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Ingo Glöckner

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Abteilung Informationstechnik, Postfach 10 01 31, 33501 Bielefeld

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Ingo Glöckner

Universität Bielefeld, Technische Fakultät, AG Technische Informatik,
Postfach 10 01 31, 33501 Bielefeld,
mailto:ingo@techfak.uni-bielefeld.de

– Second Edition –

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Abstract

In this report, a broad class of standard models of fuzzy quantification is introduced, all of which satisfy the adequacy requirements of DFS theory, an axiomatic theory of fuzzy natural language quantification. The new models arise when the known construction of DFSes in terms of three-valued cuts is separated from the fuzzy median-based aggregation used in previous work on $\mathcal{M}_{\mathcal{B}}$-DFSes. Some of the new models are beneficial compared to the known $\mathcal{M}_{\mathcal{B}}$-DFSes when the inputs are overly fuzzy and one still needs a fine-grained result ranking. The report develops the full set of criteria required to check whether a given model of fuzzy quantification based on the new construction conforms to the adequacy conditions of DFS theory; whether it propagates fuzziness in quantifiers and/or arguments and hence complies with the intuitive expectation that less detailed input should not result in more specific output; whether it is robust with respect to noise in the arguments or alternative interpretations of a fuzzy quantifier; and how it compares to other DFSes by specificity.

The present report also helps to better relate existing work on fuzzy quantification to the axiomatic framework provided by DFS theory. Recent findings indicate that the Sugeno integral and hence the 'basic' FG-count approach can be embedded into DFS theory: they can be consistently generalized to the 'hard' cases of fuzzy quantification involving multi-place, non-quantitative and/or non-monotonic quantifiers. The report proves a similar result for the Choquet integral and hence the 'basic' OWA approach, by presenting a DFS $\mathcal{F}_{\mathrm{Ch}}$ with the desired properties. It is anticipated that $\mathcal{F}_{\mathrm{Ch}}$ will see a number of applications in future software systems that profit from the use of fuzzy quantifiers.[1]

[1]The second improved edition of the report fixes a bug of the original version which affected some theorems concerned with propagation of fuzziness. Apart from these corrections, a discussion of the new model $\mathcal{F}_A$ has been added.

1 Basic concepts of DFS theory

Approximate quantifiers like *almost all* and the omnipresence of fuzzy concepts like *tall* or *rich* in natural languages (NL) pose the problem of assigning a reasonable interpretation to expressions like *almost all tall people are rich*.[2] DFS theory [6, 7] provides an axiomatic solution to the problem of ensuring an adequate interpretation. Starting from the notion of a two-valued generalized quantifier developed by the theory of generalized quantifiers (TGQ, see [1, 2, 3]), DFS theory introduces the key notions of semi-fuzzy quantifiers and fuzzy quantifiers. The benefit of introducing semi-fuzzy quantifiers is that they provide a compact description of fuzzy quantifiers. These descriptions can rely on the traditional notion of cardinality of crisp sets, which is not directly applicable to fuzzy quantifiers because these need to handle fuzzy argument sets like *tall*. The mapping from simplified descriptions, i.e. semi-fuzzy quantifiers, to corresponding fuzzy quantifiers is established through a quantifier fuzzification mechanism (QFM). DFS theory approaches the problem of reasonable interpretation by imposing formal conditions on admissable choices of QFMs. These conditions ensure that the essential properties of quantifiers and relationships between quantifiers are preserved when applying the fuzzification mechanism. They can be likened to the familiar algebraic concept of a homomorphism, i.e. of a structure preserving mapping which is compatible with a number of given constructions.

In the following we give a brief account of the basic concepts of DFS theory. The reader interested in more motivation and NL examples of the constructions or axioms is advised to consult the primary sources on DFS theory: the original presentation is [6]; the current terminology and a simplified axiom system have been introduced in [7]. Let us first define two-valued generalized quantifiers in accordance with TGQ:

Definition 1 *An* n-ary two-valued quantifier *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$*, where* $E \neq \varnothing$ *is a nonempy set called the base set or domain of* Q*,* $\mathcal{P}(E)$ *is the powerset (set of subsets) of* E*,* $n \in \mathbb{N}$ *is the arity (number of arguments) of* Q*, and* $\mathbf{2} = \{0, 1\}$ *denotes the set of two-valued truth values.*

A two-valued quantifier hence assigns a crisp quantification result $Q(Y_1, \ldots, Y_n) \in \mathbf{2}$ to each choice of crisp arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We allow for the case of nullary quantifiers ($n = 0$), which can be identified with the constants 0 and 1. Some examples of two-place quantifiers are

$$\textbf{all}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \subset Y_2$$
$$\textbf{some}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 \neq \varnothing$$
$$\textbf{no}_E(Y_1, Y_2) = 1 \Leftrightarrow Y_1 \cap Y_2 = \varnothing$$
$$\textbf{at least k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| \geq k$$
$$\textbf{more than k}_E(Y_1, Y_2) = 1 \Leftrightarrow |Y_1 \cap Y_2| > k$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$; $|\bullet|$ denotes cardinality. We usually drop the subscript E when the base set E is understood. In order to cover the approximate variety of NL quantifiers (e.g. *about 10*) and to be able to apply these quantifiers to arguments like *tall*

[2] in the present case, the result should be close to 0 (false).

and *rich*, we need to enhance this concept of quantifiers and incorporate ideas from fuzzy set theory. A *fuzzy subset* X of a given set E assigns to each element $e \in E$ a membership grade $\mu_X(e) \in \mathbf{I}$, where $\mathbf{I} = [0,1]$ is the unit interval. A fuzzy subset is hence uniquely characterised by its membership function $\mu_X : E \longrightarrow \mathbf{I}$. We shall denote the collection of all fuzzy subsets of E (i.e. its fuzzy powerset) by $\widetilde{\mathcal{P}}(E)$. We shall assume that $\widetilde{\mathcal{P}}(E)$ is an ordinary set. In the following, it will be convenient to assume that crisp subsets are a special case of fuzzy subsets, i.e. we will assume that $\mathcal{P}(E) \subseteq \widetilde{\mathcal{P}}(E)$.[3] We are now ready to introduce fuzzy quantifiers:

Definition 2 *An* n-ary fuzzy quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

A fuzzy quantifier hence assigns to each n-tuple of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ a gradual quantification result $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$. Unlike two-valued quantifiers, fuzzy quantifiers hence accept fuzzy input (we could e.g. have $X_1 = \textbf{tall} \in \widetilde{\mathcal{P}}(E)$, $X_2 = \textbf{rich} \in \widetilde{\mathcal{P}}(E)$). In addition, fuzzy quantifiers produce fuzzy (gradual) output, thus providing a more natural account of approximate quantifiers like *about ten*, *almost all*, *many* etc. However, fuzzy quantifiers pose a new problem. Consider the quantifier *more than 10 percent*, for example. Given a finite base set E, we can easily define a corresponding two-valued quantifier $\textbf{more than 10 percent} : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$, viz

$$\textbf{more than 10 percent}(Y_1, Y_2) = \begin{cases} 1 & : \quad |Y_1 \cap Y_2| > |Y_1|/10 \\ 0 & : \quad \text{else} \end{cases}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, utilizing the cardinality $|\bullet|$ of crisp sets. Unfortunately, it is not that easy to provide a straightforward definition of a corresponding fuzzy quantifier $\widetilde{\textbf{more than 10 percent}} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$. This is because X_1, X_2 in

$$\widetilde{\textbf{more than 10 percent}}(X_1, X_2)$$

are fuzzy subsets $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, i.e. we cannot utilize the familiar concept of cardinality of crisp sets to define a fuzzy quantifier. Unfortunately, there is no generally accepted notion of cardinality of fuzzy sets which could be used as a substitute for $|\bullet|$ in the fuzzy case. In order to overcome this problem, DFS theory introduces the intermediary concept of semi-fuzzy quantifiers.

Definition 3 *An* n-ary semi-fuzzy quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Q hence assigns to each n-tuple of crisp subsets $Y_1, \ldots, Y_n$ a gradual quantification result $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$. Semi-fuzzy quantifiers share the expressiveness of fuzzy

[3]Note that this subsumption relationship does not hold if one identifies fuzzy subsets and their membership functions, i.e. if one stipulates that $\widetilde{\mathcal{P}}(E) = \mathbf{I}^E$, where $\mathbf{I}^E$ denotes the set of mappings $f : E \longrightarrow \mathbf{I}$. It is hence understood that the appropriate transformations (e.g. from a crisp subset $A \subseteq E$ to its characteristic function $\chi_A \in \mathbf{2}^E \subseteq \mathbf{I}^E$) are carried out and for the sake of readability, we will omit these in our notation.

quantifiers because they can model fuzzy (gradual) quantification results. Like fuzzy quantifiers, they are hence suited to represent approximate quantifiers. On the other hand, semi-fuzzy quantifiers are defined for crisp arguments only, thus alleviating the need to provide a definition for arbitrary fuzzy arguments, which made it so hard to define fuzzy quantifiers and to justify a particular choice of their definition. Because every semi-fuzzy quantifier depends on crisp arguments only, it can be conveniently defined in terms of the crisp cardinality of its arguments and their Boolean combinations. In particular, every two-valued quantifier (like the above choice of **more than 10 percent**) is a semi-fuzzy quantifier by definition.
Because of these benefits, semi-fuzzy quantifiers are considered a suitable base representation for NL quantifiers: sufficiently expressive to capture all quantifiers in the sense of TGQ as well as approximate quantifiers, and still sufficiently simple to allow for a straightforward definition. However, semi-fuzzy quantifiers cannot be applied to fuzzy arguments like *tall* or *rich*. We hence need a mechanism which accepts a description of the target quantifier, stated as a semi-fuzzy quantifier, and returns a corresponding fuzzy quantifier which properly generalises the semi-fuzzy quantifier to the case of fuzzy arguments.

Definition 4 *A* quantifier fuzzification mechanism *(QFM) $\mathcal{F}$ assigns to each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ of the same arity $n \in \mathbb{N}$ and on the same base set E.*

Of course, this definition must be further taylored to a class of 'reasonable' fuzzification mechanisms. We expect a fuzzification mechanism to be 'systematic' or 'well-behaved' and in conformance to linguistic considerations, and it is time to spell out appropriate criteria. Perhaps the most elementary condition on a fuzzification mechanism is that it properly generalizes the original semi-fuzzy quantifier. We can express this succintly if we introduce the following notion of underlying semi-fuzzy quantifiers.

Definition 5 *Let $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ be a fuzzy quantifier. The* underlying semi-fuzzy quantifier $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n)\,,$$

for all n-tuples of crisp *subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.*

It is natural to require that $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. $\mathcal{F}(Q)$ properly generalizes Q in the sense that $\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)$ whenever all arguments are crisp.

Another adequacy constraint is based on the relationship of crisp and fuzzy membership assessments with quantification. We make this relationship explicit through the following definitions of (fuzzy) projection quantifiers:

Definition 6 *Suppose E is a base set and $e \in E$. The projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by*

$$\pi_e(Y) = \chi_Y(e)\,,$$

where $\chi_Y : E \longrightarrow \mathbf{2}$ *is the characteristic function of* $Y \in \mathcal{P}(E)$*, thus*

$$\chi_Y(e) = \begin{cases} 1 & : \quad e \in Y \\ 0 & : \quad \textit{else} \end{cases}$$

For example, we can use the crisp projection quantifier π_{John} to evaluate crisp membership assessments like *Is John married?*, which can be evaluated by computing $\pi_{\mathrm{John}}(\textbf{married})$, where **married** $\in \mathcal{P}(E)$ is the crisp subset of married people in E. A corresponding definition of fuzzy projection quantifiers is straightforward.

Definition 7 *Let a base set* E *be given and* $e \in E$*. The* fuzzy projection quantifier $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is defined by*

$$\widetilde{\pi}_e(X) = \mu_X(e)$$

for all $X \in \widetilde{\mathcal{P}}(E)$*.*

For example, we can evaluate $\widetilde{\pi}_{\mathrm{John}}(\textbf{tall})$ to assess the grade to which John is tall, and we can compute $\widetilde{\pi}_{\mathrm{John}}(\textbf{rich})$ to determine $\mu_{\textbf{rich}}(\mathrm{John})$, the degree to which John is rich. Because crisp and fuzzy projection quantifiers play the same role, viz. that of crisp/fuzzy membership assessment, we expect a reasonable choice of QFM $\mathcal{F}$ to recognize this relationship and map each crisp projection quantifier π_e to the corresponding fuzzy projection quantifier, i.e. $\widetilde{\pi}_e = \mathcal{F}(\pi_e)$.

We can also evaluate a QFM from the perspective of propositional fuzzy logic. This is because a QFM can not only be applied to semi-fuzzy quantifiers. By a canonical construction, every QFM also gives rise to induced fuzzy truth functions, i.e. to a unique choice of fuzzy conjunction, disjunction etc. In order to establish this link between logical connectives and quantifiers, we first observe that $\mathbf{2}^n \cong \mathcal{P}(\{1, \ldots, n\})$, using the bijection $\eta : \mathbf{2}^n \longrightarrow \mathcal{P}(\{1, \ldots, n\})$ defined by

$$\eta(x_1, \ldots, x_n) = \{k \in \{1, \ldots, n\} : x_k = 1\} \,,$$

for all $x_1, \ldots, x_n \in \mathbf{2}$. We can use an analogous construction in the fuzzy case. We then have $\mathbf{I}^n \cong \widetilde{\mathcal{P}}(\{1, \ldots, n\})$, based on the bijection $\widetilde{\eta} : \mathbf{I}^n \longrightarrow \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ defined by

$$\mu_{\widetilde{\eta}(x_1, \ldots, x_n)}(k) = x_k \,,$$

for all $x_1, \ldots, x_n \in \mathbf{I}$ and $k \in \{1, \ldots, n\}$. These bijections can be utilized for a translation between semi-fuzzy truth functions (i.e. mappings $f : \mathbf{2}^n \longrightarrow \mathbf{I}$) and corresponding semi-fuzzy quantifiers $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$, and similarly the translation from fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ into fuzzy truth functions $\widetilde{f} : \mathbf{I}^n \longrightarrow \mathbf{I}$.

Definition 8 *Suppose* $\mathcal{F}$ *is a QFM and* $f : \mathbf{2}^n \longrightarrow \mathbf{I}$ *is a mapping (i.e. a 'semi-fuzzy truth function') of arity* $n > 0$*. The semi-fuzzy quantifier* $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ *is defined by*

$$Q_f(Y) = f(\eta^{-1}(Y))$$

for all $Y \in \mathcal{P}(\{1, \ldots, n\})$. *The induced fuzzy truth function* $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ *is defined by*

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n))\,,$$

for all $x_1, \ldots, x_n \in \mathbf{I}$. *If* $f : \mathbf{2}^0 \longrightarrow \mathbf{I}$ *is a nullary semi-fuzzy truth function (i.e., a constant), we shall define* $\widetilde{\mathcal{F}}(f) : \mathbf{I}^0 \longrightarrow \mathbf{I}$ *by* $\widetilde{\mathcal{F}}(f)(\varnothing) = \mathcal{F}(c)(\varnothing)$, *where* $c : \mathcal{P}(\{\varnothing\})^0 \longrightarrow \mathbf{I}$ *is the constant* $c(\varnothing) = f(\varnothing)$.[4]

We shall not impose restrictions on the induced connectives at this time; these will be entailed by the remaining axioms.
Induced operations on fuzzy sets, i.e. fuzzy complement $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$, fuzzy intersection $\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ and fuzzy union $\widetilde{\cup} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$, can be defined element-wise in terms of the induced negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$, conjunction $\widetilde{\wedge} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ or disjunction $\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$, respectively. For example, the induced complement $\widetilde{\neg} X \in \widetilde{\mathcal{P}}(E)$ of $X \in \widetilde{\mathcal{P}}(E)$ is defined by

$$\mu_{\widetilde{\neg} X}(e) = \widetilde{\neg}\, \mu_X(e)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$ and $e \in E$.
Based on the induced fuzzy negation and complement, we can express important constructions on quantifiers like negation, formation of antonyms, and dualisation.

Definition 9 *The* external negation $\widetilde{\neg} Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$(\widetilde{\neg} Q)(Y_1, \ldots, Y_n) = \widetilde{\neg}(Q(Y_1, \ldots, Y_n))\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *The definition of* $\widetilde{\neg} \widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *in the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is analogous.*

For example, **no** is the negation of **some**.

Definition 10 *Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. The* antonym $Q\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of* Q *is defined by*

$$Q\neg(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. *The antonym* $\widetilde{Q}\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *of a fuzzy quantifier* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined analogously, based on the given fuzzy complement* $\widetilde{\neg}$.

For example, **no** is the antonym of **all**. The dual $Q\widetilde{\square}$ of a quantifier is the negation of the antonym, or equivalently: the antonym of the negation. Hence

[4]The special treatment of nullary truth functions is necessary to avoid the use of $Q_f : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which is not a semi-fuzzy quantifier because the base set is empty. More information on the construction of induced fuzzy truth functions may be found in [7].

Definition 11 *The* dual $Q\widetilde{\square} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $n > 0$ *is defined by*

$$Q\widetilde{\square}(Y_1, \ldots, Y_n) = \widetilde{\neg}\, Q(Y_1, \ldots, Y_{n-1}, \neg Y_n)\,,$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$*. The dual* $\widetilde{Q}\widetilde{\square} = \widetilde{\neg}\, \widetilde{Q} \widetilde{\neg}$ *of a fuzzy quantifier* $\widetilde{Q}$ *is defined analogously.*

For example, **some** is the dual of **all**. We expect that a given QFM $\mathcal{F}$ be compatible with these constructions on quantifiers. Hence $\mathcal{F}$(**no**) should be the negation of $\mathcal{F}$(**some**), $\mathcal{F}$(**no**) should be the antonym of $\mathcal{F}$(**all**) and $\mathcal{F}$(**some**) should be the dual of $\mathcal{F}$(**all**).

Apart from negation/complementation, we can also form intersections and unions of argument sets to construct new quantifiers from given ones.

Definition 12 *Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of arity* $n > 0$ *be given. We define quantifiers* $Q\cup, Q\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ *by*

$$Q\cup(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cup Y_{n+1})$$
$$Q\cap(Y_1, \ldots, Y_{n+1}) = Q(Y_1, \ldots, Y_{n-1}, Y_n \cap Y_{n+1})$$

for all $Y_1, \ldots, Y_{n+1} \in \mathcal{P}(E)$*. In the case of fuzzy quantifiers,* $\widetilde{Q}\widetilde{\cup}$ *and* $\widetilde{Q}\widetilde{\cap}$ *are defined analogously, based on the given fuzzy set operations* $\widetilde{\cup}$ *and* $\widetilde{\cap}$*, resp.*

Another important characteristic of quantifiers expresses through their monotonicity properties.

Definition 13 *A semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is said to be* nonincreasing in its i-th argument, $i \in \{1, \ldots, n\}$*, if*

$$Q(Y_1, \ldots, Y_n) \geq Q(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n)$$

whenever $Y_1, \ldots, Y_n, Y_i' \in \mathcal{P}(E)$ *such that* $Y_i \subseteq Y_i'$*.* Q *is said to be* nondecreasing in the i-th argument *if the reverse inequation holds. The definitions for fuzzy quantifiers are analogous.*

For example, **all** is nonincreasing in the first argument and nondecreasing in the second argument. We expect each reasonable choice of QFM $\mathcal{F}$ to preserve such monotonicity properties. Hence $\mathcal{F}$(**all**) should be nonincreasing in the first and nondecreasing in the second argument.

We can also utilize a QFM to construct fuzzy powerset mappings. Let us first recall the concept of a powerset mapping in the crisp case. To each mapping $f : E \longrightarrow E'$, we can associate a mapping $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ (the powerset mapping of f) which is defined by

$$\widehat{f}(Y) = \{f(e) : e \in Y\}\,,$$

for all $Y \in \mathcal{P}(E)$.[5] In order to generalise this concept to the fuzzy case, we need a mechanism which associates fuzzy powerset mappings $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ to given mappings $f : E \longrightarrow E'$. Such a mechanism is called an *extension principle*.[6] The standard extension principle, proposed by Zadeh [15], is defined by

$$\mu_{\hat{\hat{f}}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}\,, \tag{1}$$

for all $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $e' \in E'$. With each QFM, we can associate a corresponding extension principle through a canonical construction.

Definition 14 *Every QFM $\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E \longrightarrow E'$ (where $E, E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ defined by*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$.

We require that every 'reasonable' choice of $\mathcal{F}$ be compatible with its induced extension principle in the following sense. Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $f_1, \ldots, f_n : E' \longrightarrow E$ are given mappings, $E' \neq \varnothing$. We can construct the semi-fuzzy quantifier $Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by composing Q with the powerset mappings $\widehat{f}_i, \ldots, \widehat{f_n}$, i.e.

$$(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i)(Y_1, \ldots, Y_n) = Q(\widehat{f}_1(Y_1), \ldots, \widehat{f_n}(Y_n))\,, \tag{2}$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E')$. By utilizing the induced extension principle $\widehat{\mathcal{F}}$ of a QFM, we can perform a similar construction on fuzzy quantifiers, thus composing $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ with $\widehat{\mathcal{F}}(f_1), \ldots, \widehat{\mathcal{F}}(f_n)$ to form the fuzzy quantifier $\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i) : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$ defined by

$$(\widetilde{Q} \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i))(X_1, \ldots, X_n) = \widetilde{Q}(\widehat{\mathcal{F}}(f_1)(X_1), \ldots, \widehat{\mathcal{F}}(f_n)(X_n))\,,$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$. We require that a QFM $\mathcal{F}$ be compatible with this construction, i.e.

$$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)\,.$$

This condition is of particular importance because it is the only criterion to relate the behaviour of $\mathcal{F}$ on different base sets E, E'. We can combine the above conditions in order to capture our expectations on well-behaved models of fuzzy quantification in a condensed set of axioms.

[5]Often the same symbol is used to denote both the original mapping and the powerset mapping.

[6]For our purposes, it will be convenient to assume that $E, E' \neq \varnothing$.

Definition 15 *A QFM $\mathcal{F}$ is called a* determiner fuzzification scheme *(DFS) if the following conditions are satisfied for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(Z-2)
Dualisation	$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square} \quad n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup} \quad n > 0$	(Z-4)
Preservation of monotonicity	*If Q is nonincreasing in n-th arg, then $\mathcal{F}(Q)$ is nonincreasing in n-th arg,* $n > 0$	(Z-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}\limits_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(Q) \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(Z-6)

The axioms (Z-1) to (Z-6) have been shown to be independent in [7], which also shows that the present axiom set is equivalent to the original definition of DFSes in [6] which was based on nine axioms.

2 Some properties of DFSes and special subclasses

The above conditions (Z-1)–(Z-6) are intended to cover those adequacy criteria that are essential from the perspective of linguistics and fuzzy logic, and to provide a formalisation of these criteria in terms of a system of independent axioms. Due to the goal of obtaining an independent system, it was not possible to include all of these adequacy criteria directly into the axiom set, thus compromising its independence. However, it has been shown in [7] that DFSes comply with a large number of linguistic and logical adequacy criteria. The following chapter is not intended to review these results on adequacy properties of DFSes, which can be found in full detail in [7]. By contrast, the chapter focuses on those definitions and theorems only, that are necessary to understand and prove the new theorems. Unless otherwise stated, the proofs of the cited theorems can be found in [6, 7].

First we review some results on the fuzzy truth functions induced by a DFS. Let us recall the definition of a strong negation (i.e. 'reasonable' fuzzy negation operator):

Definition 16 $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *iff it satisfies*

a. $\widetilde{\neg}\, 0 = 1$ *(boundary condition)*

b. $\widetilde{\neg}\, x_1 \geq \widetilde{\neg}\, x_2$ *for all* $x_1, x_2 \in \mathbf{I}$ *such that* $x_1 < x_2$ *(i.e.* $\widetilde{\neg}$ *is monotonically decreasing)*

c. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e.* $\widetilde{\neg}$ *is involutive).*

Note. Whenever the standard negation $\neg x = 1 - x$ is being assumed, we shall drop the 'tilde'-notation. Hence the standard fuzzy complement is denoted $\neg X$, where $\mu_{\neg X}(e) = 1 - \mu_X(e)$. Similarly, the external negation of a (semi-) fuzzy quantifier with respect to the standard negation is written $\neg Q$, and the antonym of a fuzzy quantifier with respect to the standard fuzzy complement is written as $\widetilde{Q}\neg$.

We also recall the concepts of a t-norm (i.e. 'reasonable' fuzzy conjunction) and s-norm ('reasonable' fuzzy disjunction), see [10]. The fuzzy truth functions induced by a DFS are guaranteed to belong to the class of such reasonable operators:

Theorem 1 *In every DFS* $\mathcal{F}$*,*

a. $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$ *is the identity truth function;*

b. $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator;*

c. $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ *is a* t*-norm;*

d. $x_1 \,\widetilde{\vee}\, x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \,\widetilde{\wedge}\, \widetilde{\neg}\, x_2)$*, i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$*.*

Next we show that one does not loose any interesting phenomena if attention is restricted to DFSes that induce the standard negation $\neg x = 1 - x$.

Definition 17 *Suppose $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ is strong negation operator. A DFS $\mathcal{F}$ is called a $\widetilde{\neg}$-DFS if its induced negation coincides with $\widetilde{\neg}$, i.e. $\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}$. In particular, we will call $\mathcal{F}$ a $\neg$-DFS if it induces the standard negation $\neg x = 1 - x$.*

Definition 18 *Suppose $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ a bijection. For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we define*

$$\mathcal{F}^{\sigma}(Q)(X_1, \ldots, X_n) = \sigma^{-1}\mathcal{F}(\sigma Q)(\sigma X_1,\ \ldots,\ \sigma X_n)\,,$$

where σQ abbreviates $\sigma \circ Q$, and $\sigma X_i \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset with $\mu_{\sigma X_i} = \sigma \circ \mu_{X_i}$.

Theorem 2 *If $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ an increasing bijection, then $\mathcal{F}^{\sigma}$ is a DFS.*

We recall that for every strong negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ there is a monotonically increasing bijection $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ such that $\widetilde{\neg}\, x = \sigma^{-1}(1 - \sigma(x))$ for all $x \in \mathbf{I}$, see [8, Th-3.7]. The mapping σ is called the *generator* of $\widetilde{\neg}$.

Theorem 3 *Suppose $\mathcal{F}$ is a $\widetilde{\neg}$-DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ the generator of $\widetilde{\neg}$. Then $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$ is a $\neg$-DFS and $\mathcal{F} = \mathcal{F}'^{\sigma}$.*

This means that we can freely move from an arbitrary $\widetilde{\neg}$-DFS to a corresponding $\neg$-DFS and vice versa: in the following, we hence restrict attention to $\neg$-DFSes. Among these, we discern further subclasses according to their induced disjunction.

Definition 19 *A $\neg$-DFS $\mathcal{F}$ which induces a fuzzy disjunction $\widetilde{\vee}$ is called a $\widetilde{\vee}$-DFS.*

Definition 20 *A DFS $\mathcal{F}$ is called a* standard DFS *if and only if $\mathcal{F}$ is a $\max$-DFS, i.e. a DFS which induces the standard negation $\neg x = 1 - x$ and the standard disjunction $x \vee y = \max(x, y)$.*

Note. It is then apparent from earlier work [7, Th-17.a, p. 20 and Th-25, p. 25] that standard DFSes are exactly those $\neg$-DFSes which induce the standard extension principle $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$.

Theorem 4 *Suppose $\mathcal{J}$ is a non-empty index set and $(\mathcal{F}_j)_{j \in \mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes. Further suppose that $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ satisfies the following conditions:*

a. If $f \in \mathbf{I}^{\mathcal{J}}$ is constant, i.e. if there is a $c \in \mathbf{I}$ such that $f(j) = c$ for all $j \in \mathcal{J}$, then $\Psi(f) = c$.

b. $\Psi(1 - f) = 1 - \Psi(f)$, where $1 - f \in \mathbf{I}^{\mathcal{J}}$ is point-wise defined by $(1 - f)(j) = 1 - f(j)$, for all $j \in \mathcal{J}$.

c. Ψ is monotonically nondecreasing, i.e. if $f(j) \leq g(j)$ for all $j \in \mathcal{J}$, then $\Psi(f) \leq \Psi(g)$.

If we define $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ by

$$\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](Q)(X_1,\ldots,X_n) = \Psi((\mathcal{F}_j(Q)(X_1,\ldots,X_n))_{j\in\mathcal{J}})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$, then $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ is a $\widetilde{\vee}$-DFS.

Therefore convex combinations of $\widetilde{\vee}$-DFSes like the arithmetic mean, and stable symmetric sums [11] of $\widetilde{\vee}$-DFSes are again $\widetilde{\vee}$-DFSes.

The $\neg$-DFSes can be partially ordered by 'specificity' or 'fuzziness', in the sense of closeness to $\frac{1}{2}$. We define a partial order $\preceq_c \subseteq \mathbf{I}\times\mathbf{I}$ by

$$x \preceq_c y \Leftrightarrow y \le x \le \tfrac{1}{2} \text{ or } \tfrac{1}{2} \le x \le y\,, \tag{3}$$

for all $x, y \in \mathbf{I}$. $\preceq_c$ is Mukaidono's ambiguity relation, see [9]. We extend this basic definition of $\preceq_c$ for scalars to the case of DFSes in the obvious way:

Definition 21 *Suppose $\mathcal{F}, \mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is* consistently less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, iff for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}(Q)(X_1,\ldots,X_n) \preceq_c \mathcal{F}'(Q)(X_1,\ldots,X_n)\,.$$

We now wish to establish the existence of consistently least specific $\widetilde{\vee}$-DFSes. As it turns out, the greatest lower specificity bound of a collection of $\widetilde{\vee}$-DFSes can be expressed using the fuzzy median $\mathrm{med}_{\frac{1}{2}}$, which is defined as follows.

Definition 22 *The* fuzzy median $\mathrm{med}_{\frac{1}{2}} : \mathbf{I}\times\mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{med}_{\frac{1}{2}}(u_1,\, u_2) = \begin{cases} \min(u_1,\, u_2) & : \ \min(u_1,\, u_2) > \frac{1}{2} \\ \max(u_1,\, u_2) & : \ \max(u_1,\, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

$\mathrm{med}_{\frac{1}{2}}$ is an associative mean operator [4] and the only stable (i.e. idempotent) associative symmetric sum [11]. It can be generalised to an operator $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.

Definition 23 *The generalised fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ is defined by*

$$\mathrm{m}_{\frac{1}{2}}\, X = \mathrm{med}_{\frac{1}{2}}(\inf X,\ \sup X)\,,$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Now we can state the desired theorem.

Theorem 5 *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$. Then there exists a greatest lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$, i.e. a $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{glb}}$ such that $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}_j$ for all $j \in \mathcal{J}$ (i.e. $\mathcal{F}_{\mathrm{glb}}$ is a lower specificity bound), and for all other lower specificity bounds $\mathcal{F}'$, $\mathcal{F}' \preceq_c \mathcal{F}_{\mathrm{glb}}$.*
$\mathcal{F}_{\mathrm{glb}}$ is defined by

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\},$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

In particular, the theorem asserts the existence of least specific $\widetilde{\vee}$-DFSes, i.e. whenever $\widetilde{\vee}$ is an s-norm such that $\widetilde{\vee}$-DFSes exist, then there exists a least specific $\widetilde{\vee}$-DFS (just apply the above theorem to the collection of all $\widetilde{\vee}$-DFSes).
As concerns the converse issue of most specific DFSes, i.e. least upper bounds with respect to $\preceq_c$, the following definition of 'specificity consistence' turns out to provide the key concept:

Definition 24 *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes $\mathcal{F}_j$, $j \in \mathcal{J}$ where $\mathcal{J} \neq \varnothing$. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is called* specificity consistent *iff for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, either $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ or $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$, where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.*

We can now express the exact conditions under which a collection of $\widetilde{\vee}$-DFSes has a least upper specificity bound.

Theorem 6 *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$.*

a. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ has upper specificity bounds exactly if $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent.

b. If $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent, then its least upper specificity bound is the $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{lub}}$ defined by

$$\mathcal{F}_{\mathrm{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : & R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : & R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.

3 Further adequacy considerations

In the following, we shall discuss several additional adequacy criteria for approaches to fuzzy quantification. The first two criteria are concerned with the 'propagation of fuzziness', i.e. the way in which the amount of imprecision in the model's inputs affects changes of the model's outputs. To this end, let us recall the partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ defined by equation (3). We can extend $\preceq_c$ to fuzzy sets $X \in \widetilde{\mathcal{P}}(E)$, semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ as follows:

$$\begin{aligned} X \preceq_c X' &\iff \mu_X(e) \preceq_c \mu_{X'}(e) && \text{for all } e \in E; \\ Q \preceq_c Q' &\iff Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n) && \text{for all } Y_1, \ldots, Y_n \in \mathcal{P}(E); \\ \widetilde{Q} \preceq_c \widetilde{Q}' &\iff \widetilde{Q}(X_1, \ldots, X_n) \preceq_c \widetilde{Q}'(X_1, \ldots, X_n) && \text{for all } X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\,. \end{aligned}$$

Intuitively, we expect that the quantification results become less specific whenever the quantifier or the argument sets become less specific: the fuzzier the input, the fuzzier the output.

Definition 25 *We say that a QFM $\mathcal{F}$* propagates fuzziness in arguments *if and only if the following property is valid for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n, X'_1, \ldots, X'_n$: If $X_i \preceq_c X'_i$ for all $i = 1, \ldots, n$, then $\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}(Q)(X'_1, \ldots, X'_n)$.*
We say that $\mathcal{F}$ propagates fuzziness in quantifiers *if and only if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.*

Both conditions are certainly natural to require, and I consider them as desirable but optional. A more detailed discussion can be found below on page 23 and in the conclusion.

Finally, we introduce two adequacy criteria concerned with distinct aspects of the 'smoothness' or 'continuity' of a DFS. These conditions are essential for DFSes to be *practical* because it is extremely important for applications that the results of a DFS be stable with respect to slight changes in the inputs. These 'changes' can either occur in the fuzzy argument sets (e.g. due to noise), or they can affect the semi-fuzzy quantifier. For example, if a person A has a slightly different interpretation of quantifier Q than person B, then we still want them to understand each others, i.e. the quantification results obtained from the two models of the target quantifier should be very similar in such cases.
In order to express the robustness criterion with respect to slight changes in the fuzzy arguments, we first need to introduce a metric on fuzzy subsets, which serves as a numerical quantity of the similarity of the arguments. For all base sets E and all $n \in \mathbb{N}$, we define the metric $d : \widetilde{\mathcal{P}}(E)^n \times \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) = \max_{i=1}^{n} \sup\{|\mu_{X_i}(e) - \mu_{X'_i}(e)| : e \in E\}\,, \quad (4)$$

for all $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$. Based on this metric, we can now express the desired criterion for continuity in arguments.

Definition 26 *We say that a QFM $\mathcal{F}$ is* arg-continuous *if and only if $\mathcal{F}$ maps all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to continuous fuzzy quantifiers $\mathcal{F}(Q)$, i.e. for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*

and $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $d(\mathcal{F}(Q)(X_1, \ldots, X_n), \mathcal{F}(Q)(X_1', \ldots, X_n')) < \varepsilon$ *for all* $X_1', \ldots, X_n' \in \widetilde{\mathcal{P}}(E)$ *with* $d((X_1, \ldots, X_n), (X_1', \ldots, X_n')) < \delta$*.*

The second robustness criterion is intended to capture the idea that slight changes in a semi-fuzzy quantifier should not cause the quantification results to change drastically. To introduce this criterion, we must first define suitable distance measures for semi-fuzzy quantifiers and for fuzzy quantifiers. Hence for all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$d(Q, Q') = \sup\{|Q(Y_1, \ldots, Y_n) - Q'(Y_1, \ldots, Y_n)| : Y_1, \ldots, Y_n \in \mathcal{P}(E)\}\,, \quad (5)$$

and similarly for all fuzzy quantifiers $\widetilde{Q}, \widetilde{Q}' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$,

$$d(\widetilde{Q}, \widetilde{Q}') = \sup\{|\widetilde{Q}(X_1, \ldots, X_n) - \widetilde{Q}'(X_1, \ldots, X_n)| : X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\}\,. \quad (6)$$

Definition 27 *We say that a QFM* $\mathcal{F}$ *is* Q-continuous *if and only if for each semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $d(\mathcal{F}(Q), \mathcal{F}(Q')) < \varepsilon$ *whenever* $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *satisfies* $d(Q, Q') < \delta$*.*

Both condition are crucial to the utility of a DFS and should be possessed by every model employed in practical applications. They are not part of the DFS axioms because I wanted to have DFSes for general t-norms (including the discontinuous variety).

4 The class of $\mathcal{M}_{\mathcal{B}}$-DFSes

In [6], the first three models of the DFS axioms have been presented. An investigation of the common principle underlying these DFSes has led to the introduction of $\mathcal{M}_{\mathcal{B}}$-DFSes in [7], the class of DFSes defined in terms of three-valued cuts of arguments and subsequent aggregation by applying the fuzzy median. In the following, I briefly recall the definitions necessary to introduce $\mathcal{M}_{\mathcal{B}}$-QFMs and to understand how they work. This includes a characterisation of the class of $\mathcal{M}_{\mathcal{B}}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping $\mathcal{B}$, as well as a presentation of important models and special properties of $\mathcal{M}_{\mathcal{B}}$-DFSes. Unless otherwise stated, the proofs of all theorems cited in this chapter may be found in [7].
Let us first define the unrestricted class of $\mathcal{M}_{\mathcal{B}}$-QFMs, which will then be shrunk to the reasonable cases of $\mathcal{M}_{\mathcal{B}}$-DFSes by imposing conditions on the aggregation mapping. To this end, we need to introduce some notation. We recall the concept of α-cuts and strict α-cuts of fuzzy subsets:

Definition 28 *Let E be a given set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $\alpha \in \mathbf{I}$. By $X_{\geq\alpha} \in \mathcal{P}(E)$ we denote the α-cut*

$$X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\} .$$

Definition 29 *Let $X \in \widetilde{\mathcal{P}}(E)$ be given and $\alpha \in \mathbf{I}$. By $X_{>\alpha} \in \mathcal{P}(E)$ we denote the strict α-cut*

$$X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\} .$$

In terms of these α-cuts, we define the cut range $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$, which represents a three-valued cut at the 'cautiousness level' $\gamma \in \mathbf{I}$ by a set of alternatives $\{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\}$. The reason for introducing three-valued cuts is that we need a cutting mechanism compatible to complementation. α-cuts, however, have $(\neg X)_{\geq\alpha} \neq \neg(X_{\geq\alpha})$. The desired symmetry is easily obtained with three-valued cuts, defined as follows:

Definition 30 *Suppose E is some set, $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. $X_\gamma^{\min}, X_\gamma^{\max} \in \mathcal{P}(E)$ and $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ are defined by*

$$X_\gamma^{\min} = \begin{cases} X_{>\frac{1}{2}} & : \gamma = 0 \\ X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} & : \gamma > 0 \end{cases}$$
$$X_\gamma^{\max} = \begin{cases} X_{\geq\frac{1}{2}} & : \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma > 0 \end{cases}$$
$$\mathcal{T}_\gamma(X) = \{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\} .$$

Note. The relationship of cut ranges $\mathcal{T}_\gamma(X)$ and three-valued sets is discussed in [6, p. 58+] and [7, p. 39+].

How can we use these cut ranges to evaluate fuzzy quantifiers? The basic idea is that we can view the crisp range $\mathcal{T}_\gamma(X)$ as providing a set of alternatives to be checked. For example, in order to evaluate a quantifier Q at a certain cut level γ, we have to consider all choices of $Q(Y_1, \ldots, Y_n)$, where $Y_i \in \mathcal{T}_\gamma(X_i)$. The set of results obtained in this way must then be aggregated to a single result in the unit interval, which we denote as $Q_\gamma(X_1, \ldots, X_n) \in \mathbf{I}$. The generalised fuzzy median (see Def. 23) is well-suited to carry out this aggregation. The use of the fuzzy median for this purpose was originally motivated by the observation that the resulting fuzzification mechanisms embed Kleene's three-valued logic. This is useful because the targeted class of models (viz, standard DFSes) are known to embed Kleene's logic, too.
Let us hence use the crisp ranges $\mathcal{T}_\gamma(X_i)$ of the argument sets to define a family of QFMs $(\bullet)_\gamma$, indexed by the cautiousness parameter $\gamma \in \mathbf{I}$:

Definition 31 *For every $\gamma \in \mathbf{I}$, we denote by $(\bullet)_\gamma$ the QFM defined by*

$$Q_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} ,$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

None of the QFMs $(\bullet)_\gamma$ is a DFS, because the required information is spread over various cut levels. Hence in order to define DFSes based on these QFMs, we must simultaneously consider the results obtained at all levels of cautiousness γ, i.e. the γ-indexed family $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$. We can then apply various aggregation operators on these γ-indexed results to obtain new QFMs, which have a chance of being DFSes. We now define the domain on which these aggregation operators can act.

Definition 32 *$\mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^-$ and $\mathbb{B} \subseteq \mathbf{I}^{\mathbf{I}}$ are defined by*

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) > \tfrac{1}{2} \textit{ and } f(\mathbf{I}) \subseteq [\tfrac{1}{2}, 1] \textit{ and } f \textit{ nonincreasing } \}\\
\mathbb{B}^{\frac{1}{2}} &= \{\mathrm{c}_{\frac{1}{2}}\}\\
\mathbb{B}^- &= \{f \in \mathbf{I}^{\mathbf{I}} : f(0) < \tfrac{1}{2} \textit{ and } f(\mathbf{I}) \subseteq [0, \tfrac{1}{2}] \textit{ and } f \textit{ nondecreasing } \}\\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^- .
\end{aligned}$$

Note. In the definition of $\mathbb{B}^{\frac{1}{2}}$, $\mathrm{c}_{\frac{1}{2}} : \mathbf{I} \longrightarrow \mathbf{I}$ is the constant $\mathrm{c}_{\frac{1}{2}}(x) = \frac{1}{2}$ for all $x \in \mathbf{I}$. More generally, we stipulate for all $a \in \mathbf{I}$ that $\mathrm{c}_a : \mathbf{I} \longrightarrow \mathbf{I}$ be the constant mapping

$$\mathrm{c}_a(x) = a \,, \tag{7}$$

for all $x \in \mathbf{I}$.

Theorem 7

a. *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ *are given. Then*

$$(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}} \in \begin{cases} \mathbb{B}^+ & : \quad Q_0(X_1, \dots, X_n) > \frac{1}{2} \\ \mathbb{B}^{\frac{1}{2}} & : \quad Q_0(X_1, \dots, X_n) = \frac{1}{2} \\ \mathbb{B}^- & : \quad Q_0(X_1, \dots, X_n) < \frac{1}{2} \end{cases}$$

b. *For each* $f \in \mathbb{B}$ *there exists* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ *such that* $f = (Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$.

Given an aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$, we define the corresponding QFM $\mathcal{M}_\mathcal{B}$ as follows.

Definition 33 *Suppose* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given. The QFM* $\mathcal{M}_\mathcal{B}$ *is defined by*

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \dots, X_n) = \mathcal{B}((Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}) \,, \tag{8}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1 \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{M}_\mathcal{B}$-QFMs we mean the class of all QFMs $\mathcal{M}_\mathcal{B}$ defined in this way. It is apparent that if we do not impose restrictions on admissible choices of $\mathcal{B}$, the resulting QFMs will often fail to be DFSes. Hence let us state the necessary and sufficient conditions that $\mathcal{B}$ must satisfy in order to make $\mathcal{M}_\mathcal{B}$ a DFS. To express these conditions, we first need some constructions on $\mathbb{B}$.

Definition 34 *Suppose* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e., nondecreasing or nonincreasing). The mappings* $f^\flat$, $f^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$ *are defined by:*

$$f^\sharp = \begin{cases} \lim_{y \to x^+} f(y) & : \quad x < 1 \\ f(1) & : \quad x = 1 \end{cases}$$

$$f^\flat = \begin{cases} \lim_{y \to x^-} f(y) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases} \qquad \text{for all } f \in \mathbb{B}, x \in \mathbf{I}.$$

It is apparent that if $f \in \mathbb{B}$, then $f^\sharp \in \mathbb{B}$ and $f^\flat \in \mathbb{B}$. $f^\sharp$ and $f^\flat$ are obviously very 'similar' to each others (and to f) and every reasonable $\mathcal{B}$ should map $f^\flat$ and $f^\sharp$ to the same aggregation result. This turns out to be essential for $\mathcal{M}_\mathcal{B}$ to satisfy (Z-6), because $(Q_\gamma(X_1, \dots, X_n))_{\gamma \in \mathbf{I}}$ is not compatible with (Z-6) in a precise sense, but only modulo $\sharp/\flat$.

We shall further introduce several coefficients which describe certain aspects of a mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Definition 35 *For every monotonic mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$ (i.e., either nondecreasing or nonincreasing), we define*

$$f_0^* = \lim_{\gamma \to 0^+} f(\gamma) \tag{9}$$
$$f_*^{0\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{10}$$
$$f_*^{\frac{1}{2}\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \tag{11}$$
$$f_1^* = \lim_{\gamma \to 1^-} f(\gamma) \tag{12}$$
$$f_*^{1\uparrow} = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} \tag{13}$$
$$f_*^{1\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 1\}\,. \tag{14}$$

We only need $f_*^{\frac{1}{2}\downarrow}$ to define the desired conditions on $\mathcal{B}$; it turns out to be essential for ensuring a proper behaviour of $\mathcal{M}_\mathcal{B}$ in the case of three-valued quantifiers, and in particular to ensure the desired results for the two-valued projection quantifiers of (Z-2). We will use the remaining coefficients later to define examples of $\mathcal{M}_\mathcal{B}$-DFSes.

Definition 36 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathcal{B}$, we define the following conditions on $\mathcal{B}$:*

$$\mathcal{B}(f) = f(0) \quad \textit{if } f \textit{ is constant, i.e. } f(x) = f(0) \textit{ for all } x \in \mathbf{I} \tag{B-1}$$
$$\mathcal{B}(1 - f) = 1 - \mathcal{B}(f) \tag{B-2}$$
If $f(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$, then (B-3)
$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}\downarrow} & : \ f \in \mathbb{B}^+ \\ \frac{1}{2} & : \ f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}\downarrow} & : \ f \in \mathbb{B}^- \end{cases}$$
$$\mathcal{B}(f^\sharp) = \mathcal{B}(f^\flat) \tag{B-4}$$
If $f \leq g$, then $\mathcal{B}(f) \leq \mathcal{B}(g)$ (B-5)

As witnessed by the next theorem, these conditions capture precisely the requirement on $\mathcal{B}$ for $\mathcal{M}_\mathcal{B}$ to be a DFS.

Theorem 8

a. The conditions (B-1) *to* (B-5) *are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*

b. The conditions (B-1) *to* (B-5) *are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

c. The conditions (B-1) *to* (B-5) *are independent.*

In particular, $\mathcal{B}(f) = 1 - \mathcal{B}(1-f)$ for all $f \in \mathbb{B}$, and $\mathcal{B}(f) \geq \frac{1}{2}$ whenever $f \in \mathbb{B}^+$. We can hence give a more concise description of $\mathcal{M}_\mathcal{B}$-DFSes, because it is sufficient to consider their behaviour on $\mathbb{B}^+$ only:

Definition 37 *By $\mathbb{H} \subseteq \mathbf{I}^\mathbf{I}$ we denote the set of nonincreasing $f : \mathbf{I} \longrightarrow \mathbf{I}$, $f \neq 0$,*

$$\mathbb{H} = \{ f \in \mathbf{I}^\mathbf{I} : \ f \text{ nonincreasing and } f(0) > 0 \} .$$

We can associate with each $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ a $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ as follows:

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{15}$$

Theorem 9 *If $\mathcal{M}_\mathcal{B}$ is a DFS, then $\mathcal{B}$ can be defined in terms of a mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to equation* (15)*. $\mathcal{B}'$ is defined by*

$$\mathcal{B}'(f) = 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 . \tag{16}$$

We can hence focus on mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ without loosing any desired models.

Definition 38 *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathbb{H}$, we define the following conditions on $\mathcal{B}'$:*

$$\mathcal{B}'(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{C-1}$$

$$\text{If } \widehat{f}(\mathbf{I}) \subseteq \{0,1\}, \text{ then } \mathcal{B}'(f) = f_*^{0\downarrow}, \tag{C-2}$$

$$\mathcal{B}'(f^\sharp) = \mathcal{B}'(f^\flat) \quad \text{if } \widehat{f}((0,1]) \neq \{0\} \tag{C-3}$$

$$\text{If } f \leq g, \text{ then } \mathcal{B}'(f) \leq \mathcal{B}'(g) \tag{C-4}$$

A theorem analogous to Th-8 can be proven for (C-1) to (C-4):

Theorem 10

a. *The conditions* (C-1) *to* (C-4) *are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*

b. *The conditions* (C-1) *to* (C-4) *are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

c. *The conditions* (C-1) *to* (C-4) *are independent.*

Our introducing of $\mathcal{B}'$ is only a matter of convenience, because the definition of $\mathcal{B}'$ is usually shorter than the definition of the corresponding $\mathcal{B}$. We now present some examples of $\mathcal{M}_\mathcal{B}$-QFMs.

Definition 39 *By $\mathcal{M}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_\int(f) = \int_0^1 f(x)\,dx , \qquad \text{for all } f \in \mathbb{H}.$$

Theorem 11 *$\mathcal{M}$ is a standard DFS.*

$\mathcal{M}$ is Q-continuous and arg-continuous and hence a good choice for applications.

Definition 40 *By $\mathcal{M}_U$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_U(f) = \max(f_*^{1\uparrow}, f_1^*) \qquad \textit{for all } f \in \mathbb{H}, \textit{ see } (12) \textit{ and } (13).$$

Theorem 12 *Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^{1\uparrow} \oplus f_1^*\,,$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equations (8) *and* (15)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_U$ is a standard DFS. It is neither Q-continuous nor arg-continuous and hence not practical. However, $\mathcal{M}_U$ is of theoretical interest because it represents an extreme case of $\mathcal{M}_\mathcal{B}$-DFS in terms of specificity:

Theorem 13 *$\mathcal{M}_U$ is the least specific $\mathcal{M}_\mathcal{B}$-DFS.*

Let us now consider the issue of most specific $\mathcal{M}_\mathcal{B}$-DFSes.

Definition 41 *By $\mathcal{M}_S$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_S(f) = \min(f_*^{0\downarrow}, f_0^*) \qquad \textit{for all } f \in \mathbb{H}\textit{; see } (9) \textit{ and } (10).$$

Theorem 14 *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^{0\downarrow} \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the QFM $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (8) *and* (15)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_S$ is a standard DFS. $\mathcal{M}_S$ fails on both continuity conditions, but:

Theorem 15 *$\mathcal{M}_S$ is the most specific $\mathcal{M}_\mathcal{B}$-DFS.*

Definition 42 *By $\mathcal{M}_{CX}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_{CX}(f) = \sup\{\min(x, f(x)) : x \in \mathbf{I}\} \qquad \textit{for all } f \in \mathbb{H}.$$

Theorem 16 *Suppose $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a continuous t-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\}$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (8) *and* (15)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

Therefore $\mathcal{M}_{CX}$ is a standard DFS. It is Q-continuous and arg-continuous and hence a good choice for applications.
As has been shown in [7], $\mathcal{M}_{CX}$ exhibits unique properties. In fact, it is the only standard DFS which is compatible with a construction called 'fuzzy argument insertion', which ensures a compositional interpretation of adjectival restriction with fuzzy adjectives. $\mathcal{M}_{CX}$ can be shown to generalize the well-known Sugeno integral to the case of multiplace and non-monotonic quantifiers. Hence $\mathcal{M}_{CX}$ consistently generalises the basic FG-count approach of [16, 13], which is restricted to quantitative and non-decreasing one-place quantifiers. In addition, $\mathcal{M}_{CX}$ can be shown to implement the so-called 'substitution approach' to fuzzy quantification [12], i.e. the fuzzy quantifier is modelled by constructing an equivalent logical formula (involving fuzzy connectives). The reader interested in details is invited to consult [7].

Returning to $\mathcal{M}_{\mathcal{B}}$-DFSes in general, we can state that:

Theorem 17

- *All $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued arguments, i.e. whenever the arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ satisfy $\mu_{X_i}(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$;*
- *all $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$.*

This is different from general standard DFSes, which are guaranteed to coincide only for two-valued quantifiers.
An issue not yet addressed in previous publications is whether $\preceq_c$ is a genuine partial order (i.e. not a total order). As we now show, $\preceq_c$ is a genuine partial order on $\mathcal{M}_{\mathcal{B}}$-DFSes. In particular, the standard DFSes are only partially ordered by $\preceq_c$.

Theorem 18 *$\preceq_c$ is not a total order on $\mathcal{M}_{\mathcal{B}}$-DFSes.*
(Proof: A.1, p.41+)

One of the characteristic properties of $\mathcal{M}_{\mathcal{B}}$-DFSes is that they propagate fuzziness.

Theorem 19

- *Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in quantifiers.*
- *Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in arguments.*

I consider this an important adequacy criterion because it strikes me as implausible that the results should become more specific when the input (quantifier or argument) gets fuzzier. Nevertheless, there seems to be a price one has to pay for the propagation of fuzziness: as the input becomes less specific, the result of an $\mathcal{M}_{\mathcal{B}}$-DFS is likely to attain the least specific value of $\frac{1}{2}$, see Th-34 and Th-40 below. In some applications, it might be preferable to sacrifice the propagation of fuzziness, in order to obtain specific results (e.g. a fine-grained result ranking) even in those cases where the input is overly fuzzy. In addition, the study of such models is of theoretical interest, because it helps to gain a better understanding of the full class of standard DFSes. Specifically, I would like to show that standard DFSes exist which fail to propagate fuzziness in quantifiers and/or arguments, and gain some insight into the structure and properties of such models.

In order to span a broader class of DFS models, we must drop the median-based aggregation mechanism of $\mathcal{M}_{\mathcal{B}}$-DFSes. We get an idea of how to proceed if we simply expand the definition of the generalized fuzzy median and rewrite $(\bullet)_\gamma$ as

$$\begin{aligned} Q_\gamma(X_1, \dots, X_n) \;&=\; \mathrm{med}_{\frac{1}{2}}(\sup\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}, \\ &\qquad\quad \inf\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\})\,. \end{aligned} \tag{17}$$

This is apparent from Def. 23 and Def. 31. The fuzzy median can then be replaced with other connectives, e.g. the arithmetic mean $(x+y)/2$. If we view $\sup\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ and $\inf\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\}$ as mappings that depend on γ, then we can even eliminate the pointwise application of the connective and define more 'holistic' mechanisms. In the next chapter this basic idea of creating a broader class of DFS candidates is turned into precise definitions. As it turns out, most constructions of relevance to $\mathcal{M}_{\mathcal{B}}$-DFSes can easily be adapted to the more general case. In particular, we can state the necessary and sufficient conditions for the new models to be DFSes in analogy to the conditions (B-1) to (B-5).

5 The class of $\mathcal{F}_\xi$-DFSes

Based on the definition of the crisp range $\mathcal{T}_\gamma(X)$ of a three-valued cut, which provides a set of alternative choices for crisp arguments, we define the upper and lower bounds of the quantification results given these alternatives as follows:

Definition 43 *Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n$ be given. We define the upper bound mapping $\top_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ and the lower bound mapping $\bot_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ by*

$$\top_{Q,X_1,\ldots,X_n}(\gamma) = \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}$$
$$\bot_{Q,X_1,\ldots,X_n}(\gamma) = \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\,.$$

The following properties of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ are apparent:

Theorem 20 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are a choice of fuzzy arguments. Then*

1. *$\top_{Q,X_1,\ldots,X_n}$ is monotonically nondecreasing;*
2. *$\bot_{Q,X_1,\ldots,X_n}$ is monotonically nonincreasing;*
3. *$\bot_{Q,X_1,\ldots,X_n} \le \top_{Q,X_1,\ldots,X_n}$.*

(Proof: B.1, p.42+)

We can hence define the domain $\mathbb{T}$ of aggregation operators $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which combine the results of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ as follows.

