{aligned}$$

To this end, we can rely on L-76 and L-77 which establish the uniqueness and existence of an x' with the desired properties, viz.

$$\sup\{z : f(z) > z\} = \text{the unique } x' \text{ s.th. } f(y) > y \text{ for all } y < x' \text{ and } f(y) < y \text{ for all } y > x'$$

The theorem is then apparent from L-79 and L-81.

F.10 Proof of Theorem 95

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative semi-fuzzy quantifier on a finite base set $E \neq \varnothing$. Further suppose that $Y_0, \ldots, Y_m \in \mathcal{P}(E)$, $m = |E|$, are arbitrary subsets of E of cardinality $|Y_j| = j$. We will define a mapping $q' : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ by

$$q'(j) = Q(Y_j) \tag{332}$$

for all $j \in \{0, \ldots, |E|\}$, given our above choice of $Y_0, \ldots, Y_m$.
Now let $Y \in \mathcal{P}(E)$ an arbitrary subset of E. Abbreviating

$$k = |Y| \,, \tag{333}$$

we have $|Y| = |Y_k|$. Because Y, Y_k are finite sets of equal cardinality, there exists a bijection $\beta' : Y \longrightarrow Y_k$. Because $Y, Y \subseteq E$ and E finite, β' can be extended to a bijection $\beta : E \longrightarrow E$ such that $\beta|_Y = \beta'$, i.e.

$$\widehat{\beta}(Y) = \widehat{\beta'}(Y) = Y_k \,. \tag{334}$$

We conclude that

$$\begin{aligned} Q(Y) &= Q(\widehat{\beta}(Y)) && \text{by Def. 31, } Q \text{ quantitative} \\ &= Q(Y_k) && \text{by (334)} \\ &= q'(k) && \text{by (332)} \\ &= q'(|Y|) && \text{by (333)} \,. \end{aligned}$$

In particular, this proves that $Q(Y) = Q(Y') = q(k)$ for all $Y, Y' \in \mathcal{P}(E)$ such that $|Y| = |Y'| = k$. Therefore q' is independent of the chosen representatives $Y_0, \ldots, Y_m$, i.e. the mapping q mentioned in the theorem is well-defined and coincides with q'.
It remains to be shown that no such mapping $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ exists if Q is not quantitative. Hence suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier on a finite base set, and let us assume that Q is not quantitative. Then there exist $Y' \in \mathcal{P}(E)$ and an automorphism $\beta : E \longrightarrow E$ such that

$$Q(\widehat{\beta}(Y')) \neq Q(\beta) \,. \tag{335}$$

Now let us assume that there is a mapping $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|) \tag{336}$$

for all $Y \in \mathcal{P}(E)$. Observing that $|Y'| = |\widehat{\beta}(Y')|$ because β is an automorphism (i.e., permutation), we obtain

$$\begin{aligned} Q(Y') &= q(|Y'|) && \text{by (336)} \\ &= q(|\widehat{\beta}(Y')|) && \text{because } |Y'| = |\widehat{\beta}(Y')| \\ &= Q(\widehat{\beta}(Y'))\,, && \text{by (336)} \end{aligned}$$

i.e. $Q(Y') = Q(\widehat{\beta}(Y'))$, which contradicts (335). Hence (336) is false, i.e. no mapping $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ with property (336) exists if Q is not quantitative.

F.11 Proof of Theorem 96

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative semi-fuzzy quantifier on a finite base set and $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ is the mapping defined by (33), i.e. it holds that

$$Q(Y) = q(|Y|) \tag{337}$$

for all $Y \in \mathcal{P}(E)$.

a. Let us first assume that Q is convex. We have to show that q has the property claimed by the theorem. Hence let $j', j, j'' \in \{0, \dots, |E|\}$ such that $j' \le j \le j''$. We can then choose subsets $Y', Y, Y'' \in \mathcal{P}(E)$ such that $|Y'| = j'$, $|Y| = j$ and $|Y''| = j''$. In addition, we may assume that $Y' \subseteq Y \subseteq Y''$. Then

$$\begin{aligned} q(j) &= Q(Y) && \text{by (337), } |Y| = j \\ &\ge \min(Q(Y'), Q(Y'')) && \text{because } Q \text{ convex, } Y' \subseteq Y \subseteq Y'' \\ &= \min(q(j'), q(j''))\,. && \text{by (337), } |Y'| = j',\ |Y''| = j'' \end{aligned}$$

b. To see that the converse implication holds, let us assume that Q is not convex in its argument. Then there exist $Y', Y, Y'' \in \mathcal{P}(E)$ such that $Y' \subseteq Y \subseteq Y''$ and

$$Q(Y) < \min(Q(Y'), Q(Y''))\,. \tag{338}$$

Abbreviating $j = |Y|$, $j' = |Y'|$ and $j'' = |Y''|$, we clearly have $j' \le j \le j''$ because $Y' \subseteq Y \subseteq Y''$. However,

$$\begin{aligned} q(j) &= Q(Y) && \text{by (337), } |Y| = j \\ &< \min(Q(Y'), Q(Y'')) && \text{by (338)} \\ &= \min(q(j'), q(j''))\,. && \text{by (337), } j' = |Y'|,\ j'' = |Y''| \end{aligned}$$

Hence there exists a choice of $j' \le j \le j''$ such that $q(j) < \min(q(j'), q(j''))$ whenever Q is not convex, i.e. the convexity of Q is indeed necessary for the claimed property of q.

F.12 Proof of Theorem 97

Suppose $m \in \mathbb{N} \setminus \{0\}$ and $q : \{0, \dots, m\} \longrightarrow \mathbf{I}$ has the following property: whenever $j', j, j'' \in \{0, \dots, |E|\}$ and $j' \leq j \leq j''$, then

$$q(j) \geq \min(q(j'), q(j'')) \,. \tag{339}$$

We have to show that there exists $j_{\mathrm{pk}} \in \{0, \dots, m\}$ such that

$$q(j) \leq q(j') \tag{340}$$

for all $j \leq j'' \leq j_{\mathrm{pk}}$, and

$$q(j) \geq q(j') \tag{341}$$

for all $j_{\mathrm{pk}} \leq j \leq j'$.
To this end, let us define $q^+, q^- : \{0, \dots, m\} \longrightarrow \mathbf{I}$ by

$$q^+(j) = \max\{q(k) : 0 \leq k \leq j\} \tag{342}$$
$$q^-(j) = \max\{q(k) : j \leq k \leq m\} \tag{343}$$

for all $j \in \{0, \dots, m\}$. Then for all $j \in \{0, \dots, m\}$,

$$q(j) \leq \min(q^+(j), q^-(j)) \tag{344}$$

because apparently $q(j) \leq q^+(j)$, $q(j) \leq q^-(j)$. In order to prove that $q = q^+ \wedge q^-$, it remains to be shown that $q(j) \geq \min(q^+(j), q^-(j))$ for each $j \in \{0, \dots, m\}$. Given such j, we know from (342) that there exists $j^+ \in \{0, \dots, j\}$ such that

$$q^+(j) = q(j^+) \,. \tag{345}$$

Similarly, we know from (343) that there exists $j^- \in \{j, \dots, m\}$ such that

$$q^-(j) = q(j^-) \,. \tag{346}$$

Clearly $j^+ \leq j \leq j^-$ and hence

$$\begin{aligned} q(j) &\geq \min(q(j^+), q(j^-)) && \text{by (339)} \\ &= \min(q^+(j), q^-(j)) \,, && \text{by (345), (346)} \end{aligned}$$

i.e. $q(j) \geq \min(q^+(j), q^-(j))$. Combining this with (344), we conclude that

$$q(j) = \min(q^+(j), q^-(j)) \tag{347}$$

for all $j \in \{0, \dots, m\}$.
Let us also observe that q^+ is nondecreasing and q^- is nonincreasing, i.e.

$$q^+(j) \leq q^+(j') \tag{348}$$
$$q^+(j) \geq q^-(j') \tag{349}$$

whenever $j \leq j'$. This is an immediate consequence of (342) and (343).
Let us now define

$$j_{\rm pk} = \max\{j : q(j) = q^+(j)\}\,. \tag{350}$$

Because for all $j \in \{0, \ldots, m\}$, $q(j) = \min(q^+(j), q^-(j))$, we know that for all j, either $q(j) = q^+(j)$ or $q(j) = q^-(j)$. It is hence apparent from $q(j) \neq q^+(j)$ that

$$q(j) = q^-(j) \tag{351}$$

for all $j > j_{\rm pk}$. Let us now prove that $q(j) = q^+(j)$ for all $j \leq j_{\rm pk}$. In this case,

$$\begin{aligned} q^+(j) &\leq q^+(j_{\rm pk}) && \text{by (348), } j \leq j_{\rm pk} \\ &\leq q^-(j_{\rm pk}) && \text{by (350)} \\ &\leq q^-(j)\,. && \text{by (349), } j \leq j_{\rm pk} \end{aligned}$$

Hence

$$q(j) \leq \min(q^+(j), q^-(j)) = q^+(j) \tag{352}$$

whenever $j \leq j_{\rm pk}$.
Based on these observations, it is now easy to show that q satisfies (339).
If $j' \leq j \leq j_{\rm pk}$, then

$$\begin{aligned} q(j) &= q^+(j) && \text{by (352)} \\ &\geq q^+(j') && \text{by (348), } j' \leq j \\ &= q(j')\,, && \text{by (352)} \end{aligned}$$

as desired.
If $j_{\rm pk} < j \leq j'$, then

$$\begin{aligned} q(j) &= q^-(j) && \text{by (351)} \\ &\geq q^-(j') && \text{by (349)} \\ &= q(j')\,. && \text{by (351)} \end{aligned}$$

Let us now show that $q(j_{\rm pk}) > q(j_{\rm pk} + 1)$, provided that $j_{\rm pk} < m$. Hence let us assume to the contrary that $q(j_{\rm pk}) \leq q(j_{\rm pk} + 1)$. Then

$$\begin{aligned} q^+(j_{\rm pk} + 1) &= \max(q(j_{\rm pk} + 1), q^+(j_{\rm pk})) && \text{by (342)} \\ &= \max(q(j_{\rm pk} + 1), q(j_{\rm pk})) && \text{by (350)} \\ &= q(j_{\rm pk} + 1)\,, && \text{by assumption, } q(j_{\rm pk}) \leq q(j_{\rm pk} + 1) \end{aligned}$$

which contradicts equation (350). Hence our above assumption is false, and

$$q(j_{\rm pk} + 1) < q(j_{\rm pk})\,. \tag{353}$$

We can use this to treat the remaining case that $j = j_{\rm pk}$, $j' \geq j_{\rm pk}$. If $j' = j_{\rm pk}$, then trivially $q(j_{\rm pk}) \geq q(j')$. If $j' > j_{\rm pk}$, then

$$\begin{aligned} q(j') &= q^-(j') && \text{by (351)} \\ &\leq q^-(j_{\rm pk} + 1) && \text{by (349)} \\ &= q(j_{\rm pk} + 1) && \text{by (351)} \\ &< q(j_{\rm pk})\,. && \text{by (353)} \end{aligned}$$

F.13 Proof of Theorem 98

Lemma 82
Suppose $E \neq \varnothing$ is a finite base set and $X \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in \mathbf{I}$,

$$\{|Y| : Y \in \mathcal{T}_\gamma(X)\} = \{j : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}\,.$$

Proof
Let us first show that $\{|Y| : Y \in \mathcal{T}_\gamma(X)\} \subseteq \{j : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}$. From Def. 66, we know that $(X)_\gamma^{\min} \subseteq (X)_\gamma^{\max}$ and $\mathcal{T}_\gamma(X) = \{Y : (X)_\gamma^{\min} \subseteq Y \subseteq (X)_\gamma^{\max}\}$. Hence if $Y \in \mathcal{T}_\gamma(X)$, then $\min X\gamma \subseteq Y \subseteq (X)_\gamma^{\max}$ and hence $|X|_\gamma^{\min} \leq |Y| \leq |X|_\gamma^{\max}$, i.e. $j = |Y|$ satisfies $|X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}$ and is hence an element of $\{j : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}$.
To see that $\{|Y| : Y \in \mathcal{T}_\gamma(X)\} \supseteq \{j : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}$, suppose $j \in \{|X|_\gamma^{\min}, |X|_\gamma^{\min} + 1, \dots, |X|_\gamma^{\max}\}$ is given. Abbreviating $r = j - |X|_\gamma^{\min}$, we clearly have

$$\begin{aligned} |(X)_\gamma^{\max} \setminus (X)_\gamma^{\min}| &= |X|_\gamma^{\max} \setminus |X|_\gamma^{\min} && \text{by (31),(32), } (X)_\gamma^{\min} \subseteq (X)_\gamma^{\max} \\ &\geq j - |X|_\gamma^{\min} \\ &= r\,. && \text{by def. of } r \end{aligned}$$

Because $|(X)_\gamma^{\max} \setminus (X)_\gamma^{\min}| \geq r$, there is a subset $R \subseteq (X)_\gamma^{\max} \setminus (X)_\gamma^{\min}$ of cardinality $|R| = r$. The set $Y = (X)_\gamma^{\min} \cup R$ apparently satisfies $(X)_\gamma^{\min} \subseteq Y$ and $Y = (X)_\gamma^{\min} \cup R \subseteq (X)_\gamma^{\min} \cup ((X)_\gamma^{\max} \setminus (X)_\gamma^{\min}) = (X)_\gamma^{\max}$, i.e. $Y \in \mathcal{T}_\gamma(X)$ by Def. 66. The cardinality of Y is

$$\begin{aligned} |Y| &= |(X)_\gamma^{\min} \cup R| \\ &= |X|_\gamma^{\min} + |R| && \text{because } (X)_\gamma^{\min} \cap R = \varnothing \\ &= |X|_\gamma^{\min} + r && \text{because } |R| = r \\ &= |X|_\gamma^{\min} + (j - |X|_\gamma^{\min}) && \text{because } r = j - |X|_\gamma^{\min} \\ &= j\,, \end{aligned}$$

as desired.

Proof of Theorem 98

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative semi-fuzzy quantifier on a finite base set $E \neq \varnothing$. By Th-95, there exists $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|) \tag{354}$$

for all $Y \in \mathcal{P}(E)$. Now let $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Then

$$\begin{aligned} Q_\gamma^{\min}(X) &= \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (15)} \\ &= \inf\{q(|Y|) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (354)} \\ &= \inf\{q(j) : j \in \{|Y| : Y \in \mathcal{T}_\gamma(X)\}\} && \text{substitution } j = |Y| \\ &= \inf\{q(j) : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\} && \text{by L-82} \\ &= \min\{q(j) : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\}\,, && \text{because } \{j : |X|_\gamma^{\min} \leq j \leq |X|_\gamma^{\max}\} \text{ finite} \end{aligned}$$

i.e.

$$Q_\gamma^{\min}(X) = q^{\min}(|X|_\gamma^{\min}, |X|_\gamma^{\max}) \tag{355}$$

by Def. 94. Similarly

$$\begin{aligned}
Q_\gamma^{\max}(X) &= \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (16)}\\
&= \sup\{q(|Y|) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (354)}\\
&= \sup\{q(j) : j \in \{|Y| : Y \in \mathcal{T}_\gamma(X)\}\} && \text{substitution } j = |Y|\\
&= \sup\{q(j) : |X|_\gamma^{\min} \le j \le |X|_\gamma^{\max}\} && \text{by L-82}\\
&= \max\{q(j) : |X|_\gamma^{\min} \le j \le |X|_\gamma^{\max}\}\,, && \text{because } \{j : |X|_\gamma^{\min} \le j \le |X|_\gamma^{\max}\} \text{ finite}
\end{aligned}$$

i.e. by Def. 94,

$$Q_\gamma^{\max}(X) = q^{\max}(|X|_\gamma^{\min}, |X|_\gamma^{\max})\,. \tag{356}$$

Combining (355), (356), (14), we conclude that

$$Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(q^{\min}(|X|_\gamma^{\min}, |X|_\gamma^{\max}), q^{\max}(|X|_\gamma^{\min}, |X|_\gamma^{\max}))\,.$$

Alternatively, we may express $Q_\gamma(X)$ as follows.

$$\begin{aligned}
Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 67}\\
&= \mathrm{m}_{\frac{1}{2}}\{q(|Y|) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (354)}\\
&= \mathrm{m}_{\frac{1}{2}}\{q(j) : j \in \{|Y| : Y \in \mathcal{T}_\gamma(X)\}\} && \text{substitution } j = |Y|\\
&= \mathrm{m}_{\frac{1}{2}}\{q(j) : |X|_\gamma^{\min} \le j \le |X|_\gamma^{\max}\}\,. && \text{by L-82}
\end{aligned}$$

F.14 Proof of Theorem 99

Lemma 83
Suppose $m \in \mathbb{N} \setminus \{0\}$ and $q : \{0, \ldots, m\}$ has the following property: whenever $j', j, j'' \in \{0, \ldots, |E|\}$ and $j' \le j \le j''$, then

$$q(j) \ge \min(q(j'), q(j''))\,. \tag{357}$$

Further suppose that $j_{\mathrm{pk}} \in \{0, \ldots, m\}$ is as in Th-97. Then for all $j_{\min}, j_{\max} \in \{0, \ldots, m\}$ such that $j_{\min} \le j_{\max}$,

$$q^{\min}(j_{\min}, j_{\max}) = \min\{q(j) : j_{\min} \le j \le j_{\max}\} = \min(j_{\min}, j_{\max})$$

and

$$q^{\max}(j_{\min}, j_{\max}) = \max\{q(j) : j_{\min} \le j \le j_{\max}\} = \begin{cases} q(j_{\min}) & : \quad j_{\min} > j_{\mathrm{pk}} \\ q(j_{\max}) & : \quad j_{\max} < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \quad j_{\min} \le j_{\mathrm{pk}} \le j_{\max} \end{cases}$$

Proof
Apparently

$$\min\{q(j) : j_{\min} \le j \le j_{\max}\} \le \min(j_{\min}, j_{\max}) . \tag{358}$$

To see that the converse inequation also holds, let $j' \in \{j_{\min}, \dots, j_{\max}\}$ a choice of j such that

$$q(j') = \min\{q(j) : j_{\min} \le j \le j_{\max}\} . \tag{359}$$

Then $j_{\min} \le j' \le j_{\max}$ and by the assumed property (357) of q,

$$\begin{aligned} &\min\{q(j) : j_{\min} \le j \le j_{\max}\} \\ &= q(j') && \text{by (359)} \\ &\ge \min(q(j_{\min}), q(j_{\max})) . && \text{by (357)} \end{aligned}$$

Combining this with (358), we conclude that

$$\min\{q(j) : j_{\min} \le j \le j_{\max}\} = \min(q(j_{\min}), q(j_{\max})) .$$

Let us now address the second part of the lemma. If $j_{\min} > j_{\mathrm{pk}}$, then by Th-97,

$$q(j) \le q(j_{\min})$$

for all $j > q(j_{\min})$. Hence

$$\max\{q(j) : j_{\min} \le j \le j_{\max}\} = q(j_{\min}) .$$

If $j_{\max} < j_{\mathrm{pk}}$, then

$$q(j) \le q(j_{\max})$$

for all $j < q(j_{\max})$. Hence

$$\max\{q(j) : j_{\min} \le j \le j_{\max}\} = q(j_{\max}) .$$

In the remaining case that $j_{\min} \le j_{\mathrm{pk}} \le j_{\max}$, we can again apply theorem Th-97 and conclude that $q(j) \le q(j_{\mathrm{pk}})$ for all $j_{\min} \le j \le j_{\max}$. In addition, $j_{\mathrm{pk}} \in \{j : j_{\min} \le j \le j_{\max}\}$. Hence in this case,

$$\max\{q(j) : j_{\min} \le j \le j_{\max}\} = q(j_{\mathrm{pk}}) .$$

Proof of Theorem 99

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set. Further suppose that $q : \{0, \dots, |E|\} \longrightarrow \mathbf{I}$ is the mapping defined by (33) and $j_{\mathrm{pk}} \in \{0, \dots, |E|\}$ is chosen as in Th-97. Now let $X \in \widetilde{\mathcal{P}}(E)$ be a fuzzy subset of E. Then

$$\begin{aligned} &Q_\gamma^{\min}(X) \\ &= q^{\min}(|X|_\gamma^{\min}, |X|_\gamma^{\max}) && \text{by Th-98} \\ &= \min(q(|X|_\gamma^{\min}), q(|X|_\gamma^{\max})) . && \text{by L-83} \end{aligned}$$

Similarly in the case of $Q_\gamma^{\max}(X)$,

$$\begin{aligned}
&Q_\gamma^{\max}(X)\\
&= q^{\max}(|X|_\gamma^{\min}, |X|_\gamma^{\max}) && \text{by Th-98}\\
&= \begin{cases} q(|X|_\gamma^{\min}) & : \quad |X|_\gamma^{\min} > j_{\mathrm{pk}} \\ q(|X|_\gamma^{\max}) & : \quad |X|_\gamma^{\max} < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \quad |X|_\gamma^{\min} \le j_{\mathrm{pk}} \le |X|_\gamma^{\max} \end{cases} && \text{by L-83}
\end{aligned}$$

F.15 Proof of Theorem 100

Lemma 84

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex semi-fuzzy quantifier on a finite base set. Further suppose that $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. Then

$$Q_\gamma^{\min}(X) \ge \min(Q_\gamma^{\min}(X'), Q_\gamma^{\min}(X''))$$

and

$$Q_\gamma^{\max}(X) \ge \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max}(X''))\,,$$

for all $\gamma \in \mathbf{I}$.

Proof

Let us assume that $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex semi-fuzzy quantifier on a finite base set. By Th-96, there exists $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|) \tag{360}$$

for all $Y \in \mathcal{P}(E)$, and

$$q(j) \ge \min(q(j'), q(j'')) \tag{361}$$

whenever $0 \le j' \le j \le j' \le |E|$.

Now suppose that $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$.

Let us first consider $Q_\gamma^{\min}(X)$. By Th-96,

$$q(|X|_\gamma^{\min}) \ge \min(q(|X'|_\gamma^{\min}), q(|X''|_\gamma^{\min})) \tag{362}$$

$$q(|X|_\gamma^{\max}) \ge \min(q(|X'|_\gamma^{\max}), q(|X''|_\gamma^{\max})) \tag{363}$$

because $X' \subseteq X \subseteq X''$ and hence $|X'|_\gamma^{\min} \le |X|_\gamma^{\min} \le |X''|_\gamma^{\min}$ and $|X'|_\gamma^{\max} \le |X|_\gamma^{\max} \le |X''|_\gamma^{\max}$. Therefore

$$\begin{aligned}
Q_\gamma^{\min}(X) &= \min(q(|X|_\gamma^{\min}), q(|X|_\gamma^{\max})) && \text{by Th-99}\\
&\ge \min(\min(q(|X'|_\gamma^{\min}), q(|X''|_\gamma^{\min})), \min(q(|X'|_\gamma^{\max}), q(|X''|_\gamma^{\max}))) && \text{by (362), (363)}\\
&= \min(\min(q(|X'|_\gamma^{\min}), q(|X'|_\gamma^{\max})), \min(q(|X''|_\gamma^{\min}), q(|X''|_\gamma^{\max})))\\
&= \min(Q_\gamma^{\min}(X'), Q_\gamma^{\min}(X''))\,. && \text{by Th-99}
\end{aligned}$$

Now let us consider $Q_\gamma^{\max}(X)$. The proof is by contradiction. Let us assume that

$$Q_\gamma^{\max}(X) < \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max}(X'')) \,. \tag{364}$$

Then there are $Y' \in \mathcal{T}_\gamma(X')$, $Y'' \in \mathcal{T}_\gamma(X'')$ such that

$$Q(Y) < \min(Q(Y'), Q(Y'')) \tag{365}$$

for all $Y \in \mathcal{T}_\gamma(X)$. We shall discern two cases.

a. $|Y'| \leq |Y''|$. Now let

$$z = \max(|X|_\gamma^{\min}, |Y'|) \tag{366}$$

Then apparently

$$|X|_\gamma^{\min} \leq \max(|X|_\gamma^{\min}, |Y'|) = z$$

and

$$\begin{aligned} z &= \max(|X|_\gamma^{\min}, |Y'|) \\ &\leq \max(|X|_\gamma^{\min}, |X'|_\gamma^{\max}) && \text{because } Y' \subseteq (X')_\gamma^{\max} \\ &\leq \max(|X|_\gamma^{\min}, |X|_\gamma^{\max}) && \text{because } X' \subset X\text{, i.e. } |X'|_\gamma^{\max} \leq |X|_\gamma^{\max} \\ &= |X|_\gamma^{\max} \,. && \text{because } (X)_\gamma^{\min} \subseteq (X)_\gamma^{\max} \end{aligned}$$

Hence by L-82, there exists $Z \in \mathcal{T}_\gamma(X)$ such that

$$z = |Z| \,. \tag{367}$$

Let us further observe that

$$|Y'| \leq \max(|X|_\gamma^{\min}, |Y'|) = z$$

and

$$\begin{aligned} z &= \max(|X|_\gamma^{\min}, |Y'|) \\ &\leq \max(|X|_\gamma^{\min}, |Y''|) && \text{by assumption of case a., } |Y'| \leq |Y''| \\ &\leq \max(|X''|_\gamma^{\min}, |Y''|) && \text{because } X \subseteq X'' \\ &= |Y''| \,. && \text{because } (X'')_\gamma^{\min} \subseteq Y'' \end{aligned}$$

Combining this with (367), we conclude that there exists $Z \in \mathcal{T}_\gamma(X)$ such that

$$|Y'| \leq |Z| \leq |Y''| \,. \tag{368}$$

Therefore

$$\begin{aligned} Q_\gamma^{\max}(X) &= \max\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (16), } E \text{ finite} \\ &= \max\{q(|Y|) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (360)} \\ &\geq q(|Z|) && \text{because } Z \in \mathcal{T}_\gamma(X) \\ &\geq \min(q(|Y'|), q(|Y''|)) \,, && \text{by (361)} \end{aligned}$$

which contradicts the assumption (364) that $Q_\gamma^{\max}(X) < \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max} X'')$. We hence conclude that $Q_\gamma^{\max}(X) \geq \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max} X'')$.

b. $|Y'| > |Y''|$. In this case,

$$\begin{aligned} |X|_\gamma^{\min} &\le |X''|_\gamma^{\min} && \text{because } X \subseteq X'' \\ &\le |Y''| && \text{because } (X'')_\gamma^{\min} \subseteq Y'' \\ &< |Y'| && \text{by assumption of case b., } |Y''| < |Y'| \\ &\le |X'|_\gamma^{\max} && \text{because } Y' \subseteq (X')_\gamma^{\max} \\ &\le |X|_\gamma^{\max}\,. && \text{because } X' \subseteq X \text{ and hence } (X')_\gamma^{\max} \subseteq (X)_\gamma^{\max} \end{aligned}$$

Hence by L-82, there exists $Z \in \mathcal{T}_\gamma(X)$ such that $|Y'| = |Z|$, i.e.

$$\begin{aligned} Q_\gamma^{\max}(X) &= \max\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (16), } E \text{ finite} \\ &= \max\{q(|Y|) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (360)} \\ &\ge q(|Z|) && \text{because } Z \in \mathcal{T}_\gamma(X) \\ &= q(|Y'|) && \text{because } |Z| = |Y'| \\ &\ge \min(q(|Y'|), q(|Y''|))\,. \end{aligned}$$

Again, this contradicts the assumption (364) that $Q_\gamma^{\max}(X) < \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max}X'')$. It follows that $Q_\gamma^{\max}(X) \ge \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max}X'')$, as desired.

Lemma 85

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set $E \neq \varnothing$ and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. If there exists some $\gamma'' \in \mathbf{I}$ such that $Q_{\gamma''}^{\max}(X'') > Q_{\gamma''}^{\max}(X)$, then $Q_\gamma^{\max}(X) \ge Q_\gamma^{\max}(X')$ for all $\gamma \in \mathbf{I}$.

Proof By Th-96, there exist $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|) \tag{369}$$

for all $Y \in \mathcal{P}(E)$ and

$$q(j) \ge \min(q(j'), q(j'')) \tag{370}$$

whenever $j' \le j \le j''$. We may further choose $j_{\mathrm{pk}} \in \{0, \ldots, |E|\}$ as in Th-97.
Now let $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. Further suppose that there is some $\gamma'' \in \mathbf{I}$ such that

$$Q_{\gamma''}^{\max}(X'') > Q_{\gamma''}^{\max}(X)\,, \tag{371}$$

and let $\gamma' \in \mathbf{I}$.

i. $|X''|_{\gamma''}^{\max} < j_{\mathrm{pk}}$.

i.a $\gamma' \ge \gamma''$. Then

$$|X'|_{\gamma'}^{\min} \le |X|_{\gamma'}^{\min} \le |X''|_{\gamma'}^{\min} \le |X''|_{\gamma''}^{\min} \le |X''|_{\gamma''}^{\max} < j_{\mathrm{pk}}\,, \tag{372}$$

which is apparent from Def. 66, $X' \subseteq X \subseteq X''$ and $\gamma'' \le \gamma'$. For the same reasons,

$$|X'|_\gamma^{\max} \le |X|_\gamma^{\max} \le |X''|_\gamma^{\max}\,. \tag{373}$$

We shall discern two subcases.

i.a.1 $|X''|_{\gamma'}^{\max} \geq j_{\text{pk}}$.
Then by (372) and Th-99,

$$Q_{\gamma'}^{\max}(X'') = q(j_{\text{pk}})\,. \tag{374}$$

It is also apparent from Th-99 that there exist j' such that

$$Q_{\gamma'}^{\max}(X') = q(j')\,.$$

Hence by Th-97 and (374),

$$Q_{\gamma'}^{\max}(X') = q(j') \leq q(j_{\text{pk}}) = Q_{\gamma'}^{\max}(X'')\,.$$

In particular

$$\min(Q_{\gamma'}^{\max}(X'), Q_{\gamma'}^{\max}(X'')) = Q_{\gamma'}^{\max}(X') \tag{375}$$

Therefore

$$\begin{aligned} Q_{\gamma'}^{\max}(X) &\geq \min(Q_{\gamma'}^{\max}(X'), Q_{\gamma'}^{\max}(X'')) && \text{by L-84} \\ &= Q_{\gamma'}^{\max}(X')\,. && \text{by (375)} \end{aligned}$$

i.a.2 $Q_{\gamma'}^{\max}(X'') < j_{\text{pk}}$. We may then conclude from $X' \subseteq X \subseteq X''$ that

$$|X'|_{\gamma'}^{\max} \leq |X|_{\gamma'}^{\max} \leq |X''|_{\gamma'}^{\max} < j_{\text{pk}} \tag{376}$$

and hence

$$\begin{aligned} Q_{\gamma'}^{\max}(X') &= q(|X'|_{\gamma'}^{\max}) && \text{by Th-99} \\ &\leq q(|X|_{\gamma'}^{\max}) && \text{by Th-97} \\ &= Q_{\gamma'}^{\max}(X)\,. && \text{by Th-99} \end{aligned}$$

i.b $\gamma' < \gamma''$. Then

$$|X'|_{\gamma'}^{\max} \leq |X|_{\gamma'}^{\max} \leq |X''|_{\gamma'}^{\max} \leq |X''|_{\gamma''}^{\max} < j_{\text{pk}}\,, \tag{377}$$

which is apparent from Def. 66 noting that $X' \subseteq X \subseteq X''$ and $\gamma' < \gamma''$. Therefore

$$\begin{aligned} Q_{\gamma'}^{\max}(X) &= q((X)_{\gamma'}^{\max}) && \text{by Th-99} \\ &\geq q((X')_{\gamma'}^{\max}) && \text{by Th-97} \\ &= Q_{\gamma'}^{\max}(X')\,. && \text{by Th-99} \end{aligned}$$

ii. $|X''|_{\gamma''}^{\min} > j_{\text{pk}}$. Then by Th-99,

$$Q_{\gamma''}^{\max}(X'') = q(|X''|_{\gamma''}^{\min})\,. \tag{378}$$

We know from $X \subseteq X''$ that

$$|X|_{\gamma''}^{\min} \leq |X''|_{\gamma''}^{\min}\,. \tag{379}$$

Let us assume that $j_{\mathrm{pk}} \le |X|^{\min}_{\gamma''}$. Then

$$\begin{aligned} Q^{\max}_{\gamma''}(X) &= q(|X|^{\min}_{\gamma''}) && \text{by Th-99} \\ &\ge q(|X''|^{\min}_{\gamma''}) && \text{by Th-97, (379)} \\ &= Q^{\max}_{\gamma''}(X''), \end{aligned}$$

which contradicts the requirement of the lemma that $Q^{\max}_{\gamma''}(X'') > Q^{\max}_{\gamma''}(X)$. Hence the assumption that $j_{\mathrm{pk}} \le |X|^{\min}_{\gamma''}$, i.e.

$$(X)^{\min}_{\gamma''} < j_{\mathrm{pk}}. \tag{380}$$

Now suppose $(X)^{\max}_{\gamma''} \ge j_{\mathrm{pk}}$. Then

$$\begin{aligned} Q^{\max}_{\gamma''}(X) &= q(j_{\mathrm{pk}}) && \text{by Th-99, (380)} \\ &\ge q(|X''|^{\min}_{\gamma''}) && \text{by Th-97} \\ &= Q^{\max}_{\gamma''}(X''), && \text{by (378)} \end{aligned}$$

which again contradicts $Q^{\max}_{\gamma''}(X'') > Q^{\max}_{\gamma''}(X)$. Hence the assumption $j_{\mathrm{pk}} \le |X|^{\min}_{\gamma''}$ is false, and

$$(X)^{\max}_{\gamma''} < j_{\mathrm{pk}}. \tag{381}$$

In the following, we shall again discern two cases.

ii.a $\gamma' \ge \gamma''$. Then

$$|X'|^{\min}_{\gamma'} \le |X|^{\min}_{\gamma'} \le |X|^{\min}_{\gamma''} < j_{\mathrm{pk}}, \tag{382}$$

which is apparent from (31). $X' \subseteq X$, $\gamma'' \le \gamma'$ and (380). If $|X|^{\max}_{\gamma'} \ge j_{\mathrm{pk}}$, then

$$Q^{\max}_{\gamma'}(X) = q(j_{\mathrm{pk}}) \tag{383}$$

by Th-99 and (382). It is also apparent from Th-99 that there exists j' such that $Q^{\max}_{\gamma'}(X') = q(j')$. Then

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &= q(j_{\mathrm{pk}}) && \text{by (383)} \\ &\ge q(j') && \text{by Th-97} \\ &= Q^{\max}_{\gamma'}(X'). && \text{by choice of } j' \end{aligned}$$

In the remaining case that $|X|^{\max}_{\gamma'} < j_{\mathrm{pk}}$, it follows from $X' \subseteq X$ that $(X')^{\max}_{\gamma'} \le (X)^{\max}_{\gamma'} < j_{\mathrm{pk}}$ also. Therefore

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &= q(|X|^{\max}_{\gamma'}) && \text{by Th-99} \\ &\ge q(|X'|^{\max}_{\gamma'}) && \text{by } |X'|^{\max}_{\gamma'} \le |X|^{\max}_{\gamma'} < j_{\mathrm{pk}}, \text{ Th-97} \\ &= Q^{\max}_{\gamma'}(X'). && \text{by Th-99} \end{aligned}$$

ii.b $\gamma' < \gamma''$. Then

$$|X'|^{\max}_{\gamma'} \le |X|^{\max}_{\gamma'} \le |X|^{\max}_{\gamma''} < j_{\mathrm{pk}}, \tag{384}$$

because $X' \subseteq X$, $\gamma' < \gamma''$, and recalling equation (381). Therefore

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &= q(|X|^{\max}_{\gamma'}) && \text{by Th-99, (384)} \\ &\ge q(|X'|^{\max}_{\gamma'}) && \text{by } |X'|^{\max}_{\gamma'} \le |X|^{\max}_{\gamma'} \le j_{\mathrm{pk}}, \text{ Th-97} \\ &= Q^{\max}_{\gamma'}(X'). && \text{by Th-99, (384)} \end{aligned}$$

iii. $|X''|^{\min}_{\gamma''} \leq j_{\mathrm{pk}} \leq |X''|^{\max}_{\gamma''}$.

iii.a $\gamma' \geq \gamma''$. Then

$$|X''|^{\min}_{\gamma'} \leq |X''|^{\min}_{\gamma''} \leq j_{\mathrm{pk}} \leq |X''|^{\max}_{\gamma''} \leq |X''|^{\max}_{\gamma'} \,. \tag{385}$$

By Th-99, there exists j' such that $Q^{\max}_{\gamma'}(X') = q(j')$. Now

$$\begin{aligned} Q^{\max}_{\gamma'}(X'') &= q(j_{\mathrm{pk}}) && \text{by Th-99, (385)} \\ &\geq q(j') && \text{by Th-97} \\ &= Q^{\max}_{\gamma'}(X') \,, \end{aligned}$$

in particular

$$Q^{\max}_{\gamma'}(X') = \min(Q^{\max}_{\gamma'}(X'), Q^{\max}_{\gamma'}(X'')) \,. \tag{386}$$

Therefore

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &\geq \min(Q^{\max}_{\gamma'}(X'), Q^{\max}_{\gamma'}(X'')) && \text{by L-84} \\ &= Q^{\max}_{\gamma'}(X') \,. && \text{by (386)} \end{aligned}$$

iii.b $\gamma' < \gamma''$.

iii.b.1 $|X''|^{\max}_{\gamma'} < j_{\mathrm{pk}}$. Then

$$|X'|^{\max}_{\gamma'} \leq |X|^{\max}_{\gamma'} \leq |X''|^{\max}_{\gamma'} < j_{\mathrm{pk}} \tag{387}$$

and hence

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &= q(|X|^{\max}_{\gamma'}) && \text{by Th-99, (387)} \\ &\geq q(|X'|^{\max}_{\gamma'}) && \text{by } |X'|^{\max}_{\gamma'} \leq |X|^{\max}_{\gamma'} \leq j_{\mathrm{pk}}\text{, Th-97} \\ &= Q^{\max}_{\gamma'}(X') \,. && \text{by Th-99, (387)} \end{aligned}$$

iii.b.2 $|X''|^{\min}_{\gamma'} > j_{\mathrm{pk}}$.
Suppose $|X'|^{\min}_{\gamma'} \geq j_{\mathrm{pk}}$. Then

$$|X'|^{\max}_{\gamma'} \geq |X'|^{\min}_{\gamma'} \geq j_{\mathrm{pk}}$$

Hence

$$\begin{aligned} |X'|^{\min}_{\gamma''} \leq |X''|^{\min}_{\gamma''} \leq |X''|^{\min}_{\gamma'} &\leq j_{\mathrm{pk}} \\ |X'|^{\max}_{\gamma''} \geq |X'|^{\max}_{\gamma'} &\geq j_{\mathrm{pk}} \end{aligned}$$

because $\gamma' < \gamma''$, $X' \subseteq X''$. Therefore

$$|X'|^{\min}_{\gamma''} \leq j_{\mathrm{pk}} \leq |X'|^{\max}_{\gamma''} \,, \tag{388}$$

and hence

$$\begin{aligned} Q^{\max}_{\gamma''}(X') &= q(j_{\mathrm{pk}}) && \text{by Th-99, (388)} \\ &= Q^{\max}_{\gamma''}(X'') && \text{by assumptions of case iii.b and Th-99} \\ &> Q^{\max}_{\gamma''}(X)\,, && \text{by precondition the lemma.} \end{aligned}$$

which contradicts L-84. (This case is hence not possible.)
In the remaining case that $|X'|^{\min}_{\gamma'} < j_{\mathrm{pk}}$, suppose $|X|^{\min}_{\gamma'} < j_{\mathrm{pk}}$. Then $|X'|^{\min}_{\gamma'} \le |X|^{\min}_{\gamma'}$ because $X' \subseteq X$ and hence

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &= q((X)^{\min}_{\gamma'}) && \text{by Th-99, } |X|^{\min}_{\gamma'} < j_{\mathrm{pk}} \\ &\ge q((X')^{\min}_{\gamma'}) && \text{by } |X'|^{\min}_{\gamma'} \le |X|^{\min}_{\gamma'} \le j_{\mathrm{pk}}, \text{Th-97} \\ &= Q^{\max}_{\gamma'}(X')\,. && \text{by Th-99, } |X'|^{\min}_{\gamma'} < j_{\mathrm{pk}} \end{aligned}$$

Finally if $|X|^{\min}_{\gamma'} < j_{\mathrm{pk}}$ and $|X|^{\min}_{\gamma'} \ge j_{\mathrm{pk}}$, then

$$\begin{aligned} &|X|^{\min}_{\gamma''} \le |X''|^{\min}_{\gamma''} \le j_{\mathrm{pk}} \\ &|X''|^{\max}_{\gamma''} \ge |X|^{\max}_{\gamma''} \ge |X|^{\max}_{\gamma'} \ge |X|^{\max}_{\gamma'} \ge j_{\mathrm{pk}} \end{aligned}$$

because $\gamma' < \gamma''$, i.e.

$$|X|^{\min}_{\gamma''} \le j_{\mathrm{pk}} \le |X|^{\max}_{\gamma''}\,. \tag{389}$$

Hence

$$\begin{aligned} Q^{\max}_{\gamma''}(X) &= q(j_{\mathrm{pk}}) && \text{by Th-99} \\ &= Q^{\max}_{\gamma'}(X'')\,, && \text{by Th-99} \end{aligned}$$

which contradicts the precondition of the lemma that $Q^{\max}_{\gamma''}(X'') > Q^{\max}_{\gamma''}(X)$. The last subcase that $|X|^{\min}_{\gamma'} < j_{\mathrm{pk}}$ and $|X|^{\min}_{\gamma'} \ge j_{\mathrm{pk}}$ is hence not possible.

iii.b.3 $|X''|^{\min}_{\gamma'} \le j_{\mathrm{pk}} \le |X''|^{\max}_{\gamma'}$. Then

$$Q^{\max}_{\gamma'}(X'') = q(j_{\mathrm{pk}}) \tag{390}$$

by Th-99. It is also apparent from this theorem that there is a j' such that $Q^{\max}_{\gamma'}(X') = q(j')$. Hence

$$\begin{aligned} Q^{\max}_{\gamma'}(X'') &= q(j_{\mathrm{pk}}) && \text{by (390)} \\ &\ge q(j') && \text{by Th-97} \\ &= Q^{\max}_{\gamma'}(X')\,. \end{aligned}$$

In particular,

$$Q^{\max}_{\gamma'}(X') = \min(Q^{\max}_{\gamma'}(X'), Q^{\max}_{\gamma'}(X''))\,. \tag{391}$$

Therefore

$$\begin{aligned} Q^{\max}_{\gamma'}(X) &\ge \min(Q^{\max}_{\gamma'}(X'), Q^{\max}_{\gamma'}(X'')) && \text{by L-84} \\ &= Q^{\max}_{\gamma'}(X')\,. && \text{by (391)} \end{aligned}$$

Lemma 86
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a convex semi-fuzzy quantifier. Then the antonym $Q\neg : \mathcal{P}(E) \longrightarrow \mathbf{I}$ defined by Def. 11 is also convex.

Proof

Let $Y', Y, Y'' \in \mathcal{P}(E)$ such that $Y' \subseteq Y \subseteq Y''$. Then

$$\neg Y'' \subseteq \neg Y \subseteq \neg Y\,. \tag{392}$$

Therefore

$$\begin{aligned} Q\neg(Y) &= Q(\neg Y) && \text{by Def. 11} \\ &\geq \min(Q(\neg Y'), Q(\neg Y'')) && \text{by (392), } Q \text{ convex} \\ &= \min(Q\neg(Y'), Q\neg(Y''))\,. && \text{by Def. 11} \end{aligned}$$

Lemma 87

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set $E \neq \varnothing$ and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. If there exists some $\gamma' \in \mathbf{I}$ such that $Q_{\gamma'}(X') > Q_{\gamma'}(X)$, then $Q_\gamma(X) \geq Q_\gamma(X'')$ for all $\gamma \in \mathbf{I}$.

Proof

We shall reduce this case to L-85 by using the antonym $Q\neg$. Hence suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ satisfy $X' \subseteq X \subseteq X''$. Further suppose that there exists $\gamma' \in \mathbf{I}$ such that

$$Q_{\gamma'}(X') > Q_{\gamma'}(X)\,. \tag{393}$$

Abbreviating $Z = \neg X$, $Z' = \neg X'$ and $Z'' = \neg X''$, we apparently have

$$Z'' \subseteq Z \subseteq Z'\,. \tag{394}$$

In addition,

$$\begin{aligned} Q_\gamma(X) &= (Q\neg\neg)_\gamma(X) = (Q\neg)_\gamma(\neg X) = (Q\neg)_\gamma(Z) \\ Q_\gamma(X') &= (Q\neg\neg)_\gamma(X') = (Q\neg)_\gamma(\neg X') = (Q\neg)_\gamma(Z') \\ Q_\gamma(X'') &= (Q\neg\neg)_\gamma(X'') = (Q\neg)_\gamma(\neg X'') = (Q\neg)_\gamma(Z'') \end{aligned}$$

by L-31, i.e.

$$Q_\gamma^{\max}(X) = (Q\neg)_\gamma^{\max}(Z) \tag{395}$$

$$Q_\gamma^{\max}(X') = (Q\neg)_\gamma^{\max}(Z') \tag{396}$$

$$Q_\gamma^{\max}(X'') = (Q\neg)_\gamma^{\max}(Z'') \tag{397}$$

for all $\gamma \in \mathbf{I}$.
We know from L-86 that $Q\neg$ is convex; it is also apparently quantitative (see Def. 31).
In addition,

$$\begin{aligned} (Q\neg)_{\gamma'}^{\max}(Z') &= Q_{\gamma'}^{\max}(X') && \text{by (396)} \\ &> Q_{\gamma'}^{\max}(X) && \text{by (393)} \\ &= (Q\neg)_{\gamma'}^{\max}\,. && \text{by (395)} \end{aligned}$$

Therefore

$$\begin{aligned} Q^{\max}_{\gamma''}(X) &= (Q\neg)^{\max}_{\gamma''}(Z) && \text{by (395)} \\ &\geq (Q\neg)^{\max}_{\gamma''}(Z'') && \text{by L-85, (394)} \\ &= Q^{\max}_{\gamma''}(X'') && \text{by (397)}\,, \end{aligned}$$

as desired.

Lemma 88

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

a. If $Q_0(X_1, \dots, X_n) \geq \frac{1}{2}$*, then*

$$Q_\gamma(X_1, \dots, X_n) = \max(\tfrac{1}{2}, Q^{\min}_\gamma(X_1, \dots, X_n))$$

for all $\gamma \in \mathbf{I}$.

b. If $Q_0(X_1, \dots, X_n) \leq \frac{1}{2}$*, then*

$$Q_\gamma(X_1, \dots, X_n) = \min(\tfrac{1}{2}, Q^{\max}_\gamma(X_1, \dots, X_n))$$

for all $\gamma \in \mathbf{I}$.

Proof

Case a.: $Q_0(X_1, \dots, X_n) \geq \frac{1}{2}$

Let us first observe that by Def. 45,

$$\mathrm{m}_{\frac{1}{2}}(a, b) = \max(\tfrac{1}{2}, \min(a, b)) \tag{398}$$

whenever $\max(a, b) \geq \frac{1}{2}$. In addition, we may conclude from

$$\begin{aligned} \tfrac{1}{2} &\leq Q_0(X_1, \dots, X_n) \\ &= \mathrm{m}_{\frac{1}{2}}(Q^{\max}_0(X_1, \dots, X_n), Q^{\min}_{(X_1,\dots,X_n)}) && \text{by (14)} \end{aligned}$$

that

$$Q^{\max}_0(X_1, \dots, X_n) \geq \tfrac{1}{2}\,. \tag{399}$$

Now for arbitrary $\gamma \in \mathbf{I}$, $\mathcal{T}_0(X_i) \subseteq \mathcal{T}_\gamma(X_i)$ for $i = 1, \dots, n$ by Def. 66 and hence

$$\begin{aligned} Q^{\max}_\gamma(X_1, \dots, X_n) &= \sup\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (16)} \\ &\leq \sup\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_0(X_i)\} && \text{because } \mathcal{T}_0(X_i) \subseteq \mathcal{T}_\gamma(X_i), \text{ all } i \\ &= Q^{\max}_0(X_1, \dots, X_n) && \text{by (16)} \\ &\geq \tfrac{1}{2}\,. \end{aligned}$$

Hence by (398) and (399),

$$Q_\gamma(X_1, \dots, X_n) = \mathrm{m}_{\frac{1}{2}}(Q^{\max}_\gamma(X), Q^{\min}_\gamma(X)) = \max(\tfrac{1}{2}, Q^{\min}_\gamma(X_1, \dots, X_n))\,, \tag{400}$$

as desired.

Case b.: $Q_0(X_1, \ldots, X_n) \geq \frac{1}{2}$
The proof of this case is completely analogous to that of **a.** We first observe that by Def. 45,

$$\mathrm{m}_{\frac{1}{2}}(a, b) = \min(\tfrac{1}{2}, \max(a, b)) \tag{401}$$

whenever $\min(a, b) \leq \frac{1}{2}$. We may then conclude from

$$\begin{aligned} \tfrac{1}{2} &\geq Q_0(X_1, \ldots, X_n) \\ &= \mathrm{m}_{\frac{1}{2}}(Q_0^{\max}(X_1, \ldots, X_n), Q^{\min}_{(X_1,\ldots,X_n)}) && \text{by (14)} \end{aligned}$$

that

$$Q_0^{\min}(X_1, \ldots, X_n) \leq \tfrac{1}{2}\,. \tag{402}$$

Now for arbitrary $\gamma \in \mathbf{I}$, $\mathcal{T}_0(X_i) \subseteq \mathcal{T}_\gamma(X_i)$ for $i = 1, \ldots, n$ by Def. 66 and hence

$$\begin{aligned} Q_\gamma^{\min}(X_1, \ldots, X_n) &= \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)} \\ &\leq \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_0(X_i)\} && \text{because } \mathcal{T}_0(X_i) \subseteq \mathcal{T}_\gamma(X_i)\text{, all } i \\ &= Q_0^{\min}(X_1, \ldots, X_n) && \text{by (15)} \\ &\leq \tfrac{1}{2}\,. \end{aligned}$$

Hence by (401) and (402),

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\max}(X), Q_\gamma^{\min}(X)) = \min(\tfrac{1}{2}, Q_\gamma^{\max}(X_1, \ldots, X_n))\,. \tag{403}$$

Lemma 89
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set $E \neq \varnothing$ and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. Further assume that $Q_0(X') < \frac{1}{2}$, $Q_0(X) < \frac{1}{2}$ and $Q_0(X'') < \frac{1}{2}$. Then at least one of the following conditions holds:

a. $Q_\gamma(X) \geq Q_\gamma(X')$ *for all* $\gamma \in \mathbf{I}$*;*

b. $Q_\gamma(X) \geq Q_\gamma(X'')$ *for all* $\gamma \in \mathbf{I}$.

