Universität Bielefeld

Forschungsbericht der
Technischen Fakultät
Abteilung Informationstechnik

An Axiomatic Theory of Fuzzy Quantifiers in Natural Languages

Ingo Glöckner

Report 2000-03

Universität Bielefeld • Postfach 10 01 31 • 33501 Bielefeld • FRG

Impressum: Herausgeber:
Robert Giegerich, Alois Knoll, Peter Ladkin,
Helge Ritter, Gerhard Sagerer, Ipke Wachsmuth

Technische Fakultät der Universität Bielefeld,
Abteilung Informationstechnik, Postfach 10 01 31,
33501 Bielefeld, FRG

ISSN 0946-7831

An Axiomatic Theory of Fuzzy Quantifiers in Natural Languages

Ingo Glöckner

University of Bielefeld, Faculty of Technology, 33501 Bielefeld, Germany

Abstract Many applications e.g. in approximate reasoning, data summarisation, information retrieval etc. can profit from the use of *fuzzy quantifiers* like "almost all" or "many", which provide flexible means of information aggregation, and are capable of extracting meaningful linguistic summaries from large amounts of raw data. However, as will be shown by a number of counterexamples, existing approaches fail to provide a convincing interpretation of fuzzy quantifiers in the important case of two-place quantification (e.g. "about half of the blondes are tall"). The interpretation of fuzzy quantifiers should hence be based on a solid *axiomatic foundation* in order to guarantee predictable and linguistically well-motivated results. In the report, an independent axiom system for "reasonable" approaches to fuzzy quantification is introduced, that are consistent with the use of quantifiers in NL. A number of linguistic adequacy criteria are formalized and it is shown that every model of the axiom system exhibits these essential properties. However, some principled adequacy bounds for approaches to fuzzy quantification are also established, which in most cases result from the known conflict between idempotence/distributivity and the law of contradiction in the presence of fuzziness. In addition, a broad class of models of the axiomatic framework is introduced. One of these models, which generalises the Sugeno integral (and hence the FG-count approach) can be shown to possess unique adequacy properties. Its analysis unveals the first definition of fuzzy cardinality which achieves adequate results with arbitrary quantitative one-place quantifiers. It is also shown how the Choquet integral (and hence the OWA approach) can be generalized to a model of the axiomatic framework. The resulting models not only represent a significant theoretical advance in fuzzy quantification; they are also practical. Efficient histogram-based algorithms for evaluating the resulting fuzzy quantifiers are described at the end of the report.[1]

1 Introduction

Natural language (NL) quantifiers, and in particular their approximate variety ("almost all", "a few" etc.), provide flexible means for expressing accumulative properties of collections. They can even describe global (e.g., quantitative) aspects of relationships between individuals (like in "most blondes are tall", which summarises the relationship between "blonde" and "tall" people). In addition, aggregational modes of temporal or local description such as "almost always", "everywhere" are naturally modelled through quantification. Because of their suitability to describe a view of the phenomenon *as a whole*, the modelling of NL quantifiers is one of the enabling techniques for intelligent multi-criteria decision making [26], data summarisation [20], information retrieval [4], fuzzy databases [14] and other applications which might profit from natural language technology.

Following Zadeh [31,32], fuzzy set theory attempts to model NL quantifiers by operators called *fuzzy quantifiers*. We can discern the following main issues in fuzzy quantification:

[1] The proofs of all theorems cited in this work have been presented in a sequence of reports [11,9,10].

– *Interpretation:* the development of methods for *evaluating* quantifying expressions which capture the meaning of natural language quantifiers, e.g. [32,18,26,29,11];
– *Summarisation:* the development of processes for *constructing* quantifying statements ("linguistic summaries"), which succintly describe a collection of observations and/or relationships between a large number of observations (find domain concepts X and Y and a quantifier Q such that "Q X's are Y's" is true), e.g. [20];
– *Reasoning:* The development of methods which deduce further knowledge from a set of rules and/or facts involving fuzzy quantifiers, e.g. [33].

In the following, we shall focus on the interpretation task. This seems to be methodically preferable because both a convincing summarisation and appropriate rules for approximate reasoning, can only be established once the semantics of fuzzy quantifiers are better understood.

2 Existing approaches to fuzzy quantification

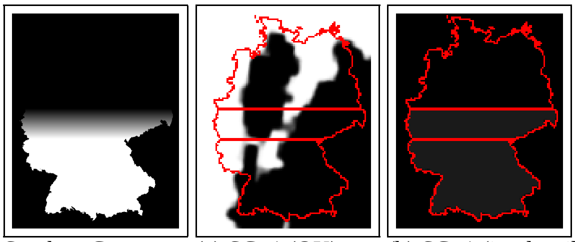
Several classes of operators have been proposed as properly representing the phenomenon of approximate or fuzzy NL quantification (a survey is provided in [16]), but there is no consensus about the proper choice, and notes on implausible behavior of these approaches are scattered over the literature [18,19,28,11]. In particular, it has been shown in [12] that none of these approaches provides acceptable results in the important case of two-place quantification, i.e. quantification restricted by a fuzzy predicate, as in "almost all blondes are lucky". In the following, we shall briefly review some of these counterexamples which enforced our decision to abandon these approaches, and to develop a fundamentally different approach to fuzzy quantification.[2]

Let us firstly consider the Σ-count approach [32]. This approach is known to accumulate "small" membership grades in an undesirable way. In the situation depicted in Fig. 1, for example, all of Southern Germany is 10 % cloudy, and the condition "about ten percent of Southern Germany are cloudy" is hence considered fully true by the Σ-count approach.[3] This is clearly implausible (to see this, consider the question *which* 10 percent are cloudy). In addition, the Σ-count approach produces discontinuous operators in the case of two-valued quantifiers like "more than 30 percent". This is inacceptable in practical applications because there is almost always some amount of noise, which can have drastic effects on the quantification results of the Σ-count approach. Because these effects also occur in the simple case of one-place quantification, we shall not consider the Σ-count approach further.

Yager [26,27] proposes an approach to fuzzy quantification based on ordered weighted averaging (OWA) operators. These perform well in the case of one-place quantification

[2] The exact formulas and quantifiers used in these counterexamples are explained in [12].

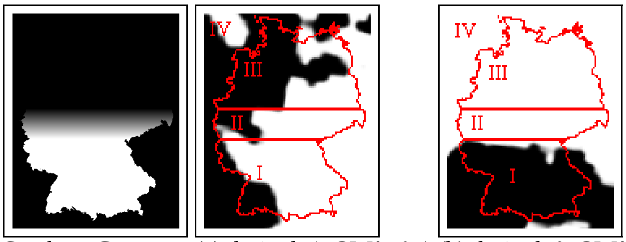
[3] In the image "SouthernGermany", pixels which fully belong to Southern Germany are depicted white. In the cloudiness image, pixels classified as cloudy depicted white. The contours of Germany, split in southern, intermediate and northern part, have been added to facilitate interpretation.



SouthernGermany (a) SC: 1 (OK) (b) SC: 1 (implausible)

Figure1. About 10 percent of Southern Germany are cloudy (SC: results of Σ-count approach)

with monotonic quantifiers. However, the formula for two-place quantification with OWA operators (proposed in [26, p. 190]) exhibits unacceptable behaviour. It can be proven that from a linguistic standpoint, the *only* quantifiers which it models adequately are the existential and universal quantifier, see [12]. To provide an example, the results of the OWA-approach in Fig. 2 reveal a undesirable dependency on cloudiness grades in regions III and IV, which do not belong to Southern Germany at all.



Southern Germany (a) desired: 1, OWA: 0.1 (b) desired: 0, OWA: 0.6

Figure2. At least 60 percent of Southern Germany are cloudy (OWA-approach)

The FG-count approach [32,25] utilizes a fuzzy measure of the cardinality of fuzzy sets to evaluate fuzzy quantifiers. This basic approach is well-behaved in the case of one-place quantifiers. Yager [27, p.72] proposes a weighting formula, which provides a definition of two-place quantification in conformance with the FG-count framework. Again, it is this formula for two-place quantification which yields implausible results. Consider the situation depicted in Fig. 3. There are no clouds at all in (the support of) SouthernGermany-1, hence we expect that "at least 5% of Southern Germany are cloudy" is false. The result of Yager's formula, however, is 0.55. The example also demonstrates that the resulting operators can be discontinuous: if we replace SouthernGermany-1 with the slightly different SouthernGermany-2, the result jumps to 0.95, although there are still no clouds in the region of interest.

Figure3. At least 5 percent of Southern Germany are cloudy (FG-count approach)

Ralescu [18] has proposed the use of the FE-count[4] for purposes of fuzzy quantification. Now consider Fig. 4, which illustrates a serious drawback of the FE-count approach. The computed "nonemptiness grade" of 1 in case (a) is adequate because (a) is a crisp nonempty region. The fuzzy image region in (b) is certainly fully nonempty to a degree of one, too, because it contains the crisp nonempty image region (a). The result of the FE-count approach, however, is 0.5. The FE-count approach hence produces counterintuitive results even in the simple case of absolute one-place quantification with a monotonic quantifier.

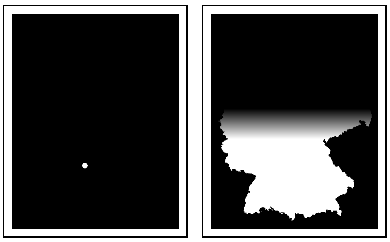


Figure4. The image region X is nonempty (FE-count approach)

The above examples clearly show the drawbacks of existing approaches with respect to linguistic adequacy. In particular, the formulas proposed for the important case of two-place quantification apparently fail to grasp the intuitive meaning of NL quantifiers. In the following, we shall present a theory of fuzzy quantification built on an axiomatic foundation. We will formalize a number of linguistic adequacy conditions on approaches to fuzzy quantification and show that the axioms of our theory incorporate

[4] the FE-count is a fuzzy measure of the cardinality of fuzzy sets, cf. [32]

(or entail) the essential conditions. A model of these axioms, called a DFS (Determiner Fuzzification Scheme), is hence immune against the pitfalls of existing approaches.

3 The axiomatic framework of DFS theory

Because of the inadequate behaviour of all existing approaches in the case of two-place quantification, we have decided to abandon both their representation of fuzzy quantifiers by fuzzy numbers, and the idea of solving the problems of fuzzy quantification by introducing an appropriate measure for the cardinality of fuzzy sets.[5] Instead, we will assume the framework provided by the current *linguistic* theory of NL quantifcation, the theory of generalized quantifiers (TGQ [1,2]), which has been developed independently of the treatment of fuzzy quantifiers in fuzzy set theory, and provides a conceptually rather different view of natural language quantifiers. We shall introduce two-valued quantifiers in concordance with TGQ:

Definition 1. *An* n-ary generalized quantifier *on a base set* $E \neq \varnothing$ *is a mapping* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{2} = \{0, 1\}$.

A two-valued quantifier hence assigns to each n-tuple of crisp subsets $X_1, \ldots, X_n \in \mathcal{P}(E)$ a two-valued quantification result $Q(X_1, \ldots, X_n) \in \mathbf{2}$. Well-known examples are

$$\begin{aligned}
&\forall_E(X) = 1 \Leftrightarrow X = E\\
&\exists_E(X) = 1 \Leftrightarrow X \neq \varnothing\\
&\mathbf{all}_E(X_1,\, X_2) = 1 \Leftrightarrow X_1 \subseteq X_2\\
&\mathbf{some}_E(X_1,\, X_2) = 1 \Leftrightarrow X_1 \cap X_2 \neq \varnothing\\
&\mathbf{at\ least\ k}_E(X_1,\, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq k\,.
\end{aligned}$$

Whenever the base set is clear from the context, we drop the subscript E; $|\bullet|$ denotes cardinality. For finite E, we can define proportional quantifiers like

$$\begin{aligned}
&[\mathbf{rate} \geq r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| \geq r\,|X_1|\\
&[\mathbf{rate} > r](X_1, X_2) = 1 \Leftrightarrow |X_1 \cap X_2| > r\,|X_1|
\end{aligned}$$

for $r \in \mathbf{I}$, $X_1, X_2 \in \mathcal{P}(E)$. For example, "at least 30 percent of the X's are Y's" can be expressed as $[\mathbf{rate} \geq 0.3](X, Y)$, while $[\mathbf{rate} > 0.4]$ is suited to model "more than 40 percent". By the *scope* of an NL quantifier we denote the argument occupied by the verbal phrase (e.g. "sleep" in "all men sleep"); by convention, the scope is the last argument of a quantifier. The first argument of a two-place quantifier is its *restriction*. The two-place use of a two-place quantifier, like in "most X's are Y's" is called its *restricted use*, while its one-place use (relative to the whole domain E, like in "most

[5] As we shall see later, it is possible to recover the cardinality-based approach to fuzzy quantification in the case of quantitative one-place quantifiers, see (Th-60).

elements of the domain are Y") is its *unrestricted use*. For example, the unrestricted use of **all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is modelled by $\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$, which has $\forall(X) = \mathbf{all}(E, X)$. TGQ has classified the wealth of quantificational phenomena in natural languages in order to unveil universal properties shared by quantifiers in all natural languages, or to single out classes of quantifiers with specific properties (we shall describe some of these properties below). However, an extension to the continuous-valued case, in order to better capture the meaning of approximate quantifiers like "many" or "about ten", has not been an issue for TGQ. In addition, TGQ has ignored the problem of providing a convincing interpretation for quantifying statements in the presence of fuzziness, i.e. in the frequent case that the arguments of the quantifier are occupied by concepts like "tall" or "cloudy" which do not possess sharply defined boundaries.

Hence let us introduce the fuzzy framework. Suppose E is a given set. A fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ of a set E assigns to each $e \in E$ a membership degree $\mu_X(e) \in \mathbf{I} = [0, 1]$; we denote by $\widetilde{\mathcal{P}}(E)$ the set of all fuzzy subsets (fuzzy powerset) of E.

Definition 2. *An n-ary* fuzzy quantifier *$\widetilde{Q}$ on a base set $E \neq \varnothing$ is a mapping $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of fuzzy subsets $X_1, \ldots, X_n$ of E assigns a gradual result $\widetilde{Q}(X_1, \ldots, X_n) \in \mathbf{I}$.*[6]

An example is $\widetilde{\mathbf{some}}(X_1, X_2) = \sup\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}$, for all $X \in \widetilde{\mathcal{P}}(E)$. How can we justify that this operator is a good model of the NL quantifier "some"? How can we describe characteristics of fuzzy quantifiers and how can we locate a fuzzy quantifier based on a description of desired properties? Fuzzy quantifiers are possibly too rich a set of operators to investigate this question directly. Few intuitions apply to the behaviour of quantifiers in the case that the arguments are fuzzy, and the familiar concept of *cardinality* of crisp sets, which makes it easy to define quantifiers on crisp arguments, is no longer available.

We therefore have to introduce some kind of *simplified description* of the essential aspects of a fuzzy quantifier. In order to comply with linguistic theory, this representation should be rich enough to embed all two-valued quantifiers of TGQ.

Definition 3. *An n-ary* semi-fuzzy quantifier *on a base set $E \neq \varnothing$ is a mapping $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which to each n-tuple of* crisp *subsets of E assigns a gradual result $Q(X_1, \ldots, X_n) \in \mathbf{I}$.*

Semi-fuzzy quantifiers are half-way between two-valued quantifiers and fuzzy quantifiers because they have crisp input and fuzzy (gradual) output. In particular, every two-valued quantifier of TGQ is a semi-fuzzy quantifier by definition. To provide an

[6] This definition closely resembles Zadeh's [33, pp.756] alternative view of fuzzy quantifiers as fuzzy second-order predicates, but models these as mappings in order to simplify notation. In addition, we permit for arbitrary $n \in \mathbb{N}$.

example, a possible definition of the semi-fuzzy quantifier **almost all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is

$$\textbf{almost all}(X_1, X_2) = \begin{cases} f_{\text{almost all}}(\frac{|X_1 \cap X_2|}{|X_1|}) & : \quad X_1 \neq \varnothing \\ 1 & : \quad \text{else} \end{cases} \tag{1}$$

where $f_{\text{almost all}}(z) = S(z, 0.7, 0.9)$, using Zadeh's S-function (see Fig. 5). Unlike

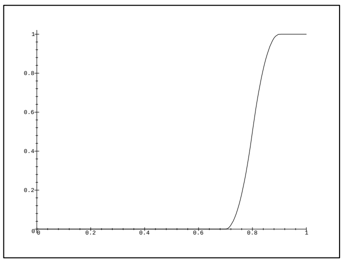


Figure5. A possible definition of $f_{\text{almost all}}$

the representations chosen by existing approaches to fuzzy quantification, semi-fuzzy quantifiers can express genuine multiplace quantification (arbitrary n); they are not restricted to the absolute and proportional types; they are not necessarily quantitative (in the sense of automorphism-invariance); and there is no a priori restriction to finite domains. Compared to fuzzy quantifiers, the main benefit of introducing semi-fuzzy quantifiers is conceptual simplicity due to the restriction to crisp argument sets, which usually makes it easy to understand the input-output behavior of a semi-fuzzy quantifier. Most importantly, we have the familiar concept of crisp cardinality available, which is of invaluable help in defining the quantifiers of interest. Being half-way between two-valued generalized quantifiers and fuzzy quantifiers, semi-fuzzy quantifiers do not accept fuzzy input, and we have to make use of a fuzzification mechanism which transports these to fuzzy quantifiers.

Definition 4. *A* quantifier fuzzification mechanism $\mathcal{F}$ *assigns to each semi-fuzzy quantifier* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a corresponding fuzzy quantifier* $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *of the same arity* n *and on the same base set* E*.*

By viewing approaches to fuzzy quantification as instances of quantifier fuzzification mechanisms (QFM), we are able to explore the linguistic adequacy of these approaches by investigating preservation and homomorphism properties of the corresponding fuzzification mappings [11,12,9]. To this end, we first need to introduce several concepts related to (semi-) fuzzy quantifiers.

Definition 5. *Suppose* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is a fuzzy quantifier. By* $\mathcal{U}(\widetilde{Q}) : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the* underlying semi-fuzzy quantifier, *viz.*

$$\mathcal{U}(\widetilde{Q})(Y_1, \ldots, Y_n) = \widetilde{Q}(Y_1, \ldots, Y_n) \tag{2}$$

for all crisp *subsets* $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

Every reasonable QFM $\mathcal{F}$ should correctly generalise the semi-fuzzy quantifiers to which it is applied, i.e. for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, we should have

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n) \tag{3}$$

for all *crisp* arguments $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, or equivalently: $\mathcal{U}(\mathcal{F}(Q)) = Q$.