Definition 44 *$\mathbb{T} \subseteq \mathbf{I}^\mathbf{I} \times \mathbf{I}^\mathbf{I}$ is defined by*

$$\mathbb{T} = \{(\top, \bot) : \ \top : \mathbf{I} \longrightarrow \mathbf{I} \textit{ nondecreasing, } \bot : \mathbf{I} \longrightarrow \mathbf{I} \textit{ nonincreasing, } \bot \le \top\}\,.$$

It is apparent from Th-20 that $(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) \in \mathbb{T}$, regardless of the semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the choice of fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. In addition, it can be shown that $\mathbb{T}$ is the minimal set which embeds all such pairs of mappings.

Theorem 21 *Let $(\top, \bot) \in \mathbb{T}$ be given. We define semi-fuzzy quantifiers $Q', Q'', Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by*

$$Q'(Y) = \top(\sup Y') \tag{18}$$
$$Q''(Y) = \bot(\inf Y'') \tag{19}$$
$$Q(Y) = \begin{cases} Q''(Y) & : \quad Y' = \varnothing \\ Q'(Y) & : \quad \textit{else} \end{cases} \tag{20}$$

where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \quad (21)$$
$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \quad (22)$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$.
Further suppose that the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ *is defined by*

$$\mu_X(c, z) = \begin{cases} \frac{1}{2} - \frac{1}{2}z & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2}z & : \quad c = 1 \end{cases} \quad (23)$$

for all $(c, z) \in \mathbf{2} \times \mathbf{I}$.
Then $\top = \top_{Q,X}$ *and* $\bot = \bot_{Q,X}$.
(Proof: B.2, p.44+)

Based on the aggregation operator $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, we define a corresponding QFM $\mathcal{F}_\xi$ in the obvious way.

Definition 45 *For every mapping* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, *the QFM* $\mathcal{F}_\xi$ *is defined by*

$$\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}), \quad (24)$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and all fuzzy subsets* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

The class of QFMs defined in this way will be called the class of $\mathcal{F}_\xi$-QFMs. Apparently, it contains a number of models that do not fulfill the DFS axioms. We hence impose five elementary conditions on the aggregation mapping ξ which provide a characterisation of the well-behaved models, i.e. of the class of $\mathcal{F}_\xi$-DFSes.

Definition 46 *For all* $(\top, \bot) \in \mathbb{T}$, *we impose the following conditions on aggregation mappings* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$.

If $\top = \bot$, *then* $\xi(\top, \bot) = \top(0)$ (X-1)

$\xi(1 - \bot, 1 - \top) = 1 - \xi(\top, \bot)$ (X-2)

If $\top = \mathrm{c}_1$ *and* $\bot(\mathbf{I}) \subseteq \{0, 1\}$, *then* $\xi(\top, \bot) = \frac{1}{2} + \frac{1}{2}\bot_*^{0\downarrow}$ (X-3)

$\xi(\top^\flat, \bot) = \xi(\top^\sharp, \bot)$ (X-4)

If $(\top', \bot') \in \mathbb{T}$ *such that* $\top \leq \top'$ *and* $\bot \leq \bot'$, *then* $\xi(\top, \bot) \leq \xi(\top', \bot')$ (X-5)

Let us now show that (X-1) to (X-5) are sufficient for $\mathcal{F}_\xi$ to be a DFS.

Theorem 22 *If* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5), *then* $\mathcal{F}_\xi$ *is a standard DFS.*
(Proof: B.3, p.47+)

Theorem 23 *The conditions* (X-1) *to* (X-5) *on* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *are necessary for* $\mathcal{F}_\xi$ *to be a DFS.*
(Proof: B.4, p.62+)

Hence the 'X-conditions' are necessary and sufficient for $\mathcal{F}_\xi$ to be a DFS, and all $\mathcal{F}_\xi$-DFSes are indeed standard DFSes. The criteria can also be shown to be independent. To facilitate the independence proof, we first relate $\mathcal{M}_\mathcal{B}$-QFMs to the broader class of $\mathcal{F}_\xi$-QFMs:

Theorem 24 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is a given aggregation mapping. Then $\mathcal{M}_\mathcal{B} = \mathcal{F}_\xi$, where $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is defined by*

$$\xi(\top, \bot) = \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) \tag{25}$$

for all $(\top, \bot) \in \mathbb{T}$, and $\mathrm{med}_{\frac{1}{2}}(\top, \bot)$ abbreviates

$$\mathrm{med}_{\frac{1}{2}}(\top, \bot)(\gamma) = \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) \,,$$

for all $\gamma \in \mathbf{I}$.
(Proof: B.5, p.74+)

Hence all $\mathcal{M}_\mathcal{B}$-QFMs are $\mathcal{F}_\xi$-QFMs, and all $\mathcal{M}_\mathcal{B}$-DFSes are $\mathcal{F}_\xi$-DFSes. The next theorem helps us to prove that the 'X-conditions' are independent, because the 'B-conditions' have already been shown to been independent in [7]:

Theorem 25 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given and $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is defined by equation* (25)*. Then*

1. (B-1) *is equivalent to* (X-1)*;*
2. (B-2) *is equivalent to* (X-2)*;*
3. *(a)* (B-3) *entails* (X-3)*;*
   (b) the conjunction of (X-2) *and* (X-3) *entails* (B-3)*;*
4. *(a)* (B-4) *entails* (X-4)*;*
   (b) the conjunction of (X-2) *and* (X-4) *entails* (B-4)*;*
5. (B-5) *is equivalent to* (X-5)*.*

(Proof: B.6, p.74+)

Theorem 26 *The conditions* (X-1) *to* (X-5) *are independent.*
(Proof: B.7, p.84+)

Let us now give examples of 'genuine' $\mathcal{F}_\xi$-DFSes (i.e. models that go beyond the special case of $\mathcal{M}_\mathcal{B}$-DFSes).

Definition 47 *The QFM $\mathcal{F}_{\mathrm{Ch}} = \mathcal{F}_{\xi_{\mathrm{Ch}}}$ is defined in terms of $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ by*

$$\xi_{\mathrm{Ch}}(\top, \bot) = \tfrac{1}{2} \int_0^1 \top(\gamma)\, d\gamma + \tfrac{1}{2} \int_0^1 \bot(\gamma)\, d\gamma \,,$$

for all $(\top, \bot) \in \mathbb{T}$.

Note. Both integrals are guaranteed to exist because $\top$ and $\bot$ are monotonic mappings (i.e., $\top$ nondecreasing and $\bot$ nonincreasing).

Theorem 27 *$\mathcal{F}_{\mathrm{Ch}}$ is a standard DFS.*
(Proof: B.8, p.85+)

The DFS $\mathcal{F}_{\mathrm{Ch}}$ is of special interest because of its close relationship to the well-known Choquet integral, which is defined as follows.

Definition 48 *Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing semi-fuzzy quantifier and $X \in \widetilde{\mathcal{P}}(E)$. The* Choquet integral *$(Ch) \int X\, dQ$ is defined by*

$$(Ch) \int X\, dQ = \int_0^1 Q(X_{\geq \alpha})\, d\alpha\,.$$

Theorem 28 *Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nondecreasing. Then for all $X \in \widetilde{\mathcal{P}}(E)$,*

$$(Ch) \int X\, dQ = \mathcal{F}_{\mathrm{Ch}}(Q)(X)\,.$$

(Proof: B.9, p.89+)

Hence $\mathcal{F}_{\mathrm{Ch}}$ coincides with the Choquet integral on fuzzy quantifiers whenever the latter is defined. In order to relate this result with previous work on fuzzy quantification, we first need to introduce some more notation.

Definition 49 *Let a finite base set $E \neq \varnothing$ of cardinality $|E| = m$ be given. For a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$, we denote by $\mu_{[j]}(X) \in \mathbf{I}$, $j = 1, \ldots, m$, the j-th largest membership value of X (including duplicates).*
More formally, consider an ordering of the elements of E such that $E = \{e_1, \ldots, e_m\}$ and $\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)$. Then define $\mu_{[j]}(X) = \mu_X(e_j)$. It is apparent that the results do not depend on the chosen ordering if ambiguities exist.
We stipulate that $\mu_{[0]}(X) = 1$ and that $\mu_{[j]}(X) = 0$ whenever $j > m$.

As a corollary of the above theorem, we then obtain (cf. [5]):

Theorem 29 *Suppose $E \neq \varnothing$ is a finite base set, $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is a nondecreasing mapping such that $q(0) = 0$, $q(|E|) = 1$, and $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by $Q(Y) = q(|Y|)$ for all $Y \in \mathcal{P}(E)$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}_{\mathrm{Ch}}(Q)(X) = \sum_{j=1}^{|E|} (q(j) - q(j-1)) \cdot \mu_{[j]}(X)\,,$$

i.e. $\mathcal{F}_{\mathrm{Ch}}$ consistently generalises Yager's OWA approach [14].
(Proof: B.10, p.90+)

Definition 50 *The QFM $\mathcal{F}_S$ is defined in terms of $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ by*

$$\xi_S(\top, \bot) = \begin{cases} \min(\top_1^*, \frac{1}{2} + \frac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}) & : \quad \bot(0) > \frac{1}{2} \\ \max(\bot_1^*, \frac{1}{2} - \frac{1}{2}\top_*^{\geq\frac{1}{2}\downarrow}) & : \quad \top(0) < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$, where the coefficients $f_*^{\leq\frac{1}{2}\downarrow}, f_*^{\geq\frac{1}{2}\downarrow} \in \mathbf{I}$ are defined by

$$f_*^{\leq\frac{1}{2}\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \leq \tfrac{1}{2}\} \tag{26}$$

$$f_*^{\geq\frac{1}{2}\downarrow} = \inf\{\gamma \in \mathbf{I} : f(\gamma) \geq \tfrac{1}{2}\}\,, \tag{27}$$

for all $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Theorem 30 *$\mathcal{F}_S$ is a standard DFS.*
(Proof: B.11, p.93+)

A third model of interest is the following QFM $\mathcal{F}_A$:

Definition 51 *The QFM $\mathcal{F}_A$ is defined in terms of $\xi_A : \mathbb{T} \longrightarrow \mathbf{I}$ by*

$$\xi_A(\top, \bot) = \begin{cases} \min(\bot_0^*, \frac{1}{2} + \frac{1}{2}\bot_*^{0\downarrow}) & : \quad \bot_0^* > \frac{1}{2} \\ \max(\top_0^*, \frac{1}{2} - \frac{1}{2}\top_*^{1\downarrow}) & : \quad \top_0^* < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

for all $(\top, \bot) \in \mathbb{T}$.

Theorem 31 *$\mathcal{F}_A$ is a standard DFS.*
(Proof: B.12, p.99+)

Turning to properties of $\mathcal{F}_\xi$-DFSes, we shall first investigate the precise conditions under which an $\mathcal{F}_\xi$-DFS propagates fuzziness in quantifiers and/or in arguments.

Definition 52 *We say that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$* propagates fuzziness *if and only if*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\top \preceq_c \top'$ and $\bot \preceq_c \bot'$.

Theorem 32 *An $\mathcal{F}_\xi$-QFM propagates fuzziness in quantifiers if and only if ξ propagates fuzziness.*
(Proof: B.13, p.104+)

If $\mathcal{F}_\xi$ is a DFS, then ξ's propagating of fuzziness is equivalent to the following condition, which is much easier to check:

Theorem 33 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5). *Then* ξ *propagates fuzziness if and only if*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

for all $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) > \frac{1}{2}$.
(Proof: B.14, p.106+)

For proofs that a given $\mathcal{F}_\xi$-DFS does not propagate fuzziness in quantifiers, the following necessary condition can be of interest.

Theorem 34 *Let* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *be a mapping which satisfies* (X-1) *to* (X-5). *If* ξ *propagates fuzziness, then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $(\top, \bot) \in \mathbb{T}$ *such that* $\top(0) \geq \frac{1}{2} \geq \bot(0)$.
(Proof: B.15, p.110+)

It is this condition which explains why the results of $\mathcal{M}_\mathcal{B}$-DFSes tend to attain $\frac{1}{2}$ when the input is overly fuzzy. If one really needs different quantification results for $(\top, \bot), (\top', \bot')$ with $\bot(0) \leq \frac{1}{2} \leq \top(0)$ and $\bot'(0) \leq \frac{1}{2} \leq \top'(0)$, one obviously must resort to $\mathcal{F}_\xi$-DFSes that do not propagate fuzziness in quantifiers.
As concerns our examples of $\mathcal{F}_\xi$-DFSes, we can attest the following.

Theorem 35 $\mathcal{F}_{\mathrm{Ch}}$ *does not propagate fuzziness in quantifiers.*
(Proof: B.16, p.110+)

Hence $\mathcal{F}_{\mathrm{Ch}}$ is a 'genuine' $\mathcal{F}_\xi$-DFS (i.e. not an $\mathcal{M}_\mathcal{B}$-DFS) by Th-19. In particular, this proves that the $\mathcal{F}_\xi$-DFSes indeed form a more general class of DFSes than $\mathcal{M}_\mathcal{B}$-DFSes.
For the DFS $\mathcal{F}_S$, we have a positive result.

Theorem 36 $\mathcal{F}_S$ *propagates fuzziness in quantifiers.*
(Proof: B.17, p.111+)

Turning to $\mathcal{F}_A$, we have

Theorem 37 $\mathcal{F}_A$ *does not propagate fuzziness in quantifiers.*
(Proof: B.18, p.111+)

We can also state the necessary and sufficient conditions on ξ for $\mathcal{F}_\xi$ to propagate fuzziness in arguments. To this end, we first introduce the following property of ξ.

Definition 53 *We say that* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ propagates unspecificity *if and only if*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ *satisfy* $\top \geq \top'$ *and* $\bot \leq \bot'$.

Theorem 38 *An $\mathcal{F}_\xi$-QFM propagates fuzziness in arguments if and only if ξ propagates unspecificity.*
(Proof: B.19, p.112+)

If $\mathcal{F}_\xi$ is sufficiently well-behaved (in particular, if $\mathcal{F}_\xi$ is a DFS), it is possible to state the following equivalent condition:

Theorem 39 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2), (X-4) *and* (X-5). *Then the following conditions are equivalent:*

a. *ξ propagates unspecificity;*

b. *for all $(\top, \bot) \in \mathbb{T}$ with $\bot(0) \geq \frac{1}{2}$, $\xi(\top, \bot) = \xi(\mathrm{c}_1, \bot)$.*

(Proof: B.20, p.115+)

We can also establish a necessary condition which facilitates the proof that a given $\mathcal{F}_\xi$-DFSes does not propagate fuzziness in arguments:

Theorem 40 *If an $\mathcal{F}_\xi$-DFS propagates fuzziness in arguments, then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $(\top, \bot) \in \mathbb{T}$ such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$.
(Proof: B.21, p.118+)

For example, we can use this condition to prove that

Theorem 41 *$\mathcal{F}_{\mathrm{Ch}}$ does not propagate fuzziness in arguments.*
(Proof: B.22, p.119+)

As concerns $\mathcal{F}_S$, we have the following result.

Theorem 42 *$\mathcal{F}_S$ does not propagate fuzziness in arguments.*
(Proof: B.23, p.119+)

Note. Hence $\mathcal{F}_S$ is a 'genuine' $\mathcal{F}_\xi$ DFS as well, which is apparent from Th-19.
Turning to $\mathcal{F}_A$, which failed to propagate fuzziness in quantifiers, it is easily observed that $\mathcal{F}_A$ still propagates fuzziness in its arguments:

Theorem 43 *$\mathcal{F}_A$ propagates fuzziness in arguments.*
(Proof: B.24, p.119+)

In particular, the conditions of propagating fuzziness in quantifiers and arguments are independent in the case of $\mathcal{F}_\xi$-DFSes, as stated in the following corollary.

Theorem 44 *The conditions of propagating fuzziness in quantifiers and in arguments are independent for $\mathcal{F}_\xi$-DFSes.*
(Proof: B.25, p.120+)

Finally, we can justify the subclass of $\mathcal{M}_\mathcal{B}$-DFSes which are exactly those $\mathcal{F}_\xi$-DFSes that propagate fuzziness in both quantifiers and arguments.

Theorem 45 *Suppose an $\mathcal{F}_\xi$-DFS propagates fuzziness in both quantifiers and arguments. Then $\mathcal{F}_\xi$ is an $\mathcal{M}_\mathcal{B}$-DFS.*
(Proof: B.26, p.120+)

Note. The converse implication is already known from Th-19.
Next we shall investigate the exact conditions under which an $\mathcal{F}_\xi$-QFM is Q-continuous or arg-continuous. To be able to discuss Q-continuous $\mathcal{F}_\xi$-QFMs, we introduce a metric $d : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$. For all nondecreasing mappings $\top, \top' : \mathbf{I} \longrightarrow \mathbf{I}$, we define

$$d(\top, \top') = \sup\{|\top(\gamma) - \top'(\gamma)| : \gamma \in \mathbf{I}\}\,. \tag{28}$$

We proceed similarly for nondecreasing mappings $\bot, \bot' : \mathbf{I} \longrightarrow \mathbf{I}$. In this case,

$$d(\bot, \bot') = \sup\{|\bot(\gamma) - \bot'(\gamma)| : \gamma \in \mathbf{I}\}\,. \tag{29}$$

Finally, we define $d : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$ by

$$d((\top, \bot), (\top', \bot')) = \max(d(\top, \top'), d(\bot, \bot'))\,, \tag{30}$$

for all $(\top, \bot), (\top', \bot') \in \mathbb{T}$. It is apparent that d is indeed a metric. We will utilize d to express a condition on ξ which characterises the Q-continuous $\mathcal{F}_\xi$-QFMs.

Theorem 46 *Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping which satisfies* (X-5). *Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi$ is Q-continuous;

b. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ satisfy $d((\top, \bot), (\top', \bot')) < \delta$.

(Proof: B.27, p.122+)

If ξ is sufficiently well-behaved, then the above condition can simplified into the following criterion, which is easier to check.

Theorem 47 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2) *and* (X-5). *Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi$ is Q-continuous;

b. for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\xi(\top', \bot) - \xi(\top, \bot) < \varepsilon$ whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ satisfy $d(\top, \top') < \delta$ and $\top \le \top'$.

(Proof: B.28, p.128+)

We have the following results for the examples of $\mathcal{F}_\xi$-DFSes.

Theorem 48 *$\mathcal{F}_{\mathrm{Ch}}$ is Q-continuous.*
(Proof: B.29, p.130+)

Theorem 49 *$\mathcal{F}_S$ is not Q-continuous.*
(Proof: B.30, p.131+)

Theorem 50 *$\mathcal{F}_A$ is not Q-continuous.*
(Proof: B.31, p.132+)

As concerns continuity in arguments, we first need to introduced another distance measure $d' : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$, which can be used to characterise the arg-continuous $\mathcal{F}_\xi$-QFMs in terms of conditions on ξ. For all nondecreasing mappings $\top, \top' : \mathbf{I} \longrightarrow \mathbf{I}$, we define

$$d'(\top, \top') = \sup\{\inf\{\gamma' : \min(\top(\gamma'), \top'(\gamma')) \geq \max(\top(\gamma), \top'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\}\,. \tag{31}$$

Similarly for nonincreasing mappings $\bot, \bot' : \mathbf{I} \longrightarrow \mathbf{I}$,

$$d'(\bot, \bot') = \sup\{\inf\{\gamma' : \max(\bot(\gamma'), \bot'(\gamma')) \leq \min(\bot(\gamma), \bot'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\}\,. \tag{32}$$

Finally, we define $d' : \mathbb{T} \times \mathbb{T} \longrightarrow \mathbf{I}$ by

$$d'((\top, \bot), (\top', \bot')) = \max(d'(\top, \top'), d'(\bot, \bot'))\,, \tag{33}$$

for all $(\top, \bot), (\top', \bot') \in \mathbb{T}$. It is easily checked that d' is a 'pseudo-metric', i.e. it is symmetric and satisfies the triangle inequation, but $d'((\top, \bot), (\top', \bot')) = 0$ does not imply that $(\top, \bot) = (\top', \bot')$. However, d' is a metric modulo $\sharp\flat$, i.e. on the equivalence classes of $(\top, \bot) \sim (\top', \bot') \Leftrightarrow (\top^{\flat\sharp}, \bot^{\flat\sharp}) = ({\top'}^{\flat\sharp}, {\bot'}^{\flat\sharp})$. Hence $d'((\top, \bot), (\top', \bot')) = 0$ entails that $(\top, \bot) \sim (\top', \bot')$, i.e. $\xi(\top, \bot) = \xi(\top', \bot')$ whenever ξ satisfies (X-2), (X-4) and (X-5). Based on d', we can now assert the following.

Theorem 51 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2), (X-4) *and* (X-5). *Then the following conditions are equivalent:*

a. *$\mathcal{F}_\xi$ is arg-continuous;*

b. *for all $(\top, \bot) \in \mathbb{T}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top', \bot') \in \mathbb{T}$ satisfies $d'((\top, \bot), (\top', \bot')) < \delta$.*

(Proof: B.32, p.133+)

In some cases, the following sufficient condition can shorten the proof that a given $\mathcal{F}_\xi$ is arg-continuous.

Theorem 52 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2) *and* (X-5) *Then $\mathcal{F}_\xi$ is arg-continuous if the following condition holds: For all $\varepsilon > 0$ there exists $\delta > 0$ such that $\xi(\top', \bot) - \xi(\top, \bot) < \varepsilon$ whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ satisfy $d'(\top, \top') < \delta$ and $\top \leq \top'$.*
(Proof: B.33, p.146+)

Based on these theorems, it is easy to prove the following.

Theorem 53 *$\mathcal{F}_{\mathrm{Ch}}$ is arg-continuous.*
(Proof: B.34, p.148+)

Theorem 54 *$\mathcal{F}_S$ is not arg-continuous.*
(Proof: B.35, p.149+)

Theorem 55 *$\mathcal{F}_A$ is not arg-continuous.*
(Proof: B.36, p.150+)

Hence $\mathcal{F}_{\mathrm{Ch}}$ is continuous both in quantifiers and arguments; which is important for applications. The second example, $\mathcal{F}_S$, fails on both continuity conditions and is hence not practical. (We will see below that $\mathcal{F}_S$ is of theoretical interest because it represents a boundary case of $\mathcal{F}_\xi$-DFSes).
We are also interested in the specificity of $\mathcal{F}_\xi$-DFSes. The following theorem facilitates the proof that a given $\mathcal{F}_\xi$-QFM is less specific than another $\mathcal{F}_\xi$-QFM by relating the specificity order on $\mathcal{F}_\xi$ to the specificity order on ξ:

Theorem 56 *Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ be given mappings. Then the following conditions are equivalent:*

a. *$\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$;*

b. *$\xi \preceq_c \xi'$.*

(Proof: B.37, p.151+)

In the case of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers, it is sufficient to check a simpler condition.

Theorem 57 *Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ be given mappings which satisfy* (X-1) *to* (X-5) *and suppose that ξ, ξ' have the additional property that $\xi(\top, \bot) = \xi'(\top, \bot) = \frac{1}{2}$ whenever $(\top, \bot) \in \mathbb{T}$ with $\top(0) \geq \frac{1}{2} \geq \bot(0)$. Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$;

b. for all $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) > \frac{1}{2}$, $\xi(\top, \bot) \leq \xi'(\top, \bot)$.

(Proof: B.38, p.151+)

As regards least specific $\mathcal{F}_\xi$-DFSes, we can prove the following:

Theorem 58 *$\mathcal{M}_U$ is the least specific $\mathcal{F}_\xi$-DFS.*
(Proof: B.39, p.153+)

Turning to the issue of most specific models, I first state a theorem for establishing or rejecting specificity consistence. This is useful because specificity consistence is tightly coupled to the existence of least upper specificity bounds, see Th-6.

Theorem 59 *Consider a pair of mappings $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$. The QFMs $\mathcal{F}_\xi$ and $\mathcal{F}_{\xi'}$ are specificity consistent if and only if ξ, ξ' are specificity consistent, i.e. for all $(\top, \bot) \in \mathbb{T}$, either $\{\xi(\top, \bot), \xi'(\top, \bot)\} \subseteq [0, \frac{1}{2}]$ or $\{\xi(\top, \bot), \xi'(\top, \bot)\} \subseteq [\frac{1}{2}, 1]$.*
(Proof: B.40, p.156+)

An investigation of a possible most specific $\mathcal{F}_\xi$-DFS reveals the following.

Theorem 60 *The class of $\mathcal{F}_\xi$-DFSes is not specificity consistent.*
(Proof: B.41, p.157+)

Hence by Th-6, a "most specific $\mathcal{F}_\xi$-DFS" does not exist. However, we obtain a positive result if we restrict attention to the class of $\mathcal{F}_\xi$-DFSes which propagate fuzziness in quantifiers or arguments. This is apparent from the following observation.

Theorem 61 *Suppose $\mathbb{F}$ is a collection of $\mathcal{F}_\xi$-DFSes $\mathcal{F}_\xi \in \mathbb{F}$ with the property that $\xi(\top, \bot) = \frac{1}{2}$ whenever $(\top, \bot) \in \mathbb{T}$ is such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$. Then $\mathbb{F}$ is specificity consistent.*
(Proof: B.42, p.159+)

We then have the following corollaries.

Theorem 62 *The class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers is specificity consistent.*
(Proof: B.43, p.160+)

Theorem 63 *The class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in arguments is specificity consistent.*
(Proof: B.44, p.160+)

By Th-6, the $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers have a least upper specificity bound which, as it turns out, also propagates fuzziness in quantifiers.

Theorem 64 *$\mathcal{F}_S$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in quantifiers.*
(Proof: B.45, p.160+)

Similarly, we can conclude from Th-63 that there is a most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments.

Theorem 65 *$\mathcal{F}_A$ is the most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments.*
(Proof: B.46, p.161+)

6 Conclusion

It has been the goal of this report to broaden the class of known models of fuzzy quantification. There are several reasons why I wanted to explore standard DFSes beyond the class of $\mathcal{M}_{\mathcal{B}}$-DFSes introduced in [7]. The first reason is concerned with propagation of fuzziness. $\mathcal{M}_{\mathcal{B}}$-DFSes are particularly well-behaved because they propagate fuzziness in quantifiers as well as in arguments: the fuzzier the input, the fuzzier the output. In most cases, this is the expected and desirable behaviour because one usually does not want the results to become more precise when there is less precision in the input. However, I anticipate that there are applications in which it is preferable to sacrifice propagation of fuzziness, in order to prevent the results from attaining the least specific value of $\frac{1}{2}$. This might be the case, for example, when the input is overly fuzzy and one still needs a fine-grained result ranking. In these cases, one could profit from models that do not propagate fuzziness. The second reason stems from the intent to relate the present approach with existing work on fuzzy quantification. I have already shown in [7] that there exists a DFS $\mathcal{M}_{CX}$ which generalizes the Sugeno integral and hence the 'basic' FG-count approach[7] to arbitrary semi-fuzzy quantifiers, which can be multiplace and/or non-quantitative and need not be monotonic. However, a similar result concerning the Choquet integral and hence the 'basic' OWA approach[8] was still missing. In order to embed the Choquet integral into the framework of DFS theory, it was necessary to go beyond $\mathcal{M}_{\mathcal{B}}$-DFSes because the Choquet integral does not propagate fuzziness. Last but not least, the study of a broader class of standard models is interesting in its own right, because it helps to gain new insight into the structure of fuzzy quantification that might eventually lead to a complete classification of standard DFSes.

In the report, I have first reviewed the basic concepts of DFS theory and cited a few additional definitions and theorems about properties of DFSes the familiarity with which is necessary to understand the new theorems and to carry out their proofs. In addition, a couple of special adequacy properties have been introduced, which are desirable but not required for general DFSes. Apart from the criteria of propagating fuzziness in quantifiers and/or arguments, the most important extra requirement is certainly that of robustness with respect to slight changes in the parameters. This stability consideration is covered by the criterion of arg-continuity, which accounts for differences in the arguments, and by the criterion of Q-continuity, which accounts for differences in the interpretation of quantifiers. These conditions are essential for practical applications because they ensure a certain insensitivity with respect to noise.

After defining these properties, the class of $\mathcal{M}_{\mathcal{B}}$-DFSes has been reviewed. The construction of these models in terms of three-valued cuts has provided a suitable starting point for the generalisation to a broader class of models, the class of $\mathcal{F}_{\xi}$-DFSes. To this end, $Q_\gamma(X_1, \dots, X_n)$ and $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ have been replaced with a pair of mappings $(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) \in \mathbb{T}$, and a corresponding aggregation operator $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which maps such pairs into quantification results $\xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n})$.

I have presented the essential criteria that make it easy to check whether a given

[7] i.e. the formula for quantitative nondecreasing one-place quantifiers.

[8] again, the formula for quantitative regular nondecreasing one-place quantifiers.

$\mathcal{F}_\xi$-QFM is a DFS; whether it propagates fuzziness in quantifiers and/or in arguments; whether it is Q-continuous and/or arg-continuous; whether it is specificity consistent with other $\mathcal{F}_\xi$-DFSes; and how it compares to these DFSes by specificity.

In particular, I have shown that the class of $\mathcal{F}_\xi$-DFSes is broad enough to contain DFSes which are rather different from $\mathcal{M}_\mathcal{B}$-DFSes. Among the $\mathcal{F}_\xi$-DFSes, some models neither propagate fuzziness in arguments nor in quantifiers; some models propagate fuzziness in quantifiers, but not in arguments, while others propagate fuzziness in arguments, but not in quantifiers, and some propagate fuzziness both in quantifiers and arguments. The latter class of $\mathcal{F}_\xi$-DFSes has been proven be exactly the class of $\mathcal{M}_\mathcal{B}$-DFSes. The present report has hence succeeded in defining a class of DFSes which (unlike $\mathcal{M}_\mathcal{B}$-DFSes) fail to propagate fuzziness. The report also explains why the models which do not propagate fuzziness have a chance of performing better than those that propagate fuzziness in situations where the inputs are overly fuzzy. This can be mainly attributed to the property described in Th-34 and Th-40: if an $\mathcal{F}_\xi$-DFS propagates fuzziness in quantifiers or in arguments, then $\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) = \frac{1}{2}$ whenever $\top_{Q,(X_1,\ldots,X_n)} \geq \frac{1}{2}$ and $\bot_{Q,(X_1,\ldots,X_n)} \leq \frac{1}{2}$. Hence there is a certain range in which the results of an $\mathcal{F}_\xi$-DFS are constantly $\frac{1}{2}$, which can be undesirable if one needs a fine-grained result ranking. Because both types of propagating fuzziness cause this kind of behaviour, one must resort to models that fail on both conditions if one needs specific results even when there is a lot of fuzziness in the inputs.

The DFS $\mathcal{F}_{\mathrm{Ch}}$ is a promising choice in such situations because it also fulfills the continuity requirements. It is anticipated that $\mathcal{F}_{\mathrm{Ch}}$ will find a number of uses in real-world applications that utilize fuzzy quantifiers. However, $\mathcal{F}_{\mathrm{Ch}}$ is also an interesting model from a scientific perspective because $\mathcal{F}_{\mathrm{Ch}}$ can be shown to embed the Choquet integral, thus generalizing it to the case of non-monotonic and multi-place quantifiers. The report hence also succeeds in relating DFS theory with existing work on fuzzy quantification because the Choquet integral is known to embed the OWA approach.

Concerning theoretical aspects of fuzzy quantification, the report proves that there are standard DFSes beyond $\mathcal{M}_\mathcal{B}$-DFSes, and it also substantiates the existence of standard DFS which do not propagate fuzziness in arguments and/or quantifiers. In particular, it has been proven that the conditions of propagating fuzziness in quantifiers and arguments are mutually independent. Apart from propagation of fuzziness, some novel results concerning the specificity order $\preceq_c$ have also been established. In particular, I have shown that $\preceq_c$ is not a total order on standard DFSes, not even in the 'simple' case of $\mathcal{M}_\mathcal{B}$-DFSes. Most importantly, I have shown that there is no most specific standard DFS. However, there is a most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in quantifiers, viz $\mathcal{F}_S$, and there is also a most specific $\mathcal{F}_\xi$-DFS that propagates fuzziness in arguments, viz $\mathcal{F}_A$. Apparently $\mathcal{F}_S$ and $\mathcal{F}_A$ are not practical models because they fail on both continuity conditions, but this seems to be typical for boundary cases with respect to specificity.

The present report has shed some light on theoretical aspects of fuzzy quantification, but a number of issues remain unresolved. Most importantly, there is no evidence yet concerning the question whether there are standard DFSes beyond $\mathcal{F}_\xi$-DFSes and if so, how the $\mathcal{M}_\mathcal{B}$-DFSes and $\mathcal{F}_\xi$-DFSes are located within the 'full' class of standard

DFSes. The ultimate goal is to identify this full class, to uncover the structure of its models, and to characterise its natural subclasses. Further theoretical work is required to clarify these matters, and it is hoped that some of the techniques presented in this report will help to accomplish this endeavour.

Appendix

Any proposition which occurs in the main text is called a *theorem*, and any proposition which only occurs in the proofs a *lemma*. Theorems are referred to as Th-n, where n is the number of the theorem, while lemmata are referred to as L-n, where n is the number of the lemma. Equations which are embedded in proofs are referred to as (n), where n is the number of the equation.

A Proof of theorems in chapter 4

A.1 Proof of Theorem 18

Lemma 1 *Suppose* $\mathcal{B}'_1, \mathcal{B}'_2 : \mathbb{H} \longrightarrow \mathbf{I}$ *are given. Further suppose that* $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{BB}$ *are the mappings associated with* $\mathcal{B}'_1$ *and* $\mathcal{B}'_2$*, resp., according to equation* (15)*, and* $\mathcal{M}_{\mathcal{B}_1}$*,* $\mathcal{M}_{\mathcal{B}_2}$ *are the corresponding QFMs defined by Def. 33. Then* $\mathcal{M}_{\mathcal{B}_1} \preceq_c \mathcal{M}_{\mathcal{B}_2}$ *iff* $\mathcal{B}'_1 \leq \mathcal{B}'_2$.

Proof See [7, Th-86, p.61].

Proof of Theorem 18

We recall the $\mathcal{M}_\mathcal{B}$-DFSes $\mathcal{M}_U$ and $\mathcal{M}_S$ defined by Def. 40 and Def. 41. We stipulate $\mathcal{M}_{\mathcal{B}'_1} = (\mathcal{M}_U + \mathcal{M}_S)/2$. Being a convex combination of standard DFSes, $\mathcal{M}_{\mathcal{B}'_1}$ is also a standard DFS by Th-4. In addition, it is apparent from Def. 33 and (16) that

$$\mathcal{B}'_1(f) = \tfrac{\mathcal{B}'_U(f)+\mathcal{B}'_S(f)}{2}\,, \tag{34}$$

where

$$\mathcal{B}'_U(f) = \max(f_*^{1\uparrow}, f_1^*) \tag{35}$$

$$\mathcal{B}'_S(f) = \min(f_*^{0\downarrow}, f_0^*)\,, \tag{36}$$

for all $f \in \mathbb{H}$, see Def. 40 and Def. 41.
Now let us define mappings $f, g \in \mathbb{H}$ as follows.

$$f(\gamma) = \begin{cases} 1 - \frac{3}{2}\gamma & : \quad \gamma \leq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma & : \quad \gamma > \frac{1}{2} \end{cases} \tag{37}$$

$$g(\gamma) = \begin{cases} 1 - \frac{1}{2}\gamma & : \quad \gamma \leq \frac{1}{2} \\ \frac{3}{2} - \frac{3}{2}\gamma & : \quad \gamma > \frac{1}{2} \end{cases} \tag{38}$$

for all $\gamma \in \mathbf{I}$. Then by (10), (9), (13) and (12),

$$f_*^{0\downarrow} = 1$$
$$f_0^* = 1$$
$$f_*^{1\uparrow} = 0$$
$$f_1^* = 0$$
$$g_*^{0\downarrow} = 1$$
$$g_0^* = 1$$
$$g_*^{1\uparrow} = 0$$
$$g_1^* = 0\,.$$

Hence $\mathcal{B}'_U(f) = \mathcal{B}'_U(g) = 0$, $\mathcal{B}'_S(f) = \mathcal{B}'_S(g) = 1$ by (35), (36) and in turn,

$$\mathcal{B}'_1(f) = \tfrac{1}{2}$$
$$\mathcal{B}'_1(g) = \tfrac{1}{2}\,.$$

Now let us consider the results of $\mathcal{B}'_{\int}$, see Def. 39. In the case of f,

$$\begin{aligned}
\mathcal{B}'_{\int}(f) &= \int_0^1 f(\gamma)\,d\gamma && \text{by Def. 39}\\
&= \int_0^{\frac{1}{2}} (1 - \tfrac{3}{2}\gamma)\,d\gamma + \int_{\frac{1}{2}}^1 (\tfrac{1}{2} - \tfrac{1}{2}\gamma)\,d\gamma && \text{by (37)}\\
&= \tfrac{3}{8}
\end{aligned}$$

and in the case of g,

$$\begin{aligned}
\mathcal{B}'_{\int}(g) &= \int_0^1 g(\gamma)\,d\gamma && \text{by Def. 39}\\
&= \int_0^{\frac{1}{2}} (1 - \tfrac{1}{2}\gamma)\,d\gamma + \int_{\frac{1}{2}}^1 (\tfrac{3}{2} - \tfrac{3}{2})\,d\gamma && \text{by (38)}\\
&= \tfrac{5}{8}\,.
\end{aligned}$$

Hence there exist $f, g \in \mathbb{B}$ with $\mathcal{B}'_{\int}(f) = \frac{3}{8} < \frac{1}{2} = \mathcal{B}'_1(f)$ and $\mathcal{B}'_{\int}(g) = \frac{5}{8} > \frac{1}{2} = \mathcal{B}'_1(g)$. By L-1, $\mathcal{M} \not\preceq_c \mathcal{M}_{\mathcal{B}'_1}$ and $\mathcal{M}_{\mathcal{B}'_1} \not\preceq_c \mathcal{M}$, i.e. $\preceq_c$ is a genuine *partial* order.

B Proof of theorems in chapter 5

B.1 Proof of Theorem 20

Lemma 2 *Let $E \neq \varnothing$ be a given base set and $X \in \widetilde{\mathcal{P}}(E)$. Then*

$$\mathcal{T}_\gamma(X) \subseteq \mathcal{T}_{\gamma'}(X)$$

whenever $\gamma \leq \gamma'$.

Proof If $\gamma = \gamma' = 0$, this is trivial. If $\gamma = 0$ and $\gamma' > 0$, then

$$X_0^{\min} = X_{>\frac{1}{2}} \supseteq X_{\geq \frac{1}{2} + \frac{1}{2}\gamma'} = X_{\gamma'}^{\min} \tag{39}$$

and

$$X_0^{\max} = X_{\geq \frac{1}{2}} \subseteq X_{> \frac{1}{2} - \frac{1}{2}\gamma'} = X_{\gamma'}^{\max} \tag{40}$$

which is apparent from Def. 30, Def. 28 and Def. 29. Hence

$$\begin{aligned} \mathcal{T}_0(X) &= \{Y : X_0^{\min} \subseteq Y \subseteq X_0^{\max}\} && \text{by Def. 30} \\ &\subseteq \{Y : X_{\gamma'}^{\min} \subseteq Y \subseteq X_{\gamma'}^{\max}\} && \text{by (39), (40)} \\ &= \mathcal{T}_{\gamma'}(X)\,. && \text{by Def. 30} \end{aligned}$$

Finally if $0 < \gamma \leq \gamma'$, then

$$X_{\gamma'}^{\min} = X_{\geq \frac{1}{2} + \frac{1}{2}\gamma'} \subseteq X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = X_{\gamma}^{\min} \tag{41}$$

and

$$X_{\gamma}^{\max} = X_{> \frac{1}{2} - \frac{1}{2}\gamma} \subseteq X_{> \frac{1}{2} - \frac{1}{2}\gamma'} = X_{\gamma'}^{\max} \tag{42}$$

by Def. 30, Def. 28 and Def. 29. Therefore

$$\begin{aligned} \mathcal{T}_\gamma(X) &= \{Y : X_\gamma^{\min} \subseteq Y \subseteq X_\gamma^{\max}\} && \text{by Def. 30} \\ &\subseteq \{Y : X_{\gamma'}^{\min} \subseteq Y \subseteq X_{\gamma'}^{\max}\} && \text{by (41), (42)} \\ &= \mathcal{T}_{\gamma'}(X)\,. && \text{by Def. 30} \end{aligned}$$

Proof of Theorem 20

Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given.

1. In order to prove that $\top_{Q,X_1,\dots,X_n}$ is nondecreasing, let $\gamma, \gamma' \in \mathbf{I}$, $\gamma \leq \gamma'$. Then $\mathcal{T}_\gamma(X_i) \subseteq \mathcal{T}_{\gamma'}(X_i)$ for $i = 1, \dots, n$ by L-2. In particular,

$$\begin{aligned} &\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n)\} \\ &\subseteq \{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_{\gamma'}(X_1), \dots, Y_n \in \mathcal{T}_{\gamma'}(X_n)\}\,. \end{aligned} \tag{43}$$

Therefore

$$\begin{aligned} \top_{Q,X_1,\dots,X_n}(\gamma) &= \sup\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\ &\leq \sup\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_{\gamma'}(X_1), \dots, Y_n \in \mathcal{T}_{\gamma'}(X_n)\} && \text{by (43)} \\ &= \top_{Q,X_1,\dots,X_n}(\gamma')\,, && \text{by Def. 43} \end{aligned}$$

i.e. $\top_{Q,X_1,\dots,X_n}$ is nondecreasing.

2. Turning to $\bot_{Q,X_1,\dots,X_n}$, we again choose $\gamma \leq \gamma' \in \mathbf{I}$. Then

$$\begin{aligned}\bot_{Q,X_1,\dots,X_n}(\gamma) &= \inf\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43}\\ &\geq \inf\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_{\gamma'}(X_1),\dots,Y_n \in \mathcal{T}_{\gamma'}(X_n)\} && \text{by (43)}\\ &= \bot_{Q,X_1,\dots,X_n}(\gamma'), && \text{by Def. 43}\end{aligned}$$

which proves that $\bot_{Q,X_1,\dots,X_n}$ is nonincreasing.

3. It is apparent from Def. 30 that $X_\gamma^{\min} \subseteq X_\gamma^{\max}$ for arbitrary fuzzy subsets $X \in \widetilde{\mathcal{P}}(E)$ and cutting parameters $\gamma \in \mathbf{I}$. Given the semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, fuzzy subsets $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$, we hence know that $\mathcal{T}_\gamma(X_i) \neq \varnothing$ for all $i = 1,\dots,n$. In particular,

$$\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} \neq \varnothing\,. \tag{44}$$

Recalling that $\inf Z \leq \sup Z$ whenever $Z \subseteq \mathbf{I}$ is nonempty, we conclude that

$$\begin{aligned}\bot_{Q,X_1,\dots,X_n}(\gamma) &= \inf\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43}\\ &\leq \sup\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by (44)}\\ &= \top_{Q,X_1,\dots,X_n}(\gamma)\,. && \text{by Def. 43}\end{aligned}$$

Because $\gamma \in \mathbf{I}$ was arbitrary, this means that $\bot_{Q,X_1,\dots,X_n} \leq \top_{Q,X_1,\dots,X_n}$, as desired.

B.2 Proof of Theorem 21

Lemma 3 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is nondecreasing in its i-th argument ($i \in \{1,\dots,n\}$) and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$\top_{Q,X_1,\dots,X_n}(\gamma)$$
$$= \sup\{Q(Y_1,\dots,Y_{i-1},(X_i)_\gamma^{\max},Y_{n+1},\dots,Y_n) : Y_i \in \{1,\dots,i-1,i+1,\dots,n\}\}$$
$$\bot_{Q,X_1,\dots,X_n}(\gamma)$$
$$= \inf\{Q(Y_1,\dots,Y_{i-1},(X_i)_\gamma^{\min},Y_{n+1},\dots,Y_n) : Y_i \in \{1,\dots,i-1,i+1,\dots,n\}\}$$

for all $\gamma \in \mathbf{I}$. Similarly if Q is nonincreasing in its i-th argument, then

$$\top_{Q,X_1,\dots,X_n}(\gamma)$$
$$= \sup\{Q(Y_1,\dots,Y_{i-1},(X_i)_\gamma^{\min},Y_{n+1},\dots,Y_n) : Y_i \in \{1,\dots,i-1,i+1,\dots,n\}\}$$
$$\bot_{Q,X_1,\dots,X_n}(\gamma)$$
$$= \inf\{Q(Y_1,\dots,Y_{i-1},(X_i)_\gamma^{\max},Y_{n+1},\dots,Y_n) : Y_i \in \{1,\dots,i-1,i+1,\dots,n\}\}$$

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ be given. We shall assume that Q is nondecreasing in its i-th argument, where $i \in \{1,\dots,n\}$. Further let $\gamma \in \mathbf{I}$

be given. By Def. 30, $Y_i \subseteq (X_i)^{\max}_\gamma$ for all $Y_i \in \mathcal{T}_\gamma(X_i)$. Because Q is nondecreasing in i, we conclude that

$$Q(Y_1, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\max}_\gamma, Y_{i+1}, \ldots, Y_n)$$

for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$. Hence by Def. 43,

$$\begin{aligned}&\top_{Q,X_1,\ldots,X_n}\\ &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\\ &\leq \{Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\max}_\gamma, Y_{n+1}, \ldots, Y_n) : Y_i \in \{1, \ldots, i-1, i+1, \ldots, n\}\}\end{aligned}$$

Noticing that $(X_i)^{\max}_\gamma \in \mathcal{T}_\gamma(X_i)$ by Def. 30, the converse inequation also holds, i.e.

$$\begin{aligned}&\top_{Q,X_1,\ldots,X_n}\\ &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\\ &= \sup\{Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\max}_\gamma, Y_{n+1}, \ldots, Y_n) : Y_i \in \{1, \ldots, i-1, i+1, \ldots, n\}\}\,,\end{aligned}$$

as desired.
Concerning $\bot_{Q,X_1,\ldots,X_n}$, we may proceed similarly. First we observe that by Def. 30, $(X_i)^{\min}_\gamma \subseteq Y_i$ for all $Y_i \in \mathcal{T}_\gamma(X_i)$. Because Q is nondecreasing in i, we conclude that

$$Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\min}_\gamma, Y_{i+1}, \ldots, Y_n) \leq Q(Y_1, \ldots, Y_n)$$

for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$. Again by Def. 43,

$$\begin{aligned}&\bot_{Q,X_1,\ldots,X_n}\\ &= \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\\ &\geq \{Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\min}_\gamma, Y_{n+1}, \ldots, Y_n) : Y_i \in \{1, \ldots, i-1, i+1, \ldots, n\}\}\end{aligned}$$

Because $(X_i)^{\min}_\gamma \in \mathcal{T}_\gamma(X_i)$ by Def. 30, the converse inequation also holds, i.e. we get the desired

$$\begin{aligned}&\bot_{Q,X_1,\ldots,X_n}\\ &= \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}\\ &= \inf\{Q(Y_1, \ldots, Y_{i-1}, (X_i)^{\min}_\gamma, Y_{n+1}, \ldots, Y_n) : Y_i \in \{1, \ldots, i-1, i+1, \ldots, n\}\}\,.\end{aligned}$$

The proof for a quantifier which is nonincreasing in its i-th argument is analogous.

Lemma 4 *Let* $(\top, \bot) \in \mathbb{T}$ *be given. Then*

$$\top(\gamma) \geq \bot(\gamma')\,,$$

for all $\gamma, \gamma' \in \mathbf{I}$.

Proof Suppose $(\top, \bot) \in \mathbb{T}$ and $\gamma, \gamma' \in \mathbf{I}$. From Def. 44, we know that $\top$ is nondecreasing, that $\bot$ is nondecreasing and that $\bot \leq \top$. Therefore

$$\begin{aligned} \top(\gamma) &\geq \top(0) && \text{because } \top \text{ nondecreasing} \\ &\geq \bot(0) && \text{because } \bot \leq \top \\ &\geq \bot(\gamma') \,. && \text{because } \bot \text{ nonincreasing} \end{aligned}$$

Lemma 5 *Suppose* $(\top, \bot) \in \mathbb{T}$ *is given. The semi-fuzzy quantifier* $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *defined by equation* (20) *is nondecreasing in its argument.*

Proof Let $(\top, \bot) \in \mathbb{T}$ be given. We further assume that $Q', Q'', Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ are defined by (18), (19) and (20), respectively.
Now let a choice of $Y_1, Y_2 \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$ be given where $Y_1 \subseteq Y_2$. We define crisp subsets $Y_1', Y_2', Y_1'', Y_2'' \in \mathcal{P}(\mathbf{I})$ according to (21) and (22), resp. Then

$$\begin{aligned} Y_1' &= \{z \in \mathbf{I} : (0, z) \in Y_1\} \\ Y_1'' &= \{z \in \mathbf{I} : (1, z) \in Y_1\} \\ Y_2' &= \{z \in \mathbf{I} : (0, z) \in Y_2\} \\ Y_2'' &= \{z \in \mathbf{I} : (1, z) \in Y_2\} \,. \end{aligned}$$

It is hence apparent from $Y_1 \subseteq Y_2$ that $Y_1' \subseteq Y_2'$ and $Y_1'' \subseteq Y_2''$. Therefore $\sup Y_1' \leq \sup Y_2'$ and because $\top$ is nondecreasing by Def. 44,

$$Q'(Y_1) = \top(\sup Y_1') \leq \top(\sup Y_2') = Q'(Y_2) \,,$$

i.e. Q' is nondecreasing. Similarly, we conclude from $Y_1'' \subseteq Y_2''$ that $\inf Y_1'' \geq \inf Y_2''$. Hence because $\bot$ is nonincreasing by Def. 44,

$$Q''(Y_1) = \bot(\inf Y_1'') \leq \bot(\inf Y_2'') = Q''(Y_2) \,,$$

i.e. Q'' is nondecreasing as well. Finally, let us utilize that by L-4,

$$\top(\gamma) \geq \bot(\gamma')$$

for all $\gamma, \gamma' \in \mathbf{I}$. Hence

$$Q'(Y) = \top(\sup Y') \geq \bot(\inf Y'') = Q''(Y) \,,$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where $Y', Y'' \in \mathcal{P}(\mathbf{I})$ are defined by (21) and (22).
Summarizing, we now know that Q' and Q'' are nondecreasing and that $Q' \geq Q''$. In order to finish the proof, we separate the following cases.

a. $Y_2' = \varnothing$**.** Then $Y_1' = \varnothing$ also because $Y_1' \subseteq Y_2'$. Hence

$$\begin{aligned} Q(Y_1) &= Q''(Y_1) && \text{by (20) because } Y_1' = \varnothing \\ &\leq Q''(Y_2) && \text{because } Q'' \text{ nondec and } Y_1 \subseteq Y_2 \\ &= Q(Y_2) \,. && \text{by (20) because } Y_2' = \varnothing \end{aligned}$$

b. $Y'_1 = \varnothing$ **and** $Y'_2 \neq \varnothing$**.** In this case,

$$\begin{aligned}
Q(Y_1) &= Q''(Y_1) && \text{by (20) because } Y'_1 = \varnothing \\
&\leq Q'(Y_1) && \text{because } Q'' \leq Q' \\
&\leq Q'(Y_2) && \text{because } Q' \text{ nondec and } Y_1 \subseteq Y_2 \\
&= Q(Y_2)\,. && \text{by (20) because } Y'_2 = \varnothing
\end{aligned}$$

c. $Y'_1 \neq \varnothing$**.** Then $Y'_2 \neq \varnothing$ as well because $Y_1 \subseteq Y_2$. Therefore

$$\begin{aligned}
Q(Y_1) &= Q'(Y_1) && \text{by (20) because } Y'_1 \neq \varnothing \\
&\leq Q'(Y_2) && \text{because } Q' \text{ nondec and } Y_1 \subseteq Y_2 \\
&= Q(Y_2)\,, && \text{by (20) because } Y'_2 \neq \varnothing
\end{aligned}$$

as desired.

Proof of Theorem 21

Suppose $(\top, \bot) \in \mathbb{T}$ are given and $Q', Q'', Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ are defined as stated in the theorem. Then by Def. 30,

$$\begin{aligned}
X_0^{\min} &= X_{\geq \frac{1}{2}} = \{1\} \times (0, 1] \\
X_0^{\max} &= X_{> \frac{1}{2}} = (\{0\} \times \{0\}) \cup (\{1\} \times \mathbf{I})
\end{aligned}$$

and for $\gamma > 0$,

$$\begin{aligned}
X_\gamma^{\min} &= X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = \{1\} \times [\gamma, 1] \\
X_\gamma^{\max} &= X_{> \frac{1}{2} - \frac{1}{2}\gamma} = (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I})
\end{aligned}$$

Hence for $\gamma = 0$, $\top_{Q,X}(0) = \sup\{Q(Y) : Y \in \mathcal{T}_0(X)\} = \top(0)$, which is apparent because $Z = X_0^{\max} = (\{0\} \times \{0\}) \cup (\{1\} \times \mathbf{I}) \in \mathcal{T}_0(X)$ reaches the maximum $Q(Z) = Q'(Z) = \top(\sup\{0\}) = \top(0)$; see L-3 and L-5. By the same lemmata, $\bot_{Q,X}(0) = \inf\{Q(Y) : Y \in \mathcal{T}_0(X)\} = \bot(0)$ because $Z' = X_0^{\min} = \{1\} \times (0, 1] \in \mathcal{T}_0(X)$ reaches the minimum $Q(Z') = Q''(Z') = \bot(\inf(0, 1]) = \bot(0)$, as desired.
In the case that $\gamma > 0$, we again apply L-3 and L-5 and conclude that $\top_{Q,X}(\gamma) = \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} = \top(\gamma)$, because $Z = X_\gamma^{\max} = (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I}) \in \mathcal{T}_\gamma(X)$ reaches the maximum $Q(Z) = Q'(Z) = \top(\sup[0, \gamma)) = \top(\gamma)$. Similarly, we obtain that $\bot_{Q,X}(\gamma) = \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} = \bot(\gamma)$ because $Z' = X_\gamma^{\min} = \{1\} \times [\gamma, 1] \in \mathcal{T}_\gamma(X)$ attains the minimum $Q(Z') = Q''(Z') = \bot(\inf[\gamma, 1]) = \bot(\gamma)$.
Because $\gamma \in \mathbf{I}$ was arbitrary, this proves that $\top = \top_{Q,X}$ and $\bot = \bot_{Q,X}$, as desired.

B.3 Proof of Theorem 22

Lemma 6 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \breve{\mathcal{P}}(E) \subseteq \widetilde{\mathcal{P}}(E)$ is a choice of three-valued argument sets. Then $\top_{Q,X_1,\ldots,X_n}$ and*

$\bot_{Q,X_1,\ldots,X_n}$ *are constant mappings, i.e.*

$$\top_{Q,(X_1,\ldots,X_n)}(\gamma) = \top_{Q,(X_1,\ldots,X_n)}(0)$$
$$\bot_{Q,(X_1,\ldots,X_n)}(\gamma) = \bot_{Q,(X_1,\ldots,X_n)}(0)$$

for all $\gamma \in \mathbf{I}$.

Proof It is apparent from definition Def. 30 that $\mathcal{T}_\gamma(X) = \mathcal{T}(X)$ for all $\gamma \in \mathbf{I}$ whenever $X \in \breve{\mathcal{P}}(E) \subseteq \widetilde{\mathcal{P}}(E)$ is a three-valued subset of E. In particular,

$$\mathcal{T}_\gamma(X) = \mathcal{T}_0(X) \tag{45}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
&\top_{Q,X_1,\ldots,X_n}(\gamma) \\
&= \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_0(X_1),\ldots,Y_n \in \mathcal{T}_0(X_n)\} && \text{by (45)} \\
&= \top_{Q,X_1,\ldots,X_n}(0) && \text{by Def. 43}\,.
\end{aligned}$$

By the same reasoning, it can be shown that $\bot_{Q,X_1,\ldots,X_n}(\gamma) = \bot_{Q,X_1,\ldots,X_n}(0)$ for all $\gamma \in \mathbf{I}$.