Proof
Let us first apply L-88 to obtain that

$$Q_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\max}(X), Q_\gamma^{\min}(X)) = \min(\tfrac{1}{2}, Q_\gamma^{\max}(X)) \tag{404}$$

and for the same reasons,

$$Q_\gamma(X') = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\max}(X'), Q_\gamma^{\min}(X')) = \min(\tfrac{1}{2}, Q_\gamma^{\max}(X')) \tag{405}$$

and

$$Q_\gamma(X'') = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\max}(X''), Q_\gamma^{\min}(X'')) = \min(\tfrac{1}{2}, Q_\gamma^{\max}(X'')) \tag{406}$$

for all $\gamma \in \mathbf{I}$.

Case 1. Let us suppose that condition a. of the lemma does not hold, i.e. the negated condition is fulfilled: There exists some $\gamma' \in \mathbf{I}$ such that

$$Q_{\gamma'}(X) < Q_{\gamma'}(X')\,. \tag{407}$$

We have to show that condition b. holds in this situation. Firstly, from

$$Q_{\gamma'}(X) = \min(\tfrac{1}{2}, Q^{\max}_{\gamma'}(X)) < \min(\tfrac{1}{2}, Q^{\max}_{\gamma'}(X')) \leq \tfrac{1}{2}$$

we obtain that $Q_{\gamma'}(X) = Q^{\max}_{\gamma'}(X)$. Hence

$$Q_{\gamma'}(X) = Q^{\max}_{\gamma'}(X) < \min(\tfrac{1}{2}, Q^{\max}_{\gamma'}(X') \leq Q^{\max}_{\gamma'}(X')\,.$$

We can hence apply L-87 to obtain that

$$Q^{\max}_{\gamma}(X) \geq Q^{\max}_{\gamma}(X'') \tag{408}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} Q_\gamma(X) &= \min(\tfrac{1}{2}, Q^{\max}_{\gamma}(X)) && \text{by (404)}\\ &\geq \min(\tfrac{1}{2}, Q^{\max}_{\gamma}(X'')) && \text{by (408)}\\ &= Q_\gamma(X'')\,. && \text{by (406)} \end{aligned}$$

Case 2. Now let us assume that condition b. of the lemma does not hold, i.e. the negated condition is fulfilled: There exists some $\gamma'' \in \mathbf{I}$ such that

$$Q_{\gamma''}(X) < Q_{\gamma''}(X'')\,. \tag{409}$$

We have to show that under these circumstances, condition a. is guaranteed to hold. Firstly, from

$$Q_{\gamma''}(X) = \min(\tfrac{1}{2}, Q^{\max}_{\gamma''}(X)) < \min(\tfrac{1}{2}, Q^{\max}_{\gamma''}(X'')) \leq \tfrac{1}{2}$$

we obtain that $Q_{\gamma''}(X) = Q^{\max}_{\gamma''}(X)$. Hence

$$Q_{\gamma''}(X) = Q^{\max}_{\gamma''}(X) < \min(\tfrac{1}{2}, Q^{\max}_{\gamma''}(X'') \leq Q^{\max}_{\gamma''}(X'')\,.$$

We can hence apply L-85 to obtain that

$$Q^{\max}_{\gamma}(X) \geq Q^{\max}_{\gamma}(X') \tag{410}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} Q_\gamma(X) &= \min(\tfrac{1}{2}, Q^{\max}_{\gamma}(X)) && \text{by (404)}\\ &\geq \min(\tfrac{1}{2}, Q^{\max}_{\gamma}(X')) && \text{by (410)}\\ &= Q_\gamma(X')\,. && \text{by (405)} \end{aligned}$$

As the following lemma shows, all $\mathcal{M}_\mathcal{B}$-DFSes weakly preserve convexity in most (but not necessary all) cases:

Lemma 90

Suppose $\mathcal{M}_\mathcal{B}$ is an $\mathcal{M}_\mathcal{B}$-DFS. Further suppose that $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set $E \neq \varnothing$ and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. If $Q_0(X') \leq \frac{1}{2}$ or $Q_0(X) \leq \frac{1}{2}$ or $Q_0(X'') \leq \frac{1}{2}$, then

$$\mathcal{M}_\mathcal{B}(Q)(X) \geq \min(\mathcal{M}_\mathcal{B}(Q)(X'), \mathcal{M}_\mathcal{B}(Q)(X''))\,.$$

Proof

Suppose $\mathcal{M}_{\mathcal{B}}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS. Then by Th-62, we know that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-1) to (B-5). In addition, we know from Th-63 that $\mathcal{B}$ can be defined in terms of $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to (23).
Now let $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ be a quantitative convex quantifier on a finite base set $E \neq \varnothing$ and $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ a choice of base sets such that $X' \subseteq X \subseteq X''$ and

$$\min\{Q_0(X'), Q_0(X), Q_0(X'')\} \leq \tfrac{1}{2}\,. \tag{411}$$

We will discern the following cases.

Case 1: $Q_0(X) > \frac{1}{2}$
Then by (411), there exists $Z \in \{X', X''\}$ such that $Q_0(Z) \leq \frac{1}{2}$. Hence $(Q_\gamma(X))_{\gamma\in\mathbf{I}} \in \mathbb{B}^+$ and $(Q_\gamma(Z))_{\gamma\in\mathbf{I}} \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$ (see Def. 68). Therefore

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(Q)(X) &= \mathcal{B}((Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 69}\\
&\geq \tfrac{1}{2} && \text{by (23) and } (Q_\gamma(X))_{\gamma\in\mathbf{I}} \in \mathbb{B}^+\\
&\geq \mathcal{B}((Q_\gamma(Z))_{\gamma\in\mathbf{I}}) && \text{by (23) and } (Q_\gamma(Z))_{\gamma\in\mathbf{I}} \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}\\
&= \mathcal{M}_{\mathcal{B}}(Q)(Z)\,. && \text{by Def. 69}
\end{aligned}$$

Case 2: $Q_0(X) = \frac{1}{2}$ Suppose that

$$\min(Q_0(X'), Q_0(X'')) > \tfrac{1}{2}\,. \tag{412}$$

Then

$$\begin{aligned}
\tfrac{1}{2} &< \min(Q_0(X'), Q_0(X'')) && \text{by (412)}\\
&= \min(\max(\tfrac{1}{2}, Q_0^{\min}(X')), \max(\tfrac{1}{2}, Q_0^{\min}(X''))) && \text{by L-88}\\
&= \max(\tfrac{1}{2}, \min(Q_0^{\min}(X'), Q_0^{\min}(X'')))\\
&\leq \max(\tfrac{1}{2}, Q_0^{\min}(X)) && \text{by L-84}\\
&= \tfrac{1}{2}\,, && \text{by L-88}
\end{aligned}$$

i.e. $\frac{1}{2} < \frac{1}{2}$, a contradiction. This proves that condition (412) is never satisfied; by contrast, we always have

$$\min(Q_0(X'), Q_0(X'')) \leq \tfrac{1}{2}$$

Hence there exists $Z \in \{X'.X''\}$ such that $Q_0(Z) \leq \frac{1}{2}$, and

$$\begin{aligned}
\mathcal{M}_{\mathcal{B}}(Q)(X) &= \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) && \text{by Def. 68 because } Q_0(X) = \tfrac{1}{2}\\
&= \tfrac{1}{2} && \text{by (23)}\\
&\geq \mathcal{B}((Q_\gamma(Z))_{\gamma\in\mathbf{I}}) && \text{by (23) as } (Q_\gamma(Z))_{\gamma\in\mathbf{I}} \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}\\
&= \mathcal{M}_{\mathcal{B}}(Q)(Z) && \text{by Def. 69}\\
&\geq \min(\mathcal{M}_{\mathcal{B}}(Q)(X'), \mathcal{M}_{\mathcal{B}}(Q)(X''))\,. && \text{because } Z \in \{X', X''\}
\end{aligned}$$

Case 3: $Q_0(X) < \frac{1}{2}$ Let us assume that

$$\min(Q_0(X'), Q_0(X'')) \geq \tfrac{1}{2}\,. \tag{413}$$

Then

$$\begin{aligned} Q_0(X) &= \min(\tfrac{1}{2}, Q_0^{\max}(X)) && \text{by L-88} \\ &\geq \min(\tfrac{1}{2}, \min(Q_0^{\max}(X'), Q_0^{\max} X'')) && \text{by L-84} \\ &\geq \tfrac{1}{2}\,, && \text{by (413),Def. 45} \end{aligned}$$

which contradicts the assumption of case 3. that $Q_0(X) < \frac{1}{2}$. Hence condition (413) is false; by contrast, we have

$$\min(Q_0(X'), Q_0(X'')) < \tfrac{1}{2}\,,$$

i.e. there exists $Z, Z' \in \{X', X''\}$ such that $Q_0(Z) < \frac{1}{2}$, and $Z' \neq Z$.

a. If $Q_0(Z') \geq \frac{1}{2}$, then

$$\min(Q_\gamma^{\max}(Z), Q_\gamma^{\max}(Z')) = Q_\gamma^{\max}(Z) \tag{414}$$

and hence

$$\begin{aligned} Q_\gamma^{\max}(X) &\geq \min(Q_\gamma^{\max}(X'), Q_\gamma^{\max}(X'')) && \text{by L-84} \\ &= Q_\gamma^{\max}(Z) && \text{by (414), } \{X', X''\} = \{Z, Z'\}\,, \end{aligned}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} Q_\gamma(X) &= \min(\tfrac{1}{2}, Q_\gamma^{\max}(X)) && \text{by L-88} \\ &\geq \min(\tfrac{1}{2}, Q_\gamma^{\max}(Z)) && \text{because } Q_\gamma^{\max}(Z) \leq Q_\gamma^{\max}(X) \\ &= Q_\gamma(Z) && \text{by L-88} \end{aligned}$$

From this we obtain the desired result

$$\begin{aligned} \mathcal{M}_{\mathcal{B}}(Q)(X) &= \mathcal{B}((Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 69} \\ &\geq \mathcal{B}((Q_\gamma(Z))_{\gamma\in\mathbf{I}}) && \text{by (B-5)} \\ &= \mathcal{M}_{\mathcal{B}}(Q)(Z) && \text{by Def. 69} \\ &\geq \min(\mathcal{M}_{\mathcal{B}}(Q)(X'), \mathcal{M}_{\mathcal{B}}(Q)(X''))\,. && \text{because } Z \in \{X', X''\} \end{aligned}$$

b. $Q_0(Z') < \frac{1}{2}$, i.e.

$$\max\{Q_0(X'), Q_0(X), Q_0(X'')\} < \tfrac{1}{2}\,. \tag{415}$$

Then by L-89, there exists $W \in \{X', X''\}$ such that

$$Q_\gamma(X) \geq Q_\gamma(W) \tag{416}$$

for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned} \mathcal{M}_{\mathcal{B}}(Q)(X) &= \mathcal{B}((Q_\gamma(X))_{\gamma\in\mathbf{I}}) && \text{by Def. 69} \\ &\geq \mathcal{B}((Q_\gamma(W))_{\gamma\in\mathbf{I}}) && \text{by (B-5)} \\ &= \mathcal{M}_{\mathcal{B}}(Q)(W) && \text{by Def. 69} \\ &\geq \min(\mathcal{M}_{\mathcal{B}}(Q)(X'), \mathcal{M}_{\mathcal{B}}(Q)(X''))\,, && \text{because } W \in \{X', X''\} \end{aligned}$$

which finishes the proof of the lemma.

Lemma 91
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set $E \neq \varnothing$, and let $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$. If $Q_0(X') > \frac{1}{2}$, $Q_0(X) > \frac{1}{2}$ and $Q_0(X'') > \frac{1}{2}$, then

$$Q_\gamma(X) \geq \min(Q_\gamma(X'), Q_\gamma(X'')),$$

for all $\gamma \in \mathbf{I}$.

Proof
Assume $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex quantifier on a finite base set. Further assume that $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ is a chopice of argument sets such that

$$X' \subseteq X \subseteq X''. \tag{417}$$

and

$$\min\{Q_0(X'), Q_0(X), Q_0(X'')\} > \tfrac{1}{2}. \tag{418}$$

Then

$$\begin{aligned}
Q_\gamma(X) &= \max(\tfrac{1}{2}, Q_\gamma^{\min}(X)) && \text{by L-88, (418)}\\
&\geq \max(\tfrac{1}{2}, \min(Q_\gamma^{\min}(X'), Q_\gamma^{\min}(X''))) && \text{by L-84, (417)}\\
&= \min(\max(\tfrac{1}{2}, Q_\gamma^{\min}(X')), \max(\tfrac{1}{2}, Q_\gamma^{\min}(X''))) && \text{by distributivity of } \min, \max\\
&= \min(Q_\gamma(X'), Q_\gamma(X'')). && \text{by (418)}
\end{aligned}$$

Lemma 92
Suppose $f, f', f'' \in \mathbb{H}$ are such that

$$f(\gamma) \geq \min(f'(\gamma), f''(\gamma))$$

for all $\gamma \in \mathbf{I}$. Then

$$\mathcal{B}'_{CX}(f) \geq \min(\mathcal{B}'_{CX}(f'), \mathcal{B}'_{CX}(f'')).$$

Proof
Let us assume that $f, f', f'' \in \mathbb{H}$ are given such that

$$f(\gamma) \geq \min(f'(\gamma), f''(\gamma)) \tag{419}$$

for all $\gamma \in \mathbf{I}$. Let us abbreviate

$$\omega = \mathcal{B}'_{CX}(f) \tag{420}$$
$$\omega' = \mathcal{B}'_{CX}(f') \tag{421}$$
$$\omega'' = \mathcal{B}'_{CX}(f'') \tag{422}$$

The proof is by contradiction. Hence let us assume that $\mathcal{B}'_{CX}(f) < \min(\mathcal{B}'_{CX}(f'), \mathcal{B}'_{CX}(f''))$, i.e.

$$\omega < \min(\omega', \omega''). \tag{423}$$

Hence there exists $\gamma \in \mathbf{I}$ such that

$$\omega < \gamma < \min(\omega, \omega')\,.$$

Then by Th-94,

$$f(\gamma) < z < f'(\gamma)$$
$$f(\gamma) < z < f''(\gamma)\,,$$

i.e.

$$f(\gamma) < \min(f'(\gamma), f''(\gamma))$$

which contradicts requirement (419) on f, f', f''. We conclude that assumption (423) is false, and the opposite condition $\mathcal{B}'_{CX}(f) \geq \min(\mathcal{B}'_{CX}(f'), \mathcal{B}'_{CX}(f''))$ is true, as desired.

Proof of Theorem 100

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative convex semi-fuzzy quantifier on a finite base set. Further suppose that $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ is a choice of argument sets such that $X' \subseteq X \subseteq X''$. For the proof, we shall discern two situations.

Case a.: $\min\{Q_0(X'), Q_0(X), Q_0(X'')\} > \frac{1}{2}$. Let us abbreviate

$$g(\gamma) = Q_\gamma(X) \tag{424}$$
$$g'(\gamma) = Q_\gamma(X') \tag{425}$$
$$g''(\gamma) = Q_\gamma(X'') \tag{426}$$

for all $\gamma \in \mathbf{I}$. By the assumption of case a., $g(0) > 0$, $g'(0) > 0$ and $g''(0) > 0$, i.e. $g, g', g'' \in \mathbb{B}^+$. We can apply L-91 to conclude that

$$g(\gamma) \geq \min(g'(\gamma), g''(\gamma))\,, \tag{427}$$

for all $\gamma \in \mathbf{I}$.
Because $g, g', g'' \in \mathbb{B}^+$, we can define $f, f', f'' \in \mathbb{H}$ by

$$f = 2g - 1 \tag{428}$$
$$f' = 2g' - 1 \tag{429}$$
$$f'' = 2g'' - 1 \tag{430}$$

We then obtain from (427) that

$$f(\gamma) \geq \min(f'(\gamma), f''(\gamma)) \tag{431}$$

for all $\gamma \in \mathbf{I}$. Applying L-92, we conclude that

$$\mathcal{B}'_{CX}(f) \geq \min(\mathcal{B}'_{CX}(f'), \mathcal{B}'_{CX}(f''))\,. \tag{432}$$

Therefore

$$\begin{aligned}
\mathcal{M}_{CX}(Q)(X) &= \mathcal{B}_{CX}(g) && \text{by Def. 69, (424)}\\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(f) && \text{by (23), (428)}\\
&\geq \tfrac{1}{2} + \tfrac{1}{2}\min(\mathcal{B}'_{CX}(f'), \mathcal{B}'_{CX}(f'')) && \text{by (432)}\\
&= \min(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(f'), \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(f'')) &&\\
&= \min(\mathcal{B}_{CX}(g'), \mathcal{B}_{CX}(g'')) && \text{by (23), (429), (430)}\\
&= \min(\mathcal{M}_{CX}(Q)(X'), \mathcal{M}_{CX}(Q)(X''))\,. && \text{by Def. 69, (425), (426)}
\end{aligned}$$

Case b.: $\min\{Q_0(X'), Q_0(X), Q_0(X'')\} \leq \frac{1}{2}$.
Recalling that $\mathcal{M}_{CX}$ is an $\mathcal{M}_{\mathcal{B}}$-DFS by Th-93, we can directly apply L-90 to obtain that

$$\mathcal{M}_{CX}(Q)(X) \geq \min(\mathcal{M}_{CX}(Q)(X'), \mathcal{M}_{CX}(Q)(X'')) \,,$$

as desired.

F.16 Proof of Theorem 101

Lemma 93
Suppose $f \in \mathbb{B}^+$ and there exist $m \in \mathbb{N} \setminus \{0,\}$, $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$ such that $f(0) = f(\gamma_1)$, and

$$f(\gamma) = f(\gamma_j)$$

for all $\gamma \in (\gamma_{j-1}, \gamma_j]$, $j = 1, \ldots, m$. Further suppose that $\gamma_* \in (0,1]$ is given, and $f_1, f_2 \in \mathbb{B}$ are defined by

$$f_1(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \gamma_* \\ \frac{1}{2} & : \quad \gamma > \gamma_* \end{cases}$$
$$f_2(\gamma) = f(\gamma_*)$$

for all $\gamma \in \mathbf{I}$. Then there exists a convex quantitative semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ and a choice of fuzzy subsets $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ such that $X' \subseteq X \subseteq X''$ and

$$f(\gamma) = Q_\gamma(X)$$
$$f_1(\gamma) = Q_\gamma(X')$$
$$f_2(\gamma) = Q_\gamma(X'')$$

for all $\gamma \in \mathbf{I}$.

Proof We can assume without loss of generality that $m > 1$ (if $m = 1$, simply add another γ_j). We can further assume that there exists $j_* \in \{1, \ldots, m\}$ such that $\gamma_* = \gamma_{j_*}$ (otherwise, simply add γ_{j_*} to the γ_j's). Let us abbreviate

$$k = m + 1 + j_* \,. \tag{433}$$

We shall define $q : \{0, \ldots, k\} \longrightarrow \mathbf{I}$ as follows.

$$q(j) = \begin{cases} \frac{1}{2} & : \quad j = 0 \\ f(\gamma_{m-j+1}) & : \quad 1 \leq j \leq m \\ 1 & : \quad j = m+1 \\ f(\gamma_{j-(m+1)}) & : \quad m+2 \leq j \leq k = m+1+j_* \end{cases} \tag{434}$$

for all $j \in \{0, \ldots, k\}$. We will define a one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on the finite base set $E = \{1, \ldots, k\}$ by

$$Q(Y) = q(|Y|) \tag{435}$$

for all $Y \in \mathcal{P}(E)$. Recalling that $f \in \mathbb{B}^+$ is nonincreasing by Def. 68 and observing that $\gamma_{j'} > \gamma_j$ whenever $j' > j$, it is apparent from Th-96 that Q is quantitative and convex. In addition,

$$j_{\text{pk}} = m + 1 \tag{436}$$

is obviously a proper choice for j_{pk} in Th-97.
Let us now define fuzzy subsets $X', X, X'' \in \widetilde{\mathcal{P}}(E)$ as follows.

$$\mu_{X'}(j) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma_{j_*} & : \ 1 \le j \le m+1 \\ 0 & : \ j > m+1 \end{cases} \tag{437}$$

$$\mu_X(j) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma_{j_*} & : \ 1 \le j \le j_* \\ \frac{1}{2} + \frac{1}{2}\gamma_j & : \ j_* + 1 \le j \le m \\ \frac{1}{2} + \frac{1}{2}\gamma_{j_*} & : \ j = m+1 \\ \frac{1}{2} & : \ j = m+2 \\ \frac{1}{2} - \frac{1}{2}\gamma_{j-(m+2)} & : \ m+3 \le j \le k = m+1+j_* \end{cases} \tag{438}$$

$$\mu_{X''}(j) = 1 \tag{439}$$

for all $j \in E = \{1, \ldots, k\}$. Then clearly

$$X' \subseteq X \subseteq X''. \tag{440}$$

Let us now consider $Q_\gamma(X')$. If $\gamma \le \gamma_*$, then $|X'|_\gamma^{\min} = |X'|_\gamma^{\max} = m + 1$ by (437), Def. 66 and hence

$$\begin{aligned} Q_\gamma(X') &= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X'), Q_\gamma^{\max}(X')) && \text{by (14)} \\ &= \mathrm{m}_{\frac{1}{2}}(\min(q(m+1), q(m+1)), q(m+1)) && \text{by Th-99, } |X'|_\gamma^{\min} = |X'|_\gamma^{\max} = m+1 = j_{\text{pk}} \\ &= q(m+1) && \text{by Def. 45} \\ &= 1\,. && \text{by (434)} \end{aligned}$$

In the remaining case that $\gamma > \gamma_*$, $|X'|_\gamma^{\min} = 0$ and $|X'|_\gamma^{\max} = m + 1$ by (437) and Def. 66. Therefore

$$\begin{aligned} Q_\gamma(X') &= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X'), Q_\gamma^{\max}(X')) && \text{by (14)} \\ &= \mathrm{m}_{\frac{1}{2}}(\min(q(0), q(m+1)), q(m+1)) && \text{by Th-99, } |X'|_\gamma^{\min} = 0,\ |X'|_\gamma^{\max} = m+1 = j_{\text{pk}} \\ &= \mathrm{m}_{\frac{1}{2}}(\min(\tfrac{1}{2}, 1), 1) && \text{by (434)} \\ &= \mathrm{m}_{\frac{1}{2}}(\tfrac{1}{2}, 1) \\ &= \tfrac{1}{2}\,. && \text{by Def. 45} \end{aligned}$$

Hence $Q_\gamma(X') = f_1(\gamma)$ for all $\gamma \in \mathbf{I}$, as desired.
Let us now consider $Q_\gamma(X'')$. Because $X'' = E$ is crisp by (439),

$$\begin{aligned} Q_\gamma(X'') &= Q(X'') && \text{by L-44} \\ &= Q(E) && \text{because } X'' = E \\ &= q(|E|) && \text{by (435)} \\ &= q(k) && \text{because } |E| = |\{1, \ldots, k\}| = k \\ &= \gamma_*\,, && \text{by (434), } \gamma_{j_*} = \gamma_* \end{aligned}$$

i.e. $Q_\gamma(X'') = \gamma_* = f_2(\gamma)$ for all $\gamma \in \mathbf{I}$.
Finally, let us consider $Q_\gamma(X)$. If $\gamma = 0$, then by Def. 66 and (438),

$$(X)_0^{\min} = (X)_{>\frac{1}{2}} = \{1, \ldots, m+1\}$$
$$(X)_0^{\max} = (X)_{\geq\frac{1}{2}} = \{1, \ldots, m+2\}\,.$$

Hence by (31), (32), $|X|_0^{\min} = m+1$ and $|X|_0^{\max} = m+2$, i.e.

$$\begin{aligned}
Q_0(X) &= \mathrm{m}_{\frac{1}{2}}(Q_0^{\min}(X), Q_0^{\max}(X)) && \text{by (14)}\\
&= \mathrm{m}_{\frac{1}{2}}(\min(q(m+1), q(m+2)), q(m+1)) && \text{by Th-99, } |X|_0^{\min} = m+1 = j_{\mathrm{pk}}\\
& && \text{and } |X|_0^{\max} = m+2\\
&= \mathrm{m}_{\frac{1}{2}}(\min(1, f(\gamma_1)), f(\gamma_1)) && \text{by (435)}\\
&= \mathrm{m}_{\frac{1}{2}}(\min(1, f(0)), f(0)) && \text{by assumption, } f(0) = f(\gamma_1)\\
&= f(0)\,. && \text{by L-46}
\end{aligned}$$

If $0 < \gamma \leq \gamma_*$, there exists $j' \in \{1, \ldots, j_*\}$ such that $\gamma \in (\gamma_{j'-1}, \gamma_{j'}]$. Then by Def. 66 and (438),

$$(X)_\gamma^{\min} = (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{1, \ldots, m+1\}$$
$$(X)_\gamma^{\max} = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{1, \ldots, j'+m+1\}\,.$$

Therefore $|X|_\gamma^{\min} = m+1$ and $|X|_\gamma^{\max} = j'+m+1$. We conclude that

$$\begin{aligned}
Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X), Q_\gamma^{\max}(X)) && \text{by (14)}\\
&= \mathrm{m}_{\frac{1}{2}}(\min(q(m+1), q(j'+m+1)), q(m+1)) && \text{by Th-99, } |X|_\gamma^{\min} = m+1 = j_{\mathrm{pk}}\\
& && \text{and } |X|_\gamma^{\max} = j'+m+1\\
&= \mathrm{m}_{\frac{1}{2}}(\min(1, f(\gamma_{j'})), 1) && \text{by (435)}\\
&= \mathrm{m}_{\frac{1}{2}}(f(\gamma_{j'}), 1)\\
&= f(\gamma_{j'}) && \text{by L-46}\\
&= f(\gamma)\,. && \text{by assumption on } f,\ \gamma \in (\gamma_{j'-1}, \gamma_{j'}]
\end{aligned}$$

Finally if $\gamma > \gamma_*$, then there exists $j' > j_*$ such that $\gamma \in (\gamma_{j'-1}, \gamma_j]$. Again by Def. 66 and (438),

$$(X)_\gamma^{\min} = (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{j', \ldots, m\}$$
$$(X)_\gamma^{\max} = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{1, \ldots, k\}\,.$$

Therefore $|X|_\gamma^{\min} = m - j' + 1$, $|X|_\gamma^{\max} = k$, and hence

$Q_\gamma(X)$	
$= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X), Q_\gamma^{\max}(X))$	by (14)
$= \mathrm{m}_{\frac{1}{2}}(\min(q(m-j'+1), q(k)), q(m+1))$	by Th-99 and
	$|X|_\gamma^{\min} = m - j' + 1 \le m + 1 = j_{\mathrm{pk}} \le k = |X|_\gamma^{\max}$
$= \mathrm{m}_{\frac{1}{2}}(\min(f(\gamma_{j'}), f(\gamma_{j_*})), 1)$	by (435)
$= \mathrm{m}_{\frac{1}{2}}(f(\gamma_{j'}), 1)$	because f noninc., $j' > j_*$
$= f(\gamma_{j'})$	by L-46
$= f(\gamma)\,.$	by assumption on f, $\gamma \in (\gamma_{j'-1}, \gamma_{j'}]$

Lemma 94

Suppose $g \in \mathbb{H}$ and there exist $m \in \mathbb{N} \setminus \{0,\}$, $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \gamma_m = 1$ such that $g(0) = g(\gamma_1)$, and

$$g(\gamma) = g(\gamma_j)$$

for all $\gamma \in (\gamma_{j-1}, \gamma_j]$, $j = 1, \ldots, m$. Further suppose that $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given, and that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (23) *and Def. 69. If $\mathcal{M}_\mathcal{B}$ is a DFS and weakly preserves convexity, then*

$$\mathcal{B}'(g) \ge \mathcal{B}'_{CX}(g)\,.$$

Proof

Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is a mapping such that the QFM $\mathcal{M}_\mathcal{B}$ defined by (23) and Def. 69 is a DFS and weakly preserves convexity. In particular, we know from Th-62 that $\mathcal{B}$ satisfies (B-1) and (B-3).

Further suppose that $g \in \mathbb{H}$ has the properties assumed in the lemma. We shall define define $f \in \mathbb{B}^+$ by $f = \frac{1}{2} + \frac{1}{2}g$. f apparently exhibits the properties required by L-93. Hence for all $\gamma_* \in (0, 1]$, there exists a quantitative convex semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ and fuzzy subsets $X', X, X'' \in \widetilde{\mathcal{P}}(E)$, $X' \subseteq X \subseteq X''$ such that

$$f(\gamma) = Q_\gamma(X), \tag{441}$$
$$f_1(\gamma) = Q_\gamma(X') \tag{442}$$

and

$$f_2(\gamma) = Q_\gamma(X'')\,, \tag{443}$$

using the abbreviations of L-93. Because $\mathcal{M}_\mathcal{B}$ weakly preserves convexity,

$\mathcal{B}(f) = \mathcal{M}_\mathcal{B}(Q)(X)$	by (441), Def. 69
$\ge \min(\mathcal{M}_\mathcal{B}(Q)(X'), \mathcal{M}_\mathcal{B}(Q)(X''))$	because $\mathcal{M}_\mathcal{B}$ weakly convex
$= \min(\mathcal{B}(f_1), \mathcal{B}(f_2))$	by (442), (443), Def. 69
$= \min(\frac{1}{2} + \frac{1}{2}\gamma_*, f(\gamma_*))\,,$	by (B-1),(B-3)

i.e.

$$\mathcal{B}(f) \geq \min(\tfrac{1}{2} + \tfrac{1}{2}\gamma_*, f(\gamma_*)) \,. \tag{444}$$

Hence

$$\begin{aligned}
\mathcal{B}'(g) &= 2\mathcal{B}(f) - 1 && \text{by (24), } f = \tfrac{1}{2} + \tfrac{1}{2}g \\
&\geq 2 \cdot \min(\tfrac{1}{2} + \tfrac{1}{2}\gamma_*, f(\gamma_*)) - 1 && \text{by (444)} \\
&= \min(2(\tfrac{1}{2} + \tfrac{1}{2}\gamma_*) - 1, 2f(\gamma_*) - 1) \\
&= \min(\gamma_*, g(\gamma_*)) \,, && \text{because } g = 2f - 1
\end{aligned}$$

Because $\gamma_* \in (0, 1]$ was chosen arbitrarily, we conclude that

$$\begin{aligned}
\mathcal{B}'(g) &\geq \sup\{\min(\gamma, g(\gamma)) : \gamma \in (0, 1]\} \\
&= \sup\{\min(\gamma, g(\gamma)) : \gamma \in (0, 1]\} \cup \{0\} \\
&= \sup\{\min(\gamma, g(\gamma)) : \gamma \in \mathbf{I}\} \\
&= \mathcal{B}'_{CX}(g) \,. && \text{by Th-94} \,.
\end{aligned}$$

Proof of Theorem 101

Suppose $\mathcal{M}_\mathcal{B}$ is a DFS which weakly preserves convexity. Then by L-94, $\mathcal{B}'(f) \geq \mathcal{B}'_{CX}(f)$ for all $f \in \mathbb{H}$. Hence by Th-86, $\mathcal{M}_{CX} \preceq_c \mathcal{M}_\mathcal{B}$.

F.17 Proof of Theorem 102

Lemma 95
Suppose $J \neq \varnothing$ is an index set and $(A_j)_{j\in J}$ is a J-indexed family of subsets $A_j \in \mathcal{P}(\mathbf{I}) \setminus \{\varnothing\}$. Then

$$\mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} = \mathrm{m}_{\frac{1}{2}} \bigcup_{j\in J} A_j \,.$$

Proof
The proof is based on the observation that $\mathrm{m}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ as defined by Def. 45 satisfies the equation

$$\mathrm{m}_{\frac{1}{2}}(a, b) = \begin{cases} \max(\frac{1}{2}, \min(a, b)) & : \quad \max(a, b) \geq \frac{1}{2} \\ \min(\frac{1}{2}, \max(a, b)) & : \quad \min(a, b) \leq \frac{1}{2} \end{cases} \tag{445}$$

Hence if $X \in \mathcal{P}(\mathbf{I})$, $X \neq \varnothing$ (i.e. $\inf X \leq \sup X$),

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X, \sup X) = \begin{cases} \max(\frac{1}{2}, \inf X) & : \quad \sup X \geq \frac{1}{2} \\ \min(\frac{1}{2}, \sup X) & : \quad \inf X \leq \frac{1}{2} \end{cases} \tag{446}$$

by Def. 46 and (445). In order to profit from this reformulation of $\mathrm{m}_{\frac{1}{2}}$, let us abbreviate

$$\begin{aligned}
J^+ &= \{j \in J : \sup A_j \geq \tfrac{1}{2}\} \\
J^- &= \{j \in J : \inf A_j \leq \tfrac{1}{2}\}
\end{aligned}$$

Apparently $J = J^+ \cup J^-$. The sets J^+ and J^- are not necessarily disjoint but this will not pose problems because of the idempotency of $\mathrm{m}_{\frac{1}{2}}$.
Now for every $j \in J^+$,

$$\mathrm{m}_{\frac{1}{2}} A_j = \max(\tfrac{1}{2}, \inf A_j) \tag{447}$$

and for every $j \in J^-$,

$$\mathrm{m}_{\frac{1}{2}} A_j = \min(\tfrac{1}{2}, \sup A_j)\,, \tag{448}$$

which is obvious from (446) and the definition of J^+ and J^-.
In the following, we will treat separately three cases.

Case a.: $J^- = \varnothing$, i.e. $J = J^+$.
Then

$$\begin{aligned}
&\mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} \\
&= \mathrm{m}_{\frac{1}{2}}\{\max(\tfrac{1}{2}, \inf A_j) : j \in J\} && \text{by (447), } J = J^+ \\
&= \max(\tfrac{1}{2}, \inf\{\max(\tfrac{1}{2}, \inf A_j) : j \in J\}) && \text{by (446)} \\
&= \max(\tfrac{1}{2}, \max(\tfrac{1}{2}, \inf\{\inf A_j : j \in J\})) && \text{by distributivity} \\
&= \max(\tfrac{1}{2}, \inf\{\inf A_j : j \in J\}) && \text{because } \max \text{ idempotent} \\
&= \max(\tfrac{1}{2}, \inf \bigcup_{j \in J} A_j)\,,
\end{aligned}$$

i.e.

$$\mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} = \max(\tfrac{1}{2}, \inf \bigcup_{j \in J} A_j)\,. \tag{449}$$

Because $J = J^+$ and $J \neq \varnothing$ by assumption of the lemma, we know that there exists some $j_0 \in J^+$, i.e. a choice of $j_0 \in J$ such that $\sup A_{j_0} \geq \frac{1}{2}$. Clearly

$$\begin{aligned}
\sup \bigcup_{j \in J} A_j &\geq \sup A_{j_0} && \text{because } j_0 \in J \text{, i.e. } A_{j_0} \subseteq \bigcup_{j \in J} A_j \\
&\geq \tfrac{1}{2}\,. && \text{because } j_0 \in J^+
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathrm{m}_{\frac{1}{2}} \bigcup_{j \in J} A_j &= \max(\tfrac{1}{2}, \inf \bigcup_{j \in J} A_j) && \text{by (446)} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\}\,. && \text{by (449)}
\end{aligned}$$

Case b.: $J^+ = \varnothing$, i.e. $J = J^-$.
By reasoning analogous to that in case a.,

$$\begin{aligned}
\mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} &= \mathrm{m}_{\frac{1}{2}}\{\min(\tfrac{1}{2}, \sup A_j) : j \in J\} \\
&= \min(\tfrac{1}{2}, \sup \bigcup_{j \in J} A_j) \\
&= \mathrm{m}_{\frac{1}{2}} \bigcup_{j \in J} A_j
\end{aligned}$$

provided that $J = J^-$.

Case c.: $J^+ \neq \varnothing$ **and** $J^- \neq \varnothing$**.**
In the remaining case that $J^+ \neq \varnothing$ and $J^- \neq \varnothing$, there exist $j^+ \in J^+, j^- \in J^-$. Therefore

$$\begin{aligned}\sup\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} &\geq \mathrm{m}_{\frac{1}{2}} A_{j^+} && \text{because } j^+ \in J\\ &= \max(\tfrac{1}{2}, \inf A_j) && \text{by (447), } j^+ \in J^+,\ A_{j^+} \neq \varnothing\\ &\geq \tfrac{1}{2}\,,\end{aligned}$$

and similarly

$$\begin{aligned}\inf\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} &\leq \mathrm{m}_{\frac{1}{2}} A_{j^-} && \text{because } j^- \in J\\ &= \min(\tfrac{1}{2}, \sup A_{j^-}) && \text{by (448), } j \in J^-,\ A_{j^-} \neq \varnothing\\ &\leq \tfrac{1}{2}\,.\end{aligned}$$

Hence

$$\begin{aligned}&\mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\}\\ &\qquad = \min(\tfrac{1}{2}, \sup\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\}) && \text{by (446), } \inf\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} \leq \tfrac{1}{2}\\ &\qquad = \tfrac{1}{2}\,. && \text{because } \sup\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\} \geq \tfrac{1}{2}\end{aligned}$$

Considering $\mathrm{m}_{\frac{1}{2}} \bigcup\limits_{j\in J} A_j$, we have $A_{j^+} \subseteq \bigcup\limits_{j\in J} A_j$, hence

$$\sup \bigcup_{j\in J} A_j \geq \sup A_{j^+} \geq \tfrac{1}{2}\,,$$

and similarly

$$\inf \bigcup_{j\in J} A_j \leq \inf A_{j^-} \leq \tfrac{1}{2}$$

because $j^- \in J$ and hence $A_{j^-} \subseteq \bigcup\limits_{j\in J} A_j$. Therefore

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} \bigcup_{j\in J} A_j &= \min(\tfrac{1}{2}, \sup \bigcup_{j\in J} A_j) && \text{by (446), } \inf \bigcup_{j\in J} \leq \tfrac{1}{2}\\ &= \tfrac{1}{2} && \text{because } \sup \bigcup_{j\in J} A_j \geq \tfrac{1}{2}\\ &= \mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}} A_j : j \in J\}\,,\end{aligned}$$

as desired.

Lemma 96
Suppose $(V_j)_{\gamma\in\mathbf{I}}$ *is an* $\mathbf{I}$*-indexed family of subsets* $V_\gamma \subseteq \mathbb{H}$, $\gamma \in \mathbf{I}$ *such that* $V_0 \neq \varnothing$ *and* $V_\gamma \subseteq V_{\gamma'}$ *whenever* $\gamma \leq \gamma'$*. Define* $g, h : \mathbf{I} \longrightarrow \mathbf{I}$ *by*

$$\begin{aligned}g(\gamma) &= \inf\{f(\gamma) : f \in V_\gamma\}\\ h(\gamma) &= \inf\{\mathcal{B}'_{CX}(f) : f \in V_\gamma\}\end{aligned}$$

for all $\gamma \in \mathbf{I}$.
We require that $h(0) > 0$*. Then* $g, h \in \mathbb{H}$ *and* $\mathcal{B}'_{CX}(g) = \mathcal{B}'_{CX}(h)$.

Proof
Let us first show that $g \in \mathbb{H}$. By Def. 75, we have to show that g is nonincreasing and $g(0) > 0$. Hence let $\gamma \leq \gamma'$. Then

$$\begin{aligned} g(\gamma) &= \inf\{f(\gamma) : f \in V_\gamma\} \\ &\geq \inf\{f(\gamma') : f \in V_\gamma\} && \text{because all } f \in V_\gamma \subseteq \mathbb{H} \text{ are nonincreasing} \\ &\geq \inf\{f(\gamma') : f \in V_{\gamma'}\} && \text{because } V_\gamma \subseteq V_{\gamma'} \\ &= g(\gamma')\,. \end{aligned}$$

This proves that g is nonincreasing; let us now show that $g(0) > 0$. By assumption of the lemma,

$$h(0) = \inf\{\mathcal{B}'_{CX}(f) : f \in V_0\} > 0\,.$$

Hence for all $f \in V_0$,

$$\mathcal{B}'_{CX}(f) \geq h(0) > 0\,. \tag{450}$$

Because $\mathcal{M}_{CX}$ is a DFS (see Th-93, we know from Th-62 and Th-53 that $\mathcal{B}'_{CX}$ satisfies (C-1) and (C-4). Let us also observe that each $f \in V_0 \subseteq \mathbb{H}$ is nonincreasing by Def. 75 and hence $f(\gamma) \leq f(0) = \mathrm{c}_{f(0)}(\gamma)$, i.e. $f \leq \mathrm{c}_{f(0)}$. Therefore

$$\begin{aligned} f(0) &= \mathcal{B}'_{CX}(\mathrm{c}_{f(0)}) && \text{by (C-1)} \\ &\geq \mathcal{B}'_{CX}(f) && \text{by (C-4), } f \leq \mathrm{c}_{f(0)} \\ &\geq h(0)\,, && \text{by (450)} \end{aligned}$$

i.e.

$$f(0) \geq h(0) \tag{451}$$

for all $f \in V_0$. We conclude that

$$\begin{aligned} g(0) &= \inf\{f(0) : f \in V_0\} \\ &\geq \inf\{h(0) : f \in V_0\} && \text{by (451)} \\ &= h(0) \\ &> 0\,. && \text{by assumption of lemma} \end{aligned}$$

Hence g is nonincreasing and $g(0) > 0$, i.e. $g \in \mathbb{H}$ by Def. 75.
Let us next show that $h \in \mathbb{B}$. Hence let $\gamma \leq \gamma'$. Then

$$\begin{aligned} h(\gamma) &= \inf\{\mathcal{B}'_{CX}(f) : f \in V_\gamma\} \\ &\leq \inf\{\mathcal{B}'_{CX}(f) : f \in V_{\gamma'}\} && \text{because } V_\gamma \subseteq V_{\gamma'} \\ &= h(\gamma')\,, \end{aligned}$$

i.e. h is nonincreasing. We already know by assumption of the lemma that $h(0) > 0$. Hence $h \in \mathbb{H}$ by Def. 75.
Let us abbreviate

$$D_\gamma = \{f \in V_\gamma : \mathcal{B}'_{CX}(f) < \gamma\} \tag{452}$$

for all γ. Because $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \leq \gamma'$, it is apparent that

$$D_\gamma \subseteq D_{\gamma'} \tag{453}$$

whenever $\gamma \leq \gamma'$.
I will now show that $\mathcal{B}'_{CX}(g) = \inf\{\gamma : D_\gamma \neq \varnothing\}$. To this end, let us first observe that

$$\mathcal{B}'_{CX}(g) \leq \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}\,. \tag{454}$$

This is because by (453), we have $D_{\gamma'} \neq \varnothing$ for all $\gamma' > \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}$, i.e. there is some $f_0 \in V_{\gamma'}$ such that $\mathcal{B}'_{CX}(f_0) < \gamma'$. It follows by Th-94 that

$$f_0(\gamma') < \gamma'\,, \tag{455}$$

and hence

$$\begin{aligned} g(\gamma') &= \inf\{f(\gamma') : f \in V_{\gamma'}\} \\ &\leq \inf\{f_0(\gamma')\} && \text{because } f_0 \in V_{\gamma'} \\ &< \gamma' && \text{by (455)} \end{aligned}$$

Again from Th-94, we conclude that $\mathcal{B}'_{CX}(g) \leq \gamma'$. Because $\gamma' > \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}$ was arbitrarily chosen, this means that $\mathcal{B}'_{CX}(g) \leq \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}$, i.e. (454) holds.
To show that

$$\mathcal{B}'_{CX}(g) \geq \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}\,, \tag{456}$$

let us choose γ', γ'' such that

$$0 \leq \gamma' < \gamma'' < \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}\,. \tag{457}$$

Because $D_{\gamma''} = \varnothing$ by (457) and (453), we know that for all $f \in V_{\gamma''}$,

$$\mathcal{B}'_{CX}(f) \geq \gamma''\,. \tag{458}$$

Because $\gamma' < \gamma''$ by (457), we also know that $V_{\gamma'} \subseteq V_{\gamma''}$, i.e. for all $\gamma \in V_{\gamma'}$,

$$\begin{aligned} \mathcal{B}'_{CX}(f) &\geq \gamma'' && \text{by (458)} \\ &> \gamma' && \text{by (457)}\,. \end{aligned}$$

By applying Th-94, we conclude from $\gamma' < \mathcal{B}'_{CX}(f)$ that

$$f(\gamma') > \gamma'$$

for all $f \in V_{\gamma'}$. Therefore

$$\begin{aligned} g(\gamma') &= \inf\{f(\gamma') : f \in V_{\gamma'}\} \\ &\geq \gamma'\,, \end{aligned}$$

i.e. $\mathcal{B}'_{CX}(g) \geq \gamma'$ by Th-94. Because $\gamma' < \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}$ was arbitrarily chosen, this means that

$$\mathcal{B}'_{CX}(g) \geq \sup[0, \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}) = \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}\,,$$

i.e. (456) holds. Combining this with (454), we conclude that

$$\mathcal{B}'_{CX}(g) = \inf\{\gamma : D_\gamma \neq \varnothing\}\,. \tag{459}$$

It remains to be shown that $\mathcal{B}'_{CX}(h) = \mathcal{B}'_{CX}(g)$. By Th-94, we can prove this by showing that

a. $h(\gamma') > \gamma'$ for all $\gamma' < \mathcal{B}'_{CX}(g)$, and

b. $h(\gamma') < \gamma'$ for all $\gamma' > \mathcal{B}'_{CX}(g)$.

ad a. We may assume that $\mathcal{B}'_{CX}(g) > 0$ because condition a. becomes vacuous in the case that $\mathcal{B}'_{CX}(g) = 0$. To see that condition a. holds in the nontrivial case $\mathcal{B}'_{CX}(g) > 0$, let

$$\gamma' < \gamma'' < \mathcal{B}'_{CX}(g)\,. \tag{460}$$

By (459), $\gamma'' < \inf\{\gamma \in \mathbf{I} : D_\gamma \neq \varnothing\}$, i.e. $D_{\gamma''} \neq \varnothing$. By (452), this means that

$$\mathcal{B}'_{CX}(f) \geq \gamma''\,. \tag{461}$$

for all $f \in V_{\gamma''}$. Therefore

$$\begin{aligned} h(\gamma') &= \inf\{\mathcal{B}'_{CX}(f) : f \in V_{\gamma'}\} \\ &\geq \inf\{\mathcal{B}'_{CX}(f) : f \in V_{\gamma''}\} && \text{because } V_{\gamma'} \subseteq V_{\gamma''} \\ &\geq \gamma'' && \text{by (461)} \\ &> \gamma\,. && \text{by (460)} \end{aligned}$$

Because $\gamma' < \mathcal{B}'_{CX}(g)$ was arbitrarily chosen, this finishes the proof of part a.

ad b. We may assume that $\mathcal{B}'_{CX}(g) < 1$ because condition b. becomes vacuous in the case that $\mathcal{B}'_{CX}(g) = 1$. To see that condition b. holds in the nontrivial case $\mathcal{B}'_{CX}(g) < 1$, let $\gamma' > \mathcal{B}'_{CX}(g)$. By (459), this means that $D_{\gamma'} \neq \varnothing$, i.e. there exists some $f' \in V_{\gamma'}$ such that

$$\mathcal{B}'_{CX}(f') < \gamma'\,. \tag{462}$$

Therefore

$$\begin{aligned} h(\gamma) &= \inf\{\mathcal{B}'_{CX}(f) : f \in V_{\gamma'}\} \\ &\leq \inf\{\mathcal{B}'_{CX}(f')\} && \text{because } f' \in D_{\gamma'} \subseteq V_{\gamma'} \\ &= \mathcal{B}'_{CX}(f') \\ &< \gamma\,. && \text{by (462)} \end{aligned}$$

This finishes the proof that condition b. holds.

In the following, we will generalize this statement to $\mathbf{I}$-indexed families $(V_\gamma)_{\gamma\in\mathbf{I}}$ where $V_\gamma \subseteq \mathbb{B}$.

Lemma 97

Suppose $(V_\gamma)_{\gamma\in\mathbf{I}}$ is an $\mathbf{I}$-indexed family of subsets $V_\gamma \subseteq \mathbb{B}$, all $\gamma \in \mathbf{I}$, such that $V_0 \neq \varnothing$ and $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \leq \gamma'$. Define $g : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(\gamma) = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\}\,,$$

for all $\gamma \in \mathbf{I}$. Then $g \in \mathbb{B}$.

Proof

Case a.: $g(0) > \frac{1}{2}$**.**
In this case,

$$\begin{aligned} g(0) &= \mathrm{m}_{\frac{1}{2}}\{f(0) : f \in V_0\} && \text{by definition of } g \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{f(0) : f \in V_0\}, \sup\{f(0) : f \in V_0\}) && \text{by Def. 46} \\ &> \tfrac{1}{2}\,, && \text{by assumption of case a.} \end{aligned}$$

i.e. by Def. 45,

$$g(0) = \inf\{f(0) : f \in V_0\} > \tfrac{1}{2}\,.$$

In particular, $f(0) > \frac{1}{2}$ for all $f \in V_0$, i.e. $V_0 \subseteq \mathbb{B}^+$. Hence

$$\begin{aligned} \sup\{f(\gamma) : f \in V_\gamma\} &\geq \sup\{f(\gamma) : f \in V_0\} && \text{because } V_0 \subseteq V_\gamma,\ 0 \leq \gamma \\ &\geq \sup\{\tfrac{1}{2} : f \in V_0\} && \text{because } f(\gamma) \geq \tfrac{1}{2} \text{ for all } f \in \mathbb{B}^+, \text{ and } V_0 \subseteq \mathbb{B}^+ \\ &= \tfrac{1}{2}\,. && \text{because } V_0 \neq \varnothing \end{aligned}$$

Recalling that $\mathrm{m}_{\frac{1}{2}}(a, b) = \mathrm{m}_{\frac{1}{2}}(\frac{1}{2}, \min(a, b))$ whenever $\max(a, b) \geq \frac{1}{2}$ (see Def. 45), we conclude that

$$\begin{aligned} g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{f(\gamma) : f \in V_\gamma\}, \sup\{f(\gamma) : f \in V_\gamma\}) && \text{by Def. 46} \\ &= \max(\tfrac{1}{2}, \inf\{f(\gamma) : f \in V_\gamma\})\,, \end{aligned}$$

i.e.

$$g(\gamma) = \max(\tfrac{1}{2}, \inf\{f(\gamma) : f \in V_\gamma\}) \tag{463}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (463) that g is nonincreasing and $g(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. In addition, we know that $g(0) > \frac{1}{2}$ by assumption of case a. Therefore by Def. 68, $g \in \mathbb{B}^+$; in particular, $g \in \mathbb{B}$.

Case b.: $g(0) < \frac{1}{2}$**.**
The proof that g is nondecreasing and $g(\gamma) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ is analogous to that of case a., this time using the relationship $\mathrm{m}_{\frac{1}{2}}(a, b) = \min(\frac{1}{2}, \max(a, b))$, which holds whenever $\min(a, b) \leq \frac{1}{2}$.