Let us now consider a special case of quantifiers, called projection quantifiers. Suppose E is a set of persons and John $\in E$. We can then express the membership assessment "Is John contained in Y?", where Y is a crisp subset $Y \in \mathcal{P}(E)$, by computing $\chi_Y(\text{John})$, where $\chi_Y : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is the characteristic function

$$\chi_Y(e) = \begin{cases} 1 & : \quad e \in Y \\ 0 & : \quad e \notin Y\,. \end{cases} \tag{4}$$

for all $Y \in \mathcal{P}(E)$, $e \in E$. Similarly, we can evaluate the fuzzy membership assessment "To which grade is John contained in X", where $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E, by computing $\mu_X(\text{John})$. Abstracting from argument sets, we obtain the following definitions of projection quantifiers:

Definition 6. *Suppose* $E \neq \varnothing$ *is given and* $e \in E$. *The* projection quantifier $\pi_e : \mathcal{P}(E) \longrightarrow \mathbf{2}$ *is defined by* $\pi_e(Y) = \chi_Y(e)$, *for all* $Y \in \mathcal{P}(E)$.
Similarly, the fuzzy projection quantifier $\widetilde{\pi}_e : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ *is defined by* $\widetilde{\pi}_e(X) = \mu_X(e)$, *for all* $X \in \widetilde{\mathcal{P}}(E)$.

It is apparent from the relationship of these quantifiers with crisp / fuzzy membership assessments that $\widetilde{\pi}_e$ is the proper fuzzy counterpart of π_e, and we should have $\mathcal{F}(\pi_e) = \widetilde{\pi}_e$ in every reasonable QFM. Hence for the crisp subset **married** $\in \mathcal{P}(E)$,

$$\pi_{\text{John}}(\textbf{married}) = \begin{cases} 1 & : \quad \text{John} \in \textbf{married} \\ 0 & : \quad \text{else} \end{cases}$$

and we should also have that $\mathcal{F}(\pi_{\text{John}})(\textbf{lucky}) = \widetilde{\pi}_{\text{John}}(\textbf{lucky}) = \mu_{\textbf{lucky}}(\text{John})$, where **lucky** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of lucky people.

We expect that our framework not only provides an interpretation for quantifiers, but also for the propositional part of the logic. We therefore need to associate a suitable choice of fuzzy conjunction, fuzzy disjunction etc. with a given QFM $\mathcal{F}$. By a canonical construction, which we describe now, $\mathcal{F}$ induces a unique fuzzy operator for each of the propositional connectives. As the starting point for the construction of induced connectives, let us observe that $\mathbf{2}^n \cong \mathcal{P}(\{1, \ldots, n\})$, using the bijection $\eta : \mathbf{2}^n \longrightarrow \mathcal{P}(\{1, \ldots, n\})$ defined by $\eta(x_1, \ldots, x_n) = \{k \in \{1, \ldots, n\} : x_k = 1\}$, for all $x_1, \ldots, x_n \in \mathbf{2}$. An analogous construction is possible in the fuzzy case, where we have $\mathbf{I}^n \cong \widetilde{\mathcal{P}}(\{1, \ldots, n\})$, using the bijection $\widetilde{\eta} : \mathbf{I}^n \longrightarrow \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ defined by $\mu_{\widetilde{\eta}(x_1,\ldots,x_n)}(k) = x_k$ for all $x_1, \ldots, x_n \in \mathbf{I}$ and $k \in \{1, \ldots, n\}$.

Definition 7 (Induced fuzzy truth functions).
Suppose $\mathcal{F}$ is a QFM and f is a semi-fuzzy truth function (i.e. a mapping $f : \mathbf{2}^n \longrightarrow \mathbf{I}$) of arity $n > 0$. The semi-fuzzy quantifier $Q_f : \mathcal{P}(\{1, \ldots, n\}) \longrightarrow \mathbf{I}$ is defined by

$$Q_f(X) = f(\eta^{-1}(X))$$

for all $X \in \mathcal{P}(\{1, \ldots, n\})$. The induced fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots, x_n) = \mathcal{F}(Q_f)(\widetilde{\eta}(x_1, \ldots, x_n)),$$

for all $x_1, \ldots, x_n \in \mathbf{I}$. If $f : \mathbf{I}^0 \longrightarrow \mathbf{I}$ is a nullary semi-fuzzy truth function (i.e., a constant), we shall define $\widetilde{\mathcal{F}}(f) : 2^0 \longrightarrow \mathbf{I}$ by $\widetilde{\mathcal{F}}(f)(\varnothing) = \mathcal{F}(c)(\varnothing)$, where $c : \mathcal{P}(\{\varnothing\})^0 \longrightarrow \mathbf{I}$ is the constant $c(\varnothing) = f(\varnothing)$.[7]

Whenever $\mathcal{F}$ is understood from context, we shall abbreviate $\widetilde{\mathcal{F}}(f)$ as $\widetilde{f}$. By pointwise application of the induced negation $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$, conjunction $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$, and disjunction $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$, $\mathcal{F}$ also induces a unique choice of fuzzy complement $\widetilde{\neg}$, fuzzy intersection $\widetilde{\cap}$, and fuzzy union $\widetilde{\cup}$. In the following, we will assume that an arbitrary but fixed choice of these connectives is given.

Definition 8. *Suppose a semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is given. The* external negation $\widetilde{\neg} Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and the* antonym $Q\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are defined by*

$$(\widetilde{\neg} Q)(X_1, \ldots, X_n) = \widetilde{\neg}(Q(X_1, \ldots, X_n)) \tag{5}$$

$$Q\neg(X_1, \ldots, X_n) = Q(X_1, \ldots, X_{n-1}, \neg X_n) \qquad n > 0 \tag{6}$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$, where $\neg X_n$ denotes complementation. By the dual $Q\widetilde{\square} : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *of Q we denote $Q\widetilde{\square} = \widetilde{\neg} Q\neg$, $n > 0$. The definitions of negation $\widetilde{\neg}\widetilde{Q}$, antonym $\widetilde{Q}\widetilde{\neg}$ and dual $\widetilde{Q}\widetilde{\square}$ of a fuzzy quantifier are analogous.*

For example, **less than n** is the negation of **at least n**, **no** is the antonym of **all**, and **some** is the dual of **all**. It is straightforward to require that a QFM be compatible with

[7] The special treatment of nullary semi-fuzzy truth functions is necessary because in this case, we would have $Q_f : \mathcal{P}(\varnothing) \longrightarrow \mathbf{I}$, which does not conform to our definition of semi-fuzzy quantifiers and fuzzy quantifiers based on nonempty base-sets. Rather than adapting the definition of semi-fuzzy and fuzzy quantifiers in such a way as to allow for empty base sets, and hence cover Q_f for nullary f, too, we prefer to treat the case of nullary truth functions by a different construction, see [9].

these constructions, i.e. we desire that $\mathcal{F}(\textbf{less than n})$ be the negation of $\mathcal{F}(\textbf{at least n})$, $\mathcal{F}(\textbf{no})$ be the antonym of $\mathcal{F}(\textbf{all})$, and $\mathcal{F}(\textbf{some})$ be the dual of $\mathcal{F}(\textbf{all})$. We hence say that $\mathcal{F}$ preserves negation, antonymy, and dualisation, if $\mathcal{F}(\neg Q) = \neg\mathcal{F}(Q)$, $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\neg$ and $\mathcal{F}(Q\widetilde{\Box}) = \mathcal{F}(Q)\widetilde{\Box}$, resp.

Definition 9. *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$. The semi-fuzzy quantifier $Q\cup \in: \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by*

$$\begin{aligned} &Q\cup(X_1,\ \dots,\ X_{n+1}) \\ &= Q(X_1,\ \dots,\ X_{n-1},\ X_n \cup X_{n+1})\,, \end{aligned}$$

for all $X_1,\ \dots,\ X_{n+1} \in \mathcal{P}(E)$. In the case of fuzzy quantifiers, $\widetilde{Q}\widetilde{\cup} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously.

In order to allow for a compositional interpretation of composite quantifiers like "all X's are Y's or $Z's$", we require that a QFM $\mathcal{F}$ be compatible with unions of the argument sets. For example, the semi-fuzzy quantifier $\textbf{all}\cup$, which has $\textbf{all}\cup(X,Y,Z) = \textbf{all}(X, Y \cup Z)$ for crisp $X,Y,Z \in \mathcal{P}(E)$, should be mapped to $\mathcal{F}(\textbf{all})\widetilde{\cup}$, i.e. for all fuzzy subsets $X,Y,Z \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(\textbf{all}\cup)(X,Y,Z) = \mathcal{F}(\textbf{all})(X, Y \mathbin{\widetilde{\cup}} Z)$.

Definition 10. *A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is said to be* nonincreasing *in its i-th argument ($i \in \{1,\ \dots,\ n\}$, $n > 0$) iff for all $X_1,\ \dots,\ X_n,\ X'_i \in \mathcal{P}(E)$ such that $X_i \subseteq X'_i$,*

$$Q(X_1,\dots,X_n) \geq Q(X_1,\ \dots,\ X_{i-1},\ X'_i,\ X_{i+1},\ \dots,\ X_n)\,.$$

Nondecreasing monotonicity of Q is defined by changing '$\geq$' to '$\leq$' in the above inequation. On fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we use an analog definition, where $X_1,\ \dots,\ X_n,\ X'_i \in \widetilde{\mathcal{P}}(E)$, and "$\subseteq$" is the fuzzy inclusion relation.

For example, **all** is nonincreasing in the first and nondecreasing in the second argument. **most** is nondecreasing in its second argument, etc. It is natural to require that monotonicity properties of a quantifier in its arguments be preserved when applying a QFM $\mathcal{F}$. For example, we expect that $\mathcal{F}(\textbf{all})$ is nonincreasing in the first and nondecreasing in the second argument.

Every mapping $f : E \longrightarrow E'$ uniquely determines a powerset function $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$, which is defined by $\widehat{f}(X) = \{f(e) : e \in X\}$, for all $X \in \mathcal{P}(E)$. The underlying mechanism which transports f to $\widehat{f}$ can be generalized to the case of fuzzy sets.

Definition 11 (Induced extension principle).
Suppose $\mathcal{F}$ is a QFM. $\mathcal{F}$ induces an extension principle $\widehat{\mathcal{F}}$ which to each $f : E \longrightarrow E'$ (where $E,\ E' \neq \varnothing$) assigns the mapping $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ defined by

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(e') = \mathcal{F}(\chi_{\widehat{f}(\bullet)}(e'))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $e' \in E'$.

The *standard extension principle* $(\hat{\hat{\bullet}})$ is defined by $\mu_{\hat{\hat{(f)}}(X)}(e') = \sup\{\mu_X(e) : e \in f^{-1}(e')\}$, cf. [30]. Now suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $f_1, \dots, f_n : E' \longrightarrow E$ are given ($E' \neq \varnothing$). We can define a semi-fuzzy quantifier

$$Q' = Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i : \mathcal{P}(E')^n \longrightarrow \mathbf{I},$$
$$Q'(Y_1, \dots, Y_n) = Q(\widehat{f}_1(Y_1), \dots, \widehat{f}_n(Y_n)),$$

for $Y_1, \dots, Y_n \in \mathcal{P}(E')$. We can also construct a fuzzy quantifier $\widetilde{Q}' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ by

$$\widetilde{Q}' = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i) : \mathcal{P}(E')^n \longrightarrow \mathbf{I},$$
$$\widetilde{Q}'(X_1, \dots, X_n) = \mathcal{F}(Q)(\widehat{\mathcal{F}}(f_1)(Y_1), \dots, \widehat{\mathcal{F}}(f_n)(X_n)),$$

for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. It is natural to require that a QFM $\mathcal{F}$ be compatible with its induced extension principle, i.e. $\mathcal{F}(Q') = \widetilde{Q}'$, or equivalently

$$\mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i)\,. \tag{7}$$

Equation (7) hence establishes a relation between powerset functions and the induced extension principle $\widehat{\mathcal{F}}$. It is of particular importance to DFS theory because it is the only axiom which relates the behaviour of $\mathcal{F}$ on different domains E, E'.

The following definition of the DFS axioms summarises our above considerations on reasonable QFMs.

Definition 12 (DFS: Determiner Fuzzification Scheme).
A QFM $\mathcal{F}$ is called a determiner fuzzification scheme (DFS) *iff the following axioms are satisfied for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$:*

Correct generalisation	$\mathcal{U}(\mathcal{F}(Q)) = Q$ *if* $n \leq 1$	(Z-1)
Projection quantifiers	$\mathcal{F}(Q) = \widetilde{\pi}_e$ *if there exists* $e \in E$ *s.th.* $Q = \pi_e$	(Z-2)
Dualisation	$\mathcal{F}(Q\widetilde{\square}) = \mathcal{F}(Q)\widetilde{\square}$ $\quad n > 0$	(Z-3)
Internal joins	$\mathcal{F}(Q\cup) = \mathcal{F}(Q)\widetilde{\cup}$ $\quad n > 0$	(Z-4)
Preservation of monotonicity	*Q noninc. in n-th arg $\Rightarrow$ $\mathcal{F}(Q)$ noninc. in n-th arg,* $n > 0$	(Z-5)
Functional application	$\mathcal{F}(Q \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i) = \mathcal{F}(Q) \circ \mathop{\times}_{i=1}^{n} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \dots, f_n : E' \longrightarrow E$, $E' \neq \varnothing$.	(Z-6)

As has been shown in [9], the axioms (Z-1) to (Z-6) form an independent axiom set.

4 Properties of DFSes

4.1 Correct generalisation

Let us firstly establish that $\mathcal{F}(Q)$ coincides with the original semi-fuzzy quantifier Q when all arguments are crisp sets, i.e. that $\mathcal{F}(Q)$ consistently extends Q.

Theorem 1. *Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier. Then $\mathcal{U}(\mathcal{F}(Q)) = Q$, i.e. for all* crisp *subsets $Y_1, \ldots, Y_n \in \mathcal{P}(E)$,*

$$\mathcal{F}(Q)(Y_1, \ldots, Y_n) = Q(Y_1, \ldots, Y_n)\,.$$

For example, if E is a set of persons, and **women**, **married** $\in \mathcal{P}(E)$ are the crisp sets of "women" and "married persons" in E, then

$$\mathcal{F}(\textbf{some})(\textbf{women}, \textbf{married}) = \textbf{some}(\textbf{women}, \textbf{married})\,,$$

i.e. the "fuzzy some" obtained by applying $\mathcal{F}$ coincides with the (original) "crisp some" whenever the latter is defined, which is of course highly desirable.

4.2 Properties of the induced truth functions

Let us now turn to the fuzzy truth functions induced by a DFS. As for negation, the standard choice in fuzzy logic is certainly $\neg : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\neg x = 1 - x$ for all $x \in \mathbf{I}$. The essential properties of this and other reasonable negation operators are captured by the following definition.

Definition 13. $\tilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *iff it satisfies*

- *a.* $\tilde{\neg}\, 0 = 1$ *(boundary condition)*
- *b.* $\tilde{\neg}\, x_1 \geq \tilde{\neg}\, x_2$ *for all $x_1, x_2 \in \mathbf{I}$ such that $x_1 < x_2$ (i.e. $\tilde{\neg}$ is monotonically decreasing)*
- *c.* $\tilde{\neg} \circ \tilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e. $\tilde{\neg}$ is involutive).*

With conjunction, there are several common choices in fuzzy logic (although the standard is certainly $\wedge = \min$). All of these belong to the class of t-norms, which seems to capture what one would expect of a reasonable conjunction operator. The dual concept of t-norm is that of an s-norm, which expresses the essential properties of fuzzy disjunction operators (cf. Schweizer & Sklar [21]).

Theorem 2. *In every DFS $\mathcal{F}$,*

- *a.* $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$ *is the identity truth function;*
- *b.* $\tilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator;*

c. $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ *is a t-norm;*
d. $x_1 \,\widetilde{\vee}\, x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \,\widetilde{\wedge}\, \widetilde{\neg}\, x_2)$*, i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$*,*
e. $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \,\widetilde{\vee}\, x_2$

The fuzzy disjunction induced by $\mathcal{F}$ is therefore definable in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$, and the fuzzy implication induced by $\mathcal{F}$ is definable in terms of $\widetilde{\vee}$ and $\widetilde{\neg}$ (and hence also in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$). A similar point can be made about all other two-place logical connectives except for antivalence xor and equivalence $\leftrightarrow$ (see remarks on p. 23 and p. 24).

4.3 Preservation of argument structure

We shall now discuss homomorphism properties of DFSes with respect to operations on the argument sets.

Definition 14 (Argument transpositions).
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier,* $n > 0$ *and* $i \in \{0, \ldots, n\}$*. By* $Q\tau_i : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the semi-fuzzy quantifier defined by*

$$Q\tau_i(X_1, \ldots, X_n) = Q(X_1,\ \ldots,\ X_{i-1},\ X_n,\ X_{i+1},\ \ldots,\ X_{n-1},\ X_i)\,,$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$*. In the case of fuzzy quantifiers* $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*, we define* $\widetilde{Q}\tau_i : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *analogously.*

Theorem 3. *Every DFS* $\mathcal{F}$ *is compatible with argument transpositions, i.e. whenever* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $i \in \{1, \ldots, n\}$ *are given, then* $\mathcal{F}(Q\tau_i) = \mathcal{F}(Q)\tau_i$*.*

Because every permutation can be expressed as a sequence of transpositions, (Th-3) ensures that $\mathcal{F}$ commutes with arbitrary permutations of the arguments of a quantifier. In particular, it guarantees that symmetry properties of a quantifier Q carry over to its fuzzified analogon $\mathcal{F}(Q)$. Hence $\mathcal{F}(\textbf{some})(\textbf{rich}, \textbf{young}) = \mathcal{F}(\textbf{some})(\textbf{young}, \textbf{rich})$, i.e. the meaning of “some rich people are young” and “some young people are rich” coincide.

Theorem 4. *Every DFS* $\mathcal{F}$ *is compatible with the formation of antonyms, i.e. whenever* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier of arity* $n > 0$*, then* $\mathcal{F}(Q\neg) = \mathcal{F}(Q)\widetilde{\neg}$*.*

The theorem guarantees e.g. that $\mathcal{F}(\textbf{all})(\textbf{rich}, \widetilde{\neg}\,\textbf{lucky}) = \mathcal{F}(\textbf{no})(\textbf{rich}, \textbf{lucky})$. Let us note that by (Th-3), the theorem generalises to arbitrary argument positions.

Theorem 5. *Every DFS* $\mathcal{F}$ *is compatible with the negation of quantifiers, i.e. whenever* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a semi-fuzzy quantifier, then* $\mathcal{F}(\widetilde{\neg}\, Q) = \widetilde{\neg}\, \mathcal{F}(Q)$*.*

Hence $\mathcal{F}(\textbf{at most 10})(\textbf{young}, \textbf{rich}) = \widetilde{\neg}\, \mathcal{F}(\textbf{more than 10})(\textbf{young}, \textbf{rich})$.