Lemma 7 *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier and* $X_1,\ldots,X_n \in \mathcal{P}(E)$ *is a choice of crisp argument sets. Then*

$$\top_{Q,X_1,\ldots,X_n} = \bot_{Q,X_1,\ldots,X_n} = Q(X_1,\ldots,X_n)\,.$$

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a choice of crisp arguments $X_1,\ldots,X_n \in \mathcal{P}(E)$ be given. It is then apparent from Def. 30 that

$$\mathcal{T}_0(X_i) = \{X_i\}\,, \tag{46}$$

for all $i = 1,\ldots,n$. Therefore

$$\begin{aligned}
&\top_{Q,X_1,\ldots,X_n}(0) \\
&= \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \{X_1\},\ldots,Y_n \in \{X_n\}\} && \text{by Def. 43 and (46)} \\
&= \sup\{Q(X_1,\ldots,X_n)\} \\
&= Q(X_1,\ldots,X_n)\,.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\bot_{Q,X_1,\ldots,X_n}(0) \\
&= \inf\{Q(Y_1,\ldots,Y_n) : Y_1 \in \{X_1\},\ldots,Y_n \in \{X_n\}\} && \text{by Def. 43 and (46)} \\
&= \inf\{Q(X_1,\ldots,X_n)\} \\
&= Q(X_1,\ldots,X_n)\,.
\end{aligned}$$

We may then apply L-6 and conclude that

$$\begin{aligned}\top_{Q,X_1,\dots,X_n}(\gamma) &= \top_{Q,X_1,\dots,X_n}(0)\\ &= Q(X_1,\dots,X_n)\\ &= \bot_{Q,X_1,\dots,X_n}(0)\\ &= \bot_{Q,X_1,\dots,X_n}(\gamma)\end{aligned}$$

for all $\gamma \in \mathbf{I}$, as desired.

Lemma 8 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-1)*, then the QFM $\mathcal{F}_\xi$ defined by Def. 45 satisfies $\mathcal{U}(\mathcal{F}_\xi(Q)) = Q$ for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

Note. In particular, $\mathcal{F}_\xi$ satisfies (Z-1), which weakens the lemma to the case $n \leq 1$.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \dots, X_n \in \mathcal{P}(E)$ be a choice of crisp subsets of E. We may apply L-7 and conclude that

$$\top_{Q,X_1,\dots,X_n}(\gamma) = \bot_{Q,X_1,\dots,X_n}(\gamma) = Q(X_1,\dots,X_n)\,, \tag{47}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}\mathcal{F}_\xi(Q)(X_1,\dots,X_n) &= \xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) && \text{by Def. 45}\\ &= \top_{Q,X_1,\dots,X_n}(0) && \text{by (47) and (X-1)}\\ &= Q(X_1,\dots,X_n)\,. && \text{by (47)}\end{aligned}$$

Lemma 9 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2) *and* (X-3)*. Then $\mathcal{F}_\xi$ coincides with $\mathcal{M}$ on two-valued quantifiers, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued quantifier and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ are fuzzy arguments, then*

$$\mathcal{F}_\xi(Q)(X_1,\dots,X_n) = \mathcal{M}(Q)(X_1,\dots,X_n)\,.$$

Note. In particular, $\mathcal{F}_\xi$ induces the standard negation $\neg x = 1-x$, the standard conjunction $x \wedge y = \min(x, y)$, the standard disjunction $x \vee y = \max(x, y)$ and the standard extension principle $\widehat{\mathcal{F}_\xi} = (\hat{\hat{\bullet}})$, which is apparent because all of these are obtained from two-valued quantifiers, and $\mathcal{M}$ is known to be a standard DFS by Th-11.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ be given and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. Because Q is two-valued, i.e. $Q(Y_1, \dots, Y_n) \in \mathbf{2} = \{0, 1\}$, it is apparent that $\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n)\} \subseteq \mathbf{2}$. Hence by Def. 43, $\top_{Q,X_1,\dots,X_n}(\gamma) \in \mathbf{2}$ and $\bot_{Q,X_1,\dots,X_n}(\gamma) \in \mathbf{2}$ for all $\gamma \in \mathbf{I}$. It is then apparent from Th-20 (i.e. $\bot_{Q,X_1,\dots,X_n} \leq \top_{Q,X_1,\dots,X_n}$ and the fact that $\top_{Q,X_1,\dots,X_n}$ is nondecreasing and $\bot_{Q,X_1,\dots,X_n}$ nonincreasing) that there are only the following possibilities:

a. $\top_{Q,X_1,\ldots,X_n}(0) = 1$**.**
Then $\top_{Q,X_1,\ldots,X_n}(\gamma) = 1$ for all $\gamma \in \mathbf{I}$ because $\top_{Q,X_1,\ldots,X_n}$ is nondecreasing by Th-20, i.e.

$$\top_{Q,X_1,\ldots,X_n} = \mathrm{c}_1 \,. \tag{48}$$

In addition, $\bot_{Q,X_1,\ldots,X_n}(\gamma) \in \{0,1\}$ by our above reasoning, i.e.

$$\widehat{\bot}_{Q,X_1,\ldots,X_n}(\mathbf{I}) \subseteq \{0,1\}\,. \tag{49}$$

We may hence apply (X-3) and conclude that

$$\xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) = \tfrac{1}{2} + \tfrac{1}{2}(\top_{Q,X_1,\ldots,X_n})_*^{0\downarrow}\,. \tag{50}$$

In this case,

$$\begin{aligned} Q_\gamma(X_1,\ldots,X_n) &= \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by (17)}\\ &= \mathrm{med}_{\frac{1}{2}}(1, \bot_{Q,X_1,\ldots,X_n}(\gamma)) && \text{by (48)}\\ &= \begin{cases} 1 & : \quad \bot_{Q,X_1,\ldots,X_n}(\gamma) = 1\\ \frac{1}{2} & : \quad \text{else}\end{cases} && \text{by (49), Def. 22}\end{aligned}$$

Abbreviating $f(\gamma) = Q_\gamma(X_1,\ldots,X_n)$, we hence obtain

$$f_*^{\frac{1}{2}\downarrow} = \bot_{Q,X_1,\ldots,X_n}\,. \tag{51}$$

Therefore

$$\begin{aligned} &\mathcal{M}(Q)(X_1,\ldots,X_n)\\ &= \mathcal{B}_\int(f) && \text{by Def. 33, definition of } f\\ &= \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}\downarrow} && \text{by (B-3) [Th-11, Th-8]}\\ &= \tfrac{1}{2} + \tfrac{1}{2}(\bot_{Q,X_1,\ldots,X_n})_*^{0\downarrow} && \text{by (51)}\\ &= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by (50)}\\ &= \mathcal{F}_\xi(Q)(X_1,\ldots,X_n)\,. && \text{by Def. 45}\end{aligned}$$

b. $\bot_{Q,X_1,\ldots,X_n}(0) = 0$**.**
The proof of this case is analogous: First we use the fact that $\bot_{Q,X_1,\ldots,X_n}$ is nonincreasing by Th-20 to conclude that $\bot_{Q,X_1,\ldots,X_n}(\gamma) = 0$ for all $\gamma \in \mathbf{I}$, i.e. $\bot_{Q,X_1,\ldots,X_n} = \mathrm{c}_0$. Again, we conclude from $\top_{Q,X_1,\ldots,X_n}(\gamma) \in \mathbf{2}$ for all $\gamma \in \mathbf{I}$ that $\widehat{\top}_{Q,X_1,\ldots,X_n}(\mathbf{I}) \subseteq \{\mathbf{2}\}$. In the following, we shall abbreviate $\top' = 1 - \bot_{Q,X_1,\ldots,X_n}$ and $\bot' = 1 - \top_{Q,X_1,\ldots,X_n}$. Clearly $(\top', \bot') \in \mathbb{T}$, $\top' = \mathrm{c}_1$ and $\widehat{\bot'}(\mathbf{I}) \subseteq \mathbf{2}$. We may hence apply (X-3), which yields

$$\xi(\top', \bot') = \tfrac{1}{2} + \tfrac{1}{2}{\bot'}_*^{0\downarrow} \tag{52}$$

Therefore

$$
\begin{aligned}
&\mathcal{F}_\xi(Q)(X_1,\ldots,X_n) \\
&= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by Def. 45} \\
&= \xi(1-\bot', 1-\top') && \text{by definition of } \top', \bot' \\
&= 1-\xi(\top', \bot') && \text{by (X-2)} \\
&= 1-(\tfrac{1}{2}+\tfrac{1}{2}{\bot'}_*^{0\downarrow}) && \text{by (52)} \\
&= \tfrac{1}{2}-\tfrac{1}{2}{\bot'}_*^{0\downarrow}
\end{aligned}
$$

i.e.

$$\mathcal{F}_\xi(Q)(X_1,\ldots,X_n) = \tfrac{1}{2}-\tfrac{1}{2}{\bot'}_*^{0\downarrow}\,. \tag{53}$$

Abbreviating $f(\gamma) = Q_\gamma(X_1,\ldots,X_n)$, we obviously have

$$
\begin{aligned}
f(\gamma) &= \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\ldots,X_n}(\gamma), \bot_{Q,X_1,\ldots,X_n}(\gamma)) && \text{by (17)} \\
&= \mathrm{med}_{\frac{1}{2}}(\top_{Q,X_1,\ldots,X_n}(\gamma), 0) && \text{because } \bot_{Q,X_1,\ldots,X_n} = \mathrm{c}_0 \\
&= \begin{cases} 0 & : \quad \top_{Q,X_1,\ldots,X_n} = 0 \\ \frac{1}{2} & : \quad \text{else} \end{cases} && \text{by Def. 22 and } \top_{Q,X_1,\ldots,X_n} \in \mathbf{2}
\end{aligned}
$$

Therefore

$$\mathcal{B}_{\int}(f) = \tfrac{1}{2}-\tfrac{1}{2}f_*^{\frac{1}{2}\downarrow} = \tfrac{1}{2}-\tfrac{1}{2}{\bot'}_*^{0\downarrow} \tag{54}$$

where the first equation is apparent from (B-3), which holds by Th-11 and Th-8, and the second equation is apparent from (10). Finally

$$
\begin{aligned}
&\mathcal{M}(Q)(X_1,\ldots,X_n) \\
&= \mathcal{B}_{\int}(f) && \text{by Def. 33, definition of } f \\
&= \tfrac{1}{2}-\tfrac{1}{2}{\bot'}_*^{0\downarrow} && \text{by (54)} \\
&= \mathcal{F}_\xi(Q)(X_1,\ldots,X_n)\,. && \text{by (53)}
\end{aligned}
$$

Lemma 10 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2) *and* (X-3)*, then $\mathcal{F}_\xi$ satisfies* (Z-2)*.*

Proof Let $E \neq \varnothing$ be a given base set and $e \in E$ an arbitrary element of E. The projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is two-valued by Def. 6. Therefore

$$
\begin{aligned}
\mathcal{F}_\xi(\pi_e) &= \mathcal{M}(\pi_e) && \text{by L-9} \\
&= \widetilde{\pi}_e\,. && \text{by Th-11}
\end{aligned}
$$

Lemma 11 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

a. $\top_{\neg Q, X_1, \ldots, X_n} = 1 - \bot_{Q, X_1, \ldots, X_n}$;

b. $\bot_{\neg Q, X_1, \ldots, X_n} = 1 - \top_{Q, X_1, \ldots, X_n}$,

where $\neg x = 1 - x$ is the standard negation.

Proof Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Further let $\gamma \in \mathbf{I}$.

a. The first claim of the lemma is obvious from $\sup\{1 - a : a \in A\} = 1 - \inf A$ for all $A \in \mathcal{P}(\mathbf{I})$:

$$
\begin{aligned}
&\top_{\neg Q, X_1, \ldots, X_n} \\
&= \sup\{\neg Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \sup\{1 - Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && (\neg x = 1 - x) \\
&= 1 - \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} \\
&= 1 - \bot_{Q, X_1, \ldots, X_n}(\gamma)\,. && \text{by Def. 43}
\end{aligned}
$$

b. The second claim of the lemma is entailed by the first one because

$$
\begin{aligned}
&\bot_{\neg Q, X_1, \ldots, X_n} \\
&= 1 - (1 - \bot_{\neg Q, X_1, \ldots, X_n}) \\
&= 1 - \top_{\neg\neg Q, X_1, \ldots, X_n} && \text{by part \textbf{a.} of the lemma} \\
&= 1 - \top_{Q, X_1, \ldots, X_n}\,, && \text{because } \neg x = 1 - x \text{ involutive}
\end{aligned}
$$

which finishes the proof.

Lemma 12 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$. Then for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$
\begin{aligned}
\top_{Q\neg, X_1, \ldots, X_n} &= \top_{Q, X_1, \ldots, X_{n-1}, \neg X_n} \\
\bot_{Q\neg, X_1, \ldots, X_n} &= \bot_{Q, X_1, \ldots, X_{n-1}, \neg X_n}\,,
\end{aligned}
$$

where $\neg X_n \in \widetilde{\mathcal{P}}(E)$ is the standard fuzzy complement $\mu_{X_n}(e) = 1 - \mu_{X_n}(e)$, for all $e \in E$.

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given ($n > 0$) and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. We already know from the proof of [6, L-22, p.127] ($\gamma > 0$) and [7, L-30, p.110] ($\gamma = 0$) that

$$\mathcal{T}_\gamma(\neg X_n) = \{\neg Y : Y \in \mathcal{T}_\gamma(X_n)\}\,, \tag{55}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
&\top_{Q\neg,X_1,\dots,X_n}(\gamma) \\
&= \sup\{(Q\neg)(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \sup\{Q(Y_1,\dots,Y_{n-1},\neg Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 10} \\
&= \sup\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), \\
&\quad Y_n \in \mathcal{T}_\gamma(\neg X_n)\} && \text{by (55)} \\
&= \top_{Q,X_1,\dots,X_{n-1},\neg X_n}(\gamma)\,, && \text{by Def. 43}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, and similarly

$$\begin{aligned}
&\bot_{Q\neg,X_1,\dots,X_n}(\gamma) \\
&= \inf\{(Q\neg)(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \inf\{Q(Y_1,\dots,Y_{n-1},\neg Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 10} \\
&= \inf\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), \\
&\quad Y_n \in \mathcal{T}_\gamma(\neg X_n)\} && \text{by (55)} \\
&= \bot_{Q,X_1,\dots,X_{n-1},\neg X_n}(\gamma)\,. && \text{by Def. 43}
\end{aligned}$$

Lemma 13 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2) *and* (X-3). *Then* $\mathcal{F}_\xi$ *satisfies* (Z-3).

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given ($n > 0$) and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$. By L-9 and Th-11, we know that $\mathcal{F}_x i$ induces the standard fuzzy negation $\neg x = 1 - x$. By Def. 11, dualisation based on the standard negation/complement can be decomposed as Because $Q\square = \neg Q\neg$. Hence by L-11 and L-12,

$$\top_{Q\square,X_1,\dots,X_n} = 1 - \bot_{Q,X_1,\dots,X_{n-1},\neg X_n}\,. \tag{56}$$

and

$$\bot_{Q\square,X_1,\dots,X_n} = 1 - \top_{Q,X_1,\dots,X_{n-1},\neg X_n}\,. \tag{57}$$

Hence

$$\begin{aligned}
&\mathcal{F}_\xi(Q)(X_1,\dots,X_n) \\
&= \xi(\top_{Q\square,X_1,\dots,X_n},\bot_{Q\square,X_1,\dots,X_n}) && \text{by Def. 45} \\
&= \xi(1-\bot_{Q,X_1,\dots,X_{n-1},\neg X_n},1-\top_{Q,X_1,\dots,X_{n-1},\neg X_n}) && \text{by (56), (57)} \\
&= 1-\xi(\top_{Q,X_1,\dots,X_{n-1},\neg X_n},\bot_{Q,X_1,\dots,X_{n-1},\neg X_n}) && \text{by (X-2)} \\
&= 1-\mathcal{F}_\xi(Q)(X_1,\dots,X_{n-1},\neg X_n) && \text{by Def. 45} \\
&= \neg\mathcal{F}_\xi(Q)(X_1,\dots,X_{n-1},\neg X_n) && \text{by L-9 and Th-11} \\
&= \mathcal{F}_\xi(Q)\square(X_1,\dots,X_n)\,. && \text{by Def. 11}
\end{aligned}$$

Because $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ were arbitrarily chosen, we conclude that $\mathcal{F}_\xi(Q\square) = \mathcal{F}_\xi(Q)\square$, as desired.

Lemma 14 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2) *and* (X-3). *Then* $\mathcal{F}_\xi$ *satisfies* (Z-4).

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given, $n > 0$. Further let $X_1, \dots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$ be a given choice of fuzzy arguments, and let $\gamma \in \mathbf{I}$. It has been shown in the proof of [6, L-23, p.128] that

$$\mathcal{T}_\gamma(X_n \cap X_{n+1}) = \{Y_n \cap Y_{n+1} : Y_n \in \mathcal{T}_\gamma(X_n), Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\} \tag{58}$$

whenever $\gamma > 0$. The equation also holds if $\gamma = 0$, as has been shown in [7, L-32, p.112]. Recalling that $\mathcal{T}_\gamma(\neg Z) = \{\neg Y : Y \in \mathcal{T}_\gamma(Z)\}$, which has been shown to hold for $\gamma > 0$ in the proof of [6, L-22, p.127] ($\gamma > 0$) and for $\gamma = 0$ in [7, L-30, p.110], we may apply DeMorgan's law and conclude from (58) that

$$\mathcal{T}_\gamma(X_n \cup X_{n+1}) = \{Y_n \cup Y_{n+1} : Y_n \in \mathcal{T}_\gamma(X_n), Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\}\,, \tag{59}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
&\top_{Q\cup, X_1, \dots, X_{n+1}}(\gamma) \\
&= \sup\{Q\cup(Y_1, \dots, Y_{n+1}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\} && \text{by Def. 43} \\
&= \sup\{Q(Y_1, \dots, Y_{n-1}, Y_n \cup Y_{n+1}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n), \\
&\quad Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\} && \text{by Def. 12} \\
&= \sup\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), \\
&\quad Y_n \in \mathcal{T}_\gamma(X_n \cup X_{n+1})\} && \text{by (59)} \\
&= \top_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}(\gamma)\,, && \text{by Def. 43}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$\top_{Q\cup, X_1, \dots, X_{n+1}} = \top_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}\,. \tag{60}$$

Analogously,

$$\begin{aligned}
&\bot_{Q\cup, X_1, \dots, X_{n+1}}(\gamma) \\
&= \inf\{Q\cup(Y_1, \dots, Y_{n+1}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\} && \text{by Def. 43} \\
&= \inf\{Q(Y_1, \dots, Y_{n-1}, Y_n \cup Y_{n+1}) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n), \\
&\quad Y_{n+1} \in \mathcal{T}_\gamma(X_{n+1})\} && \text{by Def. 12} \\
&= \inf\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), \\
&\quad Y_n \in \mathcal{T}_\gamma(X_n \cup X_{n+1})\} && \text{by (59)} \\
&= \bot_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}(\gamma)\,, && \text{by Def. 43}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$\bot_{Q\cup, X_1, \dots, X_{n+1}} = \bot_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}\,. \tag{61}$$

Hence

$$\begin{aligned}
&\mathcal{F}_\xi(Q\cup)(X_1, \dots, X_{n+1}) \\
&= \xi(\top_{Q\cup, X_1, \dots, X_{n+1}}, \bot_{Q\cup, X_1, \dots, X_{n+1}}) && \text{by Def. 45} \\
&= \xi(\top_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}, \bot_{Q, X_1, \dots, X_{n-1}, X_n \cup X_{n+1}}) && \text{by (60), (61)} \\
&= \mathcal{F}_\xi(Q)(X_1, \dots, X_{n-1}, X_n \cup X_{n+1})\,. && \text{by Def. 45}
\end{aligned}$$

Because $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$ were arbitrarily chosen, we conclude that $\mathcal{F}_\xi(Q\cup) = \mathcal{F}_\xi(Q)\cup$.

Lemma 15 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-5)*, then $\mathcal{F}_\xi$ satisfies* (Z-5)*.*

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be nonincreasing in its n-th argument, $n > 0$. Further let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$, $X_n \subseteq X'_n$. Then for all $\gamma \in \mathbf{I}$,

$$\begin{aligned} &\top_{Q,X_1,\ldots,X_n}(\gamma) \\ &= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 45} \\ &= \sup\{Q(Y_1, \ldots, Y_{n+1}, {X_n}_\gamma^{\min}) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\} \end{aligned}$$

(because Q nonincreasing in n-th arg and $(X_n)_\gamma^{\min} \subseteq Y$ for all $Y_n \in \mathcal{T}_\gamma(X_n)$)

$$\geq \sup\{Q(Y_1, \ldots, Y_{n+1}, {X'_n}_\gamma^{\min}) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1})\}$$

(because Q nonincreasing in n-th argument and $(X_n)_\gamma^{\min} \subseteq (X'_n)_\gamma^{\min}$)

$$= \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_{n-1} \in \mathcal{T}_\gamma(X_{n-1}), Y_n \in \mathcal{T}_\gamma(X'_n)\}$$

(because Q nonincreasing in n-th argument and $(X'_n)_\gamma^{\min} \subseteq Y$ for all $Y_n \in \mathcal{T}_\gamma(X'_n)$)

$$= \top_{Q,X_1,\ldots,X_{n-1},X'_n}(\gamma)\,,$$

i.e.

$$\top_{Q,X_1,\ldots,X_n} \geq \top_{Q,X_1,\ldots,X_{n-1},X'_n}\,. \tag{62}$$

By similar reasoning based on ${X_n}_\gamma^{\max}$ and ${X'_n}_\gamma^{\max}$, one shows that

$$\bot_{Q,X_1,\ldots,X_n} \geq \bot_{Q,X_1,\ldots,X_{n-1},X'_n}\,. \tag{63}$$

Therefore

$$\begin{aligned} \mathcal{F}_\xi(Q)(X_1, \ldots, X_n) &= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by Def. 45} \\ &\geq \xi(\top_{Q,X_1,\ldots,X_{n-1},X'_n}, \bot_{Q,X_1,\ldots,X_{n-1},X'_n}) && \text{by (X-5)} \\ &= \mathcal{F}_\xi(Q)(X_1, \ldots, X_{n-1}, X'_n)\,. \end{aligned}$$

Hence $\mathcal{F}_\xi(Q)$ is nonincreasing in its n-th argument, as desired.

The following chain of lemmata is targeted at the proof that (X-2), (X-3), (X-4) and (X-5) are sufficient for $\mathcal{F}_\xi$ to satisfy (Z-6). For similar reasons as in [6, p. 132] and [7, p. 116], we shall introduce a modified definition of $\top_{Q,X_1,\ldots,X_n}$ and $\bot_{Q,X_1,\ldots,X_n}$ which is apparently compatible with functional application (Z-6). We will then show that the original definition gives rise to the same QFM as the modified definition, thus inheriting its compliance with (Z-6).

Definition 54 *Let a semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *be given and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$*.* $\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}, \bot^{\blacktriangledown}_{Q,X_1,\ldots,X_n} : \mathbf{I} \longrightarrow \mathbf{I}$ *are defined by*

$$\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma) = \sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} \quad (64)$$

$$\bot^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma) = \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} \quad (65)$$

where

$$\mathcal{T}^{\blacktriangledown}_{\gamma}(X) = \{Y : X^{\blacktriangledown \min}_{\gamma} \subseteq Y \subseteq X^{\blacktriangledown \max}_{\gamma}\} \quad (66)$$

$$X^{\blacktriangledown \min}_{\gamma} = X_{>\frac{1}{2}+\frac{1}{2}\gamma} \quad (67)$$

$$X^{\blacktriangledown \max}_{\gamma} = X^{\max}_{\gamma} = \begin{cases} X_{\geq\frac{1}{2}} & : \gamma = 0 \\ X_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma > 0 \end{cases} \quad (68)$$

for all $\gamma \in \mathbf{I}$*.*

Lemma 16 *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier,* E' *is some non-empty base set,* $f_1, \ldots, f_n : E' \longrightarrow E$ *are mappings and and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$*. Then for all* $\gamma \in (0, 1]$*,*

$$\top^{\blacktriangledown}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\ldots,X_n}(\gamma) = \top^{\blacktriangledown}_{Q,\hat{\hat{f}}_1,\ldots,\hat{\hat{f}}_n}(\gamma)$$

$$\bot^{\blacktriangledown}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\ldots,X_n}(\gamma) = \bot^{\blacktriangledown}_{Q,\hat{\hat{f}}_1,\ldots,\hat{\hat{f}}_n}(\gamma)\,.$$

Proof Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $f_1, \ldots, f_n : E' \longrightarrow E$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E')$ be given and $\gamma \in (0, 1]$. We first recall that by equation (*) in the proof of [6, L-27, p.134],

$$\mathcal{T}^{\blacktriangledown}_{\gamma}(\hat{\hat{f}}_i(X_i)) = \{\widehat{f}_i(Y) : Y \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_i)\} \quad (69)$$

because $\gamma > 0$. Hence

$$\begin{aligned}
&\top^{\blacktriangledown}_{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\ldots,X_n}(\gamma) \\
&= \sup\{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i Y_1, \ldots, Y_n : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} && \text{by (64)} \\
&= \sup Q(\widehat{f_1}(Y_1), \ldots, \widehat{f_n}(Y_n)) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} && \text{by (2)} \\
&= \sup Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(\hat{\hat{f}}_1(X_1)), \ldots, \mathcal{T}^{\blacktriangledown}_{\gamma}(\hat{\hat{f}}_n(X_n))\} && \text{by (69)} \\
&= \top^{\blacktriangledown}_{Q,\hat{\hat{f}}_1(X_1),\ldots,\hat{\hat{f}}_n(X_n)}(\gamma)\,. && \text{by (64)}
\end{aligned}$$

By analogous reasoning,

$$\begin{aligned}
&\bot^{\blacktriangledown}_{Q\circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i, X_1,\dots,X_n}(\gamma) \\
&= \inf\{Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i Y_1,\dots,Y_n : Y_1 \in \mathcal{T}^{\blacktriangledown}_\gamma(X_1),\dots,Y_n \in \mathcal{T}^{\blacktriangledown}_\gamma(X_n)\} && \text{by (65)} \\
&= \inf Q(\widehat{f}_1(Y_1),\dots,\widehat{f_n}(Y_n)) : Y_1 \in \mathcal{T}^{\blacktriangledown}_\gamma(X_1),\dots,Y_n \in \mathcal{T}^{\blacktriangledown}_\gamma(X_n)\} && \text{by (2)} \\
&= \inf Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_\gamma(\hat{\hat{f}}_1(X_1)),\dots,\mathcal{T}^{\blacktriangledown}_\gamma(\hat{\hat{f}}_n(X_n))\} && \text{by (69)} \\
&= \bot^{\blacktriangledown}_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}(\gamma)\,. && \text{by (65)}
\end{aligned}$$

Lemma 17 *For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$,*

$$(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}) \in \mathbb{T}\,.$$

Proof We have to show that for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$, $\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}$ is nondecreasing; $\bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}$ is nonincreasing; and

$$\top^{\blacktriangledown}_{Q,X_1,\dots,X_n} \geq \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}\,.$$

The proof of these properties is entirely analogous to that of Th-20 for the original definitions $\top_{Q,X_1,\dots,X_n}$, $\bot_{Q,X_1,\dots,X_n}$.

Lemma 18 *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$.*

a. For all $\gamma \in \mathbf{I}$,

$$\begin{aligned}
&\top_{Q,X_1,\dots,X_n}(\gamma) \leq \top^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \\
&\top_{Q,X_1,\dots,X_n}(\gamma') \geq \top^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \qquad \textit{for all } \gamma' > \gamma\,.
\end{aligned}$$

b. For all $\gamma \in \mathbf{I}$,

$$\begin{aligned}
&\bot_{Q,X_1,\dots,X_n}(\gamma) \geq \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \\
&\bot_{Q,X_1,\dots,X_n}(\gamma') \leq \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \qquad \textit{for all } \gamma' > \gamma\,.
\end{aligned}$$

Proof Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, a choice of fuzzy arguments $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ be given and $\gamma \in \mathbf{I}$. We first observe that for all $X \in \widetilde{\mathcal{P}}(E)$, $X^{\blacktriangledown\min}_\gamma \subseteq X^{\min}_\gamma$, which is apparent from Def. 30, (67). Furthermore $X^{\blacktriangledown\max}_\gamma = X^{\max}_\gamma$ by (68). Hence by Def. 30,

$$\mathcal{T}_\gamma(X) \subseteq \mathcal{T}^{\blacktriangledown}_\gamma(X)\,. \tag{70}$$

It is also apparent from Def. 30 and (66) that

$$\mathcal{T}_{\gamma'}(X) \supseteq \mathcal{T}^{\blacktriangledown}_\gamma(X) \tag{71}$$

whenever $\gamma' > \gamma$, see proof of [6, L-30, p.136+].

a. Considering $\top_{Q,X_1,\ldots,X_n}$ and $\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}$, we immediately have

$$\begin{aligned}
&\top_{Q,X_1,\ldots,X_n}(\gamma) \\
&= \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&\le \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}^{\blacktriangledown}_\gamma(X_n)\} && \text{by (70)} \\
&= \top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma)\,. && \text{by (64)}
\end{aligned}$$

In addition if $\gamma' > \gamma$, then

$$\begin{aligned}
&\top_{Q,X_1,\ldots,X_n}(\gamma') \\
&= \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_{\gamma'}(X_1),\ldots,Y_n \in \mathcal{T}_{\gamma'}(X_n)\} && \text{by Def. 43} \\
&\ge \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}^{\blacktriangledown}_\gamma(X_n)\} && \text{by (71)} \\
&= \top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma)\,. && \text{by (64)}
\end{aligned}$$

b. The proofs for $\bot_{Q,X_1,\ldots,X_n}$ vs. $\bot^{\blacktriangledown}_{Q,X_1,\ldots,X_n}$ are analogous, reversing all inequations and replacing '$\sup$' with '$\inf$'.

Lemma 19 *a. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nonincreasing, then*

$$f^\sharp \le f \le f^\flat\,.$$

b. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is a constant mapping, then $f^\sharp = f^\flat = f$.

c. If $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing, then

$$f^\flat \le f \le f^\sharp\,.$$

Proof See [7, L-39, p.117].

Lemma 20 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-4) *and* (X-5)*, then*

$$\xi(\top^\flat, \bot) = \xi(\top, \bot) = \xi(\top^\sharp, \bot)\,,$$

for all $(\top, \bot) \in \mathbb{T}$.

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ fulfills (X-4) and (X-5), and let $(\top, \bot) \in \mathbb{T}$ be given. By Def. 44, we know that $\top$ is nondecreasing. Hence $\top^\flat \le \top \le \top^\sharp$. By (X-5),

$$\xi(\top^\flat, \bot) \le \xi(\top, \bot) \le \xi(\top^\sharp, \bot)\,.$$

On the other hand, $\xi(\top^\flat, \bot) = \xi(\top^\sharp, \bot)$ by (X-4). Hence $\xi(\top^\flat, \bot) = \xi(\top, \bot) = \xi(\top^\sharp, \bot)$, as desired.

Lemma 21 *If $\top, \top' : \mathbf{I} \longrightarrow \mathbf{I}$ are nondecreasing mappings and $\top|_{(0,1)} = \top'|_{(0,1)}$, then ${\top^\flat}^\sharp = {\top'^\flat}^\sharp$.*

Proof The proof of the lemma is identical to that of [7, L-41, p.118].

Lemma 22 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-4) *and* (X-5). *If* $(\top, \bot), (\top', \bot) \in \mathbb{T}$ *and* $\top|_{(0,1)} = \top'|_{(0,1)}$, *then* $\xi(\top, \bot) = \xi(\top', \bot)$.

Proof This is now trivial:

$$\begin{aligned} \xi(\top, \bot) &= \xi(\top^{\flat\sharp}, \bot) && \text{by (X-4)} \\ &= \xi(\top'^{\flat\sharp}, \bot) && \text{by L-21} \\ &= \xi(\top', \bot)\,. && \text{by (X-4)} \end{aligned}$$

Lemma 23 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2), (X-4) *and* (X-5). *If* $(\top, \bot), (\top', \bot') \in \mathbb{T}$ *with* $\top|_{(0,1)} = \top'|_{(0,1)}$ *and* $\bot|_{(0,1)} = \bot'|_{(0,1)}$, *then* $\xi(\top, \bot) = \xi(\top', \bot')$.

Proof Straightforward.

$$\begin{aligned} \xi(\top, \bot) &= \xi(\top', \bot) && \text{by L-22} \\ &= 1 - \xi(1 - \bot, 1 - \top') && \text{by (X-2)} \\ &= 1 - \xi(1 - \bot', 1 - \top') && \text{by L-22} \\ &= \xi(\top', \bot')\,. && \text{by (X-2)} \end{aligned}$$

Lemma 24 *If* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-4) *and* (X-5), *then*

$$\xi(\top_{Q,X_1,\dots,X_n}, \bot) = \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot)$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ *and* $\bot : \mathbf{I} \longrightarrow \mathbf{I}$ *such that* $(\top_{Q,X_1,\dots,X_n}, \bot) \in \mathbb{T}$.

Proof Let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, and a choice of fuzzy arguments $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$ be given. Further let $\bot : \mathbf{I} \longrightarrow \mathbf{I}$ be a mapping such that $(\top_{Q,(X_1,\dots,X_n)}, \bot) \in \mathbb{T}$. To prove the claim of the lemma, we first observe that

$$\xi(\top_{Q,X_1,\dots,X_n}, \bot) \leq \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot)\,, \tag{72}$$

which is apparent from L-18.a and (X-5). Concerning the converse inequation, let $\gamma \in [0, 1)$. Then

$$\begin{aligned} (\top_{Q,X_1,\dots,X_n})^{\sharp}(\gamma) &= \lim_{\gamma' \to \gamma^+} \top_{Q,X_1,\dots,X_n}(\gamma) && \text{by Def. 34} \\ &\geq \top^{\blacktriangledown}_{Q,X_1,\dots,X_n}(\gamma) \end{aligned}$$

as is easily seen from L-18.a. It is then apparent from this inequation that

$$\top' \geq \top^{\blacktriangledown}_{Q,X_1,\ldots,X_n} \tag{73}$$

if we define $\top' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\top'(\gamma) = \begin{cases} (\top_{Q,X_1,\ldots,X_n})^{\sharp}(\gamma) & : \quad \gamma < 1 \\ 1 & : \quad \gamma = 1 \end{cases} \tag{74}$$

for all $\gamma \in \mathbf{I}$. In turn,

$$\begin{aligned} \xi(\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}, \bot) &\leq \xi(\top', \bot) && \text{by (73), (X-5)} \\ &= \xi((\top_{Q,X_1,\ldots,X_n})^{\sharp}, \bot && \text{by (74), L-22} \\ &= \xi(\top_{Q,X_1,\ldots,X_n}, \bot)\,, && \text{by (X-4)} \end{aligned}$$

i.e.

$$\xi(\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}, \bot) \leq \xi(\top_{Q,X_1,\ldots,X_n}, \bot)\,. \tag{75}$$

Combining (72) and (75), we obtain the intended $\xi(\top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}, \bot) = \xi(\top_{Q,X_1,\ldots,X_n}, \bot)$.

Lemma 25 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

a. $\top^{\blacktriangledown}_{\neg Q,X_1,\ldots,X_n} = 1 - \bot^{\blacktriangledown}_{Q,X_1,\ldots,X_n}$;

b. $\bot^{\blacktriangledown}_{\neg Q,X_1,\ldots,X_n} = 1 - \top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}$.

Proof Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Further let $\gamma \in \mathbf{I}$.

a. The first claim of the lemma is obvious from $\sup\{1 - a : a \in A\} = 1 - \inf A$ for all $A \in \mathcal{P}(\mathbf{I})$:

$$\begin{aligned} &\top^{\blacktriangledown}_{\neg Q,X_1,\ldots,X_n} \\ &= \sup\{\neg Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} && \text{by (64)} \\ &= \sup\{1 - Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} && (\neg x = 1 - x) \\ &= 1 - \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_1), \ldots, Y_n \in \mathcal{T}^{\blacktriangledown}_{\gamma}(X_n)\} \\ &= 1 - \bot^{\blacktriangledown}_{Q,X_1,\ldots,X_n}(\gamma)\,. && \text{by (65)} \end{aligned}$$

b. The second claim of the lemma is entailed by the first one because

$$\begin{aligned} &\bot^{\blacktriangledown}_{\neg Q,X_1,\ldots,X_n} \\ &= 1 - (1 - \bot^{\blacktriangledown}_{\neg Q,X_1,\ldots,X_n}) \\ &= 1 - \top^{\blacktriangledown}_{\neg\neg Q,X_1,\ldots,X_n} && \text{by part } \textbf{a.} \text{ of the lemma} \\ &= 1 - \top^{\blacktriangledown}_{Q,X_1,\ldots,X_n}\,, && \text{because } \neg x = 1 - x \text{ involutive} \end{aligned}$$

as desired.

Lemma 26 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2), (X-4) *and* (X-5), *then*

$$\xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) = \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

Proof Straightforward.

$$\begin{aligned}
&\xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) \\
&= \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) && \text{by L-24} \\
&= 1 - (1 - \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n})) \\
&= 1 - \xi(1 - \bot_{Q,X_1,\dots,X_n}, 1 - \top^{\blacktriangledown}_{Q,X_1,\dots,X_n}) && \text{by (X-2)} \\
&= 1 - \xi(\top_{\neg Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{\neg Q,X_1,\dots,X_n}) && \text{by L-11, L-25} \\
&= 1 - \xi(\top^{\blacktriangledown}_{\neg Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{\neg Q,X_1,\dots,X_n}) && \text{by L-24} \\
&= 1 - \xi(1 - \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}, 1 - \top^{\blacktriangledown}_{Q,X_1,\dots,X_n}) && \text{by L-25} \\
&= 1 - (1 - \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n})) && \text{by (X-2)} \\
&= \xi(\top^{\blacktriangledown}_{Q,X_1,\dots,X_n}, \bot^{\blacktriangledown}_{Q,X_1,\dots,X_n}) \,.
\end{aligned}$$

Lemma 27 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-2), (X-3), (X-4) *and* (X-5). *Then $\mathcal{F}_\xi$ satisfies* (Z-6).

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a mapping which satisfies (X-2), (X-3), (X-4) and (X-5). Further suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $f_1, \dots, f_n : E' \longrightarrow E$ are mappings ($E' \neq \varnothing$) and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E')$. Then

$$\begin{aligned}
&\mathcal{F}_\xi(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \dots, X_n) \\
&= \xi(\top_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1,\dots,X_n}, \bot_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1,\dots,X_n}) && \text{by Def. 45} \\
&= \xi(\top^{\blacktriangledown}_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1,\dots,X_n}, \bot^{\blacktriangledown}_{Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i, X_1,\dots,X_n}) && \text{by L-26} \\
&= \xi(\top^{\blacktriangledown}_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}, \bot^{\blacktriangledown}_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}) && \text{by L-16, L-23} \\
&= \xi(\top_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}, \bot_{Q,\hat{\hat{f}}_1(X_1),\dots,\hat{\hat{f}}_n(X_n)}) && \text{by L-26} \\
&= \mathcal{F}_\xi(Q)(\hat{\hat{f}}_1(X_1), \dots, \hat{\hat{f}}_n(X_n)) \,. && \text{by Def. 45}
\end{aligned}$$

This finishes the proof because $\mathcal{F}_\xi$ induces the standard extension principle ($\hat{\hat{\bullet}}$), which is apparent from L-9 and Th-11.

Proof of Theorem 22

The theorem is a corollary of L-8, L-10, L-13, L-14, L-15 and L-27, which ensure that $\mathcal{F}_\xi$ satisfies (Z-1), (Z-2), (Z-3), (Z-4), (Z-5) and (Z-6), respectively.

B.4 Proof of Theorem 23

Lemma 28 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ does not satisfy* (X-1)*, then $\mathcal{F}_\xi$ does not satisfy* (Z-1)*.*

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ fails on (X-1), i.e. there exists $a \in \mathbf{I}$ such that

$$\xi(\mathrm{c}_a, \mathrm{c}_a) \neq a\,. \tag{76}$$

We define a nullary semi-fuzzy quantifier $Q_a : \mathcal{P}(\{*\})^0 \longrightarrow \mathbf{I}$ by

$$Q_a(\varnothing) = a\,. \tag{77}$$

Then $\top_{Q_a,\varnothing}(\gamma) = Q_a(\varnothing) = a$ and $\bot_{Q_a,\varnothing}(\gamma) = Q_a(\varnothing) = a$ for all $\gamma \in \mathbf{I}$ by L-7, i.e.

$$\top_{Q_a,\varnothing} = \bot_{Q_a,\varnothing} = \mathrm{c}_a\,. \tag{78}$$

Therefore

$$\begin{aligned} \mathcal{F}_\xi(Q_a)(\varnothing) &= \xi(\top_{Q_a,\varnothing}, \bot_{Q_a,\varnothing}) && \text{by Def. 45} \\ &= \xi(\mathrm{c}_a, \mathrm{c}_a) && \text{by (78)} \\ &\neq a && \text{by (76)} \\ &= Q_a(\varnothing)\,. && \text{by (77)} \end{aligned}$$

Hence $\mathcal{F}_\xi$ violates (Z-1) because $\mathcal{U}(\mathcal{F}_\xi(Q_a)) \neq Q_a$.

Lemma 29 *If $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ does not satisfy* (X-5)*, then $\mathcal{F}_\xi$ does not satisfy* (Z-5)*.*

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ fails on (X-5). Then there exist $(\top, \bot), (\top', \bot') \in \mathbb{T}$ such that $\top \leq \top'$, $\bot \leq \bot'$ and

$$\xi(\top, \bot) > \xi(\top', \bot')\,. \tag{79}$$

By Th-21, there exist semi-fuzzy quantifiers $Q_1, Q_2 : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ such that

$$\begin{aligned} \top_{Q_1,X} &= \top \\ \bot_{Q_1,X} &= \bot \\ \top_{Q_2,X} &= \top' \\ \bot_{Q_2,X} &= \bot' \end{aligned}$$

We shall assume that Q_1 and Q_2 are defined according to equation (20) and that X is defined according to (23). It is then apparent from $\top \leq \top'$ and $\bot \leq \bot'$ that $Q_1 \leq Q_2$.

In addition, Q_1 and Q_2 are nondecreasing in their argument by L-5. Based on Q_1 and Q_2, we define semi-fuzzy quantifiers $Q'_1, Q'_2 : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by $Q'_1 = Q_1\neg$, $Q'_2 = Q_2\neg$, i.e. Q'_1 and Q'_2 are the antonyms of Q_1 and Q_2 with respect to standard fuzzy complementation. Hence Q'_1 and Q'_2 are nonincreasing in their argument, which is apparent from the fact that Q_1 and Q_2 are nondecreasing and from the definition of antonyms, see Def. 10. We shall further define $X' \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ by $X' = \neg X$, i.e. X' is the standard complement of the fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ defined by (23). Then by L-12 and Th-21,

$$\begin{aligned}
\top_{Q'_1,X'} &= \top_{Q_1\neg,\neg X,=} \top_{Q_1,\neg\neg X} = \top_{Q_1,X} = \top \\
\bot_{Q'_1,X'} &= \bot_{Q_1\neg,\neg X,=} \bot_{Q_1,\neg\neg X} = \bot_{Q_1,X} = \top \\
\top_{Q'_2,X'} &= \top_{Q_2\neg,\neg X,=} \top_{Q_2,\neg\neg X} = \top_{Q_2,X} = \top' \\
\bot_{Q'_2,X'} &= \bot_{Q_2\neg,\neg X,=} \bot_{Q_2,\neg\neg X} = \bot_{Q_2,X} = \top'
\end{aligned}$$

We shall define a semi-fuzzy quantifier $C : \mathcal{P}(\mathbf{2} \times \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$C(Y) = \begin{cases} Q'_2(\{(c,z) : (1,c,z) \in Y\}) & : \quad Y \cap (\{0\} \times \mathbf{2} \times \mathbf{I}) = \varnothing \\ Q'_1(\{(c,z) : (0,c,z) \in Y\}) & : \quad Y \cap (\{0\} \times \mathbf{2} \times \mathbf{I}) \neq \varnothing \end{cases}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{2} \times \mathbf{I})$. It is apparent from the fact that Q'_1 and Q'_2 are nonincreasing and $Q'_1 \leq Q'_2$ that C is nonincreasing as well.
Now we define $Z, Z' \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{2} \times \mathbf{I})$, $Z \subseteq Z'$ by

$$\mu_Z(a,c,z) = \begin{cases} \mu_{X'}(c,z) & : \quad a = 1 \\ 0 & : \quad \text{else} \end{cases}$$

$$\mu_{Z'}(a,c,z) = \begin{cases} 1 & : \quad a = 1 \\ \mu_{X'}(c,z) & : \quad a = 0 \end{cases}$$

It is then apparent from the above equations and by L-3 from the nonincreasing monotonicity of C and Q'_2 that

$$\top_{C,Z}(\gamma) = C(Z^{\min}_\gamma) = Q'_2({X'}^{\min}_\gamma) = \top_{Q'_2,X'}(\gamma) = \top'(\gamma)$$

for all $\gamma \in \mathbf{I}$, i.e. $\top_{C,Z} = \top'$. By similar reasoning,

$$\bot_{C,Z}(\gamma) = C(Z^{\max}_\gamma) = Q'_2({X'}^{\max}_\gamma) = \bot_{Q'_2,X'}(\gamma) = \bot'(\gamma)$$

and for Z',

$$\top_{C,Z'}(\gamma) = C({Z'}^{\min}_\gamma) = Q'_1({X'}^{\min}_\gamma) = \top_{Q'_1,X'}(\gamma) = \top(\gamma)\,,$$

$$\bot_{C,Z'}(\gamma) = C({Z'}^{\max}_\gamma) = Q'_1({X'}^{\max}_\gamma) = \bot_{Q'_1,X'}(\gamma) = \bot(\gamma)\,.$$

Hence $\bot_{C,Z} = \bot'$, $\top_{C,Z'} = \top$ and $\bot_{C,Z'} = \bot$. Consequently

$$\begin{aligned}
\mathcal{F}_\xi(C)(Z) &= \xi(\top_{C,Z}, \bot_{C,Z}) && \text{by Def. 45} \\
&= \xi(\top', \bot') && \text{because } \top_{C,Z} = \top',\ \bot_{C,Z} = \bot' \\
&< \xi(\top, \bot) \text{by (79)} && \\
&= \xi(\top_{C,Z'}, \bot_{C,Z'}) && \text{because } \top_{C,Z'} = \top',\ \bot_{C,Z'} = \bot' \\
&= \mathcal{F}_\xi(C)(Z')\,. && \text{by Def. 45}
\end{aligned}$$

Hence there exists a nonincreasing quantifier C and fuzzy arguments $Z \subseteq Z'$ such that $\mathcal{F}(C)(Z) < \mathcal{F}(C)(Z')$, i.e. $\mathcal{F}_\xi$ does not preserve the nonincreasing monotonicity of C in its arguments and hence violates (Z-5).

Lemma 30 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-5)*. If* ξ *does not fulfill* (X-3)*, then* $\mathcal{F}_\xi$ *violates* (Z-2)*.*

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-5) but fails on (X-3). Then there exists $(\mathrm{c}_1, \bot) \in \mathbb{T}$ such that $\widehat{\bot}(\mathbf{I}) \subseteq \mathbf{2}$ and

$$\xi(\mathrm{c}_1, \bot) \neq \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow} . \tag{80}$$

We shall discern two cases.

a. $\xi(\mathrm{c}_1, \bot) > \frac{1}{2} + \frac{1}{2}\bot_*^{0\downarrow}$**,** i.e. $\bot_*^{0\downarrow} < 2\xi(\mathrm{c}_1, \bot) - 1$. We may hence choose $z \in \mathbf{I}$ such that

$$\bot_*^{0\downarrow} < z < 2\xi(\mathrm{c}_1, \bot) - 1 . \tag{81}$$

We shall further define $\bot' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\bot'(\gamma) = \begin{cases} 1 & : \quad \gamma \leq z \\ 0 & : \quad \gamma > z \end{cases} \tag{82}$$

for all $\gamma \in \mathbf{I}$. Hence by (10),

$$\bot'^{0\downarrow}_* = z$$

and by (81),

$$\tfrac{1}{2} + \tfrac{1}{2}\bot'^{0\downarrow}_* = \tfrac{1}{2} + \tfrac{1}{2}z < \xi(\mathrm{c}_1, \bot) . \tag{83}$$

In addition, it apparently holds that $(\mathrm{c}_1, \bot') \in \mathbb{T}$ and $\bot' \geq \bot$.
Now let $E = \{*\}$ and consider the projection quantifier $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{2}$. Further let $X \in \widetilde{\mathcal{P}}(\{*\})$ be the fuzzy subset defined by $\mu_X(*) = \frac{1}{2} + \frac{1}{2}z$. Because π_* is nondecreasing in its argument, we conclude from L-3 and Def. 30 that $\top_{\pi_*,X}(\gamma) = 1 = \mathrm{c}_1(\gamma)$ and $\bot_{\pi_*,X}(\gamma) = \bot'(\gamma)$ for all $\gamma \in \mathbf{I}$, i.e.

$$\top_{\pi_*,X} = \mathrm{c}_1 , \tag{84}$$

$$\bot_{\pi_*,X} = \bot' . \tag{85}$$

Therefore

$$\begin{aligned}
\mathcal{F}_x i(\pi_*)(X) &= \xi(\mathrm{c}_1, \bot') && \text{by Def. 45, (84), (85)} \\
&\geq \xi(\mathrm{c}_1, \bot) && \text{by (X-5) because } \bot' \geq \bot \\
&> \tfrac{1}{2} + \tfrac{1}{2}z && \text{by (83)} \\
&= \mu_X(*) && \text{by definition of } X \\
&= \widetilde{\pi}_*(X) . && \text{by Def. 7}
\end{aligned}$$

Hence $\mathcal{F}_\xi(\pi_*)(X) \neq \widetilde{\pi}_*(X)$, i.e. $\mathcal{F}_\xi$ does not satisfy (Z-2).

b. $\xi(\mathrm{c}_1, \bot) < \frac{1}{2} + \frac{1}{2}\bot_*^{0\downarrow}$, i.e. $\bot_*^{0\downarrow} > 2\xi(\mathrm{c}_1, \bot) - 1$. In this case, we choose $z \in \mathbf{I}$ such that

$$2\xi(\mathrm{c}_1, \bot) - 1 < z < \bot_*^{0\downarrow} .$$

The proof based on this choice of z is analogous to that of **a.**, reversing inequations.

Lemma 31 *Let* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *be given. If* $\mathcal{F}_\xi$ *is a DFS, then* $\mathcal{F}_\xi$ *induces the standard negation* $\widetilde{\mathcal{F}}_\xi(\neg) = \neg$.

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{F}_\xi$ is a DFS. Then ξ satisfies (X-3) by L-30. In the following, we shall abbreviate $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$. In addition, let us recall that Def. 8, $\widetilde{\neg}(x) = Q'(X)$ for all $x \in \mathbf{I}$, where $Q' : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$ is defined by

$$Q'(Y) = \neg\eta^{-1}(Y) \tag{86}$$

for all $Y \in \mathcal{P}(\{1\})$, and $X \in \widetilde{\mathcal{P}}(\{1\})$ is defined by $X = \widetilde{\eta}(x)$, i.e.

$$\mu_X(1) = x . \tag{87}$$

Now let $x \in [0, \frac{1}{2})$. We can apply L-3 because $\neg$, and hence Q', is nonincreasing. Therefore

$$\top_{Q',X} = Q'(X_\gamma^{\min}) = 1$$

and

$$\bot_{Q',X} = Q'(X_\gamma^{\max}) = \begin{cases} 1 & : \quad \gamma \le 1 - 2x \\ 0 & : \quad \gamma > 1 - 2x \end{cases}$$

for all $\gamma \in \mathbf{I}$, i.e. $\top_{Q',X} = \mathrm{c}_1$, $\widehat{\bot}_{Q',X}(\mathbf{I}) \subseteq \mathbf{2}$ and $(\bot_{Q',X})_*^{0\downarrow} = 1 - 2x$. Therefore

$$\begin{aligned}
\widetilde{\neg}\, x &= \mathcal{F}_\xi(Q')(X) && \text{by Def. 8, (87)} \\
&= \xi(\top_{Q',X}, \bot_{Q',X}) && \text{by Def. 45} \\
&= \tfrac{1}{2} + \tfrac{1}{2}(1 - 2x) && \text{by (X-3)} \\
&= 1 - x .
\end{aligned}$$

This proves that

$$\widetilde{\neg}\, x = 1 - x , \tag{88}$$

for all $x \in [0, \frac{1}{2})$. Now let $x \in (\frac{1}{2}, 1]$. By assumption, $\mathcal{F}_\xi$ is a DFS, i.e. $\widetilde{\neg}$ is a strong negation operator by Th-1. In particular, $\widetilde{\neg}$ is an involutive bijection by Def. 16. Because $\widetilde{\neg}$ is involutive, it holds that $x = \widetilde{\neg}\,\widetilde{\neg}\, x$. On the other hand, $x \in (\frac{1}{2}, 1]$ implies that $1 - x \in [0, \frac{1}{2})$. Hence by (88), $x = 1 - (1 - x) = \widetilde{\neg}(1 - x)$. Combining both equations, we have $\widetilde{\neg}\,\widetilde{\neg}\, x = \widetilde{\neg}(1 - x)$. But $\widetilde{\neg}$ is an injection, i.e. we can cancel the leftmost $\widetilde{\neg}$ to obtain the desired $\widetilde{\neg}\, x = 1 - x$. This proves that $\widetilde{\neg}\, x = 1 - x$ for all $x \in \mathbf{I} \setminus \{\frac{1}{2}\}$. It is then apparent from the fact that $\widetilde{\neg}$ is a bijection that it fulfills $\widetilde{\neg}\,\frac{1}{2} = \frac{1}{2}$, which finishes the proof that $\widetilde{\neg} = \neg$.

Lemma 32 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is a mapping such that* $\mathcal{F}_\xi$ *induces the standard negation. If* ξ *does not satisfy* (X-2)*, then* $\mathcal{F}_\xi$ *does not satisfy* (Z-3).

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping such that $\mathcal{F}_\xi$ induces the standard negation $\widetilde{\mathcal{F}}_\xi(\neg) = \neg$, $\neg x = 1x$. Further suppose that ξ violates (X-2), i.e. there exist $(\top, \bot) \in \mathbb{T}$ such that

$$\xi(1 - \bot, 1 - \top) \neq 1 - \xi(\top, \bot)\,. \tag{89}$$

By Th-21 there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ with

$$\top_{Q,X} = \top \tag{90}$$

$$\bot_{Q,X} = \bot\,. \tag{91}$$

Hence

$$\begin{aligned} \top_{Q\Box,\neg X} &= \top_{\neg Q\neg,\neg X} && \text{by Def. 11} \\ &= 1 - \bot_{Q,\neg\neg X} && \text{by L-11 and L-12} \\ &= 1 - \bot_{Q,X} && \text{because } \neg\neg X = X\,, \end{aligned}$$

i.e. by (90),

$$\top_{Q\Box,\neg X} = 1 - \bot\,. \tag{92}$$

For the same reasons, $\bot_{Q\Box,\neg X} = 1 - \top_{Q,X}$ and by (91),

$$\bot_{Q\Box,\neg X} = 1 - \top\,. \tag{93}$$

Hence

$$\begin{aligned} \mathcal{F}_\xi(Q\Box)(\neg X) &= \xi(\top_{Q\Box,\neg X}, \bot_{Q\Box,\neg X}) && \text{by Def. 45} \\ &= \xi(1 - \bot, 1 - \top) && \text{by (92), (93)} \\ &\neq 1 - \xi(\top, \bot) && \text{by (89)} \\ &= 1 - \xi(\top_{Q,X}, \bot_{Q,X}) && \text{by (90), (91)} \\ &= \neg\mathcal{F}_\xi(Q)(X) && \text{by Def. 45} \\ &= \neg\mathcal{F}_\xi(Q)(\neg\neg X) && \text{because } \neg\neg X = X \\ &= \mathcal{F}_\xi(Q)\Box(\neg X)\,, && \text{by Def. 11} \end{aligned}$$

i.e. $\mathcal{F}_\xi$ violates (Z-3).