Case c.: $g(0) = \frac{1}{2}$**.**
In this case, let us abbreviate

$$\begin{aligned} V_0^+ &= V_0 \cap \mathbb{B}^+ \\ V_0^{\frac{1}{2}} &= V_0 \cap \mathbb{B}^{\frac{1}{2}} \\ V_0^- &= V_0 \cap \mathbb{B}^-\,. \end{aligned}$$

If $V_0^{\frac{1}{2}} \neq \varnothing$, then $\mathrm{c}_{\frac{1}{2}} \in V_0$, i.e. $\mathrm{c}_{\frac{1}{2}} \in V_\gamma$ for all $\gamma \in \mathbf{I}$ because $V_\gamma \supseteq V_0$, for all $\gamma \in \mathbf{I}$. Therefore

$$\inf\{f(\gamma) : f \in V_\gamma\} \leq \inf\{\mathrm{c}_{\frac{1}{2}}(\gamma)\} = \tfrac{1}{2}$$
$$\sup\{f(\gamma) : f \in V_\gamma\} \geq \sup\{\mathrm{c}_{\frac{1}{2}}(\gamma)\} = \tfrac{1}{2}\,,$$

i.e.

$$\begin{aligned} g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{f(\gamma) : f \in V_\gamma\}, \sup\{f(\gamma) : f \in V_\gamma\}) && \text{by Def. 46} \\ &= \tfrac{1}{2} \end{aligned}$$

(see Def. 45: if $\min(a,b) \leq \frac{1}{2}$,$\max(a.b) \geq \frac{1}{2}$, then $\mathrm{m}_{\frac{1}{2}}(a,b) = \frac{1}{2}$). This shows that $f(\gamma) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$, i.e. $f = \mathrm{c}_{\frac{1}{2}} \in \mathbb{B}^{\frac{1}{2}}$.

In the case that $V_0^{\frac{1}{2}} = \varnothing$, we shall discern the following situations:

i. $V_0^+ \neq \varnothing, V_0^- \neq \varnothing$.
Let $f^+ \in V_0^+$, $f^- \in V_0^-$. Then $f^+, f^- \in V_\gamma$ for all $\gamma \in \mathbf{I}$ because $V_0 \subseteq V_\gamma$ as $0 \leq \gamma$. Hence

$$\begin{aligned} &\inf\{f(\gamma) : f \in V_\gamma\} \leq \inf\{f^-(\gamma)\} = f^-(\gamma) \leq \tfrac{1}{2} && \text{because } f^- \in \mathbb{B}^- \\ &\sup\{f(\gamma) : f \in V_\gamma\} \geq \sup\{f^+(\gamma)\} = f^+(\gamma) \geq \tfrac{1}{2} && \text{because } f^+ \in \mathbb{B}^+\,, \end{aligned}$$

i.e.

$$f(\gamma) = \mathrm{m}_{\frac{1}{2}}(\inf\{f(\gamma) : f \in V_\gamma\}, \sup\{f(\gamma) : f \in V_\gamma\}) = \tfrac{1}{2}\,.$$

Hence $f = \mathrm{c}_{\frac{1}{2}} \in \mathbb{B}^{\frac{1}{2}}$.

ii. $V_0^+ = \varnothing, V_0^- \neq \varnothing$.
Because $V_0^- \neq \varnothing$, there is some $f^- \in V_0$ such that $f^- \in \mathbb{B}^-$. Because $V_0 \subseteq V_\gamma$ for all $\gamma \in \mathbf{I}$, $f^- \in V_\gamma$ for all γ and hence

$$\inf\{f(\gamma) : f \in V_\gamma\} \leq f^-(\gamma) \leq \tfrac{1}{2}\,, \tag{464}$$

for all $\gamma \in \mathbf{I}$.
Let us now consider $\sup\{f(\gamma) : f \in V_\gamma\}$. We know that $g(0) = \frac{1}{2}$, i.e. $\sup\{f(0) : f \in V_0\} \geq \frac{1}{2}$ by Def. 46 and Def. 45. Because $V_0^+ = V_0^{\frac{1}{2}} = \varnothing$, i.e. $V_0 = V^-$, this means that

$$\sup\{f(0) : f \in V_0\} = \tfrac{1}{2}\,. \tag{465}$$

Now let $\varepsilon > 0$. By (465), there exists $f^+ \in V_0 = V_0^-$ such that $f(0) > \frac{1}{2} - \varepsilon$; because $V_\gamma \supseteq V_0$, apparently $f^+ \in V_\gamma$ for all γ. Therefore

$$\begin{aligned} \sup\{f(\gamma) : f \in V_\gamma\} &\geq \sup\{f^+(\gamma)\} && \text{because } f^+ \in V_\gamma \\ &= f^+(\gamma) \\ &\geq f^+(0) && \text{because } f^+ \in V_0^- \subseteq \mathbb{B}^- \text{ is nondecreasing} \\ &> \tfrac{1}{2} - \varepsilon\,. \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily, this proves that

$$\sup\{f(\gamma) : f \in V_\gamma\} \geq \tfrac{1}{2} \tag{466}$$

for all $\gamma \in \mathbf{I}$. It is then obvious from (464), (466) and Def. 45 that

$$\begin{aligned} f(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{f(\gamma) : f \in V_\gamma\}, \sup\{f(\gamma) : f \in V_\gamma\}) && \text{by Def. 46} \\ &= \tfrac{1}{2}\,. && \text{by Def. 45} \end{aligned}$$

Hence $g = \mathrm{c}_{\frac{1}{2}} \in \mathbb{B}^{\frac{1}{2}}$.

iii. $V_0^+ \neq \varnothing$, $V_0^- = \varnothing$.
The proof of this case is analogous to that of case ii.

The case that $V_0^+ = V_0^- = \varnothing$ is not possible if $V_0^{\frac{1}{2}} = \varnothing$ because by assumption, $V_0 = V_0^+ \cup V_0^{\frac{1}{2}} \cup V_0^- \neq \varnothing$.

Lemma 98

Suppose $(V_\gamma)_{\gamma\in\mathbf{I}}$ is an $\mathbf{I}$-indexed family of subsets $V_\gamma \subseteq \mathbb{B}$, all $\gamma \in \mathbf{I}$, such that $V_0 \neq \varnothing$ and $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \leq \gamma'$. Define $h : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$h(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}\,,$$

for all $\gamma \in \mathbf{I}$. Then $h \in \mathbb{B}$.

Proof

a. Suppose $h(0) > \frac{1}{2}$. Let us observe from Def. 45 that $\mathrm{m}_{\frac{1}{2}}(a, b) > \frac{1}{2}$ only if $\min(a, b) > \frac{1}{2}$ and that in this case $\mathrm{m}_{\frac{1}{2}}(a, b) = \min(a, b)$. We can hence conclude from

$$\begin{aligned} h(0) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_0\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in V_0\}, \sup\{\mathcal{B}_{CX}(f) : f \in V_0\}) && \text{by Def. 46} \\ &> \tfrac{1}{2} \end{aligned}$$

that actually

$$h(0) = \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} > \tfrac{1}{2}\,. \tag{467}$$

Because $V_0 \neq \varnothing$, $\sup\{f(0) : f \in V_0\} \geq \inf\{f(0) : f \in V_0\} > \frac{1}{2}$ and hence

$$\begin{aligned} \sup\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &\geq \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} && \text{because } V_0 \subseteq V_\gamma \\ &\geq \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} && \text{because } V_0 \neq \varnothing \\ &> \tfrac{1}{2}\,. && \text{by (467)} \end{aligned}$$

Noting that by Def. 45, $\max(a, b) \geq \frac{1}{2}$ implies that $\mathrm{m}_{\frac{1}{2}}(a, b) = \max(\frac{1}{2}, \min(a, b)$, we conclude that

$$h(\gamma) = \max(\tfrac{1}{2}, \inf\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}) \tag{468}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (468) that h is nonincreasing in γ and $h(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. Furthermore, $h(0) > \frac{1}{2}$ by assumption of case a., i.e. $h \in \mathbb{B}^+$.

b. By similar reasoning, and using the relationship $\mathrm{m}_{\frac{1}{2}}(a,b) = \min(\frac{1}{2}, \max(a,b))$ which holds whenever $\min(a,b) \leq \frac{1}{2}$, it is shown that $h(0) < \frac{1}{2}$ implies that h is nondecreasing and $h(\gamma) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$, i.e. $h \in \mathbb{B}^-$ by Def. 68.

c. If $h(0) = \frac{1}{2}$, i.e.

$$\begin{aligned} h(0) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_0\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in V_0\}, \sup\{\mathcal{B}_{CX}(f) : f \in V_0\}) && \text{by Def. 46} \\ &= \tfrac{1}{2}\,, \end{aligned}$$

then by Def. 45,

$$\begin{aligned} \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} \leq \tfrac{1}{2} \\ \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} \geq \tfrac{1}{2} \end{aligned}$$

Therefore

$$\begin{aligned} \inf\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &\leq \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} && \text{because } V_0 \subseteq V_\gamma \\ &\leq \tfrac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \sup\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &\geq \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} && \text{because } V_0 \subseteq V_\gamma \\ &\geq \tfrac{1}{2}\,, \end{aligned}$$

i.e.

$$\begin{aligned} h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}, \sup\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}) && \text{by Def. 46} \\ &= \tfrac{1}{2}\,. && \text{by above inequations and Def. 45} \end{aligned}$$

Lemma 99

Suppose $(V_\gamma)_{\gamma \in \mathbf{I}}$ is an $\mathbf{I}$-indexed family of subsets $V_\gamma \subseteq \mathbb{B}$, $\gamma \in \mathbf{I}$ such that $V_0 \neq \varnothing$ and $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \leq \gamma'$. If

$$\mathrm{m}_{\frac{1}{2}}\{f(0) : f \in V_0\} > \tfrac{1}{2}\,,$$

then there is an $\mathbf{I}$-indexed family $(W_\gamma)_{\gamma \in \mathbf{I}}$ of subsets $W_\gamma \subseteq \mathbb{B}^+$, $\gamma \in \mathbf{I}$ such that $W_0 \neq \varnothing$, $W_\gamma \subseteq W_{\gamma'}$ whenever $\gamma \leq \gamma'$, and

$$\begin{aligned} \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\} \\ \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\}\,, \end{aligned}$$

for all $\gamma \in \mathbf{I}$.

Proof
Let us define $f^* : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$f^*(\gamma) = \begin{cases} 1 & : \quad \gamma = 0 \\ \frac{1}{2} & : \quad \gamma > 0 \end{cases} \tag{469}$$

for all $\gamma \in \mathbf{I}$. Obviously f^* is nonincreasing, $f^*(0) = 1 > \frac{1}{2}$, and $f^*(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. Hence by Def. 68,

$$f^* \in \mathbb{B}^+ . \tag{470}$$

We shall now define $(W_\gamma)_{\gamma \in \mathbf{I}}$ by

$$W_\gamma = \begin{cases} V_\gamma & : \quad V_\gamma \subseteq \mathbb{B}^+ \\ (V_\gamma \cap \mathbb{B}^+) \cup \{f^*\} & : \quad V_\gamma \not\subseteq \mathbb{B}^+ \end{cases} \tag{471}$$

for all $\gamma \in \mathbf{I}$. Because $f^* \in \mathbb{B}^+$ by (470), it is obvious from (471) that $W_\gamma \subseteq \mathbb{B}^+$ for all $\gamma \in \mathbf{I}$.
By assumption, $\mathrm{m}_{\frac{1}{2}}\{f(0) : f \in V_0\} > \frac{1}{2}$; in particular, $f(0) > \frac{1}{2}$ for all $f \in V_0$. By Def. 68, $f \in \mathbb{B}$ and $f(0) > \frac{1}{2}$ means that $f \in \mathbb{B}^+$. Hence $f \in \mathbb{B}^+$ for all $f \in V_0 \subseteq \mathbb{B}$, or equivalently:

$$V_0 \subseteq \mathbb{B}^+ . \tag{472}$$

By (471), $W_0 = V_0 \neq \varnothing$.
Now let $\gamma \leq \gamma'$. It is apparent from the fact that $V_\gamma \subseteq V_{\gamma'}$ whenever $\gamma \leq \gamma'$ that $V_\gamma \not\subseteq \mathbb{B}^+$ entails $V_{\gamma'} \not\subseteq \mathbb{B}^+$ for all $\gamma' > \gamma$ because $V_{\gamma'} \supseteq V_\gamma$. Therefore

- if $\gamma \leq \gamma'$ and $V_{\gamma'} \subseteq \mathbb{B}^+$, $W_\gamma = V_\gamma \subseteq V_{\gamma'} \subseteq W_{\gamma'}$.
- if $V_\gamma \not\subseteq \mathbb{B}^+$, then $V_{\gamma'} \not\subseteq \mathbb{B}^+$, too. Therefore $W_\gamma = (V_\gamma \cap \mathbb{B}^+) \cup \{f^*\} \subseteq (V_{\gamma'} \cap \mathbb{B}^+) \cup \{f^*\} = W_{\gamma'}$.
- if $V_\gamma \subseteq \mathbb{B}^+$ and $V_{\gamma'} \not\subseteq \mathbb{B}^+$, then $W_\gamma = V_\gamma = V_\gamma \cap \mathbb{B}^+ \subseteq V_{\gamma'} \cap \mathbb{B}^+ \subseteq (V_{\gamma'} \cap \mathbb{B}^+) \cup \{f^*\} = W_{\gamma'}$.

Summarizing, $W_\gamma \subseteq W_{\gamma'}$ whenever $\gamma \leq \gamma'$.
Let us now prove the remaining claims of the lemma. We shall discern two cases.

Case a.: $V_\gamma \subseteq \mathbb{B}^+$.
Then by (471), $V_\gamma = W_\gamma$. Hence trivially

$$\mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\}$$

and

$$\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\} .$$

Case b.: $V_\gamma \not\subseteq \mathbb{B}^+$.
Then we know from (472) that $\gamma > 0$, i.e.

$$f^*(\gamma) = \tfrac{1}{2} \tag{473}$$

by (469). Hence

$$\begin{aligned}
&\mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\}\\
&\quad = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in (V_\gamma \cap \mathbb{B}^+) \cup \{f^*\}\} && \text{by (471), } V_\gamma \not\subseteq \mathbb{B}^+\\
&\quad = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in (V_\gamma \cap \mathbb{B}^+)\} \cup \{f^*(\gamma)\}\\
&\quad = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in (V_\gamma \cap \mathbb{B}^+)\} \cup \{\tfrac{1}{2}\} && \text{by (473)}\\
&\quad = \tfrac{1}{2}\,,
\end{aligned}$$

i.e.

$$\mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\} = \tfrac{1}{2}\,, \tag{474}$$

because $\mathrm{m}_{\frac{1}{2}} X = \frac{1}{2}$ whenever $\frac{1}{2} \in X$, which is apparent from Def. 46 and Def. 45.
Now $V_\gamma \cap \mathbb{B}^+ \supseteq V_0 \cap \mathbb{B}^+ = V_0 \neq \varnothing$ by (472).
Therefore

$$\begin{aligned}
x &= \sup\{f(\gamma) : \gamma \in V_\gamma\}\\
&\geq \sup\{f(\gamma) : \gamma \in V_\gamma \cap \mathbb{B}^+\} && \text{because } V_\gamma \cap \mathbb{B}^+ \subseteq V_\gamma\\
&\geq \tfrac{1}{2}\,. && \text{because } V_\gamma \cap \mathbb{B}^+ \neq \varnothing\text{, see above}
\end{aligned}$$

Similarly,

$$\begin{aligned}
y &= \inf\{f(\gamma) : \gamma \in V_\gamma\}\\
&\leq \inf\{f(\gamma) : \gamma \in V_\gamma \cap \mathbb{B}^-\} && \text{because } V_\gamma \cap \mathbb{B}^- \subseteq V_\gamma\\
&\leq \tfrac{1}{2}\,. && \text{because } V_\gamma \cap \mathbb{B}^- \neq \varnothing
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\}\\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{f(\gamma) : f \in V_\gamma\}, \sup\{f(\gamma) : f \in V_\gamma\}) && \text{by Def. 46}\\
&= \mathrm{m}_{\frac{1}{2}}(y, x) && \text{see abbreviations above}\\
&= \tfrac{1}{2} && \text{by Def. 45 because } x \geq \tfrac{1}{2}, y \leq \tfrac{1}{2}\\
&= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\}\,. && \text{by (474)}
\end{aligned}$$

It remains to be shown that

$$\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\}\,.$$

To this end, let us first observe that $(f^*)^\sharp = \mathrm{c}_{\frac{1}{2}}$ and hence

$$\mathcal{B}_{CX}(f^*) = \mathcal{B}_{CX}(\mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2} \tag{475}$$

because $\mathcal{B}_{CX}$ satisfies (B-4) and (B-1). Therefore

$$\begin{aligned}
\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\} &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in (V_\gamma \cap \mathbb{B}^+) \cup \{f^*\}\} && \text{by (471), } V_\gamma \not\subseteq \mathbb{B}^+\\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma \cap \mathbb{B}^+\} \cup \{\mathcal{B}_{CX}(f^*)\}\\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma \cap \mathbb{B}^+\} \cup \{\tfrac{1}{2}\} && \text{by (475)}\\
&= \tfrac{1}{2}\,,
\end{aligned}$$

i.e.

$$\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\} = \tfrac{1}{2}\,, \tag{476}$$

again because $\frac{1}{2} \in X$ implies that $\mathrm{m}_{\frac{1}{2}} X = \frac{1}{2}$.
Let us now observe that

$$\begin{aligned}
\sup\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &\geq \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} && \text{because } V_0 \subseteq V_\gamma\\
&\geq \tfrac{1}{2}\,. && \text{because } V_0 \subseteq \mathbb{B}^+ \text{ by (472)}
\end{aligned}$$

Furthermore, $V_\gamma \not\subseteq \mathbb{B}^+$ by assumption of case b., i.e. there exists some $f_0 \in V_\gamma$ such that $f_0 \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$. In particular, $\mathcal{B}_{CX}(f) \leq \frac{1}{2}$ (see Def. 68, (23)). Hence

$$\begin{aligned}
\inf\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &\leq \mathcal{B}_{CX}(f_0) && \text{because } f_0 \in V_\gamma\\
&\leq \tfrac{1}{2}\,. && \text{because } f_0 \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}
\end{aligned}$$

By Def. 45, $\mathrm{m}_{\frac{1}{2}}(a,b) = \frac{1}{2}$ whenever $\max(a,b) \geq \frac{1}{2}$ and $\min(a,b) \leq \frac{1}{2}$. Therefore

$$\begin{aligned}
\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}, \sup\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}) && \text{by Def. 46}\\
&= \tfrac{1}{2}\\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\}\,. && \text{by (476)}
\end{aligned}$$

Lemma 100

Suppose $(Z_\gamma)_{\gamma \in \mathbf{I}}$ is an $\mathbf{I}$-indexed family of subsets $Z_\gamma \subseteq \mathbb{B}^+$ such that $Z_0 \neq \varnothing$ and $Z_\gamma \subseteq Z_{\gamma'}$ whenever $\gamma \leq \gamma'$. Define $g, h : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\begin{aligned}
g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in Z_\gamma\}\\
h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in Z_\gamma\}\,,
\end{aligned}$$

for all $\gamma \in \mathbf{I}$.
If $h(0) = \frac{1}{2}$, then $\widehat{g}((0,1]) = \{\frac{1}{2}\}$.

Proof

Let $\gamma' > 0$. We can show that $g(\gamma') = \frac{1}{2}$ by proving that for all $\varepsilon \in (0, \gamma')$, $g(\gamma') < \frac{1}{2} + \varepsilon$.
Hence let $\varepsilon, \delta \in \mathbf{I}$ such that

$$0 < \delta < \varepsilon < \gamma'\,. \tag{477}$$

Because $Z_0 \subseteq \mathbb{B}^+$, $\mathcal{B}'_{CX}(f) \geq \frac{1}{2}$ for all $f \in Z_0$ by (23). Therefore

$$\begin{aligned}
\tfrac{1}{2} &= h(0) && \text{by assumption} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}'_{CX}(f) : f \in Z_0\} && \text{by def. of } h \\
&= \inf\{\mathcal{B}'_{CX}(f) : f \in Z_0\}\,. && \text{because } \mathcal{B}'_{CX}(f) \geq \tfrac{1}{2} \text{ for all } f \in Z_0
\end{aligned}$$

Hence there exists some $f_0 \in Z_0$ such that

$$\mathcal{B}'_{CX}(f_0) \in [\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{\delta}{2})\,. \tag{478}$$

By Th-94 and (23), we conclude that

$$f_0(\gamma) < \tfrac{1}{2} + \tfrac{1}{2}\gamma \tag{479}$$

for all $\gamma > 2\mathcal{B}_{CX}(f_0) - 1$. Because $\mathcal{B}_{CX}(f_0) < \frac{1}{2} + \frac{1}{2}\delta$ by (478) and $\delta < \varepsilon$ by (477), this means that

$$\mathcal{B}_{CX}(f_0) < \tfrac{1}{2} + \tfrac{1}{2}\delta < \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\,,$$

i.e.

$$2\mathcal{B}_{CX}(f_0) - 1 < 2(\tfrac{1}{2} + \tfrac{1}{2}\varepsilon) - 1 < \varepsilon\,.$$

Hence by (479),

$$f_0(\varepsilon) < \tfrac{1}{2} + \tfrac{1}{2}\varepsilon\,. \tag{480}$$

Furthermore $\gamma' > \varepsilon$ by (477), and hence

$$\begin{aligned}
f_0(\gamma') &\leq f_0(\varepsilon) && \text{because } \gamma' > \varepsilon,\ f_0 \in \mathbb{B}^+ \text{ nonincreasing by Def. 68} \\
&< \tfrac{1}{2} + \tfrac{1}{2}\varepsilon && \text{by (479)} \\
&< \tfrac{1}{2} + \varepsilon && \text{because } \varepsilon > 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
g(\gamma') &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma') : f \in Z_{\gamma'}\} && \text{by def. of } g \\
&= \inf\{f(\gamma') : f \in Z_{\gamma'}\} && \text{because } f(\gamma') \geq \tfrac{1}{2} \text{ for all } f \in Z_{\gamma'} \subseteq \mathbb{B}^+ \\
&\leq \inf\{f_0(\gamma')\} && \text{because } f_0 \in Z_0 \subseteq Z_{\gamma'} \\
&= f_0(\gamma') \\
&< \tfrac{1}{2} + \varepsilon\,.
\end{aligned}$$

Because ε was chosen arbitrarily, we conclude that

$$g(\gamma') \leq \tfrac{1}{2}\,. \tag{481}$$

On the other hand, we know that $f(\gamma') \geq \frac{1}{2}$ for all $f \in V_{\gamma'} \subseteq \mathbb{B}^+$ (see Def. 68) and hence $g(\gamma) = \mathrm{m}_{\frac{1}{2}}\{f(\gamma') : f \in V_{\gamma'}\} \geq \frac{1}{2}$. Combining this with (481) yields $g(\gamma') = \frac{1}{2}$.
Because $\gamma' > 0$ was arbitrarily chosen, we may conclude that $g(\gamma') = \frac{1}{2}$ for all $\gamma' > 0$, as desired.

Lemma 101
Suppose $(W_\gamma))\gamma \in \mathbf{I}$ is an $\mathbf{I}$-indexed family of subsets $W_\gamma \subseteq \mathbb{B}^+$ such that $W_0 \neq \varnothing$ and $W_\gamma \subseteq W_{\gamma'}$ whenever $\gamma \leq \gamma'$. Define $g, h : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\begin{aligned}
g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\} \\
h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\}\,,
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. If $h(0) > \frac{1}{2}$, then also $g(0) > \frac{1}{2}$.

Proof
By the above definition of h, $h(0) > \frac{1}{2}$ means that

$$\begin{aligned} h(0) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_0\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in W_0\}, \sup\{\mathcal{B}_{CX}(f) : f \in W_0\}) && \text{by Def. 46} \\ &> \tfrac{1}{2}\,. \end{aligned}$$

It is obvious from Def. 45 that $\mathrm{m}_{\frac{1}{2}}(a, b) > \frac{1}{2}$ is possible only if $\min(a, b) > \frac{1}{2}$ and that in this case, $\mathrm{m}_{\frac{1}{2}}(a, b) = \min(a, b)$. We hence conclude that

$$h(0) = \inf\{\mathcal{B}_{CX}(f) : f \in W_0\} > \tfrac{1}{2}\,. \tag{482}$$

By assumption, $W_0 \subseteq \mathbb{B}^+$. Hence every $f \in W_0$ is contained in $\mathbb{B}^+$ and by Def. 68, f is nonincreasing. In particular, $f(\gamma) \le f(0) = \mathrm{c}_{f(0)}$ for all $\gamma \in \mathbf{I}$, i.e. $f \le \mathrm{c}_{f(0)}$. Therefore

$$\begin{aligned} f(0) &= \mathcal{B}_{CX}(\mathrm{c}_{f(0)}) && \text{by (B-1)} \\ &\ge \mathcal{B}_{CX}(f)\,. && \text{by (B-5), } \mathrm{c}_{f(0)} \ge f \end{aligned}$$

Hence

$$\begin{aligned} \inf\{f(0) : f \in W_0\} &\ge \inf\{\mathcal{B}_{CX}(f) : f \in W_0\} && \text{because } f(0) \ge \mathcal{B}_{CX}(f) \text{ for all } f \in W_0 \\ &= h(0) && \text{by (482)} \\ &> \tfrac{1}{2}\,, && \text{by assumption of the lemma} \end{aligned}$$

i.e.

$$\inf\{f(0) : f \in W_0\} > \tfrac{1}{2} \tag{483}$$

Because $W_0 \neq \varnothing$, $\sup\{f(0) : f \in W_0\} \ge \inf\{f(0) : f \in W_0\} > \frac{1}{2}$. Observing that by Def. 45, $\min(a, b) > \frac{1}{2}$ implies that $\mathrm{m}_{\frac{1}{2}}(a, b) = \min(a, b)$, it is then apparent that

$$\begin{aligned} \tfrac{1}{2} &< \inf\{f(0) : f \in W_0\} && \text{by (483)} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{f(0) : f \in W_0\}, \sup\{f(0) : f \in W_0\}) && \text{by above reasoning} \\ &= \mathrm{m}_{\frac{1}{2}}\{f(0) : f \in W_0\} && \text{by Def. 46} \\ &= g(0)\,, \end{aligned}$$

i.e. $g(0) > \frac{1}{2}$, as desired.
Because of these lemmata, we can now restrict attention to the following special case which connects $\mathcal{B}_{CX}$ to the lemma L-96 on $\mathcal{B}'_{CX}$.

Lemma 102
Suppose $(Z_\gamma))\gamma \in \mathbf{I}$ is an $\mathbf{I}$-indexed family of subsets $Z_\gamma \subseteq \mathbb{B}^+$ such that $Z_0 \neq \varnothing$ and $Z_\gamma \subseteq Z_{\gamma'}$ whenever $\gamma \le \gamma'$. Define $g, h : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\begin{aligned} g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in Z_\gamma\} \\ h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in Z_\gamma\}\,, \end{aligned}$$

for all $\gamma \in \mathbf{I}$. We require that $h(0) > \frac{1}{2}$. Then $g \in \mathbb{B}^+$, $h \in \mathbb{B}^+$ and $\mathcal{B}_{CX}(g) = \mathcal{B}_{CX}(h)$.

Proof
We already know from L-97 that $g \in \mathbb{B}$. By L-101, we conclude from $h(0) > \frac{1}{2}$ that $g(0) > \frac{1}{2}$ and hence $g \in \mathbb{B}^+$ by Def. 68.
Let us now observe that for all $\gamma \in \mathbf{I}$, $Z_\gamma \subseteq \mathbb{B}^+$ and hence $f(\gamma) \geq \frac{1}{2}$ for all $f \in Z_\gamma \neq \varnothing$. Therefore $X = \{f(\gamma) : f \in Z_\gamma\}$ is a nonempty subset of $[\frac{1}{2}, 1]$, i.e. $\sup X \geq \frac{1}{2}$ and $\inf X \geq \frac{1}{2}$. We compute

$$\begin{aligned} g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in Z_\gamma\} && \text{by definition of } g \\ &= \mathrm{m}_{\frac{1}{2}}\, X && \text{because } X = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in Z_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\sup X, \inf X) && \text{by Def. 46} \\ &= \inf X\,, && \text{by Def. 45 because } \sup X \geq \tfrac{1}{2}, \inf X \geq \tfrac{1}{2} \end{aligned}$$

i.e.

$$g(\gamma) = \inf\{f(\gamma) : f \in Z_\gamma\}\,. \tag{484}$$

Let us now consider h. We already know from L-98 that $h \in \mathbb{B}$. By assumption of the lemma, $h(0) > \frac{1}{2}$ and hence $h \in \mathbb{B}^+$ by Def. 68.
Because $Z_1 \subseteq \mathbb{B}^+$, $\mathcal{B}_{CX}(f) \geq \frac{1}{2}$ for all $f \in Z_1$; this is a consequence of (23). Hence for all $\gamma \in \mathbf{I}$, $Y = \{\mathcal{B}_{CX}(f) : f \in Z_\gamma\} \subseteq \{\mathcal{B}_{CX}(f) : f \in Z_1\} \subseteq [\frac{1}{2}, 1]$. In particular, $\inf Y \geq \frac{1}{2}$ and $\sup Y \geq \frac{1}{2}$. Hence by similar reasoning as in the case of g,

$$h(\gamma) = \mathrm{m}_{\frac{1}{2}}\, Y = \inf Y = \inf\{\mathcal{B}_{CX}(f) : f \in Z_\gamma\} \tag{485}$$

for all $\gamma \in \mathbf{I}$.
Let us define an $\mathbf{I}$-indexed family $(Z'_\gamma)_{\gamma \in \mathbf{I}}$ by

$$Z'_\gamma = \{2f - 1 : f \in Z_\gamma\} \tag{486}$$

for all $\gamma \in \mathbf{I}$. It is apparent from $Z_\gamma \subseteq \mathbb{B}^+$ that $Z'_\gamma \subseteq \mathbb{H}$; see Def. 68 and Def. 75. It is also clear that $(Z'_\gamma)_{\gamma \in \mathbf{I}}$ satisfies all requirements stated in L-96. Therefore

$$\begin{aligned} \mathcal{B}_{CX}(g) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(2g - 1) && \text{by (23), } g \in \mathbb{B}^+ \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(2(\inf\{f(\gamma) : f \in Z_\gamma\})_{\gamma \in \mathbf{I}} - 1) && \text{by (484)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf\{2f(\gamma) - 1 : f \in Z_\gamma\})_{\gamma \in \mathbf{I}}) \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf\{(2f - 1)(\gamma) : f \in Z_\gamma\})_{\gamma \in \mathbf{I}}) \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf\{f'(\gamma) : f' \in Z'_\gamma\})_{\gamma \in \mathbf{I}}) && \text{by (486)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf \mathcal{B}'_{CX}(f') : f' \in Z'_\gamma\})_{\gamma \in \mathbf{I}}) && \text{by L-96} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf\{\mathcal{B}'_{CX}(2f - 1) : f \in Z_\gamma\})_{\gamma \in \mathbf{I}}) && \text{by (486)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}((\inf\{2\mathcal{B}_{CX}(f) - 1 : f \in Z_\gamma\})_{\gamma \in \mathbf{I}}) && \text{by (24)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(2(\inf\{\mathcal{B}_{CX}(f) : f \in Z_\gamma\})_{\gamma \in \mathbf{I}} - 1) \\ &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_{CX}(2h - 1) && \text{by (485)} \\ &= \mathcal{B}_{CX}(h)\,. && \text{by (23), } h \in \mathbb{B}^+ \end{aligned}$$

Lemma 103

Suppose that $(V_\gamma)_{\gamma\in\mathbf{I}}$ *is an* $\mathbf{I}$*-indexed family of subsets* $V_\gamma \subseteq \mathbb{B}$ *such that* $V_0 \neq \varnothing$ *and* $V_\gamma \subseteq V_{\gamma'}$ *whenever* $\gamma \leq \gamma'$. *Define* $g, h : \mathbf{I} \longrightarrow \mathbf{I}$ *by*

$$g(\gamma) = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\}$$
$$h(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\}\,,$$

for all $\gamma \in \mathbf{I}$. *Then* $g, h \in \mathbb{B}$ *and*

$$\mathcal{B}(g) = \mathcal{B}(h)\,.$$

Proof Clearly $g \in \mathbb{B}$ by L-97 and $h \in \mathbb{B}$ by L-98. We shall discern the following cases.

Case a.: $g(0) > \frac{1}{2}$

Then by L-99, there exists an $\mathbf{I}$-indexed family $(W_\gamma)_{\gamma\in\mathbf{I}}$ of subsets $W_\gamma \subseteq \mathbb{B}^+$, all $\gamma \in \mathbf{I}$, such that $W_0 \neq \varnothing$, $W_\gamma \subseteq W_{\gamma'}$ whenever $\gamma \leq \gamma'$, and

$$g(\gamma) = \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in W_\gamma\} \tag{487}$$
$$h(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in W_\gamma\} \tag{488}$$

for all $\gamma \in \mathbf{I}$. We shall discern two more subcases in dependence on $h(0)$.

i. $h(0) = \frac{1}{2}$.
Then $h = \mathrm{c}_{\frac{1}{2}}$ by Def. 68 because $h \in \mathbb{B}$. By L-100, $\widehat{g}((0,1]) = \{\frac{1}{2}\}$, i.e. $g^\sharp = \mathrm{c}_{\frac{1}{2}}$ (see Def. 71) and hence

$$\begin{aligned}
\mathcal{B}_{CX}(g) &= \mathcal{B}_{CX}(g^\sharp) && \text{because } \mathcal{B}_{CX} \text{ satisfies (B-4)}\\
&= \mathcal{B}_{CX}(\mathrm{c}_{\frac{1}{2}}) && \text{because } g^\sharp = \mathrm{c}_{\frac{1}{2}}\\
&= \mathcal{B}_{CX}(h)\,. && \text{because } h = \mathrm{c}_{\frac{1}{2}}
\end{aligned}$$

ii. $h(0) > \frac{1}{2}$.
In this case, the preconditions of L-102 are fulfilled, which permits us to conclude from (487) and (488) that $\mathcal{B}_{CX}(g) = \mathcal{B}_{CX}(h)$.

Case b.: $g(0) = \frac{1}{2}$

Because $g \in \mathbb{B}$, $g(0) = \frac{1}{2}$ is possible only if $g = \mathrm{c}_{\frac{1}{2}}$. We hence conclude that

$$\mathcal{B}_{CX}(g) = \tfrac{1}{2}\,, \tag{489}$$

because $\mathcal{B}_{CX}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$ by (23).

Let us now consider $\mathcal{B}_{CX}(h)$. In order to prove that $\mathcal{B}_{CX}(h) = \frac{1}{2}$, too, we first need some observations on g. By assumption of case b., $g(0) = \frac{1}{2}$, i.e.

$$\begin{aligned}
g(0) &= \mathrm{m}_{\frac{1}{2}}\{f(0) : f \in V_0\}\\
&= \mathrm{m}_{\frac{1}{2}}(\inf\{f(0) : f \in V_0\}, \sup\{f(0) : f \in V_0\}) && \text{by Def. 46}\\
&= \tfrac{1}{2}\,.
\end{aligned}$$

It is apparent from Def. 45 that $\mathrm{m}_{\frac{1}{2}}(a, b) = \frac{1}{2}$ if and only if $\min(a, b) \leq \frac{1}{2}$ and $\max(a, b) \geq \frac{1}{2}$; and because $V_0 \neq \varnothing$, we know that $\inf\{f(0) : f \in V_0\} \leq \sup\{f(0) : f \in V_0\}$. Therefore $g(0) = \frac{1}{2}$ is possible only if

$$\inf\{f(0) : f \in V_0\} \leq \tfrac{1}{2} \tag{490}$$
$$\sup\{f(0) : f \in V_0\} \geq \tfrac{1}{2}\,. \tag{491}$$

Let us now use (490) to prove that

$$\inf\{\mathcal{B}_{CX}(f) : f \in V_0\} \leq \tfrac{1}{2}\,. \tag{492}$$

i. $V_0 \setminus \mathbb{B}^+ \neq \varnothing$.
Then there exists some $f_0 \in V_0$ such that $f_0 \in \mathbb{B}^-$ or $f_0 \in \mathbb{B}^{\frac{1}{2}}$. In any case, $\mathcal{B}_{CX}(f_0) \leq \frac{1}{2}$ by (23) and hence

$$\begin{aligned} \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} &\leq \mathcal{B}_{CX}(f_0) && \text{because } f_0 \in V_0 \\ &\leq \tfrac{1}{2}\,, && \text{by (23) because } f_0 \in \mathbb{B}^- \cup \mathbb{B}^+ \end{aligned}$$

i.e. (492) holds.

ii. $V_0 \setminus \mathbb{B}^+ = \varnothing$, i.e. $V_0 \subseteq \mathbb{B}^+$.
Let $\varepsilon > 0$. Because $\inf\{f(0) : f \in V_0\} \leq \frac{1}{2}$ by (490), there exists some $f' \in V_0$ such that

$$f'(0) < \inf\{f(0) : f \in V_0\} + \varepsilon\,.$$

In particular,

$$f'(0) < \tfrac{1}{2} + \varepsilon\,, \tag{493}$$

which is obvious from (490). Because $V_0 \subseteq \mathbb{B}^+$ by assumption of case ii., we know that $f_0 \in \mathbb{B}^+$. Hence f_0 is nonincreasing by Def. 68. In particular, $f_0(\gamma) \leq f_0(0) = \mathrm{c}_{f_0(0)}(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e. $f_0 \leq \mathrm{c}_{f_0(0)}$. Therefore

$$\begin{aligned} \inf\{\mathcal{B}_{CX}(f) : f \in V_0\} &\leq \mathcal{B}_{CX}(f_0) && \text{because } f_0 \in V_0 \\ &\leq \mathcal{B}_{CX}(\mathrm{c}_{f_0(0)}) && \text{by (B-5)} \\ &= f_0(0) && \text{by (B-1)} \\ &< \tfrac{1}{2} + \varepsilon && \text{by (493)}\,. \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitarily, we conclude that $\inf\{\mathcal{B}_{CX}(f) : f \in V_0\} \leq \frac{1}{2}$, i.e. (492) is satisfied.

We can utilize (491) in an analogous way to prove that

$$\sup\{\mathcal{B}_{CX}(f) : f \in V_0\} \geq \tfrac{1}{2}\,. \tag{494}$$

We shall again discern two cases.

i. $V_0 \setminus \mathbb{B}^- \neq \varnothing$.

Then there exists some $f_0 \in V_0$ such that $f_0 \in \mathbb{B}^+$ or $f_0 \in \mathbb{B}^{\frac{1}{2}}$. In any case, $\mathcal{B}_{CX}(f_0) \geq \frac{1}{2}$ by (23) and hence

$$\begin{aligned} \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} &\geq \mathcal{B}_{CX}(f_0) && \text{because } f_0 \in V_0 \\ &\geq \tfrac{1}{2}\,, && \text{by (23) because } f_0 \in \mathbb{B}^+ \cup \mathbb{B}^+ \end{aligned}$$

i.e. (494) holds.

ii. $V_0 \setminus \mathbb{B}^- = \varnothing$, i.e. $V_0 \subseteq \mathbb{B}^-$.
Let $\varepsilon > 0$. Because $\sup\{f(0) : f \in V_0\} \geq \frac{1}{2}$ by (491), there exists some $f' \in V_0$ such that

$$f'(0) > \sup\{f(0) : f \in V_0\} - \varepsilon\,.$$

In particular,

$$f'(0) > \tfrac{1}{2} - \varepsilon\,, \tag{495}$$

which is obvious from (491). Because $V_0 \subseteq \mathbb{B}^-$ by assumption of case ii., we know that $f_0 \in \mathbb{B}^-$. Hence f_0 is nondecreasing by Def. 68. In particular, $f_0(\gamma) \geq f_0(0) = \mathrm{c}_{f_0(0)}(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e. $f_0 \geq \mathrm{c}_{f_0(0)}$. Therefore

$$\begin{aligned} \sup\{\mathcal{B}_{CX}(f) : f \in V_0\} &\geq \mathcal{B}_{CX}(f_0) && \text{because } f_0 \in V_0 \\ &\geq \mathcal{B}_{CX}(\mathrm{c}_{f_0(0)}) && \text{by (B-5)} \\ &= f_0(0) && \text{by (B-1)} \\ &> \tfrac{1}{2} - \varepsilon && \text{by (495)}\,. \end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitarily, we conclude that $\sup\{\mathcal{B}_{CX}(f) : f \in V_0\} \geq \frac{1}{2}$, i.e. (494) is satisfied.

We can summarize these results as follows.

$$\begin{aligned} h(0) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_0\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\mathcal{B}_{CX}(f) : f \in V_0\}, \sup\{\mathcal{B}_{CX}(f) : f \in V_0\}) && \text{by Def. 46} \\ &= \tfrac{1}{2}\,. && \text{by Def. 45, (492) and (494)} \end{aligned}$$

Because $h \in \mathbb{B}$, $h(0) = \frac{1}{2}$ is possible only if $h = \mathrm{c}_{\frac{1}{2}}$ (see Def. 68). Hence

$$\begin{aligned} \mathcal{B}_{CX}(h) &= \mathcal{B}_{CX}(\mathrm{c}_{\frac{1}{2}}) && \text{because } h = \mathrm{c}_{\frac{1}{2}} \\ &= \tfrac{1}{2} && \text{by (23)} \\ &= \mathcal{B}_{CX}(g)\,. && \text{by (489)} \end{aligned}$$

Case c.: $g(0) < \frac{1}{2}$
Then

$$\begin{aligned} g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}\{1 - (1 - f(\gamma)) : f \in V_\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}\{1 - f'(\gamma) : f' \in V'_\gamma\} && \text{where } V'_\gamma = \{1 - f : f \in V_\gamma\} \\ &= 1 - \mathrm{m}_{\frac{1}{2}}\{f'(\gamma) : f' \in V'_\gamma\} && \text{apparent from Def. 46} \\ &= 1 - g'(\gamma)\,, \end{aligned}$$

i.e.

$$g = 1 - g' \tag{496}$$

where we have abbreviated

$$g'(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f') : f' \in V'_\gamma\}$$

for all $\gamma \in \mathbf{I}$. Similarly

$$\begin{aligned}
h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(1-(1-f)) : f \in V_\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(1-f') : f' \in V'_\gamma\} && \text{where } V'_\gamma = \{1-f : f \in V_\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{1-\mathcal{B}_{CX}(f') : f' \in V'_\gamma\} && \text{because } \mathcal{B}_{CX} \text{ satisfies (B-2)} \\
&= 1-\mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f') : f' \in V'_\gamma\} && \text{apparent from Def. 46} \\
&= 1-h'(\gamma)\,,
\end{aligned}$$

i.e.

$$h = 1 - h' \tag{497}$$

where

$$h'(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f') : f' \in V'_\gamma\}\,,$$

for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned}
\mathcal{B}_{CX}(g) &= \mathcal{B}_{CX}(1-g') && \text{by (496)} \\
&= 1-\mathcal{B}_{CX}(g') && \text{because } \mathcal{B}_{CX} \text{ satisfies (B-2)} \\
&= 1-\mathcal{B}_{CX}(h') && \text{by case a. of the proof of this lemma} \\
&= \mathcal{B}_{CX}(1-h') && \text{because } \mathcal{B}_{CX} \text{ satisfies (B-2)} \\
&= \mathcal{B}_{CX}(h)\,, && \text{by (497)}
\end{aligned}$$

as desired.

Proof of Theorem 102

Let us define $g : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$g(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}}\{Q(Y_1,\ldots,Y_n) : Y_n \in \mathcal{T}_\gamma(X_n)\} : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\}\,, \tag{498}$$

for all $\gamma \in \mathbf{I}$. It is then apparent that

$$\begin{aligned}
&Q_\gamma(X_1,\ldots,X_n) \\
&= \mathrm{m}_{\frac{1}{2}}\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_{n-1} \in X_{n-1}\gamma, X_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 67} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}}\{Q(Y_1,\ldots,Y_n) : Y_n \in \mathcal{T}_\gamma(X_n)\} \\
&\qquad : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by L-95, } \mathcal{T}_\gamma(X_n) \neq \varnothing \\
&= g(\gamma)\,. && \text{by (498)}
\end{aligned}$$

In particular, $g \in \mathbb{B}$ and

$$\mathcal{M}_{CX}(Q)(X_1, \dots, X_n) = \mathcal{B}_{CX}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) = \mathcal{B}_{CX}(g) \,. \tag{499}$$

For each choice of $Y_1, \dots, Y_n \in \mathcal{P}(E)$, let us define $f_{Y_1,\dots,Y_{n-1}} : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$f_{Y_1,\dots,Y_{n-1}}(\gamma) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_{n-1}, Y_n) : Y_n \in \mathcal{T}_\gamma(X_n)\} \,, \tag{500}$$

for all $\gamma \in \mathbf{I}$. It is then apparent from the fact that

$$f_{Y_1,\dots,Y_{n-1}}(\gamma) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_{n-1}, Y_n) : Y_n \in \mathcal{T}_\gamma(X_n)\} = Q_\gamma(Y_1, \dots, Y_n, X_n) \tag{501}$$

that $f_{Y_1,\dots,Y_{n-1}} \in \mathbb{B}$.
Now let us define a family $(V_\gamma)_{\gamma \in \mathbf{I}}$ of subsets $V_\gamma \subseteq \mathbb{B}$ as follows:

$$V_\gamma = \{f_{Y_1,\dots,Y_n}(\gamma) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} \,. \tag{502}$$

Because for $i \in \{1, \dots, n-1\}$, $\mathcal{T}_0(X_i) \neq \varnothing$ and $\mathcal{T}_\gamma(X_i) \subseteq \mathcal{T}_{\gamma'}(X_i)$ whenever $\gamma \leq \gamma'$, it is apparent that V_γ satisfies the conditions of L-103. In addition, it is apparent from the definition of g that

$$\begin{aligned}
g(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_n \in \mathcal{T}_\gamma(X_n)\} : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by (498)} \\
&= \mathrm{m}_{\frac{1}{2}}\{f_{Y_1,\dots,Y_{n-1}}(\gamma) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by (500)} \\
&= \mathrm{m}_{\frac{1}{2}}\{f(\gamma) : f \in V_\gamma\} \,. && \text{by (502)}
\end{aligned}$$

Let us now define $h : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$h(\gamma) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} \,, \tag{503}$$

for all $\gamma \in \mathbf{I}$. Then by L-103,

$$\mathcal{B}_{CX}(g) = \mathcal{B}_{CX}(h) \,. \tag{504}$$

Let us now take a closer look at h.

$$\begin{aligned}
h(\gamma) &= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f) : f \in V_\gamma\} && \text{by (503)} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}(f_{Y_1,\dots,Y_{n-1}}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by (502)} \\
&= \mathrm{m}_{\frac{1}{2}}\{\mathcal{B}_{CX}((Q_\gamma(Y_1, \dots, Y_{n-1}, X_n))_{\gamma \in \mathbf{I}}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by (501)} \\
&= \mathrm{m}_{\frac{1}{2}}\{(Q \,\widetilde{\triangleleft}\, X_n)(Y_1, \dots, Y_{n-1}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} && \text{by Def. 89} \\
&= (Q \,\widetilde{\triangleleft}\, X_n)_\gamma(X_1, \dots, X_{n-1}) && \text{by Def. 67} \,,
\end{aligned}$$

i.e.

$$h(\gamma) = (Q \,\widetilde{\triangleleft}\, X_n)_\gamma(X_1, \dots, X_{n-1}) \,, \tag{505}$$

for all $\gamma \in \mathbf{I}$. We can summarize our results as follows:

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X_1, \dots, X_n) \\
&= \mathcal{B}_{CX}(g) && \text{by (499)} \\
&= \mathcal{B}_{CX}(h) && \text{by (504)} \\
&= \mathcal{B}_{CX}(((Q \,\widetilde{\triangleleft}\, X_n)_\gamma(X_1, \dots, X_{n-1}))_{\gamma \in \mathbf{I}}) && \text{by (505)} \\
&= \mathcal{M}_{CX}(Q \,\widetilde{\triangleleft}\, X_n)(X_1, \dots, X_{n-1}) \,,
\end{aligned}$$

as desired.

F.18 Proof of Theorem 103

Lemma 104
Let $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *be given. Then*

$$d(Q_\gamma^{\min}, Q'^{\min}_\gamma) \le d(Q, Q')$$
$$d(Q_\gamma^{\max}, Q'^{\max}_\gamma) \le d(Q, Q')$$

for all $\gamma \in \mathbf{I}$.