Definition 15 (Internal meets).
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$. The semi-fuzzy quantifier $Q\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by

$$Q\cap(X_1, \ldots, X_{n+1}) = Q(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1}),$$

for all $(X_1, \ldots, X_{n+1}) \in \mathcal{P}(E)^{n+1}$. In the case of a fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\widetilde{Q}\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously.

Theorem 6. *Every DFS $\mathcal{F}$ is compatible with the intersection of arguments, i.e. whenever $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$, then $\mathcal{F}(Q\cap) = \mathcal{F}(Q)\widetilde{\cap}$.*

For example, $\mathcal{F}(\mathbf{some}) = \mathcal{F}(\exists)\widetilde{\cap}$, because the two-place quantifier **some** can be expressed as **some** $= \exists\cap$. Let us also remark that by (Th-3), this property generalises to intersections in arbitrary argument positions.

Definition 16. *Suppose $\mathcal{F}$ is some QFM. We say that $\mathcal{F}$* is compatible with cylindrical extensions *iff the following condition holds for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Whenever $n' \in \mathbb{N}$, $n' \geq n$; $i_1, \ldots, i_n \in \{1, \ldots, n'\}$ such that $1 \leq i_1 < i_2 < \cdots < i_n \leq n'$, and $Q' : \mathcal{P}(E)^{n'} \longrightarrow \mathbf{I}$ is defined by*

$$Q'(Y_1, \ldots, Y_{n'}) = Q(Y_{i_1}, \ldots, Y_{i_n})$$

for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$, then

$$\mathcal{F}(Q')(X_1, \ldots, X_{n'}) = \mathcal{F}(Q)(X_{i_1}, \ldots, X_{i_n}),$$

for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

This property of being compatible with cylindrical extensions is very fundamental. It simply states that vacuous argument positions of a quantifier can be eliminated. For example, if $Q' : \mathcal{P}(E)^4 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and if there exists a semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ such that $Q'(Y_1, Y_2, Y_3, Y_4) = Q(Y_3)$ for all $Y_1, \ldots, Y_4 \in \mathcal{P}(E)$, then we know that Q' does not really depend on all arguments; it is apparent that the choice of Y_1, Y_2 and Y_4 has no effect on the quantification result. It is hence straightforward to require that $\mathcal{F}(Q')(X_1, X_2, X_3, X_4) = \mathcal{F}(Q)(X_3)$ for all $X_1, \ldots, X_4 \in \widetilde{\mathcal{P}}(E)$, i.e. $\mathcal{F}(Q')$ is also independent of X_1, X_2, X_4, and it can be computed from $\mathcal{F}(Q)$.

Theorem 7. *Every DFS $\mathcal{F}$ is compatible with cylindrical extensions.*

Let us now introduce another very fundamental adequacy condition on QFMs. Suppose $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset. The *support* $\mathrm{spp}(X) \in \mathcal{P}(E)$ and the *core*, $\mathrm{core}(X) \in \mathcal{P}(E)$ are defined by

$$\mathrm{spp}(X) = \{e \in E : \mu_X(e) > 0\} \tag{8}$$

$$\mathrm{core}(X) = \{e \in E : \mu_X(e) = 1\}. \tag{9}$$

spp(X) contains all elements which potentially belong to X and core(X) contains all elements which fully belong to X. The interpretation of a fuzzy subset X is hence ambiguous only with respect to crisp subsets Y in the context range

$$\mathrm{cxt}(X) = \{Y \in \mathcal{P}(E) : \mathrm{core}(X) \subseteq Y \subseteq \mathrm{spp}(Y)\}\,. \tag{10}$$

For example, let $E = \{a, b, c\}$ and suppose $X \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset

$$\mu_X(e) = \begin{cases} 1 & : \quad x = a \text{ or } x = b \\ \frac{1}{2} & : \quad x = c \end{cases} \tag{11}$$

The corresponding context range is

$$\mathrm{cxt}(X) = \{Y : \{a, b\} \subseteq Y \subseteq \{a, b, c\}\} = \{\{a, b\}, \{a, b, c\}\}\,.$$

Now let us consider $\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Because $\exists(\{a, b\}) = \exists(\{a, b, c\}) = 1$, $\exists(Y) = 1$ for all crisp subsets in the context of X. We hence expect that $\mathcal{F}(\exists)(X) = 1$: regardless of whether we assume that $c \in X$ or $c \notin X$, the quantification result is always equal to one.

Definition 17. *Assume that $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ are given. We say that Q and Q' are contextually equal relative to $(X_1, \ldots, X_n)$, in symbols: $Q \sim_{(X_1,\ldots,X_n)} Q'$, if and only if $Q|_{\mathrm{cxt}(X_1)\times\cdots\times\mathrm{cxt}(X_n)} = Q'|_{\mathrm{cxt}(X_1)\times\cdots\times\mathrm{cxt}(X_n)}$, i.e. $Q(Y_1, \ldots, Y_n) = Q'(Y_1, \ldots, Y_n)$ for all $Y_1 \in \mathrm{cxt}(X_1), \ldots, Y_n \in \mathrm{cxt}(X_n)$.*

It is apparent that for each $E \neq \varnothing$, $n \in \mathbb{N}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $\sim_{(X_1,\ldots,X_n)}$ is an equivalence relation on the set of all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

Definition 18. *A QFM $\mathcal{F}$ is said to be* contextual *iff for all $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and every choice of fuzzy argument sets $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$:*

$$Q \sim_{(X_1,\ldots,X_n)} Q' \quad \Rightarrow \quad \mathcal{F}(Q)(X_1, \ldots, X_n) = \mathcal{F}(Q')(X_1, \ldots, X_n)\,.$$

As illustrated by our motivating example, it is highly desirable that a QFM satisfies this very elementary and fundamental adequacy condition.

Theorem 8. *Every DFS $\mathcal{F}$ is contextual.*

Definition 19 (Argument insertion).
Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier, $n > 0$, and $A \in \mathcal{P}(E)$. By $Q \triangleleft A : \mathcal{P}(E)^{n-1} \longrightarrow \mathbf{I}$ we denote the semi-fuzzy quantifier defined by

$$Q \triangleleft A(X_1, \ \ldots, \ X_{n-1}) = Q(X_1, \ \ldots, \ X_{n-1}, A)$$

for all $X_1, \ \ldots, \ X_{n-1} \in \mathcal{P}(E)$. (Analogous definition of $\widetilde{Q} \triangleleft A$ for fuzzy quantifiers).

Theorem 9. *Suppose $\mathcal{F}$ is a contextual QFM which is compatible with cylindrical extensions. Then $\mathcal{F}$ is compatible with argument insertions, i.e. $\mathcal{F}(Q \triangleleft A) = \mathcal{F}(Q) \triangleleft A$ for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and all crisp subsets $A \in \mathcal{P}(E)$. In particular, every DFS is compatible with argument insertions.*
(Proof: A.1, p.47+)

The main application of argument insertion is that of modelling *adjectival restriction* by a crisp adjective. For example, if **married** $\in \mathcal{P}(E)$ is extension of the crisp adjective "married", then $Q' = \mathbf{many}\tau_1 \cap \triangleleft \mathbf{married}\tau_1$, i.e. $Q'(X, Y) = \mathbf{many}(\mathbf{married} \cap X, Y)$, models the composite quantifier "many married X's are Y's". By the above theorem, we then have $\mathcal{F}(Q')(\mathbf{rich}, \mathbf{lucky}) = \mathcal{F}(\mathbf{many})(\mathbf{married} \,\widetilde{\cap}\, \mathbf{rich}, \mathbf{lucky})$, as desired. Let us remark that adjectival restriction with a *fuzzy* adjective cannot be modelled directly. This is because we cannot insert a fuzzy argument A into a semi-fuzzy quantifier.

4.4 Monotonicity properties

Theorem 10. *Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Then Q is monotonically nondecreasing (nonincreasing) in its i-th argument ($i \leq n$) if and only if $\mathcal{F}(Q)$ is monotonically nondecreasing (nonincreasing) in its i-th argument.*

For example, **some** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is monotonically nondecreasing in both arguments. By the theorem, then, $\mathcal{F}(\mathbf{some}) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is nondecreasing in both arguments also. In particular,

$$\mathcal{F}(\mathbf{some})(\mathbf{young\_men},\ \mathbf{very\_tall}) \leq \mathcal{F}(\mathbf{some})(\mathbf{men},\ \mathbf{tall})\,,$$

i.e. "some young men are very tall" entails "some men are tall", if **young_men** $\subseteq$ **men** and **very_tall** $\subseteq$ **tall**.

Let us now state that every DFS preserves monotonicity properties of semi-fuzzy quantifiers even if these hold only locally.

Definition 20. *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $U, V \in \mathcal{P}(E)^n$ are given. We say that* Q is locally nondecreasing in the range (U, V) *iff for all $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \mathcal{P}(E)$ such that $U_i \subseteq X_i \subseteq X'_i \subseteq V_i$ ($i = 1, \ldots, n$), we have $Q(X_1, \ldots, X_n) \leq Q(X'_1, \ldots, X'_n)$. We will say that Q is locally nonincreasing in the range (U, V) if under the same conditions, $Q(X_1, \ldots, X_n) \leq Q(X'_1, \ldots, X'_n)$. On fuzzy quantifiers, local monotonicity is defined analogously, but $X_1, \ldots, X_n, X'_1, \ldots, X'_n$ are taken from $\widetilde{\mathcal{P}}(E)$, and '$\subseteq$' is the fuzzy inclusion relation.*

Theorem 11. *Suppose $\mathcal{F}$ is a DFS, $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a semi-fuzzy quantifier and $U, V \in \mathcal{P}(E)^n$. Then Q is locally nondecreasing (nonincreasing) in the range (U, V) iff $\mathcal{F}(Q)$ is locally nondecreasing (nonincreasing) in the range (U, V).*

DFSes can also be shown to be *monotonic* in the sense of preserving inequations between quantifiers. Let us firstly define a partial order $\leq$ on (semi-)fuzzy quantifiers.

Definition 21. *Suppose* $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers. Let us write* $Q \le Q'$ *iff for all* $X_1, \dots, X_n \in \mathcal{P}(E)$, $Q(X_1, \dots, X_n) \le Q'(X_1, \dots, X_n)$. *On fuzzy quantifiers, we define* $\le$ *analogously, where* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.

Theorem 12. *Suppose* $\mathcal{F}$ *is a DFS and* $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers. Then* $Q \le Q'$ *if and only if* $\mathcal{F}(Q) \le \mathcal{F}(Q')$.

Definition 22. *Suppose* $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *are semi-fuzzy quantifiers and* $U, V \in \mathcal{P}(E)^n$. *We say that* Q_1 *is (not necessarily strictly) smaller than* Q_2 *in the range* (U, V), *in symbols:* $Q_1 \le_{(U,V)} Q_2$, *iff for all* $X_1, \dots, X_n \in \mathcal{P}(E)$ *such that* $U_1 \subseteq X_1 \subseteq V_1, \dots, U_n \subseteq X_n \subseteq V_n$,

$$Q_1(X_1, \dots, X_n) \le Q_2(X_1, \dots, X_n)\,.$$

On semi-fuzzy quantifiers, we define $Q_1 \le_{(U,V)} Q_2$ *analogously, but* $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$, *and '*$\subseteq$*' denotes the fuzzy inclusion relation.*

Every DFS preserves inequations between quantifiers even if these hold only locally.

Theorem 13. *Suppose* $\mathcal{F}$ *is a DFS,* $Q_1, Q_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $U, V \in \mathcal{P}(E)^n$. *Then*

$$Q_1 \le_{(U,V)} Q_2 \Leftrightarrow \mathcal{F}(Q_1) \le_{(U,V)} \mathcal{F}(Q_2)\,.$$

4.5 Properties of the induced extension principle

Theorem 14. *Suppose* $\mathcal{F}$ *is a DFS and* $\widehat{\mathcal{F}}$ *the extension principle induced by* $\mathcal{F}$. *Then for all* $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ *(where* $E \ne \varnothing$, $E' \ne \varnothing$, $E'' \ne \varnothing$*),*

a. $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$
b. $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$

The induced extension principles of all DFSes coincide on injective mappings.

Theorem 15. *Suppose* $\mathcal{F}$ *is a DFS and* $f : E \longrightarrow E'$ *is an injection. Then for all* $X \in \widetilde{\mathcal{P}}(E)$, $v \in E'$,

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(v) = \begin{cases} \mu_X(f^{-1}(v)) & : \quad v \in \mathrm{Im}\, f \\ 0 & : \quad v \notin \mathrm{Im}\, f \end{cases}$$

We shall now introduce an important property of natural language quantifiers known as *having extension:* if $E \subseteq E'$ are base sets, the interpretation of the quantifier of interest in E is $Q_E : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, and its interpretation in E' is $Q_{E'} : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$, then

$$Q_E(X_1, \dots, X_n) = Q_{E'}(X_1, \dots, X_n)\,, \tag{12}$$

for all $X_1 \dots X_n \in \mathcal{P}(E)$. For example, suppose E is a set of men and that **married**, **have_children** $\in \mathcal{P}(E)$ are subsets of E. Further suppose that we extend E to a larger base set E' which, in addition to men, also contains, say, their shoes. We should then expect that $\mathbf{most}_E(\mathbf{married}, \mathbf{have\_children}) = \mathbf{most}_{E'}(\mathbf{married}, \mathbf{have\_children})$, because the shoes we have added to E are neither men, nor do they have children. The cross-domain property of having extension expresses some kind of context insensitivity: given $X_1, \dots, X_n \in \mathcal{P}(E)$, we can add an arbitrary number of objects to our original domain E without altering the quantification result. Alternatively, we can drop elements of E which are irrelevant to all argument sets (i.e. not contained in the union of $X_1, \dots, X_n$). Having extension is hence an insensitivity property of NL quantifiers with respect to the choice of the full domain E, which is often to some degree arbitrary. An analogous definition of having extension for fuzzy quantifiers is easily obtained from (12); in this case, the property must hold for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$. It is natural to require that the fuzzy quantifiers corresponding to given semi-fuzzy quantifiers which have extension also possess this property.

Definition 23. *A QFM $\mathcal{F}$ is said to* preserve extension *if each pair of semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $Q' : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ such that $E \subseteq E'$ and $Q'|_{\mathcal{P}(E)^n} = Q$, i.e. $Q(X_1, \dots, X_n) = Q'(X_1, \dots, X_n)$ for all $X_1, \dots, X_n \in \mathcal{P}(E)$, is mapped to fuzzy quantifiers $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{F}(Q') : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$ with $\mathcal{F}(Q')|_{\widetilde{\mathcal{P}}(E)^n} = \mathcal{F}(Q)$, i.e. $\mathcal{F}(Q)(X_1, \dots, X_n) = \mathcal{F}(Q')(X_1, \dots, X_n)$, for all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$.*

Theorem 16. *Every DFS $\mathcal{F}$ preserves extension.*
(Proof: A.2, p.47+)

This is apparent from (Z-6) and (Th-15).

Definition 24. *A semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is called* quantitative *iff for all automorphisms*[8] *$\beta : E \longrightarrow E$ and all $Y_1, \dots, Y_n \in \mathcal{P}(E)$,*

$$Q(Y_1, \dots, Y_n) = Q(\widehat{\beta}(Y_1),\ \dots,\ \widehat{\beta}(Y_n))\,.$$

Similarly, a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is said to be quantitative *iff for all automorphisms $\beta : E \longrightarrow E$ and all $X_1, \dots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\widetilde{Q}(X_1, \dots, X_n) = \widetilde{Q}(\hat{\hat{\beta}}(X_1),\ \dots,\ \hat{\hat{\beta}}(X_n))\,,$$

where $\hat{\hat{\beta}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ is obtained by applying the standard extension principle.

By (Th-15), the induced extension principles of all DFSes coincide on injective mappings. Therefore, the explicit mention of the standard extension principle in the above definition does *not* limit its applicability to any particular choice of extension principle.

[8] i.e. bijections of E into itself

Theorem 17. *Suppose $\mathcal{F}$ is a DFS. For all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, Q is quantitative if and only if $\mathcal{F}(Q)$ is quantitative.*

For example, the quantitative quantifiers **all**, **some** and **at least n** are mapped to quantitative semi-fuzzy quantifiers $\mathcal{F}(\textbf{all})$, $\mathcal{F}(\textbf{some})$ and $\mathcal{F}(\textbf{at least n})$, respectively. On the other hand, the non-quantitative projection quantifier **john** $= \pi_{\mathrm{John}}$ is mapped to the fuzzy projection quantifier $\mathcal{F}(\textbf{john}) = \widetilde{\pi}_{\mathrm{John}}$, which is also non-quantitative.
We will now establish that a DFS is compatible with exactly one extension principle.

Theorem 18. *Suppose $\mathcal{F}$ is a DFS and $\mathcal{E}$ an extension principle such that for every semi-fuzzy quantifier $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ and all $f_1 : E \longrightarrow E', \ \ldots, f_n : E \longrightarrow E'$, $\mathcal{F}(Q \circ \overset{n}{\underset{i=1}{\times}} \widehat{f}_i) = \mathcal{F}(Q) \circ \overset{n}{\underset{i=1}{\times}} \mathcal{E}(f_i)$. Then $\widehat{\mathcal{F}} = \mathcal{E}$.*

The extension principle $\widehat{\mathcal{F}}$ of a DFS $\mathcal{F}$ is uniquely determined by the fuzzy existential quantifiers $\mathcal{F}(\exists) = \mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ induced by $\mathcal{F}$. The converse can also be shown: the fuzzy existential quantifiers obtained from a DFS $\mathcal{F}$ are uniquely determined by its extension principle $\widehat{\mathcal{F}}$.

Theorem 19. *Suppose $\mathcal{F}$ is a DFS.*

a. For every mapping $f : E \longrightarrow E'$ and all $e' \in E'$, $\mu_{\widehat{\mathcal{F}}(f)(\bullet)}(e') = \mathcal{F}(\exists)\widetilde{\cap}\triangleleft f^{-1}(e')$.

b. If $E \neq \varnothing$ and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$, then $\mathcal{F}(\exists) = \widetilde{\pi}_\varnothing \circ \widehat{\mathcal{F}}(!)$, where $! : E \longrightarrow \{\varnothing\}$ is the mapping defined by $!(x) = \varnothing$ for all $x \in E$.

A notion closely related to extension principles is that of *fuzzy inverse images*. In the case of crisp sets, inverse images $f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$ of a given $f : E \longrightarrow E'$ are defined by $f^{-1}(V) = \{e \in E : f(e) \in V\}$, for all $V \in \mathcal{P}(E')$. Generalising this concept, every QFM $\mathcal{F}$ induces fuzzy inverse images by means of the following construction.