Lemma 33 *Suppose* $(\top, \bot) \in \mathbb{T}$ *are given. We define* $\top_1 : \mathbf{I} \longrightarrow \mathbf{I}$ *by*

$$\top_1(\gamma) = \begin{cases} \top^\sharp(\gamma) & : \gamma > 0 \\ \top(0) & : \gamma = 0 \end{cases} \tag{94}$$

for all $\gamma \in \mathbf{I}$.

a. There exist $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$ *such that*

$$\begin{aligned} &\top_{Q\circ\widehat{g},X} = \top_1 \\ &\top_{Q,\hat{\hat{g}}(X)} = \top^\flat \\ &\bot_{Q\circ\widehat{g},X} = \bot_{Q,\hat{\hat{g}}(X)} = \bot\,. \end{aligned}$$

b. *There exist* $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ *and* $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$ *such that*

$$\begin{aligned} \top_{Q \circ \widehat{g}, X} &= \top_1 \\ \top_{Q, \hat{\hat{g}}(X)} &= \top^{\sharp} \\ \bot_{Q \circ \widehat{g}, X} = \bot_{Q, \hat{\hat{g}}(X)} &= \bot \,. \end{aligned}$$

Proof We shall define $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ by

$$g(c, z_1, z_2) = (c, z_1) \tag{95}$$

for all $c \in \mathbf{2}$ and $z_1, z_2 \in \mathbf{I}$. We shall further define $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$ by

$$\mu_X(c, z_1, z_2) = \begin{cases} \frac{1}{2} + \frac{1}{2} z_2 & : \quad c = 0,\ z_2 < z_1 \\ \frac{1}{2} z_2 & : \quad c = 0,\ z_1 = 0,\ z_2 < 1 \\ \frac{1}{2} - \frac{1}{2} z_1 & : \quad c = 1 \\ 0 & : \quad \text{else} \end{cases} \tag{96}$$

for all $c \in \mathbf{2}$, $z_1, z_2 \in \mathbf{I}$.
Now we shall investigate the cut ranges of X at different choices of the cutting parameter $\gamma \in \mathbf{I}$. Firstly if $\gamma = 0$, we conclude from (96) and Def. 30 that

$$X_0^{\min} = X_{> \frac{1}{2}} = \{0\} \times \{(z_1, z_2) : z_2 < z_1\}$$

and

$$X_0^{\max} = X_{\geq \frac{1}{2}} = (\{0\} \times \{(z_1, z_2) : z_2 < z_1\}) \cup (\{1\} \times \{0\} \times \mathbf{I}) \,.$$

If $\gamma > 0$, then

$$X_\gamma^{\min} = X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} = \{0\} \times \{(z_1, z_2) : \gamma \leq z_2 < z_1\}$$

and

$$\begin{aligned} X_\gamma^{\max} &= X_{> \frac{1}{2} - \frac{1}{2}\gamma} \\ &= (\{0\} \times \{(z_1, z_2) : z_2 < z_1\}) \cup (\{0\} \times \{0\} \times (1 - \gamma, 1)) \cup (\{1\} \times [0, \gamma) \times \mathbf{I}) \,. \end{aligned}$$

In turn, we obtain for $\widehat{g}(X_\gamma^{\min})$ and $\widehat{g}(X_\gamma^{\max})$ that

$$\widehat{g}(X_0^{\min}) = \{0\} \times (0, 1] \tag{97}$$

$$\widehat{g}(X_0^{\max}) = (\{0\} \times (0, 1]) \cup \{(1, 0)\} \tag{98}$$

and if $\gamma > 0$,

$$\widehat{g}(X_\gamma^{\min}) = \{0\} \times (\gamma, 1] \tag{99}$$

$$\widehat{g}(X_\gamma^{\max}) = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma)) \,. \tag{100}$$

which is obvious from (95) and the above results on $X_\gamma^{\min}$, $X_\gamma^{\max}$.
In the following, we shall abbreviate $V = \hat{\hat{g}}(X)$. It is apparent from (1) that

$$\mu_V(c, z_1) = \begin{cases} \frac{1}{2} + \frac{1}{2}z_1 & : \quad c = 0 \\ \frac{1}{2} - \frac{1}{2}z_1 & : \quad c = 1 \end{cases} \tag{101}$$

for all $c \in \mathbf{2}$ and $z_1 \in \mathbf{I}$. Hence by Def. 30,

$$V_0^{\min} = V_{>\frac{1}{2}} = \{0\} \times (0, 1] \tag{102}$$

$$V_0^{\max} = V_{\geq\frac{1}{2}} = (\{0\} \times \mathbf{I}) \cup \{(1, 0)\} \tag{103}$$

and if $\gamma > 0$,

$$V_\gamma^{\min} = V_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{0\} \times [\gamma, 1] \tag{104}$$

$$V_\gamma^{\max} = V_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma)) \,. \tag{105}$$

a. Let $(\top, \bot) \in \mathbb{T}$ be given. We define $Q, Q', Q'' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ as follows.

$$Q'(Y) = \begin{cases} \top^\sharp(y_\ell) & : \quad y_\ell \neq 0, y_\ell \notin Y' \\ \top^\flat(y_\ell) & : \quad y_\ell \neq 0, y_\ell \in Y' \\ \top(0) & : \quad y_\ell = 0 \end{cases} \tag{106}$$

$$Q''(Y) = \bot(y_u) \tag{107}$$

$$Q(Y) = \begin{cases} Q'(Y) & : \quad Y'' = \varnothing \\ Q''(Y) & : \quad Y'' \neq \varnothing \end{cases} \tag{108}$$

where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \tag{109}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \tag{110}$$

$$y_u = \sup Y'' \tag{111}$$

$$y_\ell = \inf Y' \,, \tag{112}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$.
It is obvious from L-19 and the fact that $\top^\sharp$ and $\top^\flat$ are nondecreasing that Q' is nonincreasing. Similarly, Q'' is clearly nonincreasing. Observing that $\top^\sharp(\gamma) \geq \top(0)$, $\top^\flat(\gamma) \geq \top(0)$ and $\top(0) > \bot(\gamma')$ for all $\gamma, \gamma' \in \mathbf{I}$, we conclude that $Q' \geq Q''$. We then obtain from (108) that Q is nonincreasing. Therefore

$$\begin{aligned} \top_{Q,\hat{\hat{g}}(X)}(0) &= Q(V_0^{\min}) && \text{by L-3 and } V = \hat{\hat{g}}(X) \\ &= Q(\{0\} \times (0, 1]) && \text{by (102)} \\ &= Q'(\{0\} \times (0, 1]) && \text{by (108)} \\ &= \top(0) && \text{by (106)} \\ &= \top^\flat(0) \,. && \text{by Def. 34} \end{aligned}$$

Similarly

$$\begin{aligned}
\bot_{Q,\hat{\hat{g}}(X)}(0) &= Q(V_0^{\max}) && \text{by L-3 and } V = \hat{\hat{g}}(X) \\
&= Q((\{0\} \times \mathbf{I}) \cup \{(1,0)\}) && \text{by (103)} \\
&= Q''((\{0\} \times \mathbf{I}) \cup \{(1,0)\}) && \text{by (108)} \\
&= \bot(0) && \text{by (107)}
\end{aligned}$$

In the case that $\gamma > 0$,

$$\begin{aligned}
\top_{Q,\hat{\hat{g}}(X)}(\gamma) &= Q(V_\gamma^{\min}) && \text{by L-3 and } V = \hat{\hat{g}}(X) \\
&= Q(\{0\} \times [\gamma, 1]) && \text{by (104)} \\
&= Q'(\{0\} \times [\gamma, 1]) && \text{by (108)} \\
&= \top^\flat(\gamma) && \text{by (106)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q,\hat{\hat{g}}(X)}(\gamma) &= Q(V_\gamma^{\max}) && \text{by L-3 and } V = \hat{\hat{g}}(X) \\
&= Q((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{by (105)} \\
&= Q''((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{by (108)} \\
&= \bot(\gamma)\,. && \text{by (107)}
\end{aligned}$$

Hence

$$\top_{Q,\hat{\hat{g}}(X)} = \top^\flat \tag{113}$$

and

$$\bot_{Q,\hat{\hat{g}}(X)} = \bot\,. \tag{114}$$

Turning to $\top_{Q\circ\widehat{g},X}$ and $\bot_{Q\circ\widehat{g},X}$, we utilize that $Q \circ \widehat{g}$ is nonincreasing because Q is nonincreasing. Therefore

$$\begin{aligned}
\top_{Q\circ\widehat{g},X}(0) &= (Q \circ \widehat{g})(X_0^{\min}) && \text{by L-3} \\
&= Q(\widehat{g}(X_0^{\min})) && \text{by (2)} \\
&= Q(\{0\} \times (0, 1]) && \text{by (97)} \\
&= Q'(\{0\} \times (0, 1]) && \text{by (108)} \\
&= \top(0) && \text{by (106)} \\
&= \top_1(0) && \text{by (94)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q\circ\widehat{g},X}(0) &= (Q \circ \widehat{g})(X_0^{\max}) && \text{by L-3} \\
&= Q(\widehat{g}(X_0^{\max})) && \text{by (2)} \\
&= Q((\{0\} \times (0, 1]) \cup \{(1, 0)\}) && \text{by (98)} \\
&= Q''((\{0\} \times (0, 1]) \cup \{(1, 0)\}) && \text{by (108)} \\
&= \bot(0)\,. && \text{by (107)}
\end{aligned}$$

For $\gamma > 0$, then,

$$\begin{aligned}
\top_{Q\circ\widehat{g},(}X)(\gamma) &= (Q \circ \widehat{g})(X_\gamma^{\min}) && \text{by L-3} \\
&= Q(\widehat{g}(X_\gamma^{\min})) && \text{by (2)} \\
&= Q(\{0\} \times (\gamma, 1]) && \text{by (99)} \\
&= Q'(\{0\} \times (\gamma, 1]) && \text{by (108)} \\
&= \top^\sharp(\gamma) && \text{by (106)} \\
&= \top_1(\gamma) && \text{by (94)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q\circ\widehat{g},(}X)(\gamma) &= (Q \circ \widehat{g})(X_\gamma^{\max}) && \text{by L-3} \\
&= Q(\widehat{g}(X_\gamma^{\max})) && \text{by (2)} \\
&= Q((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{by (100)} \\
&= Q''((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{by (108)} \\
&= \bot(\gamma)\,. && \text{by (107)}
\end{aligned}$$

Hence

$$\top_{Q\circ\widehat{g},X} = \top_1 \tag{115}$$

and

$$\bot_{Q\circ\widehat{g},X} = \bot\,. \tag{116}$$

As shown by (113), (114), (115) and (116), the presented choices for Q, g and X are suitable for proving part **a.** of the lemma.

b. For the proof of **b.**, let again $(\top, \bot) \in \mathbb{T}$ be given. We will assume the same choice of $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$. In this case, however, we define $Q, Q', Q'' : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ as follows. For all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$,

$$Q'(Y) = \begin{cases} \top(0) & : \ y_\ell = 0,\ y_\ell \notin Y' \\ \top^\sharp(y_\ell) & : \ \text{else} \end{cases} \tag{117}$$

$$Q''(Y) = \bot(y_u) \tag{118}$$

$$Q(Y) = \begin{cases} Q'(Y) & : \ Y'' = \varnothing \\ Q''(Y) & : \ Y'' \neq \varnothing \end{cases} \tag{119}$$

where Y', Y'', y_ℓ and y_u are defined as in **a.** Unlike the quantifiers used in the proof of the first part of the lemma, the above choice of Q', and hence Q, is not necessarily monotonic. We can hence not apply L-3. Nevertheless it is easy to figure out which choices of argument sets $Y \in \mathcal{T}_\gamma(X)$ attain the maximum $\top_{Q,Z}(\gamma)$ and the minimum $\bot_{Q,Z}(\gamma)$ for a given choice of $Z \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$. This is because Q'' is nonincreasing and $Q' \geq Q''$. In addition, Q' is 'almost' nonincreasing (we just have to take care if

$y_\ell = 0$).
Let us first consider the case that $\gamma = 0$. Then

$$\begin{aligned}
\top_{Q,\hat{\hat{g}}(X)}(0) &= \sup\{Q(Y) : Y \in \mathcal{T}_0(V)\} && \text{by Def. 43 and } V = \hat{\hat{g}}(X) \\
&= Q(\{0\} \times \mathbf{I}) && \text{see (102), (103)} \\
& Q'(\{0\} \times \mathbf{I}) && \text{by (119)} \\
& \top^\sharp(0) && \text{by (117)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q,\hat{\hat{g}}(X)}(0) &= \inf\{Q(Y) : Y \in \mathcal{T}_0(V)\} && \text{by Def. 43 and } V = \hat{\hat{g}}(X) \\
&= Q((\{0\} \times \mathbf{I}) \cup \{(1,0)\}) && \text{see (102), (103)} \\
& Q''((\{0\} \times \mathbf{I}) \cup \{(1,0)\}) && \text{by (119)} \\
& \bot(0)\,. && \text{by (118)}
\end{aligned}$$

Similarly

$$\begin{aligned}
\top_{Q\circ\widehat{g},X}(0) &= \sup\{Q(\widehat{g}(Y)) : Y \in \mathcal{T}_0(X)\} && \text{by Def. 43, (2)} \\
&= \sup\{Q(Z) : Z \in \widehat{g}(\mathcal{T}_0(X))\} && \text{by [6, L-26, p.133]} \\
&= Q(\{0\} \times (0,1]) && \text{see (97)} \\
&= Q'(\{0\} \times (0,1]) && \text{by (119)} \\
&= \top(0) && \text{by (117)} \\
&= \top_1(0) && \text{by (94)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q\circ\widehat{g},X}(0) &= \inf\{Q(\widehat{g}(Y)) : Y \in \mathcal{T}_0(X)\} && \text{by Def. 43, (2)} \\
&= \sup\{Q(Z) : Z \in \widehat{g}(\mathcal{T}_0(X))\} && \text{by [6, L-26, p.133]} \\
&= Q((\{0\} \times (0,1]) \cup \{(1,0)\}) && \text{see (98)} \\
&= Q'((\{0\} \times (0,1]) \cup \{(1,0)\}) && \text{by (119)} \\
&= \bot(0)\,. && \text{by (118)}
\end{aligned}$$

Now let us assume that $\gamma > 0$. Then

$$\begin{aligned}
\top_{Q,\hat{\hat{g}}(X)}(\gamma) &= \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(V)\} && \text{by Def. 43 and } V = \hat{\hat{g}}(X) \\
&= Q(\{0\} \times [\gamma,1]) && \text{see (104), (105)} \\
& Q'(\{0\} \times [\gamma,1]) && \text{by (119)} \\
& \top^\sharp(\gamma) && \text{by (117)}
\end{aligned}$$

and

$$\begin{aligned}
\bot_{Q,\hat{\hat{g}}(X)}(\gamma) &= \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(V)\} && \text{by Def. 43 and } V = \hat{\hat{g}}(X) \\
&= Q((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0,\gamma))) && \text{see (104), (105)} \\
& Q''((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0,\gamma))) && \text{by (119)} \\
& \bot(\gamma)\,. && \text{by (118)}
\end{aligned}$$

Finally

$$
\begin{aligned}
\top_{Q\circ\widehat{g},X}(\gamma) &= \sup\{Q(\widehat{g}(Y)) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 43, (2)}\\
&= \sup\{Q(Z) : Z \in \widehat{g}(\mathcal{T}_\gamma(X))\} && \text{by [6, L-26, p.133]}\\
&= Q(\{0\} \times (\gamma, 1]) && \text{see (99)}\\
&= Q'(\{0\} \times (\gamma, 1]) && \text{by (119)}\\
&= \top^\sharp(\gamma) && \text{by (117)}\\
&= \top_1(0) && \text{by (94)}
\end{aligned}
$$

and

$$
\begin{aligned}
\bot_{Q\circ\widehat{g},X}(\gamma) &= \inf\{Q(\widehat{g}(Y)) : Y \in \mathcal{T}_\gamma(X)\} && \text{by Def. 43, (2)}\\
&= \sup\{Q(Z) : Z \in \widehat{g}(\mathcal{T}_\gamma(X))\} && \text{by [6, L-26, p.133]}\\
&= Q((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{see (100)}\\
&= Q'((\{0\} \times \mathbf{I}) \cup (\{1\} \times [0, \gamma))) && \text{by (119)}\\
&= \bot(\gamma) && \text{by (118)}
\end{aligned}
$$

as desired.

Lemma 34 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a given aggregation mapping and $\mathcal{F}_\xi$ is the QFM defined in terms of ξ. Further assume that ξ satisfies* (X-2) *and* (X-3)*. If $\mathcal{F}_\xi$ satisfies* (Z-6)*, then ξ satisfies* (X-4)*.*

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies conditions (X-2) and (X-3). Further assume that $\mathcal{F}_\xi$ satisfies (Z-6).
Let a choice of $(\top, \bot) \in \mathbb{T}$ be given. We have to show that $\xi(\top^\sharp, \bot) = \xi(\top^\flat, \bot)$. To this end, we first observe that by L-9 and Th-11, $\mathcal{F}_\xi$ induces the standard extension principle. By L-33, there exists $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$, $g : \mathbf{2} \times \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{2} \times \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I} \times \mathbf{I})$ such that $\top_{Q\circ\widehat{g},X} = \top_1$, $\top_{Q,\hat{\hat{g}}(X)} = \top^\flat$ and $\bot_{Q\circ\widehat{g},X} = \bot_{Q,\hat{\hat{g}}(X)} = \bot$. We may hence conclude from the fact that $\mathcal{F}_\xi$ satisfies (Z-6) and induces the standard extension principle that

$$
\begin{aligned}
\xi(\top^\flat, \bot) &= \xi(\top_{Q,\hat{\hat{g}}(X)}, \bot_{Q,\hat{\hat{g}}(X)}) && \text{by L-33.a}\\
&= \mathcal{F}_\xi(Q)(\hat{\hat{g}}(X)) && \text{by Def. 45}\\
&= \mathcal{F}_\xi(Q \circ \widehat{g})(X) && \text{by (Z-6)}\\
&= \xi(\top_{Q\circ\widehat{g},X}, \bot_{Q\circ\widehat{g},X}) && \text{by Def. 45}\\
&= \xi(\top_1, \bot)\,, && \text{by L-33.a}
\end{aligned}
$$

where $\top_1$ is defined by (94). In a similar way, we can use quantifier Q, fuzzy subset X and mapping g of L-33.b to prove that

$$\begin{aligned}
\xi(\top^\sharp, \bot) &= \xi(\top_{Q,\hat{\hat{g}}(X)}, \bot_{Q,\hat{\hat{g}}(X)}) && \text{by L-33.b} \\
&= \mathcal{F}_\xi(Q)(\hat{g}(X)) && \text{by Def. 45} \\
&= \mathcal{F}_\xi(Q \circ \widehat{g})(X) && \text{by (Z-6)} \\
&= \xi(\top_{Q\circ\widehat{g},X}, \bot_{Q\circ\widehat{g},X}) && \text{by Def. 45} \\
&= \xi(\top_1, \bot)\,, && \text{by L-33.b}
\end{aligned}$$

Hence $\xi(\top^\flat, \bot) = \xi(\top_1, \bot) = \xi(\top^\sharp, \bot)$, i.e. (X-4) holds, as desired.

Proof of Theorem 23

Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is given. We have to show that $\mathcal{F}_\xi$ is not a DFS if one of the conditions (X-1) to (X-5) is violated.

a. If ξ violates (X-1), then $\mathcal{F}_\xi$ does not satisfy (Z-1) by L-28.

b. If ξ violates (X-5), then $\mathcal{F}_\xi$ violates (Z-5) by L-29.

In the following, we can hence assume that ξ satisfies (X-1) and (X-5) (otherwise, $\mathcal{F}_\xi$ fails to be a DFS by a. or b.). Under these circumstances,

c. if ξ fails on (X-3), then $\mathcal{F}_\xi$ violates (Z-2) by L-30.

In the following, we may hence assume that ξ satisfies (X-3) because otherwise, $\mathcal{F}_\xi$ is not a DFS anyway.

d. if $\mathcal{F}_\xi$ does not induce the standard negation $\neg x = 1 - x$, then $\mathcal{F}_\xi$ is not a DFS by L-31.

In the following, we can hence assume that $\mathcal{F}_\xi$ induces the standard negation. Then by L-32,

e. if ξ does not satisfy (X-2), then $\mathcal{F}_\xi$ does not satisfy (Z-3).

Hence (X-2) is also a necessary condition for $\mathcal{F}_\xi$ to be a DFS and we may assume that all ‘x-conditions’ except (possibly) for (X-4) hold. Then by L-34,

f. if ξ does not satisfy (X-4), then $\mathcal{F}_\xi$ does not satisfy (Z-6).

This finishes the proof that all conditions (X-1) to (X-5) are necessary for $\mathcal{F}_\xi$ to be a DFS.

B.5 Proof of Theorem 24

Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be a given aggregation mapping. Further suppose that $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given. Then

$$\begin{aligned}
&Q_\gamma(X_1, \ldots, X_n) \\
&= \mathrm{med}_{\frac{1}{2}}(\sup\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}, \\
&\qquad \inf\{Q(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}) && \text{by (17)} \\
&= \mathrm{med}_{\frac{1}{2}}(\top_{Q,(X_1,\ldots,X_n)}(\gamma), \bot_{Q,(X_1,\ldots,X_n)}(\gamma)) && \text{by Def. 43},
\end{aligned}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} = \mathrm{med}_{\frac{1}{2}}(\top_{Q,(X_1,\ldots,X_n)}, \bot_{Q,(X_1,\ldots,X_n)})\,. \tag{120}$$

Therefore

$$\begin{aligned}
\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) &= \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}) && \text{by Def. 33} \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top_{Q,(X_1,\ldots,X_n)}, \bot_{Q,(X_1,\ldots,X_n)})) && \text{by (120)} \\
&= \xi(\top_{Q,(X_1,\ldots,X_n)}, \bot_{Q,(X_1,\ldots,X_n)})\,, && \text{by (25)}
\end{aligned}$$

as desired.

B.6 Proof of Theorem 25

Lemma 35 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given and $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is the mapping defined by* (25)*. $\mathcal{B}$ satisfies* (B-1) *if and only if ξ satisfies* (X-1)

Proof Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be given and let ξ be the corresponding mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. To see that (X-1) entails (B-1), let us assume that ξ satisfies (X-1). Further suppose that $f \in \mathbb{B}$ is a given constant, i.e. $f = \mathrm{c}_a$ for some $a \in \mathbf{I}$. It is apparent from Def. 22 that

$$\mathrm{med}_{\frac{1}{2}}(\mathrm{c}_a, \mathrm{c}_a) = \mathrm{c}_a\,. \tag{121}$$

Therefore

$$\begin{aligned}
\mathcal{B}(\mathrm{c}_a) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\mathrm{c}_a, \mathrm{c}_a)) && \text{by (121)} \\
&= \xi(\mathrm{c}_a, \mathrm{c}_a) && \text{by (25)} \\
&= a\,. && \text{by (X-1)}
\end{aligned}$$

Now we prove the converse claim that (B-1) entails (X-1). Hence let $(\top, \bot) \in \mathbb{T}$ such that $\top = \bot$. It then follows from the fact that $\top$ is nondecreasing while $\bot$ is nonincreasing that $\top = \bot$ is indeed constant, i.e. $\top = \bot = \mathrm{c}_a$ for $a = \top(0)$. Hence

$$\begin{aligned} \xi(\top, \bot) &= \xi(\mathrm{c}_a, \mathrm{c}_a) && \text{because } \top = \bot = \mathrm{c}_a \\ &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\mathrm{c}_a, \mathrm{c}_a)) && \text{by (25)} \\ &= \mathcal{B}(\mathrm{c}_a) && \text{by idempotence of } \mathrm{med}_{\frac{1}{2}} \\ &= a && \text{by (B-1)} \\ &= \top(0)\,. && \text{because } \top = \mathrm{c}_a \end{aligned}$$

Lemma 36 *For all $f \in \mathbb{B}$ there exist $(\top, \bot) \in \mathbb{T}$ such that $f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)$. More specifically,*

a. If $f \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$, then $f = \mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1, f)$.

b. If $f \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$, then $f = \mathrm{med}_{\frac{1}{2}}(f, \mathrm{c}_0)$.

Proof Let $f \in \mathbb{B}$ be given.

a.: $f \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$. Then f is nonincreasing by Def. 32 and hence $(\mathrm{c}_1, f) \in \mathbb{T}$. Again from Def. 32, we know that $f(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$; hence

$$\begin{aligned} \mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1(\gamma), f(\gamma)) &= \mathrm{med}_{\frac{1}{2}}(1, f(\gamma)) && \text{by (7)} \\ &= f(\gamma)\,. && \text{by Def. 22 and } f(\gamma) \geq \tfrac{1}{2} \end{aligned}$$

b.: $f \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$. In this case, f is nondecreasing by Def. 32 and $f \geq \mathrm{c}_0$, hence $(f, \mathrm{c}_0) \in \mathbb{T}$. We conclude from Def. 32 that $f(\gamma) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} \mathrm{med}_{\frac{1}{2}}(f(\gamma), \mathrm{c}_0(\gamma)) &= \mathrm{med}_{\frac{1}{2}}(f(\gamma), 1) && \text{by (7)} \\ &= f(\gamma)\,. && \text{by Def. 22 and } f(\gamma) \leq \tfrac{1}{2} \end{aligned}$$

Lemma 37 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given and ξ is the corresponding mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$. Then $\mathcal{B}$ satisfies* (B-2) *if and only if ξ satisfies* (X-2).

Proof Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be given and suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is defined by (25).
We shall first assume that $\mathcal{B}$ satisfies (B-2) and consider a choice of $(\top, \bot) \in \mathbb{T}$. Then

$$\begin{aligned}
&\xi(1-\bot, 1-\top) \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(1-\bot, 1-\top)) && \text{by (25)} \\
&= \mathcal{B}(1-\mathrm{med}_{\frac{1}{2}}(\bot, \top)) && \text{because } \mathrm{med}_{\frac{1}{2}} \text{ symmetrical w.r.t. } \neg \\
&= \mathcal{B}(1-\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{because } \mathrm{med}_{\frac{1}{2}} \text{ commutative} \\
&= 1-\mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{by (B-2)} \\
&= 1-\xi(\top, \bot)\,, && \text{by (25)}
\end{aligned}$$

i.e. ξ satisfies (X-2).
Conversely, suppose that ξ satisfies (X-2) and let $f \in \mathbb{B}$ be given. By L-36, there exists a choice of $(\top, \bot) \in \mathbb{T}$ such that

$$f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)\,. \tag{122}$$

We compute

$$\begin{aligned}
&\mathcal{B}(1-f) \\
&= \mathcal{B}(1-\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{by (122)} \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(1-\top, 1-\bot)) && \text{because } \mathrm{med}_{\frac{1}{2}} \text{ symmetric w.r.t. } \neg \\
&= \xi(1-\bot, 1-\top) && \text{by (25) and } \mathrm{med}_{\frac{1}{2}} \text{ commutative} \\
&= 1-\xi(\top, \bot) && \text{by (X-2)} \\
&= 1-\mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{by (25)} \\
&= 1-\mathcal{B}(f)\,. && \text{by (122)}
\end{aligned}$$

Lemma 38 *Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be a given aggregation mapping and let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be defined by* (25).

a. if $\mathcal{B}$ satisfies (B-3), *then ξ satisfies* (X-3).

b. if ξ satisfies (X-2) *and* (X-3), *then $\mathcal{B}$ satisfies* (B-3).

Proof

a. Suppose $\mathcal{B}$ satisfies (B-3), and let a choice of $(\mathrm{c}_1, \perp) \in \mathbb{T}$ be given such that $\widehat{\perp}(\mathbf{I}) \subseteq \mathbf{2}$. We shall abbreviate

$$f = \mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1, \perp) \tag{123}$$

Then

$$\begin{aligned} f(\gamma) &= \mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1(\gamma), \perp(\gamma)) && \text{by (123)} \\ &= \mathrm{med}_{\frac{1}{2}}(1, \perp(\gamma)) && \text{by (7)} \\ &= \begin{cases} 1 & : \quad \perp(\gamma) = 1 \\ \frac{1}{2} & : \quad \perp(\gamma) = 0 \end{cases} \end{aligned}$$

for all $\gamma \in \mathbf{I}$, where the last step is apparent from Def. 22 and the fact that $\perp$ is two-valued. Because $\perp$ is nonincreasing and $\mathrm{med}_{\frac{1}{2}}$ is nondecreasing in its arguments, it follows that f is nonincreasing as well. Therefore $f \in \mathbb{B}^+$, and f has one of the following forms:

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma \le \perp_*^{0\downarrow} \\ \frac{1}{2} & : \quad \gamma > \perp_*^{0\downarrow} \end{cases}$$

or

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma < \perp_*^{0\downarrow} \\ \frac{1}{2} & : \quad \gamma \ge \perp_*^{0\downarrow} \end{cases}$$

In any case, $\widehat{f}(\mathbf{I}) \subseteq \{\frac{1}{2}, 1\}$ and $f_*^{\frac{1}{2}\downarrow} = \perp_*^{0\downarrow}$. Therefore

$$\begin{aligned} \xi(\mathrm{c}_1, \perp) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1, \perp)) && \text{by (25)} \\ &= \mathcal{B}(f) && \text{by (123)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}\downarrow} && \text{by (B-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \perp_*^{0\downarrow}, && \text{because } f_*^{\frac{1}{2}} = \perp_*^{0\downarrow} \end{aligned}$$

i.e. ξ satisfies (X-3).

b. Suppose ξ satisfies (X-2) and (X-3). Further let a choice of $f \in \mathbb{B}$ be given that satisfies $\widehat{f}(\mathbf{I}) \subseteq \{0, \frac{1}{2}, 1\}$.

If $f \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$, then actually $\widehat{f}(\mathbf{I}) \subseteq \{\frac{1}{2}, 1\}$, which is apparent from Def. 32. In addition, we know from this definition that f is nonincreasing. Hence f has one of the following forms:

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma \le f_*^{\frac{1}{2}\downarrow} \\ \frac{1}{2} & : \quad \gamma > f_*^{\frac{1}{2}\downarrow} \end{cases}$$

or

$$f(\gamma) = \begin{cases} 1 & : \quad \gamma < f_*^{\frac{1}{2}\downarrow} \\ \frac{1}{2} & : \quad \gamma \ge f_*^{\frac{1}{2}\downarrow} \end{cases}$$

In any case,

$$f = \mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1, g)\,, \tag{124}$$

provided we define

$$g(\gamma) = \begin{cases} 1 & : \quad f(\gamma) = 1 \\ 0 & : \quad f(\gamma) = \frac{1}{2} \end{cases}$$

(This is apparent from Def. 22). In addition, $\widehat{g}(\mathbf{I}) \subseteq \mathbf{2}$ and by (10), (11),

$$g_*^{0\downarrow} = f_*^{\frac{1}{2}\downarrow}\,. \tag{125}$$

Therefore

$$\begin{aligned} \mathcal{B}(f) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\mathrm{c}_1, g)) && \text{by (124)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} g_*^{0\downarrow} && \text{by (X-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}\downarrow}\,, && \text{by (125)} \end{aligned}$$

i.e. (B-3) holds. Note that this properly covers the case $f = \mathrm{c}_{\frac{1}{2}}$ because in this case, $f_*^{\frac{1}{2}\downarrow} = 0$, i.e. $\mathcal{B}(f) = \frac{1}{2}$, as desired.

In the remaining case that $f \in \mathbb{B}^-$, we may proceed as follows:

$$\begin{aligned} \mathcal{B}(f) &= \mathcal{B}(1 - (1 - f)) \\ &= 1 - \mathcal{B}(1 - f) && \text{by (X-2), L-37} \\ &= 1 - (\tfrac{1}{2} + \tfrac{1}{2}(1 - f)_*^{\frac{1}{2}\downarrow}) && \text{already proven: } 1 - f \in \mathbb{B}^+ \\ &= 1 - (\tfrac{1}{2} + \tfrac{1}{2} f_*^{\frac{1}{2}\downarrow}) && \text{apparent from (11)} \\ &= \tfrac{1}{2} - \tfrac{1}{2} f_*^{\frac{1}{2}\downarrow}\,, \end{aligned}$$

i.e. (B-3) also holds if $f \in \mathbb{B}^-$.

Lemma 39 *Suppose $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ is given and ξ is the corresponding mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$.*

a. If $\mathcal{B}$ satisfies (B-4)*, then ξ satisfies* (X-4).

b. If ξ satisfies (X-2) *and* (X-4)*, then $\mathcal{B}$ satisfies* (B-4).

Proof

a. Suppose $\mathcal{B}$ satisfies (B-4) and let a choice of $(\top, \bot) \in \mathbb{T}$ be given.

a.1: $\top(0) < \frac{1}{2}$. Because $\top$ is nondecreasing, $\bot \leq \top$, and because $\bot$ is nonincreasing, this means that $\bot(\gamma) < \frac{1}{2}$; to see this, consider $\bot(\gamma) \leq \bot(0) \leq \top(0) < \frac{1}{2}$. Hence by Def. 22,

$$f(\gamma) = \text{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) = \begin{cases} \top(\gamma) & : \quad \top(\gamma) < \frac{1}{2} \\ \frac{1}{2} & : \quad \top(\gamma) \geq \frac{1}{2} \end{cases} \tag{126}$$

Let us now abbreviate

$$\gamma_* = \inf\{\gamma \in \mathbf{I} : \top(\gamma) \geq \tfrac{1}{2}\} \, . \tag{127}$$

Clearly

$$f(\gamma) = \top(\gamma) \tag{128}$$

for all $\gamma \in [0, \gamma_*)$; this is apparent from (126) and (127). Hence

$$\begin{aligned} f^\sharp(\gamma) &= \lim_{\gamma' \to \gamma^+} f(\gamma') && \text{by Def. 34} \\ &= \lim_{\gamma' \to \gamma^+} \top(\gamma') && \text{by (128)} \\ &= \top^\sharp(\gamma) && \text{by Def. 34} \end{aligned}$$

i.e.

$$f^\sharp(\gamma) = \text{med}_{\frac{1}{2}}(\top^\sharp(\gamma), \bot(\gamma)) \, , \tag{129}$$

where the last equation is valid because $\top^\sharp(\gamma) < \frac{1}{2}$ for $\gamma \in [0, \gamma_*)$.
If $\gamma \in (\gamma_*, 1]$, then $\top(\gamma) \geq \frac{1}{2}$ and $f(\gamma) = \frac{1}{2}$ by (126) and Def. 22. Therefore

$$f^\sharp(\gamma) = \lim_{\gamma' \to \gamma^+} f(\gamma) = \lim_{\gamma' \to \gamma^+} \tfrac{1}{2} = \tfrac{1}{2} \tag{130}$$

for all $\gamma \in [\gamma_*, 1)$, see Def. 34. Turning to $\top^\sharp$, it follows from $\top(\gamma) \geq \frac{1}{2}$ for all $\gamma > \gamma_*$ that

$$\top^\sharp(\gamma) = \lim_{\gamma' \to \gamma^+} \top(\gamma') \geq \tfrac{1}{2} \tag{131}$$

for all $\gamma \in [\gamma_*, 1)$. Therefore

$$
\begin{aligned}
f^\sharp(\gamma) &= \tfrac{1}{2} && \text{by (130)}\\
&= \operatorname{med}_{\frac{1}{2}}(\top^\sharp(\gamma), \bot(\gamma))\,, && \text{by (131), } \bot(\gamma) < \tfrac{1}{2}\text{, Def. 22}
\end{aligned}
$$

for all $\gamma \in [\gamma + *, 1)$. Finally for $\gamma = 1$,

$$
\begin{aligned}
\operatorname{med}_{\frac{1}{2}}(\top^\sharp(1), \bot(1)) &= \operatorname{med}_{\frac{1}{2}}(\top(1), \bot(1)) && \text{by Def. 34}\\
&= f(1) && \text{by (126)}\\
&= f^\sharp(1)\,. && \text{by Def. 34}
\end{aligned}
$$

Hence

$$
f^\sharp = \operatorname{med}_{\frac{1}{2}}(\top^\sharp, \bot)\,. \tag{132}
$$

Similarly, it can be shown that $f^\flat = \operatorname{med}_{\frac{1}{2}}(\top^\flat, \bot)$: if $\gamma = 0$, then

$$
f^\flat(0) = f(0) = \operatorname{med}_{\frac{1}{2}}(\top(0), \bot(0)) = \operatorname{med}_{\frac{1}{2}}(\top^\flat(0), \bot(0))\,.
$$

This is immediate from Def. 34 and (126). If $0 < \gamma \leq \gamma_*$, then

$$
f^\flat(\gamma) = \lim_{\gamma' \to \gamma^-} f(\gamma') = \lim_{\gamma' \to \gamma^-} \top(\gamma') = \top^\flat(\gamma) = \operatorname{med}_{\frac{1}{2}}(\top^\flat(\gamma), \bot(\gamma))\,,
$$

which is clear from (128) and $\top^\flat(\gamma) \leq \frac{1}{2}$. Finally in the case that $\gamma > \gamma_*$, we have

$$
f^\flat(\gamma) = \lim_{\gamma' \to \gamma^-} f(\gamma') = \lim_{\gamma' \to \gamma^-} \tfrac{1}{2} = \tfrac{1}{2} = \operatorname{med}_{\frac{1}{2}}(\top^\flat(\gamma), \bot(\gamma))\,.
$$

This is obvious from Def. 22 if we recall that

$$
\top^\flat(\gamma) = \lim_{\gamma' \to \gamma^-} \top(\gamma') \geq \tfrac{1}{2}
$$

for all $\gamma > \gamma_*$; see (127). Summarising, we have shown that

$$
f^\flat = \operatorname{med}_{\frac{1}{2}}(\top^\flat, \bot)\,. \tag{133}
$$

Therefore

$$
\begin{aligned}
\xi(\top^\sharp, \bot) &= \mathcal{B}(\operatorname{med}_{\frac{1}{2}}(\top^\sharp, \bot)) && \text{by (25)}\\
&= \mathcal{B}(f^\sharp) && \text{by (132)}\\
&= \mathcal{B}(f^\flat) && \text{by (B-4)}\\
&= \mathcal{B}(\operatorname{med}_{\frac{1}{2}}(\top^\flat, \bot)) && \text{by (133)}\\
&= \xi(\top^\flat, \bot)\,, && \text{by (25)}
\end{aligned}
$$

i.e. (X-4) holds, as desired.

a.2: $\bot(0) > \frac{1}{2}$**.** In this case, $\top(\gamma) \geq \top(0) \geq \bot(0) > \frac{1}{2}$ for all γ. Hence by the definition of fuzzy median Def. 22,

$$f(\gamma) = \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) = \begin{cases} \bot(\gamma) & : \quad \bot(\gamma) > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot(\gamma) \leq \frac{1}{2} \end{cases}$$

For the same reasons,

$$\mathrm{med}_{\frac{1}{2}}(\top^\sharp(\gamma), \bot(\gamma)) = \begin{cases} \bot(\gamma) & : \quad \bot(\gamma) > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot(\gamma) \leq \frac{1}{2} \end{cases} = f(\gamma)\,,$$

and also

$$\mathrm{med}_{\frac{1}{2}}(\top^\flat(\gamma), \bot(\gamma)) = \begin{cases} \bot(\gamma) & : \quad \bot(\gamma) > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot(\gamma) \leq \frac{1}{2} \end{cases} = f(\gamma)\,,$$

i.e.

$$\mathrm{med}_{\frac{1}{2}}(\top^\sharp, \bot) = \mathrm{med}_{\frac{1}{2}}(\top^\flat, \bot)\,. \tag{134}$$

Therefore

$$\begin{aligned} \xi(\top^\sharp, \bot) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top^\sharp, \bot)) && \text{by (25)} \\ &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top^\flat, \bot)) && \text{by (134)} \\ &= \xi(\top^\flat, \bot)\,. && \text{by (25)} \end{aligned}$$

a.3: $\top(0) \geq \frac{1}{2}$ **and** $\bot(0) \leq \frac{1}{2}$**.** In this case, we conclude from the fact that $\top$ is nondecreasing and that $\bot$ is nonincreasing that $\top(\gamma) \geq \frac{1}{2}$ and $\bot(\gamma) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. It is then straighforward from Def. 34 that $\top^\sharp(\gamma) \geq \frac{1}{2}$ and $\top^\flat(\gamma) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$. Hence by Def. 22,

$$\mathrm{med}_{\frac{1}{2}}(\top^\sharp(\gamma), \bot(\gamma)) = \tfrac{1}{2} = \mathrm{med}_{\frac{1}{2}}(\top^\flat(\gamma), \bot(\gamma)) \tag{135}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned} \xi(\top^\sharp, \bot) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top^\sharp, \bot)) && \text{by (25)} \\ &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top^\flat, \bot)) && \text{by (135)} \\ &= \xi(\top^\flat, \bot)\,. && \text{by (25)} \end{aligned}$$

This finishes the proof of part **a.** of the lemma.

b. Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is defined in terms of $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ according to equation (25) and satisfies (X-2) and (X-4). In order to show that $\mathcal{B}$ satisfies (B-4), let us consider a choice of $f \in \mathbb{B}$. We shall discern two cases.

b.1: $f \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$**.** Then also $f^\sharp \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$ and $f^\flat \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$, see Def. 34 and Def. 32. Therefore

$$\begin{aligned}
\mathcal{B}(f^\sharp) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(f^\sharp, \mathrm{c}_0)) && \text{by L-36}\\
&= \xi(f^\sharp, \mathrm{c}_0) && \text{by (25)}\\
&= \xi(f^\flat, \mathrm{c}_0) && \text{by (X-4)}\\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(f^\flat, \mathrm{c}_0)) && \text{by (25)}\\
&= \mathcal{B}(f^\flat)\,, && \text{by L-36}
\end{aligned}$$

as desired.

b.2: $f \in \mathbb{B}^+$**.** In this case, clearly $f^\sharp \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$ and $f^\flat \in \mathbb{B}^+$. Hence

$$\begin{aligned}
\mathcal{B}(f^\sharp) &= \mathcal{B}(1-(1-f^\sharp))\\
&= 1-\mathcal{B}(1-f^\sharp) && \text{by (X-2), L-37}\\
&= 1-\mathcal{B}((1-f)^\sharp) && \text{apparent from Def. 34}\\
&= 1-\mathcal{B}((1-f)^\flat) && \text{by part \textbf{b.1} of this lemma}\\
&= \mathcal{B}(1-(1-f^\flat)) && \text{by (X-2), L-37, Def. 34}\\
&= \mathcal{B}(f^\flat)\,.
\end{aligned}$$

Hence (B-4) is valid for $\mathcal{B}$, which we intended to show.

Lemma 40 *Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be a mapping and let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be defined by* (25)*. Then $\mathcal{B}$ satisfies* (B-5) *if and only if ξ satisfies* (X-5)*.*

Proof Suppose $\mathcal{B}$ satisfies (B-5) and let a choice of $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given such that $\top \leq \top'$ and $\bot \leq \bot'$. Then

$$\begin{aligned}
\xi(\top, \bot) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{by (25)}\\
&\leq \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top', \bot'))\\
&= \xi(\top', \bot')\,, && \text{by (25)}
\end{aligned}$$

where the middle inequation holds because $\mathrm{med}_{\frac{1}{2}}(\top, \bot) \leq \mathrm{med}_{\frac{1}{2}}(\top', \bot')$ (by the monotonicity of $\mathrm{med}_{\frac{1}{2}}$) and because $\mathcal{B}$ satisfies (B-5).
Considering the converse implication, suppose ξ satisfies (X-5) and let a choice of $f, g \in \mathbb{B}$ be given such that $f \leq g$. We shall discern three cases.

a.: $f, g \in \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}}$**.** We can then apply L-36 to conclude that

$$f = \mathrm{med}_{\frac{1}{2}}(c_1, f) \tag{136}$$

$$g = \mathrm{med}_{\frac{1}{2}}(c_1, g)\,. \tag{137}$$

Hence

$$\begin{aligned}
\mathcal{B}(f) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(c_1, f)) && \text{by (136)} \\
&= \xi(c_1, f) && \text{by (25)} \\
&\leq \xi(c_1, g) && \text{by (X-5) and } f \leq g \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(c_1, g)) && \text{by (25)} \\
&= \mathcal{B}(g)\,. && \text{by (137)}
\end{aligned}$$

b.: $f \in \mathbb{B}^-$**,** $g \in \mathbb{B}^+$**.** In this case,

$$\begin{aligned}
\mathcal{B}(f) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(f, c_0)) && \text{by L-36} \\
&= \xi(f, c_0) && \text{by (25)} \\
&\leq \xi(c_1, g) && \text{by (X-5)} \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(c_1, g)) && \text{by (25)} \\
&= \mathcal{B}(g)\,. && \text{by L-36}
\end{aligned}$$

c.: $f, g \in \mathbb{B}^- \cup \mathbb{B}^{\frac{1}{2}}$**.** Then

$$\begin{aligned}
\mathcal{B}(f) &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(f, c_0)) && \text{by L-36} \\
&= \xi(f, c_0) && \text{by (25)} \\
&\leq \xi(g, c_0) && \text{by (X-5) and } f \leq g \\
&= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(g, c_0)) && \text{by (25)} \\
&= \mathcal{B}(g)\,. && \text{by L-36}
\end{aligned}$$

Proof of Theorem 25

Let $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ be given and let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be defined by (25). All claims of the theorem are immediate from the above lemmata; viz

1. The equivalence of (B-1) and (X-1) has been shown in L-35;
2. The equivalence of (B-2) and (X-2) has been shown in L-37;

3. the claimed relationship between (B-3) and (X-2)/(X-3) has been established in L-38;
4. the claimed relationship between (B-4) and (X-2)/(X-4) has been established in L-39;
5. the equivalence of (B-5) and (X-5) has been established in L-40.

B.7 Proof of Theorem 26

We already know from Th-8 that the conditions (B-1) to (B-5) are independent, i.e. for all $i \in \{1, \ldots, 5\}$, there exists a choice of $\mathcal{B}_i : \mathbb{B} \longrightarrow \mathbf{I}$ which satisfies all (B-j), $i \neq j$, but fails on (B-i). Now let us show the independence of the conditions (X-i), $i = 1, \ldots, 5$, based on the above choices of the $\mathcal{B}_i$'s.

Independence of (X-1)**.** To see that (X-1) is independent of the remaining (X-j), consider $\mathcal{B}_1$. We shall define $\xi_1 : \mathbb{T} \longrightarrow \mathbf{I}$ in terms of $\mathcal{B}_1$ according to equation (25). Then ξ_1 fails on (X-1) by Th-25.1 and ξ_1 satisfies (X-2), (X-3), (X-4) and (X-5) by Th-25.2, Th-25.3(a), Th-25.4(a) and Th-25.5, resp.

Independence of (X-2)**.** In this case, we define $\xi_2 : \mathbb{T} \longrightarrow \mathbf{I}$ in terms of $\mathcal{B}_2$. Clearly ξ_2 fails on (X-2) by Th-25.2 and ξ_2 satisfies (X-1), (X-3), (X-4) and (X-5) by Th-25.1, Th-25.3(a), Th-25.4(a) and Th-25.5, resp.

Independence of (X-3)**.** Now we define ξ_3 in terms of $\mathcal{B}_3$. Then ξ_3 satisfies (X-1), (X-2), (X-4) and (X-5) by Th-25.1, Th-25.2, Th-25.4(a) and Th-25.5, resp. Because ξ_3 satisfies (X-2) and $\mathcal{B}_3$ fails on (B-3), we obtain from Th-25.3(b) by contraposition that ξ_3 fails on (X-3).

Independence of (X-4)**.** In this case we define ξ_4 in terms of $\mathcal{B}_4$. Then ξ_4 satisfies (X-1), (X-2), (X-3) and (X-5) by Th-25.1, Th-25.2, Th-25.3(a) and Th-25.5, resp. Because ξ_4 satisfies (X-2) and $\mathcal{B}_4$ fails on (B-4), we obtain from Th-25.4(b) by contraposition that ξ_4 fails on (X-4).

Independence of (X-5)**.** Finally we use $\mathcal{B}_5$ to define ξ_5. Then ξ_5 violates (X-5) by Th-25.5, but it satisfies all other conditions by Th-25.1, Th-25.2, Th-25.3(a) and Th-25.4(a).

B.8 Proof of Theorem 27

Lemma 41 *For all monotonic mappings* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *(i.e. either nondecreasing or nonincreasing),*

$$\int_0^1 f^\flat(\gamma)\, d\gamma = \int_0^1 f^\sharp(\gamma)\, d\gamma\,.$$

Proof We will reuse earlier proofs for that. We already know that the QFM $\mathcal{M}$ with $\mathcal{B}_\int(f) = \int_0^1 f(\gamma)\, d\gamma$ for all $\gamma \in \mathbb{B}$ is a DFS, see Def. 39 and Th-11. Hence $\mathcal{B}_\int$ satisfies (B-4) by Th-8, i.e.

$$\int_0^1 f^\sharp(\gamma)\, d\gamma = \mathcal{B}_\int(f^\sharp) = \mathcal{B}_\int(f^\flat) = \int_0^1 f^\flat(\gamma)\, d\gamma \tag{138}$$

for all $f \in \mathbb{B}$. In the following, we discern the case of nondecreasing and nonincreasing mappings.

a.: nondecreasing mappings. Suppose $\top : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing. Then it is apparent from Def. 32 that $\frac{1}{2}\top \in \mathbb{B}$. In addition, we have

$$\begin{aligned}
(\tfrac{1}{2}\top)^\flat(\gamma) &= \lim_{\gamma' \to \gamma^-} \tfrac{1}{2}\top(\gamma') && \text{by Def. 34}\\
&= \tfrac{1}{2} \lim_{\gamma' \to \gamma^-} \top(\gamma') \\
&= \tfrac{1}{2}(\top^\flat) && \text{by Def. 34}
\end{aligned}$$

for $\gamma > 0$; the case $\gamma = 0$ is trivial. Similarly

$$\begin{aligned}
(\tfrac{1}{2}\top)^\sharp(\gamma) &= \lim_{\gamma' \to \gamma^+} \tfrac{1}{2}\top(\gamma') && \text{by Def. 34}\\
&= \tfrac{1}{2} \lim_{\gamma' \to \gamma^+} \top(\gamma') \\
&= \tfrac{1}{2}(\top^\sharp) && \text{by Def. 34}
\end{aligned}$$

for $\gamma < 1$; the case $\gamma = 1$ is trivial. Hence

$$(\tfrac{1}{2}\top)^\flat = \tfrac{1}{2}(\top^\flat) \tag{139}$$

and

$$(\top^\sharp)^\sharp = \tfrac{1}{2}(\top^\sharp)\,. \tag{140}$$

In turn,

$$\begin{aligned}
\int_0^1 \top^\sharp(\gamma)\,d\gamma &= \int_0^1 2\cdot \tfrac{1}{2}\top^\sharp(\gamma)\,d\gamma \\
&= 2\int_0^1 \left(\tfrac{1}{2}\top\right)^\sharp(\gamma)\,d\gamma && \text{by (140)} \\
&= 2\int_0^1 \left(\tfrac{1}{2}\top\right)^\flat(\gamma)\,d\gamma && \text{by (138)} \\
&= \int_0^1 2\cdot \tfrac{1}{2}\top^\flat(\gamma)\,d\gamma && \text{by (139)} \\
&= \int_0^1 \top^\flat(\gamma)\,d\gamma\,.
\end{aligned}$$

b.: nonincreasing mappings. Suppose $\perp : \mathbf{I} \longrightarrow \mathbf{I}$ is nonincreasing. Then $1-\perp$ is nondecreasing. In addition,

$$\begin{aligned}
(1-\perp)^\flat(\gamma) &= \lim_{\gamma'\to\gamma^-} 1-\perp(\gamma') && \text{by Def. 34} \\
&= 1-\lim_{\gamma'\to\gamma^-}\perp(\gamma') \\
&= 1-\perp^\flat(\gamma) && \text{by Def. 34}
\end{aligned}$$

for $\gamma > 0$, and

$$\begin{aligned}
(1-\perp)^\sharp(\gamma) &= \lim_{\gamma'\to\gamma^+} 1-\perp(\gamma') && \text{by Def. 34} \\
&= 1-\lim_{\gamma'\to\gamma^+}\perp(\gamma') \\
&= 1-\perp^\sharp(\gamma) && \text{by Def. 34}
\end{aligned}$$

for $\gamma < 1$, the remaining cases are again trivial. Hence

$$(1-\perp)^\flat = 1-\perp^\flat \qquad\qquad (1-\perp)^\sharp = 1-\perp^\sharp\,. \tag{141}$$

We can now proceed as follows.

$$\begin{aligned}
\int_0^1 \perp^\sharp(\gamma)\,d\gamma &= 1-\int_0^1 1-\perp^\sharp(\gamma)\,d\gamma \\
&= 1-\int_0^1 (1-\perp)^\sharp(\gamma)\,d\gamma && \text{by (141)} \\
&= 1-\int_0^1 (1-\perp)^\flat(\gamma)\,d\gamma && \text{by part } \textbf{a.} \text{ of the proof} \\
&= 1-\int_0^1 1-\perp^\flat(\gamma)\,d\gamma && \text{by (141)} \\
&= \int_0^1 \perp^\flat(\gamma)\,d\gamma\,,
\end{aligned}$$

as desired.

Proof of Theorem 27

By Th-22, we can prove that $\mathcal{F}_{\mathrm{Ch}}$ is a standard DFS by showing that the mapping $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ defined in Def. 47 satisfies the conditions (X-1) to (X-5). We will consider these conditions in turn.

ξ_{Ch} **satisfies** (X-1)**.** To see this, consider $(\mathrm{c}_a, \mathrm{c}_a) \in \mathbb{T}$, where $a \in \mathbf{I}$ is a given constant. Apparently

$$\begin{aligned}
\xi_{\mathrm{Ch}}(\mathrm{c}_a, \mathrm{c}_a) &= \tfrac{1}{2}\int_0^1 \mathrm{c}_a(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \mathrm{c}_a(\gamma)\,d\gamma && \text{by Def. 47}\\
&= \int_0^1 a\,d\gamma && \text{by (7)}\\
&= a\,,
\end{aligned}$$

i.e. (X-1) holds, as desired.