Proof We shall abbreviate $\delta = d(Q, Q')$. Then by Def. 81,

$$|Q(Y_1, \ldots, Y_n) - Q'(Y_1, \ldots, Y_n)| \le \delta\,, \tag{506}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Hence

$$\begin{aligned} Q_\gamma^{\min}(X_1, \ldots, X_n) &= \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)} \\ &\ge \inf\{Q'(Y_1, \ldots, Y_n) - \delta : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (506)} \\ &= \inf\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} - \delta \\ &= Q'^{\min}_\gamma(X_1, \ldots, X_n) - \delta\,. \end{aligned}$$

Analogously,

$$\begin{aligned} Q_\gamma^{\min}(X_1, \ldots, X_n) &= \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)} \\ &\le \inf\{Q'(Y_1, \ldots, Y_n) + \delta : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (506)} \\ &= \inf\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} + \delta \\ &= Q'^{\min}_\gamma(X_1, \ldots, X_n) + \delta\,. \end{aligned}$$

Hence $|Q_\gamma^{\min}(X_1, \ldots, X_n) - Q'^{\min}_\gamma(X_1, \ldots, X_n)| \le \delta$. Because $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ were arbitrarily chosen, we conclude that

$$d(Q_\gamma^{\min}, Q'^{\min}_\gamma) = \sup |Q_\gamma^{\min}(X_1, \ldots, X_n) - Q'^{\min}_\gamma(X_1, \ldots, X_n)| \le \delta\,.$$

In the case of $Q_\gamma^{\max}$, we can proceed analogously:

$$\begin{aligned} Q_\gamma^{\max}(X_1, \ldots, X_n) &= \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (16)} \\ &\ge \sup\{Q'(Y_1, \ldots, Y_n) - \delta : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (506)} \\ &= \sup\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} - \delta \\ &= Q'^{\max}_\gamma(X_1, \ldots, X_n) - \delta \end{aligned}$$

and

$$\begin{aligned} Q_\gamma^{\max}(X_1, \ldots, X_n) &= \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (16)} \\ &\le \sup\{Q'(Y_1, \ldots, Y_n) + \delta : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (506)} \\ &= \sup\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} + \delta \\ &= Q'^{\max}_\gamma(X_1, \ldots, X_n) + \delta\,. \end{aligned}$$

Proof of Theorem 103

Case a.: $Q_0(X_1,\ldots,X_n) > \frac{1}{2}$**.**

a.1 $Q_0'(X_1,\ldots,X_n) > \frac{1}{2}$**.** In this case,

$$\begin{aligned}
Q_\gamma(X_1,\ldots,X_n) &= \max(\tfrac{1}{2}, Q_\gamma^{\min}(X_1,\ldots,X_n)) && \text{by L-88}\\
&\le \max(\tfrac{1}{2}, Q'^{\min}_\gamma(X_1,\ldots,X_n)+\delta) && \text{by L-104}\\
&\le \max(\tfrac{1}{2}+\delta, Q'^{\min}_\gamma(X_1,\ldots,X_n)+\delta) && \text{by monotonicity of } \max\\
&= \max(\tfrac{1}{2}, Q'^{\min}_\gamma(X_1,\ldots,X_n))+\delta\\
&= Q'_\gamma(X_1,\ldots,X_n)+\delta && \text{by L-88}
\end{aligned}$$

and similarly

$$\begin{aligned}
Q_\gamma(X_1,\ldots,X_n) &= \max(\tfrac{1}{2}, Q_\gamma^{\min}(X_1,\ldots,X_n)) && \text{by L-88}\\
&\ge \max(\tfrac{1}{2}, Q'^{\min}_\gamma(X_1,\ldots,X_n)-\delta) && \text{by L-104}\\
&\ge \max(\tfrac{1}{2}-\delta, Q'^{\min}_\gamma(X_1,\ldots,X_n)-\delta) && \text{by monotonicity of } \max\\
&= \max(\tfrac{1}{2}, Q'^{\min}_\gamma(X_1,\ldots,X_n))-\delta\\
&= Q'_\gamma(X_1,\ldots,X_n)-\delta && \text{by L-88}
\end{aligned}$$

This proves that $|Q_\gamma(X_1,\ldots,X_n) - Q'_\gamma(X_1,\ldots,X_n)| \le \delta$.

a.2 $Q_0'(X_1,\ldots,X_n) = \frac{1}{2}$**.** In this case, we conclude from Th-39 that

$$Q'_\gamma(X_1,\ldots,X_n) = \tfrac{1}{2} \tag{507}$$

for all $\gamma \in \mathbf{I}$. We can then utilise (14) and Def. 45 to conclude that

$$Q'^{\min}_\gamma(X_1,\ldots,X_n) \le \tfrac{1}{2} \tag{508}$$
$$Q'^{\max}_\gamma(X_1,\ldots,X_n) \ge \tfrac{1}{2} \tag{509}$$

for all $\gamma \in \mathbf{I}$. In addition, we know that $Q_0(X_1,\ldots,X_n) > \frac{1}{2}$; hence

$$\begin{aligned}
Q_0(X_1,\ldots,X_n) &= Q_0^{\min}(X_1,\ldots,X_n) && \text{by L-88}\\
&\le Q'^{\min}_0(X_1,\ldots,X_n)+\delta && \text{by L-104}\\
&\le \tfrac{1}{2}+\delta\,, && \text{by (508)}
\end{aligned}$$

i.e.

$$Q_0(X_1,\ldots,X_n) \le \tfrac{1}{2}+\delta\,. \tag{510}$$

Therefore

$$\begin{aligned}
Q_\gamma(X_1,\ldots,X_n) &\le Q_0(X_1,\ldots,X_n) && \text{by Th-39}\\
&\le \tfrac{1}{2}+\delta && \text{by (510)}\\
&= Q'_\gamma(X_1,\ldots,X_n)+\delta\,. && \text{by (507)}
\end{aligned}$$

On the other hand, we know from Th-39 and $Q_0(X_1,\ldots,X_n) > \frac{1}{2}$ that

$$Q_\gamma(X_1,\ldots,X_n) \ge \tfrac{1}{2} = Q'_\gamma(X_1,\ldots,X_n) \ge Q'_\gamma(X_1,\ldots,X_n)-\delta\,.$$

Hence $|Q_\gamma(X_1,\ldots,X_n) - Q'_\gamma(X_1,\ldots,X_n)| \le \delta$.

a.3 $Q'_0(X_1, \ldots, X_n) < \frac{1}{2}$**.** In this case,

$$\begin{aligned}
&|Q_0(X_1, \ldots, X_n) - Q'_0(X_1, \ldots, X_n)| \\
&= Q_0(X_1, \ldots, X_n) - Q'_0(X_1, \ldots, X_n) && \text{because } Q_0(X_1, \ldots, X_n) > \tfrac{1}{2} > Q'_0(X_1, \ldots, X_n) \\
&= Q_0^{\min}(X_1, \ldots, X_n) - Q'^{\max}_0(X_1, \ldots, X_n) && \text{by L-88} \\
&\leq Q_0^{\min}(X_1, \ldots, X_n) - Q'^{\min}_0(X_1, \ldots, X_n) && \text{because } Q'^{\min}_0 \leq Q'^{\max}_0\text{, cf. (15), (16)} \\
&= |Q_0^{\min}(X_1, \ldots, X_n) - Q'^{\min}_0(X_1, \ldots, X_n)| && \text{as } Q_0^{\min}(X_1, \ldots, X_n) > \tfrac{1}{2} > Q'^{\min}_0(X_1, \ldots, X_n) \\
&\leq \delta\,, && \text{by L-104}
\end{aligned}$$

which proves the claim of the theorem when $\gamma = 0$. Let us abbreviate

$$a = Q_0(X_1, \ldots, X_n) - \tfrac{1}{2} \tag{511}$$
$$a' = \tfrac{1}{2} - Q'_0(X_1, \ldots, X_n)\,. \tag{512}$$

Then clearly

$$\begin{aligned}
|Q_0(X_1, \ldots, X_n) - Q'_0(X_1, \ldots, X_n)| &= Q_0(X_1, \ldots, X_n) - Q'_0(X_1, \ldots, X_n) \\
&= (Q_0(X_1, \ldots, X_n) - \tfrac{1}{2}) + (\tfrac{1}{2} - Q'_0(X_1, \ldots, X_n)) \\
&= a + a'\,.
\end{aligned}$$

In particular,

$$a + a' = |Q_0(X_1, \ldots, X_n) - Q'_0(X_1, \ldots, X_n)| \leq \delta\,. \tag{513}$$

Now in the case that $\gamma > 0$,

$$Q_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2} \leq Q_0(X_1, \ldots, X_n) - \tfrac{1}{2} = a \tag{514}$$
$$\tfrac{1}{2} - Q'_\gamma(X_1, \ldots, X_n) \leq \tfrac{1}{2} - Q'_0(X_1, \ldots, X_n) = a' \tag{515}$$

by Th-39, (511) and (512). Therefore

$$\begin{aligned}
&|Q_\gamma(X_1, \ldots, X_n) - Q'_\gamma(X_1, \ldots, X_n)| \\
&= Q_\gamma(X_1, \ldots, X_n) - Q'_\gamma(X_1, \ldots, X_n) && \text{as } Q_\gamma(X_1, \ldots, X_n) \geq \tfrac{1}{2} \geq Q'_\gamma(X_1, \ldots, X_n) \\
& && \text{by Th-39} \\
&= (Q_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) + (\tfrac{1}{2} - Q'_\gamma(X_1, \ldots, X_n)) \\
&\leq a + a' && \text{by (514), (515)} \\
&\leq \delta\,. && \text{by (513)}
\end{aligned}$$

Case b.: $Q_0(X_1, \ldots, X_n) = \frac{1}{2}$**.**

b.1 $Q'_0(X_1, \ldots, X_n) > \frac{1}{2}$**.** Analogous to **a.2**.

b.2 $Q'_0(X_1, \ldots, X_n) = \frac{1}{2}$**.** Trivial because by Th-39, $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2} = Q'_\gamma(X_1, \ldots, X_n)$ for all $\gamma \in \mathbf{I}$.

b.3 $Q'_0(X_1,\ldots,X_n) < \frac{1}{2}$**.** Then

$$\begin{aligned}
&|Q_\gamma(X_1,\ldots,X_n) - Q'_\gamma(X_1,\ldots,X_n)| \\
&= \tfrac{1}{2} - Q'_\gamma(X_1,\ldots,X_n) && \text{by Th-39}\\
&\le \tfrac{1}{2} - Q'_0(X_1,\ldots,X_n) && \text{by Th-39}\\
&= \tfrac{1}{2} - {Q'}_0^{\max}(X_1,\ldots,X_n) && \text{by L-88}\\
&\le Q_0^{\max}(X_1,\ldots,X_n) - {Q'}_0^{\max}(X_1,\ldots,X_n) && \text{by L-88, } Q_0^{\max}(X_1,\ldots,X_n) \ge \tfrac{1}{2}\\
&\le \delta\,. && \text{by L-104}
\end{aligned}$$

Case c.: $Q_0(X_1,\ldots,X_n) < \frac{1}{2}$**.**

c.1 $Q'_0(X_1,\ldots,X_n) > \frac{1}{2}$**.** See case **a.3**.

c.2 $Q'_0(X_1,\ldots,X_n) = \frac{1}{2}$**.** Analogous to **b.3**.

c.3 $Q'_0(X_1,\ldots,X_n) < \frac{1}{2}$**.** In this case,

$$\begin{aligned}
Q_\gamma(X_1,\ldots,X_n) &= \min(\tfrac{1}{2}, Q_\gamma^{\max}(X_1,\ldots,X_n)) && \text{by L-88}\\
&\le \min(\tfrac{1}{2}, {Q'}_\gamma^{\max} + \delta) && \text{by L-104}\\
&\le \min(\tfrac{1}{2} + \delta, {Q'}_\gamma^{\max} + \delta) &&\\
&= \min(\tfrac{1}{2}, {Q'}_\gamma^{\max}) + \delta &&\\
&= {Q'}_\gamma(X_1,\ldots,X_n) + \delta\,. && \text{by L-88}
\end{aligned}$$

Similarly

$$\begin{aligned}
Q_\gamma(X_1,\ldots,X_n) &= \min(\tfrac{1}{2}, Q_\gamma^{\max}(X_1,\ldots,X_n)) && \text{by L-88}\\
&\ge \min(\tfrac{1}{2}, {Q'}_\gamma^{\max} - \delta) && \text{by L-104}\\
&\ge \min(\tfrac{1}{2} - \delta, {Q'}_\gamma^{\max} - \delta) &&\\
&= \min(\tfrac{1}{2}, {Q'}_\gamma^{\max}) - \delta &&\\
&= {Q'}_\gamma(X_1,\ldots,X_n) - \delta\,. && \text{by L-88}
\end{aligned}$$

Hence $|Q_\gamma(X_1,\ldots,X_n) - Q'_\gamma(X_1,\ldots,X_n)| \le \delta$, as desired.

F.19 Proof of Theorem 104

Let us first introduce a metric $d : \mathbb{B} \times \mathbb{B} \longrightarrow \mathbf{I}$, which we define by

$$d(f,g) = \sup\{|f(\gamma) - g(\gamma)| : \gamma \in \mathbf{I}\}\,, \tag{516}$$

for all $f, g \in \mathbb{B}$.

Lemma 105
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ has the following property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mathcal{B}(f) - \mathcal{B}(g)| < \varepsilon$ whenever $f, g \in \mathbb{B}$ satisfy $d(f,g) < \delta$. Then $\mathcal{M}_{\mathcal{B}}$ is Q-continuous.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $\varepsilon > 0$. By the assumption of the lemma, there exists $\delta > 0$ such that

$$|\mathcal{B}(f) - \mathcal{B}(g)| < \varepsilon \tag{517}$$

whenever $d(f, g) < \delta$, $f, g \in \mathbb{B}$. Now let $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier such that

$$d(Q, Q') < \delta\,. \tag{518}$$

Further let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We shall abbreviate

$$f(\gamma) = Q_\gamma(X_1, \ldots, X_n) \tag{519}$$
$$g(\gamma) = Q'_\gamma(X_1, \ldots, X_n) \tag{520}$$

for all $\gamma \in \mathbf{I}$. By Th-103, $d(Q, Q') < \delta$ entails that

$$|Q_\gamma(X_1, \ldots, X_n) - Q'_\gamma(X_1, \ldots, X_n)| \le d(Q, Q') \tag{521}$$

for all $\gamma \in \mathbf{I}$. Hence

$$\begin{aligned} d(f, g) &= \sup\{|Q_\gamma(X_1, \ldots, X_n) - Q'_\gamma(X_1, \ldots, X_n)| && \text{by (516), (519), (520)} \\ &\le d(Q, Q') && \text{by (521)} \\ &< \delta\,. && \text{by (518)} \end{aligned}$$

Therefore

$$\begin{aligned} &|\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(Q')(X_1, \ldots, X_n)| \\ &= |\mathcal{B}(f) - \mathcal{B}(g)| && \text{by Def. 69, (519), (520)} \\ &< \varepsilon\,, && \text{by (517) and } d(f, g) < \delta \end{aligned}$$

which finishes the proof.

Lemma 106
Suppose $\mathcal{B}' : \mathbb{B} \longrightarrow \mathbf{I}$ has the following property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < \delta$. Then $\mathcal{M}_\mathcal{B}$ is Q-continuous.

Proof It is sufficient to show that $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$, as defined by (23), satisfies the conditions of Lemma L-106. Hence let $\varepsilon > 0$. By the assumption on $\mathcal{B}'$, there exists $\delta > 0$ such that

$$|\mathcal{B}'(f) - \mathcal{B}'(g)| < 2\varepsilon \tag{522}$$

for all $f, g \in \mathbb{H}$ such that $d(f, g) < \delta$. Now let $p, q \in \mathbb{B}$ such that

$$d(p, q) < \min(\tfrac{1}{2}\delta, \varepsilon)\,. \tag{523}$$

In order to prove that $|\mathcal{B}(p) - \mathcal{B}(q)| < \varepsilon$, we shall discern the following cases:

a.: $p, q \in \mathbb{B}^+$**.** Then apparently

$$d(2p-1, 2q-1) = 2d(p,q) < \delta\,. \tag{524}$$

By (23), $\mathcal{B}(p) = \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2p-1)$ and $\mathcal{B}(q) = \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2q-1)$. Hence

$$\begin{aligned} |\mathcal{B}(p) - \mathcal{B}(q)| &= |(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2p-1)) - (\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(2q-1))| && \text{by (23), } p,q \in \mathbb{B}^+ \\ &= \tfrac{1}{2}|\mathcal{B}'(2q-1) - \mathcal{B}'(2p-1)| \\ &< \varepsilon\,. && \text{by (522), (524)} \end{aligned}$$

b.: $p, q \in \mathbb{B}^-$**.** Then $1-p, 1-q \in \mathbb{B}^+$. In addition, it is apparent from (516) that $d(1-p, 1-q) = d(p,q)$. Hence

$$\begin{aligned} |\mathcal{B}(p) - \mathcal{B}(q)| &= |(1-\mathcal{B}(p)) - (1-\mathcal{B}(q))| \\ &= |\mathcal{B}(1-p) - \mathcal{B}(1-q)| && \text{by (B-2)} \\ &< \varepsilon\,. && \text{by case a. of the proof} \end{aligned}$$

c.: neither $p, q \in \mathbb{B}^+$ **nor** $p, q \in \mathbb{B}^-$**.** In this case, we know from Def. 68 that either $p \le q$ or $p \ge q$. Without loss of generality, we shall assume that $p \ge q$, i.e. $p \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$ and $q \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$. We also know from Def. 68 that $\mathrm{c}_{q(0)} \le q \le p \le \mathrm{c}_{p(0)}$. Therefore

$$q(0) = \mathcal{B}(\mathrm{c}_{q(0)}) \le \mathcal{B}(q) \le \mathcal{B}(p) \le \mathcal{B}(\mathrm{c}_{p(0)}) = p(0) \tag{525}$$

which is apparent from (B-5). Hence

$$\begin{aligned} |\mathcal{B}(p) - \mathcal{B}(q)| &= \mathcal{B}(p) - \mathcal{B}(q) && \text{by (B-5) and } p \ge q \\ &\le p(0) - q(0) && \text{by (525)} \\ &\le d(p,q) && \text{by (516)} \\ &< \varepsilon\,, && \text{by (523)} \end{aligned}$$

as desired.

Lemma 107
Let $f \in \mathbb{B}^+$ be a given mapping, $a \in \mathbf{I}$ and $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ the semi-fuzzy quantifier defined by

$$Q(Y) = \min(1, \sup Y + a)\,, \tag{526}$$

for all $Y \in \mathcal{P}(\mathbf{I})$. Then $Q_\gamma(X) = g^\flat(\gamma)$ for all $\gamma > 0$, where $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ is the fuzzy set defined by

$$\mu_X(z) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma \in \mathbf{I} : f(\gamma) \ge z\}\,, \tag{527}$$

for all $z \in \mathbf{I}$, and

$$g(\gamma) = \min(1, f(\gamma) + a) \tag{528}$$

for all $\gamma \in \mathbf{I}$.

Proof Let $\gamma > 0$, i.e.

$$(X)_\gamma^{\min} = (X)_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = \{z \in \mathbf{I} : \sup\{\gamma \in \mathbf{I} : f(\gamma) \geq z\} \geq \gamma\} . \tag{529}$$

We will first show that $f^\flat(\gamma) \leq \sup (X)_\gamma^{\min}$. Hence suppose $\gamma' < \gamma$. Then $f(\gamma') \geq \inf\{f(\gamma') : \gamma' < \gamma\} = f^\flat(\gamma)$ by Def. 71 and Th-43. Hence $f^\flat(\gamma) \in (X)_\gamma^{\min}$ by (529), i.e.

$$\sup (X)_\gamma^{\min} \geq f^\flat(\gamma) . \tag{530}$$

Now suppose that $z \in (X)_\gamma^{\min}$, i.e.

$$f(\gamma') \geq z \tag{531}$$

for all $\gamma' < \gamma$ by (529). Hence

$$\begin{aligned} f^\flat(\gamma) &= \inf\{f(\gamma') : \gamma' < \gamma\} && \text{by Def. 71, Th-43} \\ &\geq z , && \text{by (531)} \end{aligned}$$

i.e.

$$f^\flat(\gamma) \geq \sup (X)_\gamma^{\min} .$$

Combining this with (530), we conclude that

$$f^\flat(\gamma) = \sup (X)_\gamma^{\min} \tag{532}$$

for all $\gamma > 0$. Hence

$$\begin{aligned} Q_\gamma^{\min}(X) &= \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} && \text{by (15)} \\ &= Q((X)_\gamma^{\min}) && \text{because } Q \text{ is nondecreasing} \\ &= \min(1, \sup (X)_\gamma^{\min} + a) && \text{by (526)} \\ &= \min(1, f^\flat(\gamma) + a) && \text{by (532)} \\ &= g^\flat(\gamma) , \end{aligned}$$

i.e.

$$Q_\gamma^{\min}(X) = g^\flat(\gamma) \tag{533}$$

where the last step is apparent from (528) and Def. 71. Hence

$$\begin{aligned} Q_\gamma(X) &= \mathrm{m}_{\frac{1}{2}}(Q_X^{\min}\gamma, Q_X^{\max}\gamma) && \text{by (14)} \\ &= Q_X^{\min}\gamma && \text{because } \tfrac{1}{2} \geq g^\flat(\gamma) = (X)_\gamma^{\min} \leq (X)_\gamma^{\max} \\ &= g^\flat(\gamma) , && \text{by (533)} \end{aligned}$$

which finishes the proof of the lemma.

Lemma 108
Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given such that $\mathcal{M}_\mathcal{B}$ is a DFS. If $\mathcal{M}_\mathcal{B}$ is Q-continuous, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{B}(f) - \mathcal{B}(g)| < \varepsilon$ whenever $f, g \in \mathbb{B}$ satisfy $d(f, g) < \delta$.

Proof The proof is by contraposition. Suppose that there exists $\varepsilon > 0$ such that for all $\delta > 0$, there is a choice of $f, g \in \mathbb{B}$ such that $d(f, g) < \delta$ and $|\mathcal{B}(f) - \mathcal{B}(g)| \geq \varepsilon$. We have to show that $\mathcal{M}_{\mathcal{B}}$ is not Q-continuous.
Let us assume that ε is such a critical choice with respect to $\mathcal{B}$. We shall denote by $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ the quantifier defined by

$$Q(Y) = \sup Y\,, \tag{534}$$

for all $Y \in \mathcal{P}(\mathbf{I})$. We will show that for all $\delta > 0$, there exists a semi-fuzzy quantifier $Q' : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ such that $d(Q, Q') < \delta$, but $d(\mathcal{M}_{\mathcal{B}}(Q), \mathcal{M}_{\mathcal{B}}(Q')) \geq \varepsilon$.
Hence let $\delta > 0$. By our choice of ε, there exist $f, g \in \mathbb{B}$ such that $d(f, g) < \delta$ and $|\mathcal{B}(f) - \mathcal{B}(g)| \geq \varepsilon$. Without loss of generality, we may assume that

$$f \leq g\,. \tag{535}$$

This is because $d(f, g) = d(\min(f, q), \max(f, g))$, as is easily seen from (516), and because $|\mathcal{B}(\max(f, g)) - \mathcal{B}(\min(f, g))| \geq |\mathcal{B}(f) - \mathcal{B}(g)|$, which is apparent from $\mathcal{B}(\min(f, g)) \leq \mathcal{B}(f) \leq \mathcal{B}(\max(f, g))$ and $\mathcal{B}(\min(f, g)) \leq \mathcal{B}(g) \leq \mathcal{B}(\max(f, g))$, see (B-5).

We may also assume that $\delta < \varepsilon$. It is hence legitimate to exclude the case that $f \in \mathbb{B}^{-} \cup \mathbb{B}^{\frac{1}{2}}$ and $g \in \mathbb{B}^{+} \cup \mathbb{B}^{\frac{1}{2}}$, because in this case, $|\mathcal{B}(f) - \mathcal{B}(g)| = \mathcal{B}(g) - \mathcal{B}(f) \leq g(0) - f(0) \leq \delta < \varepsilon$ (see proof of lemma L-106, case **c.**).
It is hence sufficient to consider the following cases:

a.: $f, g \in \mathbb{B}^{+}$**.** In this case, let us define the semi-fuzzy quantifier $Q' : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q'(Y) = \min(1, \sup Y + d(f, g)) \tag{536}$$

for all $Y \in \mathbf{I}$. It is then apparent from (534) that

$$d(Q, Q') = \sup\{|Q'(Y) - Q(Y)| : Y \in \mathcal{P}(\mathbf{I})\} = d(f, g) < \delta\,.$$

Now let us consider the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by equation (527). We can apply L-107, which yields

$$Q_\gamma(X) = f^\flat(\gamma) \tag{537}$$
$$Q'_\gamma(X) = h^\flat(\gamma) \tag{538}$$

for all $\gamma > 0$, where $h \in \mathbb{B}$ is defined by

$$h(\gamma) = \min(1, f(\gamma) + d(f, g)) \tag{539}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (539) and (516) that $h \geq f$. Therefore

$$\begin{aligned}
|\mathcal{M}_{\mathcal{B}}(Q')(X) - \mathcal{M}_{\mathcal{B}}(Q)(X)| &= |\mathcal{B}((Q'_\gamma(X))_{\gamma\in\mathbf{I}}) - \mathcal{B}((Q_\gamma(X))_{\gamma\in\mathbf{I}})| && \text{by Def. 69}\\
&= |\mathcal{B}(h) - \mathcal{B}(f)| && \text{by (537), (538) and L-41}\\
&= \mathcal{B}(h) - \mathcal{B}(f) && \text{by (B-5) and } h \geq f\\
&\geq \mathcal{B}(g) - \mathcal{B}(f) && \text{by (B-5) and } h \geq g \geq f\\
&= |\mathcal{B}(g) - \mathcal{B}(f)| && \text{by (535)}\\
&\geq \varepsilon\,.
\end{aligned}$$

b.: $f, g \in \mathbb{B}^-$. In this case, $1 - f, 1 - g \in \mathbb{B}^+$ satisfy $d(1 - f, 1 - g) = d(f, g) < \delta$ and

$$|\mathcal{B}(1 - f) - \mathcal{B}(1 - g)| = |(1 - \mathcal{B}(f)) - (1 - \mathcal{B}(g))| = |\mathcal{B}(f) - \mathcal{B}(g)| \geq \varepsilon$$

Hence there exist $f' = 1 - f, g' = 1 - g$ which satisfy the conditions of case **a.**, i.e. the construction developed in case **a.** can be applied.
As a corollary, we have:

Lemma 109
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given such that $\mathcal{M}_\mathcal{B}$ is a DFS. If $\mathcal{M}_\mathcal{B}$ is Q-continuous, then for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < \delta$.

Proof Let $\varepsilon > 0$ be given. We already know by L-108 that there exists $\delta > 0$ such that

$$|\mathcal{B}(p) - \mathcal{B}(q)| < \tfrac{1}{2}\varepsilon \tag{540}$$

whenever $p, q \in \mathbb{B}$ satisfy $d(p, q) < \delta$. Recalling Th-63 and (24), we obtain that whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < 2\delta$, then

$$\begin{aligned} |\mathcal{B}'(f) - \mathcal{B}'(g)| &= |(2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1) - (2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}g) - 1)| && \text{by (24)} \\ &= 2|\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - \mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}g)| \\ &< \varepsilon\,, \end{aligned}$$

because $d(\frac{1}{2} + \frac{1}{2}f, \frac{1}{2} + \frac{1}{2}g) = \frac{1}{2}d(f, g) < \delta$ and hence $|\mathcal{B}(\frac{1}{2} + \frac{1}{2}f) - \mathcal{B}(\frac{1}{2} + \frac{1}{2}g)| < \frac{1}{2}\varepsilon$ by (540).

Proof of Theorem 104

The theorem is an immediate consequence of lemma L-105, which shows that the condition on $\mathcal{B}'$ is sufficient for $\mathcal{M}_\mathcal{B}$ to be Q-continuous, and of lemma L-109, which proves that the condition is also necessary.

F.20 Proof of Theorem 105

Lemma 110
Let $E \neq \varnothing$ be some base set and $X, X' \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma, \gamma' \in \mathbf{I}$ such that $\gamma' > \gamma + 2d(X, X')$, $\mathcal{T}_\gamma(X) \subseteq \mathcal{T}_{\gamma'}(X')$.

Proof Let us first show that $(X')^{\min}_{\gamma'} \subseteq (X)^{\min}_{\gamma}$. Hence let $e \in (X')^{\min}_{\gamma'} = (X)_{\geq \frac{1}{2} + \frac{1}{2}\gamma'}$ (by Def. 66 because $\gamma' > 0$). Then

$$\begin{aligned} \mu_{X'}(e) &\geq \tfrac{1}{2} + \tfrac{1}{2}\gamma' \\ &> \tfrac{1}{2} + \tfrac{1}{2}(\gamma + 2d(X, X')) && \text{by assumption that } \gamma' > \gamma + 2d(X, X') \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma + d(X, X')\,, \end{aligned}$$

i.e.

$$\mu_{X'}(e) > \tfrac{1}{2} + \tfrac{1}{2}\gamma + d(X, X')\,. \tag{541}$$

We know from (25) that $|\mu_X(e) - \mu_{X'}(e)| \leq d(X, X')$. Hence

$$\begin{aligned} \mu_X(e) &\geq \mu_{X'}(e) - d(X, X') \\ &> \tfrac{1}{2} + \tfrac{1}{2}\gamma + d(X, X') - d(X, X') && \text{by (541)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma\,, \end{aligned}$$

in particular $e \in (X)^{\min}_\gamma$, cf. Def. 66.
Similarly, we prove that $(X)^{\max}_\gamma \subseteq (X')^{\max}_{\gamma'}$. Hence let $e \in (X)^{\max}_\gamma$. By Def. 66, we know that

$$\mu_X(e) \geq \tfrac{1}{2} - \tfrac{1}{2}\gamma\,. \tag{542}$$

In addition, we know from (25) that $|\mu_X(e) - \mu_{X'}(e)| \leq d(X, X')$ and hence $\mu_{X'}(e) \geq \mu_X(e) - d(X, X')$. Therefore

$$\begin{aligned} \mu_{X'}(e) &\geq \mu_X(e) - d(X, X') \\ &\geq \tfrac{1}{2} - \tfrac{1}{2}\gamma - d(X, X') && \text{by (542)} \\ &= \tfrac{1}{2} - \tfrac{1}{2}(\gamma + 2d(X, X')) \\ &> \tfrac{1}{2} - \tfrac{1}{2}\gamma'\,. && \text{by assumption that } \gamma' > \gamma + 2d(X, X') \end{aligned}$$

It is then apparent from Def. 66 and $\gamma' > 0$ that $e \in (X')_{> \frac{1}{2} + \frac{1}{2}\gamma'} = (X')^{\max}_{\gamma'}$.
Summarising these results, we have shown that $(X')^{\min}_{\gamma'} \subseteq (X)^{\min}_\gamma$ and $(X)^{\max}_\gamma \subseteq (X')^{\max}_{\gamma'}$, i.e. $\mathcal{T}_\gamma(X) \subseteq \mathcal{T}_{\gamma'}(X')$ by Def. 66.

Lemma 111
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ a choice of fuzzy argument sets. Then

$$\begin{aligned} (\neg Q)^{\min}_\gamma(X_1, \ldots, X_n) &= \neg(Q^{\max}_\gamma(X_1, \ldots, X_n)) \\ (\neg Q)^{\max}_\gamma(X_1, \ldots, X_n) &= \neg(Q^{\min}_\gamma(X_1, \ldots, X_n))\,, \end{aligned}$$

for all $\gamma \in \mathbf{I}$.

Proof Trivial. Firstly

$$\begin{aligned} (\neg Q)^{\min}_\gamma(X_1, \ldots, X_n) &= \inf\{\neg Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)} \\ &= \inf\{1 - Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by } \neg x = 1 - x \\ &= 1 - \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} \\ &= 1 - Q^{\max}_\gamma(X_1, \ldots, X_n) && \text{by (16)} \\ &= \neg(Q^{\max}_\gamma(X_1, \ldots, X_n))\,. && \text{by } \neg x = 1 - x \end{aligned}$$

Similarly

$$\begin{aligned} (\neg Q)^{\max}_\gamma(X_1, \ldots, X_n) &= \neg\neg(\neg Q)^{\max}_\gamma(X_1, \ldots, X_n) && \text{because } \neg \text{ involution} \\ &= \neg(\neg\neg Q)^{\min}_\gamma(X_1, \ldots, X_n) && \text{by above reasoning} \\ &= \neg(Q^{\min}_\gamma(X_1, \ldots, X_n))\,. && \text{because } \neg \text{ involution} \end{aligned}$$

Lemma 112
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*,* $\delta > 0$ *and* $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ *such that*

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta\,.$$

Then for all $\gamma, \gamma' \in \mathbf{I}$ *such that* $\gamma' \geq \gamma + 2\delta$*,*

a. $\min(Q^{\max}_{\gamma'}(X_1, \ldots, X_n), Q^{\max}_{\gamma'}(X'_1, \ldots, X'_n)) \geq \max(Q^{\max}_{\gamma}(X_1, \ldots, X_n), Q^{\max}_{\gamma}(X'_1, \ldots, X'_n))$.

b. $\max(Q^{\min}_{\gamma'}(X_1, \ldots, X_n), Q^{\min}_{\gamma'}(X'_1, \ldots, X'_n)) \leq \min(Q^{\min}_{\gamma}(X_1, \ldots, X_n), Q^{\min}_{\gamma}(X'_1, \ldots, X'_n))$.

Proof

ad a.) Without loss of generality, we may assume that $Q^{\max}_{\gamma}(X_1, \ldots, X_n) \geq Q^{\max}_{\gamma}(X'_1, \ldots, X'_n)$, i.e.

$$\sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} \geq Q^{\max}_{\gamma}(X'_1, \ldots, X'_n) \tag{543}$$

by (16). Hence let $\varepsilon > 0$. By (543), there exist $Y^*_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y^*_n \in \mathcal{T}_\gamma(X_n)$ such that

$$Q(Y^*_1, \ldots, Y^*_n) > Q^{\max}_{\gamma}(X'_1, \ldots, X'_n) - \varepsilon\,. \tag{544}$$

Now consider $\gamma' \geq \gamma + 2\delta$. It is apparent from (25) that

$$d(X_i, X'_i) \leq d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta\,.$$

We can hence apply L-110 to conclude that $\mathcal{T}_\gamma(X_i) \subseteq \mathcal{T}_{\gamma'}(X'_i)$ for $i = 1, \ldots, n$, i.e.

$$Y^*_i \in \mathcal{T}_{\gamma'}(X'_i)\,, \tag{545}$$

for all $i = 1, \ldots, n$. Therefore

$$\begin{aligned} Q^{\max}_{\gamma'}(X'_1, \ldots, X'_n) &= \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_{\gamma'}(X'_i)\} && \text{bu (16)} \\ &\geq Q(Y^*_1, \ldots, Y^*_n) && \text{by (545)} \\ &> Q^{\max}_{\gamma}(X_1, \ldots, X_n) - \varepsilon && \text{by (544)} \\ &= \max(Q^{\max}_{\gamma}(X_1, \ldots, X_n), Q^{\max}_{\gamma}(X'_1, \ldots, X'_n)) - \varepsilon\,. && \text{by assumption of a.} \end{aligned}$$

$\varepsilon \to 0$ yields

$$Q^{\max}_{\gamma'}(X'_1, \ldots, X'_n) \geq \max(Q^{\max}_{\gamma}(X_1, \ldots, X_n), Q^{\max}_{\gamma}(X'_1, \ldots, X'_n))\,. \tag{546}$$

In addition, $\mathcal{T}_{\gamma'}(X_i) \supseteq \mathcal{T}_\gamma(X_i)$ for all $i = 1, \ldots, n$ because $\gamma' \geq \gamma$ and hence

$$\begin{aligned} Q^{\max}_{\gamma'}(X_1, \ldots, X_n) &= \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_{\gamma'}(X_i)\} && \text{by (16)} \\ &\geq \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_{\gamma}(X_i)\} && \text{because } \mathcal{T}_\gamma(X_i) \subseteq \mathcal{T}_{\gamma'}(X_i) \\ & && \text{for all } i \\ &= Q^{\max}_{\gamma}(X_1, \ldots, X_n) && \text{by (16)} \\ &= \max(Q^{\max}_{\gamma}(X_1, \ldots, X_n), Q^{\max}_{\gamma}(X'_1, \ldots, X'_n)) && \text{by assumption of a.} \end{aligned}$$

Combining this with (546),

$$\min(Q^{\max}_{\gamma'}(X_1, \ldots, X_n), Q^{\max}_{\gamma'}(X'_1, \ldots, X'_n)) \geq \max(Q^{\max}_{\gamma}(X_1, \ldots, X_n), Q^{\max}_{\gamma}(X'_1, \ldots, X'_n))\,,$$

as desired.

ad b.) In this case,

$$\begin{aligned}
&\max(Q^{\min}_{\gamma'}(X_1,\dots,X_n), Q^{\min}_{\gamma'}(X'_1,\dots,X'_n)) \\
&= \max(\neg\neg Q^{\min}_{\gamma'}(X_1,\dots,X_n), \neg\neg Q^{\min}_{\gamma'}(X'_1,\dots,X'_n)) && \text{because } \neg = 1-x \text{ involutive} \\
&= \max(\neg(\neg Q)^{\max}_{\gamma'}(X_1,\dots,X_n), \neg(\neg Q)^{\max}_{\gamma'}(X'_1,\dots,X'_n)) && \text{by L-111} \\
&= \neg\min((\neg Q)^{\max}_{\gamma'}(X_1,\dots,X_n), (\neg Q)^{\max}_{\gamma'}(X'_1,\dots,X'_n)) && \text{by De Morgan's law} \\
&\le \neg\max((\neg Q)^{\max}_{\gamma}(X_1,\dots,X_n), (\neg Q)^{\max}_{\gamma}(X_1,\dots,X_n)) && \text{by part a. of the lemma} \\
&= \neg\max(\neg Q^{\min}_{\gamma}(X_1,\dots,X_n), \neg Q^{\min}_{\gamma}(X'_1,\dots,X'_n)) && \text{by L-111} \\
&= \min(\neg\neg Q^{\min}_{\gamma}(X_1,\dots,X_n), \neg\neg Q^{\min}_{\gamma}(X'_1,\dots,X'_n)) && \text{by De Morgan's law} \\
&= \min(Q^{\min}_{\gamma}(X_1,\dots,X_n), Q^{\min}_{\gamma}(X'_1,\dots,X'_n))\,. && \text{because } \neg \text{ involution}
\end{aligned}$$

Lemma 113

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $\delta > 0$, and $X_1,\dots,X_n,X'_1,\dots,X'_n \in \widetilde{\mathcal{P}}(E)$ satisfy $d((X_1,\dots,X_n),(X'_1,\dots,X'_n)) < \delta$. If $Q_0(X_1,\dots,X_n) > \frac{1}{2}$ and $Q_0(X'_1,\dots,X'_n) > \frac{1}{2}$, then

$$\max(Q_{\gamma'}(X_1,\dots,X_n), Q_{\gamma'}(X'_1,\dots,X'_n)) \le \min(Q_{\gamma}(X_1,\dots,X_n), Q_{\gamma}(X'_1,\dots,X'_n))$$

whenever $\gamma' \ge \gamma + 2\delta$.

Proof We compute

$$\begin{aligned}
&\max(Q_{\gamma'}(X_1,\dots,X_n), Q_{\gamma'}(X'_1,\dots,X'_n)) \\
&= \max(\max(\tfrac{1}{2}, Q^{\min}_{\gamma'}(X_1,\dots,X_n)), \max(\tfrac{1}{2}, Q^{\min}_{\gamma'}(X'_1,\dots,X'_n))) && \text{by L-88} \\
&= \max(\tfrac{1}{2}, \max(Q^{\min}_{\gamma'}(X_1,\dots,X_n), Q^{\min}_{\gamma'}(X'_1,\dots,X'_n))) \\
&\le \max(\tfrac{1}{2}, \min(Q^{\min}_{\gamma}(X_1,\dots,X_n), Q^{\min}_{\gamma}(X'_1,\dots,X'_n))) && \text{by L-112} \\
&= \min(\max(\tfrac{1}{2}, Q^{\min}_{\gamma}(X_1,\dots,X_n)), \max(\tfrac{1}{2}, Q^{\min}_{\gamma}(X'_1,\dots,X'_n))) && \text{by distributivity of } \min, \max \\
&= \min(Q_{\gamma}(X_1,\dots,X_n), Q_{\gamma}(X'_1,\dots,X'_n))\,. && \text{by L-88}
\end{aligned}$$

Lemma 114

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $\delta > 0$ and $X_1,\dots,X_n,X'_1,\dots,X'_n \in \widetilde{\mathcal{P}}(E)$ such that

$$d((X_1,\dots,X_n),(X'_1,\dots,X'_n)) < \delta\,.$$

We shall abbreviate

$$\begin{aligned}
f(\gamma) &= Q_{\gamma}(X_1,\dots,X_n) \\
g(\gamma) &= Q_{\gamma}(X'_1,\dots,X'_n)\,,
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. If $f \in \mathbb{B}^+$ and $g \in \mathbb{B}^+$, then $d'(f',g') < 2\delta$, where $f' = 2f-1$, $g' = 2g-1$, and d' is defined by equation (35).

Proof Assume to the contrary that $d'(f', g') \geq 2\delta$. Then by (35), there exists $\gamma \in \mathbf{I}$ such that $\inf\{\gamma' - \gamma : \gamma' \in \mathbf{I}, \max(f'(\gamma'), g'(\gamma')) \leq \min(f'(\gamma), g'(\gamma))\} \geq 2\delta$, i.e.

$$\inf\{\gamma' : \max(f'(\gamma'), g'(\gamma')) \leq \min(f'(\gamma), g'(\gamma))\} \geq \gamma + 2\delta$$

and by utilizing $f = \frac{1}{2} + \frac{1}{2}f'$, $g = \frac{1}{2} + \frac{1}{2}g$,

$$\inf\{\gamma' : \max(f(\gamma'), g(\gamma')) \leq \min(f(\gamma), g(\gamma))\} \geq \gamma + 2\delta\,.$$

In other words: if $\gamma' < \gamma + 2\delta$, then

$$\max(f(\gamma'), g(\gamma')) > \min(f(\gamma), g(\gamma))\,. \tag{547}$$

By assumption, $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. We may hence choose $\kappa \in \mathbf{I}$ such that $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \kappa < \delta$. In addition, we set $\gamma' = \gamma + 2\kappa$. Applying lemma L-113, we deduce that

$$\max(f(\gamma'), g(\gamma')) \leq \min(f(\gamma), g(\gamma))\,.$$

This contradicts (547) because $\gamma' = \gamma + 2\kappa < \gamma + 2\delta$. Hence the initial assumption that $d'(f', g') \geq 2\delta$ is false; by contradiction, we conclude that $d'(f', g') < 2\delta$, as desired.

Lemma 115
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$. We shall abbreviate

$$f(\gamma) = Q_\gamma(X_1, \ldots, X_n)$$
$$g(\gamma) = Q_\gamma(X'_1, \ldots, X'_n)\,,$$

for all $\gamma \in \mathbf{I}$. If neither $f, g \in \mathbb{B}^+$ nor $f, g \in \mathbb{B}^-$, then $|\mathcal{B}(f) - \mathcal{B}(g)| < 2\delta$ whenever $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$, and $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-3) *and* (B-5)*.*

Proof
Without loss of generality, we may assume that $f(0) \geq g(0)$. Hence only the following possibilities are left:

a. $f \in \mathbb{B}^+$, $g \in \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^-$.

b. $f \in \mathbb{B}^{\frac{1}{2}}$, $g \in \mathbb{B}^{\frac{1}{2}}$.

c. $f \in \mathbb{B}^{\frac{1}{2}}$, $g \in \mathbb{B}^-$.

We shall consider these cases in turn.

Case a.: $f \in \mathbb{B}^+$, $g \in \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^-$. We conclude from L-88 that

$$f(\gamma) = \max(\tfrac{1}{2}, Q_\gamma^{\min}(X_1, \ldots, X_n)) \quad (548)$$

$$g(\gamma) = \min(\tfrac{1}{2}, Q_\gamma^{\max}(X_1', \ldots, X_n')) \,. \quad (549)$$

Because $d((X_1, \ldots, X_n), (X_1', \ldots, X_n')) < \delta$, we may choose some κ such that

$$d((X_1, \ldots, X_n), (X_1', \ldots, X_n')) < \kappa < \delta \,. \quad (550)$$

Let us consider $\gamma' = 2\kappa$. Because $\frac{1}{2} < f(0) = Q_0(X_1, \ldots, X_n)$, we know from (549) that $Q_0^{\max}(X_1, \ldots, X_n) > \frac{1}{2}$. It is then apparent from Def. 66 that $\mathcal{T}_0(X_i) \subseteq \mathcal{T}_{\gamma'}(X_i)$ for $i = 1, \ldots, n$ and hence $Q_{\gamma'}^{\max}(X_1, \ldots, X_n) \geq Q_0^{\max}(X_1, \ldots, X_n) > \frac{1}{2}$ by (16). We can apply lemma L-112 to deduce that

$$Q_{\gamma'}^{\max}(X_1', \ldots, X_n') \geq Q_0^{\max}(X_1, \ldots, X_n) > \tfrac{1}{2} \,. \quad (551)$$

Similarly, because $\frac{1}{2} \geq g(0) = Q_0(X_1', \ldots, X_n')$, we know that $Q_0^{\min}(X_1', \ldots, X_n') \leq \frac{1}{2}$. By analogous reasoning as above, $Q_{\gamma'}^{\min}(X_1', \ldots, X_n') \leq Q_0^{\min}(X_1', \ldots, X_n')$ and by L-112,

$$Q_{\gamma'}^{\min}(X_1, \ldots, X_n) \leq Q_0^{\min}(X_1', \ldots, X_n') \leq \tfrac{1}{2} \,. \quad (552)$$

We conclude from (548) and (552) that $f(\gamma') = \frac{1}{2}$. Similarly, we know from (549) and (551) that $g(\gamma') = \frac{1}{2}$. It is then apparent from Def. 68 that $f(\gamma) = g(\gamma) = \frac{1}{2}$ for all $\gamma \geq \gamma'$. It follows that

$$h^\ell \leq g \leq f \leq h^u \,, \quad (553)$$

where

$$h^\ell(\gamma) = \begin{cases} 0 & : \quad \gamma \leq \gamma' \\ \frac{1}{2} & : \quad \gamma > \gamma' \end{cases}$$
$$h^u(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \gamma' \\ \frac{1}{2} & : \quad \gamma > \gamma' \end{cases}$$

By (B-3)

$$\mathcal{B}(h^\ell) = \tfrac{1}{2} - \tfrac{1}{2}\gamma' = \tfrac{1}{2} - \tfrac{1}{2}(2\kappa) = \tfrac{1}{2} - \kappa$$
$$\mathcal{B}(h^u) = \tfrac{1}{2} + \tfrac{1}{2}\gamma' = \tfrac{1}{2} + \tfrac{1}{2}(2\kappa) = \tfrac{1}{2} + \kappa \,.$$

Hence by (553) and (B-5),

$$\tfrac{1}{2} - \kappa \leq \mathcal{B}(g) \leq \mathcal{B}(f) \leq \tfrac{1}{2} + \kappa \,,$$

i.e. $|\mathcal{B}(f) - \mathcal{B}(g)| \leq 2\kappa < 2\delta$ by (550).

Case b.: $f \in \mathbb{B}^{\frac{1}{2}}$, $g \in \mathbb{B}^{\frac{1}{2}}$. Trivial because $\mathcal{B}(f) = \mathcal{B}(g) = \frac{1}{2}$ by (B-3).

Case c.: $f \in \mathbb{B}^{\frac{1}{2}}$, $g \in \mathbb{B}^-$. Because $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$, we may again choose κ such that $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \kappa < \delta$ and set $\gamma' = 2\kappa$.

Because $f \in \mathbb{B}^{\frac{1}{2}}$, we apparently have $Q_\gamma(X_1, \ldots, X_n) = f(\gamma) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. In particular, we may conclude from (14) and $Q_0(X_1, \ldots, X_n) = \frac{1}{2}$ that $Q_0^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$. By similar reasoning as in case **a.**, it is easily shown that $Q_{\gamma'}^{\max}(X'_1, \ldots, X'_n) \geq Q_0^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$. Hence by L-88, $Q_{\gamma'}(X'_1, \ldots, X'_n) = g(\gamma') = \frac{1}{2}$. We conclude from Def. 68 that $g(\gamma) = \frac{1}{2}$ for all $\gamma \geq \gamma'$. Abbreviating

$$h^\ell(\gamma) = \begin{cases} 0 & : \quad \gamma \leq \gamma' \\ \frac{1}{2} & : \quad \gamma > \gamma' \end{cases}$$

for all $\gamma \in \mathbf{I}$, we have

$$h^\ell \leq g \leq f = \mathrm{c}_{\frac{1}{2}}$$

and therefore

$$\begin{aligned} \tfrac{1}{2} - \kappa &= \tfrac{1}{2} - \tfrac{1}{2}\gamma' && \text{by } \gamma' = 2\kappa \\ &= \mathcal{B}(h^\ell) && \text{by (B-3)} \\ &\leq \mathcal{B}(g) && \text{by (B-5)} \\ &\leq \mathcal{B}(f) && \text{by (B-5), } g \leq f \\ &= \tfrac{1}{2}\,. && \text{by (B-3)} \end{aligned}$$

Hence $|\mathcal{B}(f) - \mathcal{B}(g)| \leq \kappa < \delta$, as desired.

Lemma 116
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *satisfies* (C-2) *and* (C-4). *If* $\mathcal{B}'$ *has the property that for all* $f \in \mathbb{H}$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ *whenever* $d'(f, g) < \delta$, *then* $\mathcal{M}_\mathcal{B}$ *is arg-continuous.*

Proof Suppose that $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies the conditions of the lemma. Further let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$. We shall abbreviate

$$f(\gamma) = Q_\gamma(X_1, \ldots, X_n)\,, \tag{554}$$

for all $\gamma \in \mathbf{I}$. In the following, we shall discern three cases.

Case a.: $f \in \mathbb{B}^+$. It is then convenient to define $f' \in \mathbb{H}$ by

$$f'(\gamma) = 2f(\gamma) - 1 \tag{555}$$

for all $\gamma \in \mathbf{I}$. By the assumption on $\mathcal{B}'$, there exists $\delta' > 0$ such that

$$|\mathcal{B}'(f') - \mathcal{B}'(g')| < 2\varepsilon \tag{556}$$

whenever $g' \in \mathbb{H}$ satisfies $d'(f', g') < \delta'$. We set

$$\delta = \min(\tfrac{\delta'}{2}, \tfrac{\varepsilon}{2})\,. \tag{557}$$

Now let $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. We shall abbreviate

$$g(\gamma) = Q_\gamma(X'_1, \ldots, X'_n) \tag{558}$$

for all $\gamma \in \mathbf{I}$.
If $g \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$, then

$$\begin{aligned}
&|\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(Q)(X'_1, \ldots, X'_n)| \\
&= |\mathcal{B}(f) - \mathcal{B}(g)| && \text{by Def. 69, (554) and (558)} \\
&< 2\delta && \text{by L-115} \\
&\leq \varepsilon\,. && \text{by (557)}
\end{aligned}$$

In the remaining case that $g \in \mathbb{B}^+$, we may define $g' \in \mathbb{H}$ by

$$g'(\gamma) = 2g(\gamma) - 1\,, \tag{559}$$

for all $\gamma \in \mathbf{I}$. Because $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$, we conclude from L-114 that $d'(f', g') < 2\delta \leq \delta'$. Hence by our choice of δ',

$$|\mathcal{B}'(f) - \mathcal{B}'(g)| < 2\varepsilon \tag{560}$$

and in turn,

$$\begin{aligned}
&|\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(Q)(X'_1, \ldots, X'_n)| \\
&= |\mathcal{B}(f) - \mathcal{B}(g)| && \text{by Def. 69, (554) and (558)} \\
&= |(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(f')) - (\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(g'))| && \text{by (555), (559), (23)} \\
&= \tfrac{1}{2}|\mathcal{B}'(f') - \mathcal{B}'(g')| \\
&< \varepsilon\,. && \text{by (560)}
\end{aligned}$$

Case b.: $f \in \mathbb{B}^{\frac{1}{2}}$**.** In this case, let $\delta = \frac{\varepsilon}{2}$. Whenever $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$, we may apply lemma L-115 and deduce that $|\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(Q)(X'_1, \ldots, X'_n)| = |\mathcal{B}(f) - \mathcal{B}(g)| < 2\delta = \varepsilon$.