Definition 25. *Suppose $\mathcal{F}$ is a quantifier fuzzification mechanism and $f : E \longrightarrow E'$ is some mapping. $\mathcal{F}$ induces a* fuzzy inverse image mapping $\widehat{\mathcal{F}}^{-1}(f) : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ *which to each $Y \in \widetilde{\mathcal{P}}(E')$ assigns the fuzzy subset $\widehat{\mathcal{F}}^{-1}(f)$ defined by*

$$\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mathcal{F}(\chi_{f^{-1}(\bullet)}(e))(Y)\,.$$

If $\mathcal{F}$ is a DFS, then its induced fuzzy inverse images coincide with the apparent "reasonable" definition:

Theorem 20. *Suppose $\mathcal{F}$ is a DFS, $f : E \longrightarrow E'$ is a mapping and $Y \in \widetilde{\mathcal{P}}(E')$. Then for all $e \in E$, $\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(e) = \mu_Y(f(e))$.*

4.6 Properties with respect to the standard quantifiers

Theorem 21. *Suppose $\mathcal{F}$ is a DFS and $E \neq \varnothing$ is some base set. Then*

$$\mathcal{F}(\forall)(X) = \inf \left\{ \mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\},$$

$$\mathcal{F}(\exists)(X) = \sup \left\{ \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}.$$

In particular, $\mathcal{F}(\forall)$ is a T-quantifier and $\mathcal{F}(\exists)$ is an S-quantifier in the sense of Thiele [23]. If E is finite, i.e. $E = \{e_1, \ldots, e_m\}$ where the e_i are pairwise distinct, then $\mathcal{F}(\forall)(X) = \mathop{\widetilde{\wedge}}_{i=1}^{m} \mu_X(e_i)$ and $\mathcal{F}(\exists)(X) = \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(e_i)$. Hence the fuzzy universal (existential) quantifiers are reasonable in the sense that the important relationship between $\forall$ and $\wedge$ ($\exists$ and $\vee$, resp.), which holds in the finite case, is preserved by a DFS. The theorem also shows that in every DFS, the fuzzy existential and fuzzy universal quantifiers are uniquely determined by the induced fuzzy disjunction and conjunction.

Theorem 22. *Suppose $\mathcal{F}$ is a DFS, $\widehat{\mathcal{F}}$ its induced extension principle and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$.*

a. $\widehat{\mathcal{F}}$ is uniquely determined by $\widetilde{\vee}$;
b. $\widetilde{\vee}$ is uniquely determined by $\widehat{\mathcal{F}}$, viz. $x_1 \widetilde{\vee} x_2 = (\widetilde{\pi}_\varnothing \circ \widehat{\mathcal{F}}(!))(X)$ for all $x_1, x_2 \in \mathbf{I}$, where $X \in \widetilde{\mathcal{P}}(\{1, 2\})$ is defined by $\mu_X(1) = x_1$ and $\mu_X(2) = x_2$, and $!$ is the unique mapping $! : \{1, 2\} \longrightarrow \{\varnothing\}$.

(Proof: A.3, p.48+)

In particular, if $\widehat{\mathcal{F}} = (\hat{\hat{\bullet}})$ is the standard extension principle, then $\widetilde{\vee} = \max$. Because I did not want QFMs in which $\widetilde{\vee} \neq \max$ to be a priori excluded from consideration, I have stated axiom (Z-6) in terms of the extension principle $\widehat{\mathcal{F}}$ induced by $\mathcal{F}$, rather than requiring the compatibility of $\mathcal{F}$ to the standard extension principle.

4.7 Special subclasses of DFSes

We will now turn to subclasses of DFS models which satisfy some additional requirements.

Definition 26. *Let $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ a strong negation operator. A DFS $\mathcal{F}$ is called a $\widetilde{\neg}$-DFS if its induced negation coincides with $\widetilde{\neg}$, i.e. $\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}$. In particular, we will call $\mathcal{F}$ a $\neg$-DFS if it induces the standard negation $\neg x = 1 - x$.*

Definition 27. *Suppose $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ a bijection. For every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, we define*

$$\mathcal{F}^\sigma(Q)(X_1, \ldots, X_n) = \sigma^{-1}\mathcal{F}(\sigma Q)(\sigma X_1, \ldots, \sigma X_n),$$

where σQ abbreviates $\sigma \circ Q$, and $\sigma X_i \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset with $\mu_{\sigma X_i} = \sigma \circ \mu_{X_i}$.

Theorem 23. *If $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ an increasing bijection, then $\mathcal{F}^\sigma$ is a DFS.*

It is well-known [15, Th-3.7] that for every strong negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ there is a monotonically increasing bijection $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ such that $\widetilde{\neg}\, x = \sigma^{-1}(1 - \sigma(x))$ for all $x \in \mathbf{I}$. The mapping σ is called the *generator* of $\widetilde{\neg}$.

Theorem 24. *Suppose $\mathcal{F}$ is a $\widetilde{\neg}$-DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ the generator of $\widetilde{\neg}$. Then $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$ is a $\neg$-DFS and $\mathcal{F} = \mathcal{F}'^{\sigma}$.*

This means that we can freely move from an arbitrary $\widetilde{\neg}$-DFS to a corresponding $\neg$-DFS and vice versa: We would not gain anything (really) new when considering other types of DFSes, and can hence restrict attention to $\neg$-DFSes.

Definition 28. *A $\neg$-DFS $\mathcal{F}$ which induces a fuzzy disjunction $\widetilde{\vee}$ is called a $\widetilde{\vee}$-DFS.*

Theorem 25. *Suppose $\mathcal{J}$ is a non-empty index set and $(\mathcal{F}_j)_{j \in \mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes. Further suppose that $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ satisfies the following conditions:*

a. If $f \in \mathbf{I}^{\mathcal{J}}$ is constant, i.e. if there is a $c \in \mathbf{I}$ such that $f(j) = c$ for all $j \in \mathcal{J}$, then $\Psi(f) = c$.
b. $\Psi(1 - f) = 1 - \Psi(f)$, where $1 - f \in \mathbf{I}^{\mathcal{J}}$ is point-wise defined by $(1 - f)(j) = 1 - f(j)$, for all $j \in \mathcal{J}$.
c. Ψ is monotonically increasing, i.e. if $f(j) \leq g(j)$ for all $j \in \mathcal{J}$, then $\Psi(f) \leq \Psi(g)$.

If we define $\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}]$ by

$$\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}](Q)(X_1, \ldots, X_n) = \Psi((\mathcal{F}_j(Q)(X_1, \ldots, X_n))_{j \in \mathcal{J}})$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, then $\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}]$ is a $\widetilde{\vee}$-DFS.

In particular, convex combinations (e.g., arithmetic mean) and stable symmetric sums [22] of $\widetilde{\vee}$-DFSes are again $\widetilde{\vee}$-DFSes. The $\neg$-DFSes can be partially ordered by "specificity" or "cautiousness", in the sense of closeness to $\frac{1}{2}$. We shall define a partial order $\preceq_c \subseteq \mathbf{I} \times \mathbf{I}$ by

$$x \preceq_c y \Leftrightarrow y \leq x \leq \tfrac{1}{2} \text{ or } \tfrac{1}{2} \leq x \leq y\,, \qquad (13)$$

for all $x, y \in \mathbf{I}$.[9]

Definition 29. *Suppose $\mathcal{F}, \mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is* consistently less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, iff for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(Q)(X_1, \ldots, X_n) \preceq_c \mathcal{F}'(Q)(X_1, \ldots, X_n)$.*

[9] $\preceq_c$ is Mukaidono's ambiguity relation, see [17].

We now wish to establish the existence of consistently least specific $\widetilde{\vee}$-DFSes. In order to be able to state the theorem, we firstly need to introduce the *fuzzy median* $\mathrm{med}_{\frac{1}{2}}$.

Definition 30. *The* fuzzy median $\mathrm{med}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{med}_{\frac{1}{2}}(u_1,\, u_2) = \begin{cases} \min(u_1,\, u_2) & : \quad \min(u_1,\, u_2) > \frac{1}{2} \\ \max(u_1,\, u_2) & : \quad \max(u_1,\, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \quad \text{else} \end{cases}$$

$\mathrm{med}_{\frac{1}{2}}$ is an associative mean operator [3] and the only stable (i.e. idempotent) associative symmetric sum [22]. The fuzzy median can be generalised to an operator $\mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.

Definition 31. *The generalised fuzzy median* $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{med}_{\frac{1}{2}}(\inf X,\ \sup X)\,, \qquad \textit{for all } X \in \mathcal{P}(\mathbf{I}).$$

Theorem 26. *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-index collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$. Then there exists a greatest lower specificity bound on $(\mathcal{F}_j)_{j\in\mathcal{J}}$, i.e. a $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{glb}}$ such that $\mathcal{F}_{\mathrm{glb}} \preceq_c \mathcal{F}_j$ for all $j \in \mathcal{J}$ (i.e. $\mathcal{F}_{\mathrm{glb}}$ is a lower specificity bound), and for all other lower specificity bounds $\mathcal{F}'$, $\mathcal{F}' \preceq_c \mathcal{F}_{\mathrm{glb}}$. $\mathcal{F}_{\mathrm{glb}}$ is defined by*

$$\mathcal{F}_{\mathrm{glb}}(Q)(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}\,,$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

In particular, the theorem asserts the existence of least specific $\widetilde{\vee}$-DFSes, i.e. whenever $\widetilde{\vee}$ is an s-norm such that $\widetilde{\vee}$-DFSes exist, then there exists a least specific $\widetilde{\vee}$-DFS (just apply the above theorem to the collection of all $\widetilde{\vee}$-DFSes). We shall now address the issue of most specific DFSes:

Definition 32. *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes $\mathcal{F}_j$, $j \in \mathcal{J}$ where $\mathcal{J} \neq \varnothing$. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is called* specificity consistent *iff for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$, either $R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}]$ or $R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1]$, where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.*

Theorem 27. *Suppose $\widetilde{\vee}$ is an s-norm and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is a $\mathcal{J}$-indexed collection of $\widetilde{\vee}$-DFSes where $\mathcal{J} \neq \varnothing$.*

a. $(\mathcal{F}_j)_{j\in\mathcal{J}}$ has upper specificity bounds exactly if $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent.
b. If $(\mathcal{F}_j)_{j\in\mathcal{J}}$ is specificity consistent, then its least upper specificity bound is the $\widetilde{\vee}$-DFS $\mathcal{F}_{\mathrm{lub}}$ defined by

$$\mathcal{F}_{\mathrm{lub}}(Q)(X_1, \ldots, X_n) = \begin{cases} \sup R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [\frac{1}{2}, 1] \\ \inf R_{Q,X_1,\ldots,X_n} & : \quad R_{Q,X_1,\ldots,X_n} \subseteq [0, \frac{1}{2}] \end{cases}$$

where $R_{Q,X_1,\ldots,X_n} = \{\mathcal{F}_j(Q)(X_1, \ldots, X_n) : j \in \mathcal{J}\}$.

Definition 33. *A DFS $\mathcal{F}$ such that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is called a $(\widetilde{\neg}, \widetilde{\vee})$-DFS.*

Definition 34. *Suppose $\widetilde{\wedge}, \widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ are given. For all semi-fuzzy quantifiers $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the conjunction $Q \,\widetilde{\wedge}\, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the disjunction $Q \,\widetilde{\vee}\, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of Q and Q' are defined by*

$$(Q \,\widetilde{\wedge}\, Q')(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n) \,\widetilde{\wedge}\, Q'(X_1, \ldots, X_n)$$
$$(Q \,\widetilde{\vee}\, Q')(X_1, \ldots, X_n) = Q(X_1, \ldots, X_n) \,\widetilde{\vee}\, Q'(X_1, \ldots, X_n)$$

for all $X_1, \ldots, X_n \in \mathcal{P}(E)$. For fuzzy quantifiers, $\widetilde{Q} \,\widetilde{\wedge}\, \widetilde{Q}'$ and $\widetilde{Q} \,\widetilde{\vee}\, \widetilde{Q}'$ are defined analogously.

In the following, we shall be concerned with $(\widetilde{\neg}, \max)$-DFSes, i.e. DFSes which induce the standard disjunction $\widetilde{\mathcal{F}}(\vee) = \vee = \max$. In the case of $(\widetilde{\neg}, \max)$-DFSes, we can establish a theorem on conjunctions and disjunctions of (semi-)fuzzy quantifiers.

Theorem 28. *Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $Q, Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,*

a. $\mathcal{F}(Q \wedge Q') \leq \mathcal{F}(Q) \wedge \mathcal{F}(Q')$
b. $\mathcal{F}(Q \vee Q') \geq \mathcal{F}(Q) \vee \mathcal{F}(Q')$.

We have so far not made any claims about the interpretation of $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow)$ and $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor})$ in a given DFS $\mathcal{F}$.

Theorem 29. *Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $x_1, x_2 \in \mathbf{I}$,*

a. $x_1 \,\widetilde{\leftrightarrow}\, x_2 = (x_1 \wedge x_2) \vee (\widetilde{\neg}\, x_1 \wedge \widetilde{\neg}\, x_2)$
b. $x_1 \,\widetilde{\mathrm{xor}}\, x_2 = (x_1 \wedge \widetilde{\neg}\, x_2) \vee (\widetilde{\neg}\, x_1 \wedge x_2)$.

Definition 35. *By a* standard DFS *we denote a $(\neg, \max)$-DFS.*

Standard DFS induce the standard connectives of fuzzy logic and by (Th-22), they also induce the standard extension principle. It has been remarked in [11, p.49] that the propositional part of a standard DFS coincides with the well-known K-standard sequence logic of Dienes [6]. In particular, the three-valued fragment is Kleene's three-valued logic. Standard DFSes represent a "boundary" case of DFSes because they induce the smallest fuzzy existential quantifiers, the smallest extension principle, and the largest fuzzy universal quantifiers.

4.8 Further adequacy criteria and theoretical adequacy bounds

In the following, we shall discuss a number of additional adequacy criteria for approaches to fuzzy quantification.

Definition 36. *We say that a QFM $\mathcal{F}$ is* arg-continuous *if and only if $\mathcal{F}$ maps all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to continuous fuzzy quantifiers $\mathcal{F}(Q)$, i.e. for all $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\mathcal{F}(Q)(X_1, \ldots, X_n), \mathcal{F}(Q)(X'_1, \ldots, X'_n)) < \varepsilon$ for all $X'_1, \ldots, X'_n \in \widetilde{\mathcal{P}}(E)$ with $d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) < \delta$; where*

$$d((X_1, \ldots, X_n), (X'_1, \ldots, X'_n)) = \max_{i=1}^{n} \sup\{|\mu_{X_i}(e) - \mu_{X'_i}(e)| : e \in E\} .$$

Definition 37. *We say that a QFM $\mathcal{F}$ is* Q-continuous *if and only if for each semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(\mathcal{F}(Q), \mathcal{F}(Q')) < \varepsilon$ whenever $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ satisfies $d(Q, Q') < \delta$; where*

$$d(Q, Q') = \sup\{|Q(Y_1, \ldots, Y_n) - Q'(Y_1, \ldots, Y_n)| : Y_1, \ldots, Y_n \in \mathcal{P}(E)\}$$
$$d(\mathcal{F}(Q), \mathcal{F}(Q')) = \sup\{|\mathcal{F}(Q)(X_1, \ldots, X_n) - \mathcal{F}(Q')(X_1, \ldots, X_n)| : X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\} .$$

Arg-continuity means that a small change in the membership grades $\mu_{X_i}(e)$ of the argument sets does not change $\mathcal{F}(Q)(X_1, \ldots, X_n)$ drastically; it hence expresses an important robustness condition with respect to noise. Q-continuity captures an important aspect of robustness with respect to imperfect knowledge about the precise definition of a quantifier; i.e. slightly different definitions of Q will produce similar quantification results. Both conditions are crucial to the utility of a DFS and should be possessed by every practical model. They are not part of the DFS axioms because we wanted to have DFSes for general t-norms (including the discontinuous variety).

Theorem 30. *Suppose a DFS $\mathcal{F}$ has the property that*

$$\mathcal{F}(Q \,\widetilde{\wedge}\, Q') = \mathcal{F}(Q) \,\widetilde{\wedge}\, \mathcal{F}(Q') \tag{14}$$

for all semi-fuzzy quantifiers Q, $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, where $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$. Then $\widetilde{\wedge}$ satisfies $x \,\widetilde{\wedge}\, \widetilde{\neg}\, x = 0$ for all $x, y \in \mathbf{I}$.

A DFS $\mathcal{F}$ which is homomorphic with respect to conjunctions (or equivalently, disjunctions) of quantifiers therefore induces a t-norm which respects the law of contradiction. This is clearly unacceptable since it would exclude many interesting t-norms; in particular, the standard choice $\widetilde{\mathcal{F}}(\wedge) = \min$. We have therefore *not* required in general that a DFS be homomorphic with respect to conjunctions/disjunctions of quantifiers.

The reader will certainly have noticed our special treatment of the propositional connectives $\leftrightarrow$ and xor. The difficulties in proving properties of DFSes with respect to these connectives are caused by the fact that the definition of $\leftrightarrow, \mathrm{xor} : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ involves *multiple occurrences* of propositional variables.

Definition 38. *Suppose $Q : \mathcal{P}(E)^m \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $\xi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ is a mapping. By $Q\xi : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the semi-fuzzy quantifier defined by $Q\xi(Y_1, \ldots, Y_n) = Q(Y_{\xi(1)}, \ldots, Y_{\xi(n)})$, for all $Y_1, \ldots, Y_n \in \mathcal{P}(E)$. We use an analog definition for fuzzy quantifiers.*

The interesting case is that of non-injective ξ, which inserts the same variable in two (or more) argument positions of the original quantifier Q.

Theorem 31. *Suppose $\mathcal{F}$ is a DFS which is compatible with the duplication of variables, i.e. whenever $Q : \mathcal{P}(E)^m \longrightarrow \mathbf{I}$ and $\xi : \{1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ for some $n \in \mathbb{N}$, then $\mathcal{F}(Q\xi) = \mathcal{F}(Q)\xi$. Then the induced conjunction $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ satisfies $x \,\widetilde{\wedge}\, \widetilde{\neg}\, x = 0$ for all $x, y \in \mathbf{I}$.*

Again, we find this too restrictive and therefore have *not* required that $\mathcal{F}$ be homomorphic with respect to the duplication of variables.

Definition 39. *Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary semi-fuzzy quantifier such that $n > 0$. Q is said to be* convex[10] in its i-th argument, *where $i \in \{1, \ldots, n\}$, iff*

$$Q(X_1, \ldots, X_n) \geq \min(\, Q(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n), \\ Q(X_1, \ldots, X_{i-1}, X_i'', X_{i+1}, \ldots, X_n))$$

whenever $X_1, \ldots, X_n, X_i', X_i'' \in \mathcal{P}(E)$ and $X_i' \subseteq X_i \subseteq X_i''$.
Convexity of a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ in the i-th argument is defined analogously, where $X_1, \ldots, X_n, X_i', X_i'' \in \widetilde{\mathcal{P}}(E)$, and '$\subseteq$' is the fuzzy inclusion relation.