ξ_{Ch} **satisfies** (X-2)**.** Let $(\top, \bot) \in \mathbb{T}$ be given. Then

$$\begin{aligned}
&\xi_{\mathrm{Ch}}(1-\bot, 1-\top)\\
&= \tfrac{1}{2}\int_0^1 (1-\bot(\gamma))\,d\gamma + \tfrac{1}{2}\int_0^1 (1-\top(\gamma))\,d\gamma && \text{by Def. 47}\\
&= \tfrac{1}{2}(\int_0^1 1\,d\gamma - \int_0^1 \bot(\gamma)\,d\gamma + \int_0^1 1\,d\gamma - \int_0^1 \top(\gamma)\,d\gamma) && \text{by additivity of } \textstyle\int\\
&= \tfrac{1}{2}(2 - \int_0^1 \bot(\gamma)\,d\gamma - \int_0^1 \top(\gamma)\,d\gamma\\
&= 1 - (\tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \top(\gamma)\,d\gamma\\
&= 1 - \xi_{\mathrm{Ch}}(\top, \bot)\,. && \text{by Def. 47}
\end{aligned}$$

ξ_{Ch} **satisfies** (X-3) To this end, let $(\mathrm{c}_1, \bot) \in \mathbb{T}$ such that $\widehat{\bot}(\mathbf{I}) \subseteq \{0, 1\}$. Because $\bot$ is nonincreasing, this means that $\bot$ has one of the following two forms:

$$\bot(\gamma) = \begin{cases} 1 & : \quad \gamma \le \bot_*^{0\downarrow} \\ 0 & : \quad \gamma > \bot_*^{0\downarrow} \end{cases}$$

or

$$\bot(\gamma) = \begin{cases} 1 & : \quad \gamma < \bot_*^{0\downarrow} \\ 0 & : \quad \gamma \ge \bot_*^{0\downarrow} \end{cases}$$

for all $\gamma \in \mathbf{I}$, see (10). In any case,

$$\begin{aligned}\int_0^1 \bot(\gamma)\,d\gamma &= \int_0^{\bot_*^{0\downarrow}} \bot(\gamma)\,d\gamma + \int_{\bot_*^{0\downarrow}}^1 \bot(\gamma)\,d\gamma \\ &= \int_0^{\bot_*^{0\downarrow}} 1\,d\gamma + \int_{\bot_*^{0\downarrow}}^1 0\,d\gamma \qquad \text{see above} \\ &= \bot_*^{0\downarrow}\,,\end{aligned}$$

i.e.

$$\inf_0^1 \bot(\gamma)\,d\gamma = \bot_*^{0\downarrow}\,. \tag{142}$$

Therefore

$$\begin{aligned}\xi_{\mathrm{Ch}}(\mathrm{c}_1, \bot) &= \tfrac{1}{2}\int_0^1 \mathrm{c}_1\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma && \text{by Def. 47} \\ &= \tfrac{1}{2}\int_0^1 1\,d\gamma + \tfrac{1}{2}\bot_*^{0\downarrow} && \text{by (7), (142)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}\,,\end{aligned}$$

i.e. ξ_{Ch} satisfies (X-3).

ξ_{Ch} **satisfies (X-4).** Let $(\top, \bot) \in \mathbb{T}$ be given. Then

$$\begin{aligned}\xi_{\mathrm{Ch}}(\top^\sharp, \bot) &= \tfrac{1}{2}\int_0^1 \top^\sharp(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma && \text{by Def. 47} \\ &= \tfrac{1}{2}\int +0^1 \top^\flat(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma && \text{by L-41} \\ &= \xi_{\mathrm{Ch}}(\top^\flat, \bot)\,, && \text{by Def. 47}\end{aligned}$$

i.e. ξ_{Ch} fulfills (X-4).

ξ_{Ch} **satisfies (X-5).** To see this, let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\top \leq \top'$ and $\bot \leq \bot'$. Then

$$\begin{aligned}\xi_{\mathrm{Ch}}(\top, \bot) &= \tfrac{1}{2}\int_0^1 \top(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma && \text{by Def. 47} \\ &\leq \tfrac{1}{2}\int_0^1 \top'(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot'(\gamma)\,d\gamma && \text{by monotonicity of } \textstyle\int \\ &= \xi_{\mathrm{Ch}}(\top', \bot')\,,\end{aligned}$$

i.e. (X-5) holds. This finishes the proof that $\mathcal{F}_{\mathrm{Ch}}$ is a DFS.

B.9 Proof of Theorem 28

Lemma 42 *Let $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ be nondecreasing mappings such that for all $x \in \mathbf{I}$,*

$$\begin{aligned} &f(x) \le g(x) \\ &f(y) \ge g(x) && \text{for all } y > x. \end{aligned}$$

Then for all $a \le b$, $a, b \in \mathbf{I}$,

$$\int_a^b f(x)dx = \int_a^b g(x)dx\,.$$

Proof

$$\begin{aligned} \int_a^b f(x)\,dx &= (b-a)\int_0^1 f((b-a)x+a)\,dx \\ &= (b-a)\int_0^1 g((b-a)x+a)\,dx && \text{by [6, L-34, p. 140]} \\ &= \int_a^b g(x)\,dx\,. \end{aligned}$$

Lemma 43 *Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing semi-fuzzy quantifier. Then for all $a, b \in \mathbf{I}$ with $a \le b$,*

$$\int_a^b Q(X_{\ge\alpha})d\alpha = \int_a^b Q(X_{>\alpha})d\alpha\,.$$

Proof Consider $\alpha \in \mathbf{I}$. It is apparent from Def. 28 and Def. 29 that

$$\begin{aligned} &X_{>\alpha} \subseteq X_{\ge\alpha} \\ &X_{>\alpha'} \supseteq X_{\ge\alpha} && \text{for all } \alpha' > \alpha. \end{aligned}$$

In turn because Q is nondecreasing, we obtain from Def. 13 that

$$\begin{aligned} &Q(X_{>\alpha}) \le Q(X_{\ge\alpha}) \\ &Q(X_{>\alpha'}) \ge Q(X_{\ge\alpha}) && \text{for all } \alpha' > \alpha. \end{aligned}$$

Hence $Q(X_{>\bullet}), Q(X_{\ge\bullet}) : \mathbf{I} \longrightarrow \mathbf{I}$ satisfy the requirements of L-42, from which we obtain the desired

$$\int_a^b Q(X_{>\alpha})d\alpha = \int_a^b Q(X_{\ge\alpha})d\alpha\,.$$

Proof of Theorem 28

Let $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ be a given nondecreasing quantifier. Now consider a choice of fuzzy argument set $X \in \widetilde{\mathcal{P}}(E)$. We compute

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Ch}}(Q)(X) \\
&= \tfrac{1}{2}\int_0^1 \top_{Q,X}(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot_{Q,X}(\gamma)\,d\gamma && \text{by Def. 47} \\
&= \tfrac{1}{2}\int_0^1 Q(X_\gamma^{\max})\,d\gamma + \tfrac{1}{2}\int_0^1 Q(X_\gamma^{\min})\,d\gamma && \text{by L-3} \\
&= \tfrac{1}{2}\int_0^1 Q(X_{>\frac{1}{2}-\frac{1}{2}\gamma})\,d\gamma + \tfrac{1}{2}\int_0^1 Q(X_{\ge\frac{1}{2}+\frac{1}{2}\gamma})\,d\gamma && \text{by Def. 30 and L-23} \\
&= \tfrac{1}{2}\cdot 2\cdot\int_0^{\frac{1}{2}} Q(X_{>\alpha})d\alpha + \tfrac{1}{2}\cdot 2\cdot\int_{\frac{1}{2}}^1 Q(X_{\ge\alpha})d\alpha && \text{by appropriate substitutions} \\
&= \int_0^{\frac{1}{2}} Q(X_{\ge\alpha}) + \int_{\frac{1}{2}}^1 Q(X_{\ge\alpha}) && \text{by L-43} \\
&= \int_0^1 Q(X_{\ge\alpha}) \\
&= (Ch)\int X\,dQ\,. && \text{by Def. 48}
\end{aligned}$$

B.10 Proof of Theorem 29

The claim made in the theorem has already been established in [5]. Nevertheless, we prove it again here to gain better insight into the relationship of the Choquet integral and OWA operators.

Lemma 44 *Let $E \neq \varnothing$ be a finite base set with cardinality $m = |E|$. Further let $q : \{0, \ldots, m\} \longrightarrow \mathbf{I}$ be a nondecreasing mapping such that*

$$q(0) = 0$$

and

$$q(m) = 1\,,$$

and let $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ be the semi-fuzzy quantifier defined by

$$Q(Y) = q(|Y|) \tag{143}$$

for all $Y \in \mathcal{P}(E)$. The for all $X \in \widetilde{\mathcal{P}}(E)$ and $j \in \{1, \ldots, m\}$,

$$\int_{\mu_{[j+1]}(X)}^{\mu_{[j]}(X)} Q(X_{\ge\alpha})\,d\alpha = (\mu_{[j]}(X) - \mu_{[j+1]}(X))\cdot q(j)\,.$$

Proof We first consider the case that $\mu_{[j]}(X) = \mu_{[j+1]}(X)$. Then

$$\begin{aligned}
&\int_{\mu_{[j+1]}(X)}^{\mu_{[j]}(X)} Q(X_{\geq\alpha})\, d\alpha \\
&= \int_{\mu_{[j]}(X)}^{\mu_{[j]}(X)} Q(X_{\geq\alpha})\, d\alpha \\
&= 0 \\
&= (\mu_{[j]}(X) - \mu_{[j]}(X)) \cdot q(j) \\
&= (\mu_{[j]}(X) - \mu_{[j+1]}(X)) \cdot q(j)\,.
\end{aligned}$$

In the remaining case that $\mu_{[j]}(X) > \mu_{[j+1]}(X)$, we choose an arbitrary ordering of the elements of E such that $E = \{e_1, \ldots, e_m\}$ and $\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)$, i.e.

$$\mu_X(e_k) = \mu_{[k]}(X) \tag{144}$$

by Def. 49. Hence $\mu_X(e_k) \geq \mu_{[j]}(X)$ for $1 \leq k \leq j$ and $\mu_X(e_k) < \mu_{[j]}(X)$ for $k > j$, which is apparent from our assumption that $\mu_{[j+1]}(X) < \mu_{[j]}(X)$. We therefore obtain for all $\alpha \in (\mu_{[j+1]}(X), \mu_{[j]}(X)]$ that

$$\begin{aligned}
X_{\geq\alpha} &= \{e \in E : \mu_X(e) \geq \alpha\} \\
&= \{e_k \in E : \mu_{[k]}(X) \geq \alpha\} \\
&= \{e_k : k = 1, \ldots, j\}\,.
\end{aligned}$$

In turn

$$|X_{\geq\alpha}| = |\{e_k : k = 1, \ldots, j\}| = j\,, \tag{145}$$

which holds because all e_i are distinct (this is apparent from $|E| = m$ and $E = \{e_1, \ldots, e_m\}$). We hence obtain

$$\begin{aligned}
&\int_{\mu_{[j+1]}(X)}^{\mu_{[j]}(X)} Q(X_{\geq\alpha})\, d\alpha \\
&= \int_{\mu_{[j+1]}(X)}^{\mu_{[j]}(X)} q(|X_{\geq\alpha}|)\, d\alpha && \text{by (143)} \\
&= \int_{\mu_{[j+1]}(X)}^{\mu_{[j]}(X)} q(j)\, d\alpha && \text{by (145)} \\
&= (\mu_{[j]}(X) - \mu_{[j+1]}(X)) \cdot q(j)\,,
\end{aligned}$$

as desired.

Proof of Theorem 29

In order to prove that $\mathcal{F}_{\mathrm{Ch}}$ generalizes the OWA approach, we proceed as follows. Suppose $E \neq \varnothing$ is a finite base set of cardinality $|E| = m$ and $q : \{0, \ldots, m\} \longrightarrow \mathbf{I}$ is a nondecreasing mapping with

$$q(0) = 0 \tag{146}$$

and

$$q(m) = 1\,. \tag{147}$$

Further assume that $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by

$$Q(Y) = q(|Y|)\,, \tag{148}$$

for all $Y \in \mathcal{P}(E)$. Now consider $X \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
&\mathcal{F}_{\mathrm{Ch}}(Q)(X) \\
&= (Ch)\int X\,dQ && \text{by Th-28} \\
&= \int_0^1 Q(X_{\geq\alpha})\,d\alpha && \text{by Def. 48} \\
&= \sum_{k=0}^{m} \int_{\mu_{[m+1-k]}(X)}^{\mu_{[m-k]}(X)} Q(X_{\geq\alpha})\,d\alpha \\
&= \sum_{k=0}^{m} (\mu_{[m-k]}(X) - \mu_{[m-k+1]}(X)) \cdot q(m-k) && \text{by L-44} \\
&= \sum_{k=0}^{m} \mu_{[m-k]}(X) q(m-k) \\
&\quad - \sum_{k=0}^{m} \mu_{[m-k+1]}(X) q(m-k) \\
&= \sum_{k=0}^{m-1} \mu_{[m-k]}(X) q(m-k) + \mu_{[0]}(X) q(0) \\
&\quad - \sum_{k=0}^{m-1} \mu_{[m-k]}(X) q(m-k-1) - \mu_{[m+1]}(X) q(m) && \text{by substitution} \\
&= \sum_{k=0}^{m-1} \mu_{[m-k]}(X) q(m-k) \\
&\quad - \sum_{k=0}^{m-1} \mu_{[m-k]}(q)(m-k-1) && \text{by (146) and } \mu_{[m+1]}(X) = 0 \\
&= \sum_{k=0}^{m-1} \mu_{[m-k]}(X) \cdot (q(m-k) - q(m-k-1)) \\
&= \sum_{j=1}^{m} \mu_{[j]}(X)(q(j) - q(j-1))\,, && \text{by substitution } j = m-k
\end{aligned}$$

i.e. $\mathcal{F}_{\mathrm{Ch}}(Q)(X)$ coincides with the result of the OWA approach, as desired.

B.11 Proof of Theorem 30

Lemma 45 *Suppose* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e. nondecreasing or non-increasing). Then* $(1-f)^*_1 = 1 - f^*_1$.

Proof Trivial:

$$\begin{aligned}(1-f)^*_1 &= \lim_{\gamma\to 1^-}(1-f)(\gamma) && \text{by (12)}\\ &= \lim_{\gamma\to 1^-} 1 - f(\gamma)\\ &= 1 - \lim_{\gamma\to 1^-} f(\gamma)\\ &= 1 - f^*_1\,. && \text{by (12)}\end{aligned}$$

where the limites are guaranteed to exist because f is monotonic.

Lemma 46 *For all* $(\top,\bot) \in \mathbb{T}$,

a. $(1-\bot)^{\geq\frac{1}{2}\downarrow}_* = \bot^{\leq\frac{1}{2}\downarrow}_*$;

b. $(1-\top)^{\leq\frac{1}{2}\downarrow}_* = \top^{\geq\frac{1}{2}\downarrow}_*$.

Proof We first show that **a.** holds.

$$\begin{aligned}(1-\bot)^{\geq\frac{1}{2}\downarrow}_* &= \inf\{\gamma\in\mathbf{I} : (1-\bot)(\gamma) \geq \tfrac{1}{2}\} && \text{by (27)}\\ &= \inf\{\gamma\in\mathbf{I} : 1-\bot(\gamma) \geq \tfrac{1}{2}\}\\ &= \inf\{\gamma\in\mathbf{I} : \bot(\gamma) \leq \tfrac{1}{2}\}\\ &= \bot^{\leq\frac{1}{2}\downarrow}_*\,. && \text{by (26)}\end{aligned}$$

Similarly for **b.**,

$$\begin{aligned}(1-\top)^{\leq\frac{1}{2}\downarrow}_* &= (1-(1-\top))^{\geq\frac{1}{2}\downarrow}_* && \text{by } \mathbf{a.}\\ &= \top^{\geq\frac{1}{2}\downarrow}_*\,.\end{aligned}$$

Lemma 47 *For all monotonic mappings* $f : \mathbf{I} \longrightarrow \mathbf{I}$,

1. $(1-f)^\sharp = 1 - f^\sharp$;

2. $(1-f)^\flat = 1 - f^\flat$.

Proof Let $f : \mathbf{I} \longrightarrow \mathbf{I}$ be a monotonic mapping and let $\gamma < 1$. Then

$$\begin{aligned}(1-f)^{\sharp}(\gamma) &= \lim_{\gamma' \to \gamma^+} 1 - f(\gamma') && \text{by Def. 34}\\ &= 1 - \lim_{\gamma' \to \gamma^+} f(\gamma') \\ &= 1 - f^{\sharp}(\gamma)\,. && \text{by Def. 34}\end{aligned}$$

The case that $\gamma = 1$ is trivial.
As to the second claim, we obtain for $\gamma > 0$ that

$$\begin{aligned}(1-f)^{\flat}(\gamma) &= \lim_{\gamma' \to \gamma^-} 1 - f(\gamma') && \text{by Def. 34}\\ &= 1 - \lim_{\gamma' \to \gamma^-} f(\gamma') \\ &= 1 - f^{\flat}(\gamma)\,. && \text{by Def. 34}\end{aligned}$$

The remaining case that $\gamma = 0$ is again trivial.

Lemma 48 *Let* $(\top, \bot) \in \mathbb{T}$ *be given. Then*

a. $(\bot^{\sharp})^*_1 = (\bot^{\flat})^*_1$;

b. $(\top^{\sharp})^*_1 = (\top^{\flat})^*_1$.

Proof Suppose $(\top, \bot) \in \mathbb{T}$ is given.

a. We discern two cases in the proof of part **a.** of the lemma.

- If $\widehat{\bot}((0,1]) = \{0\}$, then $\bot^{\sharp} = \mathrm{c}_0$ and $\bot^{\flat} = \bot$ by Def. 34. Hence

$$\begin{aligned}{\bot^{\sharp}}^*_1 &= (\mathrm{c}_0)^*_1 \\ &= \lim_{\gamma \to 1^-} 0 && \text{by (12)}\\ &= 0 \\ &= \lim_{\gamma \to 1^-} \bot \\ &= \lim_{\gamma \to 1^-} \bot^{\flat} \\ &= (\bot^{\flat})^*_1\,. && \text{by (12)}\end{aligned}$$

- If $\widehat{\bot}((0,1]) \neq \{0\}$, then $(\bot^{\sharp})^*_1 = (\bot^{\flat})^*_1$ by [7, L-73.b, p. 186].

b. The proof of the second part of the lemma can be reduced to that of the first part as follows.

$$\begin{aligned}
(\top^\sharp)^*_1 &= (1-(1-\top^\sharp))^*_1 && \\
&= (1-(1-\top)^\sharp)^*_1 && \text{by L-47} \\
&= 1-((1-\top)^\sharp)^*_1 && \text{by L-45} \\
&= 1-((1-\top)^\flat)^*_1 && \text{by part } \mathbf{a.} \\
&= (1-(1-\top)^\flat)^*_1 && \text{by L-45} \\
&= (1-(1-\top^\flat))^*_1 && \text{by L-47} \\
&= (\top^\flat)^*_1 \,.
\end{aligned}$$

Lemma 49 *Let* $\top : \mathbf{I} \longrightarrow \mathbf{I}$ *be a nondecreasing mapping. Then*

$$(\top^\sharp)^{\geq\frac{1}{2}\downarrow}_* = (\top^\flat)^{\geq\frac{1}{2}\downarrow}_* \,.$$

Proof Let us abbreviate $\bot = \max(1-2\top, 0)$. It is apparent from Def. 34 and from the continuity of the involved operations that

$$\bot^\sharp = \max(1-2\cdot\top^\sharp, 0)$$

and

$$\bot^\flat = \max(1-2\cdot\top^\flat, 0)\,.$$

We hence obtain from (10) and (27) that

$$(\bot^\sharp)^{0\downarrow}_* = (\top^\sharp)^{\geq\frac{1}{2}\downarrow}_* \tag{149}$$

$$(\bot^\flat)^{0\downarrow}_* = (\top^\flat)^{\geq\frac{1}{2}\downarrow}_* \tag{150}$$

Now if $\widehat{\bot}((0,1]) = \{0\}$, then $\bot^\sharp = c_0$ and $\bot^\flat = \bot$ with $\bot^{0\downarrow}_* = 0$. Hence in this case

$$(\top^\sharp)^{\geq\frac{1}{2}\downarrow}_* = (\bot^\sharp)^{0\downarrow}_* = (c_0)^{0\downarrow}_* = 0 = \bot^{0\downarrow}_* = (\bot^\flat)^{0\downarrow}_* = (\top^\flat)^{\geq\frac{1}{2}\downarrow}_* \,.$$

In the remaining case that $\widehat{\top}((0,1]) \neq \{0\}$, we obtain the desired

$$(\top^\sharp)^{\geq\frac{1}{2}\downarrow}_* = (\bot^\sharp)^{0\downarrow}_* = (\bot^\flat)^{0\downarrow}_* = (\top^\flat)^{\geq\frac{1}{2}\downarrow}_*$$

from [7, L-54.b, p. 140] and (149), (150).

Lemma 50 *Let* $(\top, \bot) \in \mathbb{T}$ *be given. Then*

a. if $\bot(0) \leq \top(0) < \frac{1}{2}$, *then* $\xi_S(\top, \bot) \leq \frac{1}{2}$*;*

b. if $\bot(0) \leq \frac{1}{2} \leq \top(0)$, *then* $\xi_S(\top, \bot) = \frac{1}{2}$*;*

c. if $\frac{1}{2} < \bot(0) \leq \top(0)$, *then* $\xi_S(\top, \bot) \geq \frac{1}{2}$.

Proof Immediate from the definition of $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ in Def. 50.

Proof of Theorem 30

We will utilize Th-22 and prove that $\mathcal{F}_S$ is a standard DFS by showing that the corresponding mapping $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ defined in Def. 50 fulfills the conditions (X-1) to (X-5). In order to shorten the proof, we will first show that (X-2) is satisfied.

ξ_S **satisfies** (X-2). To see this, consider $(\top, \bot) \in \mathbb{T}$. We discern three cases as follows.

1. $1 - \top(0) > \frac{1}{2}$, i.e. $\top(0) < \frac{1}{2}$. Then

$$\begin{aligned}
\xi(1-\bot, 1-\top) &= \min((1-\bot)^*_1, \tfrac{1}{2} + \tfrac{1}{2}(1-\top)^{\leq\frac{1}{2}\downarrow}_*) && \text{by Def. 50}\\
&= \min(1-\bot^*_1, 1-(\tfrac{1}{2} - \tfrac{1}{2}\top^{\geq\frac{1}{2}\downarrow}_*)) && \text{by L-45, L-46}\\
&= 1-\max(\bot^*_1, \tfrac{1}{2} - \tfrac{1}{2}\top^{\geq\frac{1}{2}\downarrow}_*) && \text{by De Morgan's law}\\
&= 1-\xi_S(\top,\bot)\,. && \text{by Def. 50}
\end{aligned}$$

2. $1 - \bot(0) < \frac{1}{2}$, i.e. $\bot(0) > \frac{1}{2}$. Then similarly

$$\begin{aligned}
\xi(1-\bot, 1-\top) &= \max((1-\top)^*_1, \tfrac{1}{2} + \tfrac{1}{2}(1-\bot)^{\geq\frac{1}{2}\downarrow}_*) && \text{by Def. 50}\\
&= \max(1-\top^*_1, 1-(\tfrac{1}{2} - \tfrac{1}{2}\bot^{\leq\frac{1}{2}\downarrow}_*)) && \text{by L-45, L-46}\\
&= 1-\min(\top^*_1, \tfrac{1}{2} - \tfrac{1}{2}\bot^{\leq\frac{1}{2}\downarrow}_*) && \text{by De Morgan's law}\\
&= 1-\xi_S(\top,\bot)\,. && \text{by Def. 50}
\end{aligned}$$

3. $1 - \top(0) \leq \frac{1}{2} \leq 1 - \bot(0)$. Then by Def. 50,

$$\xi_S(1-\bot, 1-\top) = \tfrac{1}{2} = 1 - \tfrac{1}{2} = 1 - \xi_S(\top,\bot)\,,$$

because $\bot(0) \leq \frac{1}{2} \leq \top(0)$.

ξ_S **satisfies** (X-1). Because (X-2) is valid for ξ_S, we only need to consider $\mathrm{c}_a : \mathbf{I} \longrightarrow \mathbf{I}$ where $a \in [\frac{1}{2}, 1]$. It is apparent from (12) that for every such constant, $(\mathrm{c}_a)^*_1 = a$. In the following, we discern two cases.

1. if $a > \frac{1}{2}$, then

$$\begin{aligned}
(\mathrm{c}_a)^{\leq\frac{1}{2}\downarrow}_* &= \inf\{\gamma \in \mathbf{I} : \mathrm{c}_a(\gamma) \leq \tfrac{1}{2}\} && \text{by (26)}\\
&= \inf\{\gamma \in \mathbf{I} : a \leq \tfrac{1}{2}\} && \text{by (7)}\\
&= \inf \varnothing && \text{because } a > \tfrac{1}{2}\\
&= 1\,. && \text{(by the usual convention)}
\end{aligned}$$

Hence

$$\begin{aligned}\xi_S(\mathrm{c}_a, \mathrm{c}_a) &= \min((\mathrm{c}_a)^*_1, \tfrac{1}{2} + \tfrac{1}{2}(\mathrm{c}_a)^{\leq\frac{1}{2}\downarrow}_*) && \text{by Def. 50}\\ &= \min(a, 1)\\ &= a\,.\end{aligned}$$

2. In the case that $a = \frac{1}{2}$, it is immediate from Def. 50 that

$$\xi_S(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2}$$

because $\mathrm{c}_{\frac{1}{2}}(0) = \frac{1}{2}$.

ξ_S **satisfies** (X-3). Let $(\mathrm{c}_1, \perp) \in \mathbb{T}$ be given with $\widehat{\perp}(\mathbf{I}) \subseteq \{0, 1\}$. Then

$$(\mathrm{c}_1)^*_1 = \lim_{\gamma\to 1^-} 1 = 1 \tag{151}$$

and

$$\begin{aligned}\perp^{\leq\frac{1}{2}\downarrow}_* &= \inf\{\gamma \in \mathbf{I} : \perp(\gamma) \leq \tfrac{1}{2}\} && \text{by (26)}\\ &= \inf\{\gamma \in \mathbf{I} : \perp(\gamma) = 0\}\end{aligned}$$

because $\perp(\gamma) \in \{0, 1\}$ by assumption, i.e.

$$\perp^{\leq\frac{1}{2}\downarrow}_* = \perp^{0\downarrow}_*\,. \tag{152}$$

Now we treat separately the two possible cases that $\perp(0) = 1$ or $\perp(0) = 0$.
If $\perp(0) = 1$, then in particular $\perp(0) > \frac{1}{2}$. Therefore

$$\begin{aligned}\xi_S(\mathrm{c}_1, \perp) &= \min((\mathrm{c}_1)^*_1, \tfrac{1}{2} + \tfrac{1}{2}\perp^{\leq\frac{1}{2}\downarrow}_*) && \text{by Def. 50 and } \perp(0) > \tfrac{1}{2}\\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2}\perp^{0\downarrow}_*) && \text{by (151), (152)}\\ &= \tfrac{1}{2} + \tfrac{1}{2}\perp^{0\downarrow}_*\,.\end{aligned}$$

In the remaining case that $\perp(0) = 0$, we conclude from the fact that $\perp$ is nonincreasing that $\perp = \mathrm{c}_0$ and

$$\perp^{0\downarrow}_* = \inf\{\gamma \in \mathbf{I} : 0 = 0\} = \inf \mathbf{I} = 0\,. \tag{153}$$

Therefore

$$\begin{aligned}\xi_S(\mathrm{c}_1, \perp) &= \xi_S(\mathrm{c}_1, \mathrm{c}_0) && \text{because } \perp = \mathrm{c}_0\\ &= \tfrac{1}{2} && \text{by Def. 50}\\ &= \tfrac{1}{2} + \tfrac{1}{2}\perp^{0\downarrow}_*\,. && \text{by (153)}\end{aligned}$$

ξ_S **satisfies** (X-4). Let $(\top, \bot) \in \mathbb{T}$ be given. Then by L-48 and L-49,

$$\min((\top^\sharp)^*_1, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}) = \min((\top^\flat)^*_1, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow})$$
$$\max(\bot^*_1, \tfrac{1}{2} - \tfrac{1}{2}(\top^\sharp)_*^{\geq\frac{1}{2}\downarrow}) = \max(\bot^*_1, \tfrac{1}{2} - \tfrac{1}{2}(\top^\flat)_*^{\geq\frac{1}{2}\downarrow})\,.$$

It is apparent from these two equations and Def. 50 that

$$\xi_S(\top^\sharp, \bot) = \xi_S(\top^\flat, \bot)$$

holds in all cases except for one case, which must be checked separately: the case that $\top^\flat(0) < \frac{1}{2}$ and $\top^\sharp(0) \geq \frac{1}{2}$. We then have $\bot(0) \leq \top^\flat(0) < \frac{1}{2}$ and hence $\xi_S(\top^\sharp, \bot) = \frac{1}{2}$ by Def. 50. In addition, we conclude from $\top^\sharp(0) \geq \frac{1}{2}$ that $\top(\gamma) \geq \frac{1}{2}$ for all $\gamma > 0$. In turn, we conclude from Def. 34 that $\top^\flat(\gamma) \geq \frac{1}{2}$ for all $\gamma > 0$. Hence $(\top^\flat)_*^{\geq\frac{1}{2}\downarrow} = 0$, and

$$\begin{aligned}\xi_S(\top^\flat, \bot) &= \max(\bot^*_1, \tfrac{1}{2} - \tfrac{1}{2}(\top^\flat)_*^{\geq\frac{1}{2}\downarrow})\\ &= \max(\bot^*_1, \tfrac{1}{2})\\ &= \tfrac{1}{2}\,.\end{aligned}$$

ξ_S **satisfies** (X-5). It is apparent from L-50 and the fact that (X-2) is valid that only one critical case must be checked, viz the case that $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\top \leq \top'$, $\bot \leq \bot'$ and $\top(0) > \frac{1}{2}$. To see that (X-5) holds in this case, we first observe that

$$\begin{aligned}\top^*_1 &= \lim_{\gamma\to 1^-} \top(\gamma) && \text{by (12)}\\ &\leq \lim_{\gamma\to 1^-} \top'(\gamma) && \text{by monotonicity of } \lim\\ &= \top'^*_1\,. && \text{by (12)}\end{aligned}$$

We further notice that

$$\begin{aligned}\bot_*^{\leq\frac{1}{2}\downarrow} &= \inf\{\gamma \in \mathbf{I} : \bot(\gamma) \leq \tfrac{1}{2}\} && \text{by (26)}\\ &\leq \inf\{\gamma \in \mathbf{I} : \bot'(\gamma) \leq \tfrac{1}{2}\}\\ &= \bot'^{\leq\frac{1}{2}\downarrow}_* && \text{by (26)}\end{aligned}$$

because $\bot \leq \bot'$ and hence

$$\{\gamma \in \mathbf{I} : \bot(\gamma) \leq \tfrac{1}{2}\} \supseteq \{\gamma \in \mathbf{I} : \bot'(\gamma) \leq \tfrac{1}{2}\}\,.$$

Finally, we notice that $\bot'(0) \geq \bot(0) > \frac{1}{2}$. Therefore

$$\begin{aligned}\xi_S(\top, \bot) &= \min(\top^*_1, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}) && \text{by Def. 50}\\ &\leq \min(\top'^*_1, \tfrac{1}{2} + \tfrac{1}{2}\bot'^{\leq\frac{1}{2}\downarrow}_*) && \text{by above inequations}\\ &= \xi_S(\top', \bot')\,, && \text{by Def. 50}\end{aligned}$$

i.e. ξ_S satisfies (X-5).

B.12 Proof of Theorem 31

Lemma 51 *For all monotonic mappings* $f : \mathbf{I} \longrightarrow \mathbf{I}$,

$$(1-f)_0^* = 1 - (f_0^*)\,.$$

Proof Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotonic mapping (i.e. either nondecreasing or nonincreasing). Then

$$\begin{aligned}
(1-f)_0^* &= \lim_{\gamma\to 0^+} (1-f)(\gamma) && \text{by (9)}\\
&= \lim_{\gamma\to 0^+} (1-f(\gamma))\\
&= 1 - \lim_{\gamma\to 0^+} f(\gamma)\\
&= 1 - (f_0^*)\,. && \text{by (9)}
\end{aligned}$$

Lemma 52 *Let* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *be a given mapping. Then*

$$\begin{aligned}
(1-f)_*^{0\downarrow} &= f_*^{1\downarrow}\\
(1-f)_*^{1\downarrow} &= f_*^{0\downarrow}
\end{aligned}$$

Proof As to the first equation, we compute

$$\begin{aligned}
(1-f)_*^{0\downarrow} &= \inf\{\gamma : 1 - f(\gamma) = 0\} && \text{by (10)}\\
&= \inf\{\gamma : f(\gamma) = 1\}\\
&= f_*^{1\downarrow}\,. && \text{by (14)}
\end{aligned}$$

The second equation is apparent from the first one because $(1-f)_*^{1\downarrow} = (1-(1-f))_*^{0\downarrow} = f_*^{0\downarrow}$.

Lemma 53 *For all monotonic mappings* $f : \mathbf{I} \longrightarrow \mathbf{I}$,

$$(f^\sharp)_0^* = (f^\flat)_0^*\,.$$

Proof Let $f : \mathbf{I} \longrightarrow \mathbf{I}$ be a monotonic mapping (i.e. either nondecreasing or nonincreasing). We shall discern the following cases.

a.: $f = \mathrm{c}_0$**.** Then $f^\sharp = f^\flat = \mathrm{c}_0$ by Def. 34 and hence trivially $(f^\sharp)_0^* = (f^\flat)_0^*$.

b.: f is nonincreasing, $f(0) > 0$ and $\widehat{f}((0,1]) = \{0\}$. Then $f^\sharp = c_0$ and $f^\flat = f$ by Def. 34. Hence

$$\begin{aligned}
(f^\sharp)^*_0 &= \lim_{\gamma\to 0^+} f^\sharp(\gamma) && \text{by (9)}\\
&= \lim_{\gamma\to 0^+} 0 && \text{because } f^\sharp = c_0\\
&= 0 &&\\
&= f^*_0 && \text{by [7, L-59.a, p.148]}\\
&= {f^\flat}^*_0 . && \text{because } f = f^\flat
\end{aligned}$$

c.: f is nonincreasing, $f(0) > 0$ and $\widehat{f}((0,1]) \neq \{0\}$. In this case, the desired $(f^\sharp)^*_0 = (f^\flat)^*_0$ has already been proven in [7, L-59.b, p.148].

d.: f is nondecreasing. This case can be reduced to **a**–**c.** by applying L-51 and L-47.

Lemma 54 *Let $f : \mathbf{I} \longrightarrow \mathbf{I}$ be some mapping.*

a. If f is nonincreasing, then

$$(f^\sharp)^{0\downarrow}_* = (f^\flat)^{0\downarrow}_*$$

b. If f is nondecreasing, then

$$(f^\sharp)^{1\downarrow}_* = (f^\flat)^{1\downarrow}_*$$

Proof Let $f : \mathbf{I} \longrightarrow \mathbf{I}$ be the given mapping.

Proof of part a. In order to show that the equation in part **a.** of the lemma holds, let us assume that f is nonincreasing. We discern the following cases **a.1**–**a.3**.

a.1: $f = c_0$. Then $f^\sharp = f^\flat = c_0$ by Def. 34 and hence trivially $(f^\sharp)^{0\downarrow}_* = (f^\flat)^{0\downarrow}_*$.

a.2: $f(0) > 0$ and $\widehat{f}((0,1]) = \{0\}$. Then $f^\sharp = c_0$ and $f^\flat = f$ by Def. 34. Hence

$$\begin{aligned}
(f^\sharp)^{0\downarrow}_* &= \inf\{\gamma : f^\sharp(\gamma) = 0\} && \text{by (10)}\\
&= \inf\{\gamma : 0 = 0\} && \text{because } f^\sharp = c_0\\
&= \inf[0,1] &&\\
&= 0 &&\\
&= f^{0\downarrow}_* && \text{by [7, L-54.a, p.140]}\\
&= {f^\flat}^{0\downarrow}_* . && \text{because } f = f^\flat
\end{aligned}$$

a.3: $f(0) > 0$ **and** $\widehat{f}((0,1]) \neq \{0\}$**.** In this case, the desired equation $(f^\sharp)_*^{0\downarrow} = (f^\flat)_*^{0\downarrow}$ has already been proven in [7, L-54.b, p.140].

Proof of part b.: This case can be reduced to part **a.** of the lemma by applying L-52 and L-47.

Lemma 55 *Let* $(\top, \bot) \in \mathbb{T}$ *be given. Then*

a. *if* $\bot_0^* \leq \top_0^* < \frac{1}{2}$, *then* $\xi_A(\top, \bot) \leq \frac{1}{2}$;

b. *if* $\bot_0^* \leq \frac{1}{2} \leq \top_0^*$, *then* $\xi_A(\top, \bot) = \frac{1}{2}$;

c. *if* $\frac{1}{2} < \bot_0^* \leq \top_0^*$, *then* $\xi_A(\top, \bot) \geq \frac{1}{2}$.

Proof Immediate from the definition of $\xi_A : \mathbb{T} \longrightarrow \mathbf{I}$ in Def. 51.

Proof of Theorem 31

$\mathcal{F}_A$ **satisfies** (X-1)**.** Suppose $(\top, \bot) \in \mathbb{T}$ satisfies $\top = \bot$, i.e. $\top = \bot = \mathrm{c}_x$ for some $x \in \mathbf{I}$. Then

$$[\mathrm{c}_x]_0^* = \lim_{\gamma \to 0^+} \mathrm{c}_x(\gamma) = \lim_{\gamma \to 0^+} x = x \tag{154}$$

$$[\mathrm{c}_x]_*^{0\downarrow} = \begin{cases} 0 & : \quad x = 0 \\ 1 & : \quad x > 0 \end{cases} \tag{155}$$

$$[\mathrm{c}_x]_*^{1\downarrow} = \begin{cases} 0 & : \quad x = 1 \\ 1 & : \quad x < 1 \end{cases} \tag{156}$$

by (9), (10) and (14). In the following, I discern the following cases.
If $x > \frac{1}{2}$, then

$$\begin{aligned} \xi_A(\mathrm{c}_x, \mathrm{c}_x) &= \min([\mathrm{c}_x]_0^*, \tfrac{1}{2} + \tfrac{1}{2}[\mathrm{c}_x]_*^{0\downarrow}) && \text{by Def. 51} \\ &= \min(x, 1) && \text{by (154), (155)} \\ &= x\,. \end{aligned}$$

For $x = \frac{1}{2}$, we obtain

$$\begin{aligned} \xi_A(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_{\frac{1}{2}}) &= \tfrac{1}{2} && \text{by Def. 51} \\ &= x\,. \end{aligned}$$

Finally for $x < \frac{1}{2}$, we have

$$\begin{aligned} \xi_A(\mathrm{c}_x, \mathrm{c}_x) &= \max([\mathrm{c}_x]_0^*, \tfrac{1}{2} - \tfrac{1}{2}[\mathrm{c}_x]_*^{1\downarrow}) && \text{by Def. 51} \\ &= \max(x, \tfrac{1}{2} - \tfrac{1}{2} \cdot 1) && \text{by (154), (156)} \\ &= x\,. \end{aligned}$$

Hence (X-1) holds, as desired.

$\mathcal{F}_A$ **satisfies** (X-2)**.** Let $(\top, \bot) \in \mathbb{T}$ be given. We notice that

$$(1-\top)^*_0 > \tfrac{1}{2} \Leftrightarrow \top^*_0 < \tfrac{1}{2} \tag{157}$$

and

$$(1-\bot)^*_0 < \tfrac{1}{2} \Leftrightarrow \bot^*_0 > \tfrac{1}{2}\,. \tag{158}$$

In the following, I discern three cases.

1. $(1-\top)^*_0 > \frac{1}{2}$. Then

$$\begin{aligned}
\xi_A(1-\bot, 1-\top) &= \min((1-\top)^*_0, \tfrac{1}{2} + \tfrac{1}{2}(1-\top)^{0\downarrow}_*) && \text{by Def. 51}\\
&= \min(1-\top^*_0, \tfrac{1}{2} + \tfrac{1}{2}\top^{1\downarrow}_*) && \text{by L-51 and L-52}\\
&= \min(1-\top^*_0, 1 - (\tfrac{1}{2} - \tfrac{1}{2}\top^{1\downarrow}_*))\\
&= 1 - \max(\top^*_0, \tfrac{1}{2} - \tfrac{1}{2}\top^{1\downarrow}_*) && \text{by De Morgan's law}\\
&= 1 - \xi_A(\top, \bot)\,. && \text{by Def. 51 and (157)}
\end{aligned}$$

2. $(1-\bot)^*_0 < \frac{1}{2}$. Then

$$\begin{aligned}
\xi_A(1-\bot, 1-\top) &= \max((1-\bot)^*_0, \tfrac{1}{2} - \tfrac{1}{2}(1-\bot)^{1\downarrow}_*) && \text{by Def. 51}\\
&= \max(1-\bot^*_0, \tfrac{1}{2} - \tfrac{1}{2}\bot^{0\downarrow}_*) && \text{by L-51 and L-52}\\
&= \max(1-\bot^*_0, 1 - (\tfrac{1}{2} + \tfrac{1}{2}\bot^{0\downarrow}_*))\\
&= 1 - \min(\bot^*_0, \tfrac{1}{2} + \tfrac{1}{2}\bot^{0\downarrow}_*) && \text{by De Morgan's law}\\
&= 1 - \xi_A(\top, \bot)\,. && \text{by Def. 51 and (158)}
\end{aligned}$$

3. If $(1-\top)^*_0 \leq \frac{1}{2}$ and $(1-\bot)^*_0 \geq \frac{1}{2}$, then $\bot^*_0 \leq \frac{1}{2}$ and $\top^*_0 \geq \frac{1}{2}$. Therefore $\xi_A(1-\bot, 1-\top) = \frac{1}{2} = 1 - \frac{1}{2} = 1 - \xi_A(\top, \bot)$ by Def. 51.

$\mathcal{F}_A$ **satisfies** (X-3)**.** To see that (X-3) holds, consider a choice of $(\mathrm{c}_1, \bot) \in \mathbb{T}$ with $\widehat{\bot}(\mathbf{I}) \subseteq \{0, 1\}$. Then

$$[\mathrm{c}_1]^*_0 = 1 \tag{159}$$

$$\bot^*_0 \in \{0, 1\}\,. \tag{160}$$

It is hence sufficient to discern the following two cases.

a. $\bot^*_0 = 0$, i.e. $\lim_{\gamma \to 0^+} \bot(\gamma) = 0$. Because $\bot$ is nonincreasing, this entails that $\bot(\gamma) = 0$ for all $\gamma \in (0, 1]$. Hence

$$\bot^{0\downarrow}_* = \inf\{\gamma : \bot(\gamma) = 0\} = \inf(0, 1] = 0\,. \tag{161}$$

by (10).

$$\begin{aligned}
\xi_A(\mathrm{c}_1, \bot) &= \tfrac{1}{2} && \text{by Def. 51}\\
&= \tfrac{1}{2} + \tfrac{1}{2} \cdot 0\\
&= \tfrac{1}{2} + \tfrac{1}{2}\bot^{0\downarrow}_* && \text{by (161)}
\end{aligned}$$

b. $\bot_0^* = 1$. In this case,

$$\begin{aligned}
\xi_A(\mathrm{c}_1, \bot) &= \min(\bot_0^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by Def. 51} \\
&= \min(1, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{because } \bot_0^* = 1 \\
&= \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow} \,.
\end{aligned}$$

This finishes the proof that ξ_A satisfies (X-3).

$\mathcal{F}_A$ **satisfies** (X-4). Let us define $f_A : \mathbf{I}^4 \longrightarrow \mathbf{I}$ by

$$f_A(a, b, c, d) = \begin{cases} \min(a, \frac{1}{2} + \frac{1}{2}b) & : \quad c \geq a > \frac{1}{2} \\ \max(c, \frac{1}{2} - \frac{1}{2}d) & : \quad a \leq c < \frac{1}{2} \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

for all $a, b, c, d \in \mathbf{I}$. It is then apparent from Def. 51 that

$$\xi_A(\top, \bot) = f_A(\bot_0^*, \bot_*^{0\downarrow}, \top_0^*, \top_*^{1\downarrow}) \,, \tag{162}$$

for all $(\top, \bot) \in \mathbb{T}$.
Now consider a choice of $(\top, \bot) \in \mathbb{T}$. Then

$$\begin{aligned}
\xi_A(\top^\sharp, \bot) &= f_A(\bot_0^*, \bot_*^{0\downarrow}, (\top^\sharp)_0^*, (\top^\sharp)_*^{1\downarrow}) && \text{by (162)} \\
&= f_A(\bot_0^*, \bot_*^{0\downarrow}, (\top^\flat)_0^*, (\top^\flat)_*^{1\downarrow}) && \text{by L-53 and L-54} \\
&= \xi_A(\top^\flat, \bot) \,. && \text{by (162)}
\end{aligned}$$

$\mathcal{F}_A$ **satisfies** (X-5). It is apparent from L-55 and the fact that (X-2) is valid that only one critical case must be checked, viz the case that $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$\top \leq \top' \tag{163}$$

$$\bot \leq \bot' \tag{164}$$

and

$$\bot_0^* > \tfrac{1}{2} \,. \tag{165}$$

To see that (X-5) holds in this case, we first observe that

$$\bot'^*_0 = \lim_{\gamma \to 0^+} \bot'(\gamma) \geq \lim_{\gamma \to 0^+} \bot(\gamma) = \bot_0^* > \tfrac{1}{2} \tag{166}$$

by (9) and the monotonicity of $\lim$. In addition, we observe that

$$\{\gamma : \bot(\gamma) = 0\} \supseteq \{\gamma : \bot'(\gamma) = 0\}$$

because $\bot \leq \bot'$. Therefore

$$\bot_*^{0\downarrow} = \inf\{\gamma : \bot(\gamma) = 0\} \leq \inf\{\gamma : \bot'(\gamma) = 0\} = \bot'^{0\downarrow}_* \,. \tag{167}$$

We hence can proceed as follows.

$$\begin{aligned}\xi_A(\top,\bot) &= \min(\bot_0^*, \tfrac{1}{2}+\tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by Def. 51 and (165)}\\ &\le \min(\bot'^*_0, \tfrac{1}{2}+\tfrac{1}{2}\bot'^{0\downarrow}_*) && \text{by (166) and (167)}\\ &= \xi_A(\top',\bot'), && \text{by Def. 51 and (166)}\end{aligned}$$

i.e. ξ_A satisfies (X-5), as desired.

B.13 Proof of Theorem 32

Lemma 56 *Whenever $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers with $Q \preceq_c Q'$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ is a choice of fuzzy arguments, then*

$$\top_{Q,X_1,\ldots,X_n} \preceq_c \top_{Q',X_1,\ldots,X_n}$$

and

$$\bot_{Q,X_1,\ldots,X_n} \preceq_c \bot_{Q',X_1,\ldots,X_n}.$$

Proof Let $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given semi-fuzzy quantifiers such that $Q \preceq_c Q'$ and let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ be a choice of fuzzy argument sets.

a.: $\top_{Q,X_1,\ldots,X_n} \preceq_c \top_{Q',X_1,\ldots,X_n}$. Let $\gamma \in \mathbf{I}$ is given. We shall discern three cases.

a.1: $\top_{Q,X_1,\ldots,X_n}(\gamma) < \frac{1}{2}$. Recalling that

$$\top_{Q,X_1,\ldots,X_n}(\gamma) = \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\}$$

by Def. 43, $\top_{Q,X_1,\ldots,X_n}(\gamma) < \frac{1}{2}$ entails that

$$Q(Y_1,\ldots,Y_n) < \tfrac{1}{2}$$

for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$. We may then conclude from $Q \preceq_c Q'$ that

$$Q'(Y_1,\ldots,Y_n) \le Q(Y_1,\ldots,Y_n) \tag{168}$$

for all $Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)$. Hence

$$\begin{aligned}\top_{Q',X_1,\ldots,X_n}(\gamma) &= \sup\{Q'(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43}\\ &\le \sup\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \ldots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by (168)}\\ &= \top_{Q,X_1,\ldots,X_n}(\gamma). && \text{by Def. 43}\end{aligned}$$

In turn, we conclude from $\top_{Q',X_1,\ldots,X_n}(\gamma) \le \top_{Q,X_1,\ldots,X_n}(\gamma) < \frac{1}{2}$ that $\top_{Q,X_1,\ldots,X_n}(\gamma) \preceq_c \top_{Q',X_1,\ldots,X_n}(\gamma)$.

a.2: $\top_{Q,X_1,\dots,X_n}(\gamma) > \frac{1}{2}$**.** Then

$$\sup\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} > \tfrac{1}{2} \tag{169}$$

by Def. 43. Now let $\varepsilon > 0$, $\varepsilon < \top_{Q,X_1,\dots,X_n}(\gamma) - \frac{1}{2}$. By (169), there exist $Y_1' \in \mathcal{T}_\gamma(X_1),\dots,Y_n' \in \mathcal{T}_\gamma(X_n)$ such that

$$Q(Y_1',\dots,Y_n') > \top_{Q,X_1,\dots,X_n}(\gamma) - \varepsilon > \tfrac{1}{2}\,.$$

From $Q \preceq_c Q'$ and Def. 43, we conclude that

$$\top_{Q',X_1,\dots,X_n}(\gamma) \geq Q'(Y_1',\dots,Y_n') \geq Q(Y_1',\dots,Y_n') > \top_{Q,X_1,\dots,X_n}(\gamma) - \varepsilon\,.$$

$\varepsilon \to 0$ yields $\top_{Q',X_1,\dots,X_n}(\gamma) \geq \top_{Q,X_1,\dots,X_n}(\gamma) > \frac{1}{2}$. Hence $\top_{Q,X_1,\dots,X_n}(\gamma) \preceq_c \top_{Q',X_1,\dots,X_n}(\gamma)$, as desired.

a.3: $\top_{Q,X_1,\dots,X_n}(\gamma) = \frac{1}{2}$**.** This case is trivial since $\top_{Q',X_1,\dots,X_n}(\gamma) \succeq_c \frac{1}{2} = \top_{Q,X_1,\dots,X_n}$.

b.: $\bot_{Q,X_1,\dots,X_n} \preceq_c \bot_{Q',X_1,\dots,X_n}$**.** The proof of this case can be reduced to that of **a.**, noticing that $x \preceq_c y$ is equivalent to $1 - x \preceq_c 1 - y$. Hence

$$\begin{aligned}
\bot_{Q,X_1,\dots,X_n} &= 1 - \top_{\neg Q,X_1,\dots,X_n} && \text{by L-11 and } \neg x = 1 - x \text{ involution}\\
&\preceq_c 1 - \top_{\neg Q',X_1,\dots,X_n} && \text{by part } \textbf{a.} \text{ of the lemma}\\
&= \bot_{Q',X_1,\dots,X_n}\,. && \text{by L-11 and } \neg x = 1 - x \text{ involution}
\end{aligned}$$

Proof of Theorem 32

Let an $\mathcal{F}_\xi$-QFM be given.

a. We first prove that it is sufficient for $\mathcal{F}_\xi$ to propagate fuzziness in quantifiers if $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ propagates fuzziness. Hence suppose ξ propagates fuzziness. Further let $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ with $Q \preceq_c Q'$ and $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$ be given. Then

$$\begin{aligned}
\mathcal{F}_\xi(Q)(X_1,\dots,X_n) &= \xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) && \text{by Def. 45}\\
&\preceq_c \xi(\top_{Q',X_1,\dots,X_n}, \bot_{Q',X_1,\dots,X_n}) && \text{by L-56, Def. 52}\\
&= \mathcal{F}_\xi(Q')(X_1,\dots,X_n)\,, && \text{by Def. 45}
\end{aligned}$$

i.e. $\mathcal{F}_\xi$ propagates fuzziness in quantifiers, as desired.

b. Next we prove that ξ's propagating fuzziness is a necessary condition for $\mathcal{F}_\xi$ to propagate fuzziness in quantifiers. Hence let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be given and assume that ξ does not propagate fuzziness, i.e. there exist $(\top,\bot), (\top',\bot') \in \mathbb{T}$ such that $\top \preceq_c \top'$ and $\bot \preceq_c \bot'$, but

$$\xi(\top,\bot) \not\preceq_c \xi(\top',\bot')\,. \tag{170}$$

We define semi-fuzzy quantifiers $Q_1, Q_2 : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ by

$$Q_1(Y) = \begin{cases} \bot(\inf Y'') & : & Y' = \varnothing \\ \top(\sup Y') & : & Y' \neq \varnothing \end{cases}$$

$$Q_2(Y) = \begin{cases} \bot'(\inf Y'') & : & Y' = \varnothing \\ \top'(\sup Y') & : & Y'' \neq \varnothing \end{cases}$$

for all $Y \in \mathcal{P}(\mathbf{2} \times \mathbf{I})$, where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\}$$
$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \,.$$

Clearly $Q_1 \preceq_c Q_2$ because $\top \preceq_c \top'$ and $\bot \preceq_c \bot'$. Now define $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ by (23). Then $\top_{Q_1,X} = \top$, $\bot_{Q_1,X} = \bot$, $\top_{Q_2,X} = \top'$ and $\bot_{Q_2,X} = \bot'$ by Th-21. Therefore

$$\begin{aligned} \mathcal{F}_\xi(Q_1)(X) &= \xi(\top_{Q_1,X}, \bot_{Q_1,X}) && \text{by Def. 45} \\ &= \xi(\top, \bot) && \text{by Th-21} \\ &\npreceq_c \xi(\top', \bot') && \text{by (170)} \\ &= \xi(\top_{Q_2,X}, \bot_{Q_2,X}) && \text{by Th-21} \\ &= \mathcal{F}_\xi(Q_2)(X) \,. && \text{by Def. 45} \end{aligned}$$

Hence $\mathcal{F}_\xi(Q_1)(X) \npreceq_c \mathcal{F}_\xi(Q_2)(X)$ although $Q_1 \preceq_c Q_2$, i.e. $\mathcal{F}_\xi$ fails to propagate fuzziness in quantifiers. This proves that the condition on ξ is indeed necessary for $\mathcal{F}_\xi$ to propagate fuzziness in quantifiers.

B.14 Proof of Theorem 33

Lemma 57 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2)*, and it holds that*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

for all $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) > \frac{1}{2}$*. Then*

$$\xi(\top, \bot) = \xi(\min(\top, \tfrac{1}{2}), \bot)$$

for all $(\top, \bot) \in \mathbb{T}$ *with* $\top(0) < \frac{1}{2}$*.*

Proof Trivial.

$$\begin{aligned} \xi(\top, \bot) &= \xi(1 - (1 - \top), 1 - (1 - \bot)) \\ &= 1 - \xi(1 - \bot, 1 - \top) && \text{by (X-2)} \\ &= 1 - \xi(1 - \bot, \max(\tfrac{1}{2}, 1 - \top)) && \text{by assumption of the lemma} \\ &= \xi(1 - \max(\tfrac{1}{2}, 1 - \top), 1 - (1 - \bot)) && \text{by (X-2)} \\ &= \xi(\min(1 - \tfrac{1}{2}, 1 - (1 - \top)), \bot) \\ &= \xi(\min(\tfrac{1}{2}, \top), \bot) \,. \end{aligned}$$

Lemma 58 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5) *and*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

whenever $\bot(0) > \frac{1}{2}$. *Then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $\top(0) \geq \frac{1}{2} \geq \bot(0)$.

Proof Again trivial. Let $(\top, \bot) \in \mathbb{T}$ be given with $\top(0) \geq \frac{1}{2} \geq \bot(0)$. Then $\top(\gamma) \geq \top(0) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ because $\top$ is nondecreasing, i.e. $\top \geq \frac{1}{2}$. Therefore

$$\begin{aligned}
\xi(\top, \bot) &\geq \xi(\top, \mathrm{c}_0) && \text{by (X-5)} \\
&\geq \xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) && \text{by (X-5)} \\
&= \xi(\min(\mathrm{c}_1, \tfrac{1}{2}), \mathrm{c}_0) \\
&= \xi(\mathrm{c}_1, \mathrm{c}_0) && \text{by L-57} \\
&= \tfrac{1}{2} && \text{by (X-3)}\,.
\end{aligned}$$

By similar reasoning, we obtain from $\bot \leq \mathrm{c}_{\frac{1}{2}}$ that

$$\begin{aligned}
\xi(\top, \bot) &\leq \xi(\mathrm{c}_1, \bot) && \text{by (X-5)} \\
&\leq \xi(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) && \text{by (X-5)} \\
&= \xi(\mathrm{c}_1, \max(\mathrm{c}_0, \tfrac{1}{2})) \\
&= \xi(\mathrm{c}_1, \mathrm{c}_0) && \text{by assumption of the lemma} \\
&= \tfrac{1}{2}\,. && \text{by (X-3)}
\end{aligned}$$

Hence $\xi(\top, \bot) = \frac{1}{2}$, as desired.