Case c.: $f \in \mathbb{B}^-$**.** In this case, consider $\neg Q$. Defining $h(\gamma) = (\neg Q)_\gamma(X_1, \ldots, X_n)$, we know from L-29 that $h = 1 - f \in \mathbb{B}^+$. By part a. of the lemma, there exists $\delta > 0$ such that

$$|\mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(\neg Q)(X'_1, \ldots, X'_n)| < \varepsilon \tag{561}$$

whenever $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. Hence

$$\begin{aligned}
&|\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(Q)(X'_1, \ldots, X'_n)| \\
&= |(1 - \mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n)) - (1 - \mathcal{M}_\mathcal{B}(\neg Q)(X'_1, \ldots, X'_n))| && \text{by Def. 69, (B-2)} \\
&= |1 - \mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n) - 1 + \mathcal{M}_\mathcal{B}(\neg Q)(X'_1, \ldots, X'_n)| \\
&= |\mathcal{M}_\mathcal{B}(\neg Q)(X_1, \ldots, X_n) - \mathcal{M}_\mathcal{B}(\neg Q)(X'_1, \ldots, X'_n)| \\
&< \varepsilon\,. && \text{by (561)}
\end{aligned}$$

Lemma 117

Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *satisfies* (C-2), (C-3.b) *and* (C-4).

a. if $\mathcal{M}_{\mathcal{B}}$ *is arg-continuous, then for all* $f \in \mathbb{H}$ *and for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $g \in \mathbb{H}$ *with* $g \geq f$ *and* $d'(f, g) < \delta$*, it holds that* $\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon$*.*

b. if $\mathcal{M}_{\mathcal{B}}$ *is arg-continuous, then for all* $f \in \mathbb{H}$ *and for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for all* $g \in \mathbb{H}$ *with* $g \leq f$ *and* $d'(f, g) < \delta$*, it holds that* $\mathcal{B}'(f) - \mathcal{B}'(g) < \varepsilon$*.*

Proof

a. Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies (C-2), (C-3.b) and (C-4). Further suppose that there exists $f \in \mathbb{H}$, $\varepsilon > 0$ such that for all $\delta > 0$, there is some $g \in \mathbb{H}$ such that

$$f \leq g \tag{562}$$

$$d'(f, g) < \delta \tag{563}$$

$$\mathcal{B}'(g) - \mathcal{B}'(f) \geq \varepsilon\,. \tag{564}$$

We have to show that $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous. Hence let us consider the quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ defined by

$$Q(Y) = \sup Y\,, \tag{565}$$

for all $Y \in \mathcal{P}(\mathbf{I})$. We define $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ by

$$\mu_X(z) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) \geq z\}\,, \tag{566}$$

for all $z \in \mathbf{I}$. (f is as above and we shall assume the same choice of $\varepsilon > 0$). Now let $\delta > 0$. By assumption, there exists $g \in \mathbb{H}$ such that (562), (563) and (564) hold. We define $X' \in \widetilde{\mathcal{P}}(\mathbf{I})$ by

$$\mu_{X'}(z) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\}\,, \tag{567}$$

for all $z \in \mathbf{I}$. Let us now investigate $d(X, X')$. We know from (562) that $f \leq g$. Therefore (35) simplifies to

$$d'(f, g) = \sup\{\inf\{\gamma' - \gamma : g(\gamma') \leq f(\gamma)\} : \gamma \in \mathbf{I}\}\,. \tag{568}$$

Because $f \leq g$, $\frac{1}{2} + \frac{1}{2}f(\gamma) \geq z$ entails that $\frac{1}{2} + \frac{1}{2}g(\gamma) \geq z$. Hence by (566) and (567),

$$\mu_{X'}(z) \geq \mu_X(z)\,, \tag{569}$$

for all $z \in \mathbf{I}$. Let us abbreviate

$$\gamma^* = \sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) \geq z\}\,. \tag{570}$$

Apparently

$$\tfrac{1}{2} + \tfrac{1}{2}f(\gamma) < z \tag{571}$$

for all $\gamma > \gamma^*$. Now let us consider

$$\gamma' > \gamma^* + d'(f, g)\,.$$

Clearly $\gamma' - d'(f, g) > \gamma^*$, i.e. we can choose $\gamma'' \in (\gamma^*, \gamma' - d'(f, g))$. We compute

$$\begin{aligned} \tfrac{1}{2} + \tfrac{1}{2}g(\gamma') &\leq \tfrac{1}{2} + \tfrac{1}{2}f(\gamma'') && \text{by (568), } \gamma' - \gamma'' > d'(f, g) \\ &< z\,. && \text{by } \gamma'' > \gamma^* \text{ and (571)} \end{aligned}$$

Hence

$$\gamma' \notin \{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\}\,.$$

Because $\gamma' > \gamma^* + d'(f, g)$ was arbitrarily chosen, we conclude that

$$\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\} \subseteq [0, \gamma^* + d'(f, g)]\,,$$

in particular

$$\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\} \leq \gamma^* + d'(f, g)\,. \tag{572}$$

Hence

$$\begin{aligned} \mu_{X'}(z) &\leq \tfrac{1}{2} + \tfrac{1}{2}(\gamma^* + d'(f, g)) && \text{by (567), (572)} \\ &= \mu_X(z) + \tfrac{1}{2}d'(f, g)\,, && \text{by (566) and (570)} \end{aligned}$$

i.e.

$$\mu_{X'}(z) \leq \mu_X(z) + \tfrac{1}{2}d'(f, g)\,, \tag{573}$$

for all $z \in \mathbf{I}$. Hence

$$\begin{aligned} d(X, X') &= \sup\{|\mu_X(z) - \mu_{X'}(z)| : z \in \mathbf{I}\} && \text{by (25)} \\ &= \sup\{\mu_{X'}(z) - \mu_X(z) : z \in \mathbf{I}\} && \text{by (569)} \\ &\leq \sup\{\mu_X(z) + \tfrac{1}{2}d'(f, g) - \mu_X(z) : z \in \mathbf{I}\} && \text{by (573)} \\ &= \tfrac{1}{2}d'(f, g) \\ &< \delta\,. && \text{by (563)} \end{aligned}$$

On the other hand,

$$\begin{aligned} |\mathcal{M}_{\mathcal{B}}(Q)(X) - \mathcal{M}_{\mathcal{B}}(Q)(X')| &= |(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(f^\flat)) - (\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(g^\flat))| && \text{by Def. 69, (23), L-107} \\ &= \tfrac{1}{2}|\mathcal{B}'(f^\flat) - \mathcal{B}'(g^\flat)| \\ &= \tfrac{1}{2}|\mathcal{B}'(f) - \mathcal{B}'(g)| && \text{by (C-3.b), (C-4)} \\ &= \tfrac{1}{2}(\mathcal{B}'(g) - \mathcal{B}'(f)) && \text{by (C-4), } g \geq f \\ &\geq \tfrac{\varepsilon}{2}\,. && \text{by (564)} \end{aligned}$$

Hence there exists $\varepsilon' = \varepsilon/2$, a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ and a choice of fuzzy argument set $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ such that for all $\delta > 0$, there exists $X' \in \widetilde{\mathcal{P}}(E)$ with $d(X, X') < \delta$ and $|\mathcal{M}_{\mathcal{B}}(Q)(X) - \mathcal{M}_{\mathcal{B}}(Q)(X')| \geq \varepsilon'$. We conclude that $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous.

b. The proof of this case is analogous to that of case **a.**: Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ satisfies (C-2), (C-3.b) and (C-4). Further suppose that there exists $f \in \mathbb{H}$, $\varepsilon > 0$ such that for all $\delta > 0$, there is some $g \in \mathbb{H}$ such that

$$f \geq g \tag{574}$$
$$d'(f,g) < \delta \tag{575}$$
$$\mathcal{B}'(f) - \mathcal{B}'(g) \geq \varepsilon\,. \tag{576}$$

In order to show that $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous, we again consider the quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ defined by

$$Q(Y) = \sup Y\,, \tag{577}$$

for all $Y \in \mathcal{P}(\mathbf{I})$, and the fuzzy argument set $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by

$$\mu_X(z) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) \geq z\}\,, \tag{578}$$

for all $z \in \mathbf{I}$. (f is as above and we shall assume the same choice of $\varepsilon > 0$). Now let $\delta > 0$. By assumption, there exists $g \in \mathbb{H}$ such that (574), (575) and (576) hold. We define $X' \in \widetilde{\mathcal{P}}(\mathbf{I})$ by

$$\mu_{X'}(z) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\}\,, \tag{579}$$

for all $z \in \mathbf{I}$. We know from (574) that $f \geq g$. Therefore (35) simplifies to

$$d'(f,g) = \sup\{\inf\{\gamma' - \gamma : f(\gamma') \leq g(\gamma)\} : \gamma \in \mathbf{I}\}\,. \tag{580}$$

Because $f \geq g$, $\frac{1}{2} + \frac{1}{2}g(\gamma) \geq z$ entails that $\frac{1}{2} + \frac{1}{2}f(\gamma) \geq z$. Hence by (578) and (579),

$$\mu_{X'}(z) \leq \mu_X(z)\,, \tag{581}$$

for all $z \in \mathbf{I}$. We abbreviate

$$\gamma^* = \sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}g(\gamma) \geq z\}\,. \tag{582}$$

Clearly

$$\tfrac{1}{2} + \tfrac{1}{2}g(\gamma) < z \tag{583}$$

for all $\gamma > \gamma^*$. Now let us consider

$$\gamma' > \gamma^* + d'(f,g)\,.$$

Clearly $\gamma' - d'(f,g) > \gamma^*$, i.e. we can choose $\gamma'' \in (\gamma^*, \gamma' - d'(f,g))$. We compute

$$\begin{aligned} \tfrac{1}{2} + \tfrac{1}{2}f(\gamma') &\leq \tfrac{1}{2} + \tfrac{1}{2}g(\gamma'') && \text{by (580), } \gamma' - \gamma'' > d'(f,g) \\ &< z\,. && \text{by } \gamma'' > \gamma^* \text{ and (583)} \end{aligned}$$

Hence

$$\gamma' \notin \{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2}f(\gamma) \geq z\}\,.$$

Because $\gamma' > \gamma^* + d'(f, g)$ was arbitrarily chosen, we conclude that

$$\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2} f(\gamma) \geq z\} \subseteq [0, \gamma^* + d'(f, g)]\,,$$

in particular

$$\sup\{\gamma \in \mathbf{I} : \tfrac{1}{2} + \tfrac{1}{2} f(\gamma) \geq z\} \leq \gamma^* + d'(f, g)\,. \tag{584}$$

Hence

$$\begin{aligned} \mu_X(z) &\leq \tfrac{1}{2} + \tfrac{1}{2}(\gamma^* + d'(f, g)) && \text{by (578), (584)} \\ &= \mu_{X'}(z) + \tfrac{1}{2} d'(f, g)\,, && \text{by (579) and (582)} \end{aligned}$$

i.e.

$$\mu_X(z) \leq \mu_{X'}(z) + \tfrac{1}{2} d'(f, g)\,, \tag{585}$$

for all $z \in \mathbf{I}$. Hence

$$\begin{aligned} d(X, X') &= \sup\{|\mu_X(z) - \mu_{X'}(z)| : z \in \mathbf{I}\} && \text{by (25)} \\ &= \sup\{\mu_X(z) - \mu_{X'}(z) : z \in \mathbf{I}\} && \text{by (581)} \\ &\leq \sup\{\mu_{X'}(z) + \tfrac{1}{2} d'(f, g) - \mu_{X'}(z) : z \in \mathbf{I}\} && \text{by (585)} \\ &= \tfrac{1}{2} d'(f, g) \\ &< \delta\,. && \text{by (575)} \end{aligned}$$

On the other hand,

$$\begin{aligned} |\mathcal{M}_{\mathcal{B}}(Q)(X) - \mathcal{M}_{\mathcal{B}}(Q)(X')| &= |(\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(f^\flat)) - (\tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'(g^\flat))| && \text{by Def. 69, (23), L-107} \\ &= \tfrac{1}{2}|\mathcal{B}'(f^\flat) - \mathcal{B}'(g^\flat)| \\ &= \tfrac{1}{2}|\mathcal{B}'(f) - \mathcal{B}'(g)| && \text{by (C-3.b), (C-4)} \\ &= \tfrac{1}{2}(\mathcal{B}'(f) - \mathcal{B}'(g)) && \text{by (C-4), } f \geq g \\ &\geq \tfrac{\varepsilon}{2}\,. && \text{by (576)} \end{aligned}$$

Hence there exists a choice of $\varepsilon' = \varepsilon/2$, a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy argument set $X \in \widetilde{\mathcal{P}}(\mathbf{I})$ such that for all $\delta > 0$, there exists $X' \in \widetilde{\mathcal{P}}(E)$ with $d(X, X') < \delta$ and $|\mathcal{M}_{\mathcal{B}}(Q)(X) - \mathcal{M}_{\mathcal{B}}(Q)(X')| \geq \varepsilon'$. Therefore $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous.

Lemma 118
Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *satisfies* (C-2), (C-3.b) *and* (C-4). *Then the following conditions 1. and 2. (i.e. conjunction of 2.a and 2.b) are equivalent:*

1. *for all* $f \in \mathbb{H}$ *and for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $g \in \mathbb{H}$, $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ *whenever* $d'(f, g) < \delta$.

2.a *for all* $f \in \mathbb{H}$ *and for all* $\varepsilon > 0$, *there exists* $\delta_a > 0$ *such that* $\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon$ *whenever* $d'(f, g) < \delta_a$ *and* $f \leq g$.

2.b *for all* $f \in \mathbb{H}$ *and for all* $\varepsilon > 0$, *there exists* $\delta_b > 0$ *such that* $\mathcal{B}'(f) - \mathcal{B}'(g) < \varepsilon$ *whenever* $d'(f, g) < \delta_b$ *and* $g \leq f$.

Proof The implication 1. $\Rightarrow$ 2. holds trivially. To see that the reverse implication also holds, suppose $\mathcal{B}'$ satisfies 2.a and 2.b. Further let $f \in \mathbb{H}$ be given and $\varepsilon > 0$. By assumption, there exists $\delta_a > 0$ such that

$$\mathcal{B}'(g) - \mathcal{B}'(f) < \tfrac{\varepsilon}{2} \tag{586}$$

whenever $f \leq g$ and $d'(f, g) < \delta_a$. Furthermore, there exists $\delta_b > 0$ such that

$$\mathcal{B}'(f) - \mathcal{B}'(g) < \tfrac{\varepsilon}{2} \tag{587}$$

whenever $g \leq f$ and $d'(f, g) < \delta_b$. Let us set $\delta = \min(\delta_a, \delta_b)$. Now let $g \in \mathbb{H}$ such that $d'(f, g) < \delta$. We shall abbreviate $\ell = \min(f, g)$ and $u = \max(f, g)$. It is apparent from (35) that

$$d'(f, \ell) \leq d'(f, g) < \delta \tag{588}$$

$$d'(f, u) \leq d'(f, g) < \delta\,. \tag{589}$$

In addition, we clearly have $\ell \leq f \leq u$ and $\ell \leq g \leq u$. Hence by (C-4),

$$\mathcal{B}'(\ell) \leq \mathcal{B}'(f) \leq \mathcal{B}'(u)$$
$$\mathcal{B}'(\ell) \leq \mathcal{B}'(g) \leq \mathcal{B}'(u)\,.$$

From this we deduce that

$$\begin{aligned}
|\mathcal{B}'(f) - \mathcal{B}'(g)| &\leq \mathcal{B}'(u) - \mathcal{B}'(\ell) \\
&= |\mathcal{B}'(u) - \mathcal{B}'(\ell)| && \text{because } \mathcal{B}'(u) \geq \mathcal{B}'(\ell) \\
&\leq |\mathcal{B}'(u) - \mathcal{B}'(f)| + |\mathcal{B}'(f) - \mathcal{B}'(\ell)| && \text{by triangular equation} \\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} && \text{by (586)-(589)} \\
&= \varepsilon\,.
\end{aligned}$$

Proof of Theorem 105

We already know from L-116 that the condition on $\mathcal{B}'$ is sufficient for $\mathcal{M}_{\mathcal{B}}$ to be arg-continuous. In addition, we know from L-117 that the conditions 2.a and 2.b of lemma L-118 are necessary for $\mathcal{M}_{\mathcal{B}}$ to be arg-continuous. But L-118 states the equivalence of these conditions on $\mathcal{B}'$ with the condition imposed by the theorem. We hence conclude that the condition on $\mathcal{B}'$ is also necessary for $\mathcal{M}_{\mathcal{B}}$ to be arg-continuous.

F.21 Proof of Theorem 106

By lemma L-118, the conjunction of the conditions 2.a and 2.b is equivalent to condition 1 of the lemma, which in turn is sufficient for $\mathcal{M}_{\mathcal{B}}$ to be arg-continuous by L-116. Therefore we only have to show that conditions 2.a. and 2.b of lemma L-118 are entailed by the condition of the theorem. This is apparent in the case of 2.a. Hence let us show that the condition on $\mathcal{B}'$ imposed by the theorem also entails 2.b. To this end, consider some $f \in \mathbb{H}$ and a choice of $\varepsilon > 0$. By the assumption of the theorem, there exists $\delta > 0$ such that

$$\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon \tag{590}$$

whenever $f \leq g$ and $d'(f, g) < \delta$. We claim that $\delta_b = \delta$ is a valid choice for δ_b in 2.b. Hence let $g \leq f$ such that $d'(f, g) < \delta$. We may directly apply (590) (reversing the roles of f, g) to deduce that $\mathcal{B}'(f) - \mathcal{B}'(g) < \varepsilon$, as desired.

F.22 Proof of Theorem 107

Let $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ be an s-norm and define $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ by (Th-87.a), i.e.

$$\mathcal{B}'(f) = f_*^1 \oplus f_1^* \tag{591}$$

for all $f \in \mathbb{H}$. We shall define the QFM $\mathcal{M}_\mathcal{B}$ in terms of $\mathcal{B}'$ according to (23) and Def. 69 as usual. Let us recall that $\mathcal{M}_\mathcal{B}$ is a DFS by Th-87.

a.: $\mathcal{M}_\mathcal{B}$ is not Q-continuous.
By theorem Th-104, $\mathcal{M}_\mathcal{B}$ fails to be Q-continuous iff there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exist $f, g \in \mathbb{H}$ such that $d(f, g) < \delta$ and $|\mathcal{B}'(f) - \mathcal{B}'(g)| \geq \varepsilon$.
Hence let us consider $\varepsilon = \frac{1}{2}$ and let $\delta > 0$. Further define $f, g_\delta \in \mathbb{H}$ by

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \frac{1}{2} \\ 0 & : \quad \gamma > \frac{1}{2} \end{cases} \tag{592}$$

$$g_\delta(\gamma) = \begin{cases} 1 - \frac{\delta}{2} & : \quad \gamma \leq \frac{1}{2} \\ 0 & : \quad \gamma > \frac{1}{2} \end{cases} \tag{593}$$

for all $\gamma \in \mathbf{I}$. Then by (34),

$$d(f, g_\delta) = \sup\{|f(\gamma) - g_\delta(\gamma)| : \gamma \in \mathbf{I}\} = \frac{\delta}{2} < \delta\,. \tag{594}$$

Furthermore, it is apparent from (592) and the definitions of the coefficients (21) and (22) that

$$f_*^1 = \frac{1}{2}$$
$$f_1^* = 0\,,$$

hence

$$\mathcal{B}'(f) = \frac{1}{2} \oplus 0 = \frac{1}{2}\,. \tag{595}$$

Similarly from (593),

$$g_{\delta}{}_*^1 = 0$$
$$g_{\delta}{}_1^* = 0\,,$$

i.e.

$$\mathcal{B}'(g_\delta) = 0 \oplus 0 = 0\,. \tag{596}$$

Hence $d(f, g_\delta) < \delta$ by (594) and

$$|\mathcal{B}'(f) - \mathcal{B}'(g)| = \frac{1}{2} = \varepsilon$$

by (595) and (596), which finishes the proof that $\mathcal{M}_\mathcal{B}$ is not Q-continuous.

b.: $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous.
We know from Th-87 that $\mathcal{M}_{\mathcal{B}}$ is a DFS; in particular, we can apply lemma L-117. According to part **b.** of lemma L-117, we can prove that $\mathcal{M}_{\mathcal{B}}$ is not continuous in arguments by showing that there exists $\varepsilon > 0$ and $f \in \mathbb{H}$ such that for all $\delta > 0$, there exists $g \in \mathbb{H}$ such that $g \leq f$, $d'(f,g) < \delta$ and $\mathcal{B}'(f) - \mathcal{B}'(g) \geq \varepsilon$.
To this end, consider $\varepsilon = \frac{1}{2}$ and define $f \in \mathbb{H}$ by

$$f(\gamma) = \tfrac{1}{2} \tag{597}$$

for all $\gamma \in \mathbf{I}$. By (22) and (21),

$$f_*^1 = 0$$
$$f_1^* = \tfrac{1}{2}$$

i.e.

$$\mathcal{B}'(f) = 0 \oplus \tfrac{1}{2} = \tfrac{1}{2}\,. \tag{598}$$

Now let $\delta > 0$ and define $g \in \mathbb{H}$ by

$$g(\gamma) = \begin{cases} \frac{1}{2} & : \quad \gamma \leq 1 - \frac{\delta}{2} \\ 0 & : \quad \gamma > 1 - \frac{\delta}{2} \end{cases} \tag{599}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (35) and (597), (599) that

$$d'(f,g) = \tfrac{\delta}{2} < \delta\,. \tag{600}$$

In the case of g, the coefficients (22) and (21) become

$$f_*^1 = 0$$
$$f_1^* = 0\,,$$

i.e.

$$\mathcal{B}'(g) = 0 \oplus 0 = 0\,. \tag{601}$$

Hence the choice of $f, g \in \mathbb{H}$ satisfies $g \leq f$ (apparent), $d'(f,g) < \delta$ by (598) and

$$\mathcal{B}'(f) - \mathcal{B}'(g) = \tfrac{1}{2} - 0 = \tfrac{1}{2} = \varepsilon$$

by (600) and (601), which proves that $\mathcal{M}_{\mathcal{B}}$ is not arg-continuous.

F.23 Proof of Theorem 108

Let $\odot$ be a t-norm and define $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ by

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*\,, \tag{602}$$

for all $f \in \mathbb{H}$. We shall define the QFM $\mathcal{M}_{\mathcal{B}}$ in terms of $\mathcal{B}'$ according to (23) and Def. 69 as usual. By Th-91, the QFM $\mathcal{M}_{\mathcal{B}}$ is a DFS.

a.: $\mathcal{M}_\mathcal{B}$ is not Q-continuous.
By theorem Th-104, we can prove that $\mathcal{M}_\mathcal{B}$ is not Q-continuous by showing that there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exist $f, g \in \mathbb{H}$ with $d(f, g) < \delta$ and $|\mathcal{B}'(f) - \mathcal{B}'(g)| \geq \varepsilon$.
Hence let $\varepsilon = \frac{1}{2}$ and $\delta > 0$. We define $f, g \in \mathbb{H}$ by

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \frac{1}{2} \\ 0 & : \quad \gamma > \frac{1}{2} \end{cases} \tag{603}$$

$$g(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \frac{1}{2} \\ \frac{\delta}{2} & : \quad \gamma > \frac{1}{2} \end{cases} \tag{604}$$

for all $\gamma \in \mathbf{I}$. By (19) and (18),

$$f_*^0 = \tfrac{1}{2}$$
$$f_0^* = 1$$
$$g_*^0 = 1$$
$$g_0^* = 1\,,$$

i.e.

$$\mathcal{B}'(f) = \tfrac{1}{2} \odot 1 = \tfrac{1}{2} \tag{605}$$

$$\mathcal{B}'(g) = 1 \odot 1 = 1\,. \tag{606}$$

Hence there exist $f, g \in \mathbb{H}$ such that $d(f, g) = \frac{\delta}{2} < \delta$ by (34), (603) and (604), but $|\mathcal{B}'(f) - \mathcal{B}'(g)| = |\frac{1}{2} - 1| = \frac{1}{2} = \varepsilon$, which proves that $\mathcal{M}_\mathcal{B}$ is not Q-continuous.

b.: $\mathcal{M}_\mathcal{B}$ is not arg-continuous.
In this case, we can apply lemma L-117 because $\mathcal{M}_\mathcal{B}$ is a DFS. According to part **a.** of the lemma, $\mathcal{M}_\mathcal{B}$ is not arg-continuous if there exists $\varepsilon > 0$ and $f \in \mathbb{H}$ such that for all $\delta > 0$, there exists $g \in \mathbb{H}$ with $g \geq f$, $d'(f, g) < \delta$ and $\mathcal{B}'(g) - \mathcal{B}'(f) \geq \varepsilon$.
Hence let $\varepsilon = \frac{1}{2}$ define $f \in \mathbb{H}$ by

$$f(\gamma) = \tfrac{1}{2} \tag{607}$$

for all $\gamma \in \mathbf{I}$.
Now consider $\delta > 0$. We define $g \in \mathbb{H}$ by

$$g(\gamma) = \begin{cases} 1 & : \quad \gamma < \frac{\delta}{2} \\ \frac{1}{2} & : \quad \text{else} \end{cases} \tag{608}$$

for all $\gamma \in \mathbf{I}$. Clearly $g \geq f$ and $d'(f, g) = \frac{\delta}{2} < \delta$ by (35). However, we have

$$f_*^0 = 1$$
$$f_0^* = \tfrac{1}{2}$$
$$g_*^0 = 1$$
$$g_0^* = 1$$

and hence

$$\mathcal{B}'(f) = 1 \odot \tfrac{1}{2} = \tfrac{1}{2}$$
$$\mathcal{B}'(g) = 1 \odot 1 = 1\,,$$

i.e. $\mathcal{B}'(g) - \mathcal{B}'(f) = 1 - \frac{1}{2} = \frac{1}{2} = \varepsilon$, which finishes the proof that $\mathcal{M}_\mathcal{B}$ is not arg-continuous.

F.24 Proof of Theorem 109

Lemma 119
The mapping $(\bullet)^\diamondsuit$ *is monotonic, i.e. whenever* $f, g \in \mathbb{H}$ *and* $f \leq g$*, then also* $f^\diamondsuit \leq g^\diamondsuit$*.*

Proof Let $f, g \in \mathbb{H}$ be given mappings such that $f \leq g$. Further let $v \in \mathbf{I}$. Because $f \leq g$, we apparently have

$$\{\gamma \in \mathbf{I} : f(\gamma) < v\} \supseteq \{\gamma \in \mathbf{I} : g(\gamma) < v\} . \tag{609}$$

Hence

$$\begin{aligned} f^\diamondsuit(v) &= \inf\{\gamma \in \mathbf{I} : f(\gamma) < v\} && \text{by (36)} \\ &\leq \inf\{\gamma \in \mathbf{I} : g(\gamma) < v\} && \text{by (609)} \\ &= g^\diamondsuit(v) . && \text{by (36)} \end{aligned}$$

Lemma 120
For all $f \in \mathbb{H}$*,*

$$\int_0^1 f(\gamma)\, d\gamma = \int_0^1 f^\diamondsuit(v)\, dv .$$

Proof See [9, L-57,p.180].

Lemma 121
For all $f, g \in \mathbb{H}$ *such that* $f \leq g$*,*

$$d'(f, g) = d(f^\diamondsuit, g^\diamondsuit) .$$

Proof
Suppose $f, g \in \mathbb{H}$ are given and $f \leq g$. Then (35) simplifies to

$$d'(f, g) = \sup\{\inf\{\gamma' - \gamma : g(\gamma') \leq f(\gamma)\} : \gamma \in \mathbf{I}\} . \tag{610}$$

By applying lemma L-119, we conclude from $f \leq g$ that $f^\diamondsuit \leq g^\diamondsuit$, too. Hence by (34),

$$d(f^\diamondsuit, g^\diamondsuit) = \sup\{g^\diamondsuit(v) - f^\diamondsuit(v) : v \in \mathbf{I}\} . \tag{611}$$

We shall show that $d(f^\diamondsuit, g^\diamondsuit) = d'(f, g)$ by proving both inequations $d(f^\diamondsuit, g^\diamondsuit) \leq d'(f, g)$ and $d(f^\diamondsuit, g^\diamondsuit) \geq d'(f, g)$.

a.: $d(f^\diamondsuit, g^\diamondsuit) \leq d'(f, g)$**.**
By (611) and (36) it is sufficient to prove that for all $v \in \mathbf{I}$,

$$d'(f, g) \geq g^\diamondsuit(v) - f^\diamondsuit(v) = \inf\{\gamma : g(\gamma) < v\} - \inf\{\gamma : f(\gamma) < v\} . \tag{612}$$

Hence let $v \in \mathbf{I}$ be given. We abbreviate

$$\gamma' = \inf\{\gamma : g(\gamma) < v\} \quad (613)$$
$$\gamma_0 = \inf\{\gamma : f(\gamma) < v\}\,. \quad (614)$$

Consider $\delta > 0$, $\gamma_0 + \delta \leq 1$. It is apparent from (614) and the fact that $f \in \mathbb{H}$ is nonincreasing that

$$f(\gamma_0 + \delta) < v\,. \quad (615)$$

Similarly, we may conclude from (613) and the fact that $g \in \mathbb{H}$ is nonincreasing that

$$g(\gamma'') \geq v \quad (616)$$

whenever $\gamma'' < \gamma'$. Combining (615) and (616), it is apparent that

$$\inf\{\gamma'' - \gamma_0 - \delta : g(\gamma'') \leq f(\gamma_0 + \delta)\} \geq \gamma' - \gamma_0 - \delta\,, \quad (617)$$

for all $\delta > 0$. Therefore

$$\begin{aligned}
&d'(f,g) \\
&= \sup\{\inf\{\gamma' - \gamma : g(\gamma') \leq f(\gamma)\} : \gamma \in \mathbf{I}\} && \text{by (610)} \\
&\geq \sup\{\inf\{\gamma'' - \gamma_0 - \delta : g(\gamma'') \leq f(\gamma_0 + \delta)\} : \delta > 0, \delta + \gamma_0 \leq 1\} && \text{restriction of considered } \gamma\text{'s} \\
&\geq \gamma' - \gamma_0 && \text{by (617)} \\
&= \inf\{\gamma : g(\gamma) < v\} - \inf\{\gamma : f(\gamma) < v\} && \text{by (613), (614)} \\
&= g^{\diamondsuit}(v) - f^{\diamondsuit}(v)\,.
\end{aligned}$$

Because $v \in \mathbf{I}$ was arbitrarily chosen, this proves that

$$d'(f,g) \geq g^{\diamondsuit}(v) - f^{\diamondsuit}(v)$$

for all $v \in \mathbf{I}$, i.e.

$$d'(f,g) \geq \sup\{g^{\diamondsuit}(v) - f^{\diamondsuit}(v) : v \in \mathbf{I}\} = d(f^{\diamondsuit}, g^{\diamondsuit})\,,$$

as desired.

b.: $d(f^{\diamondsuit}, g^{\diamondsuit}) \geq d'(f,g)$**.**
By (610), it is sufficient to show that for all $\gamma \in \mathbf{I}$,

$$d(f^{\diamondsuit}, g^{\diamondsuit}) \geq \inf\{\gamma' - \gamma : g(\gamma') \leq f(\gamma)\}\,.$$

Hence let $\gamma \in \mathbf{I}$. We abbreviate

$$\xi = \inf\{\gamma' - \gamma : g(\gamma') \leq f(\gamma)\}\,. \quad (618)$$

Now let $\varepsilon > 0$ and

$$\gamma' = \gamma + \xi - \varepsilon\,. \quad (619)$$

Then

$$g(\gamma') > f(\gamma) \tag{620}$$

which is apparent from (618) and the fact that

$$\gamma' - \gamma = \gamma + \xi - \varepsilon - \gamma = \xi - \varepsilon < \xi\,.$$

Hence by (620),

$$f^{\Diamond}(g(\gamma')) = \inf\{x \in \mathbf{I} : f(x) < g(\gamma')\} \le \gamma\,. \tag{621}$$

Similarly

$$g^{\Diamond}(g(\gamma')) = \inf\{x : g(x) < g(\gamma')\} \ge \gamma'\,, \tag{622}$$

which is apparent from the fact that $g \in \mathbb{H}$ is nonincreasing. Summarising (621) and (622), we obtain

$$g^{\Diamond}(g(\gamma')) - f^{\Diamond}(g(\gamma')) \ge \gamma' - \gamma\,. \tag{623}$$

Hence

$$\begin{aligned} d(f^{\Diamond}, g^{\Diamond}) &= \sup\{g^{\Diamond}(v) - f^{\Diamond}(v) : v \in \mathbf{I}\} && \text{by (611)} \\ &\ge g^{\Diamond}(g(\gamma')) - f^{\Diamond}(g(\gamma')) && \text{substituiting } v = g(\gamma') \\ &\ge \gamma' - \gamma && \text{by (623)} \\ &= \xi - \varepsilon\,. && \text{by (619)} \end{aligned}$$

Hence $d(f^{\Diamond}, g^{\Diamond}) \ge \xi - \varepsilon$ for all $\varepsilon > 0$, i.e.

$$\begin{aligned} d(f^{\Diamond}, g^{\Diamond}) &\ge \xi \\ &= \inf\{\gamma' - \gamma : g(\gamma') \le f(\gamma)\} && \text{by (618)} \end{aligned}$$

for the given $\gamma \in \mathbf{I}$. Because $\gamma \in \mathbf{I}$ was chosen arbitrarily, we deduce that

$$d(f^{\Diamond}, g^{\Diamond}) \ge \sup\{\inf\{\gamma' - \gamma : g(\gamma') \le f(\gamma)\} : \gamma \in \mathbf{I}\}\,,$$

i.e. $d(f^{\Diamond}, g^{\Diamond}) \ge d'(f, g)$ by (610), which finishes the proof of the lemma.

Lemma 122
Let $f, g \in \mathbb{H}$ *and define* $u = \max(f, g)$, $\ell = \min(f, g)$. *Then*

a. $u^{\Diamond} = \max(f^{\Diamond}, g^{\Diamond})$*;*

b. $\ell^{\Diamond} = \min(f^{\Diamond}, g^{\Diamond})$.

Proof

a. In order to prove that $u^\Diamond = \max(f^\Diamond, g^\Diamond)$, let $v \in \mathbf{I}$ be given; we have to show that $u^\Diamond(v) = \max(f^\Diamond(v), g^\Diamond(v))$. To this end, let us introduce some abbreviations. By (36),

$$u^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : \max(f(\gamma), g(\gamma)) < v\} = \inf A \tag{624}$$

$$f^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : f(\gamma) < v\} = \inf F \tag{625}$$

$$g^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : g(\gamma) < v\} = \inf G\,, \tag{626}$$

where

$$A = \{\gamma \in \mathbf{I} : \max(f(\gamma), g(\gamma)) < v\} \tag{627}$$

$$F = \{\gamma \in \mathbf{I} : f(\gamma) < v\} \tag{628}$$

$$G = \{\gamma \in \mathbf{I} : g(\gamma) < v\}\,. \tag{629}$$

We know that $f, g \in \mathbb{H}$, hence f and g are nonincreasing by Def. 75. It is apparent from the monotonicity of f and (628) that F is a half-open or closed interval of the form

$$F = [f^\Diamond(v), 1] \qquad \text{or} \qquad F = (f^\Diamond(v), 1]\,. \tag{630}$$

Similarly, we conclude from the nonincreasing monotonicity of g and (629) that G is a half-open or closed interval of the form

$$G = [g^\Diamond(v), 1] \qquad \text{or} \qquad G = (g^\Diamond(v), 1]\,. \tag{631}$$

It is apparent from equations (627), (628) and (629) that $A = F \cap G$. Utilizing (630) and (631), we observe that A is a half-open or closed interval of the form

$$A = [\max(f^\Diamond(v), g^\Diamond(v)), 1] \qquad \text{or} \qquad A = (\max(f^\Diamond(v), g^\Diamond(v)), 1]\,. \tag{632}$$

In any case,

$$\begin{aligned} u^\Diamond(v) &= \inf A && \text{by (624)} \\ &= \max(f^\Diamond(v), g^\Diamond(v))\,, && \text{by (632)} \end{aligned}$$

as desired.

b.: $\ell^\Diamond = \min(f^\Diamond, g^\Diamond)$**.**
Let us consider some $v \in \mathbf{I}$; we will show that $\ell^\Diamond(v) = \min(f^\Diamond(v), g^\Diamond(v))$. We shall use similar abbreviations as in case a.: again by (36),

$$\ell^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : \max(f(\gamma), g(\gamma)) < v\} = \inf B \tag{633}$$

$$f^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : f(\gamma) < v\} = \inf F \tag{634}$$

$$g^\Diamond(v) = \inf\{\gamma \in \mathbf{I} : g(\gamma) < v\} = \inf G\,, \tag{635}$$

where

$$B = \{\gamma \in \mathbf{I} : \min(f(\gamma), g(\gamma)) < v\} \tag{636}$$

$$F = \{\gamma \in \mathbf{I} : f(\gamma) < v\} \tag{637}$$

$$G = \{\gamma \in \mathbf{I} : g(\gamma) < v\}\,. \tag{638}$$

It is easily observed that (630) and (631) are still valid. In this case, however, we have $B = F \cup G$, which is apparent from (636), (637) and (638). Utilizing (630) and (631), this means that B is a half-open or closed interval of the form

$$B = [\min(f^{\Diamond}(v), g^{\Diamond}(v)), 1] \qquad \text{or} \qquad B = (\min(f^{\Diamond}(v), g^{\Diamond}(v)), 1] \,. \tag{639}$$

Therefore

$$\begin{aligned} \ell^{\Diamond}(v) &= \inf B && \text{by (633)} \\ &= \min(f^{\Diamond}(v), g^{\Diamond}(v)) \,. && \text{by (639)} \end{aligned}$$

Proof of Theorem 109

Let $f, g \in \mathbb{H}$ be given. We first observe that

$$d'(f, g) = d'(\min(f, g), \max(f, g)) \tag{640}$$

and

$$d(f^{\Diamond}, g^{\Diamond}) = d(\min(f^{\Diamond}, g^{\Diamond}), \max(f^{\Diamond}, g^{\Diamond})) \tag{641}$$

which is apparent from (35) and (34), respectively. We compute

$$\begin{aligned} d'(f, g) &= d'(\min(f, g), \max(f, g)) && \text{by (640)} \\ &= d((\min(f, g))^{\Diamond}, (\max(f, g))^{\Diamond}) && \text{by L-121} \\ &= d(\min(f^{\Diamond}, g^{\Diamond}), \max(f^{\Diamond}, g^{\Diamond})) && \text{by L-122} \\ &= d(f^{\Diamond}, g^{\Diamond}) \,, && \text{by (641)} \end{aligned}$$

which finishes the proof of the theorem.

F.25 Proof of Theorem 110

a.: $\mathcal{M}$ is continuous in quantifiers.
By theorem Th-104, $\mathcal{M}$ is Q-continuous iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_0^1 f(\gamma)\, d\gamma - \int_0^1 g(\gamma)\, d\gamma \right| < \varepsilon \tag{642}$$

whenever $f, g \in \mathbb{H}$ satisfy $d(f, g) < \delta$. Hence let $\varepsilon > 0$ be given. I will prove that $\delta = \varepsilon$ satisfies (642). Hence let $f, g \in \mathbb{H}$ such that

$$d(f, g) < \varepsilon \,. \tag{643}$$

Then

$$\begin{aligned}
&\left|\int_0^1 f(\gamma)\,d\gamma - \int_0^1 g(\gamma)\,d\gamma\right| \\
&= \left|\int_0^1 f(\gamma) - g(\gamma)\,d\gamma\right| \\
&\le \int_0^1 |f(\gamma) - g(\gamma)|\,d\gamma \\
&\le \int_0^1 \sup\{|f(\gamma') - g(\gamma')| : \gamma' \in \mathbf{I}\}\,d\gamma && \text{monotonicity of } \textstyle\int \\
&= \sup\{|f(\gamma') - g(\gamma')| : \gamma' \in \mathbf{I}\} \\
&= d(f,g) && \text{by (34)} \\
&< \varepsilon\,, && \text{by (643)}
\end{aligned}$$

as desired.

b.: $\mathcal{M}$ is continuous in arguments.
By part a. of the theorem, we already know that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\int_0^1 p(\gamma)\,d\gamma - \int_0^1 q(\gamma)\,d\gamma\right| < \varepsilon\,. \tag{644}$$

whenever $d(p,q) < \delta$. In order to utilize this for the proof that $\mathcal{M}$ is arg-continuous as well, we apply theorem Th-105: $\mathcal{M}$ is arg-continuous if for all $\varepsilon > 0$ and all $f \in \mathbb{H}$, there exists $\delta > 0$ such that

$$\left|\int_0^1 f(\gamma) - \int_0^1 g(\gamma)\right| < \varepsilon \tag{645}$$

whenever $d'(f,g) < \delta$. Hence let $\varepsilon > 0$ and $f \in \mathbb{H}$. We already know from part a. that there exists $\delta > 0$ such that (644) holds for the given ε. Now let $g \in \mathbb{H}$ such that

$$d'(f,g) < \delta\,. \tag{646}$$

Then

$$\begin{aligned}
&\left|\int_0^1 f(\gamma)\,d\gamma - \int_0^1 g(\gamma)\,d\gamma\right| \\
&= \left|\int_0^1 f^\Diamond(\gamma)\,d\gamma - \int_0^1 g^\Diamond(\gamma)\,d\gamma\right| && \text{by L-120} \\
&< \varepsilon\,,
\end{aligned}$$

where the last step holds by (644) because

$$\begin{aligned}
d(f^\Diamond, g^\Diamond) &= d'(f,g) && \text{by Th-109} \\
&< \delta\,. && \text{by (646)}
\end{aligned}$$

F.26 Proof of Theorem 111

Suppose $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a uniform continuous t-norm, i.e. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|x_1 \odot y_1 - x_2 \odot y_2| < \varepsilon$ whenever $x_1, x_2, y_1, y_2 \in \mathbf{I}$ satisfy $\|(x_1, y_1) - (x_2, y_2)\| < \delta$, where $\|\bullet\|$ is the euclidian distance. In this case, it is apparent that $\odot$ satisfies the following property: for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x \odot y_1 - x \odot y_2| < \varepsilon\,, \qquad (647)$$

whenever $x, y_1, y_2 \in \mathbf{I}$ such that $|y_1 - y_2| < \delta$.
We shall define $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to equation (Th-93.a), i.e.

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\}\,, \qquad (648)$$

for all $f \in \mathbb{H}$. As usual, we define the QFM $\mathcal{M}_\mathcal{B}$ in terms of $\mathcal{B}'$ according to (23) and Def. 69. From theorem Th-93, we know that $\mathcal{M}_\mathcal{B}$ is a DFS.

a. $\mathcal{M}_\mathcal{B}$ is Q-continuous
Because $\mathcal{M}_\mathcal{B}$ is a DFS, we may apply theorem Th-104, i.e. $\mathcal{M}_\mathcal{B}$ is Q-continuous iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|\mathcal{B}'(f) - \mathcal{B}'(g)| < \varepsilon$ whenever $d(f, g) < \delta$.
Hence let $\varepsilon > 0$. By (647), there exists $\delta > 0$ such that

$$|x \odot y_1 - x \odot y_2| < \tfrac{\varepsilon}{2} \qquad (649)$$

whenever $x, y_1, y_2 \in \mathbf{I}$ satisfy $|y_1 - y_2| < \delta$. Now let $f, g \in \mathbb{H}$ such that $d(f, g) < \delta$. Hence by (34) and (649),

$$|\gamma \odot f(\gamma) - \gamma \odot g(\gamma)| < \tfrac{\varepsilon}{2} \qquad (650)$$

for all $\gamma \in \mathbf{I}$. In the following, we shall assume without loss of generality that $\mathcal{B}'(f) \geq \mathcal{B}'(g)$ (the proof in the case that $\mathcal{B}'(f) < \mathcal{B}'(g)$ is analogous). Then

$$
\begin{aligned}
&|\mathcal{B}'(f) - \mathcal{B}'(g)| \\
&= \mathcal{B}'(f) - \mathcal{B}'(g) && \text{by assumption that } \mathcal{B}'(f) \geq \mathcal{B}'(g) \\
&= \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\} - \sup\{\gamma \odot g(\gamma) : \gamma \in \mathbf{I}\} && \text{by (648)} \\
&= \sup\{\gamma \odot f(\gamma) - \sup\{\gamma' \odot g(\gamma') : \gamma' \in \mathbf{I}\} : \gamma \in \mathbf{I}\} \\
&\leq \sup\{\gamma \odot f(\gamma) - \gamma \odot g(\gamma) : \gamma \in \mathbf{I}\} \\
&\leq \sup\{|\gamma \odot f(\gamma) - \gamma \odot g(\gamma)| : \gamma \in \mathbf{I}\} \\
&\leq \tfrac{\varepsilon}{2} && \text{by (650)} \\
&< \varepsilon\,,
\end{aligned}
$$

as desired.

b. $\mathcal{M}_\mathcal{B}$ is arg-continuous
Because $\mathcal{M}_\mathcal{B}$ is a DFS, we may apply theorem Th-106. Hence $\mathcal{M}_\mathcal{B}$ is arg-continuous if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{B}'(g) - \mathcal{B}'(f) < \varepsilon$ whenever $f \leq g$ and $d'(f, g) < \delta$.
Hence let $\varepsilon > 0$. By (647), there exists δ' such that

$$|x \odot y_1 - x \odot y_2| < \tfrac{\varepsilon}{2} \qquad (651)$$

whenever $x, y_1, y_2 \in \mathbf{I}$ with $|y_1 - y_2| < \delta'$. Let us choose some $\delta > 0$ such that

$$\delta < \min(\delta', \tfrac{\varepsilon}{2})\,. \tag{652}$$

Now let $f, g \in \mathbb{H}$ such that $f \leq g$ and $d'(f, g) < \delta$. By (648), there exists $\gamma' \in \mathbf{I}$ such that

$$\gamma' \odot g(\gamma') > \mathcal{B}'(g) - \tfrac{\varepsilon}{2}\,. \tag{653}$$

In the case that $\gamma' < \delta$, then

$$\begin{aligned}
\mathcal{B}'(g) &< \gamma' \odot g(\gamma') + \tfrac{\varepsilon}{2} && \text{by (653)}\\
&\leq \gamma' \odot 1 + \tfrac{\varepsilon}{2} && \text{by monotonicity of } \odot\\
&= \gamma' + \tfrac{\varepsilon}{2}\\
&< \delta + \tfrac{\varepsilon}{2} && \text{by assumption of this case}\\
&< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} && \text{by (652)}\\
&< \varepsilon\,.
\end{aligned}$$

Because $\mathcal{B}'(f) \geq 0$, we conclude that

$$\mathcal{B}'(g) - \mathcal{B}'(f) \leq \mathcal{B}'(g) < \varepsilon\,,$$

as desired.

In the remaining case that $\gamma' \geq \delta$, let $\gamma = \gamma' - \delta \geq 0$. Clearly $\gamma' - \gamma = \delta > d'(f, g)$ by assumption on f, g. Hence

$$f(\gamma) \geq g(\gamma') \tag{654}$$

by (35) and

$$\begin{aligned}
\mathcal{B}'(f) &\geq \gamma \odot f(\gamma) && \text{by (648)}\\
&\geq \gamma \odot g(\gamma') && \text{by (654), monotonicity of } \odot\\
&> \gamma' \odot g(\gamma') - \tfrac{\varepsilon}{2}\,,
\end{aligned}$$

where the last step holds because $|\gamma \odot g(\gamma') - \gamma' \odot (\gamma')| < \frac{\varepsilon}{2}$ by (651) and $|\gamma' - \gamma| = \delta < \delta'$, see (652). Because $\odot$ is monotonic and $\gamma' > \gamma$, $|\gamma \odot g(\gamma') - \gamma' \odot (\gamma')| = \gamma' \odot g(\gamma') - \gamma \odot g(\gamma') < \frac{\varepsilon}{2}$, i.e. $\gamma \odot g(\gamma') > \gamma' \odot g(\gamma') - \frac{\varepsilon}{2}$. In turn, we conclude from $\mathcal{B}'(f) > \gamma' \odot g(\gamma') - \frac{\varepsilon}{2}$ that

$$\begin{aligned}
\mathcal{B}'(g) - \mathcal{B}'(f) &< \mathcal{B}'(g) - \gamma' \odot g(\gamma') + \tfrac{\varepsilon}{2}\\
&< \gamma' \odot g(\gamma') + \tfrac{\varepsilon}{2} - \gamma' \odot g(\gamma') + \tfrac{\varepsilon}{2} && \text{by (653)}\\
&= \varepsilon\,,
\end{aligned}$$

which finishes the proof that $\mathcal{M}_{\mathcal{B}}$ is arg-continuous.

F.27 Proof of Theorem 112

Lemma 123

Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers and $Q \preceq_c Q'$. Then $Q_\gamma(X_1, \ldots, X_n) \preceq_c Q'_\gamma(X_1, \ldots, X_n)$ for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$.

Proof We shall discern three cases.