For example, **between 10 and 20** is convex in both arguments, and **about 30 percent** is convex in the second argument.

Definition 40. *A QFM $\mathcal{F}$ is said to* preserve convexity of n-ary quantifiers, *where $n \in \mathbb{N} \setminus \{0\}$, if and only if every n-ary semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ which is convex in its i-th argument is mapped to a fuzzy quantifier $\mathcal{F}(Q)$ which is also convex in its i-th argument. $\mathcal{F}$ is said to* preserve convexity *if $\mathcal{F}$ preserves the convexity of n-ary quantifiers for all $n > 0$.*

Theorem 32. *Suppose $\mathcal{F}$ is a contextual QFM with the following properties: for every base set $E \neq \varnothing$,*

a. the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;
b. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;
c. If $X \in \widetilde{\mathcal{P}}(E)$ and there exists $e \in E$ such that $\mu_X(e) < 1$, then $\mathcal{F}(\sim\!\forall)(X) > 0$, where $\sim\!\forall : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by

$$(\sim\!\forall)(Y) = \begin{cases} 1 & : \quad X \neq E \\ 0 & : \quad X = E \end{cases}$$

[10] In TGQ, those quantifiers which we call 'convex' are usually dubbed 'continuous'. I have decided to change terminology because of the possible ambiguity of 'continuous', which could also mean 'smooth'.

Then $\mathcal{F}$ does not preserve convexity of one-place quantifiers $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on finite base sets $E \neq \varnothing$. In particular, no DFS preserves convexity.

This means that even if we restrict to the simple case of one-place quantifiers on finite base sets, there is still no QFM $\mathcal{F}$ which both satisfies the very elementary adequacy conditions imposed by the theorem, and preserves convexity under the simplifying assumptions. Because contextuality is a rather fundamental condition, it seems better to weaken our requirements on the preservation of convexity, rather than compromising contextuality or the other elementary conditions.

Theorem 33. *Suppose $\mathcal{F}$ is a contextual QFM which is compatible with cylindrical extensions and satisfies the following properties: for all base sets $E \neq \varnothing$,*

a. *the quantifier $\mathbb{O} : \mathcal{P}(E) \longrightarrow \mathbf{I}$, defined by $\mathbb{O}(Y) = 0$ for all $Y \in \mathcal{P}(E)$, is mapped to the fuzzy quantifier defined by $\mathcal{F}(\mathbb{O})(X) = 0$ for all $X \in \widetilde{\mathcal{P}}(E)$;*
b. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists some $e \in E$ such that $\mu_X(e) > 0$, then $\mathcal{F}(\exists)(X) > 0$;*
c. *If $X \in \widetilde{\mathcal{P}}(E)$ and there exists and there exists some $e \in E$ such that $\mu_X(e') = 0$ for all $e' \in E \setminus \{e\}$ and $\mu_X(e) < 1$, then $\mathcal{F}(\sim\exists)(X) > 0$, where $\sim\exists : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is the quantifier defined by*

$$(\sim\exists)(Y) = \begin{cases} 1 & : \quad X = \varnothing \\ 0 & : \quad X \neq \varnothing \end{cases}$$

Then $\mathcal{F}$ does not preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$ even on finite base sets. In particular, no DFS preserves the convexity of quantitative semi-fuzzy quantifiers of arity $n > 1$.

This leaves open the possibility that certain DFSes will preserve the convexity of quantitative semi-fuzzy quantifiers of arity $n = 1$:

Definition 41. *A QFM $\mathcal{F}$ is said to* weakly preserve convexity *iff $\mathcal{F}$ preserves the convexity of quantitative one-place quantifiers on finite domains.*

In this case, we get positive results on the existence of DFSes that weakly preserve convexity. An example of a conforming DFS will be given below. Let us remark that weak preservation of convexity is strong enough a condition to cover many NL quantifiers of interest, e.g. **between 10 and 20**, **about 50** and others.

One of the pervasive properties of NL quantifiers is *conservativity*. We shall call $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *conservative* if

$$Q(X_1, X_2) = Q(X_1, X_1 \cap X_2) \tag{15}$$

for all $X_1, X_2 \in \mathcal{P}(E)$. To give an example, if E is a set of persons, **married** $\in \mathcal{P}(E)$ is the subset of married persons, and **have_children** $\in \mathcal{P}(E)$ is the set of persons who have children, then the conservative semi-fuzzy quantifier **almost all** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ has

$$\textbf{almost all}(\textbf{married}, \textbf{have\_children}) = \textbf{almost all}(\textbf{married}, \textbf{married} \cap \textbf{have\_children})$$

i.e. the meanings of "almost all married persons have children" and "almost all married persons are married persons who have children" coincide. Like having extension, conservativity expresses an aspect of context insensitivity: if an element of the domain is irrelevant to the restriction (first argument) of a two-place quantifier, then it does not affect the quantification result at all. For example, every conservative $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ apparently has $Q(X_1, X_2 \cap X_1) = Q(X_1, X_2 \cup \neg X_1)$, i.e. if $e \notin X_1$, then it does not matter whether $e \in X_2$ or $e \notin X_2$. A corresponding fuzzy quantifier $\mathcal{F}(Q) : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ should at least possess the following property of *weak conservativity*:

$$\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q)(X_1, \mathrm{spp}(X_1) \cap X_2)\,, \tag{16}$$

for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $\mathrm{spp}(X_1)$ is the support of X_1, see (8). This definition is sufficiently strong to capture the context insensitivity aspect of conservativity: an element $e \in E$ which is irrelevant to the restriction of the quantifier, i.e. $\mu_{X_1}(e) = 0$, has no effect on the quantification result, which is independent of $\mu_{X_2}(e)$.

Theorem 34. *Every DFS $\mathcal{F}$ weakly preserves conservativity, i.e. if $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is conservative, then $\mathcal{F}(Q)$ is weakly conservative.*
(Proof: A.4, p.48+)

Let us say that a fuzzy quantifier $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \mathbf{I}$ is *strongly conservative* if

$$\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q)(X_1, X_1 \mathbin{\widetilde{\cap}} X_2) \qquad \text{for all } X_1, X_2 \in \widetilde{\mathcal{P}}(E).$$

In addition to the context insensitivity aspect, strong conservativity also reflects the definition of crisp conservativity in terms of intersection with the first argument.

Theorem 35. *Assume the QFM $\mathcal{F}$ satisfies the following conditions: (a) $\widetilde{\mathcal{F}}(\mathrm{id}_2) = \mathrm{id}_{\mathbf{I}}$; (b) $\widetilde{\neg}$ is a strong negation operator; (c) $\widetilde{\wedge}$ is a t-norm; (d) $\mathcal{F}$ is compatible with internal meets, see Def. 15; (e) $\mathcal{F}$ is compatible with dualisation. Then $\mathcal{F}$ does not strongly preserve conservativity, i.e. there are conservative $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ such that $\mathcal{F}(Q)$ is not strongly conservative. In particular, no DFS strongly preserves conservativity.*
(Proof: A.5, p.48+)

Hence strong preservation of conservativity cannot be ensured in a fuzzy framework, even under assumptions which are much weaker than the DFS axioms.

In our comments on argument insertion (see p. 16) we have remarked that adjectival restriction with fuzzy adjectives cannot be modelled directly: if $A \in \widetilde{\mathcal{P}}(E)$ is a *fuzzy* subset of E, then only $\mathcal{F}(Q) \triangleleft A$ is defined, but not $Q \triangleleft A$. However, one can ask if $\mathcal{F}(Q) \triangleleft A$ can be represented by a semi-fuzzy quantifier Q', i.e. if there is a Q' such that

$$\mathcal{F}(Q) \triangleleft A = \mathcal{F}(Q')\,. \tag{17}$$

The obvious choice for Q' is the following.

Definition 42. *Suppose $\mathcal{F}$ is a QFM, $Q : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier and $A \in \widetilde{\mathcal{P}}(E)$. Then $Q \,\widetilde{\triangleleft}\, A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by*

$$Q \,\widetilde{\triangleleft}\, A = \mathcal{U}(\mathcal{F}(Q) \triangleleft A)\,,$$

i.e. $Q \,\widetilde{\triangleleft}\, A(Y_1, \ldots, Y_n) = \mathcal{F}(Q)(Y_1, \ldots, Y_n, A)$ for all crisp $Y_1, \ldots, Y_n \in \mathcal{P}(E)$.

$Q' = Q \,\widetilde{\triangleleft}\, A$ is the only choice of Q' which possibly satisfies (17), because any Q' which satisfies $\mathcal{F}(Q') = \mathcal{F}(Q) \triangleleft A$ also satisfies

$$Q' = \mathcal{U}(\mathcal{F}(Q')) = \mathcal{U}(\mathcal{F}(Q) \triangleleft A) = Q \,\widetilde{\triangleleft}\, A\,,$$

which is apparent from (Th-1). Unfortunately, $Q \,\widetilde{\triangleleft}\, A$ is not guaranteed to fulfill (17) in a QFM (not even in a DFS). Let us hence turn this equation as an adequacy condition which ensures that $Q \,\widetilde{\triangleleft}\, A$ conveys the intended meaning in a given QFM $\mathcal{F}$:

Definition 43. *Suppose $\mathcal{F}$ is a QFM. We say that $\mathcal{F}$ is* compatible with fuzzy argument insertion *iff for every semi-fuzzy quantifier $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ of arity $n > 0$ and every $A \in \widetilde{\mathcal{P}}(E)$, $\mathcal{F}(Q \,\widetilde{\triangleleft}\, A) = \mathcal{F}(Q) \triangleleft A$.*

The main application of this property in natural language is that of adjectival restriction of a quantifier by means of a fuzzy adjective. For example, suppose E is a set of people, and **lucky** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of those people in E who are lucky. Further suppose **almost all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier which models "almost all". Finally, suppose the DFS $\mathcal{F}$ is chosen as the model of fuzzy quantification. We can then construct the semi-fuzzy quantifier $Q' =$ **almost all**$\cap \,\widetilde{\triangleleft}\,$**lucky** . If $\mathcal{F}$ is compatible with fuzzy argument insertion, then the semi-fuzzy quantifier Q' is guaranteed to adequately model the composite expression "almost all X's are lucky Y's", because

$$\mathcal{F}(Q')(X_1, X_2) = \mathcal{F}(Q)(X, Y \,\widetilde{\cap}\, \textbf{lucky})$$

for all fuzzy arguments $X, Y \in \widetilde{\mathcal{P}}(E)$, which (relative to $\mathcal{F}$) is the proper expression for interpreting "almost all X's are lucky Y's" in the fuzzy case. Compatibility with fuzzy argument insertion is a very restrictive adequacy condition. We shall present the unique standard DFS which fulfills this condition on p. 33.

Finally, let us recall the specificity order $\preceq_c$ defined in (13). We can extend $\preceq_c$ to fuzzy sets $X \in \widetilde{\mathcal{P}}(E)$, semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy quantifiers $\widetilde{Q} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ as follows:

$$\begin{aligned}
X \preceq_c X' &\iff \mu_X(e) \preceq_c \mu_{X'}(e) && \text{for all } e \in E;\\
Q \preceq_c Q' &\iff Q(Y_1, \ldots, Y_n) \preceq_c Q'(Y_1, \ldots, Y_n) && \text{for all } Y_1, \ldots, Y_n \in \mathcal{P}(E);\\
\widetilde{Q} \preceq_c \widetilde{Q}' &\iff \widetilde{Q}(X_1, \ldots, X_n) \preceq_c \widetilde{Q}'(X_1, \ldots, X_n) && \text{for all } X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)\,.
\end{aligned}$$

Intuitively, we should expect that the quantification results become less specific when the quantifier or the argument sets become less specific. In other words: the fuzzier the input, the fuzzier the output.

Definition 44. *We say that a QFM $\mathcal{F}$* propagates fuzziness in arguments *if and only if the following property is satisfied for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \dots, X_n, X'_1, \dots, X'_n$: If $X_i \preceq_c X'_i$ for all $i = 1, \dots, n$, then $\mathcal{F}(Q)(X_1, \dots, X_n) \preceq_c \mathcal{F}(Q)(X'_1, \dots, X'_n)$. We say that $\mathcal{F}$* propagates fuzziness in quantifiers *if and only if $\mathcal{F}(Q) \preceq_c \mathcal{F}(Q')$ whenever $Q \preceq_c Q'$.*

5 A class of models of the axiomatic framework

We will now address the issue of models of our axiomatic framework. We introduce a class of QFMs which are defined in terms of three-valued cuts of argument sets and subsequent aggregation based on the fuzzy median. We hence generalise the construction successfully used in [11] to define DFSes. In particular, we present a characterisation of the class of $\mathcal{M}_{\mathcal{B}}$-DFSes in terms of necessary and sufficient conditions on the aggregation mapping $\mathcal{B}$. In order to define the unrestricted class of $\mathcal{M}_{\mathcal{B}}$-QFMs, let us recall some concepts introduced in [11]. We need the cut range $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ which represents a three-valued cut at the "cautiousness level" $\gamma \in \mathbf{I}$ by a set of alternatives $\{Y : (X)^{\min}_\gamma \subseteq Y \subseteq (X)^{\max}_\gamma\}$:

Definition 45. *Suppose E is some set, $X \in \widetilde{\mathcal{P}}(E)$ and $\gamma \in \mathbf{I}$. $(X)^{\min}_\gamma, (X)^{\max}_\gamma \in \mathcal{P}(E)$ and $\mathcal{T}_\gamma(X) \subseteq \mathcal{P}(E)$ are defined by*

$$(X)^{\min}_\gamma = \begin{cases} X_{>\frac{1}{2}} & : \gamma = 0 \\ X_{\geq \frac{1}{2} + \frac{1}{2}\gamma} & : \gamma > 0 \end{cases}$$

$$(X)^{\max}_\gamma = \begin{cases} X_{\geq\frac{1}{2}} & : \gamma = 0 \\ X_{> \frac{1}{2} - \frac{1}{2}\gamma} & : \gamma > 0 \end{cases}$$

$$\mathcal{T}_\gamma(X) = \{Y : (X)^{\min}_\gamma \subseteq Y \subseteq (X)^{\max}_\gamma\},$$

where $X_{\geq\alpha} = \{e \in E : \mu_X(e) \geq \alpha\}$ is the α-cut and $X_{>\alpha} = \{e \in E : \mu_X(e) > \alpha\}$ is the strict α-cut.

The basic idea is to view the crisp range $\mathcal{T}_\gamma(X)$ as providing a number of alternatives to be evaluated. For example, in order to evaluate a semi-fuzzy quantifier Q at a certain cut level γ, we have to consider all choices of $Q(Y_1, \dots, Y_n)$, where $Y_i \in \mathcal{T}_\gamma(X_i)$. The set of results obtained in this way must then be aggregated to a single result in the unit interval. The generalised fuzzy median (see Def. 31) is particularly suited to carry out this aggregation because the resulting fuzzification mechanism will then contain Kleene's three-valued logic as a special subcase (see remark on p. 23). Let us hence use the crisp ranges $\mathcal{T}_\gamma(X_i)$ of the argument sets to define a family of QFMs $(\bullet)_\gamma$, indexed by the cautiousness parameter $\gamma \in \mathbf{I}$:

Definition 46. *For every $\gamma \in \mathbf{I}$, we denote by $(\bullet)_\gamma$ the QFM defined by*

$$Q_\gamma(X_1, \dots, X_n) = \mathrm{m}_{\frac{1}{2}}\{Q(Y_1, \dots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\},$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.

None of the QFMs $(\bullet)_\gamma$ is a DFS, the fuzzy median suppresses too much structure. Nevertheless, these QFMs prove useful in defining DFSes. The basic idea is that in order to compute $\mathcal{F}(Q)(X_1, \ldots, X_n)$, we should consider the results obtained at all levels of cautiousness γ, i.e. $(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$. We can then apply various aggregation operators on these γ-indexed results to obtain new QFMs, which might be DFSes. Let us now define the domain on which the aggregation operators will act.

Definition 47. $\mathbb{B}^+, \mathbb{B}^{\frac{1}{2}}, \mathbb{B}^-$ *and* $\mathbb{B} \subseteq \mathbf{I}^\mathbf{I}$ *are defined by*

$$\begin{aligned}
\mathbb{B}^+ &= \{f \in \mathbf{I}^\mathbf{I} : f(0) > \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [\tfrac{1}{2}, 1] \text{ and } f \text{ nonincreasing }\} \\
\mathbb{B}^{\frac{1}{2}} &= \{\mathrm{c}_{\frac{1}{2}}\}\,, \quad \text{where } \mathrm{c}_{\frac{1}{2}} : \mathbf{I} \longrightarrow \mathbf{I} \text{ is the constant } \mathrm{c}_{\frac{1}{2}}(x) = \tfrac{1}{2} \text{ for all } x \in \mathbf{I} \\
\mathbb{B}^- &= \{f \in \mathbf{I}^\mathbf{I} : f(0) < \tfrac{1}{2} \text{ and } f(\mathbf{I}) \subseteq [0, \tfrac{1}{2}] \text{ and } f \text{ nondecreasing }\} \\
\mathbb{B} &= \mathbb{B}^+ \cup \mathbb{B}^{\frac{1}{2}} \cup \mathbb{B}^- \,.
\end{aligned}$$

Theorem 36.

a. *Suppose* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *are given. Then*

$$(Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}} \in \begin{cases} \mathbb{B}^+ & : \quad Q_0(X_1, \ldots, X_n) > \frac{1}{2} \\ \mathbb{B}^{\frac{1}{2}} & : \quad Q_0(X_1, \ldots, X_n) = \frac{1}{2} \\ \mathbb{B}^- & : \quad Q_0(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

b. *For each* $f \in \mathbb{B}$ *there exists* $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ *such that* $f = (Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}}$.

Relative to an aggregation operator $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$, we define the QFM $\mathcal{M}_\mathcal{B}$ which corresponds to $\mathcal{B}$ as follows.

Definition 48. *Suppose* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given. The QFM* $\mathcal{M}_\mathcal{B}$ *is defined by*

$$\mathcal{M}_\mathcal{B}(Q)(X_1, \ldots, X_n) = \mathcal{B}((Q_\gamma(X_1, \ldots, X_n))_{\gamma \in \mathbf{I}})\,, \tag{18}$$

for all semi-fuzzy quantifiers $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $X_1 \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

By the class of $\mathcal{M}_\mathcal{B}$-QFMs we mean the class of all QFMs $\mathcal{M}_\mathcal{B}$ defined in this way. It is apparent that if we do not impose restrictions on admissible choices of $\mathcal{B}$, the resulting QFMs will often fail to be DFSes. We are hence interested in stating necessary and sufficient conditions on $\mathcal{B}$ for $\mathcal{M}_\mathcal{B}$ to be a DFS. In order to do so, we first need to introduce some constructions on $\mathbb{B}$.