Lemma 59 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5). *If*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

whenever $\bot(0) > \frac{1}{2}$, *then*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot)$$

whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ *such that* $\top \preceq_c \top'$.

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfy (X-1) to (X-5) and further possess the desired property, viz

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

whenever $\bot(0) > \frac{1}{2}$. Now let $(\top, \bot), (\top, \bot') \in \mathbb{T}$ be given with $\top \preceq_c \top'$. We discern three cases.

a.: $\top'(0) > \frac{1}{2}$**.** Then $\frac{1}{2} \leq \top(0) \leq \top'(0)$ because $\top \preceq_c \top'$. Taking into account that both $\top$ and $\top'$ are nondecreasing by Def. 44, we conclude that $\frac{1}{2} \leq \top \leq \top'$ Hence

$$\xi(\top, \bot) \leq \xi(\top', \bot) \,. \qquad \text{by (X-5)}$$

In addition,

$$\begin{aligned} \xi(\top, \bot) &\geq \xi(\mathrm{c}_{\frac{1}{2}}, \bot) && \text{by (X-5)} \\ &\geq \xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) && \text{by (X-5)} \\ &= \tfrac{1}{2} \,. && \text{by L-58} \end{aligned}$$

Hence $\frac{1}{2} \leq \xi(\top, \bot) \leq \xi(\top', \bot)$, i.e. $\xi(\top, \bot) \preceq_c \xi(\top', \bot)$.

b.: $\top'(0) < \frac{1}{2}$**.** In this case, $\top'(0) \leq \top(0) \leq \frac{1}{2}$ because $\top \preceq_c \top'$. Hence

$$\top'(\gamma) \leq \top(\gamma) \leq \tfrac{1}{2}$$

whenever $\top'(\gamma) \leq \frac{1}{2}$, and

$$\top'(\gamma) \geq \top(\gamma) \geq \tfrac{1}{2}$$

if $\top'(\gamma) > \frac{1}{2}$. Hence

$$\min(\top', \tfrac{1}{2}) \leq \top \,. \qquad (171)$$

We therefore obtain that

$$\begin{aligned} \xi(\top', \bot) &= \xi(\min(\top', \tfrac{1}{2}), \bot) && \text{by L-57} \\ &\leq \xi(\top, \bot) \,. && \text{by (171), (X-5)} \end{aligned}$$

On the other hand, $\bot(0) \leq \top'(0) < \frac{1}{2}$ and

$$\begin{aligned} \xi(\top, \bot) &\leq \xi(\max(\top, \tfrac{1}{2}), \bot) && \text{by (X-5)} \\ &= \tfrac{1}{2} \,. && \text{by L-58} \end{aligned}$$

Hence $\xi(\top', \bot) \leq \xi(\top, \bot) \leq \frac{1}{2}$, i.e. $\xi(\top, \bot) \preceq_c \xi(\top', \bot)$, as desired.

c.: $\top'(0) = \frac{1}{2}$**.** In this case, $\top \preceq_c \top'$ entails that $\top(0) = \frac{1}{2}$ as well. Furthermore $\bot(0) \leq \top'(0) = \frac{1}{2}$, i.e. $\xi(\top', \bot) = \frac{1}{2}$ and $\xi(\top, \bot) = \frac{1}{2}$ by L-58. Hence trivially $\xi(\top, \bot) \preceq_c \xi(\top', \bot)$.

Lemma 60 *If* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5) *and in addition*

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

whenever $\bot(0) > \frac{1}{2}$*, then* ξ *propagates fuzziness, i.e.*

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot')$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ *such that* $\top \preceq_c \top'$ *and* $\bot \preceq_c \bot'$*.*

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a mapping which satisfies (X-1) to (X-5). We shall further assume that

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2}))$$

for all $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$.
Now let a choice of $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given such that $\top \preceq_c \top'$ and $\bot \preceq_c \bot'$. We shall treat separately two cases.

a.: $\top'(0) \geq \frac{1}{2}$**.** Then $\frac{1}{2} \leq \top \leq \top'$ because $\top(0) \preceq_c \top'(0)$ and $\top'(0) \geq \frac{1}{2}$ entail that $\frac{1}{2} \leq \top(0) \leq \top'(0)$, and because $\top, \top'$ are nondecreasing. In addition, $\bot(0) \leq \top(0) \leq \top'(0)$, i.e. $(\top', \bot) \in \mathbb{T}$. We can hence proceed as follows.

$$\begin{aligned}
\xi(\top', \bot') &= 1 - \xi(1 - \bot', 1 - \top') && \text{by (X-2)} \\
&\succeq_c 1 - \xi(1 - \bot, 1 - \top') && \text{by L-59} \\
&= \xi(\top', \bot) && \text{by (X-2)} \\
&\succeq_c \xi(\top, \bot)\,. && \text{by L-59}
\end{aligned}$$

b.: $\top'(0) < \frac{1}{2}$**.** Then $\top'(0) \leq \top(0) \leq \frac{1}{2}$ and hence $\bot'(0) \leq \top'(0) \leq \top(0)$, i.e. $(\top, \bot') \in \mathbb{T}$. Therefore

$$\begin{aligned}
\xi(\top', \bot') &\succeq_c \xi(\top, \bot') && \text{by L-59} \\
&= 1 - \xi(1 - \bot', 1 - \top) && \text{by (X-2)} \\
&\succeq_c 1 - \xi(1 - \bot, 1 - \top) && \text{by L-59} \\
&= \xi(\top, \bot)\,. && \text{by (X-2)}
\end{aligned}$$

Lemma 61 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-3). *If ξ propagates fuzziness, then*

$$\xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) = \xi(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) = \tfrac{1}{2}\,.$$

Proof Trivial: from (X-3), we know that $\xi(\mathrm{c}_1, \mathrm{c}_0) = \frac{1}{2}$. But $\mathrm{c}_{\frac{1}{2}} \preceq_c \mathrm{c}_0$ and hence $\xi(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) \preceq_c \xi(\mathrm{c}_1, \mathrm{c}_0) = \frac{1}{2}$, i.e. $\xi(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}} = \frac{1}{2}$. Similarly, we conclude from $\mathrm{c}_{\frac{1}{2}} \preceq_c \mathrm{c}_1$ that $\xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) \preceq_c \xi(\mathrm{c}_1, \mathrm{c}_0) = \frac{1}{2}$, i.e. $\xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) = \frac{1}{2}$.

Lemma 62 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies* (X-1) *to* (X-5) *but violates the condition of Th-33, i.e. a choice of $(\top, \bot) \in \mathbb{T}$ exists with $\bot(0) > \frac{1}{2}$ and*

$$\xi(\top, \bot) \neq \xi(\top, \max(\bot, \tfrac{1}{2}))\,.$$

Then ξ does not propagate fuzziness.

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a mapping which satisfies (X-1) to (X-5). Further assume that

$$\xi(\top, \bot) \neq \xi(\top, \max(\bot, \tfrac{1}{2}))\,. \tag{172}$$

for some choice of $(\top, \bot) \in \mathbb{T}$ with $\bot(0) < \frac{1}{2}$. We will discern the following two cases.

a.: $\xi(\top, \bot) \geq \frac{1}{2}$. Then $\frac{1}{2} \leq \xi(\top, \bot) \leq \xi(\top, \max(\bot, \frac{1}{2}))$ by (X-5), i.e. $\xi(\top, \bot) \preceq_c \xi(\top, \max(\bot, \frac{1}{2}))$. Because $\preceq_c$ is a partial order, we could only have the desired $\xi(\top, \bot) \succeq_c \xi(\top, \max(\bot, \frac{1}{2}))$ if $\xi(\top, \bot) = \xi(\top, \max(\bot, \frac{1}{2}))$. However, this is not the case by (172). We hence obtain $\xi(\top, \max(\bot, \frac{1}{2})) \not\preceq_c \xi(\top, \bot)$ by contraposition.

b.: $\xi(\top, \bot) < \frac{1}{2}$. We shall prove this case by contradiction and assume to the contrary that ξ propagates fuzziness. Noticing that $\top(\gamma) \geq \bot(0) > \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ by Th-20.3. Hence

$$\begin{aligned}
\xi(\top, \bot) &\geq \xi(\mathrm{c}_{\frac{1}{2}}, \bot) && \text{by (X-5)}\\
&\geq \xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) && \text{by (X-5)}\\
&= \tfrac{1}{2}\,. && \text{by L-61}
\end{aligned}$$

Hence $\xi(\top, \bot) < \frac{1}{2}$ and $\xi(\top, \bot) \geq \frac{1}{2}$, a contradition. This proves that the assumption is false, i.e. ξ does not propagate fuzziness.

Proof of Theorem 33

It has been shown in L-60 that the simplified condition is sufficient for ξ to propagate fuzziness. Conversely, L-62 states that the condition is necessary for ξ to propagate fuzziness. Hence the original and the simplified condition are equivalent provided that ξ satisfies (X-1) to (X-5).

B.15 Proof of Theorem 34

The claim of the theorem is immediate from L-58 and Th-33, which show that the condition stated in the theorem is necessary for ξ to propagate fuzziness.

B.16 Proof of Theorem 35

Lemma 63 *There exist $(\top, \bot) \in \mathbb{T}$ such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$ and $\xi_{\mathrm{Ch}}(\top, \bot) \neq \frac{1}{2}$.*

Proof Consider $(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) \in \mathbb{T}$. From Def. 47, we obtain

$$\begin{aligned}\xi_{\mathrm{Ch}}(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) &= \tfrac{1}{2}\int_0^1 \mathrm{c}_1(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \mathrm{c}_{\frac{1}{2}}(\gamma)\,d\gamma \\ &= \tfrac{1}{2}\int_0^1 1\,d\gamma + \tfrac{1}{2}\int_0^1 \tfrac{1}{2}\,d\gamma && \text{by (7)} \\ &= \tfrac{1}{2} + \tfrac{1}{4} \\ &= \tfrac{3}{4}\,.\end{aligned}$$

Hence $\xi_{\mathrm{Ch}}(\mathrm{c}_1, \mathrm{c}_{\frac{1}{2}}) = \frac{3}{4} \neq \frac{1}{2}$, although $\mathrm{c}_1(0) = 1 \geq \frac{1}{2} = \mathrm{c}_{\frac{1}{2}}(0)$, which finishes the proof of the lemma.

Proof of Theorem 35

As shown in Th-27, $\mathcal{F}_{\mathrm{Ch}}$ is a DFS. Hence $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-1) to (X-5). We can hence apply Th-34 to conclude from L-63 that ξ_{Ch} does not propagate fuzziness, which in turn means by Th-32 that $\mathcal{F}_{\mathrm{Ch}}$ does not propagate fuzziness in quantifiers.

B.17 Proof of Theorem 36

We already know from Th-30 that $\mathcal{F}_S$ is a DFS, i.e. $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ as defined in Def. 50 satisfies (X-1) to (X-5) by Th-23. It is therefore sufficient to show that the condition stated in Th-33 holds. To see this, let $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$. We observe that

$$[\max(\bot, \tfrac{1}{2})]_*^{\leq \frac{1}{2}\downarrow} = \inf\{\gamma : \max(\bot(\gamma), \tfrac{1}{2}) \leq \tfrac{1}{2}\} = \inf\{\gamma : \bot(\gamma) \leq \tfrac{1}{2}\} = \bot_*^{\leq \frac{1}{2}\downarrow} \tag{173}$$

by (26). Therefore

$$\begin{aligned}&\xi_S(\top, \max(\bot, \tfrac{1}{2})) \\ &= \min(\top_1^*, \tfrac{1}{2} + \tfrac{1}{2}[\max(\bot, \tfrac{1}{2})]_*^{\leq \frac{1}{2}\downarrow}) && \text{by Def. 50} \\ &= \min(\top_1^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq \frac{1}{2}\downarrow}) && \text{by (173)} \\ &= \xi_S(\top, \bot)\,. && \text{by Def. 50}\end{aligned}$$

Hence $\mathcal{F}_S$ propagates fuzziness in quantifiers by Th-33 and Th-32.

B.18 Proof of Theorem 37

$\mathcal{F}_A$ is already known to be a DFS by Th-31, i.e. ξ_A satisfies (X-1) to (X-5) by Th-23. We can hence utilize Th-33 to show that ξ_A does not propagate fuzziness. which in turn means by Th-32 that $\mathcal{F}_A$ does not propagate fuzziness in quantifiers.

To give an example which violates the condition of Th-33, consider $(\mathrm{c}_1, \bot) \in \mathbb{T}$ with

$$\bot(\gamma) = \begin{cases} 1 & : \quad \gamma < \frac{1}{2} \\ 0 & : \quad \gamma \geq \frac{1}{2} \end{cases} \tag{174}$$

for all $\gamma \in \mathbf{I}$. It is then apparent from (9) that

$$\bot_0^* = \lim_{\gamma \to 0^+} \bot(\gamma) = 1 \tag{175}$$

$$[\max(\bot, \tfrac{1}{2})]_0^* = \lim_{\gamma \to 0^+} \max(\bot(\gamma), \tfrac{1}{2}) = 1\,. \tag{176}$$

Similarly, we obtain from (10) that

$$\bot_*^{0\downarrow} = \inf\{\gamma : \bot(\gamma) = 0\} = \inf[\tfrac{1}{2}, 1] = \tfrac{1}{2} \tag{177}$$

$$[\max(\bot, \tfrac{1}{2})]_*^{0\downarrow} = \inf\{\gamma : \max(\bot(\gamma), \tfrac{1}{2}) = 0\} = \inf \varnothing = 1\,. \tag{178}$$

Therefore

$$\begin{aligned} \xi_A(\mathrm{c}_1, \bot) &= \min(\bot_0^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by Def. 51} \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{2}) && \text{by (175), (177)} \\ &= \tfrac{3}{4} \\ &\neq 1 \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) \\ &= \min([\max(\bot, \tfrac{1}{2})]_0^*, \tfrac{1}{2} + \tfrac{1}{2}[\max(\bot, \tfrac{1}{2})]_*^{0\downarrow}) && \text{by (176), (178)} \\ &= \xi_A(\mathrm{c}_1, \max(\bot, \tfrac{1}{2}))\,. && \text{by Def. 51} \end{aligned}$$

Hence there exists $(\mathrm{c}_1, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$ and $\xi_A(\mathrm{c}_1, \bot) \neq \xi_A(\mathrm{c}_1, \max(\bot, \frac{1}{2}))$, i.e. ξ_A does not propagate fuzziness by Th-33, which in turn means by Th-32 that $\mathcal{F}_A$ does not propagate fuzziness in quantifiers.

B.19 Proof of Theorem 38

Lemma 64 *Suppose $E \neq \varnothing$ is some set and $X, X' \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets with $X \preceq_c X'$. Then*

$$\mathcal{T}_\gamma(X') \subseteq \mathcal{T}_\gamma(X)\,,$$

for all $\gamma \in \mathbf{I}$.

Proof See [7, L-125, p.286].

Proof of Theorem 38

Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be given.

a. To show that ξ's propagating of unspecificity is sufficient for $\mathcal{F}_\xi$ to propagate fuzziness in arguments, suppose that ξ fullfils this condition, i.e.

$$\xi(\top, \bot) \preceq_c \xi(\top', \bot') \tag{179}$$

whenever $\top \geq \top'$ and $\bot \leq \bot'$, see Def. 53.
Now let a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n, X'_1, \dots, X'_n \in \widetilde{\mathcal{P}}(E)$ be given such that $X_i \preceq_c X'_i$ for all $i = 1, \dots, n$. Then

$$\begin{aligned}
&\top_{Q,X_1,\dots,X_n}(\gamma) \\
&= \sup\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&\geq \sup\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X'_1), \dots, Y_n \in \mathcal{T}_\gamma(X'_n)\} && \text{by L-64} \\
&= \top_{Q,X'_1,\dots,X'_n}(\gamma)
\end{aligned}$$

and

$$\begin{aligned}
&\bot_{Q,X_1,\dots,X_n}(\gamma) \\
&= \inf\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1), \dots, Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&\leq \inf\{Q(Y_1, \dots, Y_n) : Y_1 \in \mathcal{T}_\gamma(X'_1), \dots, Y_n \in \mathcal{T}_\gamma(X'_n)\} && \text{by L-64} \\
&= \bot_{Q,X'_1,\dots,X'_n}(\gamma)
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}
\mathcal{F}_\xi(Q)(X_1, \dots, X_n) &= \xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) && \text{by Def. 45} \\
&\preceq_c \xi(\top_{Q,X'_1,\dots,X'_n}, \bot_{Q,X'_1,\dots,X'_n}) && \text{by (179)} \\
&= \mathcal{F}_\xi(Q)(X'_1, \dots, X'_n)\,, && \text{by Def. 45}
\end{aligned}$$

i.e. $\mathcal{F}_\xi$ propagates fuzziness in arguments, as desired.

b. Let us now show that ξ's propagating unspecificity is also necessary for $\mathcal{F}_\xi$ to propagate fuzziness in arguments. Hence suppose that there exist $(\top, \bot), (\top', \bot') \in \mathbb{T}$ such that $\top \geq \top'$, $\bot \leq \bot'$, but

$$\xi(\top, \bot) \not\preceq_c \xi(\top', \bot')\,. \tag{180}$$

We shall define a semi-fuzzy quantifier $Q : \mathcal{P}(\{*\} \cup \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ as follows (where $\{*\}$ is an arbitrary singleton with $* \notin \mathbf{2} \times \mathbf{I}$):

$$Q(Y) = \begin{cases} \top'(\sup Y') & : \; * \notin Y, Y' \neq \varnothing \\ \top(\sup Y') & : \; * \in Y, Y' \neq \varnothing \\ \bot'(\inf Y'') & : \; * \notin Y, Y' = \varnothing \\ \bot(\inf Y'') & : \; * \in Y, Y' = \varnothing \end{cases}$$

for all $Y \in \mathcal{P}(\{*\} \cup \mathbf{2} \times \mathbf{I})$, where

$$\begin{aligned}
Y' &= \{z \in \mathbf{I} : (0, z) \in Y\} \\
Y'' &= \{z \in \mathbf{I} : (1, z) \in Y\}\,.
\end{aligned}$$

Further we define $X, X' \in \widetilde{\mathcal{P}}(\{*\} \cup \mathbf{2} \times \mathbf{I})$ by

$$\mu_X(e) = \begin{cases} \frac{1}{2} & : \quad e = * \\ \frac{1}{2} - \frac{1}{2}z & : \quad e = (0, z) \\ \frac{1}{2} + \frac{1}{2}z & : \quad e = (1, z) \end{cases}$$

and

$$\mu_{X'}(e) = \begin{cases} 0 & : \quad e = * \\ \frac{1}{2} - \frac{1}{2}z & : \quad e = (0, z) \\ \frac{1}{2} + \frac{1}{2}z & : \quad e = (1, z) \end{cases}$$

for all $e \in \{*\} \cup \mathbf{2} \times \mathbf{I}$. Clearly $X \preceq_c X'$. Let us now investigate the cut ranges of X and X'. In the case that $\gamma = 0$, we have

$$\begin{aligned} X_0^{\min} &= X_{>\frac{1}{2}} = \{1\} \times (0, 1] \\ X_0^{\max} &= X_{\geq\frac{1}{2}} = \{*\} \cup \{(0, 0)\} \cup (\{1\} \times \mathbf{I}) \\ {X'}_0^{\min} &= {X'}_{>\frac{1}{2}} = \{1\} \times (0, 1] \\ {X'}_0^{\max} &= {X'}_{\geq\frac{1}{2}} = \{(0, 0)\} \cup (\{1\} \times \mathbf{I}) \,. \end{aligned}$$

It is then apparent from Def. 43, Def. 30 and the above definition of Q that

$$\begin{aligned} &\top_{Q,X}(0) = \sup\{Q(Y) : Y \in \mathcal{T}_0(X)\} = Q(\{*\} \cup \{(0, 0)\} \cup (\{1\} \times \mathbf{I})) = \top(0) \\ &\bot_{Q,X}(0) = \inf\{Q(Y) : Y \in \mathcal{T}_0(X)\} = Q(\{*\} \cup (\{1\} \times (0, 1])) = \bot(0) \\ &\top_{Q,X'}(0) = \sup\{Q(Y) : Y \in \mathcal{T}_0(X')\} = Q(\{(0, 0)\} \cup (\{1\} \times \mathbf{I})) = \top'(0) \\ &\bot_{Q,X'}(0) = \inf\{Q(Y) : Y \in \mathcal{T}_0(X')\} = Q(\{1\} \times (0, 1]) = \bot'(0) \,. \end{aligned}$$

In the case that $\gamma > 0$, the cut ranges are bounded by

$$\begin{aligned} X_\gamma^{\min} &= X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{1\} \times [\gamma, 1] \\ X_\gamma^{\max} &= X_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{*\} \cup (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I}) \\ {X'}_\gamma^{\min} &= {X'}_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{1\} \times [\gamma, 1] \\ {X'}_\gamma^{\max} &= {X'}_{>\frac{1}{2}-\frac{1}{2}\gamma} = (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I}) \,. \end{aligned}$$

Therefore

$$\begin{aligned} &\top_{Q,X}(\gamma) = \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} = Q(\{*\} \cup (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I})) = \top(\gamma) \\ &\bot_{Q,X}(\gamma) = \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X)\} = Q(\{*\} \cup (\{1\} \times [\gamma, 1])) = \bot(\gamma) \\ &\top_{Q,X'}(\gamma) = \sup\{Q(Y) : Y \in \mathcal{T}_\gamma(X')\} = Q((\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I})) = \top'(\gamma) \\ &\bot_{Q,X'}(\gamma) = \inf\{Q(Y) : Y \in \mathcal{T}_\gamma(X')\} = Q(\{1\} \times [\gamma, 1]) = \bot'(\gamma) \,. \end{aligned}$$

Hence

$$\top_{Q,X} = \top \qquad \bot_{Q,X} = \bot \tag{181}$$
$$\top_{Q,X'} = \top' \qquad \bot_{Q,X'} = \bot' \tag{182}$$

We conclude that

$$\begin{aligned}
\mathcal{F}_\xi(Q)(X) &= \xi(\top_{Q,X}, \bot_{Q,X}) && \text{by Def. 45}\\
&= \xi(\top, \bot) && \text{by (181)}\\
&\not\preceq_c \xi(\top', \bot') && \text{by (180)}\\
&= \xi(\top_{Q,X'}, \bot_{Q,X'}) && \text{by (182)}\\
&= \mathcal{F}_\xi(Q)(X')\,. && \text{by Def. 45}
\end{aligned}$$

Hence $\mathcal{F}_\xi(Q)(X) \not\preceq_c \mathcal{F}_\xi(Q)(X')$ although $X \preceq_c X'$, i.e. $\mathcal{F}_\xi$ fails to propagate fuzziness in arguments.

B.20 Proof of Theorem 39

Lemma 65 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2). *Then the following are equivalent:*

a. ξ *propagates unspecificity;*

b. for all $(\top, \bot), (\top, \bot') \in \mathbb{T}$ *with* $\bot' \leq \bot$, *it holds that* $\xi(\top, \bot') \preceq_c \xi(\top, \bot)$.

Proof It is apparent from Def. 53 that **b.** is a weakening of **a.**
It remains to be shown that **a.** is entailed by **b.**
Hence let a mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ which satisfies (X-2) be given and suppose that **b.** holds. To prove that **a.** also holds, we consider a choice of $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with $\bot' \leq \bot$ and $\top \leq \top'$. It is apparent from (3) that

$$x \preceq_c y \Leftrightarrow 1 - x \preceq_c 1 - y \tag{183}$$

for all $x, y \in \mathbf{I}$. Therefore

$$\begin{aligned}
\xi(\top', \bot') &\preceq_c \xi(\top', \bot) && \text{by assumption of } \mathbf{b.}\\
&= 1 - \xi(1 - \bot, 1 - \top') && \text{by (X-2)}\\
&= \preceq_c 1 - \xi(1 - \bot, 1 - \top) && \text{by (183) and assumption of } \mathbf{b.}\\
&= \xi(\top, \bot)\,, && \text{by (X-2)}
\end{aligned}$$

as desired.

Lemma 66 *Suppose* ξ *satisfies* (X-2) *and for all* $(\top, \bot) \in \mathbb{T}$ *with* $\bot(0) \geq \frac{1}{2}$, $\xi(\top, \bot) = \xi(\mathrm{c}_1, \bot)$. *Then for all* $(\top, \bot) \in \mathbb{T}$ *with* $\top(0) \leq \frac{1}{2}$, $\xi(\top, \bot) = \xi(\top, \mathrm{c}_0)$.

Proof Let $(\top,\bot) \in \mathbb{T}$ be given and suppose $\top(0) \leq \frac{1}{2}$. Then $(1-\bot, 1-\top) \in \mathbb{T}$ satisfies $(1-\top)(0) = 1-\top(0) \geq \frac{1}{2}$. Therefore

$$\begin{aligned}
\xi(\top,\bot) &= 1-\xi(1-\bot,1-\top) && \text{by (X-2)}\\
&= 1-\xi(\mathrm{c}_1,1-\top) && \text{by assumption on } \xi\\
&= 1-\xi(1-\mathrm{c}_0,1-\top) && \text{by (3)}\\
&= \xi(\top,\mathrm{c}_0)\,, && \text{by (X-2)}
\end{aligned}$$

which proves the claim of the lemma.

Lemma 67 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2). *Then* $\xi(\mathrm{c}_1,\mathrm{c}_0) = \frac{1}{2}$.

Proof Apparent.

$$\begin{aligned}
\xi(\mathrm{c}_1,\mathrm{c}_0) &= 1-\xi(1-\mathrm{c}_0,1-\mathrm{c}_1) && \text{by (X-2)}\\
&= 1-\xi(\mathrm{c}_1,\mathrm{c}_0)\,. && \text{by (7)}
\end{aligned}$$

Therefore $2\xi(\mathrm{c}_1,\mathrm{c}_0) = 1$, i.e. $\xi(\mathrm{c}_1,\mathrm{c}_0) = \frac{1}{2}$.

Lemma 68 *Suppose* ξ *satisfies* (X-2), (X-4) *and* (X-5), *and for all* $(\top,\bot) \in \mathbb{T}$ *with* $\bot(0) \geq \frac{1}{2}$, *it holds that* $\xi(\top,\bot) = \xi(\mathrm{c}_1,\bot)$. *Then for all* $(\top,\bot) \in \mathbb{T}$ *with* $\top(0) \geq \frac{1}{2} \geq \bot(0)$, $\xi(\top,\bot) = \frac{1}{2}$.

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ has the desired properties and consider $(\top,\bot) \in \mathbb{T}$ with $\top(0) \geq \frac{1}{2} \geq \bot(0)$.
We define $\top',\bot' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\top'(\gamma) = \begin{cases} \frac{1}{2} & : \quad \gamma = 0 \\ \top(\gamma) & : \quad \gamma > 0 \end{cases} \tag{184}$$

$$\bot'(\gamma) = \begin{cases} \frac{1}{2} & : \quad \gamma = 0 \\ 0 & : \quad \gamma > 0 \end{cases} \tag{185}$$

for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
\xi(\top,\bot) &= \xi(\top',\bot) && \text{by (184) and L-23}\\
&= \xi(\top',\mathrm{c}_0) && \text{by L-66}\\
&= \xi(\top',\bot') && \text{by (185) and L-23}\\
&= \xi(\mathrm{c}_1,\bot') && \text{by assumption on } \xi\\
&= \xi(\mathrm{c}_1,\mathrm{c}_0) && \text{by (185) and L-23}\\
&= \tfrac{1}{2}\,. && \text{by L-67}
\end{aligned}$$

Lemma 69 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2). *Then* $\xi(\mathrm{c}_{\frac{1}{2}},\mathrm{c}_{\frac{1}{2}}) = \frac{1}{2}$.

Proof By (7), $c_{\frac{1}{2}} = 1 - c_{\frac{1}{2}}$. Hence by (X-2), $\xi(c_{\frac{1}{2}}, c_{\frac{1}{2}}) = \xi(1 - c_{\frac{1}{2}}, 1 - c_{\frac{1}{2}}) = 1 - \xi(c_{\frac{1}{2}}, c_{\frac{1}{2}})$. Therefore $2\xi(c_{\frac{1}{2}}, c_{\frac{1}{2}}) = 1$, i.e. $\xi(c_{\frac{1}{2}}, c_{\frac{1}{2}}) = \frac{1}{2}$.

Proof of Theorem 39

Let us first show that condition **b.** of the theorem is entailed by condition **a.**
Hence suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2), (X-4) and (X-5). Further assume that ξ fulfills condition **a.** of the theorem, i.e. ξ propagates unspecificity. Now let $(\top, \bot) \in \mathbb{T}$ with $\bot(0) \geq \frac{1}{2}$. Firstly we notice that $\top(\gamma) \geq \top(0) \geq \bot(0) \geq \frac{1}{2}$, and hence $(\top, \max(\bot, \frac{1}{2})) \in \mathbb{T}$. Apparently

$$\xi(\top, \max(\bot, \tfrac{1}{2})) \geq \xi(c_{\frac{1}{2}}, c_{\frac{1}{2}}) = \tfrac{1}{2} \tag{186}$$

by (X-5) and L-69. Furthermore $\bot \leq \max(\bot, \frac{1}{2})$ and hence

$$\xi(\top, \bot) \preceq_c \xi(\top, \max(\bot, \tfrac{1}{2})) \tag{187}$$

because ξ is assumed to propagate unspecificity. By (3), we conclude from (186) and (187) that

$$\xi(\top, \bot) \geq \tfrac{1}{2}\,. \tag{188}$$

Next we notice that $\top \leq c_1$ and therefore

$$\xi(c_1, \bot) \preceq_c \xi(\top, \bot) \tag{189}$$

because ξ propagates unspecificity. It is then apparent from (3), (188) and (189) that

$$\xi(c_1, \bot) \leq \xi(\top, \bot)\,. \tag{190}$$

On the other hand, $\top \leq c_1$ entails that

$$\xi(\top, \bot) \leq \xi(c_1, \bot) \tag{191}$$

by (X-5). Combining (190) and (191), we obtain the desired $\xi(\top, \bot) = \xi(c_1, \bot)$, i.e. condition **b.** of the theorem is indeed necessary for ξ to propagate unspecificity.

It remains to be shown that **a.** is entailed by **b.**, i.e. that the latter condition is sufficient for ξ to propagate unspecificity. This can be proven by showing that condition **b.** of the theorem entails condition **b.** of lemma L-65, which is already known to be sufficient for ξ to propagate unspecificity.
Hence let a mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be given which satisfies (X-2), (X-4) and (X-5). Further suppose that for all $(\top, \bot) \in \mathbb{T}$ with $\bot(0) \geq \frac{1}{2}$, it holds that

$$\xi(\top, \bot) = \xi(c_1, \bot)\,, \tag{192}$$

i.e. condition **b.** of the theorem holds.
Now let $(\top, \bot), (\top, \bot') \in \mathbb{T}$ be given with $\bot' \leq \bot$.
In the case that $\bot'(0) \geq \frac{1}{2}$ we firstly conclude from $\bot' \leq \bot$ that

$$\xi(\top, \bot') \leq \xi(\top, \bot) \tag{193}$$

by (X-5). We further notice that

$$\begin{aligned}\xi(\top, \bot') &\geq \xi(\top, \min(\bot', \tfrac{1}{2})) && \text{by (X-5)}\\ &= \tfrac{1}{2}\,, && \text{by L-68}\end{aligned}$$

i.e.

$$\xi(\top, \bot') \geq \tfrac{1}{2}\,. \tag{194}$$

By (3), inequations (193) and (194) entail that $\xi(\top, \bot') \preceq_c \xi(\top, \bot)$, as desired.
In the case that $\bot'(0) < \frac{1}{2}$ and $\bot(0) \geq \frac{1}{2}$, we notice that $(\top, \bot) \in \mathbb{T}$ entails that $\top(0) \geq \bot(0) \geq \frac{1}{2}$, i.e. $\xi(\top, \bot') = \frac{1}{2}$ by L-68. In particular $\xi(\top, \bot') \preceq_c \xi(\top, \bot)$.
In the case that $\bot'(0) < \frac{1}{2}$, $\bot(0) < \frac{1}{2}$ and $\top(0) \geq \frac{1}{2}$, we can again apply L-68 and conclude that $\xi(\top, \bot') = \frac{1}{2} = \xi(\top, \bot)$.
Finally in the case that $\bot'(0) < \frac{1}{2}$, $\bot(0) < \frac{1}{2}$ and $\top(0) < \frac{1}{2}$, we observe that

$$\begin{aligned}\xi(\top, \bot') &= \xi(\top, \mathrm{c}_0) && \text{by (192) and L-66}\\ &= \xi(\top, \bot)\,. && \text{by (192) and L-66}\end{aligned}$$

In particular $\xi(\top, \bot') \preceq_c \xi(\top, \bot)$, as desired.

B.21 Proof of Theorem 40

Lemma 70 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-1) and ξ propagates unspecificity. Then*

$$\xi(\top, \bot) = \tfrac{1}{2}$$

whenever $(\top, \bot) \in \mathbb{T}$ satisfies $\top(0) \geq \frac{1}{2} \geq \bot(0)$.

Proof Let $(\top, \bot) \in \mathbb{T}$ be given such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$. Then $\top \geq \mathrm{c}_{\frac{1}{2}} \geq \bot$ because $\top$ is nondecreasing and $\bot$ is nonincreasing by Def. 44. Hence by Def. 53,

$$\begin{aligned}\xi(\top, \bot) &\preceq_c \xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_{\frac{1}{2}})\\ &= \tfrac{1}{2}\,, && \text{by (X-1)}\end{aligned}$$

as desired.

Proof of Theorem 40

Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a mapping such that $\mathcal{F}_\xi$ is a DFS which propagates fuzziness in arguments. Then ξ satisfies (X-1) by Th-23 and ξ is also known to propagate unspecificity by Th-38. We can hence apply lemma L-70 which states that ξ fulfills the property claimed by the theorem.

B.22 Proof of Theorem 41

We know from Th-27 that $\mathcal{F}_{\mathrm{Ch}}$ is a DFS. Hence $\xi_{\mathrm{Ch}} : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-1) to (X-5). We can now apply Th-40 to conclude from lemma L-63 that $\mathcal{F}_{\mathrm{Ch}}$ does not propagate fuzziness in arguments.

B.23 Proof of Theorem 42

By Th-38, $\mathcal{F}_S$ propagates fuzziness in arguments if and only if $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ defined in Def. 50 propagates unspecificity. Hence let us consider $(\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}), (\mathrm{c}_1, \mathrm{c}_{\frac{3}{4}}) \in \mathbb{T}$. We compute:

$$\xi_S(\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}) = \tfrac{3}{4}$$

by (X-1) and

$$\begin{aligned} \xi_S(\mathrm{c}_1, \mathrm{c}_{\frac{3}{4}}) &= \min((\mathrm{c}_1)_1^*, \tfrac{1}{2} + \tfrac{1}{2}(\mathrm{c}_{\frac{3}{4}})_*^{\leq\frac{1}{2}\downarrow} && \text{by Def. 50} \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (12), (26)} \\ &= 1\,. \end{aligned}$$

Hence $\xi_S(\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}) = \frac{3}{4} \preceq_c 1 = \xi_S(\mathrm{c}_1, \mathrm{c}_{\frac{3}{4}})$, which contradicts propagation of unspecificity, see Def. 53.

B.24 Proof of Theorem 43

We already know from Th-31 that $\mathcal{F}_A$ is a DFS. Hence ξ_A satisfies (X-1) to (X-5), see Th-23. Therefore theorem Th-39 is applicable, and we can prove that ξ propagates unspecificity by showing that $\xi_A(\top, \bot) = \xi_A(\mathrm{c}_1, \bot)$ whenever $\bot(0) \geq \frac{1}{2}$.
Hence let $(\top, \bot) \in \mathbb{T}$ be given and suppose that $\bot(0) \geq \frac{1}{2}$. Because $\bot(0) \geq \frac{1}{2}$, we clearly have $\top(\gamma) \geq \top(0) \geq \bot(0) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ and hence

$$\top_0^* = \lim_{\gamma \to 0^+} \top(\gamma) \geq \tfrac{1}{2} \tag{195}$$

by (9). In addition, we notice that

$$\bot_0^* = \lim_{\gamma \to 0^+} \bot(\gamma) \leq \bot(0) \tag{196}$$

because $\bot$ is nonincreasing. It is hence sufficient to discern the following cases.

a.: $\bot_0^* > \frac{1}{2}$. Then

$$\begin{aligned} \xi_A(\top, \bot) &= \min(\bot_0^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by Def. 51} \\ &= \xi_A(\mathrm{c}_1, \bot)\,. && \text{by Def. 51} \end{aligned}$$

b.: $\bot_0^* \le \frac{1}{2}$**.** In this case, we recall that $\top_0^* \ge \frac{1}{2}$ by (195). In addition, $[c_1]_0^* = 1 \ge \frac{1}{2}$. Therefore

$$\begin{aligned} \xi_A(\top, \bot) &= \tfrac{1}{2} && \text{by Def. 51} \\ &= \xi_A(c_1, \bot)\,, && \text{by Def. 51} \end{aligned}$$

as desired.

Hence indeed $\xi_A(\top, \bot) = \xi_A(c_1, \bot)$ whenever $\bot(0) \ge \frac{1}{2}$. We conclude from Th-39 that ξ_A propagates unspecificity. In turn, we obtain from Th-38 that $\mathcal{F}_A$ propagates fuzziness in arguments.

B.25 Proof of Theorem 44

In order to prove the independence of propagation of fuzziness in quantifiers and in arguments for $\mathcal{F}_\xi$-DFSes, we must show that there exist $\mathcal{F}_\xi$-DFSes $\mathcal{F}_{\xi_1}$ and $\mathcal{F}_{\xi_2}$ such that $\mathcal{F}_{\xi_1}$ propagates fuzziness in quantifiers, but not in arguments, and $\mathcal{F}_{\xi_2}$ propagates fuzziness in arguments, but not in quantifiers. By Th-30, Th-36 and Th-42, $\mathcal{F}_S$ is a suitable choice for $\mathcal{F}_{\xi_1}$. Finally by Th-31, Th-37 and Th-43, $\mathcal{F}_A$ is a suitable choice for $\mathcal{F}_{\xi_2}$, thus finishing the independence proof.

B.26 Proof of Theorem 45

Lemma 71 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-1) *to* (X-5)*. If* ξ *propagates both fuzziness and unspecificity, then*

$$\xi(\top, \bot) = \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))\,,$$

for all $(\top, \bot) \in \mathbb{T}$*, where* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{B}(f) = \begin{cases} \xi(c_1, f) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \xi(f, c_0) & : \quad f \in \mathbb{B}^- \end{cases} \tag{197}$$

for all $f \in \mathbb{B}$*.*

Proof Let $(\top, \bot) \in \mathbb{T}$ be given. We abbreviate $f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)$.

a.: $f(0) > \frac{1}{2}$. It is then apparent from Def. 22 that $\top(0) \geq \bot(0) > \frac{1}{2}$. Again from Def. 22, we conclude that

$$\begin{aligned} f(\gamma) &= \begin{cases} \bot(\gamma) & : \quad \bot(\gamma) > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot(\gamma) \leq \frac{1}{2} \end{cases} \\ &= \max(\bot(\gamma), \tfrac{1}{2}) \end{aligned}$$

for all $\gamma \in \mathbf{I}$ because $\top(\gamma) \geq \top(0) > \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ by Def. 44. Therefore

$$\xi(\top, \bot) = \xi(\top, \max(\bot, \tfrac{1}{2})) = \xi(\top, f) \tag{198}$$

by Th-33 because ξ is assumed to propagate fuzziness. Apparently $\top \leq \mathrm{c}_1$ and hence

$$\xi(\top, f) \leq \xi(\mathrm{c}_1, f) \tag{199}$$

by (X-5). On the other hand, $\top \leq \mathrm{c}_1$ entails that $\xi(\mathrm{c}_1, f) \preceq_c \xi(\top, f)$, because ξ is assumed to propagate unspecificity. Apparently $\xi(\top, f) \geq \xi(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) = \frac{1}{2}$ by (X-5) and L-61, recalling that ξ propagates fuzziness. But $\xi(\top, f) \geq \frac{1}{2}$ and $\xi(\mathrm{c}_1, f) \preceq_c \xi(\top, f)$ entail that $\xi(\mathrm{c}_1, f) \leq \xi(\top, f)$. Combining this with (199), we see that $\xi(\mathrm{c}_1, f) = \xi(\top, f)$. Hence by (198) and (197), $\xi(\top, \bot) = \xi(\mathrm{c}_1, f) = \mathcal{B}(f)$, as desired.

b.: $f(0) = \frac{1}{2}$. Then $\top(0) \geq \frac{1}{2}$ and $\bot(0) \leq \frac{1}{2}$ by Def. 22. Hence

$$\begin{aligned} \xi(\top, \bot) &= \tfrac{1}{2} && \text{by L-70} \\ &= \mathcal{B}(f)\,, && \text{by (197)} \end{aligned}$$

because clearly $f(\gamma) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ in this case, i.e. $f = \mathrm{c}_{\frac{1}{2}} \in \mathbb{B}^{\frac{1}{2}}$.

c.: $f(0) < \frac{1}{2}$. This can be reduced to the proof of **a.** because

$$\begin{aligned} \xi(\top, \bot) &= 1 - \xi(1 - \bot, 1 - \top) && \text{by (X-2)} \\ &= 1 - \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(1 - \bot, 1 - \top)) && \text{by } \textbf{a.} \\ &= 1 - \mathcal{B}(1 - \mathrm{med}_{\frac{1}{2}}(\bot, \top)) && \text{because } \mathrm{med}_{\frac{1}{2}} \text{ symmetric w.r.t. } \neg \\ &= 1 - \mathcal{B}(1 - \mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{because } \mathrm{med}_{\frac{1}{2}} \text{ commutative} \\ &= 1 - (1 - \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))) && \text{apparent from (197)} \\ &= \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))\,, \end{aligned}$$

as desired.

Proof of Theorem 45

Let an $\mathcal{F}_\xi$-DFS be given. We further assume that $\mathcal{F}_\xi$ propagates fuzziness both in quantifiers and arguments. Because $\mathcal{F}_\xi$ propagates fuzziness in quantifiers, we can conclude from Th-32 that ξ propagates fuzziness. Similarly because $\mathcal{F}_\xi$ propagates fuzziness in arguments, we know from Th-38 that ξ propagates unspecificity. We can therefore apply lemma L-71 which ensures that there exists $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ such that

$$\xi(\top, \bot) = \mathcal{B}(\mathrm{med}_{\frac{1}{2}}(\top, \bot))$$

for all $(\top, \bot) \in \mathbb{T}$. Hence $\mathcal{F}_\xi$ is an $\mathcal{M}_\mathcal{B}$-DFS because by Th-24, $\mathcal{M}_\mathcal{B} = \mathcal{F}_\xi$.

B.27 Proof of Theorem 46

Lemma 72 *Suppose $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are semi-fuzzy quantifiers and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then*

$$d(\top_{Q,X_1,\ldots,X_n}, \top_{Q',X_1,\ldots,X_n}) \leq d(Q, Q')$$
$$d(\bot_{Q,X_1,\ldots,X_n}, \bot_{Q',X_1,\ldots,X_n}) \leq d(Q, Q')$$
$$d((\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}), (\top_{Q',X_1,\ldots,X_n}, \bot_{Q',X_1,\ldots,X_n})) \leq d(Q, Q')$$

for all $\gamma \in \mathbf{I}$

Proof The first two inequations have been proven in [7, L-104, p.252]. The third inequation is apparent from the first two because

$$\begin{aligned}&d((\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}), (\top_{Q',X_1,\ldots,X_n}, \bot_{Q',X_1,\ldots,X_n}))\\ &= \max(d(\top_{Q,X_1,\ldots,X_n}, \top_{Q',X_1,\ldots,X_n}), d(\bot_{Q,X_1,\ldots,X_n}, \bot_{Q',X_1,\ldots,X_n}))\\ &\leq d(Q, Q')\,.\end{aligned}$$

Lemma 73 *For every mapping $\xi : \mathbb{T} \longrightarrow \mathbf{I}$, condition a. of Th-46 is entailed by condition b.*

Note. We do not need to impose the condition that ξ satisfies (X-5) in this case.

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ has the following property. For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$$

whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ satisfy $d((\top, \bot), (\top', \bot')) < \delta$. We have to show that $\mathcal{F}_\xi$ is Q-continuous. Hence let some $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and let $\varepsilon > 0$. We have to show that there exists $\delta > 0$ such that

$$d(\mathcal{F}_\xi(Q), \mathcal{F}_\xi(Q')) < \varepsilon$$

whenever $d(Q, Q') < \delta$. By the assumed property of ξ, there exists $\delta > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot')| < \tfrac{\varepsilon}{2} \tag{200}$$

whenever $d((\top, \bot), (\top', \bot')) < \delta$. Now let $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier with $d(Q, Q') < \delta$. Then for each choice of $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, Then

$$\begin{aligned}
&|\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) - \mathcal{F}_\xi(Q')(X_1, \ldots, X_n)| \\
&= |\xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) - \xi(\top_{Q',X_1,\ldots,X_n}, \bot_{Q',X_1,\ldots,X_n})| && \text{by Def. 45} \\
&\leq \tfrac{\varepsilon}{2} && \text{by (200), L-72},
\end{aligned}$$

i.e.

$$|\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) - \mathcal{F}_\xi(Q')(X_1, \ldots, X_n)| \leq \tfrac{\varepsilon}{2}\,. \tag{201}$$

Hence

$$\begin{aligned}
&d(\mathcal{F}_\xi(Q), \mathcal{F}_\xi(Q')) \\
&= \sup\{|\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) - \mathcal{F}_\xi(Q')(X_1, \ldots, X_n)| : \\
&\quad X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\} && \text{by (6)} \\
&\leq \tfrac{\varepsilon}{2} && \text{by (201)} \\
&< \varepsilon\,,
\end{aligned}$$

i.e. $\mathcal{F}_\xi$ is Q-continuous, as desired.

Lemma 74 *Let us define semi-fuzzy quantifiers* $Q', Q'', Q : \mathcal{P}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T})) \longrightarrow \mathbf{I}$ *as follows. For all* $Y \in \mathcal{P}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T}))$,

$$Q'(Y) = \top_Y(\sup Y') \tag{202}$$

$$Q''(Y) = \bot_Y(\inf Y'') \tag{203}$$

$$Q(Y) = \begin{cases} Q''(Y) & : \quad Y' = \varnothing \\ Q'(Y) & : \quad Y' \neq \varnothing \end{cases} \tag{204}$$

where

$$Y' = \{z \in \mathbf{I} : (0, z) \in Y\} \tag{205}$$

$$Y'' = \{z \in \mathbf{I} : (1, z) \in Y\} \tag{206}$$

$$\top_Y = \sup\{\top : (2, (\top, \bot)) \in Y\} \tag{207}$$

$$\bot_Y = \sup\{\bot : (2, (\top, \bot)) \in Y\} \tag{208}$$

Then for all $(\top, \bot) \in \mathbb{T}$,

$$\top_{Q,X} = \top$$

and

$$\bot_{Q,X} = \bot$$

provided we define $X \in \widetilde{\mathcal{P}}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T}))$ *by*

$$\mu_X(c, z) = \begin{cases} \frac{1}{2} - \frac{1}{2}z & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2}z & : \quad c = 1 \\ 1 & : \quad c = 2,\ z = (\top, \bot) \\ 0 & : \quad \textit{else} \end{cases} \tag{209}$$

for all $(c, z) \in (\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T})$.

Proof We first consider some monotonicity properties. It is apparent from (207) and (208) that $\top_Y \leq \top_{Y'}$ and $\bot_Y \leq \bot_{Y'}$ whenever $Y \subseteq Y'$. Because $\sup Y'$ is nondecreasing in Y and $\top_Y(z) \leq \top_Y(z')$ for $z \leq z'$, we conclude that Q' is nondecreasing in Y. Similarly, $\inf Y$ is nonincreasing in Y and $\bot_Y(z) \geq \bot_Y(z')$ for $z \leq z'$, hence Q'' is nondecreasing in Y as well. Finally, we conclude from (207), (208) and the fact that $\bot \leq \top$ for all $(\top, \bot)$ with $(2, (\top, \bot)) \in Y$ that $\bot_Y \leq \top_Y$ and hence $Q'' \leq Q'$. Therefore Q is nondecreasing in Y as well, and we can utilize L-3 to simplify the computation of $\top_{Q,X}$ and $\bot_{Q,X}$.
In the following, we will assume that a choice of $(\top, \bot) \in \mathbb{T}$ is given and that the fuzzy set X is defined in terms of $\top$ and $\bot$ according to equation (209). Let us now consider the cut ranges. If $\gamma = 0$, then

$$\begin{aligned} X_0^{\min} &= X_{>\frac{1}{2}} && \text{by Def. 30} \\ &= (\{1\} \times (0, 1]) \cup \{(2, (\top, \bot))\} && \text{by Def. 29, (209)} \end{aligned}$$

and

$$\begin{aligned} X_0^{\max} &= X_{\geq\frac{1}{2}} && \text{by Def. 30} \\ &= \{(0, 0)\} \cup (\{1\} \times [0, 1]) \cup \{(2, (\top, \bot))\}\,. && \text{by Def. 28, (209)} \end{aligned}$$

If $\gamma > 0$, then

$$\begin{aligned} X_\gamma^{\min} &= X_{\geq\frac{1}{2}+\frac{1}{2}\gamma} && \text{by Def. 30} \\ &= (\{1\} \times [\gamma, 1]) \cup \{(2, (\top, \bot))\} && \text{by Def. 28, (209)} \end{aligned}$$

and

$$\begin{aligned} X_\gamma^{\max} &= X_{>\frac{1}{2}-\frac{1}{2}\gamma} && \text{by Def. 30} \\ &= (\{0\} \times [0, \gamma)) \cup (\{1\} \times \mathbf{I}) \cup \{(2, (\top, \bot))\}\,. \end{aligned}$$

Hence for $\gamma = 0$,

$$\begin{aligned} &\top_{Q,X}(0) \\ &= Q(X_0^{\max}) && \text{by L-3} \\ &= Q(\{(0, 0)\} \cup (\{1\} \times [0, 1]) \cup \{(2, (\top, \bot))\}) \\ &= \top(\sup\{0\}) && \text{by (204), (202), (205) and (207)} \\ &= \top(0)\,. \end{aligned}$$

and

$$\begin{aligned}
&\bot_{Q,X}(0) \\
&= Q(X_0^{\min}) && \text{by L-3} \\
&= Q((\{1\} \times (0,1]) \cup \{(2,(\top,\bot))\}) \\
&= \bot(\inf(0,1]) && \text{by (204), (203), (206) and (208)} \\
&= \bot(0)\,.
\end{aligned}$$

Finally for $\gamma > 0$,

$$\begin{aligned}
&\top_{Q,X}(\gamma) \\
&= Q(X_\gamma^{\max}) && \text{by L-3} \\
&= Q((\{0\} \times [0,\gamma)) \cup (\{1\} \times \mathbf{I}) \cup \{(2,(\top,\bot))\}) \\
&= \top(\sup[0,\gamma)) && \text{by (204), (202), (205) and (207)} \\
&= \top(\gamma)\,.
\end{aligned}$$

and

$$\begin{aligned}
&\bot_{Q,X}(\gamma) \\
&= Q(X_\gamma^{\min}) && \text{by L-3} \\
&= Q((\{1\} \times [\gamma,1]) \cup \{(2,(\top,\bot))\}) \\
&= \bot(\inf[\gamma,1]) && \text{by (204), (203), (206) and (208)} \\
&= \bot(\gamma)\,,
\end{aligned}$$

as desired.