Case a.: $Q_\gamma^{\min}(X_1, \ldots, X_n) > \frac{1}{2}$**.** Hence by (15),

$$\inf Q(Y_1, \ldots, Y_n) : Y \in \mathcal{T}_\gamma(X_i)\} > \tfrac{1}{2}\,,$$

i.e. for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$,

$$\tfrac{1}{2} < Q(Y_1, \ldots, Y_n) \le Q'(Y_1, \ldots, Y_n) \qquad (655)$$

because $Q \preceq_c Q'$ and hence

$$\begin{aligned}
\tfrac{1}{2} &< Q_\gamma^{\min}(X_1, \ldots, X_n) && \text{by assumption of case a.}\\
&= \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)}\\
&\le \inf\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (655)}\\
&= Q'^{\min}_\gamma(X_1, \ldots, X_n)\,, && \text{by (15)}
\end{aligned}$$

i.e.

$$Q_\gamma^{\min}(X_1, \ldots, X_n) \preceq_c Q'^{\min}_\gamma(X_1, \ldots, X_n) \qquad (656)$$

by (9). In turn,

$$\begin{aligned}
Q_\gamma(X_1, \ldots, X_n) &= Q_\gamma^{\min}(X_1, \ldots, X_n) && \text{by L-88}\\
&\preceq_c Q'^{\min}_\gamma(X_1, \ldots, X_n) && \text{by (656)}\\
&= Q'_\gamma(X_1, \ldots, X_n)\,. && \text{by L-88}
\end{aligned}$$

Case b.: $Q_\gamma^{\max}(X_1, \ldots, X_n) < \frac{1}{2}$**.** Hence by (16),

$$\sup Q(Y_1, \ldots, Y_n) : Y \in \mathcal{T}_\gamma(X_i)\} > \tfrac{1}{2}\,,$$

i.e. for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$,

$$\tfrac{1}{2} > Q(Y_1, \ldots, Y_n) \ge Q'(Y_1, \ldots, Y_n) \qquad (657)$$

because $Q \preceq_c Q'$ and hence

$$\begin{aligned}
\tfrac{1}{2} &> Q_\gamma^{\max}(X_1, \ldots, X_n) && \text{by assumption of case b.}\\
&= \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (16)}\\
&\ge \sup\{Q'(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (657)}\\
&= Q'^{\max}_\gamma(X_1, \ldots, X_n)\,, && \text{by (16)}
\end{aligned}$$

i.e.

$$Q_\gamma^{\max}(X_1, \ldots, X_n) \preceq_c Q'^{\max}_\gamma(X_1, \ldots, X_n) \qquad (658)$$

by (9). Therefore

$$\begin{aligned}
Q_\gamma(X_1, \ldots, X_n) &= Q_\gamma^{\max}(X_1, \ldots, X_n) && \text{by L-88}\\
&\preceq_c Q'^{\max}_\gamma(X_1, \ldots, X_n) && \text{by (658)}\\
&= Q'_\gamma(X_1, \ldots, X_n)\,. && \text{by L-88}
\end{aligned}$$

Case c.: $Q_\gamma^{\min}(X_1, \dots, X_n) \le \frac{1}{2}$ **and** $Q_\gamma^{\max}(X_1, \dots, X_n) \ge \frac{1}{2}$**.** Then

$$\begin{aligned} Q_\gamma(X_1, \dots, X_n) &= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\max}(X_1, \dots, X_n), Q_\gamma^{\min}(X_1, \dots, X_n)) && \text{by (14)} \\ &= \tfrac{1}{2} && \text{by Def. 45, assumptions of case c.} \\ &\preceq_c Q'_\gamma(X_1, \dots, X_n)\,. && \text{by (9)} \end{aligned}$$

Lemma 124

Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *satisfies* (B-5) *and* $\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$*. Then* $\mathcal{B}$ *propagates fuzziness, i.e. whenever* $f, g \in \mathbb{B}$ *such that* $f \preceq_c g$*, then* $\mathcal{B}(f) \preceq_c \mathcal{B}(g)$*.*

Proof Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-5) and

$$\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2}\,. \tag{659}$$

Further let a choice of $f, g \in \mathbb{B}$ be given such that $f \preceq_c g$.
Hence if $f \in \mathbb{B}^+$, then $\frac{1}{2} < f(0) \le g(0)$, in particular $g \in \mathbb{B}^+$. Therefore $g(\gamma) \ge \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. We hence deduce that $\frac{1}{2} \le f(\gamma) \le g(\gamma)$ for $\gamma \in (0, 1]$ because $f \preceq_c g$, see (9). Therefore

$$\mathrm{c}_{\frac{1}{2}} \le f \le g$$

and

$$\tfrac{1}{2} = \mathcal{B}(\mathrm{c}_{\frac{1}{2}}) \le \mathcal{B}(f) \le \mathcal{B}(g)\,, \qquad \text{by (659), (B-5)}$$

in particular

$$\mathcal{B}(f) \preceq_c \mathcal{B}(g)\,.$$

Similarly if $f \in \mathbb{B}^-$, then $f(0) < \frac{1}{2}$ and $f(0) \preceq_c g(0)$, i.e. $g(0) < f(0) < \frac{1}{2}$. We conclude that $g \in \mathbb{B}^-$ and $f(\gamma) \le \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. Hence for all $\gamma \in \mathbf{I}$, $f(\gamma) \le \frac{1}{2}$, $g(\gamma) \le \frac{1}{2}$ and $f(\gamma) \preceq_c g(\gamma)$, i.e. $g(\gamma) \le f(\gamma) \le \frac{1}{2}$. Again by (659) and (B-5),

$$\mathcal{B}(g) \le \mathcal{B}(f) \le \tfrac{1}{2}\,,$$

i.e.

$$\mathcal{B}(f) \preceq_c \mathcal{B}(g)\,.$$

In the remaining case that $f \in \mathbb{B}^{\frac{1}{2}}$, i.e. $f = \mathrm{c}_{\frac{1}{2}}$, we immediately obtain from (659) that $\mathcal{B}(f) = \frac{1}{2} \preceq_c \mathcal{B}(g)$.

Proof of Theorem 112

Suppose $\mathcal{M}_\mathcal{B}$ is an $\mathcal{M}_\mathcal{B}$-DFSes, i.e. the mapping $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-1) to (B-5) by Th-62. In particular, $\mathcal{B}(\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$ by (B-3).

Now suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $Q \preceq_c Q'$ are given and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned} \mathcal{M}_\mathcal{B}(Q)(X_1, \dots, X_n) &= \mathcal{B}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\ &\preceq_c \mathcal{B}((Q'_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) && \text{by L-123, L-124} \\ &= \mathcal{M}_\mathcal{B}(Q')(X_1, \dots, X_n)\,. && \text{by Def. 69} \end{aligned}$$

F.28 Proof of Theorem 113

Lemma 125
Let $E \neq \varnothing$ be given and $X, X' \in \widetilde{\mathcal{P}}(E)$ such that $X \preceq_c X'$. Then for all $\gamma \in \mathbf{I}$, $\mathcal{T}_\gamma(X') \subseteq \mathcal{T}_\gamma(X)$.

Proof Let us first show that $(X')^{\max}_\gamma \subseteq (X)^{\max}_\gamma$. In the case that $\gamma = 0$, we have $(X')^{\max}_0 = (X')_{\geq \frac{1}{2}}$ by Def. 66. Hence let $e \in (X')^{\max}_0 = (X')_{\geq \frac{1}{2}}$ be given, i.e. $\mu_{X'}(e) \geq \frac{1}{2}$. Then $\mu_X(e) \preceq_c \mu_{X'}(e)$ entails that $\frac{1}{2} \leq \mu_X(e) \leq \mu_{X'}(e)$, i.e. $e \in (X)_{\geq \frac{1}{2}} = (X)^{\max}_0$. Hence $(X')^{\max}_0 \subseteq (X)^{\max}_0$.
If $\gamma > 0$, then $(X')^{\max}_\gamma = (X')_{> \frac{1}{2} - \frac{1}{2}\gamma}$ by Def. 66. Hence let $e \in (X')^{\max}_\gamma$.

- if $\frac{1}{2} - \frac{1}{2}\gamma < \mu_{X'}(e) \leq \frac{1}{2}$, then $X \preceq_c X'$ entails that $\mu_{X'}(e) \leq \mu_X(e) \leq \frac{1}{2}$. In particular $\frac{1}{2} - \frac{1}{2}\gamma < \mu_{X'}(e) \leq \mu_X(e)$, i.e. $e \in (X)^{\max}_\gamma$.
- if $\mu_{X'}(e) > \frac{1}{2}$, then $\mu_X(e) \preceq_c \mu_{X'}(e)$ entails that $\frac{1}{2} \leq \mu_X(e) \leq \mu_{X'}(e)$. In particular, $\mu_X(e) > \frac{1}{2} - \frac{1}{2}\gamma$ for $\gamma > 0$, i.e. $e \in (X)^{\max}_\gamma$.

Summarising, we have shown that $(X')^{\max}_\gamma \subseteq (X)^{\max}_\gamma$ holds in the case $\gamma > 0$ as well.
It remains to be shown that $(X)^{\min}_\gamma \subseteq (X')^{\min}_\gamma$. In the case that $\gamma = 0$, we have $(X)^{\min}_0 = (X)_{> \frac{1}{2}}$ by Def. 66. Hence if $e \in (X)^{\min}_0$, then $\mu_X(e) > \frac{1}{2}$. Because $\mu_X(e) \preceq_c \mu_{X'}(e)$, we conclude that $\frac{1}{2} < \mu_X(e) \leq \mu_{X'}(e)$. Hence $\mu_{X'}(e) > \frac{1}{2}$, too, and $e \in (X')_{> \frac{1}{2}} = (X')^{\min}_0$.
Finally if $\gamma > 0$, then $(X)^{\min}_\gamma = (X)_{\geq \frac{1}{2} + \frac{1}{2}\gamma}$. Hence if $e \in (X)^{\min}_\gamma$, then $\mu_X(e) \geq \frac{1}{2} + \frac{1}{2}\gamma > \frac{1}{2}$. Then because $\mu_X(e) \preceq_c \mu_{X'}(e)$, we deduce that $\frac{1}{2} < \mu_X(e) \leq \mu_{X'}(e)$, in particular $\frac{1}{2} + \frac{1}{2}\gamma \leq \mu_X(e) \leq \mu_{X'}(e)$, i.e. $e \in (X')_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = (X')^{\min}_\gamma$. It follows that $(X)^{\min}_\gamma \subseteq (X')^{\min}_\gamma$.
Hence for all $\gamma \in \mathbf{I}$, $(X)^{\min}_\gamma \subseteq (X')^{\min}_\gamma$ and $(X')^{\max}_\gamma \subseteq (X)^{\max}_\gamma$. We conclude that

$$\begin{aligned}
\mathcal{T}_\gamma(X') &= \{Y : (X')^{\min}_\gamma \subseteq Y \subseteq (X')^{\max}_\gamma\} && \text{by Def. 66} \\
&\subseteq \{Y : (X)^{\min}_\gamma \subseteq Y \subseteq (X)^{\max}_\gamma\} && \text{because } (X)^{\min}_\gamma \subseteq (X')^{\min}_\gamma,\ (X')^{\max}_\gamma \subseteq (X)^{\max}_\gamma \\
&= \mathcal{T}_\gamma(X)\,. && \text{by Def. 66}
\end{aligned}$$

Lemma 126
Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $X_i \preceq_c X'_i$ for $i = 1, \ldots, n$. Then for all $\gamma \in \mathbf{I}$,

$$\begin{aligned}
Q^{\min}_\gamma(X_1, \ldots, X_n) &\leq Q^{\min}_\gamma(X'_1, \ldots, X'_n) \\
Q^{\max}_\gamma(X_1, \ldots, X_n) &\geq Q^{\max}_\gamma(X'_1, \ldots, X'_n)\,.
\end{aligned}$$

Proof Apparently

$$\begin{aligned}
Q^{\min}_\gamma(X_1, \ldots, X_n) &= \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (15)} \\
&\leq \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X'_i)\} && \text{by L-125} \\
&= Q^{\min}_\gamma(X'_1, \ldots, X'_n)\,. && \text{by (15)}
\end{aligned}$$

and similarly

$$\begin{aligned} Q_\gamma^{\max}(X_1,\ldots,X_n) &= \sup\{Q(Y_1,\ldots,Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (16)} \\ &\geq \sup\{Q(Y_1,\ldots,Y_n) : Y_i \in \mathcal{T}_\gamma(X_i')\} && \text{by L-125} \\ &= Q_\gamma^{\max}(X_1',\ldots,X_n')\,. && \text{by (16)} \end{aligned}$$

Lemma 127

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $X_1,\ldots,X_n,X_1',\ldots,X_n' \in \widetilde{\mathcal{P}}(E)$. If $X_1 \preceq_c X_1',\ldots,X_n \preceq_c X_n'$, then $Q_\gamma(X_1,\ldots,X_n) \preceq_c Q_\gamma'(X_1,\ldots,X_n)$ for all $\gamma \in \mathbf{I}$.

Proof

Case a.: $Q\gamma(X_1',\ldots,X_n') > \frac{1}{2}$. Then

$$Q_\gamma^{\max}(X_1',\ldots,X_n') \geq Q_\gamma^{\min}(X_1',\ldots,X_n') > \tfrac{1}{2} \tag{660}$$

and by L-126,

$$Q_\gamma^{\max}(X_1,\ldots,X_n) \geq Q_\gamma^{\max}(X_1,\ldots,X_n) > \tfrac{1}{2}\,. \tag{661}$$

Hence

$$\begin{aligned} Q_\gamma(X_1',\ldots,X_n') &= Q_\gamma^{\min}(X_1',\ldots,X_n') && \text{by L-88} \\ &= \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X_1',\ldots,X_n'), Q_\gamma^{\max}(X_1,\ldots,X_n)) && \text{by Def. 45, (660), (661)} \\ &\geq \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X_1,\ldots,X_n), Q_\gamma^{\max} Xargs) && \text{by L-126, monotonicity of } \mathrm{m}_{\frac{1}{2}} \\ &= Q_\gamma(X_1,\ldots,X_n) && \text{by (14)} \\ &\geq \tfrac{1}{2}\,, \end{aligned}$$

i.e. $Q_\gamma(X_1,\ldots,X_n) \preceq_c Q_\gamma(X_1',\ldots,X_n')$, where $Q_\gamma(X_1,\ldots,X_n) \geq \frac{1}{2}$ holds because

$$Q_\gamma(X_1,\ldots,X_n) = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(X_1,\ldots,X_n), Q_\gamma^{\max}(X_1,\ldots,X_n))$$

and $Q_\gamma^{\max}(X_1,\ldots,X_n) > \frac{1}{2}$ by (661).

Case b.: $Q\gamma(X_1',\ldots,X_n') < \frac{1}{2}$. In this case,

$$\begin{aligned} Q_\gamma(X_1',\ldots,X_n') &= 1 - (\neg Q)_\gamma(X_1',\ldots,X_n') && \text{by L-29} \\ &\leq 1 - (\neg Q)_\gamma(X_1,\ldots,X_n) && \text{by part a. of this lemma} \\ &\leq \tfrac{1}{2}\,, \end{aligned}$$

i.e. $Q_\gamma(X_1,\ldots,X_n) \preceq_c Q_\gamma(X_1',\ldots,X_n')$, as desired.

Case c.: $Q\gamma(X'_1, \ldots, X'_n) = \frac{1}{2}$**.** From (14) and Def. 45, we conclude that

$$Q^{\max}_\gamma(X'_1, \ldots, X'_n) \geq \tfrac{1}{2}$$
$$Q^{\min}_\gamma(X'_1, \ldots, X'_n) \leq \tfrac{1}{2}\,.$$

Hence by L-126,

$$Q^{\max}_\gamma(X_1, \ldots, X_n) \geq Q^{\max}_\gamma(X'_1, \ldots, X'_n) \geq \tfrac{1}{2}$$
$$Q^{\min}_\gamma(X_1, \ldots, X_n) \leq Q^{\min}_\gamma(X'_1, \ldots, X'_n) \leq \tfrac{1}{2}\,.$$

Therefore $Q_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ by (14) and Def. 45. In particular, $Q_\gamma(X_1, \ldots, X_n) \preceq_c Q_\gamma(X'_1, \ldots, X'_n)$ by (9).

Proof of Theorem 113

Suppose $\mathcal{M}_\mathcal{B}$ is an $\mathcal{M}_\mathcal{B}$-DFS, i.e. the mapping $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ satisfies (B-1) to (B-5) by Th-62. Further let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $X_1 \preceq_c X'_1, \ldots, X_n \preceq_c X'_n$. Then

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) &= \mathcal{B}(Q_\gamma(X_1, \ldots, X_n)_{\gamma \in \mathbf{I}}) && \text{by Def. 69} \\
&\preceq_c \mathcal{B}((Q_\gamma(X'_1, \ldots, X'_n))_{\gamma \in \mathbf{I}}) && \text{by L-127, L-124} \\
&= \mathcal{M}_\mathcal{B}(Q)(X'_1, \ldots, X'_n)\,. && \text{by Def. 69}
\end{aligned}$$

G Proofs of Theorems in Chapter 8

G.1 Proof of Theorem 114

To avoid redundant effort in the proof of this and the subsequent theorems, we shall first state the following lemma.

Lemma 128 *Suppose* $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is a strong negation,* $\mathcal{J}$ *is an arbitrary index set and* $(x_j)_{j \in \mathcal{J}}$ *is a* $\mathcal{J}$*-indexed collection of* $x_j \in \mathbf{I}$*. Then*

$$\inf\{\widetilde{\neg}\, x_j : j \in \mathcal{J}\} = \widetilde{\neg} \sup\{x_j : j \in \mathcal{J}\}$$
$$\sup\{\widetilde{\neg}\, x_j : j \in \mathcal{J}\} = \widetilde{\neg} \inf\{x_j : j \in \mathcal{J}\}$$

Proof Apparent because the strong negation $\widetilde{\neg}$ is nonincreasing and continuous (see e.g. [12, Th-3.1]).

Proof of Theorem 114

a. Let $\mathcal{F}$ be a given DFS, $\mathcal{J}$ an arbitary index set and $(Q^j)_{j \in \mathcal{J}}$ is a $\mathcal{J}$-indexed family of semi-fuzzy quantifiers $Q^j : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $j \in \mathcal{J}$. Further suppose that a choice of crisp argument sets $Y_1, \ldots, Y_n \in \widetilde{\mathcal{P}}(E)$ is given. Then for all $j' \in \mathcal{J}$,

$$Q^{j'}(Y_1, \ldots, Y_n) \leq \sup\{Q^j(Y_1, \ldots, Y_n) : j \in \mathcal{J}\}$$

because $j' \in \mathcal{J}$, i.e.

$$Q^{j'} \leq \sup\{Q^j : j \in \mathcal{J}\}\,.$$

By Th-8, we conclude that

$$\mathcal{F}(Q^{j'}) \leq \mathcal{F}(\sup\{Q^j : j \in \mathcal{J}\})$$

for all $j' \in \mathcal{J}$. Hence for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $j' \in \mathcal{J}$,

$$\mathcal{F}(Q^{j'})(X_1, \ldots, X_n) \leq \mathcal{F}(\sup\{Q^j : j \in \mathcal{J}\})(X_1, \ldots, X_n)\,,$$

from which we deduce

$$\sup\{\mathcal{F}(Q^j)(X_1, \ldots, X_n) : j \in \mathcal{J}\} \leq \mathcal{F}(\sup\{Q^j : j \in \mathcal{J}\})(X_1, \ldots, X_n)\,.$$

Because $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ was arbitrary,

$$\sup\{\mathcal{F}(Q^j) : j \in \mathcal{J}\} \leq \mathcal{F}(\sup\{Q^j : j \in \mathcal{J}\})\,,$$

as desired.

b. Let $\mathcal{F}$ be a given DFS, $\mathcal{J}$ an arbitary index set and $(Q^j)_{j \in \mathcal{J}}$ a $\mathcal{J}$-indexed family of semi-fuzzy quantifiers $Q^j : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $j \in \mathcal{J}$. Then

$$\begin{aligned}
\inf\{\mathcal{F}(Q^j) : j \in \mathcal{J}\} &= \widetilde{\neg}\widetilde{\neg}\inf\{\mathcal{F}(Q^j) : j \in \mathcal{J}\} && \text{because } \widetilde{\neg} \text{ involutive}\\
&= \widetilde{\neg}\sup\{\widetilde{\neg}\mathcal{F}(Q^j) : j \in \mathcal{J}\} && \text{by L-128, Def. 95}\\
&= \widetilde{\neg}\sup\{\mathcal{F}(\widetilde{\neg}Q^j) : j \in \mathcal{J}\} && \text{by (DFS 3)}\\
&\geq \widetilde{\neg}\mathcal{F}(\sup\{\widetilde{\neg}Q^j : j \in \mathcal{J}\}) && \text{by part a. of the theorem, } \widetilde{\neg} \text{ nonincreasing}\\
&= \widetilde{\neg}\mathcal{F}(\widetilde{\neg}\inf\{Q^j : j \in \mathcal{J}\}) && \text{by L-128, Def. 95}\\
&= \widetilde{\neg}\widetilde{\neg}\mathcal{F}(\inf\{Q^j : j \in \mathcal{J}\}) && \text{by (DFS 3)}\\
&= \mathcal{F}(\inf\{Q^j : j \in \mathcal{J}\})\,. && \text{because } \widetilde{\neg} \text{ is involutive}
\end{aligned}$$

G.2 Proof of Theorem 115

Lemma 129

Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ a strong negation operator. Then for all $V, W \in \mathcal{P}(E)^n$,

$$\begin{aligned}
U(\widetilde{\neg}Q, V, W) &= \widetilde{\neg}L(Q, V, W)\\
L(\widetilde{\neg}Q, V, W) &= \widetilde{\neg}U(Q, V, W)\\
(\widetilde{\neg}Q)^U_{V,W} &= \widetilde{\neg}Q^L_{V,W}\\
(\widetilde{\neg}Q)^L_{V,W} &= \widetilde{\neg}Q^U_{V,W}
\end{aligned}$$

In addition, if $\mathcal{F}$ is a DFS, then

$$\begin{aligned}
\widetilde{(\widetilde{\neg}Q)}^U_{V,W} &= \widetilde{\neg}\widetilde{Q}^L_{V,W}\\
\widetilde{(\widetilde{\neg}Q)}^L_{V,W} &= \widetilde{\neg}\widetilde{Q}^U_{V,W}\\
\widetilde{(\widetilde{\neg}Q)}^U &= \widetilde{\neg}\widetilde{Q}^L\\
\widetilde{(\widetilde{\neg}Q)}^L &= \widetilde{\neg}\widetilde{Q}^U\,.
\end{aligned}$$

Proof Clearly

$$\begin{aligned}
U(\widetilde{\widetilde{\neg}} Q, V, W) &= \sup\{\widetilde{\widetilde{\neg}} Q(Z_1, \ldots, Z_n) : V_1 \subseteq Z_1 \subseteq W_1, \ldots, V_n \subseteq Z_n \subseteq W_n\} && \text{by Def. 96}\\
&= \widetilde{\widetilde{\neg}} \inf\{Q(Z_1, \ldots, Z_n) : V_1 \subseteq Z_1 \subseteq W_1, \ldots, V_n \subseteq Z_n \subseteq W_n\} && \text{by L-128}\\
&= \widetilde{\widetilde{\neg}} L(Q, V, W)\,, && \text{by Def. 96}
\end{aligned}$$

i.e.

$$U(\widetilde{\widetilde{\neg}} Q, V, W) = \widetilde{\widetilde{\neg}} L(Q, V, W)\,. \tag{662}$$

Hence also

$$\begin{aligned}
L(\widetilde{\widetilde{\neg}} Q, V, W) &= \widetilde{\widetilde{\neg}}\widetilde{\widetilde{\neg}} L(\widetilde{\widetilde{\neg}} Q, V, W) && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive}\\
&= \widetilde{\widetilde{\neg}} U(\widetilde{\widetilde{\neg}}\widetilde{\widetilde{\neg}} Q, V, W) && \text{by (662)}\\
&= \widetilde{\widetilde{\neg}} U(Q, V, W)\,. && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive}
\end{aligned}$$

In turn, we conclude that for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\begin{aligned}
&(\widetilde{\widetilde{\neg}} Q)^U_{V,W}(Y_1, \ldots, Y_n)\\
&= \begin{cases} U(\widetilde{\widetilde{\neg}} Q, V, W) & : \ V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n\\ 1 & : \ \text{else} \end{cases} && \text{by Def. 97}\\
&= \begin{cases} \widetilde{\widetilde{\neg}} L(Q, V, W) & : \ V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n\\ \widetilde{\widetilde{\neg}} 0 & : \ \text{else} \end{cases} && \text{by (662), } \widetilde{\widetilde{\neg}} 0 = 1\\
&= \widetilde{\widetilde{\neg}} \begin{cases} L(Q, V, W) & : \ V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n\\ 0 & : \ \text{else} \end{cases}\\
&= \widetilde{\widetilde{\neg}} Q^L_{V,W}(Y_1, \ldots, Y_n)\,, && \text{by Def. 97}
\end{aligned}$$

i.e.

$$(\widetilde{\widetilde{\neg}} Q)^U_{V,W} = \widetilde{\widetilde{\neg}} Q^L_{V,W}\,. \tag{663}$$

Similarly

$$\begin{aligned}
(\widetilde{\widetilde{\neg}} Q)^L_{V,W} &= \widetilde{\widetilde{\neg}}\widetilde{\widetilde{\neg}} (\widetilde{\widetilde{\neg}} Q)^L_{V,W} && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive}\\
&= \widetilde{\widetilde{\neg}} (\widetilde{\widetilde{\neg}}\widetilde{\widetilde{\neg}} Q)^U_{V,W} && \text{by (663)}\\
&= \widetilde{\widetilde{\neg}} Q^U_{V,W} && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive.}
\end{aligned}$$

Now let $\mathcal{F}$ be a DFS. Then

$$\begin{aligned}
\widetilde{(\widetilde{\widetilde{\neg}} Q)}^U_{V,W} &= \mathcal{F}((\widetilde{\widetilde{\neg}} Q)^U_{V,W}) && \text{by Def. 98}\\
&= \mathcal{F}(\widetilde{\widetilde{\neg}} Q^L_{V,W}) && \text{by (663)}\\
&= \widetilde{\widetilde{\neg}} \mathcal{F}(Q^L_{V,W}) && \text{by (DFS 3)}\\
&= \widetilde{\widetilde{\neg}} \widetilde{Q}^L_{V,W}\,, && \text{by Def. 98}
\end{aligned}$$

i.e.

$$\widetilde{(\widetilde{\neg} Q)}^{U}_{V,W} = \widetilde{\neg}\, \widetilde{Q}^{L}_{V,W} \,. \tag{664}$$

By the usual reasoning,

$$\begin{aligned}
\widetilde{(\widetilde{\neg} Q)}^{L}_{V,W} &= \widetilde{\neg}\,\widetilde{\neg}\,\widetilde{(\widetilde{\neg} Q)}^{L}_{V,W} && \text{because } \widetilde{\neg} \text{ involutive} \\
&= \widetilde{\neg}\,\widetilde{(\widetilde{\neg}\,\widetilde{\neg} Q)}^{U}_{V,W} && \text{by (664)} \\
&= \widetilde{\neg}\,\widetilde{Q}^{U}_{V,W} && \text{because } \widetilde{\neg} \text{ involutive.}
\end{aligned}$$

Finally

$$\begin{aligned}
\widetilde{(\widetilde{\neg} Q)}^{U} &= \inf\{\widetilde{(\widetilde{\neg} Q)}^{U}_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by Def. 99} \\
&= \inf\{\widetilde{\neg}\,\widetilde{Q}^{L}_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by (664)} \\
&= \widetilde{\neg} \sup\{\widetilde{Q}^{L}_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by L-128} \\
&= \widetilde{\neg}\,\widetilde{Q}^{L} \,, && \text{by Def. 99}
\end{aligned}$$

i.e.

$$\widetilde{(\widetilde{\neg} Q)}^{U} = \widetilde{\neg}\,\widetilde{Q}^{L} \,. \tag{665}$$

Analogously,

$$\begin{aligned}
\widetilde{(\widetilde{\neg} Q)}^{L} &= \widetilde{\neg}\,\widetilde{\neg}\,\widetilde{(\widetilde{\neg} Q)}^{L} && \text{because } \widetilde{\neg} \text{ involutive} \\
&= \widetilde{\neg}\,\widetilde{(\widetilde{\neg}\,\widetilde{\neg} Q)}^{U} && \text{by (665)} \\
&= \widetilde{\neg}\,\widetilde{Q}^{U} \,. && \text{because } \widetilde{\neg} \text{ involutive}
\end{aligned}$$

Proof of Theorem 115

Let a DFS $\mathcal{F}$ and a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given.

a. We will first show that $\widetilde{Q}^L \leq \mathcal{F}(Q)$. To this end, we consider the quantifier $Q^L : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ defined by

$$Q^L = \sup\{Q^L_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i \text{ for all } i = 1, \ldots, n\} \,. \tag{666}$$

Now let $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$ a choice of crisp argument sets. Then

$$\begin{aligned}
& Q(Y_1, \ldots, Y_n) \\
&= Q^L_{Y,Y}(Y_1, \ldots, Y_n) && \text{by Def. 97} \\
&\leq \sup\{Q^L_{V,W}(Y_1, \ldots, Y_n) : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i \text{ for all } i = 1, \ldots, n\} \\
&= Q^L(Y_1, \ldots, Y_n) && \text{by (666)}\,,
\end{aligned}$$

i.e.

$$Q \leq Q^L \,. \tag{667}$$

On the other hand, suppose that $V, W \in \mathcal{P}(E)^n$ satisfy $V_i \subseteq W_i$ for all $i = 1, \dots, n$. Then by Def. 97 and Def. 96,

$$\begin{aligned}
&Q^L_{V,W}(Y_1, \dots, Y_n) \\
&= \begin{cases} \inf\{Q(Z_1, \dots, Z_n) : V_i \subseteq Z_i \subseteq W_i,\ i = 1, \dots, n\} & : \quad V_i \subseteq Y_i \subseteq W_i, i = 1, \dots, n \\ 0 & : \quad \text{else} \end{cases} \\
&\leq \begin{cases} Q(Y_1, \dots, Y_n) & : \quad V_i \subseteq Y_i \subseteq W_i, \text{all } i = 1, \dots, n \\ 0 & : \quad \text{else} \end{cases} \\
&\leq Q(Y_1, \dots, Y_n)
\end{aligned}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$, i.e.

$$Q \geq Q^L_{V,W} \,.$$

We conclude that

$$Q \geq \sup\{Q^L_{V,W} : V, W \in \mathcal{P}(E)^n, V_i \subseteq W_i \text{ for } i = 1, \dots, n\} \,,$$

i.e. $Q \geq Q^L$ by (666). Combining this with (667), we see that

$$Q = Q^L \,. \tag{668}$$

Therefore

$$\begin{aligned}
\mathcal{F}(Q) &= \mathcal{F}(Q^L) && \text{by (668)} \\
&= \mathcal{F}(\sup\{Q^L_{V,W} : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i \text{ for all } i = 1, \dots, n\}) && \text{by (666)} \\
&\geq \sup\{\mathcal{F}(Q^L_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i \text{ for all } i = 1, \dots, n\} && \text{by Th-114} \\
&= \widetilde{Q}^L \,. && \text{by Def. 99, Def. 98}
\end{aligned}$$

b. Let us now prove that $\mathcal{F}(Q) \leq \widetilde{Q}^U$.

$$\begin{aligned}
\mathcal{F}(Q) &= \mathcal{F}(\widetilde{\widetilde{\neg}}\, \widetilde{\widetilde{\neg}}\, Q) && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive} \\
&= \widetilde{\widetilde{\neg}}\, \mathcal{F}(\widetilde{\widetilde{\neg}}\, Q) && \text{by (DFS 3)} \\
&\leq \widetilde{\widetilde{\neg}}\, \widetilde{(\widetilde{\widetilde{\neg}}\, Q)}^L && \text{by part a. and } \widetilde{\widetilde{\neg}} \text{ nonincreasing} \\
&= \widetilde{\widetilde{\neg}}\, \widetilde{\widetilde{\neg}}\, \widetilde{Q}^U && \text{by L-129} \\
&= \widetilde{Q}^U \,. && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive}
\end{aligned}$$

G.3 Proof of Theorem 116

Lemma 130

Suppose $\mathcal{F}$ is a DFS, $E \neq \varnothing$ is a base set and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

a. $\mathcal{F}(\exists)(X_1 \,\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}\, X_n) = \widetilde{\widetilde{\bigvee_{i=1}^{n}}}\, \mathcal{F}(\exists)(X_i)$.

b. $\mathcal{F}(\forall)(X_1 \,\widetilde{\widetilde{\cap}} \cdots \widetilde{\widetilde{\cap}}\, X_n) = \widetilde{\widetilde{\bigwedge_{i=1}^{n}}}\, \mathcal{F}(\forall)(X_i)$.

Proof

a. Let a base set $E \neq \varnothing$ be given. Further suppose that $Y_1, \ldots, Y_n \in \mathcal{P}(E)$ for some $n \in \mathbb{N}$. Then apparently

$$\exists(Y_1 \cup \cdots \cup Y_n) = \pi_1(\widehat{!}(Y_1 \cup \cdots \cup Y_n)) = \pi_1(\widehat{!}(Y_1) \cup \cdots \cup \widehat{!}(Y_n))$$

where $\pi_1 : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$, and ! is the unique mapping $! : E \longrightarrow \{1\}$. Because the Y_i were arbitrarily chosen, we conclude that

$$\exists \underbrace{\cup \cdots \cup}_{n-1} = \pi_1 \underbrace{\cup \cdots \cup}_{n-1} \circ \mathop{\times}_{i=1}^{n} \widehat{!} . \tag{669}$$

Therefore

$$\begin{aligned} &\mathcal{F}(\exists) \underbrace{\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}}_{n-1} \\ &= \mathcal{F}(\exists \underbrace{\cup \cdots \cup}_{n-1}) && \text{by (Z-4)} \\ &= \mathcal{F}(\pi_1 \underbrace{\cup \cdots \cup}_{n-1} \circ \mathop{\times}_{i=1}^{n} \widehat{!}) && \text{by (669)} \\ &= \widetilde{\pi}_1 \underbrace{\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}}_{n-1} \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(!) , && \text{by (Z-2), (Z-4), (Z-6)} \end{aligned}$$

i.e.

$$\mathcal{F}(\exists) \underbrace{\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}}_{n-1} = \widetilde{\pi}_1 \underbrace{\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}}_{n-1} \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(!) \tag{670}$$

From this we deduce that for all $X_1, \ldots, X_n$,

$$\begin{aligned} &\mathcal{F}(\exists)(X_1 \,\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}\, X_n) \\ &\qquad = \widetilde{\pi}_1(\widehat{\mathcal{F}}(!)(X_1) \,\widetilde{\widetilde{\cup}} \cdots \widetilde{\widetilde{\cup}}\, \widehat{\mathcal{F}}(!)(X_n)) && \text{by (670)} \\ &\qquad = \mu_{\widehat{\mathcal{F}}(!)(X_1)\widetilde{\widetilde{\cup}}\cdots\widetilde{\widetilde{\cup}}\widehat{\mathcal{F}}(!)(X_n)}(1) && \text{by Def. 7} \\ &\qquad = \mu_{\widehat{\mathcal{F}}(!)(X_1)}(1) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \mu_{\widehat{\mathcal{F}}(!)(X_n)}(1) && \text{by definition of } \widetilde{\widetilde{\cup}} \\ &\qquad = \widetilde{\pi}_1(\widehat{\mathcal{F}}(!)(X_1)) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \widetilde{\pi}_1(\widehat{\mathcal{F}}(!)(X_n)) && \text{by Def. 7} \\ &\qquad = \mathcal{F}(\exists)(X_1) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \mathcal{F}(\exists)(X_n) . && \text{by Th-17} \end{aligned}$$

b. In this case,

$$
\begin{aligned}
&\mathcal{F}(\forall)(X_1 \mathbin{\tilde{\tilde{\cap}}} \cdots \mathbin{\tilde{\tilde{\cap}}} X_n) \\
&\quad = \mathcal{F}(\exists\Box)(X_1 \mathbin{\tilde{\tilde{\cap}}} \cdots \mathbin{\tilde{\tilde{\cap}}} X_n) && \text{utilizing the duality } \forall = \exists\Box \\
&\quad = \mathcal{F}(\exists)\tilde{\tilde{\Box}}(X_1 \mathbin{\tilde{\tilde{\cap}}} \cdots \mathbin{\tilde{\tilde{\cap}}} X_n) && \text{by (Z-3)} \\
&\quad = \tilde{\tilde{\neg}}\, \mathcal{F}(\exists)(\tilde{\tilde{\neg}}(X_1 \mathbin{\tilde{\tilde{\cap}}} \cdots \mathbin{\tilde{\tilde{\cap}}} X_n)) && \text{by Def. 12} \\
&\quad = \tilde{\tilde{\neg}}\, \mathcal{F}(\exists)(\tilde{\tilde{\neg}}\, X_1 \mathbin{\tilde{\tilde{\cup}}} \cdots \mathbin{\tilde{\tilde{\cup}}} \tilde{\tilde{\neg}}\, X_n) && \text{by De Morgan's law} \\
&\quad = \tilde{\tilde{\neg}} \tilde{\tilde{\bigvee_{i=1}^{n}}} \mathcal{F}(\exists)(\tilde{\tilde{\neg}}\, X_i) && \text{by part a. of the lemma} \\
&\quad = \tilde{\tilde{\bigwedge_{i=1}^{n}}} \tilde{\tilde{\neg}}\, \mathcal{F}(\exists)(\tilde{\tilde{\neg}}\, X_i) && \text{by De Morgan's law} \\
&\quad = \tilde{\tilde{\bigwedge_{i=1}^{n}}} \mathcal{F}(\exists)\tilde{\tilde{\Box}}(X_i) && \text{by Def. 12} \\
&\quad = \tilde{\tilde{\bigwedge_{i=1}^{n}}} \mathcal{F}(\exists\Box)(X_i) && \text{by (Z-3)} \\
&\quad = \tilde{\tilde{\bigwedge_{i=1}^{n}}} \mathcal{F}(\forall)(X_i)\,, && \text{by duality } \exists\Box = \forall
\end{aligned}
$$

as desired.

Lemma 131

Let a base set $E \neq \varnothing$ be given and $V, W, Y \in \mathcal{P}(E)$ where $V \subseteq W$. Then

$$((Y \cup (W \setminus V)) \triangle \neg W) = E \Leftrightarrow V \subseteq Y \subseteq W\,.$$

Proof

"⇐". Suppose $V \subseteq Y \subseteq W$. We have to show that $((Y \cup (W \setminus V)) \triangle \neg W) = E$. This is apparent from the following observations. Firstly

$$
\begin{aligned}
Y \cup (W \setminus V) &\supseteq V \cup (W \setminus V) && \text{because } V \subseteq Y \\
&= W && \text{because } V \subseteq W \text{ by assumption}
\end{aligned}
$$

and

$$
\begin{aligned}
Y \cup (W \setminus V) &\subseteq W \cup (W \setminus V) && \text{because } V \subseteq W \\
&= W\,.
\end{aligned}
$$

Hence

$$Y \cup (W \setminus V) = W\,. \tag{671}$$

Therefore

$$
\begin{aligned}
(Y \cup (W \setminus V)) \triangle \neg W &= W \triangle \neg W && \text{by (671)} \\
&= E\,. && \text{by definition of symmetrical difference}
\end{aligned}
$$

"⇒". We will prove the converse implication by contraposition. Hence suppose that $V, W \in \mathcal{P}(E)$, $V \subseteq W$ and $Y \in \mathcal{P}(E)$ is such that $V \subseteq Y \subseteq W$ fails.

- If $V \nsubseteq Y$, then there is some $e \in E$ such that $e \in V$, $e \notin Y$. Because $e \in V$ and $V \subseteq W$, $e \notin W \setminus V$. In turn because $e \notin Y$ and $e \notin W \setminus V$, we know that $e \notin Y \cup (W \setminus V)$. But $e \in V$, $V \subseteq W$ and hence $e \in W$, i.e. $e \notin \neg W$. Hence $e \notin Y \cup (W \setminus V)$ and $e \notin \neg W$, and by the definition of symmetrical difference, $e \notin (Y \cup (W \setminus V)) \triangle \neg W$. This proves that $(Y \cup (W \setminus V)) \triangle \neg W \neq E$.

- If $Y \nsubseteq W$, then there is some $e \in E$ such that $e \in Y$ and $e \notin W$. Because $e \in Y$, we know that $e \in (Y \cup (W \setminus V))$. In addition, $e \in \neg W$ because $e \notin W$. Hence from the definition of symmetrical difference, $e \notin (Y \cup (W \setminus V)) \triangle \neg W$, which proves that $(Y \cup (W \setminus V)) \triangle \neg W \neq E$.

Lemma 132
Suppose $E \neq \varnothing$ is some base set, $n \in \mathbb{N}$ and $V, W \in \mathcal{P}(E)^n$ such that $V_i \subseteq W_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned}\Xi_{V,W}(Y_1, \ldots, Y_n) &= \forall(\bigcap_{i=1}^{n}((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)) \\ &= \bigwedge_{i=1}^{n} \forall((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)\end{aligned}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Proof Let a base set $E \neq \varnothing$ and $V, W \in \mathcal{P}(E)^n$, $n \in \mathbb{N}$ be given, where $V_i \subseteq W_i$ for all $i = 1, \ldots, n$. We shall define the two-valued quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ by

$$Q(Y_1, \ldots, Y_n) = \forall(\bigcap_{i=1}^{n}((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)), \tag{672}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. By L-131, we know that $(Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i = E$ exactly if $V_1 \subseteq Y_i \subseteq W_i$. Hence

$$\bigcap_{i=1}^{n}((Y_1 \cup (W_i \setminus V_i)) \triangle \neg W_i) = E \quad \Leftrightarrow \quad \text{for all } i = 1, \ldots, n\text{: } V_i \subseteq Y_i \subseteq W_i\,.$$

Recalling (672) and the definition of $\forall$, this means that

$$Q(Y_1, \ldots, Y_n) = \forall(\bigcap_{i=1}^{n}((Y_1 \cup (W_i \setminus V_i)) \triangle \neg W_i)) = 1 \quad \Leftrightarrow \quad \text{for all } i = 1, \ldots, n\text{: } V_i \subseteq Y_i \subseteq W_i\,.$$

Hence

$$\begin{aligned}Q(Y_1, \ldots, Y_n) &= \begin{cases} 1 & : \quad V_i \subseteq Y_i \subseteq W_i \text{ for all } i = 1, \ldots, n \\ 0 & : \quad \text{else} \end{cases} \\ &= \Xi_{V,W}(Y_1, \ldots, Y_n)\,, && \text{by Def. 101}\end{aligned}$$

for all $Y_1, \ldots, Y_n$, i.e.

$$\Xi_{V,W}(Y_1, \ldots, Y_n) = \forall(\bigcap_{i=1}^{n}((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)) \tag{673}$$

as desired. The second equation in the lemma is apparent from (673) because

$$\begin{aligned}\Xi_{V,W}(Y_1, \ldots, Y_n) &= \bigwedge_{i=1}^{n} \Xi_{V_i,W_i}(Y_i) && \text{by Def. 101} \\ &= \bigwedge_{i=1}^{n} \forall((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)\,. && \text{by (673)}\end{aligned}$$

Proof of Theorem 116

Let a base set $E \neq \varnothing$ and $V, W \in \mathcal{P}(E)^n$, $n \in \mathbb{N}$ be given, where $V_i \subseteq W_i$ for all $i = 1, \dots, n$. Further suppose that $\mathcal{F}$ is a DFS and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
&\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \dots, X_n) \\
&= \mathcal{F}(\forall)(\widetilde{\widetilde{\bigcap_{i=1}^{n}}}((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i)) && \text{by L-132, (DFS 6), (Z-4), Th-5, (DFS 7)} \\
&= \widetilde{\widetilde{\bigwedge_{i=1}^{n}}} \mathcal{F}(\forall)((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i)\,. && \text{by L-130}
\end{aligned}$$

Abbreviating $Z_i = (X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i$, we obtain the desired

$$\begin{aligned}
\mu_{Z_i}(e) &= \begin{cases} (\mu_{X_i}(e) \,\widetilde{\widetilde{\vee}}\, 0) \,\widetilde{\widetilde{\mathrm{xor}}}\, 0 & : \quad e \in W_i \\ (\mu_{X_i}(e) \,\widetilde{\widetilde{\vee}}\, 1) \,\widetilde{\widetilde{\mathrm{xor}}}\, 0 & : \quad e \in W_i \setminus V_i \\ (\mu_{X_i}(e) \,\widetilde{\widetilde{\vee}}\, 0) \,\widetilde{\widetilde{\mathrm{xor}}}\, 1 & : \quad e \notin W_i \end{cases} \\
&= \begin{cases} \mu_{X_i}(e) & : \quad e \in V_i \\ 1 & : \quad e \in W_i \setminus V_i \\ \widetilde{\widetilde{\neg}}\, \mu_{X_i}(e) & : \quad e \notin W_i \end{cases}
\end{aligned}$$

for all $e \in E$.

G.4 Proof of Theorem 117

Lemma 133
The mappings $\eta : \mathbf{2} \longrightarrow \mathcal{P}(\{1\})$ and $\pi_1 : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$ defined by Def. 50 and Def. 6, resp., are related by

$$\eta = {\pi_1}^{-1}\,.$$

Similarly, the mappings $\widetilde{\eta} : \mathbf{I} \longrightarrow \widetilde{\mathcal{P}}(\{1\})$ and $\widetilde{\pi}_1 : \widetilde{\mathcal{P}}(\{1\}) \longrightarrow \mathbf{I}$ defined by Def. 51 and Def. 7 satisfy

$$\widetilde{\eta} = \widetilde{\pi}_1^{-1}\,.$$

Proof Let us first consider the crisp case. Hence let $Y \in \mathcal{P}(\{1\})$. If $Y = \{1\}$, then $\pi_1(Y) = \pi_1(\{1\}) = 1$ and $\eta(\pi_1(Y)) = \eta(1) = \{1\} = Y$. If $Y = 0$, then $\pi_1(Y) = \pi_1(\varnothing) = 0$ and $\eta(\pi_1(Y)) = \eta(0) = \varnothing = Y$. Hence $\eta = {\pi_1}^{-1}$.
Now let us turn to the fuzzy case. Let $X \in \widetilde{\mathcal{P}}(\{1\})$ a fuzzy subset, and let us abbreviate $x = \mu_X(1)$. Then $\widetilde{\pi}_1(X) = x$ and $\widetilde{\eta}(\widetilde{\pi}_1(X)) = \widetilde{\eta}(x) = X$, because X is the unique fuzzy subset of $\{1\}$ which has $\mu_X(1) = x$. Hence $\widetilde{\eta} = \widetilde{\pi}_1^{-1}$, as desired.

Lemma 134
Let $\widetilde{\widetilde{\neg}} : \mathbf{I} \times \mathbf{I}$ be a strong negation operator. Then for all $a \in \mathbf{I}$,

$$\begin{aligned}
b_a &= \widetilde{\widetilde{\neg}}\, p_{\widetilde{\widetilde{\neg}} a} \\
p_a &= \widetilde{\widetilde{\neg}}\, b_{\widetilde{\widetilde{\neg}} a}\,.
\end{aligned}$$

If $\mathcal{F}$ is a QFM such that $\widetilde{\widetilde{\neg}}$ is a strong negation operator and such that $\mathcal{F}$ satisfies (DFS 3), *then for all $a \in \mathbf{I}$,*

$$\widetilde{\widetilde{b}}_a = \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{p}}_{\widetilde{\widetilde{\neg}}\,a}$$
$$\widetilde{\widetilde{p}}_a = \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{b}}_{\widetilde{\widetilde{\neg}}\,a}$$

Proof Suppose $a \in \mathbf{I}$ and $x \in \mathbf{2}$. Then

$$\begin{aligned} p_a(x) &= \begin{cases} 1 & : \quad x = 0 \\ a & : \quad x = 1 \end{cases} && \text{by Def. 100} \\ &= \begin{cases} \widetilde{\widetilde{\neg}}\,0 & : \quad x = 0 \\ \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\,a & : \quad x = 1 \end{cases} && \text{because } \widetilde{\widetilde{\neg}}\,0 = 1,\ \widetilde{\widetilde{\neg}} \text{ involution} \\ &= \widetilde{\widetilde{\neg}} \begin{cases} 0 & : \quad x = 0 \\ \widetilde{\widetilde{\neg}}\,a & : \quad x = 1 \end{cases} \\ &= \widetilde{\widetilde{\neg}}\, b_{\widetilde{\widetilde{\neg}}\,a}(x)\,, && \text{by Def. 100} \end{aligned}$$

i.e.

$$p_a(x) = \widetilde{\widetilde{\neg}}\, b_{\widetilde{\widetilde{\neg}}\,a}(x)\,. \tag{674}$$

Similarly

$$\begin{aligned} b_a(x) &= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, b_{\widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\,a}(x) && \text{because } \widetilde{\widetilde{\neg}} \text{ involution} \\ &= \widetilde{\widetilde{\neg}}\, p_{\widetilde{\widetilde{\neg}}\,a}(x)\,. && \text{by (674)} \end{aligned}$$

Now suppose $\mathcal{F}$ is a QFM with the required properties. Because $\widetilde{\widetilde{\neg}}$ is assumed to be a strong negation operator, $\widetilde{\widetilde{p}}_a$ and $\widetilde{\widetilde{b}}_a$ are related as claimed by the first part of the lemma. From Def. 52 and (DFS 3), we deduce that $\widetilde{\widetilde{p}}_a(x) = \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{b}}_{\widetilde{\widetilde{\neg}}\,a}(x)$ and $\widetilde{\widetilde{b}}_a(x) = \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{p}}_{\widetilde{\widetilde{\neg}}\,a}(x)$ for all $a, x \in \mathbf{I}$.

Lemma 135
Let a base set $E \neq \varnothing$ be given and $e \in E$. Further let $\mathcal{F}$ be a DFS. Then $\pi_e \neg = \widetilde{\widetilde{\neg}}\,\pi_e$ and $\widetilde{\pi}_e \widetilde{\widetilde{\neg}} = \widetilde{\widetilde{\neg}}\,\widetilde{\pi}_e$.