Definition 49. *Suppose* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *is a monotonic mapping (i.e., nondecreasing or nonincreasing). The mappings* $f^\flat,\ f^\sharp : \mathbf{I} \longrightarrow \mathbf{I}$ *are defined by:*

$$f^\sharp = \begin{cases} \lim_{y\to x^+} f(y) & : \quad x < 1 \\ f(1) & : \quad x = 1 \end{cases}$$
$$f^\flat = \begin{cases} \lim_{y\to x^-} f(y) & : \quad x > 0 \\ f(0) & : \quad x = 0 \end{cases} \qquad \text{for all } f \in \mathbb{B},\ x \in \mathbf{I}.$$

It is apparent that if $f \in \mathbb{B}$, then $f^\sharp \in \mathbb{B}$ and $f^\flat \in \mathbb{B}$. We shall further introduce several coefficients which describe certain aspects of a mapping $f : \mathbf{I} \longrightarrow \mathbf{I}$.

Definition 50. *For every monotonic mapping* $f : \mathbf{I} \longrightarrow \mathbf{I}$ *(i.e., either nondecreasing or nonincreasing), we define*

$$f_0^* = \lim_{\gamma\to 0^+} f(\gamma) \tag{19}$$
$$f_*^0 = \inf\{\gamma \in \mathbf{I} : f(\gamma) = 0\} \tag{20}$$
$$f_*^{\frac{1}{2}} = \inf\{\gamma \in \mathbf{I} : f(\gamma) = \tfrac{1}{2}\} \tag{21}$$
$$f_1^* = \lim_{\gamma\to 1^-} f(\gamma) \tag{22}$$
$$f_*^1 = \sup\{\gamma \in \mathbf{I} : f(\gamma) = 1\}\,. \tag{23}$$

Based on these concepts, we can now state a number of axioms governing the behaviour of reasonable choices of $\mathcal{B}$.

Definition 51. *Suppose* $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ *is given. For all* $f, g \in \mathcal{B}$*, we define the following conditions on* $\mathcal{B}$*:*

$$\mathcal{B}(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{B-1}$$
$$\mathcal{B}(1-f) = 1 - \mathcal{B}(f) \tag{B-2}$$
$$\text{If } \widehat{f}(\mathbf{I}) \subseteq \{0,\ \tfrac{1}{2},\ 1\}, \text{ then} \tag{B-3}$$
$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2} f_*^{\frac{1}{2}} & : \quad f \in \mathbb{B}^- \end{cases}$$
$$\mathcal{B}(f^\sharp) = \mathcal{B}(f^\flat) \tag{B-4}$$
$$\text{If } f \leq g, \text{ then } \mathcal{B}(f) \leq \mathcal{B}(g) \tag{B-5}$$

Theorem 37.

a. The conditions (B-1) *to* (B-5) *are sufficient for* $\mathcal{M}_\mathcal{B}$ *to be a standard DFS.*
b. The conditions (B-1) *to* (B-5) *are necessary for* $\mathcal{M}_\mathcal{B}$ *to be a DFS.*

c. The conditions (B-1) *to* (B-5) *are independent.*

In particular, $\mathcal{B}(f) = 1 - \mathcal{B}(1-f)$ for all $f \in \mathbb{B}$, and $\mathcal{B}(f) \geq \frac{1}{2}$ whenever $f \in \mathbb{B}^+$. We can hence give a more concise description of $\mathcal{M}_\mathcal{B}$-DFSes, by considering only their behaviour on $\mathbb{B}^+$:

Definition 52. *By $\mathbb{H} \subseteq \mathbf{I}^\mathbf{I}$ we denote the set of nonincreasing $f : \mathbf{I} \longrightarrow \mathbf{I}$, $f \neq 0$, i.e.*

$$\mathbb{H} = \{ f \in \mathbf{I}^\mathbf{I} : \ f \text{ nonincreasing and } f(0) > 0 \} .$$

We can associate with each $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ a corresponding $\mathcal{B} : \mathbb{B} \longrightarrow \mathbf{I}$ as follows:

$$\mathcal{B}(f) = \begin{cases} \frac{1}{2} + \frac{1}{2}\mathcal{B}'(2f-1) & : \quad f \in \mathbb{B}^+ \\ \frac{1}{2} & : \quad f \in \mathbb{B}^{\frac{1}{2}} \\ \frac{1}{2} - \frac{1}{2}\mathcal{B}'(1-2f) & : \quad f \in \mathbb{B}^- \end{cases} \tag{24}$$

Theorem 38. *If $\mathcal{M}_\mathcal{B}$ is a DFS, then $\mathcal{B}$ can be defined in terms of a mapping $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ according to equation* (24). *$\mathcal{B}'$ is defined by*

$$\mathcal{B}'(f) = 2\mathcal{B}(\tfrac{1}{2} + \tfrac{1}{2}f) - 1 . \tag{25}$$

We can hence focus on properties of mappings $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ without loosing any of the desired models.

Definition 53. *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given. For all $f, g \in \mathbb{H}$, we define the following conditions on $\mathcal{B}'$:*

$$\mathcal{B}'(f) = f(0) \quad \text{if } f \text{ is constant, i.e. } f(x) = f(0) \text{ for all } x \in \mathbf{I} \tag{C-1}$$

$$\text{If } \widehat{f}(\mathbf{I}) \subseteq \{0,1\}, \text{ then } \mathcal{B}'(f) = f_*^0, \tag{C-2}$$

$$\mathcal{B}'(f^\sharp) = \mathcal{B}'(f^\flat) \quad \text{if } \widehat{f}((0,1]) \neq \{0\} \tag{C-3}$$

$$\text{If } f \leq g, \text{ then } \mathcal{B}'(f) \leq \mathcal{B}'(g) \tag{C-4}$$

Theorem 39. *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is given and $\mathcal{M}_\mathcal{B}$ is defined by* (18), (24).

a. The conditions (C-1) *to* (C-4) *are sufficient for $\mathcal{M}_\mathcal{B}$ to be a standard DFS.*
b. The conditions (C-1) *to* (C-4) *are necessary for $\mathcal{M}_\mathcal{B}$ to be a DFS.*

Our introducing of $\mathcal{B}'$ is mainly a matter of convenience. We can now succintly define some examples of $\mathcal{M}_\mathcal{B}$-QFMs:

Definition 54. *By $\mathcal{M}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_\int(f) = \int_0^1 f(x)\,dx , \qquad \text{for all } f \in \mathbb{H}.$$

Theorem 40. *$\mathcal{M}$ is a standard DFS.*

$\mathcal{M}$ is Q-continuous and arg-continuous and hence a practical choice for applications.

Definition 55. *By $\mathcal{M}_U$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_U(f) = \max(f_*^1, f_1^*) \qquad \textit{for all } f \in \mathbb{H}\textit{, see (22) and (23).}$$

Theorem 41. *Suppose $\oplus : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is an s-norm and $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^1 \oplus f_1^* \,,$$

for all $f \in \mathbb{H}$. Further suppose that $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to equations (18) *and* (24)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_U$ is a standard DFS. It is neither Q-continuous nor arg-continuous and hence not practical. However, $\mathcal{M}_U$ is of theoretical interest because it represents an extreme case of $\mathcal{M}_\mathcal{B}$-DFS in terms of specificity:

Theorem 42. *$\mathcal{M}_U$ is the least specific $\mathcal{M}_\mathcal{B}$-DFS.*

Let us now consider the issue of most specific $\mathcal{M}_\mathcal{B}$-DFSes.

Definition 56. *By $\mathcal{M}_S$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_S(f) = \min(f_*^0, f_0^*) \qquad \textit{for all } f \in \mathbb{H}\textit{; see (19) and (20).}$$

Theorem 43. *Suppose $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ is defined by*

$$\mathcal{B}'(f) = f_*^0 \odot f_0^*$$

for all $f \in \mathbb{H}$, where $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ is a t-norm. Further suppose that the QFM $\mathcal{M}_\mathcal{B}$ is defined in terms of $\mathcal{B}'$ according to (18) *and* (24)*. Then $\mathcal{M}_\mathcal{B}$ is a standard DFS.*

In particular, $\mathcal{M}_S$ is a standard DFS. Again, $\mathcal{M}_S$ fails on both continuity conditions, but we have:

Theorem 44.
$\mathcal{M}_S$ is the most specific $\mathcal{M}_\mathcal{B}$-DFS.

Definition 57. *By $\mathcal{M}_{CX}$ we denote the $\mathcal{M}_\mathcal{B}$-QFM defined by*

$$\mathcal{B}'_{CX}(f) = \sup\{\min(x, f(x)) : x \in \mathbf{I}\} \qquad \textit{for all } f \in \mathbb{H}.$$

Theorem 45. *Suppose* $\odot : \mathbf{I}^2 \longrightarrow \mathbf{I}$ *is a continuous t-norm and* $\mathcal{B}' : \mathbb{H} \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{B}'(f) = \sup\{\gamma \odot f(\gamma) : \gamma \in \mathbf{I}\}$$

for all $f \in \mathbb{H}$. *Further suppose that* $\mathcal{M}_{\mathcal{B}}$ *is defined in terms of* $\mathcal{B}'$ *according to* (18) *and* (24). *Then* $\mathcal{M}_{\mathcal{B}}$ *is a standard DFS.*

In particular, $\mathcal{M}_{CX}$ is a standard DFS. It is also Q-continuous and arg-continuous and hence a good choice for applications. As we shall now show, $\mathcal{M}_{CX}$ is a DFS with unique properties.

Theorem 46.

a. $\mathcal{M}_{CX}$ *weakly preserves convexity.*
b. If an $\mathcal{M}_{\mathcal{B}}$*-DFS weakly preserves convexity, then* $\mathcal{M}_{CX} \preceq_c \mathcal{M}_{\mathcal{B}}$.

Theorem 47. $\mathcal{M}_{CX}$ *is the only standard DFS which is compatible with fuzzy argument insertion.*

Definition 58. *Suppose* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is a nondecreasing semi-fuzzy quantifier and* $X \in \widetilde{\mathcal{P}}(E)$. *The* Sugeno integral $(S) \int X\, dQ$ *is defined by*

$$(S) \int X\, dQ = \sup\{\min(\alpha, Q(X_{\geq\alpha})) : \alpha \in \mathbf{I}\} .$$

Theorem 48. *Suppose* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is nondecreasing. Then for all* $X \in \widetilde{\mathcal{P}}(E)$,

$$(S) \int X\, dQ = \mathcal{M}_{CX}(Q)(X) .$$

Hence $\mathcal{M}_{CX}$ coincides with the Sugeno integral whenever the latter is defined.

Definition 59. *Suppose* $E \neq \varnothing$ *is a finite base set of cardinality* $|E| = m$. *For a fuzzy subset* $X \in \widetilde{\mathcal{P}}(E)$, *let us denote by* $\mu_{[j]}(X) \in \mathbf{I}$, $j = 1, \ldots, m$, *the* j*-th largest membership value of* X *(including duplicates).*[11] *We shall also stipulate that* $\mu_{[0]}(X) = 1$ *and* $\mu_{[j]}(X) = 0$ *whenever* $j > m$.

Using this notation, we obtain the following corollary (cf. [5]):

Theorem 49. *Suppose* $E \neq \varnothing$ *is a finite base set,* $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ *is a nondecreasing mapping and* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *is defined by* $Q(Y) = q(|Y|)$ *for all* $Y \in \mathcal{P}(E)$. *Then for all* $X \in \widetilde{\mathcal{P}}(E)$,

$$\mathcal{M}_{CX}(Q)(X) = \max\{\min(q(j), \mu_{[j]}(X)) : 0 \leq j \leq |E|\} ,$$

i.e. $\mathcal{M}_{CX}$ *consistently generalises the FG-count approach of [32,25].*

[11] More formally, we can order the elements of E such that $E = \{e_1, \ldots, e_m\}$ and $\mu_X(e_1) \geq \cdots \geq \mu_X(e_m)$ and then define $\mu_{[j]}(X) = \mu_X(e_j)$.

Let us also observe that $\mathcal{M}_{CX}$ is a concrete implementation of a so-called "substitution approach" to fuzzy quantification [24], i.e. the fuzzy quantifier is modelled by constructing an equivalent logical formula as follows.[12]

Theorem 50. *For every $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$,*

$$\begin{aligned}
&\mathcal{M}_{CX}(Q)(X_1, \ldots, X_n)\\
&= \sup\{\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}\\
&= \inf\{\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n) : V, W \in \mathcal{P}(E)^n, V_1 \subseteq W_1, \ldots, V_n \subseteq W_n\}
\end{aligned}$$

where

$$\begin{aligned}
&\widetilde{Q}^L_{V,W}(X_1, \ldots, X_n)\\
&= \min(\Xi_{V,W}(X_1, \ldots, X_n), \inf\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i, \text{ all } i\})\\
&\widetilde{Q}^U_{V,W}(X_1, \ldots, X_n)\\
&= \max(1 - \Xi_{V,W}(X_1, \ldots, X_n), \sup\{Q(Y_1, \ldots, Y_n) : V_i \subseteq Y_i \subseteq W_i, \text{ all } i\})\\
&\Xi_{V,W}(X_1, \ldots, X_n)\\
&= \min_{i=1}^{n} \min(\inf\{\mu_X(e) : e \in V_i\}, \inf\{1 - \mu_X(e) : e \notin W_i\}) .
\end{aligned}$$

Returning to $\mathcal{M}_{\mathcal{B}}$-DFSes in general, we can state that:

Theorem 51.

- *All $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued arguments, i.e. whenever $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$ with $\mu_{X_i}(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$;*
- *all $\mathcal{M}_{\mathcal{B}}$-DFSes coincide on three-valued semi-fuzzy quantifiers, i.e. $Q : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$;*
- *Every $\mathcal{M}_{\mathcal{B}}$-DFS propagates fuzziness in quantifiers and arguments.*

In addition, every $\mathcal{M}_{\mathcal{B}}$-DFS is a consistent generalisation of the fuzzification mechanism proposed by Gaines [8] as a foundation of fuzzy reasoning. A broader class of DFS models is obtained if we drop the median-based aggregation mechanism of $\mathcal{M}_{\mathcal{B}}$-DFSes [10]. To this end, let us observe that $(\bullet)_\gamma$ can be expressed as

$$Q_\gamma(X_1, \ldots, X_n) = \operatorname{med}_{\frac{1}{2}}(Q^{\min}_\gamma(X_1, \ldots, X_n), Q^{\max}_\gamma(X_1, \ldots, X_n)) , \qquad (26)$$

where we abbreviate

$$Q^{\min}_\gamma(X_1, \ldots, X_n) = \inf\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} \qquad (27)$$

$$Q^{\max}_\gamma(X_1, \ldots, X_n) = \sup\{Q(Y_1, \ldots, Y_n) : Y_i \in \mathcal{T}_\gamma(X_i)\} . \qquad (28)$$

[12] In the finite case, inf and sup reduce to logical connectives $\wedge = \max$ and $\vee = \min$ as usual. We need to allow for occurrences of constants $Q(Y_1, \ldots, Y_n) \in \mathbf{I}$ in the resulting formula because the fuzzification mechanism is applied to semi-fuzzy quantifiers, not only to two-valued quantifiers.

This is apparent from Def. 31 and Def. 46. $\mathrm{med}_{\frac{1}{2}}(x, y)$ can then be replaced with other connectives, e.g. the arithmetic mean $(x + y)/2$. An example of a DFS not based in the fuzzy median is the following:

Definition 60. *The QFM $\mathcal{F}_{Ch}$ is defined by*

$$\mathcal{F}_{Ch}(Q)(X_1, \ldots, X_n) = \tfrac{1}{2} \int_0^1 Q_\gamma^{\min}(X_1, \ldots, X_n)\, d\gamma + \tfrac{1}{2} \int_0^1 Q_\gamma^{\max}(X_1, \ldots, X_n)\, d\gamma\,,$$

for all $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

Theorem 52. *$\mathcal{F}_{Ch}$ is a standard DFS.*

The DFS $\mathcal{F}_{Ch}$ is Q-continuous and arg-continuous. However, it neither propagates fuzziness in quantifiers nor in arguments. This demonstrates that $\mathcal{F}_{Ch}$ is rather different from $\mathcal{M}_{\mathcal{B}}$-DFSes.

Definition 61. *Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is a nondecreasing semi-fuzzy quantifier and $X \in \widetilde{\mathcal{P}}(E)$. The* Choquet integral $(Ch) \int X\, dQ$ *is defined by*

$$(Ch) \int X\, dQ = \int_0^1 Q(X_{\geq \alpha})\, d\alpha\,.$$

Theorem 53. *Suppose $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is nondecreasing. Then for all $X \in \widetilde{\mathcal{P}}(E)$,*

$$(Ch) \int X\, dQ = \mathcal{F}_{Ch}(Q)(X)\,.$$

Hence $\mathcal{F}_{Ch}$ coincides with the Choquet integral whenever the latter is defined. As a corollary, we obtain (cf. [5]):

Theorem 54. *Suppose $E \neq \varnothing$ is a finite base set, $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is a nondecreasing mapping such that $q(0) = 0$, $q(|E|) = 1$, and $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by $Q(Y) = q(|Y|)$ for all $Y \in \mathcal{P}(E)$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,*

$$\mathcal{F}_{Ch}(Q)(X) = \sum_{j=1}^{|E|} (q(j) - q(j-1)) \cdot \mu_{[j]}(X)\,,$$

i.e. $\mathcal{F}_{Ch}$ consistently generalises Yager's OWA approach [26].

6 Evaluation of Fuzzy Quantifiers in DFS

Let us now discuss the computational aspects and show how the fuzzy quantifiers $\mathcal{F}(Q)$ can be efficiently implemented.

6.1 Evaluation of "simple" quantifiers

Theorem 55. *In every standard DFS $\mathcal{F}$ and for all $E \neq \varnothing$,*

$$\begin{aligned}
&\mathcal{F}(\exists)(X) = \sup\{\mu_X(e) : e \in E\}\\
&\mathcal{F}(\forall)(X) = \inf\{\mu_X(e) : e \in E\}\\
&\mathcal{F}(\textbf{all})(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}\\
&\mathcal{F}(\textbf{some})(X_1, X_2) = \sup\{\min(\mu_{X_1}(e), \mu_{X_2}(e)) : e \in E\}\\
&\mathcal{F}(\textbf{no})(X_1, X_2) = \inf\{\max(1 - \mu_{X_1}(e), 1 - \mu_{X_2}(e)) : e \in E\}\\
&\mathcal{F}(\textbf{at least k})(X_1, X_2) = \sup\{\alpha \in \mathbf{I} : |(X_1 \cap X_2)_{\geq\alpha}| \geq k\}\,,
\end{aligned}$$

for all $X, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. In particular, if E is finite, then

$$\mathcal{F}(\textbf{at least k})(X_1, X_2) = \mu_{[k]}(X_1 \cap X_2)$$

Let us remark that **more than k** $=$ **at least k+1**, **less than k** $= 1 -$ **at least k** and **at most k** $= 1 -$ **more than k**, i.e. these quantifiers can be computed from **at least k**.