Lemma 75 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be given and let $\alpha \in \mathbf{I}$. Further suppose that $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1,\dots,Y_n) = \min(1, \alpha + Q(Y_1,\dots,Y_n))\,, \tag{210}$$

for all $Y_1,\dots,Y_n \in \widetilde{\mathcal{P}}(E)$. Then for all $X_1,\dots,X_n \in \widetilde{\mathcal{P}}(E)$,

$$\top_{Q',X_1,\dots,X_n} = \min(1, \alpha + \top_{Q,X_1,\dots,X_n})$$

and

$$\bot_{Q',X_1,\dots,X_n} = \min(1, \alpha + \bot_{Q,X_1,\dots,X_n})\,.$$

Proof Trivial. Let $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
&\top_{Q',X_1,\dots,X_n} \\
&= \sup\{Q'(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \sup\{\min(1, \alpha + Q(Y_1,\dots,Y_n)) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by (210)} \\
&= \min(1, \alpha + \sup\{Q(Y_1,\dots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\dots,Y_n \in \mathcal{T}_\gamma(X_n)\})
\end{aligned}$$

(because $\min$ and $+$ are nondecreasing and continuous)

$$= \min(1, \alpha + \top_{Q,X_1,\ldots,X_n}(\gamma))\,, \qquad \text{by Def. 43}$$

and similarly

$$\begin{aligned}
&\bot_{Q',X_1,\ldots,X_n} \\
&= \inf\{Q'(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by Def. 43} \\
&= \inf\{\min(1, \alpha + Q(Y_1,\ldots,Y_n)) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\} && \text{by (210)} \\
&= \min(1, \alpha + \inf\{Q(Y_1,\ldots,Y_n) : Y_1 \in \mathcal{T}_\gamma(X_1),\ldots,Y_n \in \mathcal{T}_\gamma(X_n)\})
\end{aligned}$$

(because $\min$ and $+$ are nondecreasing and continuous)

$$= \min(1, \alpha + \bot_{Q,X_1,\ldots,X_n}(\gamma))\,. \qquad \text{by Def. 43}$$

Proof of Theorem 46

The claim that condition b. entails condition a., i.e. that b. is sufficient for $\mathcal{F}_\xi$ to be Q-continuous, has been proven in L-73. It remains to be shown that condition b. of the theorem is also necessary for $\mathcal{F}_\xi$ to be Q-continuous. We prove this by showing that that the failure of condition b. entails the failure of condition a.
Hence let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping which satisfies (X-5) but violates condition b. In order to prove that $\mathcal{F}_\xi$ is not Q-continuous, we have to show that there exists a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exists $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ with $d(Q, Q') < \delta$ and $d(\mathcal{F}_\xi(Q), \mathcal{F}_\xi(Q')) \geq \varepsilon$.
Because ξ violates condition b., there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$d((\top, \bot), (\top', \bot')) < \delta \qquad (211)$$

and

$$|\xi(\top, \bot) - \xi(\top', \bot') \geq \varepsilon\,. \qquad (212)$$

We shall keep this choice of ε and focus on $Q : \mathcal{P}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T})) \longrightarrow \mathbf{I}$ as defined by (204). Now let $\delta > 0$. By assumption, there exist $(\top, \bot), (\top', \bot') \in \mathbb{T}$ such that (211) and (212) hold. We define

$$\begin{aligned}
\top_* &= \min(\top, \top') \\
\bot_* &= \min(\bot, \bot') \\
\top^* &= \max(\top, \top') \\
\bot^* &= \max(\bot, \bot')\,.
\end{aligned}$$

Apparently $(\top_*, \bot_*) \in \mathbb{T}$ and $(\top^*, \bot^*) \in \mathbb{T}$. In addition,

$$\begin{aligned}
&d(\top_*, \top^*) \\
&= \sup\{\max(\top(\gamma), \top'(\gamma)) - \min(\top(\gamma), \top'(\gamma)) : \gamma \in \mathbf{I}\} && \text{by (28) and } \top_* \leq \top^* \\
&= \sup\{|\top(\gamma) - \top'(\gamma)| : \gamma \in \mathbf{I}\} \\
&= d(\top, \top') && \text{by (28)}
\end{aligned}$$

and similarly

$$\begin{aligned}
&d(\bot_*, \bot^*) \\
&= \sup\{\max(\top(\gamma), \top'(\gamma)) - \min(\bot(\gamma), \bot'(\gamma)) : \gamma \in \mathbf{I}\} && \text{by (29) and } \bot_* \leq \bot^* \\
&= \sup\{|\bot(\gamma) - \bot'(\gamma)| : \gamma \in \mathbf{I}\} \\
&= d(\bot, \bot')\,, && \text{by (29)}
\end{aligned}$$

i.e.

$$d((\top_*, \bot_*), (\top^*, \top^*)) = d((\top, \bot), (\top', \bot')) < \delta \tag{213}$$

by (211). Furthermore,

$$\begin{aligned}
\top_* &\leq \top \leq \top^* \\
\top_* &\leq \top' \leq \top^* \\
\bot_* &\leq \bot \leq \bot^* \\
\bot_* &\leq \bot' \leq \bot^*
\end{aligned}$$

and hence

$$\begin{aligned}
&|\xi(\top^*, \bot^*) - \xi(\top_*, \bot_*)| \\
&= \xi(\top^*, \bot^*) - \xi(\top_*, \bot_*) && \text{by (X-5)} \\
&\geq \max(\xi(\top, \bot), \xi(\top', \bot')) - \xi(\top_*, \bot_*) && \text{by (X-5)} \\
&\geq \max(\xi(\top, \bot), \xi(\top', \bot')) - \min(\xi(\top, \bot), \xi(\top', \bot')) && \text{by (X-5)} \\
&= |\xi(\top, \bot) - \xi(\top', \bot')| \\
&\geq \varepsilon\,,
\end{aligned}$$

i.e.

$$|\xi(\top^*, \bot^*) - \xi(\top_*, \bot_*)| \geq \varepsilon\,. \tag{214}$$

Now we define a fuzzy subset $X \in \widetilde{\mathcal{P}}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T}))$ by

$$\mu_X(c, z) = \begin{cases} \frac{1}{2} - \frac{1}{2}z & : \quad c = 0 \\ \frac{1}{2} + \frac{1}{2}z & : \quad c = 1 \\ 1 & : \quad c = 2,\ z = (\top_*, \bot_*) \\ 0 & : \quad \text{else} \end{cases}$$

for all $(c, z) \in (\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T})$. Then by L-74,

$$\top_{Q,X} = \top_* \tag{215}$$
$$\bot_{Q,X} = \bot_* \,. \tag{216}$$

We further define $Q' : \mathcal{P}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T})) \longrightarrow \mathbf{I}$ by

$$Q'(Y) = \min(1, d((\top, \bot), (\top', \bot')) + Q(Y)) \,,$$

for all $Y \in \mathcal{P}((\mathbf{2} \times \mathbf{I}) \cup (\{2\} \times \mathbb{T}))$. It is obvious from this definition of Q' and (5) that $d(Q, Q') \leq d((\top, \bot), (\top', \bot'))$, i.e.

$$d(Q, Q') < \delta \tag{217}$$

by (211). By L-75 and (215)/(216),

$$\top_{Q',X} = \min(1, d((\top, \bot), (\top', \bot')) + \top_*)$$
$$\bot_{Q',X} = \min(1, d((\top, \bot), (\top', \bot')) + \bot_*)$$

In turn, we conclude from $\top_* \leq \top^*$, $\bot_* \leq \bot^*$ and (213) that

$$\top^* \leq \top_{Q',X} \tag{218}$$
$$\bot^* \leq \bot_{Q',X} \,. \tag{219}$$

Hence

$$\begin{aligned}
&|\mathcal{F}_\xi(Q)(X) - \mathcal{F}_\xi(Q')(X)| \\
&= |\xi(\top_{Q,X}, \bot_{Q,X}) - \xi(\top_{Q',X}, \bot_{Q',X})| && \text{by Def. 45} \\
&= |\xi(\top_*, \bot_*) - \xi(\top_{Q',X}, \bot_{Q',X})| && \text{by (215), (216)} \\
&= \xi(\top_{Q',X}, \bot_{Q',X}) - \xi(\top_*, \bot_*) && \text{because } \top_{Q',X} \geq \top_*, \bot_{Q',X} \geq \bot_* \\
&\geq \xi(\top^*, \bot^*) - \xi(\top_*, \bot_*) && \text{by (X-5), (218), (219)} \\
&\geq \varepsilon \,. && \text{by (214)}
\end{aligned}$$

This proves that $d(\mathcal{F}_\xi(Q), \mathcal{F}_\xi(Q')) \geq \varepsilon$ although $d(Q, Q') < \delta$ by (217). Therefore $\mathcal{F}_\xi$ is not Q-continuous.

B.28 Proof of Theorem 47

Lemma 76 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2) *and* (X-5). *Then the following conditions are equivalent:*

a. $\mathcal{F}_\xi$ *is Q-continuous;*

b. for all $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|\xi(\top, \bot) - \xi(\top', \bot)| < \varepsilon$ *whenever* $(\top, \bot), (\top', \bot) \in \mathbb{T}$ *satisfy* $d(\top, \top') < \delta$.

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping such that (X-2) and (X-5) hold.

a.⇒b.: The above condition **b.** is apparently a weakening of the following condition: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top, \bot), (\top', \bot') \in \mathbb{T}$ satisfy $d((\top, \bot), (\top', \bot')) < \delta$. The latter condition has been shown to be necessary for $\mathcal{F}_\xi$ to be Q-continuous in Th-46. Being a weakening of a necessary condition, the present condition **b.** is also necessary for $\mathcal{F}_\xi$ to be Q-continuous.

b.⇒a.: To see this, let us assume that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies condition **b.** Further let $\varepsilon > 0$. By assumption, there exists $\delta' > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot)| < \tfrac{\varepsilon}{2} \tag{220}$$

whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ such that $(\top, \top') < \delta'$.
We now consider $(\top, \bot), (\top', \bot') \in \mathbb{T}$ with

$$d((\top, \bot), (\top', \bot')) < \delta' \,. \tag{221}$$

We also note that by (221) and (30),

$$d((\top, \bot), (\top', \bot)) \le d((\top, \bot), (\top', \bot)) < \delta' \tag{222}$$

$$\begin{aligned} d((1-\bot, 1-\top'), (1-\bot', 1-\top')) &= d((\top', \bot), (\top', \bot')) \\ &\le d((\top, \bot), (\top', \bot')) < \delta' \end{aligned} \tag{223}$$

Therefore

$$\begin{aligned} &|\xi(\top', \bot) - \xi(\top', \bot')| \\ &= |(1 - \xi(1-\bot, 1-\top')) - (1 - \xi(1-\bot', 1-\top'))| && \text{by (X-2)} \\ &= |\xi(1-\bot, 1-\top') - \xi(1-\bot', 1-\top')| \,, \end{aligned}$$

i.e.

$$|\xi(\top', \bot) - \xi(\top', \bot')| < \tfrac{\varepsilon}{2} \tag{224}$$

by (220) and (223). Finally

$$\begin{aligned} &|\xi(\top, \bot) - \xi(\top', \bot')| \\ &\le |\xi(\top, \bot) - \xi(\top', \bot)| + |\xi(\top', \bot) - \xi(\top', \bot')| && \text{by triangle inequation} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} && \text{by (220), (222) and (224)} \\ &= \varepsilon \,. \end{aligned}$$

Hence for all $(\top, \bot) \in \mathbb{T}$ an all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$ whenever $(\top', \bot') \in \mathbb{T}$ satisfies $d((\top, \bot), (\top', \bot')) < \delta$. Application of L-73 yields that $\mathcal{F}_\xi$ is Q-continuous.

Proof of Theorem 47

Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping such that (X-2) and (X-5) hold.

a.⇒b.: The above condition **b.** is apparently a weakening of this condition: for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|\xi(\top, \bot) - \xi(\top', \bot)| < \varepsilon$ whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ satisfy $d(\top, \top') < \delta$. The latter condition has been shown to be necessary for $\mathcal{F}_\xi$ to be Q-continuous in L-76. Being a weakening of a necessary condition, we again conclude that the present condition **b.** is necessary for $\mathcal{F}_\xi$ to be Q-continuous.

b.⇒a.: We assume that $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2) and (X-5) and also fulfills condition **b.** We proceed by showing that ξ fulfills the condition **b.** of lemma L-76. Hence let $\varepsilon > 0$ be given. By assumption, there exists $\delta > 0$ such that

$$\xi(\top^*, \bot) - \xi(\top_*, \bot) < \varepsilon \tag{225}$$

whenever $(\top_*, \bot), (\top^*, \bot) \in \mathbb{T}$ satisfy $d(\top_*, \top^*) < \delta$ and $\top_* \leq \top^*$. Now let $(\top, \bot), (\top', \bot) \in \mathbb{T}$ with $d(\top, \top') < \delta$. We abbreviate

$$\begin{aligned} \top_* &= \min(\top, \top') \\ \top^* &= \max(\top, \top') \,. \end{aligned}$$

Clearly $(\top_*, \bot), (\top^*, \bot) \in \mathbb{T}$. In addition, it is obvious from (28) that

$$d(\top_*, \top^*) = d(\top, \top') < \delta \,. \tag{226}$$

We further conclude from (X-5) that

$$\xi(\top^*, \bot) \geq \max(\xi(\top, \bot), \xi(\top', \bot)) \tag{227}$$

$$\xi(\top_*, \bot) \leq \min(\xi(\top, \bot), \xi(\top', \bot)) \,. \tag{228}$$

Hence

$$\begin{aligned} &|\xi(\top, \bot) - \xi(\top', \bot)| \\ &= \max(\xi(\top, \bot), \xi(\top', \bot)) - \min(\xi(\top, \bot), \xi(\top', \bot)) \\ &\leq \xi(\top^*, \bot) - \xi(\top_*, \bot) && \text{by (227), (228)} \\ &= |\xi(\top^*, \bot) - \xi(\top_*, \bot)| && \text{by (X-5) and } \top_* \leq \top^* \\ &< \varepsilon \,. && \text{by (225)} \end{aligned}$$

Hence condition **b.** of L-76 is satisfied, i.e. $\mathcal{F}_\xi$ is arg-continuous.

B.29 Proof of Theorem 48

In order to show that $\mathcal{F}_{\mathrm{Ch}}$ is Q-continuous, I will prove the equivalent condition **b.** of Th-47. Hence let $\varepsilon > 0$ be given. I will show that $\delta = \varepsilon$ is a suitable choice of δ.

Hence let $(\top, \bot), (\top', \bot) \in \mathbb{T}$ with $d(\top, \top') < \varepsilon$ and $\top \leq \top'$. Because $\top \leq \top'$, $d(\top, \top') < \varepsilon$ can be rewritten as

$$\sup\{\top'(\gamma) - \top(\gamma) : \gamma \in \mathbf{I}\} < \varepsilon$$

by (28). In particular

$$\top'(\gamma) \leq \top(\gamma) + \varepsilon\,. \qquad (229)$$

Then

$$\begin{aligned}
&|\xi_{\mathrm{Ch}}(\top', \bot) - \xi_{\mathrm{Ch}}(\top, \bot)| \\
&= |\tfrac{1}{2}\int_0^1 \top'(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma \\
&\quad -\tfrac{1}{2}\int_0^1 \top(\gamma)\,d\gamma - \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma| && \text{by Def. 47} \\
&= \tfrac{1}{2}|\int_0^1 \top'(\gamma)\,d\gamma - \int_0^1 \top(\gamma)\,d\gamma| \\
&= \tfrac{1}{2}(\int_0^1 \top'(\gamma)\,d\gamma - \int_0^1 \top(\gamma)\,d\gamma) && \text{because } \top' \geq \top \\
&\leq \tfrac{1}{2}(\int_0^1 \top(\gamma) + \varepsilon\,d\gamma - \int_0^1 \top(\gamma)\,d\gamma) && \text{by (229)} \\
&= \tfrac{\varepsilon}{2} \\
&< \varepsilon\,.
\end{aligned}$$

Hence $\delta = \varepsilon$ is indeed a choice of δ with the desired properties, i.e. condition **b.** of Th-47 is fulfilled, which is equivalent to $\mathcal{F}_{\mathrm{Ch}}$ being Q-continuous.

B.30 Proof of Theorem 49

We already know from Th-30 that $\mathcal{F}_S$ is a DFS and hence satisfies (X-2) and (X-5) by Th-23. Hence Th-47 is applicable, which allows the reduction of $\mathcal{F}_S$ being Q-continuous to an equivalent condition on ξ_S. Hence let us show that condition **b.** of Th-47 violated.

Let $\varepsilon \in (0, \frac{1}{2}]$ be given and let $\delta > 0$. Consider $(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0), (\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}}, \mathrm{c}_0) \in \mathbb{T}$. Apparently $\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}} \leq \mathrm{c}_{\frac{1}{2}}$ and $d(\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}}, \mathrm{c}_{\frac{1}{2}}) = \frac{\delta}{2} < \delta$. By Def. 50, we have

$$\xi_S(\mathrm{c}_{\frac{1}{2}}, \mathrm{c}_0) = \tfrac{1}{2}$$

and

$$\begin{aligned}
&\xi_S(\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}},\mathrm{c}_0)\\
&= \max((\mathrm{c}_0)^*_1, \tfrac{1}{2}-\tfrac{1}{2}(\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}})^{\geq\frac{1}{2}\downarrow}_*) && \text{by Def. 50}\\
&= \max(0, \tfrac{1}{2}-\tfrac{1}{2}\cdot 1) && \text{by (12), (27)}\\
&= 0\,.
\end{aligned}$$

Hence $\xi_S(\mathrm{c}_{\frac{1}{2}},\mathrm{c}_0) - \xi_S(\mathrm{c}_{\frac{1}{2}-\frac{\delta}{2}},\mathrm{c}_0) = \frac{1}{2} - 0 = \frac{1}{2} \geq \varepsilon$. This proves that condition **b.** of Th-47 fails, which is necessary for $\mathcal{F}_S$ to be Q-continuous. We conclude that $\mathcal{F}_S$ is not Q-continuous.

B.31 Proof of Theorem 50

We know from Th-31 and Th-23 that ξ_A satisfies (X-2) and (X-5). Hence Th-47 is applicable, and we can show that $\mathcal{F}_A$ fails to be Q-continuous by proving that there exists $\varepsilon > 0$ such that for all $\delta > 0$, there exist $(\top,\bot), (\top',\bot) \in \mathbb{T}$ with $d(\top,\top') < \delta$, $\top \leq \top'$ and $\xi_A(\top',\bot) - \xi_A(\top,\bot) \geq \varepsilon$.
Hence consider $\varepsilon = \frac{1}{4}$ and let $\delta > 0$. We define $\top, \top'$ by

$$\top(\gamma) = \begin{cases} 0 & : \gamma \leq \frac{1}{2} \\ 1-\frac{\delta}{2} & : \gamma > \frac{1}{2} \end{cases}$$

$$\top'(\gamma) = \begin{cases} 0 & : \gamma \leq \frac{1}{2} \\ 1 & : \gamma > \frac{1}{2} \end{cases}$$

for all $\gamma \in \mathbf{I}$. Then by (9),

$$\top^*_0 = \lim_{\gamma\to 0^+} \top(\gamma) = 0 \tag{230}$$

$$\top'^*_0 = \lim_{\gamma\to 0^+} \top'(\gamma) = 0 \tag{231}$$

and by (14),

$$\top^{1\downarrow}_* = \inf\{\gamma : \top(\gamma) = 1\} = \inf\varnothing = 1 \tag{232}$$

$$\top'^{1\downarrow}_* = \inf\{\gamma : \top'(\gamma) = 1\} = \inf(\tfrac{1}{2}, 1] = \tfrac{1}{2}\,. \tag{233}$$

Clearly $\top \leq \top'$ and $d(\top,\top') = \frac{\delta}{2} < \delta$. We compute

$$\begin{aligned}
\xi_A(\top,\mathrm{c}_0) &= \max(\top^*_0, \tfrac{1}{2}-\tfrac{1}{2}\top^{1\downarrow}_*) && \text{by Def. 51}\\
&= \max(0, \tfrac{1}{2}-\tfrac{1}{2}\cdot 1) && \text{by (230), (232)}\\
&= 0
\end{aligned}$$

and

$$\begin{aligned} \xi_A(\top', \mathrm{c}_0) &= \max(\top'^*_0, \tfrac{1}{2} - \tfrac{1}{2}\top'^{1\downarrow}_*) && \text{by Def. 51} \\ &= \max(0, \tfrac{1}{2} - \tfrac{1}{2} \cdot \tfrac{1}{2}) && \text{by (231), (233)} \\ &= \tfrac{1}{4}\,. \end{aligned}$$

Therefore $\xi_A(\top', \mathrm{c}_0) - \xi_A(\top, \mathrm{c}_0) = \frac{1}{4} - 0 = \frac{1}{4} = \varepsilon$. We conclude from Th-47 that $\mathcal{F}_A$ is not Q-continuous.

B.32 Proof of Theorem 51

Lemma 77 *Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a given semi-fuzzy quantifier. Further let $\delta > 0$ and $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. Then for all $\gamma, \gamma' \in \mathbf{I}$ with $\gamma' \geq \gamma + 2\delta$,*

a. $\min(\top_{Q,X_1,\ldots,X_n}(\gamma'), \top_{Q,X'_1,\ldots,X'_n}(\gamma')) \geq \max(\top_{Q,X_1,\ldots,X_n}(\gamma), \top_{Q,X'_1,\ldots,X'_n}(\gamma))$*;*

b. $\max(\bot_{Q,X_1,\ldots,X_n}(\gamma'), \bot_{Q,X'_1,\ldots,X'_n}(\gamma')) \leq \min(\bot_{Q,X_1,\ldots,X_n}(\gamma), \bot_{Q,X'_1,\ldots,X'_n}(\gamma))$*.*

Proof See [7, L-112, p.262].

Lemma 78 *Condition a. of Th-51 is entailed by condition b.*

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping such that for all $(\top, \bot) \in \mathbb{T}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon$$

whenever $d'((\top, \bot), (\top', \bot')) < \delta$. We have to prove that $\mathcal{F}_\xi$ is arg-continuous. Hence let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ be given. We have to show that there exists $\delta > 0$ such that

$$|\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) - \mathcal{F}_\xi(Q)(X'_1, \ldots, X'_n)| < \varepsilon$$

whenever $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. By the assumption on ξ, there exists $\delta' > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot')| < \varepsilon \tag{234}$$

whenever $d'((\top, \bot), (\top', \bot')) < \delta'$. We choose $\delta = \frac{\delta'}{3}$. Now let $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ be a choice of fuzzy subsets with $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$. Then

$$\begin{aligned} &d'(\top_{Q,X_1,\ldots,X_n}, \top_{Q,X'_1,\ldots,X'_n}) \leq 2\delta && \text{by L-77 and (31)} \\ &d'(\bot_{Q,X_1,\ldots,X_n}, \bot_{Q,X'_1,\ldots,X'_n}) \leq 2\delta\,, && \text{by L-77 and (33)} \end{aligned}$$

i.e.

$$d'((\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}), (\top_{Q,X'_1,\dots,X'_n}, \bot_{Q,X'_1,\dots,X'_n})) \le 2\delta = \tfrac{2}{3}\delta' < \delta' \tag{235}$$

by (33). Therefore

$$\begin{aligned} &|\mathcal{F}_\xi(Q)(X_1,\dots,X_n) - \mathcal{F}_\xi(Q)(X'_1,\dots,X'_n)| \\ &= |\xi(\top_{Q,X_1,\dots,X_n}, \bot_{Q,X_1,\dots,X_n}) - \xi(\top_{Q,X'_1,\dots,X'_n}, \bot_{Q,X'_1,\dots,X'_n})| && \text{by Def. 45} \\ &< \varepsilon\,, && \text{by (234), (235)} \end{aligned}$$

i.e. $\mathcal{F}_\xi$ is arg-continuous, which we intended to show.

Let us now prove that the condition on ξ is also necessary for $\mathcal{F}_\xi$ to be arg-continuous. To this end, we need to construct a semi-fuzzy quantifier with special properties.

Lemma 79 *Let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given. We abbreviate $d = d'(\top, \top')$, $d' = d'(\bot, \bot')$. Then*

a. for all $\beta > d$ and $\gamma \ge \beta$, $\top(\gamma) \ge \top'(\gamma - \beta)$.

b. for all $\beta' > d'$ and $\gamma \le 1 - \beta'$, $\bot(\gamma) \ge \bot'(\gamma + \beta')$.

Proof

a. We first recall (31), viz

$$d = d'(\top, \top') = \sup\{\inf\{\gamma' : \min(\top(\gamma'), \top'(\gamma')) \ge \max(\top(\gamma), \top'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\}$$

This reveals that for the given $\gamma - \beta \ge 0$,

$$\inf\{\gamma' : \min(\top(\gamma'), \top'(\gamma')) \ge \max(\top(\gamma - \beta), \top'(\gamma - \beta))\} - (\gamma - \beta) \le d\,.$$

Because $\top, \top'$ are nondecreasing, this entails that

$$\min(\top(\gamma'), \top'(\gamma')) \ge \max(\top(\gamma - \eta d), \top'(\gamma - \eta d)) \tag{236}$$

for all $\gamma' > \gamma - \beta + d$. We now recall that $\beta > d$, i.e. $\gamma = \gamma - \beta + \beta > \gamma - \beta + d$. Hence γ is an admissable choice for γ' in (236), and

$$\begin{aligned} \top(\gamma) &\ge \min(\top(\gamma), \top'(\gamma)) \\ &\ge \max(\top(\gamma - \beta), \top'(\gamma - \beta)) && \text{by (236)} \\ &\ge \top'(\gamma - \beta)\,, \end{aligned}$$

as desired.

b. Analogous. This time we recall (32):

$d' = d'(\bot, \bot') = \sup\{\inf\{\gamma' : \max(\bot(\gamma'), \bot'(\gamma')) \le \min(\bot(\gamma), \bot'(\gamma))\} - \gamma : \gamma \in \mathbf{I}\}$

Hence for $\gamma \le 1 - \beta'$,

$$\inf\{\gamma' : \max(\bot(\gamma'), \bot'(\gamma')) \le \min(\bot(\gamma), \bot'(\gamma))\} - \gamma \le d' \,.$$

Because $\bot, \bot'$ are nonincreasing, this entails that

$$\max(\bot(\gamma'), \bot'(\gamma')) \le \min(\bot(\gamma), \bot'(\gamma))\} \tag{237}$$

for all $\gamma' > \gamma + d'$. We now recall that $\beta' > d'$, i.e. $\gamma + \beta' > \gamma + d'$. We also have $\gamma + \beta' \le 1$. Hence $\gamma + \beta'$ is an admissable choice for γ' in (237), and

$$\begin{aligned} \bot(\gamma) &\ge \min(\bot(\gamma), \bot'(\gamma)) \\ &\ge \max(\bot(\gamma + \beta'), \bot'(\gamma + \beta')) && \text{by (237)} \\ &\ge \bot'(\gamma + \beta') \,. \end{aligned}$$

Lemma 80 *Let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given.*

a. for all $\beta \ge d'(\top, \top')$ and $\gamma \in (0, 1]$ with $\gamma \ge \beta$, $\top^\flat(\gamma) \ge \top'^\flat(\gamma - \beta)$.

b. for all $\beta' \ge d'(\bot, \bot')$ and $\gamma \in [0, 1)$ with $\gamma \le 1 - \beta'$, $\bot^\sharp(\gamma) \ge \bot'^\sharp(\gamma + \beta')$.

Proof Let $(\top, \bot), (\top', \bot') \in \mathbb{T}$ be given. We abbreviate $d = d'(\top, \top')$, $d' = d'(\bot, \bot')$.

a. Suppose $\beta \ge d$ and $\gamma > 0$ with $\gamma \ge \beta$. Then

$$\top'^\flat(\gamma - \beta) = \lim_{\gamma' \to (\gamma - \beta)^-} \top'(\gamma') \,, \qquad \text{by Def. 34}$$

i.e.

$$\top'^\flat(\gamma - \beta) = \sup\{\top'(\gamma') : \gamma' < \gamma - \beta\} \tag{238}$$

because $\top'$ is nondecreasing, see [7, Th-43.a, p. 44].
Now consider $\varepsilon > 0$. By (238), there exists $\gamma' < \gamma - \beta$ with

$$\top'(\gamma') > \top'^\flat(\gamma - \beta) - \varepsilon \,. \tag{239}$$

Because $\gamma' < \gamma - \beta$, we can choose $\gamma'' \in \mathbf{I}$ with

$$\gamma' < \gamma'' < \gamma - \beta \tag{240}$$

Then $\gamma'' + d < \gamma$ and

$$\begin{aligned} \top(\gamma'' + d) &\geq \top'(\gamma') && \text{by L-79 because } \gamma' < (\gamma'' + d) - d \\ &> \top'^{\flat}(\gamma - \beta) - \varepsilon && \text{by (239)}. \end{aligned}$$

Because $\gamma'' + d < \gamma$, we hence obtain that

$$\begin{aligned} \top^{\flat}(\gamma) &= \lim_{\gamma' \to \gamma^-} \top(\gamma') && \text{by Def. 34} \\ &= \sup\{\top(\gamma') : \gamma' < \gamma\} && \text{because } \top \text{ nondecreasing, see [7, Th-43.a, p. 44]} \\ &\geq \top(\gamma'' + d) && \text{because } \gamma'' + d < \gamma \\ &> \top'^{\flat}(\gamma - \beta) - \varepsilon\,. \end{aligned}$$

Because $\varepsilon > 0$ was arbitrarily chosen, this proves that $\top^{\flat}(\gamma) > \top'^{\flat}(\gamma - \beta)$, as desired.

b. Analogous to that of **a.** Suppose $\beta' \geq d'$ and $\gamma < 1$ with $\gamma \leq 1 - \beta'$ are given. Because $\gamma + \beta' < 1$, we obtain from Def. 34 that

$$\perp'^{\sharp}(\gamma + \beta') = \lim_{\gamma' \to (\gamma + \beta')^+} \perp'(\gamma')\,.$$

Hence

$$\perp'^{\sharp}(\gamma + \beta') = \sup\{\perp'(\gamma') : \gamma' > \gamma\} \tag{241}$$

because $\perp'$ is nonincreasing; see [7, Th-43.d, p. 45].
Now consider $\varepsilon > 0$. By (241), there exists $\gamma' > \gamma + \beta'$ such that

$$\perp'(\gamma') > \perp'^{\sharp}(\gamma + \beta') - \varepsilon\,. \tag{242}$$

Because $\gamma' > \gamma + \beta'$, we can choose $\gamma'' \in \mathbf{I}$ with $\gamma + \beta' < \gamma'' < \gamma'$. In particular,

$$\gamma'' - \beta' > \gamma + \beta' - \beta' = \gamma \tag{243}$$

and

$$\gamma' > \gamma'' = (\gamma'' - \beta') + \beta' \geq (\gamma'' - \beta') + d'\,. \tag{244}$$

Therefore

$$\begin{aligned} \perp^{\sharp}(\gamma) &= \lim_{\gamma' \to \gamma^+} \perp(\gamma') && \text{by Def. 34} \\ &= \sup\{\perp(\gamma') : \gamma' > \gamma\} && \text{by [7, Th-43.d, p. 45]} \\ &\geq \perp(\gamma'' - \beta') && \text{by (243)} \\ &\geq \perp(\gamma') && \text{by (244), L-79.b} \\ &> \perp'^{\sharp}(\gamma + \beta) - \varepsilon\,. && \text{by (242)} \end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, this proves the desired $\perp^{\sharp}(\gamma) \geq \perp'^{\sharp}(\gamma + \varepsilon)$.

Lemma 81 *For all* $(\top, \bot) \in \mathbb{T}$*, we define* $Q_{(\top,\bot)} : \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *by*

$$Q_{(\top,\bot)}(Y) = \begin{cases} \top^\flat(u_{(\top,\bot)}) & : & u_{(\top,\bot)} > 0 \\ \bot^\sharp(\ell_{(\top,\bot)}) & : & u_{(\top,\bot)} = 0, \ell_{(\top,\bot)} < 1 \\ 0 & : & u_{(\top,\bot)} = 0, \ell_{(\top,\bot)} = 1 \end{cases} \tag{245}$$

where

$$u_{(\top,\bot)} = \sup Y'_{(\top,\bot)} \tag{246}$$
$$\ell_{(\top,\bot)} = \inf Y''_{(\top,\bot)} \tag{247}$$
$$Y'_{(\top,\bot)} = \{z \in \mathbf{I} : ((\top,\bot), 0, z) \in Y\} \tag{248}$$
$$Y''_{(\top,\bot)} = \{z \in \mathbf{I} : ((\top,\bot), 1, z) \in Y\}\,, \tag{249}$$

for all $Y \in \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$.
Further suppose that the semi-fuzzy quantifier $Q : \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$Q(Y) = \sup\{Q_{(\top,\bot)}(Y) : (\top,\bot) \in \mathbb{T}\} \tag{250}$$

for all $Y \in \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$*. Then for every choice of* $(\top_1, \bot_1), (\top_2, \bot_2) \in \mathbb{T}$ *with*

$$\top_2(0) \geq \bot_1(0)\,, \tag{251}$$

it holds that

$$\top_{Q,X}(\gamma) = {\top_2}^\flat(\gamma)$$

and

$$\bot_{Q,X}(\gamma) = {\bot_2}^\sharp(\gamma)\,,$$

for all $\gamma \in (0,1)$*, provided we define* $X \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ *by*

$$\begin{aligned} &\mu_X((\top,\bot), c, z) \\ &= \begin{cases} \frac{1}{2} - \frac{1}{2}\min(z + d_2, 1) & : & c = 0 \\ \frac{1}{2} - \frac{1}{2}d_1 & : & c = 1 \wedge z < d_1 \wedge (\top,\bot) \neq (\top_2,\bot_2) \\ \frac{1}{2} + \frac{1}{2}z - \frac{1}{2}d_2 & : & c = 1 \wedge (z \geq d_1 \vee (\top,\bot) = (\top_2,\bot_2)) \end{cases} \end{aligned} \tag{252}$$

for all $(\top,\bot) \in \mathbb{T}$*,* $c \in \mathbf{2}$ *and* $z \in \mathbf{I}$*, where* $d_1 = d'((\top,\bot),(\top_1,\bot_1))$ *and* $d_2 = d'((\top,\bot),(\top_2,\bot_2))$.

Proof We first consider some monotonicity properties. Let $(\top,\bot) \in \mathbb{T}$ be given. If Y increases, then $Y'_{(\top,\bot)}$ increases. Hence $u_{(\top,\bot)} = \sup Y'_{(\top,\bot)}$ increases as well. Finally because $\top$ and hence $\top^\flat$ are nondecreasing, $\top^\flat(u_{(\top,\bot)})$ will also increase. Hence $Q'(Y) = \top^\flat(u_{(\top,\bot)})$ is nondecreasing.
Similarly $Y''_{(\top,\bot)}$ increases if Y increases. Hence $\ell_{(\top,\bot)} = \inf Y''_{(\top,\bot)}$ decreases and

$\perp^\sharp(\ell_{(\top,\perp)})$ increases, because $\perp$, and hence $\perp^\sharp$, is nonincreasing. Therefore $Q''(Y) = \perp^\sharp(\ell_{(\top,\perp)})$ is nondecreasing. From Th-20 we know that $0 \leq \perp^\sharp(\gamma) \leq \top^\flat(\gamma')$ for all $\gamma, \gamma' \in \mathbf{I}$. It is hence apparent from (245) and the fact that $Q'(Y) = \top^\flat(u_{(\top,\perp)})$ and $Q''(Y) = \perp^\sharp(\ell_{(\top,\perp)})$ are nondecreasing that $Q_{(\top,\perp)}(Y)$ is also nondecreasing in Y. We can hence utilize L-3 in order to simplify the expressions for $\top_{Q,X}$ and $\perp_{Q,X}$.
Now suppose that $(\top_1, \perp_1), (\top_2, \perp_2) \in \mathbb{T}$ satisfy (251) and assume $X \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ is defined by (252). Then by Def. 30, Def. 28 and (252),

$$\begin{aligned}
&X^{\min}_\gamma \\
&= X_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \\
&= \{((\top,\perp),1,z) : \tfrac{1}{2} + \tfrac{1}{2}z - \tfrac{1}{2}d_2 \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma \wedge (z \geq d_1 \vee (\top,\perp) = (\top_2,\perp_2))\} \\
&= \{((\top,\perp),1,z) : z - d_2 \geq \gamma \wedge (z \geq d_1 \vee (\top,\perp) = (\top_2,\perp_2))\} \\
&= \{((\top,\perp),1,z) : z \geq \max(\gamma + d_2, d_1)\} \cup \{((\top_2,\perp_2),1,z) : z \geq \gamma + d_2\}\,.
\end{aligned}$$

for all $\gamma \in (0,1)$. Hence in the case of $X^{\min}_\gamma$,

$$u_{(\top,\perp)} = \sup Y'_{(\top,\perp)} = \sup \varnothing = 0 \tag{253}$$

and for $(\top,\perp) \neq (\top_2,\perp_2)$,

$$\begin{aligned}
\ell_{(\top,\perp)} &= \inf Y''_{(\top,\perp)} && \text{by (247)} \\
&= \inf\{z \in \mathbf{I} : z \geq \max(\gamma + d_2, d_1)\} && \text{by (249)} \\
&= \min(\max(\gamma + d_2, d_1), 1)\,,
\end{aligned}$$

while for $(\top,\perp) = (\top_2,\perp_2)$, $d_2 = d'((\top_2,\perp_2),(\top_2,\perp_2)) = 0$ and hence

$$\begin{aligned}
&\ell_{(\top_2,\perp_2)} \\
&= \inf Y''_{(\top_2,\perp_2)} && \text{by (247)} \\
&= \inf(\{z \in \mathbf{I} : z \geq \max(\gamma + d_2, d_1)\} \\
&\qquad \cup \{z \in \mathbf{I} : z \geq \gamma + d_2\}) && \text{by (249)} \\
&= \min(\inf\{\{z \in \mathbf{I} : z \geq \max(\gamma, d_1)\}, \\
&\qquad \inf\{z \in \mathbf{I} : z \geq \gamma\}) && \text{because } d_2 = 0 \\
&= \min(\max(\gamma, d_1), \gamma) \\
&= \gamma\,. && \text{by absorption}
\end{aligned}$$

Therefore

$$\ell_{(\top,\perp)} = \begin{cases} \min(\max(\gamma + d_2, d_1), 1) & : \quad (\top,\perp) \neq (\top_2,\perp_2) \\ \gamma & : \quad (\top,\perp) = (\top_2,\perp_2) \end{cases} \tag{254}$$

for all $(\top, \bot) \in \mathbb{T}$. Concerning $X_\gamma^{\max}$, we obtain from Def. 30, Def. 29 and (252) that

$$\begin{aligned}
&X_\gamma^{\max} \\
&= X_{> \frac{1}{2} - \frac{1}{2}\gamma} \\
&= \{((\top, \bot), 0, z) : \tfrac{1}{2} - \tfrac{1}{2}z - \tfrac{1}{2}d_2 > \tfrac{1}{2} - \tfrac{1}{2}\gamma\} \\
&\qquad \cup \{((\top, \bot), 1, z) : \tfrac{1}{2} - \tfrac{1}{2}d_1 > \tfrac{1}{2} - \tfrac{1}{2}\gamma \wedge z < d_1 \wedge (\top, \bot) \neq (\top_2, \bot_2)\} \\
&\qquad \cup \{((\top, \bot), 1, z) : z \geq d_1 \wedge z > d_2 - \gamma\} \\
&\qquad \cup \{((\top_2, \bot_2), 1, z) : z \in \mathbf{I}\} \\
&= \{((\top, \bot), 0, z) : z < \gamma - d_2\} \\
&\qquad \cup \{((\top, \bot), 1, z) : z < d_1 < \gamma \wedge (\top, \bot) \neq (\top_2, \bot_2)\} \\
&\qquad \cup \{((\top, \bot), 1, z) : z \geq d_1 \wedge z > d_2 - \gamma\} \\
&\qquad \cup \{((\top_2, \bot_2), 1, z) : z \in \mathbf{I}\}\,,
\end{aligned}$$

for all $\gamma \in (0, 1]$. Hence in the case of $X_\gamma^{\max}$,

$$u_{(\top,\bot)} = \sup Y'_{(\top,\bot)} = \sup\{z \in \mathbf{I} : z < \gamma - d_2\} = \max(\gamma - d_2, 0) \tag{255}$$

for all $(\top, \bot) \in \mathbb{T}$. As concerns $\ell_{(\top,\bot)}$, we first observe that for $(\top, \bot) \neq (\top_2, \bot_2)$,

$$\begin{aligned}
\ell_{(\top,\bot)} &= \inf Y''_{(\top,\bot)} && \text{by (249)} \\
&= \inf(\{z \in \mathbf{I} : z < d_1 < \gamma\} \\
&\qquad\quad \cup \{z \in \mathbf{I} : z \geq d_1 \wedge z > d_2 - \gamma\})\,.
\end{aligned}$$

Hence in the case that $\gamma \leq d_1$,

$$\begin{aligned}
\ell_{(\top,\bot)} &= \inf\{z \in \mathbf{I} : z \geq d_1 \wedge z > d_2 - \gamma\} \\
&= \max(d_1, d_2 - \gamma)
\end{aligned}$$

and if $\gamma > d_1$, then

$$\begin{aligned}
\ell_{(\top,\bot)} &= \inf(\{z \in \mathbf{I} : z < d_1\} \\
&\qquad\quad \cup \{z \in \mathbf{I} : z \geq d_1 \wedge z > d_2 - \gamma\}) \\
&= \begin{cases} 0 & : \quad d_1 > 0 \\ \max(0, d_2 - \gamma) & : \quad d_1 = 0 \end{cases}
\end{aligned}$$

In the remaining case that $(\top, \bot) = (\top_2, \bot_2)$, we again have $d_2 = 0$ and hence

$$\begin{aligned}
\ell_{(\top_2,\bot_2)} &= \inf Y''_{(\top_2,\bot_2)} \\
&= \inf(\{z \in \mathbf{I} : z \geq d_1\} \cup \mathbf{I}) \\
&= \inf \mathbf{I} \\
&= 0\,.
\end{aligned}$$

Summarizing, we have shown that for $X_\gamma^{\max}$,

$$\ell_{(\top,\bot)} = \begin{cases} \max(d_1, d_2 - \gamma) & : \quad (\gamma \leq d_1 \vee d_1 = 0) \wedge (\top, \bot) \neq (\top_2, \bot_2) \\ 0 & : \quad \text{else} \end{cases} \tag{256}$$

for all $(\top, \bot) \in \mathbb{T}$.
Now we consider the quantification results that are obtained for $\gamma \in (0, 1)$. As regards $Q(X_\gamma^{\min})$, we first recall that $u_{(\top,\bot)} = 0$ by (253). In order to determine the result of $Q(X_\gamma^{\min})$, we discern three cases.

1. $(\top, \bot) \neq (\top_2, \bot_2)$ and $\max(\gamma+d_2, d_1) < 1$. Then by (254), $\ell_{(\top,\bot)} = \max(\gamma+d_2, d_1) < 1$. Hence by (252),

$$\begin{aligned} Q_{(\top,\bot)}(X_\gamma^{\min}) &= \bot^\sharp(\max(\gamma + d_2, d_1)) && \text{by (245)} \\ &\leq \bot^\sharp(\gamma + d_2) && \text{because } \bot \text{ nonincreasing} \\ &\leq \bot_2^\sharp(\gamma)\,, && \text{by L-80.b} \end{aligned}$$

   i.e.

$$Q_{(\top,\bot)}(X_\gamma^{\min}) \leq \bot_2^\sharp(\gamma)\,. \tag{257}$$

2. $(\top, \bot) \neq (\top_2, \bot_2)$ and $\max(\gamma + d_2, d_1) = 1$. Then $\ell_{(\top,\bot)} = 1$ by (254). We hence obtain from (245) that

$$Q_{(\top,\bot)}(X_\gamma^{\min}) = 0 \leq \bot_2^\sharp(\gamma)\,. \tag{258}$$

3. $(\top, \bot) = (\top_2, \bot_2)$. In this case, $\ell_{(\top,\bot)} = \gamma$ by (254). In addition, $\gamma < 1$ by assumption. We then obtain from (245) that

$$Q_{(\top_2,\bot_2)}(X_\gamma^{\min}) = \bot_2^\sharp(\gamma)\,. \tag{259}$$

Therefore $Q_{(\top,\bot)}(X_\gamma^{\min}) \leq \bot_2^\sharp(\gamma)$ for all $(\top, \bot) \in \mathbb{T}$ and for $(\top_2, \bot_2) \in \mathbb{T}$,

$$Q_{(\top_2,\bot_2)}(X_\gamma^{\min}) = \bot_2^\sharp(\gamma)\,.$$

Hence

$$Q(X_\gamma^{\min}) = \sup\{Q_{(\top,\bot)}(X_\gamma^{\min}) : (\top, \bot) \in \mathbb{T}\} = \bot_2^\sharp(\gamma) \tag{260}$$

by (250) and (257)–(259).
Next we consider $Q(\gamma_)^{\max}$.

1. $(\top, \bot) = (\top_2, \bot_2)$. Then $d_2 = d'((\top_2, \bot_2), (\top_2, \bot_2)) = 0$ and hence $\gamma - d_2 > 0$, because we have assumed that $\gamma > 0$. By (255), $u_{(\top_2,\bot_2)} = \gamma > 0$. Therefore

$$Q_{(\top_2,\bot_2)}(X_\gamma^{\max}) = \top_2^\flat(\gamma) \tag{261}$$

   by (245).

2. $(\top, \bot) \neq (\top_2, \bot_2)$ and $\gamma - d_2 > 0$. Then $u_{(\top,\bot)} = \gamma - d_2 > 0$ and

$$\begin{aligned} Q_{(\top,\bot)}(X_\gamma^{\max}) &= \top^\flat(\gamma - d_2) && \text{by (245)} \\ &\leq \top_2^\flat(\gamma)\,, && \text{by L-80.a} \end{aligned}$$

   i.e.

$$Q_{(\top,\bot)}(X_\gamma^{\max}) \leq \top_2^\flat(\gamma)\,. \tag{262}$$

3. $(\top, \bot) \neq (\top_2, \bot_2)$, $\gamma - d_2 = 0$ and $\gamma \leq d_1$. Then $u_{(\top,\bot)} = 0$ by (255) and $\ell_{(\top,\bot)} = \max(d_1, d_2 - \gamma)$ by (256). If $\ell_{(\top,\bot)} < 1$, then we obtain from $\ell_{(\top,\bot)} \geq d_1$ that

$$\begin{aligned} Q_{(\top,\bot)}(X_\gamma^{\max}) &= \bot^\sharp(\ell_{(\top,\bot)}) && \text{by (245)} \\ &\leq \bot^\sharp(d_1) && \text{because } \bot \text{ nonincreasing} \\ &\leq \bot_1^\sharp(0) && \text{by L-80.b} \\ &\leq \top_2^\flat(0) && \text{by (251) L-19 and } \top_2(0) = \top_2^\flat(0) \text{ by Def. 34} \\ &\leq \top_2^\flat(\gamma)\,, && \text{because } \top_2 \text{ nondecreasing} \end{aligned}$$

i.e.

$$Q(X_\gamma^{\max}) \leq \top_2^\flat(\gamma)\,. \tag{263}$$

If $\ell_{(\top,\bot)} = 1$, then

$$Q(X_\gamma^{\max}) = 0 \leq \top_2^\flat(\gamma) \tag{264}$$

by (245).

4. $(\top, \bot) \neq (\top_2, \bot_2)$, $\gamma - d_2 = 0$ and $d_1 = 0$. Then $u_{(\top,\bot)} = 0$ by (255) and $\ell_{(\top,\bot)} = \max(d_1, d_2 - \gamma) = \max(0, 0) = 0$ by (256). Hence

$$\begin{aligned} Q_{(\top,\bot)}(X_\gamma^{\max}) &= \bot^\sharp(0) \\ &\leq \bot_1^\sharp(0) && \text{by L-80.b because } d_1 = 0 \\ &\leq \top_2^\flat(0) && \text{by (251) L-19 and } \top_2(0) = \top_2^\flat(0) \text{ by Def. 34} \\ &\leq \top_2^\flat(\gamma)\,, && \text{because } \top_2^\flat \text{ nondecreasing} \end{aligned}$$

i.e.

$$Q_{(\top,\bot)}(X_\gamma^{\max}) \leq \top_2^\flat(0)\,. \tag{265}$$

5. $(\top, \bot) \neq (\top_2, \bot_2)$, $\gamma - d_2 = 0$, $\gamma > d_1$ and $d_1 > 0$. Then $u_{(\top,\bot)} = 0$ by (255) and $\ell_{(\top,\bot)} = 0$ by (256). Because again $d_1 = 0$ and $u_{(\top,\bot)} = \ell_{(\top,\bot)} = 0$, we obtain by the same reasoning as in the previous case that

$$Q_{(\top,\bot)}(X_\gamma^{\max}) \leq \top_2^\flat(\gamma)\,. \tag{266}$$

Hence $Q_{(\top,\bot)}(X_\gamma^{\max}) \leq \top_2^\flat(\gamma)$ for all $(\top, \bot) \in \mathbb{T}$ and for $(\top_2, \bot_2) \in \mathbb{T}$,

$$Q_{(\top_2,\bot_2)}(X_\gamma^{\max}) = \top_2^\flat(\gamma)\,.$$

Consequently

$$Q(X_\gamma^{\max}) = \sup\{Q_{(\top,\bot)}(X_\gamma^{\max}) : (\top, \bot) \in \mathbb{T}\} = \top_2^\flat(\gamma) \tag{267}$$

by (250) and (261)–(266). This proves that

$$\begin{aligned}\top_{Q,X}(\gamma) &= Q(X_\gamma^{\max}) && \text{by L-3}\\ &= \top_2^\flat(\gamma)\end{aligned}$$

and

$$\begin{aligned}\bot_{Q,X}(\gamma) &= Q(X_\gamma^{\min}) && \text{by L-3}\\ &= \bot_2^\sharp(\gamma)\end{aligned}$$

for all $\gamma \in (0,1)$, as desired.

Lemma 82 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *satisfies* (X-2), (X-4) *and* (X-1). *If* $\mathcal{F}_\xi$ *is arg-continuous, then* ξ *has the following property: for all* $(\top_1, \bot_1) \in \mathbb{T}$ *and all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $|\xi(\top_1,\bot_1) - \xi(\top_2,\bot_2)| < \varepsilon$ *whenever* $(\top_2,\bot_2) \in \mathbb{T}$ *satisfies* $d'((\top_1,\bot_1),(\top_2,\bot_2)) < \delta$ *and* $\top_2(0) \geq \bot_1(0)$.

Proof The following proof is by contraposition. Hence assume that the condition stated in the lemma fails, i.e. there exists $(\top_1,\bot_1) \in \mathbb{T}$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exists $(\top_2,\bot_2) \in \mathbb{T}$ with $d'((\top_1,\bot_1),(\top_2,\bot_2)) < \delta$, $\top_2(0) \geq \bot_1(0)$ and

$$|\xi(\top_1,\bot_1) - \xi(\top_2,\bot_2)| \geq \varepsilon \tag{268}$$

We have to show that $\mathcal{F}_\xi$ is not arg-continuous. To see this, consider the quantifier $Q : \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ defined by (250) and the fuzzy argument set $X \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ defined by

$$\begin{aligned}&\mu_X((\top,\bot),c,z)\\ &= \begin{cases} \frac{1}{2} - \frac{1}{2}\min(z + d_1, 1) & : \quad c = 0\\ \frac{1}{2} - \frac{1}{2}d_1 & : \quad c = 1 \wedge z < d_1 \wedge \\ \frac{1}{2} + \frac{1}{2}z - \frac{1}{2}d_1 & : \quad c = 1 \wedge z \geq d_1 \end{cases}\end{aligned} \tag{269}$$

for all $(\top,\bot) \in \mathbb{T}$, $c \in \mathbf{2}$ and $z \in \mathbf{I}$, where I have abbreviated $d_1 = d'((\top,\bot),(\top_1,\bot_1))$. Then by L-81,

$$\top_{Q,X}(\gamma) = \top_1^\flat(\gamma) \tag{270}$$

$$\bot_{Q,X}(\gamma) = \bot_1^\sharp(\gamma) \tag{271}$$

for all $\gamma \in (0,1)$.
Now consider $\delta > 0$. By the assumed property of ξ, there exists $(\top_2,\bot_2) \in \mathbb{T}$ with $\top_2(0) \geq \bot_1(0)$,

$$d'((\top_1,\bot_1),(\top_2,\bot_2)) < \tfrac{\delta}{2} \tag{272}$$

and

$$|\xi(\top_1,\bot_1) - \xi(\top_2,\bot_2)| \geq \varepsilon\,. \tag{273}$$

We now define the fuzzy set $X' \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ by

$$\mu_{X'}((\top, \bot), c, z) = \begin{cases} \frac{1}{2} - \frac{1}{2}\min(z + d_2, 1) & : \quad c = 0 \\ \frac{1}{2} - \frac{1}{2} d_1 & : \quad c = 1 \wedge z < d_1 \wedge (\top, \bot) \neq (\top_2, \bot_2) \\ \frac{1}{2} + \frac{1}{2} z - \frac{1}{2} d_2 & : \quad c = 1 \wedge (z \geq d_1 \vee (\top, \bot) = (\top_2, \bot_2)) \end{cases} \tag{274}$$

for all $(\top, \bot) \in \mathbb{T}$, $c \in \mathbf{2}$ and $z \in \mathbf{I}$, where $d_1 = d'((\top, \bot), (\top_1, \bot_1))$ and $d_2 = d'((\top, \bot), (\top_2, \bot_2))$. Again from L-81, we obtain

$$\top_{Q,X'}(\gamma) = \top_2^\flat(\gamma) \tag{275}$$

$$\bot_{Q,X'}(\gamma) = \bot_2^\sharp(\gamma) \tag{276}$$

for all $\gamma \in (0, 1)$. Hence

$$\begin{aligned} &|\mathcal{F}_\xi(Q)(X) - \mathcal{F}_\xi(Q)(X')| \\ &= |\xi(\top_{Q,X}, \bot_{Q,X}) - \xi(\top_{Q,X'}, \bot_{Q,X'})| && \text{by Def. 45} \\ &= |\xi(\top_1^\flat, \bot_1^\sharp) - \xi(\top_2^\flat, \bot_2^\sharp)| && \text{by L-23} \\ &= |\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| && \text{by (270), (271), (275), (276)} \end{aligned}$$

i.e. by (273),

$$|\mathcal{F}_\xi(Q)(X) - \mathcal{F}_\xi(Q)(X')| \geq \varepsilon\,, \tag{277}$$

by the assumed choice of $(\top_2, \bot_2) \in \mathbb{T}$. Let us now consider the distance $d(X, X')$ of the fuzzy argument sets. Hence let $(\top, \bot) \in \mathbb{T}$, $c \in \{0, 1\}$ and $z \in \mathbf{I}$. In order to shorten the proof, I abbreviate $d_u = \max(d_1, d_2)$, $d_\ell = \min(d_1, d_2)$ and $d = d'((\top_1, \bot_1), (\top_2, \bot_2))$. It is apparent from the triangle inequation that

$$d_u \leq d_\ell + d\,. \tag{278}$$

For example if $d_u = d_1$ and $d_\ell = d_2$, then

$$d_u = d'((\top, \bot), (\top_1, \bot_1)) \leq d'((\top, \bot), (\top_2, \bot_2)) + d'((\top_2, \bot_2), (\top_1, \bot_1)) = d_\ell + d\,.$$

In the following, we discern four cases.

1. $c = 0$. Then

$$\begin{aligned} &|\mu_{X'}((\top, \bot), c, z) - \mu_X((\top, \bot), c, z)| \\ &= |\tfrac{1}{2} - \tfrac{1}{2}\min(z + d_2, 1) - (\tfrac{1}{2} - \tfrac{1}{2}\min(z + d_1, 1))| && \text{by (269), (274)} \\ &= \tfrac{1}{2}|\min(z + d_1, 1) - \min(z + d_2, 1)| \\ &= \tfrac{1}{2}(\min(z + d_u, 1) - \min(z + d_\ell, 1))\,. \end{aligned}$$

If $d_\ell + z \geq 1$, then $d_u + z \geq 1$ as well and

$$\begin{aligned} &|\mu_{X'}((\top, \bot), c, z) - \mu_X((\top, \bot), c, z)| \\ &= \tfrac{1}{2}(\min(z + d_u, 1) - \min(z + d_\ell, 1)) \\ &= \tfrac{1}{2}(1 - 1) \\ &= 0\,. \end{aligned}$$

If $d_\ell + z < 1$, then $\min(d_\ell + z, 1) = d_\ell + z$ and hence

$$\begin{aligned}
&|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| \\
&= \tfrac{1}{2}(\min(z+d_u,1) - \min(z+d_\ell,1)) \\
&= \tfrac{1}{2}(\min(z+d_u,1) - z - d_\ell) \\
&\le \tfrac{1}{2}(z + d_u - z - d_\ell) \\
&\le \tfrac{1}{2}(z + d_\ell + d - z - d_\ell) && \text{by (278)} \\
&= \tfrac{1}{2}d\,.
\end{aligned}$$

2. $c = 1 \wedge z < d_1 \wedge (\top,\bot) \neq (\top_2,\bot_2)$. Then

$$\begin{aligned}
&|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| \\
&= |\tfrac{1}{2} - \tfrac{1}{2}d_1 - (\tfrac{1}{2} - \tfrac{1}{2}d_1)| && \text{by (269), (274)} \\
&= 0\,.
\end{aligned}$$

3. $c = 1 \wedge z < d_1 \wedge (\top,\bot) = (\top_2,\bot_2)$. Then $d_1 = d'((\top_2,\bot_2),(\top_1,\bot_1)) = d$ and $d_2 = d'((\top_2,\bot_2),(\top_2,\bot_2)) = 0$. In particular, $z < d$. Hence

$$\begin{aligned}
&|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| \\
&= |\tfrac{1}{2} - \tfrac{1}{2}d_1 - (\tfrac{1}{2} + \tfrac{1}{2}z)| && \text{by (269), (274)} \\
&= \tfrac{1}{2}|-d_1 - z| \\
&= \tfrac{1}{2}(d_1 + z) \\
&< \tfrac{1}{2}(d + d) && \text{because } d_1 = d \text{ and } z < d\text{, see above} \\
&= d
\end{aligned}$$

4. $c = 1 \wedge z \geq d_1$. Then

$$\begin{aligned}
&|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| \\
&= |\tfrac{1}{2} + \tfrac{1}{2}z - \tfrac{1}{2}d_1 - (\tfrac{1}{2} + \tfrac{1}{2}z - \tfrac{1}{2}d_2)| && \text{by (269), (274)} \\
&= \tfrac{1}{2}|d_2 - d_1| \\
&= \tfrac{1}{2}(d_u - d_\ell) \\
&\le \tfrac{1}{2}(d_\ell + d - d_\ell) && \text{by (278)} \\
&= \tfrac{1}{2}d\,.
\end{aligned}$$

Summarising, we obtained in any of the above cases that

$$|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| < d\,,$$

i.e.

$$|\mu_{X'}((\top,\bot),c,z) - \mu_X((\top,\bot),c,z)| < d = d'((\top_1,\bot_1),(\top_2,\bot_2)) < \tfrac{\delta}{2} \quad (279)$$

by (272). Therefore

$$\begin{aligned}
&d(X, X')\\
&= \sup\{|\mu_{X'}((\top, \bot), c, z) - \mu_X((\top, \bot), c, z)| : (\top, \bot) \in \mathbb{T}, c \in \mathbf{2}, z \in \mathbf{I}\} \quad \text{by (4)}\\
&\leq \tfrac{\delta}{2} \quad \text{by (279)}\\
&< \delta\,.
\end{aligned}$$

Combining this with (277), this proves that for the given δ, there exists X' with $d(X, X') < \delta$ and $|\mathcal{F}_\xi(Q)(X) - \mathcal{F}_\xi(Q)(X')| \geq \varepsilon$. Because $\delta > 0$ was arbitrary, we conclude that there exists $\varepsilon > 0$, $Q : \mathcal{P}(\mathbb{T} \times \mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ (viz, our above choices) such that for all $\delta > 0$, there exists $X' \in \widetilde{\mathcal{P}}(\mathbb{T} \times \mathbf{2} \times \mathbf{I})$ with $d(X, X') < \delta$ and $|\mathcal{F}_\xi(Q)(X) - \mathcal{F}_\xi(Q)(X') \geq \varepsilon$. Hence $\mathcal{F}_\xi$ is not arg-continuous by Def. 26.