Proof To see that the first equation holds, let $Y \in \mathcal{P}(E)$. Then

$$\begin{aligned} \pi_e \neg(Y) &= \pi_e(\neg Y) && \text{by Def. 11} \\ &= \chi_{\neg Y}(e) && \text{by Def. 6} \\ &= \begin{cases} 1 & : \quad e \notin Y \\ 0 & : \quad e \in Y \end{cases} \\ &= \begin{cases} \widetilde{\widetilde{\neg}}\,0 & : \quad e \notin Y \\ \widetilde{\widetilde{\neg}}\,1 & : \quad e \in Y \end{cases} && \text{because } \widetilde{\widetilde{\neg}} \text{ strong negation} \\ &= \widetilde{\widetilde{\neg}} \begin{cases} 0 & : \quad e \notin Y \\ 1 & : \quad e \in Y \end{cases} \\ &= \widetilde{\widetilde{\neg}}\,\pi_e(Y)\,. && \text{by Def. 6} \end{aligned}$$

Similarly if $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
\widetilde{\pi}_e \widetilde{\widetilde{\neg}}(X) &= \widetilde{\pi}_e(\widetilde{\widetilde{\neg}}\, X) && \text{by Def. 11}\\
&= \mu_{\widetilde{\widetilde{\neg}}\, X}(e) && \text{by Def. 7}\\
&= \widetilde{\widetilde{\neg}}\, \mu_X(e) && \text{by definition of fuzzy complement}\\
&= \widetilde{\widetilde{\neg}}\, \widetilde{\pi}_e(X)\,. && \text{by Def. 7}
\end{aligned}$$

Lemma 136

Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and $V, W \in \mathcal{P}(E)^n$. Then for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\begin{aligned}
Q^L_{V,W}(Y_1, \ldots, Y_n) &= b_{L(Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n))\\
Q^U_{V,W}(Y_1, \ldots, Y_n) &= p_{U(Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n))\,.
\end{aligned}$$

Proof Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $V, W \in \mathcal{P}(E)^n$ are given. Then for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\begin{aligned}
&Q^L_{V,W}(Y_1, \ldots, Y_n)\\
&= \begin{cases} L(Q, V, W) & : \quad V_i \subseteq Y_i \subseteq W_i, \text{ all } i = 1, \ldots, n \\ 0 & : \quad \text{else} \end{cases} && \text{by Def. 97}\\
&= \begin{cases} L(Q, V, W) & : \quad \Xi_{V,W}(Y_1, \ldots, Y_n) = 1 \\ 0 & : \quad \Xi_{V,W}(Y_1, \ldots, Y_n) = 0 \end{cases} && \text{by Def. 101}\\
&= b_{L(Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n))\,,
\end{aligned}$$

where the last equation holds because $b_{L(Q,V,W)}(0) = 0$ and $b_{L(Q,V,W)}(1) = L(Q, V, W)$ by Def. 100.
In the case of $Q^U_{V,W}$, let $\widetilde{\widetilde{\neg}}$ a strong negation. Then

$$\begin{aligned}
&Q^U_{V,W}(Y_1, \ldots, Y_n)\\
&= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, Q^U_{V,W}(Y_1, \ldots, Y_n) && \text{because } \widetilde{\widetilde{\neg}} \text{ involution}\\
&= \widetilde{\widetilde{\neg}}\,(\widetilde{\widetilde{\neg}}\, Q)^L_{V,W}(Y_1, \ldots, Y_n) && \text{by L-129}\\
&= \widetilde{\widetilde{\neg}}\, b_{L(\widetilde{\widetilde{\neg}}\, Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n)) && \text{by first part of the lemma}\\
&= \widetilde{\widetilde{\neg}}\, b_{\widetilde{\widetilde{\neg}}\, U(Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n)) && \text{by L-129}\\
&= p_{U(Q,V,W)}(\Xi_{V,W}(Y_1, \ldots, Y_n))\,. && \text{by L-134}
\end{aligned}$$

Lemma 137

Let a base set $E \neq \varnothing$ be given. Further let $n \in \mathbb{N}$ and $V, W \in \mathcal{P}(E)^n$ such that $V_i \subseteq W_i$ for $i = 1, \ldots, n$. Then for all $a \in \mathbf{I}$ and all $Y_1, \ldots, Y_n$,

$$b_a(\Xi_{V,W}(Y_1, \ldots, Y_n)) = Q_{b_a}(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i))\triangle\neg W_i)))$$

Proof In the following, we shall denote by ! the unique mapping $! : E \longrightarrow \{1\}$. Then

$$\begin{aligned}
&\Xi_{V,W}(Y_1, \dots, Y_n) \\
&= \bigwedge_{i=1}^{n} \Xi_{V_i,W_i}(Y_i) && \text{by Def. 101} \\
&= \bigwedge_{i=1}^{n} \forall((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i) && \text{by L-132} \\
&= \bigwedge_{i=1}^{n} \neg\exists(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)) && \text{because } \forall = \exists\square \\
&= \bigwedge_{i=1}^{n} \neg\pi_1(\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i))) && \text{because } \exists = \pi_1 \circ \widehat{!} \\
&= \bigwedge_{i=1}^{n} \pi_1(\neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i))) && \text{by L-135} \\
&= \pi_1(\bigcap_{i=1}^{n} \eta(\pi_1(\neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i))))) && \text{apparent from Def. 50} \\
&= \pi_1(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)))\,, && \text{by L-133}
\end{aligned}$$

i.e.

$$\Xi_{V,W}(Y_1, \dots, Y_n) = \pi_1(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)))\,. \tag{675}$$

Hence

$$\begin{aligned}
&b_a(\Xi_{V,W}(Y_1, \dots, Y_n)) \\
&= b_a(\eta^{-1}(\eta(\Xi_{V,W}(Y_1, \dots, Y_n)))) && \text{see Def. 50} \\
&= Q_{b_a}(\eta(\Xi_{V,W}(Y_1, \dots, Y_n))) && \text{see Def. 52} \\
&= Q_{b_a}(\eta(\pi_1(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i))))) && \text{by (675)} \\
&= Q_{b_a}(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i)) \triangle \neg W_i)))\,. && \text{by L-133}
\end{aligned}$$

Lemma 138

Let $E \neq \varnothing$ be given, $n \in \mathbb{N}$ and $V, W \in \mathcal{P}(E)^n$ such that $V_i \not\subseteq W_i$ for some $i \in \{1, \dots, n\}$. Further suppose that $a \in \mathbf{I}$, and $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are defined by

$$\begin{aligned}
Q(Y_1, \dots, Y_n) &= b_a(\Xi_{V,W}(Y_1, \dots, Y_n)) \\
Q'(Y_1, \dots, Y_n) &= p_a(\Xi_{V,W}(Y_1, \dots, Y_n))
\end{aligned}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Then in every DFS $\mathcal{F}$,

$$\begin{aligned}
\mathcal{F}(Q)(X_1, \dots, X_n) &= \widetilde{\widetilde{b}}_a(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n)) \\
\mathcal{F}(Q')(X_1, \dots, X_n) &= \widetilde{\widetilde{p}}_a(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n))
\end{aligned}$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof

By assumption, there is an $i \in \{1, \ldots, n\}$ such that $V_i \not\subseteq W_i$. Hence by Def. 101,

$$\Xi_{V_i,W_i}(Y_i) = 0$$

for all $Y_i \in \mathcal{P}(E)$. Hence also by Def. 101,

$$\Xi_{V,W}(Y_1, \ldots, Y_n) = \bigwedge_{i=1}^{n} \Xi_{V_i,W_i}(Y_i) = 0 \tag{676}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We deduce from Th-1 and Th-6 that

$$\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) = 0 \tag{677}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Now let us consider Q and Q'. By (676) and Def. 100,

$$\begin{aligned} Q(Y_1, \ldots, Y_n) &= b_a(\Xi_{V,W}(Y_1, \ldots, Y_n)) = b_a(0) = 0 \\ Q'(Y_1, \ldots, Y_n) &= p_a(\Xi_{V,W}(Y_1, \ldots, Y_n)) = p_a(0) = 1 \end{aligned}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We conclude that

$$\begin{aligned} \mathcal{F}(Q)(X_1, \ldots, X_n) &= 0 && \text{by Th-1, Th-6} \\ &= b_a(0) && \text{by Def. 100} \\ &= \widetilde{\widetilde{b}}_a(0) && \text{by Th-1, Def. 52} \\ &= \widetilde{\widetilde{b}}_a(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n))\,. && \text{by (677)} \end{aligned}$$

By similar reasoning,

$$\begin{aligned} \mathcal{F}(Q')(X_1, \ldots, X_n) &= 1 && \text{by Th-1, Th-6} \\ &= p_a(0) && \text{by Def. 100} \\ &= \widetilde{\widetilde{p}}_a(0) && \text{by Th-1, Def. 52} \\ &= \widetilde{\widetilde{p}}_a(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n))\,. && \text{by (677)} \end{aligned}$$

Lemma 139

Let $E \neq \varnothing$ be given, $n \in \mathbb{N}$ and $V, W \in \mathcal{P}(E)^n$. Further suppose that $a \in \mathbf{I}$ and $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are defined by

$$\begin{aligned} Q(Y_1, \ldots, Y_n) &= b_a(\Xi_{V,W}(Y_1, \ldots, Y_n)) \\ Q'(Y_1, \ldots, Y_n) &= p_a(\Xi_{V,W}(Y_1, \ldots, Y_n)) \end{aligned}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. Then in every DFS $\mathcal{F}$,

$$\begin{aligned} \mathcal{F}(Q)(X_1, \ldots, X_n) &= \widetilde{\widetilde{b}}_a(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n)) \\ \mathcal{F}(Q')(X_1, \ldots, X_n) &= \widetilde{\widetilde{p}}_a(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n)) \end{aligned}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof The case that $V_i \not\subseteq W_i$ for some $i \in \{1, \dots, n\}$ is covered by L-138. We may hence assume that $V_i \subseteq W_i$ for all $i = 1, \dots, n$.
Let us first consider $\mathcal{F}(Q)$. We already know from L-137 that in the crisp case,

$$Q(Y_1, \dots, Y_n) = Q_{b_a}(\bigcap_{i=1}^{n} \neg\widehat{!}(\neg((Y_i \cup (W_i \setminus V_i))\triangle\neg W_i)))$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Hence by (DFS 6), (DFS 5), (Z-6), (Z-4) and (DFS 7),

$$\begin{aligned}
&\mathcal{F}(Q)(X_1, \dots, X_n) \\
&= \mathcal{F}(Q_{b_a})(\widetilde{\widetilde{\bigcap}}_{i=1}^{n} \widetilde{\widetilde{\neg}}\, \widehat{\mathcal{F}}(!)(\widetilde{\widetilde{\neg}}((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i))) \\
&= \mathcal{F}(Q_{b_a})(\widetilde{\widetilde{\bigcap}}_{i=1}^{n} \widetilde{\widetilde{\neg}}\, \widetilde{\eta}(\mathcal{F}(\exists)(\widetilde{\widetilde{\neg}}((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i)))) && \text{by Th-17, L-133} \\
&= \mathcal{F}(Q_{b_a})(\widetilde{\eta}(\widetilde{\widetilde{\bigwedge}}_{i=1}^{n} \widetilde{\widetilde{\neg}}\, \mathcal{F}(\exists)(\widetilde{\widetilde{\neg}}((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i)))) && \text{apparent from Def. 51} \\
&= \mathcal{F}(Q_{b_a})(\widetilde{\eta}(\widetilde{\widetilde{\bigwedge}}_{i=1}^{n} \mathcal{F}(\forall)((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i))) && \text{by (Z-3), } \forall = \exists\Box \\
&= \widetilde{\widetilde{b}}_a(\widetilde{\widetilde{\bigwedge}}_{i=1}^{n} \mathcal{F}(\forall)((X_i \,\widetilde{\widetilde{\cup}}\, (W_i \setminus V_i)) \,\widetilde{\widetilde{\triangle}}\, \neg W_i)) && \text{by Def. 52} \\
&= \widetilde{\widetilde{b}}_a(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n))\,. && \text{by Th-116}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathcal{F}(Q')(X_1, \dots, X_n) \\
&= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\,\mathcal{F}(Q')(X_1, \dots, X_n) && \text{because } \widetilde{\widetilde{\neg}} \text{ involutive} \\
&= \widetilde{\widetilde{\neg}}\,\mathcal{F}(\widetilde{\neg}\, Q')(X_1, \dots, X_n) && \text{by (DFS 3)} \\
&= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{b}}_{\widetilde{\widetilde{\neg}}\,a}(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n)) && \text{by L-134 and first part of this lemma} \\
&= \widetilde{\widetilde{p}}_a(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n))\,. && \text{by L-134}
\end{aligned}$$

Proof of Theorem 117

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Further let $V, W \in \mathcal{P}(E)^n$ and a DFS $\mathcal{F}$ be given. We already know from L-136 that in the crisp case,

$$\begin{aligned}
Q^L_{V,W}(Y_1, \dots, Y_n) &= b_{L(Q,V,W)}(\Xi_{V,W}(Y_1, \dots, Y_n)) \\
Q^U_{V,W}(Y_1, \dots, Y_n) &= p_{U(Q,V,W)}(\Xi_{V,W}(Y_1, \dots, Y_n))
\end{aligned}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Hence by L-139,

$$\begin{aligned}
\widetilde{Q}^L_{V,W}(Y_1, \dots, Y_n) &= \widetilde{\widetilde{b}}_{L(Q,V,W)}(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n)) \\
\widetilde{Q}^U_{V,W}(Y_1, \dots, Y_n) &= \widetilde{\widetilde{p}}_{U(Q,V,W)}(\widetilde{\Xi}_{V,W}(X_1, \dots, X_n))
\end{aligned}$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, as desired.

G.5 Proof of Theorem 118

Let us denote by $C_a : \mathbf{2} \longrightarrow \mathbf{I}$ the constant mapping

$$C_a(x) = a$$

for all $x \in \{0, 1\}$. Then clearly

$$b_a(0) = 0 \leq a = C_a(0) \qquad \text{and}$$
$$b_a(1) = a = C_a(1)\,,$$

i.e. $b_a \leq C_a$. Hence by Def. 52 and Th-8, $\widetilde{\widetilde{b}}_a \leq \widetilde{\widetilde{C}}_a$, i.e. for all $x \in \mathbf{I}$,

$$\widetilde{\widetilde{b}}_a(x) = \widetilde{\widetilde{C}}_a(x) = a \tag{678}$$

by Th-1 and Th-6. Similarly $b_a(0) = 0 = \mathrm{id}_2(0)$ and $b_a(1) = a \leq 1 = \mathrm{id}_2(1)$ and hence $\widetilde{\widetilde{b}}_a \leq \widetilde{\widetilde{\mathrm{id}}}_2$, i.e.

$$\widetilde{\widetilde{b}}_a(x) \leq \widetilde{\widetilde{\mathrm{id}}}_2(x) = \mathcal{F}(\pi_1)(\widetilde{\eta}(x)) = \widetilde{\pi}_1(\widetilde{\eta}(x)) = x \tag{679}$$

by Def. 52, (Z-2) and L-133. Combining (678) and (679), i.e. $\widetilde{\widetilde{b}}_a(x) \leq a$ and $\widetilde{\widetilde{b}}_a(x) \leq x$, we deduce

$$\widetilde{\widetilde{b}}_a(x) \leq \min(a, x)\,, \tag{680}$$

for all $x \in \mathbf{I}$. Considering $\widetilde{\widetilde{p}}_a$, we have

$$\begin{aligned}
\widetilde{\widetilde{p}}_a(x) &= \widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{b}}_{\widetilde{\widetilde{\neg}}\,a}(x) && \text{by L-134}\\
&\geq \widetilde{\widetilde{\neg}}\min(\widetilde{\widetilde{\neg}}\,a, x) && \text{by (680), } \widetilde{\widetilde{\neg}} \text{ nonincreasing}\\
&= \max(\widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\,a, \widetilde{\widetilde{\neg}}\,x) && \text{because } \widetilde{\widetilde{\neg}} \text{ nonincreasing}\\
&= \max(a, \widetilde{\widetilde{\neg}}\,x)\,, && \text{because } \widetilde{\widetilde{\neg}} \text{ involution}
\end{aligned}$$

for all $a, x \in \mathbf{I}$, as desired.

G.6 Proof of Theorem 119

Lemma 140
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing one-place quantifier. Then for all $V, W \in \mathcal{P}(E)$ where $V \subseteq W$,

$$L(Q, V, W) = Q(V)$$
$$U(Q, V, W) = Q(W)\,.$$

Proof Let $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ be a monadic nondecreasing semi-fuzzy quantifier, i.e.

$$Q(Y) \leq Q(Y')$$

whenever $Y, Y' \in \mathcal{P}(E)$ such that $Y \subseteq Y'$. Now suppose that $V, W \in \mathcal{P}(E)$, $V \subseteq W$ are given. Then for all $Y \in \mathcal{P}(E)$ such that $V \subseteq Y \subseteq W$, $Q(V) \leq Q(Y) \leq Q(W)$, because Q is nondecreasing. Hence

$$\begin{aligned} L(Q, V, W) &= \inf\{Q(Y) : V \subseteq Y \subseteq W\} && \text{by Def. 96} \\ &= Q(V) \end{aligned}$$

because $Q(V) \leq Q(Y)$ for all $Y \in \{Y : V \subseteq Y \subseteq W\}$, and because $V \in \{Y : V \subseteq Y \subseteq W\}$. Similarly

$$\begin{aligned} U(Q, V, W) &= \sup\{Q(Y) : V \subseteq Y \subseteq W\} && \text{by Def. 96} \\ &= Q(W) \end{aligned}$$

because $Q(Y) \leq Q(W)$ for all $Y \in \{Y : V \subseteq Y \subseteq W\}$ and because $W \in \{Y : V \subseteq Y \subseteq W\}$.

Lemma 141
Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing one-place quantifier, and $\mathcal{F}$ is a DFS. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned} \widetilde{Q}^L(X) &= \sup\{\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,E}(X)) : V \in \mathcal{P}(E)\} \\ \widetilde{Q}^U(X) &= \inf\{\widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\Xi}_{\varnothing,W}(X)) : W \in \mathcal{P}(E)\} \,. \end{aligned}$$

Proof Let $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ be a nondecreasing one-place quantifier, and let $\mathcal{F}$ be a given DFS. For all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned} \widetilde{Q}^L(X) &= \sup\{\widetilde{Q}^L_{V,W}(X) : V, W \in \mathcal{P}(E), V \subseteq W\} && \text{by Def. 99} \\ &= \sup\{\widetilde{\widetilde{b}}_{L(Q,V,W)}(\widetilde{\Xi}_{V,W}(X)) : V, W \in \mathcal{P}(E), V \subseteq W\} && \text{by Th-117} \\ &= \sup\{\sup\{\widetilde{\widetilde{b}}_{L(Q,V,W)}(\widetilde{\Xi}_{V,W}(X)) : W \in \mathcal{P}(E),\ V \subseteq W\} : V \in \mathcal{P}(E)\} \\ &= \sup\{\sup\{\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,W}(X)) : W \in \mathcal{P}(E),\ V \subseteq W\} : V \in \mathcal{P}(E)\} && \text{by L-140} \\ &= \sup\{\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,E}(X)) : V \in \mathcal{P}(E)\} \end{aligned}$$

where the last step holds because $b_{Q(V)}(x)$ is nondecreasing in its argument, and by Th-6 and Def. 52, it follows that $\widetilde{\widetilde{b}}_{Q(V)}(x)$ is nondecreasing in x, too. Similarly, $\Xi_{V,W} \leq \Xi_{V,W'}$ whenever $W \subseteq W'$, and by Th-8, $\widetilde{\Xi}_{V,W} \leq \widetilde{\Xi}_{V,W'}$ whenever $W \subseteq W'$. Hence $\widetilde{\Xi}_{V,W}(X) \leq \widetilde{\Xi}_{V,E}(X)$ for all $W \in \mathcal{P}(E)$, $V \subseteq W$. In turn because $\widetilde{\widetilde{b}}_{Q(V)}(x)$ is nondecreasing in x, $\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,W}(X)) \leq \widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,E}(X))$ for all $W \in \mathcal{P}(E)$, $V \subseteq W$, and hence

$$\sup\{\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,W}(X)) : W \in \mathcal{P}(E),\ V \subseteq W\} = \widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,E}(X)) \,,$$

for all $V \in \mathcal{P}(E)$.
In the case of $\widetilde{Q}^U$, we obtain by analogous reasoning that

$$\begin{aligned}
\widetilde{Q}^L(X) &= \inf\{\widetilde{Q}^U_{V,W}(X) : V, W \in \mathcal{P}(E), V \subseteq W\} && \text{by Def. 99}\\
&= \inf\{\widetilde{\widetilde{p}}_{U(Q,V,W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X)) : V, W \in \mathcal{P}(E), V \subseteq W\} && \text{by Th-117}\\
&= \inf\{\inf\{\widetilde{\widetilde{p}}_{U(Q,V,W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X)) : V \in \mathcal{P}(E),\ V \subseteq W\} : W \in \mathcal{P}(E)\}\\
&= \inf\{\inf\{\widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X)) : V \in \mathcal{P}(E),\ V \subseteq W\} : W \in \mathcal{P}(E)\} && \text{by L-140}\\
&= \inf\{\widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\widetilde{\Xi}}_{\varnothing,W}(X)) : W \in \mathcal{P}(E)\}\,.
\end{aligned}$$

To see that the last step holds, we observe that $\Xi_{\varnothing,W} \geq \Xi_{V,W}$ for all $V \subseteq W$. Hence by Th-8, $\widetilde{\widetilde{\Xi}}_{\varnothing,W} \geq \widetilde{\widetilde{\Xi}}_{V,W}$ for all $V \subseteq W$. Because $p_{Q(W)}(x)$ is nonincreasing in its argument, $\widetilde{\widetilde{p}}_{Q(W)}(x)$ is nonincreasing in its argument, too (see Def. 52 and Th-6). $\widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X))$ is hence minimized by $\widetilde{\widetilde{\Xi}}_{\varnothing,W}$, i.e.

$$\inf\{\widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\widetilde{\Xi}}_{V,W}(X)) : V \subseteq W\} = \widetilde{\widetilde{p}}_{Q(W)}(\widetilde{\widetilde{\Xi}}_{\varnothing,W}(X))\,.$$

Lemma 142
Suppose $E \neq \varnothing$ is a base set, $k \in \mathbb{N}$, $\mathcal{F}$ a DFS and $X \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\widetilde{[\geq k]}^L(X) &= \sup\{\mathcal{F}(\forall)(X \mathbin{\widetilde{\widetilde{\cup}}} \neg V) : V \in \mathcal{P}(E),\ |V| = k\}\\
\widetilde{[\geq k]}^U(X) &= \inf\{\mathcal{F}(\exists)(X \mathbin{\widetilde{\widetilde{\cap}}} \neg W) : W \in \mathcal{P}(E),\ |W| < k\}\\
&= \inf\{\mathcal{F}(\exists)(X \mathbin{\widetilde{\widetilde{\cap}}} \neg W) : W \in \mathcal{P}(E),\ |W| = k-1\}
\end{aligned}$$

Proof Let $E \neq \varnothing$ a given base set. Further assume that a DFS $\mathcal{F}$, a choice of $k \in \mathbb{N}$, and a fuzzy argument set X are given. Considering the lower bound $\widetilde{[\geq k]}^L$, we compute:

$$\begin{aligned}
\widetilde{[\geq k]}^L(X) &= \sup\{\widetilde{\widetilde{b}}_{[\geq k](V)}(\widetilde{\widetilde{\Xi}}_{V,E}(X)) : V \in \mathcal{P}(E)\} && \text{by L-141}\\
&= \sup\{\widetilde{\widetilde{b}}_1(\widetilde{\widetilde{\Xi}}_{V,E}(X)) : V \in \mathcal{P}(E),\ [\geq k](V) = 1\} && \text{because } [\geq k] \text{ two-valued}\\
&= \sup\{\widetilde{\widetilde{\Xi}}_{V,E}(X) : V \in \mathcal{P}(E),\ [\geq k](V) = 1\} && \text{because } b_1 = \mathrm{id}_2\text{, i.e. } \widetilde{\widetilde{b}}_1 = \mathrm{id}_{\mathbf{I}}\\
&= \sup\{\widetilde{\widetilde{\Xi}}_{V,E}(X) : V \in \mathcal{P}(E),\ |V| \geq k\} && \text{by Def. 103}\\
&= \sup\{\mathcal{F}(\forall)(X \mathbin{\widetilde{\widetilde{\cup}}} \neg V) : V \in \mathcal{P}(E),\ |V| \geq k\} && \text{by Th-116}\\
&= \sup\{\mathcal{F}(\forall)(X \mathbin{\widetilde{\widetilde{\cup}}} \neg V) : V \in \mathcal{P}(E),\ |V| = k\}\,,
\end{aligned}$$

where the last step holds because $\forall(X \cup \neg V) \leq \forall(X \cup \neg V')$ whenever $V' \subseteq V$, a property which transfers to $\mathcal{F}(\forall)$ by (Z-4), (DFS 7) and Th-8.

Similarly in the case of the upper bound $\widetilde{[\geq k]}^{U}$,

$$\begin{aligned}
&\widetilde{[\geq k]}^{U}(X) \\
&= \inf\{\widetilde{\widetilde{p}}_{[\geq k](W)}(\widetilde{\Xi}_{\varnothing,W}(X)) : W \in \mathcal{P}(E)\} && \text{by L-141} \\
&= \inf\{\widetilde{\widetilde{p}}_{0}(\widetilde{\Xi}_{\varnothing,W}(X)) : W \in \mathcal{P}(E),\ [\geq k](W) = 0\} && \text{because } [\geq k] \text{ two-valued} \\
&= \inf\{\widetilde{\widetilde{\neg}}\, \widetilde{\Xi}_{\varnothing,W}(X) : W \in \mathcal{P}(E),\ [\geq k](W) = 0\} && \text{because } p_0 = \neg \\
&= \inf\{\widetilde{\widetilde{\neg}}\, \widetilde{\Xi}_{\varnothing,W}(X) : W \in \mathcal{P}(E),\ |W| < k\} && \text{by Def. 103} \\
&= \inf\{\widetilde{\widetilde{\neg}}\, \mathcal{F}(\forall)((X \,\widetilde{\widetilde{\cup}}\, (W \setminus \varnothing)) \,\widetilde{\widetilde{\triangle}}\, \neg W) : W \in \mathcal{P}(E),\ |W| < k\} && \text{by Th-116} \\
&= \inf\{\widetilde{\widetilde{\neg}}\, \mathcal{F}(\forall)((\widetilde{\widetilde{\neg}}\, X) \,\widetilde{\widetilde{\cup}}\, W) : W \in \mathcal{P}(E),\ |W| < k\} \\
&= \inf\{\mathcal{F}(\exists)(\widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\, X \,\widetilde{\widetilde{\cup}}\, W)) : W \in \mathcal{P}(E),\ |W| < k\} && \text{by (Z-3), } \exists = \forall\square \\
&= \inf\{\mathcal{F}(\exists)(X \,\widetilde{\widetilde{\cap}}\, \neg W) : W \in \mathcal{P}(E),\ |W| < k\} && \text{by De Morgan's law} \\
&= \inf\{\mathcal{F}(\exists)(X \,\widetilde{\widetilde{\cap}}\, \neg W) : W \in \mathcal{P}(E),\ |W| = k-1\}\,,
\end{aligned}$$

where the last step holds because in the crisp case, $\exists(Y \cap \neg W) \leq \exists(Y \cap \neg W')$ for all $Y, W, W' \in \mathcal{P}(E)$ such that $W' \subseteq W$. This carries over to $\mathcal{F}(\exists)$ by Th-8 and (DFS 6), (DFS 7).

Proof of Theorem 119

Let us first consider $\widetilde{[\geq k]}^{L}(X)$. By lemma L-142,

$$\widetilde{[\geq k]}^{L}(X) = \sup\{\mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V) : V \in \mathcal{P}(E),\ |V| = k\} \tag{681}$$

Because $E = \{v_1, \ldots, v_m\}$ is finite, we conclude from theorem Th-24 that

$$\mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V) = c_1 \,\widetilde{\widetilde{\wedge}} \cdots \widetilde{\widetilde{\wedge}}\, c_k \tag{682}$$

where $V = \{c_1, \ldots, c_k\}$, $|V| = k$, i.e. the c_j are pairwise distinct.
Because $|E| = m$, we can order the elements of E in such a way that $E = \{e_1, \ldots, e_m\}$, $\mu_X(e_1) \geq \mu_X(e_2) \geq \cdots \geq \mu_X(e_m)$. It is apparent from equation (682) and the monotonicity of the t-norm $\widetilde{\widetilde{\wedge}}$ that $\mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V)$ is maximised by $V' = \{e_1, \ldots, e_k\}$, i.e.

$$\begin{aligned}
\widetilde{[\geq k]}^{L}(X) &= \sup\{\mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V) : V \in \mathcal{P}(E),\ |V| = k\} && \text{by (681)} \\
&= \mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V') \\
&= \mu_X(e_1) \,\widetilde{\widetilde{\wedge}} \cdots \widetilde{\widetilde{\wedge}}\, \mu_X(e_k) \\
&= \mu_{[1]}(X) \,\widetilde{\widetilde{\wedge}} \cdots \widetilde{\widetilde{\wedge}}\, \mu_{[k]}(X)\,. && \text{by Def. 102}
\end{aligned}$$

In the case of the upper bound $\widetilde{[\geq k]}^U(X)$, can again utilise that E is finite, $|E| = m$.

$$\begin{aligned}
&\widetilde{[\geq k]}^U(X) \\
&= \inf\{\mathcal{F}(\exists)(X \,\widetilde{\widetilde{\cap}}\, \neg W) : W \in \mathcal{P}(E),\ |W| = k-1\} && \text{by L-142} \\
&= \inf\{\mathcal{F}(\exists)(X \,\widetilde{\widetilde{\cap}}\, Z) : Z \in \mathcal{P}(E),\ |Z| = m-k+1)\} && \text{substituting } Z = \neg W \\
&= \inf\{z_k \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, z_m : Z = \{z_k, \dots, z_m\} \in \mathcal{P}(E),\ |Z| = m-k+1\} && \text{by Th-25} \\
&= \mu_X(e_k) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \mu_X(e_m) \\
&= \mu_{[k]}(X) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \mu_{[m]}(X)\,,
\end{aligned}$$

where again $E = \{e_1, \dots, e_m\}$, $\mu_X(e_1) \geq \mu_X(e_2) \geq \cdots \geq \mu_X(e_m)$, i.e. $\mu_{[j]}(X) = \mu_X(e_j)$ for all $j = 1, \dots, m$, cf. Def. 102. It is then apparent from monotonicity considerations that $z_k \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, z_m$ is minimized by $Z = \{e_k, \dots, e_m\}$.

G.7 Proof of Theorem 120

Assume $E \neq \varnothing$ is a finite base set of cardinality $|E| = m$, $k \in \mathbb{N}$, $\mathcal{F}$ is a DFS, and $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy argument set. We shall abbreviate

$$\beta = \sup\{\alpha \in \mathbf{I} : |(X)_{\geq\alpha}| \geq k\}\,. \tag{683}$$

Now let $\varepsilon > 0$ and choose $\gamma \in (\beta - \varepsilon, \beta)$. Then $|(X)_{\geq\gamma}| \geq k$ by (683). Hence there exist $e_1, \dots, e_k \in E$ such that $|\{e_1, \dots, e_k\}| = k$, i.e. the e_j's are pairwise distinct, and

$$\mu_X(e_j) \geq \gamma \tag{684}$$

for all $j = 1, \dots, k$. Therefore

$$\begin{aligned}
&\widetilde{[\geq k]}^L(X) \\
&= \sup\{\mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg V) : |V| = k\} && \text{by L-142} \\
&\geq \mathcal{F}(\forall)(X \,\widetilde{\widetilde{\cup}}\, \neg\{e_1, \dots, e_k\}) \\
&= \min\{\mu_X(e_j) : j = 1, \dots, k\} && \text{by Th-24 and } \widetilde{\widetilde{\wedge}} = \min \text{ because } \mathcal{F} \text{ standard DFS} \\
&\geq \gamma && \text{by (684)} \\
&> \beta - \varepsilon\,. && \text{by choice of } \gamma
\end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily, we deduce that $\widetilde{[\geq k]}^L(X) \geq \beta$. Hence by Th-115,

$$\mathcal{F}([\geq k])(X) \geq \beta\,. \tag{685}$$

To see that the converse inequation holds, let again $\varepsilon > 0$ and choose some $\gamma \in (\beta, \beta + \varepsilon)$. Then by the definition of β, $|(X)_{\geq\gamma}| < k$. Hence let $W' = (X)_{\geq\gamma}$. Then

$$\begin{aligned}
&\widetilde{[\geq k]}^U(X) \\
&= \inf\{\mathcal{F}(\exists)(X \cap \neg W) : W \in \mathcal{P}(E),\ |W| < k\} && \text{by L-142 and } \widetilde{\widetilde{\cap}} = \cap \text{ because } \mathcal{F} \text{ std-DFS} \\
&\leq \mathcal{F}(\exists)(X \cap \neg W') && \text{because } |W'| < k \\
&= \mathcal{F}(\exists)(X \cap \neg(X)_{\geq\gamma})\,. && \text{by definition of } W'
\end{aligned}$$

Clearly $e \notin (X)_{\geq\gamma}$ iff $\mu_X(e) < \gamma$, hence $e \in \neg(X)_{\geq\gamma}$ iff $\mu_X(e) < \gamma$, and

$$\mu_{X\cap\neg(X)_{\geq\gamma}}(e) = \begin{cases} \mu_X(e) & : \quad \mu_X(e) < \gamma \\ 0 & : \quad \mu_X(e) \geq \gamma \end{cases} \tag{686}$$

We conclude that

$$\begin{aligned} \widetilde{[\geq k]}^U(X) &\leq \mathcal{F}(\exists)(X \cap \neg(X)_{\geq\gamma}) \\ &= \sup\{\mu_{X\cap\neg(X)_{\geq\gamma}}(e) : e \in E\} && \text{by Th-25 and } \mathcal{F} \text{ std-DFS} \\ &\leq \gamma && \text{by (686)} \\ &< \beta + \varepsilon\,. && \text{by choice of } \gamma \in (\beta, \beta+\varepsilon) \end{aligned}$$

Because $\varepsilon > 0$ was arbitrarily chosen, this means that $\widetilde{[\geq k]}^U(X) \leq \beta$. Combining this with inequation (685), we obtain

$$\begin{aligned} \beta &\leq \mathcal{F}([\geq k])(X) && \text{by (685)} \\ &\leq \widetilde{[\geq k]}^U(X) && \text{by Th-115} \\ &\leq \beta\,, \end{aligned}$$

i.e. $\mathcal{F}([\geq k])(X) = \beta = \sup\{\alpha \in \mathbf{I} : |(X)_{\geq\alpha}| \geq k\}$, see equation (683).
The claim about $\mathcal{F}([\geq k])(X)$ in the case of finite base sets $E \neq \varnothing$, $|E| = m$ is apparent from Th-119. We then have

$$\begin{aligned} \mathcal{F}([\geq k])(X) &\geq \mu_{[1]}(X) \,\widetilde{\widetilde{\wedge}} \cdots \widetilde{\widetilde{\wedge}}\, \mu_{[k]}(X) && \text{by Th-119} \\ &= \min\{\mu_{[1]}(X), \ldots, \mu_{[k]}(X)\} && \text{because } \mathcal{F} \text{ std-DFS} \\ &= \mu_{[k]}(X)\,. && \text{apparent from Def. 102} \end{aligned}$$

Similarly

$$\begin{aligned} \mathcal{F}([\geq k])(X) &\leq \mu_{[k]}(X) \,\widetilde{\widetilde{\vee}} \cdots \widetilde{\widetilde{\vee}}\, \mu_{[m]}(X) && \text{by Th-119} \\ &= \max\{\mu_{[k]}(X), \ldots, \mu_{[m]}(X)\} && \text{because } \mathcal{F} \text{ std-DFS} \\ &= \mu_{[k]}(X)\,. && \text{apparent from Def. 102} \end{aligned}$$

Hence $\mathcal{F}([\geq k])(X) = \mu_{[k]}(X)$, as desired.

G.8 Proof of Theorem 121

Suppose $\mathcal{F}$ is a QFM which satisfies conditions a. to d. of the theorem. We shall further assume that $\mathcal{F}$ is compatible with fuzzy argument insertion. Let us first consider b_a, where $a \in \mathbf{I}$. Clearly

$$\begin{aligned} b_a(x) &= \begin{cases} 0 & : \quad x = 0 \\ a & : \quad x = 1 \end{cases} && \text{by Def. 100} \\ &= x \,\widetilde{\widetilde{\wedge}}\, a && \text{by conditions a. and b.} \\ &= \widetilde{\pi}_1(\eta(x) \,\widetilde{\widetilde{\cup}}\, \widetilde{\eta}(a)) && \text{apparent from Def. 7, Def. 50,Def. 51} \\ &= \mathcal{F}(\pi_1)(\eta(x) \,\widetilde{\widetilde{\cup}}\, \widetilde{\eta}(a)) && \text{by condition d.} \\ &= \mathcal{F}(\pi_1 \cap)(\eta(x), \widetilde{\eta}(a)) && \text{by condition c.} \\ &= \mathcal{F}(\pi_1 \cap) \,\widetilde{\triangleleft}\, \widetilde{\eta}(a)(\eta(x))\,, && \text{by Def. 89} \end{aligned}$$

i.e.

$$Q_{b_a} = \mathcal{F}(\pi_1 \cap) \mathbin{\widetilde{\triangleleft}} \widetilde{\eta}(a)\,. \tag{687}$$

Hence for all $x \in \mathbf{I}$,

$$\begin{aligned}
\widetilde{\widetilde{b}}_a(x) &= \mathcal{F}(Q_{b_a})(\widetilde{\eta}(x)) && \text{by Def. 52}\\
&= \mathcal{F}(\mathcal{F}(\pi_1 \cap) \mathbin{\widetilde{\triangleleft}} \widetilde{\eta}(a))(\widetilde{\eta}(x)) && \text{by (687)}\\
&= \mathcal{F}(\pi_1 \cap)(\widetilde{\eta}(x), \widetilde{\eta}(a)) && \text{by assumption on } \mathcal{F} \text{ (fuzzy argument insertion)}\\
&= \widetilde{\pi}_1(\widetilde{\eta}(x) \mathbin{\widetilde{\widetilde{\cap}}} \widetilde{\eta}(a)) && \text{by assumptions c. and d.}\\
&= x \mathbin{\widetilde{\widetilde{\wedge}}} a\,. && \text{by Def. 7, Def. 51}
\end{aligned}$$

Considering $\widetilde{\widetilde{p}}_a(x)$, we can apply lemma L-134 to conclude that

$$\begin{aligned}
\widetilde{\widetilde{p}}_a(x) &= \widetilde{\widetilde{\neg}}\, \widetilde{\widetilde{b}}_{\widetilde{\widetilde{\neg}} a}(x) && \text{by L-134}\\
&= \widetilde{\widetilde{\neg}}(x \mathbin{\widetilde{\widetilde{\wedge}}} \widetilde{\widetilde{\neg}}\, a) && \text{by first part of theorem, see above}\\
&= \widetilde{\widetilde{\neg}}(\widetilde{\widetilde{\neg}}\,\widetilde{\widetilde{\neg}}\, x \mathbin{\widetilde{\widetilde{\wedge}}} \widetilde{\widetilde{\neg}}\, a) && \text{by condition e., } \widetilde{\widetilde{\neg}} \text{ involution}\\
&= \widetilde{\widetilde{\neg}}\, x \mathbin{\widetilde{\widetilde{\vee}}} a\,. && \text{by condition f.}
\end{aligned}$$

G.9 Proof of Theorem 122

Lemma 143
Let a base set $E \neq \varnothing$ be given, $n \in \mathbb{N}$ and $V, W \in \mathcal{P}(E)^n$ such that $V_i \subseteq W_i$ for all $i = 1, \ldots, n$. In every standard DFS $\mathcal{F}$,

$$\begin{aligned}
\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) &= \min\{\widetilde{\Xi}_{V_i,W_i}(X_i) : i = 1, \ldots, n\}\\
\widetilde{\Xi}_{V_i,W_i}(X_i) &= \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\})\,,
\end{aligned}$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof Because $\mathcal{F}$ is a standard DFS, we have $\widetilde{\widetilde{\mathcal{F}}}(\wedge) = \min$, $\widetilde{\widetilde{\mathcal{F}}}(\neg) = \neg$, where $\neg x = 1 - x$ for all $x \in \mathbf{I}$, see Def. 49. Hence by Th-116,

$$\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) = \min\{\mathcal{F}(\forall)(Z_i) : i = 1, \ldots, n\}$$

where the $Z_i \in \widetilde{\mathcal{P}}(E)$ are defined by

$$\mu_{Z_i}(e) = \begin{cases} \mu_{X_i}(e) & : \quad e \in V_i \\ 1 & : \quad e \in W_i \setminus V_i \\ 1 - \mu_{X_i}(e) & : \quad e \notin W_i \end{cases} \tag{688}$$

for all $e \in E$. Hence for all $i = 1, \ldots, n$,

$$\begin{aligned}
&\widetilde{\Xi}_{V_i,W_i}(X_i) \\
&= \mathcal{F}(\forall)(Z_i) \\
&= \inf\{\mu_{Z_i}(e) : e \in E\} && \text{by Th-24, } \widetilde{\widetilde{\mathcal{F}}}(\wedge) = \min \\
&= \min\{\inf\{\mu_{Z_i}(e) : e \in V_i\}, \inf\{\mu_{Z_i}(e) : e \in W_i \setminus V_i\}, \\
&\qquad \inf\{\mu_{Z_i}(e) : e \notin W_i\}\} \\
&= \min\{\inf\{\mu_{X_i}(e) : e \in V_i\}, 1, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}\} && \text{by (688)} \\
&= \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) \,. && \text{because } 1 \text{ identity of } \min
\end{aligned}$$

Lemma 144

Suppose $E \neq \varnothing$ is a base set, $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. In addition, suppose that $\mathcal{F}$ is a standard DFS. Then for all $\gamma \in \mathbf{I}$ and $Y \in \mathcal{P}(E)^n$ such that $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$,

$$\widetilde{\Xi}_{Y,Y}(X_1, \ldots, X_n) \geq \tfrac{1}{2} - \tfrac{1}{2}\gamma \,.$$

Proof

Case a.: $\gamma = 0$.

Let us recall that by Def. 66, $(X_i)_0^{\min} = (X)_{>\frac{1}{2}}$ and $(X_i)_0^{\max} = (X_i)_{\geq\frac{1}{2}}$. Because $Y_i \in \mathcal{T}_0(X_i)$, we know that $Y_i \subseteq (X_i)_0^{\max}$, i.e. $Y_i \subseteq (X_i)_{\geq\frac{1}{2}}$. Hence if $e \in Y_i$, then

$$\mu_{X_i}(e) \geq \tfrac{1}{2} \tag{689}$$

by Def. 64. In addition, because $Y_i \in \mathcal{T}_0(X_i)$, we know that $(X_i)_{>\frac{1}{2}} = (X_i)_0^{\min} \subseteq Y_i$. Hence if $e \notin Y_i$, then $e \notin (X_i)_{>\frac{1}{2}}$ and by Def. 65, $\mu_{X_i}(e) \leq \frac{1}{2}$, i.e.

$$1 - \mu_{X_i}(e) \geq \tfrac{1}{2} \,. \tag{690}$$

Therefore

$$\begin{aligned}
&\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) \\
&= \min\{\min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) : i = 1, \ldots, n\} && \text{by L-143} \\
&\geq \tfrac{1}{2} \,. && \text{by (689), (690)}
\end{aligned}$$

Case b.: $\gamma > 0$.

Similarly if $\gamma > 0$, $(X_i)_\gamma^{\min} = (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma}$ and $(X_i)_\gamma^{\max} = (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma}$ by Def. 66. Because $Y_i \in \mathcal{T}_\gamma(X_i)$, $Y_i \subseteq (X_i)_\gamma^{\max}$, i.e. $Y_i \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma}$. Hence if $e \in Y_i$, then

$$\mu_{X_i}(e) > \tfrac{1}{2} - \tfrac{1}{2}\gamma \tag{691}$$

by Def. 65. In addition, because $Y_i \in \mathcal{T}_\gamma(X_i)$, we know that $(X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} = (X_i)^{\min}_\gamma \subseteq Y_i$. Hence if $e \notin Y_i$, then $e \notin (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma}$ and by Def. 64, $\mu_{X_i}(e) < \frac{1}{2} + \frac{1}{2}\gamma$, i.e.

$$1 - \mu_{X_i}(e) > \tfrac{1}{2} - \tfrac{1}{2}\gamma\,. \tag{692}$$

Therefore

$$\begin{aligned}
&\widetilde{\Xi}_{V,W}(X_1, \dots, X_n) \\
&= \min\{\min(\inf\{\mu_X(e) : e \in Y\}, \inf\{1 - \mu_X(e) : e \notin Y\}) : i = 1, \dots, n\} && \text{by L-143} \\
&\geq \tfrac{1}{2} - \tfrac{1}{2}\gamma\,. && \text{by (691), (692)}
\end{aligned}$$

Lemma 145
Suppose $E \neq \varnothing$ is a base set, $n \in \mathbb{N}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. In addition, suppose that $\mathcal{F}$ is a standard DFS. Then for all $\gamma \in \mathbf{I}$,

$$\widetilde{\Xi}_{V,W}(X_1, \dots, X_n) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\,.$$

where $V, W \in \mathcal{P}(E)^n$ are defined by $V_i = (X)^{\min}_\gamma$, $W_i = (X)^{\max}_\gamma$ for $i = 1, \dots, n$.

Proof

Case a.: $\gamma = 0$**.**
If $\gamma = 0$, then $(X_i)^{\min}_0 = (X_i)_{>\frac{1}{2}}$ and $(X_i)^{\max}_0 = (X_i)_{\geq\frac{1}{2}}$ by Def. 66. Hence if $e \in V_i = (X_i)^{\min}_0 = (X_i)_{>\frac{1}{2}}$, then

$$\mu_{X_i}(e) > \tfrac{1}{2}\,. \tag{693}$$

In addition, we have $W_i = (X_i)^{\max}_0 = (X)_{\geq\frac{1}{2}}$, i.e. $e \notin W_i$ iff $\mu_{X_i}(e) < \frac{1}{2}$ by Def. 64. Hence if $e \notin W_i$, then

$$1 - \mu_{X_i}(e) > \tfrac{1}{2}\,. \tag{694}$$

Therefore

$$\begin{aligned}
\widetilde{\Xi}_{V_i,W_i}(X_i) &= \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) && \text{by L-143} \\
&\geq \tfrac{1}{2} && \text{by (693), (694)}
\end{aligned}$$

and in turn,

$$\begin{aligned}
\widetilde{\Xi}_{V,W}(X_1, \dots, X_n) &= \min\{\widetilde{\Xi}_{V_i,W_i}(X_i) : i = 1, \dots, n\} && \text{by L-143} \\
&\geq \tfrac{1}{2}\,.
\end{aligned}$$

Case b.: $\gamma > 0$.
If $\gamma > 0$, then $(X_i)^{\min}_\gamma = (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma}$ and $(X_i)^{\max}_\gamma = (X_i)_{> \frac{1}{2}-\frac{1}{2}\gamma}$ by Def. 66. Hence if $e \in V_i = (X_i)^{\min}_\gamma = (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma}$, then

$$\mu_{X_i}(e) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\,. \tag{695}$$

In addition, we have $W_i = (X_i)^{\max}_\gamma = (X)_{> \frac{1}{2}-\frac{1}{2}\gamma}$, i.e. $e \notin W_i$ iff $\mu_{X_i}(e) \leq \frac{1}{2} - \frac{1}{2}\gamma$ by Def. 65. Hence if $e \notin W_i$, then

$$1 - \mu_{X_i}(e) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\,. \tag{696}$$

Therefore

$$\begin{aligned} \widetilde{\widetilde{\Xi}}_{V_i,W_i}(X_i) &= \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\}) && \text{by L-143} \\ &\geq \tfrac{1}{2} + \tfrac{1}{2}\gamma && \text{by (695), (696)} \end{aligned}$$

and in turn,

$$\begin{aligned} \widetilde{\widetilde{\Xi}}_{V,W}(X_1, \dots, X_n) &= \min\{\widetilde{\widetilde{\Xi}}_{V_i,W_i}(X_i) : i = 1, \dots, n\} && \text{by L-143} \\ &\geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\,. \end{aligned}$$

Definition 108
For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, all $Z_1, \dots, Z_n \in \widetilde{\mathcal{P}}(E)$ and all $\gamma \in \mathbf{I}$, the semi-fuzzy quantifiers $Q^L_{\gamma,(Z_1,\dots,Z_n)}, Q^U_{\gamma,(Z_1,\dots,Z_n)} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are defined by

$$Q^L_{\gamma,(Z_1,\dots,Z_n)}(Y_1, \dots, Y_n) = \begin{cases} Q_\gamma(Y_1, \dots, Y_n) & : \; Y_i \in \mathcal{T}_\gamma(Z_i),\ \textit{for all } i = 1, \dots, n \\ 0 & : \; \textit{else} \end{cases}$$

$$Q^U_{\gamma,(Z_1,\dots,Z_n)}(Y_1, \dots, Y_n) = \begin{cases} Q_\gamma(Y_1, \dots, Y_n) & : \; Y_i \in \mathcal{T}_\gamma(Z_i),\ \textit{for all } i = 1, \dots, n \\ 1 & : \; \textit{else} \end{cases}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. If $\mathcal{F}$ is a QFM, we abbreviate

$$\begin{aligned} \widetilde{Q}^L_{\gamma,(Z_1,\dots,Z_n)} &= \mathcal{F}(Q^L_{\gamma,(Z_1,\dots,Z_n)}) \\ \widetilde{Q}^U_{\gamma,(Z_1,\dots,Z_n)} &= \mathcal{F}(Q^U_{\gamma,(Z_1,\dots,Z_n)})\,. \end{aligned}$$

Lemma 146
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $Z_1, \dots, Z_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. Let us define $V, W \in \mathcal{P}(E)^n$ by $V_i = (Z_i)^{\min}_\gamma$, $W_i = (Z_i)^{\max}_\gamma$ for all $i = 1, \dots, n$.

a. If $Q_\gamma(Z_1, \dots, Z_n) > \frac{1}{2}$, then $Q^L_{\gamma,(Z_1,\dots,Z_n)} = Q^L_{V,W}$.

b. If $Q_\gamma(Z_1, \dots, Z_n) < \frac{1}{2}$, then $Q^U_{\gamma,(Z_1,\dots,Z_n)} = Q^U_{V,W}$.