6.2 Evaluation of quantitative one-place quantifiers

We first need some observations on quantitative one-place quantifiers.

Theorem 56. *A one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set $E \neq \varnothing$ is quantitative if and only if there exists a mapping $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ such that $Q(Y) = q(|Y|)$, for all $Y \in \mathcal{P}(E)$. q is defined by*

$$q(j) = Q(Y_j) \tag{29}$$

for $j \in \{0, \ldots, |E|\}$, where $Y_j \in \mathcal{P}(E)$ is an arbitrary subset of cardinality $|Y_j| = j$.

In particular, if the quantifier has extension, then there exists $\mu_Q : \mathbb{N} \longrightarrow \mathbf{I}$ such that for all finite base sets $E \neq \varnothing$, $q(j) = \mu_Q(j)$ for all $j \in \{0, \ldots, |E|\}$.

Theorem 57. *A quantitative one-place semi-fuzzy quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set is convex if and only if there exists $j_{\mathrm{pk}} \in \{0, \ldots, m\}$ such that $q(\ell) \leq q(u)$ for all $\ell \leq u \leq j_{\mathrm{pk}}$, and $q(\ell) \geq q(u)$ for all $j_{\mathrm{pk}} \leq \ell \leq u$; where $q : \{0, \ldots, |E|\} \longrightarrow \mathbf{I}$ is the mapping defined by* (29).

Theorem 58. *A quantitative one-place semi-fuzzy quantifier* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *on a finite base set is nondecreasing (nonincreasing) if and only if the mapping* q *defined by* (29) *is nondecreasing (nonincreasing).*

Let us now simplify the formulas for $Q_\gamma^{\min}(X_1, \ldots, X_n)$ and $Q_\gamma^{\max}(X_1, \ldots, X_n)$ in the case of a quantitative Q, cf. (26), (27), (28). Given a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ of a finite base set $E \neq \varnothing$ and $\gamma \in \mathbf{I}$, let us abbreviate

$$|X|_\gamma^{\min} = |(X)_\gamma^{\min}| \tag{30}$$

$$|X|_\gamma^{\max} = |(X)_\gamma^{\max}| \tag{31}$$

For all $0 \leq \ell \leq u \leq |E|$, we further define

$$q^{\min}(\ell, u) = \min\{q(k) : \ell \leq k \leq u\} \tag{32}$$

$$q^{\max}(\ell, u) = \max\{q(k) : \ell \leq k \leq u\} \,. \tag{33}$$

Theorem 59. *For every quantitative one-place semi-fuzzy quantifier* $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *on a finite base set, all* $X \in \widetilde{\mathcal{P}}(E)$ *and* $\gamma \in \mathbf{I}$,

$$\begin{aligned} Q_\gamma^{\min}(X) &= q^{\min}(\ell, u) \\ Q_\gamma^{\max}(X) &= q^{\max}(\ell, u) \\ Q_\gamma(X) &= \operatorname{med}_{\frac{1}{2}}(q^{\min}(\ell, u), q^{\max}(\ell, u)) \,, \end{aligned}$$

abbreviating $\ell = |X|_\gamma^{\min}$ *and* $u = |X|_\gamma^{\max}$.

We can use (Th-57) and (Th-58) for some simplifications. If Q is *convex*, then

$$q^{\min}(\ell, u) = \min(q(\ell), q(u))$$

$$q^{\max}(\ell, u) = \begin{cases} q(\ell) & : \quad \ell > j_{\mathrm{pk}} \\ q(u) & : \quad u < j_{\mathrm{pk}} \\ q(j_{\mathrm{pk}}) & : \quad \ell \leq j_{\mathrm{pk}} \leq u \end{cases}$$

and if Q is *monotonic*, then

$$q^{\min}(\ell, u) = q(\ell), \quad q^{\max}(\ell, u) = q(u) \qquad \text{if } Q \text{ nondecreasing}$$

$$q^{\min}(\ell, u) = q(u), \quad q^{\max}(\ell, u) = q(\ell) \qquad \text{if } Q \text{ nonincreasing.}$$

In the case of the DFS $\mathcal{M}_{CX}$, we can use the following *fuzzy interval cardinality* to evaluate quantitative one-place quantifiers.

Definition 62. *For every fuzzy subset* $X \in \widetilde{\mathcal{P}}(E)$, *the* fuzzy interval cardinality $\|X\|_{\mathrm{iv}} \in \widetilde{\mathcal{P}}(\mathbb{N} \times \mathbb{N})$ *is defined by*

$$\mu_{\|X\|_{\mathrm{iv}}}(\ell, u) = \begin{cases} \min(\mu_{[\ell]}(X), 1 - \mu_{[u+1]}(X)) & : \quad \ell \leq u \\ 0 & : \quad \textit{else} \end{cases} \qquad \textit{for all } \ell, u \in \mathbb{N}. \tag{34}$$

Intuitively, $\mu_{\|X\|_{\mathrm{iv}}}(\ell, u)$ expresses the degree to which X has between ℓ and u elements.

Algorithm for computing $\mathcal{M}(Q)(X)$

```
INPUT: X
// initialise H, ℓ, u
H := Hist_X;
ℓ := Σ_{j=1}^{m} H[m+j];
u := ℓ + H[m];
cq := med_{1/2}(q^min(ℓ,u), q^max(ℓ,u));
if( cq == 1/2 ) return 1/2;
sum := cq;
if( cq > 1/2 )
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true; // "no change"
        // update clauses for ℓ and u
        if( H[m+j] ≠ 0 )
            { ℓ := ℓ - H[m+j]; nc:= false; }
        if( H[m-j] ≠ 0 )
            { u := u + H[m-j]; nc:= false; }
        if( nc)
            { sum := sum + cq; continue; }
        // one of ℓ or u has changed
        cq := q^min(ℓ,u);
        if( cq ≤ 1/2 ) break;
        sum := sum + cq;
        }
else
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true;
        .
        : // update clauses etc. as above
        // one of ℓ or u has changed
        cq := q^max(ℓ,u);
        if( cq ≥ 1/2 ) break;
        sum := sum + cq;
        }
return (sum + 1/2*(m-j)) / m;
END
```

Algorithm for computing $\mathcal{F}_{Ch}(Q)(X)$

```
INPUT: X
// initialise H, ℓ, u
H := Hist_X;
ℓ := Σ_{j=1}^{m} H[m+j];
u := ℓ + H[m];
cq := q^min(ℓ,u) + q^max(ℓ,u);
sum := cq;
for( j := 1; j<m; j := j + 1 ) {
    ch := false; // "change"
    // update clauses for ℓ and u
    if( H[m+j] ≠ 0 )
        { ℓ := ℓ - H[m+j]; ch := true; }
    if( H[m-j] ≠ 0 )
        { u := u + H[m-j]; ch := true; }
    if( ch )
        // one of ℓ or u has changed
        { cq := q^min(ℓ,u) + q^max(ℓ,u); }
    sum := sum + cq;
    }
return sum / m'; // where m' = 2*m
END
```

Table1. Algorithms for evaluating quantitative one-place quantifiers

Theorem 60. *For every quantitative one-place quantifier $Q : \mathcal{P}(E) \longrightarrow \mathbf{I}$ on a finite base set and all $X \in \widetilde{\mathcal{P}}(E)$,*

$$\begin{aligned}\mathcal{M}_{CX}(Q)(X) &= \max\{\min(\mu_{\|X\|_{\mathrm{iv}}}(\ell,u), q^{\min}(\ell,u)) : 0 \leq \ell \leq u \leq |E|\} \\ &= \min\{\max(1-\mu_{\|X\|_{\mathrm{iv}}}(\ell,u), q^{\max}(\ell,u)) : 0 \leq \ell \leq u \leq |E|\}\,.\end{aligned}$$

In the case of other $\mathcal{M}_{\mathcal{B}}$-DFSes and for $\mathcal{F}_{Ch}$, a histogram-based approach can be used to efficiently implement the resulting quantifiers. For simplicity of presentation, we will describe a computation procedure suited to integer-arithmetics. We hence assume that, for a fixed $m' \in \mathbb{N} \setminus \{0\}$, all membership values of fuzzy argument sets $X_1, \ldots, X_n$ satisfy

$$\mu_{X_i}(e) \in \left\{0,\ \frac{1}{m'}, \ldots, \frac{m'-1}{m'}, 1\right\} \tag{35}$$

for all $e \in E$. If $X \in \widetilde{\mathcal{P}}(E)$ satisfies (35), we can represent the required histogram of X as an $(m'+1)$-dimensional array $\mathrm{Hist}_X : \{0, \ldots, m'\} \longrightarrow \mathbb{N}$, defined by

$$\mathrm{Hist}_X[j] = \left|\{e \in E : \mu_X(e) = \frac{j}{m'}\}\right|$$

for all $j = 0, \ldots, m'$. We further assume that m' is even, (i.e. $m' = 2m$ for a given $m \in \mathbb{N} \setminus \{0\}$). The computation procedures for the DFSes $\mathcal{M}$ and $\mathcal{F}_{Ch}$ are presented in Table 1.

In the algorithm for $\mathcal{M}$, we have utilized that $Q_\gamma(X) = \max(\frac{1}{2}, q^{\min}(\ell, u))$ if $Q_0(X) > \frac{1}{2}$ and $Q_\gamma(X) = \min(\frac{1}{2}, q^{\max}(\ell, u))$ otherwise. A further simplification is possible if Q is monotonic. For example, if Q is nondecreasing, then $q^{\min}(\ell, u) = q(\ell)$ and $q^{\max}(\ell, u) = q(u)$, i.e. we can omit the updating of u in the first for-loop and likewise omit ℓ in the second for-loop.

6.3 Evaluation of absolute quantifiers and quantifiers of exception

Definition 63. *For every two-place semi-fuzzy quantifier* $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$,

- Q *is called* absolute *iff there exists a quantitative one-place quantifier* $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ *such that* $Q = Q'\cap$, *i.e.* $Q(Y_1, Y_2) = Q'(X_1 \cap X_2)$ *for all* $Y_1, Y_2 \in \mathcal{P}(E)$.
- Q *is called a* quantifier of exception *iff there exists an absolute quantifier* $Q'' : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *such that* $Q = Q''\neg$, *i.e.* $Q(Y_1, Y_2) = Q''(X_1, \neg X_2)$ *for all* $Y_1, Y_2 \in \mathcal{P}(E)$.

For example, the two-place quantifier **about 50** is an absolute quantifier. Some examples of quantifiers of exception are presented in Table 2. The DFS axioms ensure that $\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q')(X_1 \cap X_2)$, whenever $Q = Q'\cap$ is an absolute quantifier and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. Similarly if $Q = Q'\cap\neg$ is a quantifier of exception, then $\mathcal{F}(Q)(X_1, X_2) = \mathcal{F}(Q')(X_1 \cap \neg X_2)$, for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $Q' : \mathcal{P}(E) \longrightarrow \mathbf{I}$ is quantitative. We can hence use the algorithms for computing $\mathcal{F}(Q')(X)$, $\mathcal{F} \in \{\mathcal{M}_{CX}, \mathcal{M}, \mathcal{F}_{Ch}\}$ to evaluate absolute quantifiers and quantifiers of exception.

Quantifier	**Antonym (absolute)**
all	**no**
all except exactly k	**exactly k**
all except about k	**about k**
all except at most k	**at most k**

Table2. Examples of quantifiers of exception

6.4 Evaluation of proportional quantifiers

Definition 64. *A two-place semi-fuzzy quantifier* $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ *on a finite base set is called* proportional *if there exist* $v_0 \in \mathbf{I}$, $f : \mathbf{I} \longrightarrow \mathbf{I}$ *such that*

$$Q(Y_1, Y_2) = \begin{cases} f(|Y_1 \cap Y_2| / |Y_1|) & : \quad Y_1 \neq \varnothing \\ v_0 & : \quad else \end{cases} \qquad \text{for all } Y_1, Y_2 \in \mathcal{P}(E).$$

For example, we have provided a definition of **almost all** where $f(z) = S(x, 0.7, 0.9)$ and $v_0 = 1$, see equation (1). Usually f and v_0 can be chosen independently of E, i.e. Q has extension. We shall restrict our attention to those proportional quantifiers where $f : \mathbf{I} \longrightarrow \mathbf{I}$ is nondecreasing.[13] Suppose Q is such a quantifier and $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$. We are using abbreviations $Z_1 = X_1$, $Z_2 = X_1 \cap X_2$ and $Z_3 = X_1 \cap \neg X_2$; further let $\ell_k = |Z_k|_\gamma^{\min}$ and $u_k = |Z_k|_\gamma^{\max}$, $k \in \{1, 2, 3\}$, $f^{\min} = f(\ell_2/(\ell_2 + u_3))$ and $f^{\max} = f(u_2/(u_2 + \ell_3))$. Then

$$Q_\gamma(X_1, X_2) = \mathrm{med}_{\frac{1}{2}}(q^{\min}(\ell_1, \ell_2, u_1, u_3),\, q^{\max}(\ell_1, \ell_3, u_1, u_2))\,.$$

For the definitions of $q^{\min}, q^{\max} : \{0, \ldots, |E|\}^4 \longrightarrow \mathbf{I}$ and the actual algorithms for evaluating proportional quantifiers, see Table 3 and 4. As shown in Table 4.a, a slight modification of the algorithm for $\mathcal{M}(Q)(X)$ in the case of one-place quantitative quantifiers is sufficient to compute $\mathcal{M}(Q)(X_1, X_2)$ for proportional quantifiers. 4.b depicts the algorithm for evaluating $\mathcal{M}_{CX}(Q)(X_1, X_2)$ in the case of proportional quantifiers. The algorithm for computing $\mathcal{F}_{Ch}(Q)(X)$ can be adapted in a similar fashion in order to implement $\mathcal{F}_{Ch}(Q)(X_1, X_2)$ for proportional quantifiers.

[13] if f is nonincreasing, we can compute $\mathcal{F}(Q) = \neg\mathcal{F}(\neg Q)$, noting that the negation $\neg Q$ is proportional and nondecreasing.

1. $\ell_1 > 0$. Then $q^{\min} = f^{\min}$.
2. $\ell_1 = 0$.
  a. $\ell_2 + u_3 > 0$.
    Then $q^{\min} = \min(v_0,\ f^{\min})$.
  b. $\ell_2 + u_3 = 0$.
    i. $u_1 > 0$.
      Then $q^{\min} = \min(v_0, f(1))$.
    ii. $u_1 = 0$. Then $q^{\min} = v_0$.
 	
Note. If $v_0 \leq f(1)$, then $\min(v_0, f(1)) = v_0$, i.e. we need not distinguish 2.b.i and 2.b.ii.
	For $q^{\max}(\ell_1, \ell_3, u_1, u_2)$, we have:
	
	1. $\ell_1 > 0$. Then $q^{\max} = f^{\max}$.
	2. $\ell_1 = 0$.
	  a. $u_2 + \ell_3 > 0$.
	    Then $q^{\max} = \max(v_0,\ f^{\max})$.
	  b. $u_2 + \ell_3 = 0$.
	    i. $u_1 > 0$.
	      Then $q^{\max} = \max(v_0, f(0))$.
	    ii. $u_1 = 0$. Then $q^{\max} = v_0$.
	
	Note. If $f(0) \leq v_0$, then 2.b.i and 2.b.ii need not be distinguished.

Table3. Computation of $q^{\min}$ and $q^{\max}$ for proportional quantifiers

a. Algorithm for computing $\mathcal{M}(Q)(X_1, X_2)$

```
INPUT: X_1, X_2
// initialise H_k, ℓ, u
H_1 := Hist_{X_1};
H_2 := Hist_{X_1 ∩ X_2};
H_3 := Hist_{X_1 ∩ ¬X_2};
for( k := 1; k ≤ 3; k := k+1 ) {
    ℓ_k := Σ_{j=1}^{m} H_1[m+j]; u_k := ℓ_k + H_k[m];
    }
cq := med_{1/2}(q^min(ℓ_1, ℓ_2, u_1, u_3), q^max(ℓ_1, ℓ_3, u_1, u_2))
if( cq == 1/2 ) return 1/2;
sum := cq;
if( cq > 1/2 )
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true; // "no change"
        // update clauses for ℓ_1, ℓ_2, u_1, u_3
        if( H_1[m+j] ≠ 0 )
            { ℓ_1 := ℓ_1 - H_1[m+j]; nc:= false; }
        if( H_2[m+j] ≠ 0 )
            { ℓ_2 := ℓ_2 - H_2[m+j]; nc:= false; }
        if( H_1[m-j] ≠ 0 )
            { u_1 := u_1 + H_1[m-j]; nc:= false; }
        if( H_3[m-j] ≠ 0 )
            { u_3 := u_3 + H_3[m-j]; nc:= false; }
        if( nc)
            { sum := sum + cq; continue; }
        // one of ℓ_1, ℓ_2, u_1, u_3 has changed
        cq := q^min(ℓ_1, ℓ_2, u_1, u_3);
        if( cq ≤ 1/2 ) break;
        sum := sum + cq;
        }
else
    for( j := 1; j<m; j := j + 1 ) {
        nc:= true;
        // update clauses for ℓ_1, ℓ_3, u_1, u_2
        if( H_1[m+j] ≠ 0 )
            { ℓ_1 := ℓ_1 - H_1[m+j]; nc:= false; }
        if( H_3[m+j] ≠ 0 )
            { ℓ_3 := ℓ_3 - H_3[m+j]; nc:= false; }
        if( H_1[m-j] ≠ 0 )
            { u_1 := u_1 + H_1[m-j]; nc:= false; }
        if( H_2[m-j] ≠ 0 )
            { u_2 := u_2 + H_2[m-j]; nc:= false; }
        if( nc)
            { sum := sum + cq; continue; }
        // one of ℓ_1, ℓ_3, u_1, u_2 has changed
        cq := q^max(ℓ_1, ℓ_3, u_1, u_2);
        if( cq ≥ 1/2 ) break;
        sum := sum + cq;
        }
return (sum + 1/2*(m-j)) / m;
END
```

b. Algorithm for computing $\mathcal{M}_{CX}(Q)(X_1, X_2)$

```
INPUT: X_1, X_2
// initialise H_k, ℓ, u
H_1 := Hist_{X_1};
H_2 := Hist_{X_1 ∩ X_2};
H_3 := Hist_{X_1 ∩ ¬X_2};
for( k := 1; k ≤ 3; k := k+1 ) {
    ℓ_k := Σ_{j=1}^{m} H_1[m+j]; u_k := ℓ_k + H_k[m];
    }
cq := med_{1/2}(q^min(ℓ_1, ℓ_2, u_1, u_3), q^max(ℓ_1, ℓ_3, u_1, u_2))
if( cq == 1/2 ) return 1/2;
sum := cq;
if( cq > 1/2 )
{
    for( j := 1; j<m; j := j + 1 ) {
        ch := false; // "change"
        // update clauses for ℓ_1, ℓ_2, u_1, u_3
        if( H_1[m+j] ≠ 0 )
            { ℓ_1 := ℓ_1 - H_1[m+j]; ch := true; }
        if( H_2[m+j] ≠ 0 )
            { ℓ_2 := ℓ_2 - H_2[m+j]; ch := true; }
        if( H_1[m-j] ≠ 0 )
            { u_1 := u_1 + H_1[m-j]; ch := true; }
        if( H_3[m-j] ≠ 0 )
            { u_3 := u_3 + H_3[m-j]; ch := true; }
        if( ch )
            // one of ℓ_1, ℓ_2, u_1, u_3 has changed
            { cq := q^min(ℓ_1, ℓ_2, u_1, u_3); }
        if( cq ≤ m+j )
            { return (m+j)/m′ }
        }
    return 1;
}
else
    for( j := 1; j<m; j := j + 1 ) {
        ch := false;
        // update clauses for ℓ_1, ℓ_3, u_1, u_2
        if( H_1[m+j] ≠ 0 )
            { ℓ_1 := ℓ_1 - H_1[m+j]; ch := true; }
        if( H_3[m+j] ≠ 0 )
            { ℓ_3 := ℓ_3 - H_3[m+j]; ch := true; }
        if( H_1[m-j] ≠ 0 )
            { u_1 := u_1 + H_1[m-j]; ch := true; }
        if( H_2[m-j] ≠ 0 )
            { u_2 := u_2 + H_2[m-j]; ch := true; }
        if( ch )
            // one of ℓ_1, ℓ_3, u_1, u_2 has changed
            { cq := q^max(ℓ_1, ℓ_3, u_1, u_2); }
        if( cq ≥ m-j )
            { return (m-j)/m′; }
        }
return 0;
END
```