Proof of Theorem 51

Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-2), (X-4) and (X-5). In order to prove that the conditions **a.** and **b.** are equivalent, we split the equivalence in two implications, which we prove separately.

b.⇒a.: This case is already covered by L-78, i.e. condition **b.** is sufficient for $\mathcal{F}_\xi$ to be arg-continous.

a.⇒b.: In order to prove that condition **b.** is also necessary for $\mathcal{F}_\xi$ to be arg-continuous, let us assume that $\mathcal{F}_\xi$ is arg-continuous. We have to show that **b.** holds. To this end, we first recall that by L-82, the following condition holds for ξ: for all $(\top_1, \bot_1) \in \mathbb{T}$ and all $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| < \varepsilon \tag{280}$$

whenever $(\top_2, \bot_2) \in \mathbb{T}$ satisfies

$$d'((\top_1, \bot_1), (\top_2, \bot_2)) < \delta_1 \tag{281}$$

and $\top_2(0) \geq \bot_1(0)$. Because ξ satisfies (X-2), this entails that for all $(\top_1, \bot_1) \in \mathbb{T}$, there exists $\delta_2 > 0$ such that

$$|\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| < \varepsilon \tag{282}$$

whenever $(\top_2, \bot_2) \in \mathbb{T}$ satisfies

$$d'((\top_1, \bot_1), (\top_2, \bot_2)) < \delta_2 \tag{283}$$

and $\bot_2(0) \leq \top_1(0)$. We insert the proof of this simple claim here. We already know from L-82 that there exists $\delta_2 > 0$ such that

$$|\xi(1 - \bot_1, 1 - \top_1) - \xi(\top_2', \bot_2')| < \varepsilon \tag{284}$$

whenever $(\top_2', \bot_2') \in \mathbb{T}$ satisfies

$$d'((1 - \bot_1, 1 - \top_1), (\top_2', \bot_2')) < \delta_2 \tag{285}$$

and $\top_2'(0) \geq 1 - \bot_1(0)$. Now consider $(\top_2, \bot_2) \in \mathbb{T}$ with

$$d'((\top_1, \bot_1), (\top_2, \bot_2)) < \delta_2 \tag{286}$$

and $\bot_2(0) \leq \top_1(0)$. Substituting $\top_2' = 1 - \bot_2$ and $\bot_2' = 1 - \top_2$, we have

$$\begin{aligned}
&d'((1 - \bot_1, 1 - \top_1), (\top_2', \bot_2')) \\
&= d'((1 - \bot_1, 1 - \top_1), (1 - \bot_2, 1 - \top_2)) \\
&= d'((\top_1, \bot_1), (\top_2, \bot_2)) && \text{apparent from (33)} \\
&< \delta_2 \,. && \text{by (286)}
\end{aligned}$$

Hence condition (285) is fulfilled. In addition, we clearly have $\top_2'(0) = 1 - \bot_2(0) \geq 1 - \top_1(0)$ because by assumption, $\bot_2(0) \leq \top_1(0)$. Hence we conclude from (284) that

$$\begin{aligned}
&|\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| \\
&= |1 - \xi(1 - \bot_1, 1 - \top_1) - (1 - \xi(1 - \bot_2, 1 - \top_2))| && \text{by (X-2)} \\
&= |\xi(1 - \bot_1, 1 - \top_1) - \xi(\top_2', \bot_2')| \\
&< \varepsilon \,. && \text{by (284)}
\end{aligned}$$

Hence (282) holds, i.e. the the second condition on ξ is also satisfied.
We can combine these conditions as follows. Let $(\top_1, \bot_1) \in (\top, \bot)$ and $\varepsilon > 0$ be given. Further let $\delta_1 > 0$ be chosen such that (280) holds, and let $\delta_2 > 0$ be chosen such that (282) holds. Now let $\delta = \min(\delta_1, \delta_2)$ and consider $(\top_2, \bot_2) \in \mathbb{T}$ with $d'((\top_1, \bot_1), (\top_2, \bot_2)) < \delta$. If $\top_2(0) \geq \bot_1(0)$, then $|\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| < \varepsilon$ by (280). In the remaining case that $\top_2(0) < \bot_1(0)$, we have $\bot_2(0) \leq \top_2(0) < \bot_1(0)$. Hence the second condition is applicable and by (282), $|\xi(\top_1, \bot_1) - \xi(\top_2, \bot_2)| < \varepsilon$. This finishes the proof that condition **b.** of the theorem holds whenever condition **a.** holds, as desired.

B.33 Proof of Theorem 52

Lemma 83 *Suppose* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *is a mapping such that* (X-2) *and the following condition are valid. For all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$|\xi(\top, \bot) - \xi(\top', \bot)| < \varepsilon$$

whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ *such that* $d'(\top, \top') < \delta$*. Then* $\mathcal{F}_\xi$ *is arg-continuous.*

Proof Let $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ be a given mapping with the above properties. Now let $\varepsilon > 0$. By assumption, there exists $\delta' > 0$ such that

$$|\xi(\top, \bot) - \xi(\top', \bot)| < \tfrac{\varepsilon}{2} \tag{287}$$

whenever $(\top,\bot),(\top',\bot)\in\mathbb{T}$ such that $d'(\top,\top')<\delta'$.
We now consider $(\top,\bot),(\top',\bot')\in\mathbb{T}$ with

$$d'((\top,\bot),(\top',\bot'))<\delta'\,. \tag{288}$$

We also note that by (288) and (33),

$$d'((\top,\bot),(\top',\bot))\leq d'((\top,\bot),(\top',\bot))<\delta' \tag{289}$$

$$\begin{aligned}d'((1-\bot,1-\top'),(1-\bot',1-\top'))&=d'((\top',\bot),(\top',\bot'))\\&\leq d'((\top,\bot),(\top',\bot'))<\delta'\end{aligned} \tag{290}$$

Therefore

$$\begin{aligned}&|\xi(\top',\bot)-\xi(\top',\bot')|\\&=|(1-\xi(1-\bot,1-\top'))-(1-\xi(1-\bot',1-\top'))| &&\text{by (X-2)}\\&=|\xi(1-\bot,1-\top')-\xi(1-\bot',1-\top')|\,,\end{aligned}$$

i.e.

$$|\xi(\top',\bot)-\xi(\top',\bot')|<\tfrac{\varepsilon}{2} \tag{291}$$

by (287) and (290). Finally

$$\begin{aligned}&|\xi(\top,\bot)-\xi(\top',\bot')|\\&\leq|\xi(\top,\bot)-\xi(\top',\bot)|+|\xi(\top',\bot)-\xi(\top',\bot')| &&\text{by triangle inequation}\\&<\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2} &&\text{by (287), (289) and (291)}\\&=\varepsilon\,.\end{aligned}$$

Hence for all $(\top,\bot)\in\mathbb{T}$ an all $\varepsilon>0$, there exists $\delta>0$ such that $|\xi(\top,\bot)-\xi(\top',\bot')|<\varepsilon$ whenever $(\top',\bot')\in\mathbb{T}$ satisfies $d'((\top,\bot),(\top',\bot'))<\delta$. Application of L-78 yields that $\mathcal{F}_\xi$ is arg-continuous.

Proof of Theorem 52

Let $\xi:\mathbb{T}\longrightarrow\mathbf{I}$ be a given mapping which satisfies (X-2) and (X-5) and also fulfills the condition imposed by the theorem. We will show that ξ fulfills the preconditions of lemma L-83. Hence let $\varepsilon>0$ be given. By assumption, there exists $\delta>0$ such that

$$\xi(\top^*,\bot)-\xi(\top_*,\bot)<\varepsilon \tag{292}$$

whenever $(\top_*,\bot),(\top^*,\bot)\in\mathbb{T}$ satisfy $d'(\top_*,\top^*)<\delta$ and $\top_*\leq\top^*$.
Now let $(\top,\bot),(\top',\bot)\in\mathbb{T}$ with $d'(\top,\top')<\delta$. We abbreviate

$$\begin{aligned}\top_*&=\min(\top,\top')\\\top^*&=\max(\top,\top')\,.\end{aligned}$$

Clearly $(\top_*,\bot),(\top^*,\bot)\in\mathbb{T}$. In addition, it is obvious from (31) that

$$d'(\top_*,\top^*)=d'(\top,\top')<\delta\,. \tag{293}$$

In addition, we may conclude from (X-5) that

$$\xi(\top^*, \bot) \geq \max(\xi(\top, \bot), \xi(\top', \bot)) \tag{294}$$
$$\xi(\top_*, \bot) \leq \min(\xi(\top, \bot), \xi(\top', \bot)) \,. \tag{295}$$

Hence

$$\begin{aligned}
&|\xi(\top, \bot) - \xi(\top', \bot)| \\
&= \max(\xi(\top, \bot), \xi(\top', \bot)) - \min(\xi(\top, \bot), \xi(\top', \bot)) \\
&\leq \xi(\top^*, \bot) - \xi(\top_*, \bot) && \text{by (294), (295)} \\
&= |\xi(\top^*, \bot) - \xi(\top_*, \bot)| && \text{by (X-5) and } \top_* \leq \top^* \\
&< \varepsilon \,. && \text{by (292)}
\end{aligned}$$

Hence the condition of L-83 is satisfied, from which we conclude that $\mathcal{F}_\xi$ is arg-continuous.

B.34 Proof of Theorem 53

$\mathcal{F}_{\mathrm{Ch}}$ is known to satisfy (X-2) and (X-5) by Th-27 and Th-23. In order to prove that $\mathcal{F}_{\mathrm{Ch}}$ is arg-continuous, it is hence sufficient to show that ξ_{Ch} satisfies the condition stated in Th-52: we have to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\xi(\top', \bot) - \xi(\top, \bot) < \varepsilon$ whenever $(\top, \bot), (\top', \bot) \in \mathbb{T}$ satisfy $d'(\top, \top') < \delta$ and $\top \leq \top'$.
Hence let $\varepsilon > 0$ be given. I will show that $\delta = \varepsilon$ is a proper choice of δ for ξ_{Ch}. Consider $(\top, \bot), (\top', \bot) \in \mathbb{T}$ with $d'(\top, \top') < \varepsilon$ and $\top \leq \top'$. Because $\top \leq \top'$, this means that

$$\sup\{\inf\{\gamma' : \top(\gamma') \geq \top'(\gamma)\} - \gamma : \gamma \in \mathbf{I}\} < \varepsilon$$

by (31). Hence

$$\top(\gamma + \varepsilon) \geq \top'(\gamma) \tag{296}$$

for all $\gamma \in [0, 1 - \varepsilon]$. We define $\top'' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\top''(\gamma) = \begin{cases} \top(\gamma + \varepsilon) & : \quad \gamma \leq 1 - \varepsilon \\ 1 & : \quad \gamma > 1 - \varepsilon \end{cases} \tag{297}$$

Then by (296),

$$\top'' \geq \top' \,. \tag{298}$$

Now let us put things together:

$$
\begin{aligned}
&|\xi_{\mathrm{Ch}}(\top, \bot) - \xi_{\mathrm{Ch}}(\top', \bot)| \\
&= \xi_{\mathrm{Ch}}(\top', \bot) - \xi_{\mathrm{Ch}}(\top, \bot) && \text{by (X-5)} \\
&\leq \xi_{\mathrm{Ch}}(\top'', \bot) - \xi_{\mathrm{Ch}}(\top, \bot) && \text{by (298), (X-5)} \\
&= \tfrac{1}{2}(\int_0^1 \top''(\gamma)\, d\gamma + \int_0^1 \bot(\gamma)\, d\gamma \\
&\quad - \int_0^1 \top(\gamma)\, d\gamma - \int_0^1 \bot(\gamma)\, d\gamma) && \text{by Def. 47} \\
&= \tfrac{1}{2}(\int_0^1 \top''(\gamma)\, d\gamma - \int_0^1 \top(\gamma)\, d\gamma) \\
&= \tfrac{1}{2}(\int_0^{1-\varepsilon} \top''(\gamma)\, d\gamma + \int_{1-\varepsilon}^1 \top''(\gamma)\, d\gamma \\
&\quad - \int_0^{\varepsilon} \top(\gamma)\, d\gamma - \int_{\varepsilon}^1 \top(\gamma)\, d\gamma) \\
&= \tfrac{1}{2}(\int_0^{1-\varepsilon} \top(\gamma + \varepsilon)\, d\gamma + \int_{1-\varepsilon}^1 1\, d\gamma \\
&\quad - \int_0^{\varepsilon} \top(\gamma)\, d\gamma - \int_{\varepsilon}^1 \top(\gamma)\, d\gamma) && \text{by (297)} \\
&= \tfrac{1}{2}(\int_{\varepsilon}^1 \top(\gamma)\, d\gamma + \varepsilon - \int_0^{\varepsilon} \top(\gamma)\, d\gamma - \int_{\varepsilon}^1 \top(\gamma)\, d\gamma) \\
&= \tfrac{1}{2}(\varepsilon - \int_0^{\varepsilon} \top(\gamma)\, d\gamma) \\
&\leq \tfrac{\varepsilon}{2} \\
&< \varepsilon\,.
\end{aligned}
$$

Hence $\mathcal{F}_{\mathrm{Ch}}$ is arg-continuous by Th-52.

B.35 Proof of Theorem 54

We already know from Th-30 that $\mathcal{F}_S$ is a DFS. In particular, we can deduce from Th-23 that $\xi_S : \mathbb{T} \longrightarrow \mathbf{I}$ as defined in Def. 50 satisfies (X-1) to (X-5). Hence Th-51 applies, and we can prove that $\mathcal{F}_S$ fails to be arg-continuous by showing that there exist $(\top_1, \bot_1) \in \mathbb{T}$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exists $(\top_2, \bot_2) \in \mathbb{T}$ with $d'((\top_1, \bot_1), (\top_2, \bot_2)) < \delta$ and $|\xi_S(\top_1, \bot_1) - \xi_S(\top_2, \bot_2)| \geq \epsilon$.
Hence let $(\top_1, \bot_1) = (\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}})$ and let $\varepsilon = \frac{1}{4}$. From (X-1), we immediately obtain

$$\xi_S(\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}) = \tfrac{3}{4}\,. \tag{299}$$

Now consider $\delta > 0$. We choose $(\top_2, \bot_2) = (\top_2, \mathrm{c}_{\frac{3}{4}})$ and define $\top_2 : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\top_2(\gamma) = \begin{cases} \frac{3}{4} & : \quad \gamma \leq 1 - \frac{\delta}{2} \\ 1 & : \quad \gamma > 1 - \frac{\delta}{2} \end{cases} \tag{300}$$

for all $\gamma \in \mathbf{I}$. In this case, we obtain from Def. 50 that

$$\begin{aligned} \xi_S(\top_2, \mathrm{c}_{\frac{3}{4}}) &= \min((\top_2)_1^*, \tfrac{1}{2} + \tfrac{1}{2}(\mathrm{c}_{\frac{3}{4}})_*^{\leq \frac{1}{2}\downarrow}) && \text{by Def. 50 and } \mathrm{c}_{\frac{3}{4}}(0) = \tfrac{3}{4} > \tfrac{1}{2}. \\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2} \cdot 1) && \text{by (12), (26)} \\ &= 1\,, \end{aligned}$$

i.e.

$$\xi_S(\top_2, \mathrm{c}_{\frac{3}{4}}) = 1 \tag{301}$$

Hence

$$\begin{aligned} |\xi(\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}) - \xi(\top_2, \mathrm{c}_{\frac{3}{4}})| &= |\tfrac{3}{4} - 1| && \text{by (299), (301)} \\ &= \tfrac{3}{4} \\ &\geq \varepsilon\,, \end{aligned}$$

although

$$d'((\mathrm{c}_{\frac{3}{4}}, \mathrm{c}_{\frac{3}{4}}), (\top_2, \mathrm{c}_{\frac{3}{4}})) = \tfrac{\delta}{2} < \delta\,.$$

(This is apparent from (33) and (300)). Hence condition **b.** of Th-51 fails, which is necessary for $\mathcal{F}_S$ to be arg-continuous. We conclude that $\mathcal{F}_S$ is not arg-continuous.

B.36 Proof of Theorem 55

We first recall that ξ_A satisfies (X-2), (X-4) and (X-5), see Th-31 and Th-23. Hence Th-51 is applicable, and we can show that $\mathcal{F}_A$ fails to be arg-continuous by proving that there exists $(\top, \bot) \in \mathbb{T}$ and $\varepsilon > 0$ such that for all $\delta > 0$, there exist $(\top', \bot') \in \mathbb{T}$ with $d'((\top, \bot), (\top', \bot')) < \delta$ and $|\xi_A(\top, \bot) - \xi_A(\top', \bot')| \geq \varepsilon$.
Hence let $\top = \mathrm{c}_1$, $\bot = \mathrm{c}_{\frac{1}{2}}$ and $\varepsilon = \frac{1}{2}$. Now consider $\delta > 0$. Define $\bot'$ by

$$\bot'(\gamma) = \begin{cases} 1 & : \quad \gamma \leq \frac{\delta}{2} \\ \frac{1}{2} & : \quad \gamma > \frac{\delta}{2} \end{cases}$$

Then $d'((\top, \bot), (\top, \bot')) = \frac{\delta}{2} < \delta$. In addition,

$$\bot_0^* = \lim_{\gamma \to 0^+} \mathrm{c}_{\frac{1}{2}}(\gamma) = \tfrac{1}{2} \tag{302}$$

$$\bot'^*_0 = \lim_{\gamma \to 0^+} \bot'(\gamma) = 1 \tag{303}$$

$$\bot_*^{0\downarrow} = \inf\{\gamma : \mathrm{c}_{\frac{1}{2}}(\gamma) = 0\} = \inf \varnothing = 1 \tag{304}$$

$$\bot'^{0\downarrow}_* = \inf\{\gamma : \bot'(\gamma) = 0\} = \inf \varnothing = 1 \tag{305}$$

by (9) and (10), respectively. Hence

$$\begin{aligned}\xi_A(\top,\bot) &= \min(\bot_0^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by Def. 51}\\ &= \min(\tfrac{1}{2}, \tfrac{1}{2} + \tfrac{1}{2}\cdot 1) && \text{by (302), (304)}\\ &= \tfrac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\xi_A(\top,\bot') &= \min(\bot'^*_0, \tfrac{1}{2} + \tfrac{1}{2}\bot'^{0\downarrow}_*) && \text{by Def. 51}\\ &= \min(1, \tfrac{1}{2} + \tfrac{1}{2}\cdot 1) && \text{by (303), (305)}\\ &= 1\,.\end{aligned}$$

Therefore $|\xi_A(\top,\bot) - \xi_A(\top,\bot')| = \frac{1}{2} = \varepsilon$. We conclude from Th-51 that $\mathcal{F}_A$ is not continuous in arguments.

B.37 Proof of Theorem 56

Suppose $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ are given and $\mathcal{F}_\xi, \mathcal{F}_{\xi'}$ are the corresponding QFMs.

b.⇒a.: Let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ be a semi-fuzzy quantifier and let $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}\mathcal{F}_\xi(Q)(X_1,\ldots,X_n) &= \xi(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by Def. 45}\\ &\preceq_c \xi'(\top_{Q,X_1,\ldots,X_n}, \bot_{Q,X_1,\ldots,X_n}) && \text{by assumed condition } \textbf{b.}\\ &= \mathcal{F}_{\xi'}(Q)(X_1,\ldots,X_n)\,. && \text{by Def. 45}\end{aligned}$$

Hence $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$.

a.⇒b.: Let $(\top,\bot) \in \mathbb{T}$ be given. By Th-21, there exists $Q : \mathcal{P}(\mathbf{2}\times\mathbf{I}) \longrightarrow \mathbf{I}$ and $X \in \widetilde{\mathcal{P}}(\mathbf{2}\times\mathbf{I})$ such that

$$\top = \top_{Q,X} \qquad\qquad \bot = \bot_{Q,X}\,. \tag{306}$$

Therefore

$$\begin{aligned}\xi(\top,\bot) &= \xi(\top_{Q,X}, \bot_{Q,X}) && \text{by (306)}\\ &= \mathcal{F}_\xi(Q)(X) && \text{by Def. 45}\\ &\preceq_c \mathcal{F}_{\xi'}(Q)(X) && \text{by assumed condition } \textbf{a.}\\ &= \xi'(\top_{Q,X}, \bot_{Q,X}) && \text{by Def. 45}\\ &= \xi'(\top,\bot)\,. && \text{by (306)}\end{aligned}$$

B.38 Proof of Theorem 57

Lemma 84 *Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ be given. If ξ, ξ' satisfy* (X-2)*, then the following conditions are equivalent.*

a. $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$;

b. *for all* $(\top, \bot) \in \mathbb{T}$ *with* $\top(0) \geq \frac{1}{2}$, $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$.

Proof

a.⇒b.: This is apparent, because **b.** is a weakening of the condition **b.** of Th-56, which has already been shown to be necessary for $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$.

b.⇒a.: To see this, let $(\top, \bot) \in \mathbb{T}$. If $\top(0) \geq \frac{1}{2}$, then $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$ by **b.** In the remaining case that $\top(0) < \frac{1}{2}$,

$$\begin{aligned} \xi(\top, \bot) &= 1 - \xi(1 - \bot, 1 - \top) && \text{by (X-2)} \\ &\preceq_c 1 - \xi'(1 - \bot, 1 - \top) && \text{by assumed condition } \mathbf{b.} \\ &= \xi'(\top, \bot)\,, && \text{by (X-2)} \end{aligned}$$

recalling that $1 - x \preceq_c 1 - y$ if and only if $x \preceq_c y$. Hence $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$ for all $(\top, \bot) \in \mathbb{T}$, i.e. $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$ by Th-56.

Lemma 85 *Let* $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ *be a given mapping which satisfies* (X-5) *and has the additional property that* $\xi(\top, \bot) = \frac{1}{2}$ *whenever* $\top(0) \geq \frac{1}{2} \geq \bot(0)$. *Then for all* $(\top, \bot) \in \mathbb{T}$,

a. *If* $\bot(0) > \frac{1}{2}$, *then* $\xi(\top, \bot) \geq \frac{1}{2}$;

b. *If* $\top(0) < \frac{1}{2}$, *then* $\xi(\top, \bot) \leq \frac{1}{2}$.

Proof Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ satisfies (X-1) to (X-5). We shall assume that ξ propagates fuzziness. Further let $(\top, \bot) \in \mathbb{T}$ be given.

a.: $\bot(0) > \frac{1}{2}$**.** Then $\top(0) \geq \bot(0) > \frac{1}{2}$. Therefore

$$\begin{aligned} \xi(\top, \bot) &\geq \xi(\top, \min(\bot, \tfrac{1}{2})) && \text{by (X-5)} \\ &= \tfrac{1}{2} \end{aligned}$$

by the assumed property of ξ.

b.: $\top(0) < \frac{1}{2}$**.** In this case $\bot(0) \leq \top(0) < \frac{1}{2}$. Hence

$$\begin{aligned} \xi(\top, \bot) &\leq \xi(\max(\top, \tfrac{1}{2}), \bot) && \text{by (X-5)} \\ &= \tfrac{1}{2}\,, \end{aligned}$$

again by the assumed property of ξ.

Proof of Theorem 57

Let $\xi, \xi' : \mathbb{T} \longrightarrow \mathbf{I}$ be given mappings which satisfy (X-1) to (X-5) and have the additional property that

$$\xi(\top, \bot) = \xi'(\top, \bot) = \tfrac{1}{2} \tag{307}$$

whenever $(\top, \bot) \in \mathbb{T}$ is such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$.

a.⇒b.: Let $(\top, \bot) \in \mathbb{T}$. If $\bot(0) > \frac{1}{2}$, then $\xi(\top, \bot) \geq \frac{1}{2}$ and $\xi'(\top, \bot) \geq \frac{1}{2}$ by L-85 Hence in this case, the conditions $\xi(\top, \bot) \leq \xi'(\top, \bot)$ and $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$ are equivalent by (3). The imposed condition **b.** is hence a weakening of condition **b.** of L-84, which has already been shown to be necessary for $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$.

b.⇒a.: To see this, let $(\top, \bot) \in \mathbb{T}$ with $\top(0) \geq \frac{1}{2}$. If $\top(0) \geq \bot(0) > \frac{1}{2}$, then $\xi(\top, \bot) \leq \xi'(\top, \bot)$ by assumption. It is further apparent from L-85 that $\xi(\top, \bot) \geq \frac{1}{2}$. Hence $\frac{1}{2} \leq \xi(\top, \bot) \leq \xi'(\top, \bot)$, i.e. $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$ by (3). In the remaining case that $\bot(0) \leq \frac{1}{2}$, i.e. $\bot(0) \leq \frac{1}{2} \leq \top(0)$, we conclude from (307) that $\xi(\top, \bot) = \frac{1}{2} = \xi'(\top, \bot)$. In particular, $\xi(\top, \bot) \preceq_c \xi'(\top, \bot)$. This proves that condition **b.** of L-84 is satisfied, which is sufficient for $\mathcal{F}_\xi \preceq_c \mathcal{F}_{\xi'}$.

B.39 Proof of Theorem 58

Lemma 86 *Let $(\top, \bot) \in \mathbb{T}$ be given such that $\bot(0) > \frac{1}{2}$. Then $\bot_*^{1\uparrow} = \bot_*^{1\uparrow}$, where $f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)$.*

Proof We simply need to notice that

$$\begin{aligned} f(\gamma) &= \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) \\ &= \mathrm{med}_{\frac{1}{2}}(1, \bot(\gamma)) && \text{because } \top = \mathrm{c}_1 \text{, see above} \\ &= \max(\bot(\gamma), \tfrac{1}{2})\,. && \text{by Def. 22} \end{aligned}$$

Hence

$$\begin{aligned} f_*^{1\uparrow} &= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} && \text{by (13)} \\ &= \sup\{\gamma \in \mathbf{I} : \max(\bot(\gamma), \tfrac{1}{2}) = 1\} \\ &= \sup\{\gamma \in \mathbf{I} : \bot(\gamma) = 1\} \\ &= \bot_*^{1\uparrow}\,. && \text{by (13)} \end{aligned}$$

Lemma 87 *Suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a given mapping such that $\mathcal{F}_\xi$ is a DFS. Further let $(\top, \bot) \in \mathbb{T}$ such that $\bot_*^{1\uparrow} > 0$. Then*

$$\xi(\top, \bot) \geq \tfrac{1}{2} + \tfrac{1}{2} f_*^{1\uparrow}\,,$$

where $f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)$.

Proof Suppose $\bot_*^{1\uparrow} > 0$. By (13), there exists $\gamma' > 0$ such that $\bot(\gamma') = 1$. Hence for all $\gamma \in \mathbf{I}$, $\top(\gamma) \geq \bot(\gamma') = 1$ by Th-20.c, i.e. $\top = \mathrm{c}_1$. We now define $\bot' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\bot'(\gamma) = \begin{cases} 1 & : \quad \gamma < f_*^{1\uparrow} \\ 0 & : \quad \gamma \geq f_*^{1\uparrow} \end{cases} \tag{308}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (13) that $\bot \geq \bot'$. Hence

$$\begin{aligned}
\xi(\top, \bot) &\geq \xi(\top, \bot') && \text{by (X-5)} \\
&= \xi(\mathrm{c}_1, \bot') && \text{because } \top = \mathrm{c}_1\text{, see above} \\
&= \tfrac{1}{2} + \tfrac{1}{2}{\bot'}_*^{0\downarrow} && \text{by (X-3)} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}\,, && \text{by (10), (308)}
\end{aligned}$$

i.e.

$$\xi(\top, \bot) \geq \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}\,.$$

This finishes the proof because of lemma L-86.

Proof of Theorem 58

Let us recall that $\mathcal{M}_U$ is defined in terms of $\mathcal{B}'_U : \mathbb{H} \longrightarrow \mathbf{I}$, where

$$\mathcal{B}'_U(f) = \max(f_*^{1\uparrow}, f_1^*)$$

for all $f \in \mathbb{H}$, see Def. 40. Hence by (15), $\mathcal{B}_U : \mathbb{B} \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{B}_U(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'_U(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'_U(1-2f) & : \quad f \in \mathbb{B}^- \end{cases}$$

for all $f \in \mathbb{B}$. This can be simplified as follows. Firstly if $f \in \mathbb{B}^+$, then

$$\begin{aligned}
(2f-1)_*^{1\uparrow} &= \sup\{\gamma \in \mathbf{I} : (2f-1)(\gamma) = 1\} && \text{by (13)} \\
&= \sup\{\gamma \in \mathbf{I} : 2f(\gamma) - 1 = 1\} \\
&= \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\} \\
&= f_*^{1\uparrow}\,. && \text{by (13)}
\end{aligned}$$

Similarly

$$\begin{aligned}
(2f-1)_1^* &= \lim_{\gamma \to 1^-} (2f-1)(\gamma) && \text{by (12)} \\
&= \lim_{\gamma \to 1^-} 2f(\gamma) - 1 \\
&= 2(\lim_{\gamma \to 1^-} f(\gamma)) - 1 \\
&= 2f_1^* - 1\,.
\end{aligned}$$

Hence

$$\mathcal{B}_U(f) = \tfrac{1}{2} + \tfrac{1}{2}\max(f_*^{1\uparrow}, 2f_1^* - 1) = \max(\tfrac{1}{2}f_*^{1\uparrow} + \tfrac{1}{2}, f_1^*) \tag{309}$$

for all $f \in \mathbb{B}^-$. By similar reasoning,

$$\mathcal{B}_U(f) = \tfrac{1}{2} - \tfrac{1}{2}\min(f_{**}^0, f_1^*)\,, \tag{310}$$

where we have abbreviated

$$f_{**}^0 = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 0\}\,. \tag{311}$$

In order to prove that $\mathcal{M}_U$ is the least specific $\mathcal{F}_\xi$-DFS, suppose $\xi : \mathbb{T} \longrightarrow \mathbf{I}$ is a given mapping such that $\mathcal{F}_\xi$ is a DFS. We show that $\mathcal{M}_U \preceq_c \mathcal{F}_\xi$ by proving the equivalent condition of Th-56, i.e. $\xi_U \preceq_c \xi$, where

$$\xi_U(\top, \bot) = \mathcal{B}_U(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) = \mathcal{B}_U(f) \tag{312}$$

by (25), and f abbreviates

$$f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)\,. \tag{313}$$

We will discern three cases.

a.: $\bot(0) > \frac{1}{2}$. Let us define $\bot'' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\bot''(\gamma) = \begin{cases} \bot_1^* & : \quad \gamma < 1 \\ 0 & : \quad \gamma = 1 \end{cases} \tag{314}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (12) and the fact that $\bot$ is nonincreasing by Def. 44 that $\bot \geq \bot''$. Therefore

$$\begin{aligned} \xi(\top, \bot) &\geq \xi(\top, \bot'') && \text{by (X-5)} \\ &= \xi(\top, \mathrm{c}_{\bot_1^*}) && \text{by (7), (314) and L-23} \\ &\geq \xi(\mathrm{c}_{\bot_1^*}, \mathrm{c}_{\bot_1^*}) && \text{by (X-5), Th-20.c} \\ &= \bot_1^*\,, && \text{by (X-1)} \end{aligned}$$

i.e.

$$\xi(\top, \bot) \geq \bot_1^*\,. \tag{315}$$

We also observe that

$$\begin{aligned} f_1^* &= \lim_{\gamma \to 1^-} f(\gamma) && \text{by (12)} \\ &= \lim_{\gamma \to 1^-} \mathrm{med}_{\frac{1}{2}}(\top(\gamma), \bot(\gamma)) && \text{by (313)} \\ &= \lim_{\gamma \to 1^-} \max(\tfrac{1}{2}, \bot(\gamma)) && \text{by Def. 22 because } \bot \leq \top,\ \tfrac{1}{2} \leq \top \\ &= \max(\tfrac{1}{2}, \lim_{\gamma \to 1^-} \bot(\gamma)) && \text{because } \max \text{ continuous, } \tfrac{1}{2} \text{ constant} \\ &= \max(\tfrac{1}{2}, \bot_1^*)\,, && \text{by (12)} \end{aligned}$$

i.e.

$$f_1^* = \begin{cases} \bot_1^* & : \quad \bot_1^* > \frac{1}{2} \\ \frac{1}{2} & : \quad \bot_1^* \leq \frac{1}{2} \end{cases} \tag{316}$$

In the following, we treat separately the following subcases.

1. $\bot_*^{1\uparrow} > 0$ and $\bot_1^* > \frac{1}{2}$.
   Then $\xi \top \bot \geq \frac{1}{2} + \frac{1}{2} f_*^{1\uparrow}$ by L-87. In addition, $\xi(\top, \bot) \geq \bot_1^* = f_1^*$ by (315) and (316). Hence $\xi(\top, \bot) \geq \max(\frac{1}{2} + \frac{1}{2} f_*^{1\uparrow}, f_1^*) \geq \frac{1}{2}$, i.e. $\xi(\top, \bot) \succeq_c \xi_U(\top, \bot)$ by (3), (309) and (312).

2. $\bot_*^{1\uparrow} > 0$ and $\bot_1^* \leq \frac{1}{2}$.
   Then again $\xi(\top, \bot) \geq \frac{1}{2} + \frac{1}{2} f_*^{1\uparrow} \geq \frac{1}{2}$ by L-87. In addition, $f_1^* = \frac{1}{2}$ by (316). Hence $\xi(\top, \bot) \geq \frac{1}{2} + \frac{1}{2} f_*^{1\uparrow} = \max(\frac{1}{2} + \frac{1}{2} f_*^{1\uparrow}, f_1^*) \geq \frac{1}{2}$, i.e. $\xi(\top, \bot) \succeq_c \xi_U(\top, \bot)$ by (3), (309) and (312).

3. $\bot_*^{1\uparrow} = 0$ and $\bot_1^* > \frac{1}{2}$.
   In this case, $\xi(\top, \bot) \geq \bot_1^* = f_1^* > \frac{1}{2}$ by (315) and (316). In addition, $f_*^{1\uparrow} = 0$ by L-86. Hence $\xi(\top, \bot) \geq f_1^* = \max(\frac{1}{2} + \frac{1}{2} f_*^{1\uparrow}, f_1^*) \geq \frac{1}{2}$, and again $\xi(\top, \bot) \succeq_c \xi_U(\top, \bot)$ by (3), (309) and (312).

4. $\bot_*^{1\uparrow} = 0$ and $\bot_1^* \leq \frac{1}{2}$.
   Then $f_*^{1\uparrow} = 0$ by L-86 and $f_1^* = \frac{1}{2}$ by (316). Hence $\xi_U(\top, \bot) = \frac{1}{2}$ by (309) and (312). In particular, $\xi(\top, \bot) \succeq_c \frac{1}{2} = \xi_U(\top, \bot)$ by (3).

b.: $\top(0) < \frac{1}{2}$. This can be reduced to case **a.** because

$\xi(\top, \bot) = 1 - \xi(1 - \bot, 1 - \top)$	by (X-2)
$\succeq_c 1 - \xi_U(1 - \bot, 1 - \top)$	by part **a.** of theorem
$= \xi_U(\top, \bot)\,.$	by (X-2)

c.: $\bot(0) \leq \frac{1}{2} \leq \top(0)$. Then $\xi_U(\top, \bot) = \frac{1}{2}$ by Th-19 and Th-34. Hence trivially $\xi(\top, \bot) \succeq_c \frac{1}{2} = \xi_U(\top, \bot)$ by (3).

B.40 Proof of Theorem 59

Suppose $\mathcal{F}_\xi$ and $\mathcal{F}_{\xi'}$ are specificity consistent and let $(\top, \bot) \in \mathbb{T}$. By Th-21, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ such that $\top_{Q,X} = \top$ and $\bot_{Q,X} = \bot$. Hence by Def. 45,

$$\{\xi(\top, \bot), \xi'(\top, \bot)\} = \{\mathcal{F}_\xi(Q)(X), \mathcal{F}_{\xi'}(Q)(X)\}\,.$$

Hence either

$$\{\xi(\top, \bot), \xi'(\top, \bot)\} = \{\mathcal{F}_\xi(Q)(X), \mathcal{F}_{\xi'}(Q)(X)\} \subseteq [0, \tfrac{1}{2}]$$

or

$$\{\xi(\top,\bot),\xi'(\top,\bot)\} = \{\mathcal{F}_\xi(Q)(X),\mathcal{F}_{\xi'}(Q)(X)\} \subseteq [\tfrac{1}{2},1]$$

by Def. 24. This proves that the condition on ξ is entailed by specificity consistence of $\mathcal{F}_\xi$ and $\mathcal{F}_{\xi'}$.
To see that the converse implication also holds, suppose ξ and ξ' are specificity consistent as stated in the theorem. Further let $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1,\ldots,X_n \in \widetilde{\mathcal{P}}(E)$ be given. It is then apparent from Def. 45 that

$$\begin{aligned}&\{\mathcal{F}_\xi(Q)(X_1,\ldots,X_n),\mathcal{F}_{\xi'}(Q)(X_1,\ldots,X_n)\}\\&=\{\xi(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n}),\xi'(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n})\}\end{aligned}$$

Hence either

$$\begin{aligned}&\{\mathcal{F}_\xi(Q)(X_1,\ldots,X_n),\mathcal{F}_{\xi'}(Q)(X_1,\ldots,X_n)\}\\&=\{\xi(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n}),\xi'(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n})\}\subseteq[0,\tfrac{1}{2}]\end{aligned}$$

or

$$\begin{aligned}&\{\mathcal{F}_\xi(Q)(X_1,\ldots,X_n),\mathcal{F}_{\xi'}(Q)(X_1,\ldots,X_n)\}\\&=\{\xi(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n}),\xi'(\top_{Q,X_1,\ldots,X_n},\bot_{Q,X_1,\ldots,X_n})\}\subseteq[\tfrac{1}{2},1]\,,\end{aligned}$$

i.e. $\mathcal{F}_\xi,\mathcal{F}_{\xi'}$ are specificity consistent according to Def. 24.

B.41 Proof of Theorem 60

We will simply show that the DFSes $\mathcal{M}_S$ and $\mathcal{F}_{\mathrm{Ch}}$ are not specificity consistent. To this end, we define $(\top,\bot)\in\mathbb{T}$ as follows.

$$\top(\gamma) = \tfrac{3}{4} \tag{317}$$

$$\bot(\gamma) = \begin{cases}\tfrac{3}{4} & : \quad \gamma < \tfrac{1}{10}\\ 0 & : \quad \gamma \geq \tfrac{1}{10}\end{cases} \tag{318}$$

Then

$$\int_0^1 \top(\gamma)\,d\gamma = \tfrac{3}{4}$$

and

$$\int_0^1 \bot(\gamma)\,d\gamma = \tfrac{3}{4}\cdot\tfrac{1}{10} = \tfrac{3}{40}$$

by (317) and (318), i.e.

$$\xi_{\mathrm{Ch}}(\top,\bot) = \tfrac{1}{2}\int_0^1 \top(\gamma)\,d\gamma + \tfrac{1}{2}\int_0^1 \bot(\gamma)\,d\gamma = \tfrac{1}{2}\cdot\tfrac{3}{4}+\tfrac{1}{2}\cdot\tfrac{3}{40} = \tfrac{33}{80} \tag{319}$$

by Def. 47. In the case of $\mathcal{M}_S$, we abbreviate $f = \mathrm{med}_{\frac{1}{2}}(\top, \bot)$. Then by Def. 22 (317) and (318),

$$f(\gamma) = \begin{cases} \frac{3}{4} & : \quad \gamma < \frac{1}{10} \\ \frac{1}{2} & : \quad \gamma \geq \frac{1}{10} \end{cases}$$

and

$$(2f-1)(\gamma) = \begin{cases} \frac{1}{2} & : \quad \gamma < \frac{1}{10} \\ 0 & : \quad \gamma \geq \frac{1}{10} \end{cases}$$

Hence $(2f-1)^{0\downarrow}_{*} = \frac{1}{10}$ and $(2f-1)^{*}_{0} = \frac{1}{2}$ by (10) and (9). In turn,

$$\begin{aligned} \mathcal{B}_S(f) &= \tfrac{1}{2} + \tfrac{1}{2}\mathcal{B}'_S(2f-1) && \text{by (16)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\min((2f-1)^{0\downarrow}_{*}, (2f-1)^{*}_{0}) && \text{by Def. 41} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\min(\tfrac{1}{10}, \tfrac{1}{2}) && \text{see above} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \cdot \tfrac{1}{1}0 \,, \end{aligned}$$

i.e.

$$\mathcal{B}_S(f) = \tfrac{11}{20} \,. \tag{320}$$

By Th-21, there exists a semi-fuzzy quantifier $Q : \mathcal{P}(\mathbf{2} \times \mathbf{I}) \longrightarrow \mathbf{I}$ and a fuzzy subset $X \in \widetilde{\mathcal{P}}(\mathbf{2} \times \mathbf{I})$ such that

$$\top = \top_{Q,X} \qquad\qquad \bot = \bot_{Q,X} \,. \tag{321}$$

Hence

$$\begin{aligned} \mathcal{F}_{\mathrm{Ch}}(Q)(X) &= \xi_{\mathrm{Ch}}(\top_{Q,X}, \bot_{Q,X}) && \text{by Def. 45, Def. 47} \\ &= \xi_{\mathrm{Ch}}(\top, \bot) && \text{by (321)} \\ &= \tfrac{33}{80} && \text{by (319)} \\ &< \tfrac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_S(Q)(X) &= \mathcal{B}_S(\mathrm{med}_{\frac{1}{2}}(\top_{Q,X}, \bot_{Q,X})) && \text{by Def. 41, Def. 45 and (25)} \\ &= \mathcal{B}_S(\mathrm{med}_{\frac{1}{2}}(\top, \bot)) && \text{by (321)} \\ &= \mathcal{B}_S(f) && \text{see above definition of } f \\ &= \tfrac{11}{20} && \text{by (320)} \\ &> \tfrac{1}{2} \,, \end{aligned}$$

i.e. according to Def. 24 $\mathcal{F}_{\mathrm{Ch}}$ and $\mathcal{M}_S$ are not specificity consistent.

B.42 Proof of Theorem 61

Let $\mathbb{F}$ be a collection of $\mathcal{F}_\xi$-DFSes $\mathcal{F}_\xi \in \mathbf{F}$ with the property that

$$\mathcal{F}_\xi(\top, \bot) = \tfrac{1}{2} \tag{322}$$

whenever $(\top, \bot) \in \mathbb{T}$ is such that $\top(0) \geq \frac{1}{2} \geq \bot(0)$. In the following we define

$$\mathbb{X} = \{\xi : \mathbb{T} \longrightarrow \mathbf{I} \ : \ \mathcal{F}_\xi \in \mathbb{F}\} \,. \tag{323}$$

Then apparently

$$\mathbb{F} = \{\mathcal{F}_\xi : \xi \in \mathbb{X}\} \,. \tag{324}$$

For a given semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy arguments $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we abbreviate

$$R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_\xi(Q)(X_1, \ldots, X_n) : \mathcal{F}_\xi \in \mathbb{F}\} \,.$$

Abbreviating $\top = \top_{Q,X_1,\ldots,X_n}$ and $\bot = \bot_{Q,X_1,\ldots,X_n}$, we clearly have

$$R_{Q,X_1,\ldots,X_n} = \{\xi(\top, \bot) : \xi \in \mathbb{X}\} \tag{325}$$

by (324), (323) and Def. 45. In the following, we discern three cases.

a.: $\bot(0) > \frac{1}{2}$. Then for all $\xi \in \mathbb{X}$,

$$\xi(\top, \bot) \geq \tfrac{1}{2} \tag{326}$$

by L-85.a. Hence

$$\begin{aligned} R_{Q,X_1,\ldots,X_n} &= \{\xi(\top, \bot) : \xi \in \mathbb{X}\} && \text{by (325)} \\ &\subseteq [\tfrac{1}{2}, 1] \,. && \text{by (326)} \end{aligned}$$

b.: $\top(0) < \frac{1}{2}$. In this case, we obtain from L-85.b that for all $\xi \in \mathbb{X}$,

$$\xi(\top, \bot) \leq \tfrac{1}{2} \tag{327}$$

and hence

$$\begin{aligned} R_{Q,X_1,\ldots,X_n} &= \{\xi(\top, \bot) : \xi \in \mathbb{X}\} && \text{by (325)} \\ &\subseteq [0, \tfrac{1}{2}] \,. && \text{by (327)} \end{aligned}$$

c.: $\bot(0) \leq \frac{1}{2} \leq \top(0)$. Then for all $\xi \in \mathbb{X}$,

$$\xi(\top, \bot) = \tfrac{1}{2} \tag{328}$$

by (322). Therefore

$$\begin{aligned} R_{Q,X_1,\dots,X_n} &= \{\xi(\top,\bot) : \xi \in \mathbb{X}\} && \text{by (325)} \\ &= \{\tfrac{1}{2}\} && \text{by (328)} \\ &\subseteq [\tfrac{1}{2}, 1]\,. \end{aligned}$$

Summarizing these results, the given class of DFSes $\mathbb{F}$ has the property that for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, it always holds that either $R_{Q,X_1,\dots,X_n} \subseteq [0, \frac{1}{2}]$ or $R_{Q,X_1,\dots,X_n} \subseteq [\frac{1}{2}, 1]$. Hence $\mathbb{F}$ is specificity consistent by Def. 24.

B.43 Proof of Theorem 62

The claim of the theorem is apparent from Th-34, Th-32 and Th-61.

B.44 Proof of Theorem 63

The claim of the theorem is apparent from Th-40 and and Th-61.

B.45 Proof of Theorem 64

Let us denote the class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in quantifiers as $\mathbb{F}$. We conclude from Th-62 and Th-6 that $\mathbb{F}$ has a least upper specificity bound $\mathcal{F}_{\mathrm{lub}}$.
We already know from Th-30 that $\mathcal{F}_S$ is a DFS. We also know from Th-36 that $\mathcal{F}_S$ propagates fuzziness in quantifiers. Therefore $\mathcal{F}_S \preceq_c \mathcal{F}_{\mathrm{lub}}$. It remains to be shown that $\mathcal{F}_\xi \preceq_c \mathcal{F}_S$ for all $\mathcal{F}_\xi \in \mathbb{F}$.
Hence let $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$. We define $\top' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\top'(\gamma) = \begin{cases} \top_1^* & : \quad \gamma < 1 \\ 1 & : \quad \gamma = 1 \end{cases} \tag{329}$$

Because $\top$ is nondecreasing, we apparently have $\top \le \top'$, recalling that

$$\top_1^* = \lim_{\gamma \to 1^-} \top(\gamma)\,.$$

In addition, we conclude from Th-20.c and $\top(0) \le \top_1^*$ that $\bot(\gamma) \le \top_1^*$ for all $\gamma \in \mathbf{I}$, i.e. $\bot \le \mathrm{c}_{\top_1^*}$. Therefore

$$\begin{aligned} \xi(\top,\bot) &\le \xi(\top',\bot) && \text{by (X-5)} \\ &\le \xi(\top', \mathrm{c}_{\top_1^*}) && \text{by (X-5)} \\ &= \xi(\mathrm{c}_{\top_1^*}, \mathrm{c}_{\top_1^*}) && \text{by (329), (7) and L-22} \\ &= \top_1^*\,, && \text{by (X-1)} \end{aligned}$$

i.e.

$$\xi(\top,\bot) \le \top_1^*\,. \tag{330}$$

Let us further define

$$\bot'(\gamma) = \begin{cases} 1 & : \gamma \leq \bot_*^{\leq\frac{1}{2}\downarrow} \\ \frac{1}{2} & : \gamma > \bot_*^{\leq\frac{1}{2}\downarrow} \end{cases} \tag{331}$$

and

$$\bot''(\gamma) = \begin{cases} 1 & : \gamma \leq \bot_*^{\leq\frac{1}{2}\downarrow} \\ 0 & : \gamma > \bot_*^{\leq\frac{1}{2}\downarrow} \end{cases} \tag{332}$$

for all $\gamma \in \mathbf{I}$. It is apparent from (26) that $\bot \leq \bot'$. We further notice that $\bot' = \max(\bot'', \frac{1}{2})$ and $\bot''^{0\downarrow}_* = \bot_*^{\leq\frac{1}{2}\downarrow}$. Therefore

$$\begin{aligned} \xi(\top, \bot) &\leq \xi(\top, \bot') && \text{by (X-5)} \\ &\leq \xi(\mathrm{c}_1, \bot') && \text{by (X-5)} \\ &= \xi(\mathrm{c}_1, \max(\bot'', \tfrac{1}{2})) && \text{by Th-33} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\bot''^{0\downarrow}_* && \text{by (X-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}, && \text{because } \bot_*^{\leq\frac{1}{2}} = \bot''^{0\downarrow}_* \end{aligned}$$

i.e.

$$\xi(\top, \bot) \leq \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}. \tag{333}$$

Therefore

$$\begin{aligned} \xi(\top, \bot) &\leq \min(\top_1^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{\leq\frac{1}{2}\downarrow}) && \text{by (330), (333)} \\ &= \xi_S(\top, \bot). && \text{by Def. 50} \end{aligned}$$

We conclude from Th-57, Th-32 and Th-34 that $\mathcal{F}_\xi \preceq_c \mathcal{F}_S$. This finishes the proof that $\mathcal{F}_S$ is the most specific $\mathcal{F}_\xi$-DFS which propagates fuzziness in quantifiers, i.e. $\mathcal{F}_\xi = \mathcal{F}_{\mathrm{lub}}$.

B.46 Proof of Theorem 65

Let us denote the class of $\mathcal{F}_\xi$-DFSes that propagate fuzziness in arguments as $\mathbb{F}$. We conclude from Th-63 and Th-6 that $\mathbb{F}$ has a least upper specificity bound $\mathcal{F}_{\mathrm{lub}}$.
We already know from Th-31 that $\mathcal{F}_A$ is a DFS. We also know from Th-43 that $\mathcal{F}_A$ propagates fuzziness in arguments. Therefore $\mathcal{F}_A \preceq_c \mathcal{F}_{\mathrm{lub}}$. It remains to be shown that $\mathcal{F}_\xi \preceq_c \mathcal{F}_A$ for all $\mathcal{F}_\xi \in \mathbb{F}$. To this end, we can utilize the property stated in Th-40, i.e. Th-57 is applicable. According to the latter theorem, we can show that $\mathcal{F}_\xi \preceq_c \mathcal{F}_A$ by proving that $\xi(\top, \bot) \leq \xi_A(\top, \bot)$ for all $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$.
Hence let $(\top, \bot) \in \mathbb{T}$ with $\bot(0) > \frac{1}{2}$ be given. We discern two cases.

a.: $\perp_0^* \le \frac{1}{2}$**.** In this case, we define $\perp' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\perp'(\gamma) = \begin{cases} \perp_0^* & : \quad \gamma = 0 \\ \perp(\gamma) & : \quad \gamma > 0 \end{cases}$$

for all $\gamma \in \mathbf{I}$. $\perp$ is nonincreasing because $\perp$ is nonincreasing and $\perp'(0) = \perp_0^* = \lim_{\gamma \to 0^+} \perp(\gamma)$. In addition, $\perp'(0) \le \perp(0) \le \top(0)$. We conclude that $(\top, \perp') \in \mathbb{T}$. Next we notice that $\top(0) \ge \perp(0) > \frac{1}{2}$ and $\perp'(0) = \perp_0^* \le \frac{1}{2}$, by assumption of case **a.**. Therefore

$$\begin{aligned} \xi_A(\top, \perp) &= \xi_A(\top, \perp') && \text{by L-23} \\ &= \tfrac{1}{2} && \text{by Th-40} \\ &= \xi(\top, \perp') && \text{by Th-40} \\ &= \xi(\top, \perp)\,, && \text{by L-23} \end{aligned}$$

in particular we obtain the desired $\xi(\top, \perp) \le \xi_A(\top, \perp)$.

b.: $\perp_0^* > \frac{1}{2}$**.** In this case,

$$\xi_A(\top, \perp) = \min(\perp_0^*, \tfrac{1}{2} + \tfrac{1}{2} \perp_*^{0\downarrow}) \tag{334}$$

by Def. 51. We define $\perp' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\perp'(\gamma) = \begin{cases} \perp(\gamma) & : \quad \gamma < \perp_*^{0\downarrow} \\ 0 & : \quad \gamma \ge \perp_*^{0\downarrow} \end{cases} \tag{335}$$

for all $\gamma \in \mathbf{I}$. Clearly $\perp^\sharp \le \perp' \le \perp^\flat$, hence

$$\xi(\top, \perp') = \xi(\top, \perp) \tag{336}$$

by (X-4) and (X-5). We further define $\perp'' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\perp''(\gamma) = \begin{cases} 1 & : \quad \gamma \le \perp_*^{0\downarrow} \\ 0 & : \quad \gamma > \perp_*^{0\downarrow} \end{cases} \tag{337}$$

for all $\gamma \in \mathbf{I}$. It is immediate from the definition of $\perp''$ and from equation (10) that

$$\perp''^{0\downarrow}_* = \perp_*^{0\downarrow}\,. \tag{338}$$

Now we observe that $\perp' \le \perp''$ and $\top \le \mathrm{c}_1$. Therefore

$$\begin{aligned} \xi(\top, \perp) &= \xi(\top, \perp') && \text{by (336)} \\ &\le \xi(\mathrm{c}_1, \perp'') && \text{by (X-5)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \perp''^{0\downarrow}_* && \text{by (X-3)} \\ &= \tfrac{1}{2} + \tfrac{1}{2} \perp_*^{0\downarrow}\,, && \text{by (338)} \end{aligned}$$

i.e.

$$\xi(\top, \bot) \leq \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}\,. \tag{339}$$

Finally we define $\bot''' : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$\bot'''(\gamma) = \begin{cases} \bot_0^* & : \quad \gamma = 0 \\ \bot(\gamma) & : \quad \gamma > 0 \end{cases}$$

for all $\gamma \in \mathbf{I}$. Obviously

$$\begin{aligned}
\xi(\top, \bot) &= \xi(\top, \bot''') && \text{by L-23} \\
&= \xi(\mathrm{c}_1, \bot''') && \text{by Th-38 and Th-39} \\
&\leq \xi(\mathrm{c}_1, \mathrm{c}_{\bot_0^*}) && \text{by (X-5)} \\
&= \xi(\mathrm{c}_{\bot_0^*}, \mathrm{c}_{\bot_0^*}) && \text{by Th-38 and Th-39} \\
&= \bot_0^*\,, && \text{by (X-1)}
\end{aligned}$$

i.e.

$$\xi(\top, \bot) \leq \bot_0^*\,. \tag{340}$$

Hence we get the desired

$$\begin{aligned}
\xi(\top, \bot) &=\leq \min(\bot_0^*, \tfrac{1}{2} + \tfrac{1}{2}\bot_*^{0\downarrow}) && \text{by (339), (340)} \\
&= \xi_A(\top, \bot)\,. && \text{by (334)}
\end{aligned}$$

We conclude from Th-57 and Th-40 that $\mathcal{F}_\xi \preceq_c \mathcal{F}_A$. This completes the proof that $\mathcal{F}_A$ is the most specific $\mathcal{F}_\xi$-DFS which propagates fuzziness in arguments, i.e. $\mathcal{F}_\xi = \mathcal{F}_{\mathrm{lub}}$.

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