Proof

a. If $Q_\gamma(Z_1, \ldots, Z_n) > \frac{1}{2}$, then by (14),

$$Q_\gamma(Z_1, \ldots, Z_n) = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(Z_1, \ldots, Z_n), Q_\gamma^{\max}(Z_1, \ldots, Z_n)) > \tfrac{1}{2}$$

and hence

$$Q_\gamma(Z_1, \ldots, Z_n) = Q_\gamma^{\min}(Z_1, \ldots, Z_n) \tag{697}$$

by Def. 45 because $\frac{1}{2} < Q_\gamma^{\min}(Z_1, \ldots, Z_n) \le Q_\gamma^{\max}(Z_1, \ldots, Z_n)$. If we define $V, W \in \mathcal{P}(E)^n$ by $V_i = (Z_i)_\gamma^{\min}$, $W_i = (Z_i)_\gamma^{\max}$ for all $i = 1, \ldots, n$, we hence have

$$\begin{aligned}
&Q_\gamma(Z_1, \ldots, Z_n) \\
&= Q_\gamma^{\min}(Z_1, \ldots, Z_n) && \text{by (697)} \\
&= \inf\{Q(Y_1, \ldots, Y_n) : (Z_i)_\gamma^{\min} \subseteq Y_i \subseteq (Z_i)_\gamma^{\max},\ i = 1, \ldots, n\} && \text{by (15), Def. 66} \\
&= \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ i = 1, \ldots, n\} && \text{by definition of } V, W \in \mathcal{P}(E)^n \\
&= L(Q, V, W)\,, && \text{by Def. 96}
\end{aligned}$$

i.e.

$$Q_\gamma(Z_1, \ldots, Z_n) = L(Q, V, W)\,. \tag{698}$$

Hence for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\begin{aligned}
&Q^L_{\gamma,(Z_1,\ldots,Z_n)}(Y_1, \ldots, Y_n) \\
&= \begin{cases} Q_\gamma(Y_1, \ldots, Y_n) & : \ Y_i \in \mathcal{T}_\gamma(Z_i),\ \text{for all } i = 1, \ldots, n \\ 0 & : \ \text{else} \end{cases} && \text{by Def. 108} \\
&= \begin{cases} L(Q, V, W) & : \ V_i \subseteq Y_i \subseteq W_i \text{ for all } i = 1, \ldots, n \\ 0 & : \ \text{else} \end{cases} && \text{by (698), choice of } V, W \in \mathcal{P}(E)^n \\
&= Q^L_{V,W}(Y_1, \ldots, Y_n)\,. && \text{by Def. 97}
\end{aligned}$$

b. $Q_\gamma(Z_1, \ldots, Z_n) < \frac{1}{2}$. The proof of this case is analogous to that of **a.**: If $Q_\gamma(Z_1, \ldots, Z_n) < \frac{1}{2}$, then by (14), $Q_\gamma(Z_1, \ldots, Z_n) = \mathrm{m}_{\frac{1}{2}}(Q_\gamma^{\min}(Z_1, \ldots, Z_n), Q_\gamma^{\max}(Z_1, \ldots, Z_n)) < \frac{1}{2}$ and hence

$$Q_\gamma(Z_1, \ldots, Z_n) = Q_\gamma^{\max}(Z_1, \ldots, Z_n) \tag{699}$$

by Def. 45 because $\frac{1}{2} > Q_\gamma^{\max}(Z_1, \ldots, Z_n) \ge Q_\gamma^{\min}(Z_1, \ldots, Z_n)$. If we define $V, W \in \mathcal{P}(E)^n$ by $V_i = (Z_i)_\gamma^{\min}$, $W_i = (Z_i)_\gamma^{\max}$ for all $i = 1, \ldots, n$, we hence have

$$\begin{aligned}
&Q_\gamma(Z_1, \ldots, Z_n) \\
&= Q_\gamma^{\max}(Z_1, \ldots, Z_n) && \text{by (699)} \\
&= \sup\{Q(Y_1, \ldots, Y_n) : (Z_i)_\gamma^{\min} \subseteq Y_i \subseteq (Z_i)_\gamma^{\max},\ i = 1, \ldots, n\} && \text{by (16), Def. 66} \\
&= \sup\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ i = 1, \ldots, n\} && \text{by definition of } V, W \in \mathcal{P}(E)^n \\
&= U(Q, V, W)\,, && \text{by Def. 96}
\end{aligned}$$

i.e.

$$Q_\gamma(Z_1, \ldots, Z_n) = U(Q, V, W)\,. \tag{700}$$

Hence for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,

$$\begin{aligned}
&Q^U_{\gamma,(Z_1,\ldots,Z_n)}(Y_1, \ldots, Y_n) \\
&= \begin{cases} Q_\gamma(Y_1, \ldots, Y_n) & : \ Y_i \in \mathcal{T}_\gamma(Z_i), \text{ for all } i = 1, \ldots, n \\ 1 & : \ \text{else} \end{cases} && \text{by Def. 108} \\
&= \begin{cases} U(Q, V, W) & : \ V_i \subseteq Y_i \subseteq W_i \text{ for all } i = 1, \ldots, n \\ 1 & : \ \text{else} \end{cases} && \text{by (700), choice of } V, W \in \mathcal{P}(E)^n \\
&= Q^U_{V,W}(Y_1, \ldots, Y_n)\,. && \text{by Def. 97}
\end{aligned}$$

Lemma 147

Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, a choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, and $\gamma \in \mathbf{I}$ be given. We shall assume that $\mathcal{M}_{CX}$ is used, i.e. $\widetilde{Q}^L_{\gamma,(X_1,\ldots,X_n)} = \mathcal{M}_{CX}(Q^L_{\gamma,(X_1,\ldots,X_n)})$ and $\widetilde{Q}^U_{\gamma,(X_1,\ldots,X_n)} = \mathcal{M}_{CX}(Q^U_{\gamma,(X_1,\ldots,X_n)})$. If $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$, then $\widetilde{Q}^L_{\gamma,(X_1,\ldots,X_n)}(X_1, \ldots, X_n) \geq \min(Q_\gamma(X_1, \ldots, X_n), \frac{1}{2} + \frac{1}{2}\gamma)$.

Proof

Suppose that $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$. Then

$$Q^L_{\gamma,(X_1,\ldots,X_n)} = Q^L_{V,W}\,, \tag{701}$$

where $V, W \in \mathcal{P}(E)^n$ are defined by by $V_i = (X_i)^{\min}_\gamma$, $W_i = (X_i)^{\max}_\gamma$ for all $i = 1, \ldots, n$. In addition, we can apply lemma L-145 to yield that

$$\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\,. \tag{702}$$

Therefore

$$\begin{aligned}
&\widetilde{Q}^L_{\gamma,(X_1,\ldots,X_n)}(X_1, \ldots, X_n) \\
&= \mathcal{M}_{CX}(Q^L_{\gamma,(X_1,\ldots,X_n)})(X_1, \ldots, X_n) && \text{by Def. 108} \\
&= \min(Q_\gamma(X_1, \ldots, X_n), \widetilde{\Xi}_{V,W}(X_1, \ldots, X_n)) && \text{by L-139, Th-121, Th-102, Def. 108} \\
&\geq \min(Q_\gamma(X_1, \ldots, X_n), \tfrac{1}{2} + \tfrac{1}{2}\gamma)\,. && \text{by (702)}
\end{aligned}$$

Proof of Theorem 122

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We shall discern the cases that $Q_0(X_1, \ldots, X_n) = \frac{1}{2}$, $Q_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $Q_0(X_1, \ldots, X_n) < \frac{1}{2}$.

Case a.: $Q_0(X_1,\dots,X_n) = \frac{1}{2}$**.**
From (14), we know that $Q_0(X_1,\dots,X_n) = \mathrm{m}_{\frac{1}{2}}(Q_0^{\min}(X_1,\dots,X_n), Q_0^{\max}(X_1,\dots,X_n)) = \frac{1}{2}$. It is then apparent from Def. 45 and $Q_0^{\min}(X_1,\dots,X_n) \le Q_0^{\max}(X_1,\dots,X_n)$ that $Q_0^{\min}(X_1,\dots,X_n) \le \frac{1}{2}$ and $Q_0^{\max}(X_1,\dots,X_n) \ge \frac{1}{2}$. Now let $\varepsilon > 0$. We conclude from $Q_0^{\max}(X_1,\dots,X_n) \ge \frac{1}{2}$, i.e.

$$\sup\{Q(Y_1,\dots,Y_n) : Y_i \in \mathcal{T}_0(X_i)\} \ge \tfrac{1}{2}\,,$$

that there exists $Y \in \mathcal{P}(E)^n$ such that $Y_1 \in \mathcal{T}_0(X_1),\dots,Y_n \in \mathcal{T}_0(X_n)$ and

$$Q(Y_1,\dots,Y_n) > \tfrac{1}{2} - \varepsilon\,. \tag{703}$$

We can apply L-144 to deduce that

$$\widetilde{\Xi}_{Y,Y}(X_1,\dots,X_n) \ge \tfrac{1}{2}\,. \tag{704}$$

Hence

$$\begin{aligned}
&\widetilde{Q}^L(X_1,\dots,X_n)\\
&= \sup\{\widetilde{Q}^L_{V,W}(X_1,\dots,X_n) : V,W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i,\dots,V_n \subseteq W_n\} && \text{by Def. 99}\\
&\ge \widetilde{Q}^L_{Y,Y}(X_1,\dots,X_n)\\
&= \min(L(Q,Y,Y), \widetilde{\Xi}_{Y,Y}(X_1,\dots,X_n)) && \text{by Th-117, Th-121, Th-102}\\
&= \min(Q(Y_1,\dots,Y_n), \widetilde{\Xi}_{Y,Y}(X_1,\dots,X_n)) && \text{by Def. 96}\\
&\ge \min(Q(Y_1,\dots,Y_n), \tfrac{1}{2}) && \text{by (704)}\\
&> \tfrac{1}{2} - \varepsilon\,, && \text{by (703)}
\end{aligned}$$

i.e. $\widetilde{Q}^L(X_1,\dots,X_n) > \frac{1}{2} - \varepsilon$. Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that

$$\widetilde{Q}^L(X_1,\dots,X_n) \ge \tfrac{1}{2}\,. \tag{705}$$

In turn, we deduce that

$$\begin{aligned}
&\widetilde{Q}^U(X_1,\dots,X_n)\\
&= \neg\neg\widetilde{Q}^U(X_1,\dots,X_n) && \text{because } \neg x = 1 - x \text{ involution}\\
&= \neg\widetilde{(\neg Q)}^L(X_1,\dots,X_n) && \text{by L-129}\\
&\le \neg\tfrac{1}{2} && \text{from (705) because } (\neg Q)_0(X_1,\dots,X_n) = \neg Q_0(X_1,\dots,X_n) = \tfrac{1}{2}\\
&= \tfrac{1}{2}\,.
\end{aligned}$$

Hence by Th-115,

$$\tfrac{1}{2} \le \widetilde{Q}^L(X_1,\dots,X_n) \le \widetilde{Q}^U(X_1,\dots,X_n) \le \tfrac{1}{2}\,,$$

i.e. $\widetilde{Q}^L(X_1,\dots,X_n) = \widetilde{Q}^U(X_1,\dots,X_n) = \frac{1}{2}$. Again from Th-115,

$$\tfrac{1}{2} = \widetilde{Q}^L(X_1,\dots,X_n) \le \mathcal{M}_{CX}(Q)(X_1,\dots,X_n) \le \widetilde{Q}^U(X_1,\dots,X_n) = \tfrac{1}{2}\,,$$

i.e. $\mathcal{M}_{CX}(Q)(X_1,\dots,X_n) = \widetilde{Q}^L(X_1,\dots,X_n) = \widetilde{Q}^U(X_1,\dots,X_n) = \frac{1}{2}$, as desired.

Case b.: $Q_0(X_1, \ldots, X_n) > \frac{1}{2}$.
Let us first consider $\widetilde{Q}^L(X_1, \ldots, X_n)$. We shall abbreviate

$$\gamma^* = 2\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) - 1\,. \qquad (706)$$

Now let $\varepsilon > 0$ be given, and let $\gamma \in (\gamma^* - \varepsilon, \gamma^*)$. Then by (23) and Th-94,

$$Q_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} + \tfrac{1}{2}\gamma \geq \tfrac{1}{2}\,. \qquad (707)$$

Hence

$$\begin{aligned}
&\widetilde{Q}^L(X_1, \ldots, X_n) \\
&= \sup\{\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i, \ldots, V_n \subseteq W_n\} && \text{by Def. 99} \\
&\geq \widetilde{Q}^L_{V,W}(X_1, \ldots, X_n), \quad \text{where } V_i = (X_i)^{\min}_\gamma,\ W_i = (X_i)^{\max}_\gamma,\ i = 1, \ldots, n \\
&= \widetilde{Q}^L_{\gamma,(X_1,\ldots,X_n)}(X_1, \ldots, X_n) && \text{by L-146} \\
&\geq \min(Q_\gamma(X_1, \ldots, X_n), \tfrac{1}{2} + \tfrac{1}{2}\gamma) && \text{by L-147} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\gamma && \text{by (707)} \\
&> \tfrac{1}{2} + \tfrac{1}{2}(\gamma^* - \varepsilon) && \text{because } \gamma \in (\gamma^* - \varepsilon, \gamma^*) \\
&= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* - \tfrac{\varepsilon}{2} \\
&= \tfrac{1}{2} + \tfrac{1}{2}(2\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) - 1) - \tfrac{\varepsilon}{2} && \text{by (706)} \\
&= \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) - \tfrac{\varepsilon}{2}\,.
\end{aligned}$$

Recalling that $\varepsilon > 0$ was chosen arbitrarily, we conclude that

$$\widetilde{Q}^L(X_1, \ldots, X_n) \geq \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)\,.$$

Conversely, we know that $\widetilde{Q}^L(X_1, \ldots, X_n) \leq \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)$ by Th-115, and hence

$$\widetilde{Q}^L(X_1, \ldots, X_n) = \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)\,,$$

as desired.
Next we shall discuss $\widetilde{Q}^U(X_1, \ldots, X_n)$. We will show that for all $\gamma \in \mathbf{I}$,

$$\widetilde{Q}^U(X_1, \ldots, X_n) \leq \max(Q_\gamma(X_1, \ldots, X_n), \tfrac{1}{2} + \tfrac{1}{2}\gamma)\,. \qquad (708)$$

Hence let $\gamma \in \mathbf{I}$. We shall discern two cases.
If $Q_\gamma(X_1, \ldots, X_n) > \frac{1}{2} + \frac{1}{2}\gamma \geq \frac{1}{2}$, then in particular

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}(Q^{\min}_\gamma(X_1, \ldots, X_n), Q^{\max}_\gamma(X_1, \ldots, X_n)) > \tfrac{1}{2}$$

by (14), i.e.

$$Q_\gamma(X_1, \ldots, X_n) = Q^{\min}_\gamma(X_1, \ldots, X_n) = \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} > \tfrac{1}{2} + \tfrac{1}{2}\gamma \qquad (709)$$

by Def. 45, (15) and noting that $Q^{\min}_\gamma(X_1, \ldots, X_n) \leq Q^{\max}_\gamma(X_1, \ldots, X_n)$. Hence for all $Y \in \mathcal{P}(E)^n$ such that $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$,

$$Q(Y_1, \ldots, Y_n) > \tfrac{1}{2} + \tfrac{1}{2}\gamma\,. \qquad (710)$$

In turn,

$$\begin{aligned}\widetilde{Q}^U_{Y,Y}(X_1,\dots,X_n) &= \max(1-\widetilde{\Xi}_{Y,Y}(X_1,\dots,X_n), Q(Y_1,\dots,Y_n)) && \text{by Th-121, Th-117, Th-102}\\ &= Q(Y_1,\dots,Y_n)\,, && \text{by L-144, (710)}\end{aligned}$$

i.e.

$$\widetilde{Q}^U_{Y,Y}(X_1,\dots,X_n) = Q(Y_1,\dots,Y_n)\,. \tag{711}$$

Therefore

$$\begin{aligned}&\widetilde{Q}^U(X_1,\dots,X_n)\\ &= \inf\{\widetilde{Q}^U_{V,W}(X_1,\dots,X_n) : V,W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1,\dots,V_n \subseteq W_n\} && \text{by Def. 99}\\ &\le \inf\{\widetilde{Q}^U_{Y,Y}(X_1,\dots,X_n) : Y \in \mathcal{P}(E)^n, Y_i \in \mathcal{T}_\gamma(X_i), \text{all } i\}\\ &= \inf\{Q(Y_1,\dots,Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} && \text{by (711)}\\ &= Q_\gamma(X_1,\dots,X_n) && \text{by (709)}\\ &= \max(Q_\gamma(X_1,\dots,X_n), \tfrac{1}{2}+\tfrac{1}{2}\gamma)\,,\end{aligned}$$

where the last step holds because of the assumption that $Q_\gamma(X_1,\dots,X_n) > \frac{1}{2}+\frac{1}{2}\gamma$. Hence equation (708) holds whenever $Q_\gamma(X_1,\dots,X_n) > \frac{1}{2}+\frac{1}{2}\gamma$.
In the remaining case that $Q_\gamma(X_1,\dots,X_n) \le \frac{1}{2}+\frac{1}{2}\gamma$,

$$Q_\gamma(X_1,\dots,X_n) = \max(\tfrac{1}{2}, \inf\{Q(Y_1,\dots,Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} \tag{712}$$

because $Q_0(X_1,\dots,X_n) > \frac{1}{2}$, see L-88.
Hence let $\varepsilon > 0$. By (712), there exists a choice of $Y' \in \mathcal{P}(E)^n$, $Y'_1 \in \mathcal{T}_\gamma(X_1),\dots,Y'_n \in \mathcal{T}_\gamma(X_n)$, such that

$$Q(Y'_1,\dots,Y'_n) < Q_\gamma(X_1,\dots,X_n) + \varepsilon \le \tfrac{1}{2}+\tfrac{1}{2}\gamma+\varepsilon\,. \tag{713}$$

Hence

$$\begin{aligned}&\widetilde{Q}^U(X_1,\dots,X_n)\\ &= \inf\{\widetilde{Q}^U_{V,W}(X_1,\dots,X_n) : V,W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1,\dots,V_n \subseteq W_n\} && \text{by Def. 99}\\ &\le \widetilde{Q}^U_{Y',Y'}(X_1,\dots,X_n)\\ &= \max(1-\widetilde{\Xi}_{Y',Y'}(X_1,\dots,X_n), Q(Y'_1,\dots,Y'_n)) && \text{by Th-121, Th-117, Th-102}\\ &\le \tfrac{1}{2}+\tfrac{1}{2}\gamma+\varepsilon\,. && \text{by (713), L-144}\end{aligned}$$

Because $\varepsilon > 0$ was chosen arbitrarily, we deduce that

$$\begin{aligned}\widetilde{Q}^U(X_1,\dots,X_n) &\le \tfrac{1}{2}+\tfrac{1}{2}\gamma\\ &= \max(Q_\gamma(X_1,\dots,X_n), \tfrac{1}{2}+\tfrac{1}{2}\gamma)\,,\end{aligned}$$

by the assumption that $Q_\gamma(X_1,\dots,X_n) \le \frac{1}{2}+\frac{1}{2}\gamma$.
Hence equation (708) holds for all $\gamma \in \mathbf{I}$, i.e.

$$\begin{aligned}\widetilde{Q}^U(X_1,\dots,X_n) &\le \inf\{\max(Q_\gamma(X_1,\dots,X_n), \tfrac{1}{2}+\tfrac{1}{2}\gamma) : \gamma \in \mathbf{I}\}\\ &= \mathcal{M}_{CX}(Q)(X_1,\dots,X_n)\,. && \text{by (23), Th-94}\end{aligned}$$

Conversely, we already know by Th-115 that $\mathcal{M}_{CX}(Q)(X_1,\dots,X_n) \le \widetilde{Q}^U(X_1,\dots,X_n)$. Hence

$$\mathcal{M}_{CX}(Q)(X_1,\dots,X_n) = \widetilde{Q}^U(X_1,\dots,X_n)\,.$$

Case c.: $Q_0(X_1, \ldots, X_n) < \frac{1}{2}$**.** Then

$$\begin{aligned}
&\widetilde{Q}^L(X_1, \ldots, X_n) \\
&= \neg\neg\widetilde{Q}^L(X_1, \ldots, X_n) && \text{because } \neg x = 1 - x \text{ involution} \\
&= \neg\widetilde{(\neg Q)}^U(X_1, \ldots, X_n) && \text{by L-129} \\
&= \neg\mathcal{M}_{CX}(\neg Q)(X_1, \ldots, X_n) && \text{by part b. because } (\neg Q)_0(X_1, \ldots, X_n) = 1 - Q_0(X_1, \ldots, X_n) > \tfrac{1}{2} \\
&= \neg\neg\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) && \text{by (DFS 3)} \\
&= \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) && \text{because } \neg \text{ involutive}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\widetilde{Q}^U(X_1, \ldots, X_n) \\
&= \neg\neg\widetilde{Q}^U(X_1, \ldots, X_n) && \text{because } \neg x = 1 - x \text{ involution} \\
&= \neg\widetilde{(\neg Q)}^L(X_1, \ldots, X_n) && \text{by L-129} \\
&= \neg\mathcal{M}_{CX}(\neg Q)(X_1, \ldots, X_n) && \text{by part b. because } (\neg Q)_0(X_1, \ldots, X_n) = 1 - Q_0(X_1, \ldots, X_n) > \tfrac{1}{2} \\
&= \neg\neg\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) && \text{by (DFS 3)} \\
&= \mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) && \text{because } \neg \text{ involutive.}
\end{aligned}$$

G.10 Proof of Theorem 123

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We already know that

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) \\
&\widetilde{Q}^L(X_1, \ldots, X_n) && \text{by Th-122} \\
&= \sup\{\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by Def. 99}
\end{aligned}$$

and similarly

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n) \\
&= \widetilde{Q}^U(X_1, \ldots, X_n) && \text{by Th-122} \\
&= \inf\{\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\,. && \text{by Def. 99}
\end{aligned}$$

The claim of the theorem that

$$\begin{aligned}
&\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) \\
&= \min(\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i, \text{ all } i\})
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n) \\
&= \max(1 - \widetilde{\Xi}_{V,W}(X_1, \ldots, X_n), \sup\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i, \text{ all } i\})
\end{aligned}$$

is apparent from Def. 96, Th-121, Th-117 and Th-102.
Finally, we know from L-143 that

$$\begin{aligned}&\widetilde{\Xi}_{V,W}(X_1,\dots,X_n)\\&=\min_{i=1}^{n}\min(\inf\{\mu_X(e):e\in V_i\},\inf\{1-\mu_X(e):e\notin W_i\})\,.\end{aligned}$$

because $\mathcal{M}_{CX}$ is a standard DFS by Th-93.

G.11 Proof of Theorem 124

Suppose the standard DFS $\mathcal{F}$ is compatible with fuzzy argument insertion and let $Q:\mathcal{P}(E)^n\longrightarrow\mathbf{I}$ be a semi-fuzzy quantifier and $X_1,\dots,X_n\in\widetilde{\mathcal{P}}(E)$ a choice of fuzzy arguments. Let us first observe that for all $V,W\in\mathcal{P}(E)^n$ such that $V_1\subseteq W_1,\dots,V_n\subseteq W_n$,

$$\mathcal{F}(\Xi_{V,W})(X_1,\dots,X_n)=\min_{i=1}^{n}\min(\inf\{\mu_X(e):e\in V_i\},\inf\{1-\mu_X(e):e\notin W_i\}) \tag{714}$$

by Th-93. Hence by (714) and Th-123,

$$\mathcal{F}(\Xi_{V,W})(X_1,\dots,X_n)=\mathcal{M}_{CX}(\Xi_{V,W})(X_1,\dots,X_n)\,. \tag{715}$$

In turn,

$$\begin{aligned}&\mathcal{F}(Q^L_{V,W})(X_1,\dots,X_n)\\&=\min(L(Q,V,W),\mathcal{F}(\widetilde{\Xi}_{V,W})(X_1,\dots,X_n)) && \text{by Th-117, Th-121}\\&=\min(L(Q,V,W),\mathcal{M}_{CX}(\widetilde{\Xi}_{V,W})(X_1,\dots,X_n)) && \text{by (715)}\\&=\mathcal{M}_{CX}(Q^L_{V,W})(X_1,\dots,X_n)\,, && \text{by Th-117, Th-121, Th-102}\end{aligned}$$

i.e.

$$\mathcal{F}(Q^L_{V,W})(X_1,\dots,X_n)=\mathcal{M}_{CX}(Q^L_{V,W})(X_1,\dots,X_n)\,. \tag{716}$$

Therefore

$$\begin{aligned}&\mathcal{F}(Q)\\&\geq\sup\{\mathcal{F}(Q^L_{V,W}):V,W\in\mathcal{P}(E)^n,\ V_i\subseteq W_i,\dots,V_n\subseteq W_n\} && \text{by Def. 99, Th-115}\\&=\sup\{\mathcal{M}_{CX}(Q^L_{V,W}):V,W\in\mathcal{P}(E)^n,\ V_i\subseteq W_i,\dots,V_n\subseteq W_n\} && \text{by (716)}\\&=\mathcal{M}_{CX}(Q) && \text{by Def. 99, Th-122}\,,\end{aligned}$$

i.e.

$$\mathcal{F}(Q)\geq\mathcal{M}_{CX}(Q)\,. \tag{717}$$

Analogously,

$$\begin{aligned}&\mathcal{F}(Q)\\&\leq\inf\{\mathcal{F}(Q^U_{V,W}):V,W\in\mathcal{P}(E)^n,\ V_1\subseteq W_1,\dots,V_n\subseteq W_n\} && \text{by Def. 99, Th-115}\\&=\neg\sup\{\mathcal{F}((\neg Q)^L_{V,W}):V,W\in\mathcal{P}(E)^n,\ V_1\subseteq W_1,\dots,V_n\subseteq W_n\} && \text{by L-129}\\&=\neg\sup\{\mathcal{M}_{CX}((\neg Q)^L_{V,W}):V,W\in\mathcal{P}(E)^n,\ V_1\subseteq W_1,\dots,V_n\subseteq W_n\} && \text{by (716)}\\&=\neg\mathcal{M}_{CX}(\neg Q) && \text{by Def. 99, Th-122}\\&=\neg\neg\mathcal{M}_{CX}(Q) && \text{by (DFS 3)}\\&=\mathcal{M}_{CX}(Q)\,, && \text{because }\neg\text{ involutive}\end{aligned}$$

i.e.

$$\mathcal{F}(Q) \leq \mathcal{M}_{CX}(Q)\,. \tag{718}$$

Combining (717) and (718), we obtain the desired result that $\mathcal{F}(Q) = \mathcal{M}_{CX}(Q)$.

G.12 Proof of Theorem 125

In order to prove the theorem, it is apparently sufficient to show that every standard DFS coincides with $\mathcal{M}_{CX}$ on two-valued quantifiers, because $\mathcal{M}_{CX}$ is known to be a standard DFS by Th-93. Hence let $\mathcal{F}$ be a standard DFS, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ a two-valued quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We first observe that for all $V, W \in \mathcal{P}(E)^n$ such that $V_1 \subseteq W_1, \ldots, V_n \subseteq W_n$,

$$\mathcal{F}(\Xi_{V,W})(X_1, \ldots, X_n) = \min_{i=1}^{n} \min(\inf\{\mu_X(e) : e \in V_i\}, \inf\{1 - \mu_X(e) : e \notin W_i\}) \tag{719}$$

by Th-93. Hence by (719) and Th-123,

$$\mathcal{F}(\Xi_{V,W})(X_1, \ldots, X_n) = \mathcal{M}_{CX}(\Xi_{V,W})(X_1, \ldots, X_n)\,. \tag{720}$$

Because Q is two-valued, it is apparent from Def. 96 and Def. 97 that $L(Q, V, W)$ and $Q^L_{V,W}$ are two-valued as well. If $L(Q, V, W) = 1$, then

$$\begin{aligned} Q^L_{V,W}(Y_1, \ldots, Y_n) &= b_1(\Xi_{V,W}(Y_1, \ldots, Y_n)) && \text{by L-136 and } L(Q,V,W) = 1 \\ &= \Xi_{V,W}(Y_1, \ldots, Y_n) && \text{by Def. 100, } b_1 = \mathrm{id}_2 \end{aligned}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, i.e.

$$Q^L_{V,W} = \Xi_{V,W}\,. \tag{721}$$

Hence

$$\begin{aligned} \mathcal{F}(Q^L_{V,W}) &= \mathcal{F}(\Xi_{V,W}) && \text{by (721)} \\ &= \mathcal{M}_{CX}(\Xi_{V,W}) && \text{by (720)} \\ &= \mathcal{M}_{CX}(Q^L_{V,W}) && \text{by (721)} \end{aligned}$$

i.e.

$$\mathcal{F}(Q^L_{V,W})(X_1, \ldots, X_n) = \mathcal{M}_{CX}(Q^L_{V,W})(X_1, \ldots, X_n)\,. \tag{722}$$

whenever $L(Q, V, W) = 1$. In the remaining case that $L(Q, V, W) = 0$, we apparently have $Q^L_{V,W} = 0$ and hence

$$\mathcal{F}(Q^L_{V,W})(X_1, \ldots, X_n) = 0 = \mathcal{M}_{CX}(Q^L_{V,W})(X_1, \ldots, X_n) \tag{723}$$

by Th-1 and Th-6. Therefore

$$\begin{aligned} &\mathcal{F}(Q) \\ &\geq \sup\{\mathcal{F}(Q^L_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i, \ldots, V_n \subseteq W_n\} && \text{by Def. 99, Th-115} \\ &= \sup\{\mathcal{M}_{CX}(Q^L_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_i \subseteq W_i, \ldots, V_n \subseteq W_n\} && \text{by (722), (723)} \\ &= \mathcal{M}_{CX}(Q) && \text{by Def. 99, Th-122} \end{aligned}$$

i.e.

$$\mathcal{F}(Q) \geq \mathcal{M}_{CX}(Q)\,. \tag{724}$$

Analogously,

$$\begin{aligned}
&\mathcal{F}(Q) \\
&\leq \inf\{\mathcal{F}(Q^U_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by Def. 99, Th-115} \\
&= \neg \sup\{\mathcal{F}((\neg Q)^L_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by L-129} \\
&= \neg \sup\{\mathcal{M}_{CX}((\neg Q)^L_{V,W}) : V, W \in \mathcal{P}(E)^n,\ V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\} && \text{by (722), (723)} \\
&= \neg \mathcal{M}_{CX}(\neg Q) && \text{by Def. 99, Th-122} \\
&= \neg\neg \mathcal{M}_{CX}(Q) && \text{by (DFS 3)} \\
&= \mathcal{M}_{CX}(Q)\,, && \text{because } \neg \text{ involutive}
\end{aligned}$$

i.e.

$$\mathcal{F}(Q) \leq \mathcal{M}_{CX}(Q)\,. \tag{725}$$

Combining (724) and (725), we obtain the desired result that $\mathcal{F}(Q) = \mathcal{M}_{CX}(Q)$.

G.13 Proof of Theorem 126

Let a nondecreasing semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ and a fuzzy argument set $X \in \widetilde{\mathcal{P}}(E)$ be given. For all $\alpha \in \mathbf{I}$, we shall abbreviate

$$\mathcal{V}_\alpha = \{V \in \mathcal{P}(E) : \inf\{\mu_X(e) : e \in V\} \geq \alpha\}\,. \tag{726}$$

It is apparent from Def. 64 that $V \in \mathcal{V}_\alpha$ entails

$$V \subseteq (X)_{\geq\alpha}\,. \tag{727}$$

In addition, we clearly have

$$(X)_{\geq\alpha} \in \mathcal{V}_\alpha\,. \tag{728}$$

Because Q is nondecreasing and $V \subseteq (X)_{\geq\alpha}$ by (727),

$$Q(V) \leq Q((X)_{\geq\alpha})$$

for all $V \in \mathcal{V}_\alpha$, i.e.

$$\sup\{Q(V) : V \in \mathcal{V}_\alpha\} \leq Q((X)_{\geq\alpha})\,. \tag{729}$$

Because $(X)_{\geq\alpha} \in \mathcal{V}_\alpha$ by (728), we also have

$$\sup\{Q(V) : V \in \mathcal{V}_\alpha\} \geq Q((X)_{\geq\alpha})\,. \tag{730}$$

Combining (729) and (730),

$$\sup\{Q(V) : V \in \mathcal{V}_\alpha\} = Q((X)_{\geq\alpha})\,. \tag{731}$$

Hence for all $\alpha \in \mathbf{I}$,

$$\begin{aligned} &\sup\{\min(\alpha, Q(V)) : V \in \mathcal{V}_\alpha\} \\ &= \min(\alpha, \sup\{Q(V) : V \in \mathcal{V}_\alpha\}) && \text{by distributivity of } \min, \sup \\ &= \min(\alpha, Q((X)_{\geq\alpha}))\,, && \text{by (731)} \end{aligned}$$

i.e.

$$\sup\{\min(\alpha, Q(V)) : V \in \mathcal{V}_\alpha\} = \min(\alpha, Q((X)_{\geq\alpha}))\,. \tag{732}$$

Therefore

$$\begin{aligned} &\mathcal{M}_{CX}(Q)(X) \\ &= \widetilde{Q}^L(X) && \text{by Th-122} \\ &= \sup\{\widetilde{\widetilde{b}}_{Q(V)}(\widetilde{\Xi}_{V,E}(X)) : V \in \mathcal{P}(E)\} && \text{by L-141} \\ &= \sup\{\min(Q(V), \inf\{\mu_X(e) : e \in V\}) : V \in \mathcal{P}(E)\} && \text{by Th-121,} \\ & && \text{Th-102, and L-143} \\ &= \sup\{\sup\{\min(\alpha, Q(V)) : V \in \mathcal{P}(E),\ \alpha \leq \inf\{\mu_X(e) : e \in V\}\} : \alpha \in \mathbf{I}\} \\ &= \sup\{\sup\{\min(\alpha, Q(V)) : V \in \mathcal{V}_\alpha\} : \alpha \in \mathbf{I}\} && \text{by (726)} \\ &= \sup\{\min(\alpha, Q((X)_{\geq\alpha})) : \alpha \in \mathbf{I}\} && \text{by (732)} \\ &= (S)\int X\, dQ\,. && \text{by Def. 104} \end{aligned}$$

G.14 Proof of Theorem 127

Suppose $\Omega \subset \mathbf{I}$, $Q : \mathcal{P}(E)^n \longrightarrow \Omega$ and $X_1, \ldots, X_n$ satisfy the conditions of the theorem, i.e.

$$\Omega \text{ is finite} \tag{733}$$

$$\omega \in \Omega \implies 1 - \omega \in \Omega \tag{734}$$

$$\{0,\, 1\} \subseteq \Omega \tag{735}$$

$$\mu_{X_i}(e) \in \Omega, \qquad \text{for all } i = 1, \ldots, n,\, e \in E\,. \tag{736}$$

Let us first consider an arbitrary choice of $V, W \in \mathcal{P}(E)^n$ such that $V_1 \subseteq W_1, \ldots, V_n \subseteq W_n$. By Th-123,

$$\widetilde{\Xi}_{V,W}(X_1, \ldots, X_n) = \min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\})\,. \tag{737}$$

Clearly $\{\mu_{X_i}(e) : e \in V_i\} \subseteq \Omega$ by (736). Because Ω is finite, $\{\mu_{X_i}(e) : e \in V_i\}$ is finite as well. If $\{\mu_{X_i}(e) : e \in V_i\} = \varnothing$, then $\inf\{\mu_{X_i}(e) : e \in V_i\} = \inf \varnothing = 1 \in \Omega$ by (735). If $\{\mu_{X_i}(e) : e \in V_i\} \neq \varnothing$, then

$$\begin{aligned} \inf\{\mu_{X_i}(e) : e \in V_i\} &= \min\{\mu_{X_i}(e) : e \in V_i\} \\ &\in \{\mu_{X_i}(e) : e \in V_i\} && \text{because of finiteness} \\ &\in \Omega\,. && \text{because } \{\mu_{X_i}(e) : e \in V_i\} \subseteq \Omega \end{aligned}$$

By analogous reasoning and the negation closure (734) of Ω, we also have

$$\inf\{1-\mu_{X_i}(e) : e \notin W_i\} \in \Omega\,.$$

Hence from (737),

$$\widetilde{\Xi}_{V,W}(X_1,\dots,X_n) = \min_{i=1}^{n} \min(a_i,b_i) \in \Omega\,, \tag{738}$$

because from our previous reasoning, $a_i = \inf\{\mu_{X_i}(e) : e \in V_i\} \in \Omega$ and similarly $b_i = \inf\{1 - \mu_{X_i}(e) : e \notin W_i\} \in \Omega$ for all $i = 1,\dots,n$.
Let us now consider $\inf\{Q(Y_1,\dots,Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}$. Because $Q : \mathcal{P}(E)^n \longrightarrow \Omega$ is Ω-valued, clearly

$$C = \{Q(Y_1,\dots,Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\} \subseteq \Omega$$

If $C = \varnothing$, then $\inf C = 1 \in \Omega$ by (735). If $C \neq \varnothing$, then $C \subseteq \Omega$ is a nonempty finite subset of Ω and hence $\inf C = \min C \in C$, i.e. $\inf C \in \Omega$. In any case,

$$\inf\{Q(Y_1,\dots,Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\} \in \Omega \tag{739}$$

Combining Th-123, (738) and (739), we conclude that

$$\widetilde{Q}^L_{V,W}(X_1,\dots,X_n) = \min(\widetilde{\Xi}_{V,W}(X_1,\dots,X_n), \inf\{Q(Y_1,\dots,Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}) \in \Omega\,. \tag{740}$$

Finally, we know from Th-123 that

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X_1,\dots,X_n)\\
&= \sup\{\widetilde{Q}^L_{V,W}(X_1,\dots,X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1,\dots,V_n \subseteq W_n\}\\
&= \sup S\,,
\end{aligned}$$

where we have abbreviated

$$S = \sup\{\widetilde{Q}^L_{V,W}(X_1,\dots,X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1,\dots,V_n \subseteq W_n\}$$

If $S = \varnothing$, then $\mathcal{M}_{CX}(Q)(X_1,\dots,X_n) = \sup S = \sup\varnothing = 0 \in \Omega$. In the remaining case that $S \neq \varnothing$, we know from (740) that S is a nonempty subset of the finite set Ω. Hence S is finite, too, and $\sup S = \max S \in S$, i.e. $\mathcal{M}_{CX}(Q)(X_1,\dots,X_n) = \sup S \in \Omega$, as desired.

G.15 Proof of Theorem 128

Suppose $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping with the properties required by the theorem, i.e.

$$\sigma \text{ is a bijection} \tag{741}$$

$$\sigma \text{ is nondecreasing} \tag{742}$$

$$\sigma(1-x) = 1-\sigma(x), \qquad \text{for all } x \in \mathbf{I}. \tag{743}$$

We have to show that $\mathcal{M}_{\mathcal{CX}^\sigma} = \mathcal{M}_{CX}$. Hence let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ be given. By Def. 41,

$$\mathcal{M}_{\mathcal{CX}^\sigma}(Q)(X_1,\dots,X_n) = \sigma^{-1}\mathcal{M}_{CX}(\sigma Q)(\sigma X_1,\dots,\sigma X_n)\,.$$

Let us remark that being a nondecreasing bijection, σ is in fact *increasing*, i.e.

$$\sigma(x) < \sigma(y)$$

whenever $x < y$. In addition, σ is apparently *continuous*.
Now let $V, W \in \mathcal{P}(E)^n$ such that $V_1 \subseteq W_1, \ldots, V_n \subseteq W_n$. Then

$$\begin{aligned}
&\widetilde{\widetilde{\Xi}}_{V,W}(\sigma X_1, \ldots, \sigma X_n) \\
&= \min_{i=1}^{n} \min(\inf\{\sigma\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \sigma\mu_{X_i}(e) : e \notin W_i\}) && \text{see Th-123} \\
&= \min_{i=1}^{n} \min(\inf\{\sigma\mu_{X_i}(e) : e \in V_i\}, \inf\{\sigma(1 - \mu_{X_i}(e)) : e \notin W_i\}) && \text{by (743)} \\
&= \min_{i=1}^{n} \min(\sigma(\inf\{\mu_{X_i}(e) : e \in V_i\}), \sigma(\inf\{1 - \mu_{X_i}(e) : e \notin W_i\})) && \text{by (742) and continuity of } \sigma \\
&= \sigma(\min_{i=1}^{n} \min(\inf\{\mu_{X_i}(e) : e \in V_i\}, \inf\{1 - \mu_{X_i}(e) : e \notin W_i\})) && \text{by (742)} \\
&= \sigma(\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n))\,, && \text{see Th-123}
\end{aligned}$$

i.e.

$$\widetilde{\widetilde{\Xi}}_{V,W}(\sigma X_1, \ldots, \sigma X_n) = \sigma(\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n))\,. \tag{744}$$

Furthermore

$$\begin{aligned}
&\inf\{\sigma Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\} \\
&= \sigma(\inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\})
\end{aligned} \tag{745}$$

because σ is nondecreasing and continuous. Therefore

$$\begin{aligned}
&\widetilde{(\sigma Q)}^L_{V,W}(\sigma X_1, \ldots, \sigma X_n) \\
&= \min(\widetilde{\widetilde{\Xi}}_{V,W}(\sigma X_1, \ldots, \sigma X_n), \inf\{\sigma Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}) && \text{see Th-123} \\
&= \min(\sigma\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n), \sigma \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}) && \text{by (744), (745)} \\
&= \sigma \min(\widetilde{\widetilde{\Xi}}_{V,W}(X_1, \ldots, X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i,\ \text{all } i\}) && \text{by (742)} \\
&= \sigma\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n)\,, && \text{see Th-123}
\end{aligned}$$

i.e.

$$\widetilde{(\sigma Q)}^L_{V,W}(\sigma X_1, \ldots, \sigma X_n) = \sigma\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n)\,. \tag{746}$$

Finally,

$$
\begin{aligned}
&\mathcal{M}_{\mathcal{CX}}{}^{\sigma}(Q)(X_1,\ldots,X_n) \\
&= \sigma^{-1}\mathcal{M}_{CX}(\sigma Q)(\sigma X_1,\ldots,\sigma X_n) && \text{by Def. 41}\\
&= \sigma^{-1}\sup\{\widetilde{\sigma Q}^{L}_{V,W}(\sigma X_1,\ldots,\sigma X_n) : V, W \in \mathcal{P}(E)^n, \\
&\qquad V_1 \subseteq W_1,\ldots,V_n \subseteq W_n\} && \text{by Th-123}\\
&= \sigma^{-1}\sup\{\sigma\widetilde{Q}^{L}_{V,W}(X_1,\ldots,X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1,\ldots,V_n \subseteq W_n\} && \text{by (746)}\\
&= \sigma^{-1}\sigma\sup\{\widetilde{Q}^{L}_{V,W}(X_1,\ldots,X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1,\ldots,V_n \subseteq W_n\} && \text{because } \sigma \text{ nondecreasing}\\
& && \text{and continuous}\\
&= \sup\{\widetilde{Q}^{L}_{V,W}(X_1,\ldots,X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1,\ldots,V_n \subseteq W_n\} && \text{because } \sigma \text{ bijection}\\
& && \text{by (741)}\\
&= \mathcal{M}_{CX}(Q)(X_1,\ldots,X_n)\,. && \text{by Th-123}
\end{aligned}
$$

G.16 Proof of Theorem 129

Let a quantitative one-place quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set of cardinality $|E| = m$ be given. Further let $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy argument set. We will denote by $q : \{0,\ldots,m\} \longrightarrow \mathbf{I}$ the mapping defined by (33), see Th-95. Then

$$
\begin{aligned}
&\mathcal{M}_{CX}(Q)(X) \\
&= \sup\{\min(\inf\{Q(Z) : V \subseteq Z \subseteq W\}, \widetilde{\Xi}_{V,W}(X)) : V \subseteq W\} && \text{by Th-123}\\
&= \max\{\min(\min\{Q(Z) : V \subseteq Z \subseteq W\}, \widetilde{\Xi}_{V,W}(X)) : V \subseteq W\} && \text{because } E \text{ finite}\\
&= \max\{\min(\min\{q(|Z|) : V \subseteq Z \subseteq W\}, \widetilde{\Xi}_{V,W}(X)) : V \subseteq W\} && \text{by Th-95}\\
&= \max\{\min(\min\{q(j) : \ell \leq j \leq u\}, \widetilde{\Xi}_{V,W}(X)) : V \subseteq W,\ \ell = |V|,\ u = |W|\} \\
&= \max\{\min(q^{\min}(\ell,u), \widetilde{\Xi}_{V,W}(X)) : V \subseteq W,\ \ell = |V|,\ u = |W|\} && \text{by Def. 94}\\
&= \max\{\max\{\min(q^{\min}(\ell,u), \widetilde{\Xi}_{V,W}(X)) : V \subseteq W,\ |V| = \ell,\ |W| = u\} \\
&\qquad : 0 \leq \ell \leq u \leq m\} \\
&= \max\{\min(q^{\min}(\ell,u), \max\{\widetilde{\Xi}_{V,W}(X) : V \subseteq W,\ |V| = \ell,\ |W| = u\}) \\
&\qquad : 0 \leq \ell \leq u \leq m\}\,, && \text{by distributivity}
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\mathcal{M}_{CX}(Q)(X) \\
&= \max\{\min(q^{\min}(\ell,u), \max\{\widetilde{\Xi}_{V,W}(X) : V \subseteq W,\ |V| = \ell,\ |W| = u\}) : 0 \leq \ell \leq u \leq m\}
\end{aligned}
\tag{747}
$$

Now let us recall that by L-143

$$
\widetilde{\Xi}_{V,W}(X) = \min(\min\{\mu_X(e) : e \in V\}, \min\{1 - \mu_X(e) : e \notin W\})\,, \tag{748}
$$

where "inf" turns into "min" because E is finite.
The base set E is finite of cardinality $|E| = m$. Hence we can order the elements of E such that $E = \{e_1,\ldots,e_m\}$ and

$$
\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)\,. \tag{749}
$$

Apparently, $\min\{\mu_X(e) : e \in V\}$ is maximised by

$$V^* = \{e_1, \ldots, e_\ell\} \tag{750}$$

subject to the constraint $|V| = \ell$. Similarly,

$$W^* = \{e_1, \ldots, e_u\} \tag{751}$$

apparently maximises $\min\{1-\mu_X(e) : e \notin W\}$ under the constraint $|W| = u$. Because $V^* \subseteq W^*$, this choice of V^*, W^* maximises

$$\Xi_{V,W}(X) = \min(\min\{\mu_X(e) : e \in V\}, \min\{1 - \mu_X(e) : e \notin W\})$$

relative to the constraints $|V| = \ell$, $|W| = u$ and $V \subseteq W$. Therefore

$$\begin{aligned}
&\max\{\Xi_{V,W}(X) : V \subseteq W,\ |V| = \ell,\ |W| = u\} \\
&= \Xi_{V^*,W^*}(X) \\
&= \min(\min\{\mu_X(e_j) : j = 1, \ldots, \ell\}, \min\{1 - \mu_X(e_j) : j = u+1, \ldots, m\}) && \text{by (748), (750), and (751)} \\
&= \min(\mu_X(e_\ell), 1 - \mu_X(e_{u+1})) && \text{by (749)} \\
&= \min(\mu_{[\ell]}(X), 1 - \mu_{[u+1]}(X)) && \text{by (749), Def. 102} \\
&= \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)\,, && \text{by Def. 105}
\end{aligned}$$

i.e.

$$\max\{\Xi_{V,W}(X) : V \subseteq W,\ |V| = \ell,\ |W| = u\} = \mu_{\|X\|_{\mathrm{iv}}}(\ell, u) \tag{752}$$

Hence

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X) \\
&= \max\{\min(q^{\min}(\ell, u), \max\{\widetilde{\Xi}_{V,W}(X) : V \subseteq W,\ |V| = \ell,\ |W| = u\}) \\
&\qquad : 0 \le \ell \le u \le m\} && \text{by (747)} \\
&= \max\{\min(q^{\min}(\ell, u), \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)) : 0 \le \ell \le u \le m\}\,, && \text{by (752)}
\end{aligned}$$

as desired.
Similarly

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X) \\
&= \neg\neg\mathcal{M}_{CX}(Q)(X) && \text{because } \neg \text{ involutive} \\
&= \neg\mathcal{M}_{CX}(\neg Q)(X) && \text{by (DFS 3)} \\
&= \neg \max\{\min((\neg q)^{\min}(\ell, u), \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)) : 0 \le \ell \le u \le m\} && \text{by first part of the theorem} \\
&= \min\{\max(\neg(\neg q)^{\min}(\ell, u), \mu_{\|X\|_{\mathrm{iv}}}(\ell, u)) : 0 \le \ell \le u \le m\} && \text{by De Morgan's law} \\
&= \min\{\max(q^{\max}(\ell, u), \mu_{\|X\|_{\mathrm{iv}}}(\ell, u) : 0 \le \ell \le u \le m\}\,,
\end{aligned}$$

where the last step holds because

$$\begin{aligned}
\neg(\neg q)^{\min}(\ell, u) &= \neg \min\{\neg q(j) : \ell \le j \le u\} && \text{by Def. 94} \\
&= \max\{\neg\neg q(j) : \ell \le j \le u\} && \text{by De Morgan's law} \\
&= \max\{q(j) : \ell \le j \le u\} && \text{because } \neg \text{ involution} \\
&= q^{\max}(\ell, j)\,. && \text{by Def. 94}
\end{aligned}$$

G.17 Proof of Theorem 130

Assume $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative semi-fuzzy quantifier on a finite base set. By Th-95, there exists $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that

$$Q(Y) = q(|Y|) \tag{753}$$

for all $Y \in \mathcal{P}(E)$.
Now suppose Q is nondecreasing, and let $j, j' \in \{0, \ldots, |E|\}$, $j \leq j'$. We may choose $Y, Y' \in \mathcal{P}(E)$ such that $|Y| = j$, $|Y'| = j'$ and $Y \subseteq Y'$. Then

$$\begin{aligned} q(j) &= Q(Y) && \text{by } j = |Y| \text{ and (753)} \\ &\leq Q(Y') && \text{because } Y \subseteq Y' \text{ and } Q \text{ nondecreasing} \\ &= q(j')\,. && \text{by } j' = |Y'| \text{ and (753)} \end{aligned}$$

This proves that whenever Q is nondecreasing, q is nondecreasing as well.
Let us now show that the converse claim holds, i.e. whenever q is nondecreasing, Q is nondecreasing as well. Hence suppose q satisfies $q(j) \leq q(j')$ for all $j \leq j'$. Let $Y, Y' \in \mathcal{P}(E)$ be given such that $Y \subseteq Y'$. Then clearly $|Y| \leq |Y'|$ by the monotonicity of cardinality. In turn,

$$\begin{aligned} Q(Y) &= q(|Y|) && \text{by (753)} \\ &\leq q(|Y'|) && \text{because } |Y| \leq |Y'|, q \text{ nondec} \\ &= Q(Y')\,, && \text{by (753)} \end{aligned}$$

i.e. Q is nondecreasing, as desired.
The proofs for nonincreasing Q vs. nonincreasing q are analogous.

G.18 Proof of Theorem 131

Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a quantitative one-place quantifier on a finite base set and q is the mapping defined by (33). If Q is nondecreasing, then q is also nondecreasing by Th-130. Hence for all $\ell, u \in \{0, \ldots, |E|\}$, $\ell \leq u$,

$$\begin{aligned} q^{\min}(\ell, u) &= \min\{q(j) : \ell \leq j \leq u\} && \text{by Def. 94} \\ &= q(\ell) && \text{because } q \text{ nondecreasing} \end{aligned}$$

and similarly

$$\begin{aligned} q^{\max}(\ell, u) &= \max\{q(j) : \ell \leq j \leq u\} && \text{by Def. 94} \\ &= q(u) && \text{because } q \text{ nondecreasing} \end{aligned}$$

The proof is analogous in the case of a nonincreasing quantifier.

G.19 Proof of Theorem 132

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a nondecreasing quantitative one-place quantifier on a finite base set, and $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is the mapping defined by (33). Further assume that $X \in \widetilde{\mathcal{P}}(E)$ is a given fuzzy argument set. Then

$$
\begin{aligned}
&\mathcal{M}_{CX}(Q)(X) \\
&= \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell,u), q^{\min}(\ell,u)) : 0 \le \ell \le u \le |E|\} && \text{by Th-129} \\
&= \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell,u), q(\ell)) : 0 \le \ell \le u \le |E|\} && \text{by Th-131} \\
&= \max\{\max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell,u), q(\ell)) : \ell \le u \le |E|\} : 0 \le \ell \le |E|\} \\
&= \max\{\min(q(\ell), \max\{\mu_{\|X\|_{\mathrm{iv}}}(\ell,u) : \ell \le u \le |E|\}) : 0 \le \ell \le |E|\} && \text{by distributivity} \\
&= \max\{\min(q(\ell), \max\{\min(\mu_{[\ell]}(X), 1-\mu_{[u+1]}(X)) : \ell \le u \le |E|\}) \\
&\qquad : 0 \le \ell \le |E|\} && \text{by Def. 105} \\
&= \max\{\min(q(\ell), \min(\mu_{[\ell]}(X), \max\{1-\mu_{[u+1]}(X) : \ell \le u \le |E|\})) \\
&\qquad : 0 \le \ell \le |E|\} && \text{by distributivity} \\
&= \max\{\min(q(\ell), \min(\mu_{[\ell]}(X), 1)) : 0 \le \ell \le |E|\} && \text{subst. } u = |E|\text{, see Def. 102} \\
&= \max\{\min(q(\ell), \mu_{[\ell]}(X)) : 0 \le \ell \le |E|\}\,, && \text{because } 1 \text{ identity of } \min
\end{aligned}
$$

as desired.

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