Table4. Algorithms for evaluating two-place proportional quantifiers

7 Conclusion

Fuzzy quantification is a linguistic summarisation technique capable of expressing the global characteristics of a collection of individuals, or of a relation between individuals. However, our findings clearly show that existing approaches to fuzzy quantification fail to provide convincing results in the important case of two-place quantification (e.g. "many blondes are tall"). We have developed an axiomatic framework for fuzzy quantification which complies with a large number of linguistically motivated adequacy criteria. In particular, we have presented the first models of fuzzy quantification which provide an adequate account of the "hard" cases of multiplace quantifiers, non-monotonic quantifiers, and non-quantitative quantifiers, and we have shown how the resulting operators can be efficiently implemented based on histogram computations.

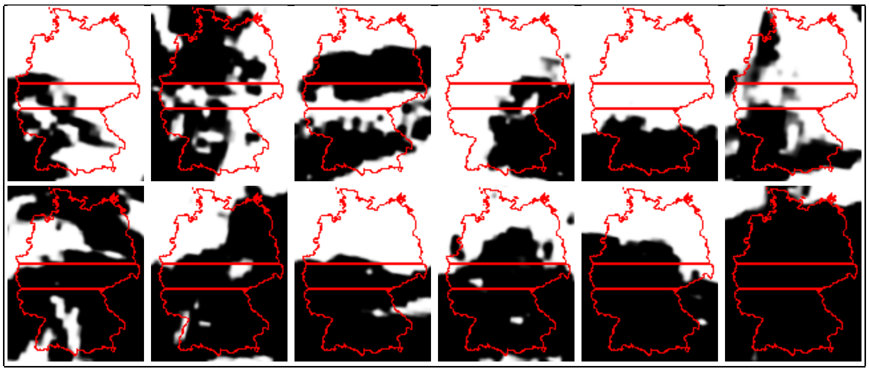


Figure6. Cloudy in Southern Germany (relevant images on top)

The model denoted $\mathcal{M}$ is being applied in a multimedia retrieval system for meteorological documents [13]. Fig. 6 depicts a ranking of cloudiness situations according to the criterion "cloudy in Southern Germany". The ranking has been computed by evaluating $\mathcal{M}(\mathbf{prop})(\mathbf{SouthernGermany}, \mathbf{cloudy})$, where $\mathbf{prop}(Y_1, Y_2) = |Y_1 \cap Y_2|/|Y_1|$. Fig. 7 presents the results of evaluating several accumulative conditions on a set of cloudiness images. These conditions are of the type "Q-times cloudy in the last days". A trapezoidal proportional quantifier $\mathbf{trp}_{a,b,c} : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$, defined by

$$\mathbf{trp}_{a,b,c}(Y_1, Y_2) = \begin{cases} t_{a,b}(|Y_1 \cap Y_2|/|Y_1|) & : \quad Y_1 \neq \varnothing \\ c & : \quad Y_1 = \varnothing \end{cases} \qquad t_{a,b}(z) = \begin{cases} 0 & : \quad z < a \\ \frac{z-a}{b-a} & : \quad a \leq z \leq b \\ 1 & : \quad z > b \end{cases}$$

has been used to model some of the quantifiers. The resulting image R has pixel intensities $R_p = \mathcal{M}(Q)(X_1, X_{2,p})$, for all pixels p where E is the set of images, $\mu_{X_1}(e)$ expresses the degree to which an image $e \in E$ belongs to the fuzzy time interval "in the last days", and $\mu_{X_{2,p}}(e)$ is the degree to which pixel p is cloudy in image e.

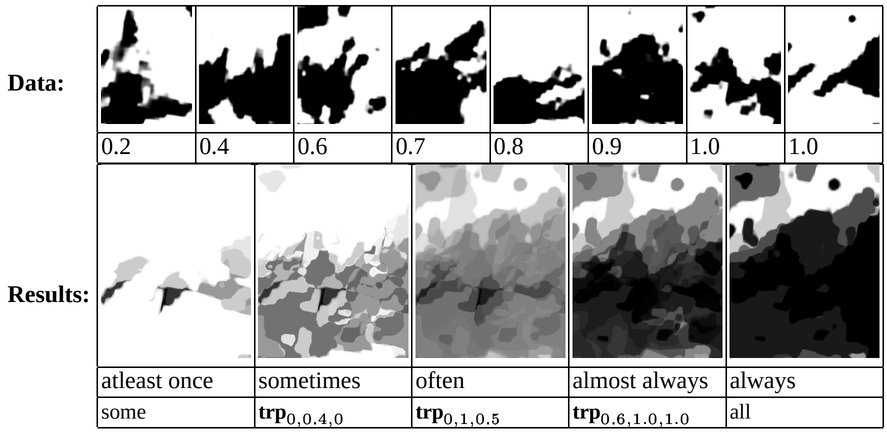


Figure7. Cloudy in the last days: Summarisation results for different choices of the quantifier Q.

References

1. J. Barwise and R. Cooper. Generalized quantifiers and natural language. *Linguistics and Philosophy*, 4:159–219, 1981.
2. J. van Benthem. Questions about quantifiers. *J. of Symb. Logic*, 49, 1984.
3. I. Bloch. Information combination operators for data fusion: a comparative review with classification. *IEEE Transactions on Systems, Man, and Cybernetics*, 26(1):52–67, 1996.
4. G. Bordogna and G. Pasi. A fuzzy information retrieval system handling users' preferences on document sections. In Dubois et al. [7].
5. P. Bosc and L. Lietard. Monotonic quantified statements and fuzzy integrals. In *Proc. of the NAFIPS/IFI/NASA '94 Joint Conference*, pages 8–12, San Antonio, Texas, 1994.
6. Z.P. Dienes. On an implication function in many-valued systems of logic. *Journal of Symbolic Logic*, 14:95–97, 1949.
7. D. Dubois, H. Prade, and R.R. Yager, editors. *Fuzzy Information Engineering*. Wiley, New York, 1997.
8. B.R. Gaines. Foundations of fuzzy reasoning. *Int. J. Man-Machine Studies*, 8:623–668, 1978.
9. I. Glöckner. Advances in DFS theory. TR2000-01, Technical Faculty, University Bielefeld, 33501 Bielefeld, Germany, Feb 2000.
10. I. Glöckner. A broad class of standard DFSes. TR2000-02, Technical Faculty, University Bielefeld, 33501 Bielefeld, Germany, 2000. to appear 2/2000.
11. I. Glöckner. DFS – an axiomatic approach to fuzzy quantification. TR97-06, Techn. Fakultät, Univ. Bielefeld, 1997.
12. I. Glöckner. A framework for evaluating approaches to fuzzy quantification. TR99-03, Techn. Fakultät, Univ. Bielefeld, 1999.
13. I. Glöckner and A. Knoll. Query evaluation and information fusion in a retrieval system for multimedia documents. In *Proceedings of Fusion'99*, pages 529–536, Sunnyvale, CA, 1999.
14. J. Kacprzyk and A. Ziólkowsi. Database queries with fuzzy linguistic quantifiers. *IEEE Trans. on Systems, Man, and Cybernetics*, 16(3):474–479, 1986.
15. G.J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications*. Prentice Hall, Upper Saddle River, NJ, 1995.
16. Y. Liu and E.E. Kerre. An overview of fuzzy quantifiers. (I). interpretations. *Fuzzy Sets and Systems*, 95:1–21, 1998.
17. M. Mukaidono. On some properties of fuzzy logic. *Syst.—Comput.—Control*, 6(2):36–43, 1975.
18. A.L. Ralescu. A note on rule representation in expert systems. *Information Sciences*, 38:193–203, 1986.
19. D. Ralescu. Cardinality, quantifiers, and the aggregation of fuzzy criteria. *Fuzzy Sets and Systems*, 69:355–365, 1995.
20. D. Rasmussen and R. Yager. A fuzzy SQL summary language for data discovery. In Dubois et al. [7], pages 253–264.
21. B. Schweizer and A. Sklar. *Probabilistic metric spaces*. North-Holland, Amsterdam, 1983.
22. W. Silvert. Symmetric summation: A class of operations on fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics*, 9:657–659, 1979.
23. H. Thiele. On T-quantifiers and S-quantifiers. In *The Twenty-Fourth International Symposium on Multiple-Valued Logic*, pages 264–269, Boston, MA, 1994.
24. R.R. Yager. Quantified propositions in a linguistic logic. *Int. J. Man-Machine Studies*, 19:195–227, 1983.
25. R.R. Yager. Approximate reasoning as a basis for rule-based expert systems. *IEEE Trans. on Systems, Man, and Cybernetics*, 14(4):636–643, Jul./Aug. 1984.

26. R.R. Yager. On ordered weighted averaging aggregation operators in multicriteria decision-making. *IEEE Trans. on Systems, Man, and Cybernetics*, 18(1):183–190, 1988.
27. R.R. Yager. Connectives and quantifiers in fuzzy sets. *Fuzzy Sets and Systems*, 40:39–75, 1991.
28. R.R. Yager. Counting the number of classes in a fuzzy set. *IEEE Trans. on Systems, Man, and Cybernetics*, 23(1):257–264, 1993.
29. R.R. Yager. Families of OWA operators. *Fuzzy Sets and Systems*, 59:125–148, 1993.
30. L.A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning. *Information Sciences*, 8,9:199–249,301–357, 1975.
31. L.A. Zadeh. A theory of approximate reasoning. In J. Hayes, D. Michie, and L. Mikulich, editors, *Mach. Intelligence*, volume 9, pages 149–194. Halstead, New York, 1979.
32. L.A. Zadeh. A computational approach to fuzzy quantifiers in natural languages. *Computers and Math. with Appl.*, 9:149–184, 1983.
33. L.A. Zadeh. Syllogistic reasoning in fuzzy logic and its application to usuality and reasoning with dispositions. *IEEE Trans. on Systems, Man, and Cybernetics*, 15(6):754–763, 1985.

A Appendix

A.1 Proof of Theorem 9

Suppose $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a semi-fuzzy quantifier of arity $n > 0$ and $A \in \mathcal{P}(E)$. We shall define the n-place quantifier $Q' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$Q'(Y_1, \dots, Y_n) = Q \triangleleft A(Y_1, \dots, Y_{n-1}) = Q(Y_1, \dots, Y_{n-1}, A)\,,$$

for all $Y_1, \dots, Y_{n-1} \in \mathcal{P}(E)$. Observing that $\mathrm{cxt}(A) = \{A\}$ because $A \in \mathcal{P}(E)$ is crisp, it is then apparent that $Q \sim_{(X_1, \dots, X_{n-1}, A)} Q'$ for all $X_1, \dots, X_{n-1} \in \widetilde{\mathcal{P}}(E)$ because

$$Q(Y_1, \dots, Y_n) = Q(Y_1, \dots, Y_{n-1}, A) = Q'(Y_1, \dots, Y_n)$$

for all $Y_1 \in \mathrm{cxt}(X_1), \dots, Y_{n-1} \in \mathrm{cxt}(X_{n-1}), Y_n \in \mathrm{cxt}(A) = \{A\}$. Hence

$$\begin{aligned}
&\mathcal{F}(Q)(X_1, \dots, X_{n-1}, A) \\
&= \mathcal{F}(Q')(X_1, \dots, X_{n-1}, A) && \text{because } \mathcal{F} \text{ is contextual} \\
&= \mathcal{F}(Q \triangleleft A)(X_1, \dots, X_{n-1}) && \text{because } \mathcal{F} \text{ preserves cylindrical extensions.}
\end{aligned}$$

A.2 Proof of Theorem 16

Suppose $\mathcal{F}$ is a DFS and $Q : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $Q : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ is a pair of semi-fuzzy quantifiers such that $E \subseteq E'$ and $Q'|_{\mathcal{P}(E)^n} = Q$, i.e.

$$Q(Y_1, \dots, Y_n) = Q'(Y_1, \dots, Y_n)\,, \tag{36}$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$. Denoting by $k : E \longrightarrow E'$ the inclusion $k(e) = e$ for al $e \in E$ and recalling the definition of crisp extension $\widehat{k}$ (see p. 10), equation (36) is apparently equivalent to

$$Q(Y_1, \dots, Y_n) = Q'(\widehat{k}(Y_1), \dots, \widehat{k}(Y_n))$$

for all $Y_1, \dots, Y_n \in \mathcal{P}(E)$, i.e. $Q = Q' \circ \mathop{\times}\limits_{i=1}^{n} \widehat{k}$. Hence

$$\begin{aligned}
\mathcal{F}(Q) &= \mathcal{F}(Q' \circ \mathop{\times}\limits_{i=1}^{n} \widehat{k}) \\
&= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \widehat{\mathcal{F}}(k) && \text{by (Z-6)} \\
&= \mathcal{F}(Q') \circ \mathop{\times}\limits_{i=1}^{n} \hat{\hat{k}}\,. && \text{by (Th-15)}
\end{aligned}$$

Therefore

$$\mathcal{F}(Q)(X_1, \dots, X_n) = \mathcal{F}(Q')(Z_1, \dots, Z_n)$$

where

$$\mu_{Z_i}(e) = \begin{cases} \mu_{X_i}(e) & : \quad e \in E \\ 0 & : \quad e \notin E \end{cases}$$

for all $i = 1, \dots, n$ and $e \in E'$, as desired.

A.3 Proof of Theorem 22

a. To see that $\widehat{\mathcal{F}}$ is uniquely determined by $\widetilde{\vee}$, simply combine the equations in (Th-19) and (Th-21).

b. The proof that $x_1 \,\widetilde{\vee}\, x_2 = (\widetilde{\pi}_\varnothing \circ \widehat{\mathcal{F}}(!))(X)$ for all $x_1,\, x_2 \in \mathbf{I}$ has been presented in [11, Th-25; p. 41].

A.4 Proof of Theorem 34

Suppose $Q : \mathcal{P}(E)^2 \longrightarrow \mathbf{I}$ is conservative, i.e.

$$Q(Y_1, Y_2) = Q(Y_1, Y_1 \cap Y_2)\,, \tag{37}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$, and suppose $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ are given. Then clearly

$$Q \sim_{(X_1,X_2)} Q \cap\!\triangleleft \mathrm{spp}(X_1) \tag{38}$$

because

$$Y_1 \in \mathrm{cxt}(X_1) \implies Y_1 \subseteq \mathrm{spp}(X_1) \tag{39}$$

by (10) and hence

$$\begin{aligned}
Q(Y_1, Y_2) &= Q(Y_1, Y_1 \cap Y_2) && \text{by (37)}\\
&= Q(Y_1, (Y_1 \cap \mathrm{spp}(X_1)) \cap Y_2) && \text{by (39)}\\
&= Q(Y_1, Y_1 \cap (\mathrm{spp}(X_1) \cap Y_2)) && \text{by associativity of } \cap\\
&= Q(Y_1, \mathrm{spp}(X_1) \cap Y_2) && \text{by (37)}\\
&= Q(Y_1, Y_2 \cap \mathrm{spp}(X_1))\,, && \text{by commutativity of } \cap
\end{aligned}$$

for all $Y_1 \in \mathrm{cxt}(X_1), Y_2 \in \mathrm{cxt}(X_2)$. Therefore

$$\begin{aligned}
\mathcal{F}(Q)(X_1, X_2) &= \mathcal{F}(Q \cap\!\triangleleft \mathrm{spp}(X_1))(X_1, X_2) && \text{by (38), (Th-8)}\\
&= \mathcal{F}(Q)(X_1, X_2 \,\widetilde{\cap}\, \mathrm{spp}(X_1)) && \text{by (Th-6), (Th-9)}\\
&= \mathcal{F}(Q)(X_1, \mathrm{spp}(X_1) \,\widetilde{\cap}\, X_2)\,, && \text{because } \widetilde{\wedge} \text{ is } t\text{-norm by (Th-2)}
\end{aligned}$$

as desired.

A.5 Proof of Theorem 35

Suppose $\mathcal{F}$ is a QFM which satisfies the conditions a. to e. of the theorem. The proof is by contradiction, i.e. we shall assume that $\mathcal{F}$ strongly preserves conservativity.

Let us first observe that for all $x \in \mathbf{I}$,

$$\begin{aligned}
x &= \mathrm{id}_{\mathbf{I}}(x) \\
&= \widetilde{\mathcal{F}}(\mathrm{id}_2) && \text{by condition a.} \\
&= \mathcal{F}(Q_{\mathrm{id}_2})(\widetilde{\eta}(x)) && \text{by Def. 7} \\
&= \mathcal{F}(\pi_1)(X)\,,
\end{aligned}$$

where $X \in \widetilde{\mathcal{P}}(\{1\})$ is the fuzzy subset defined by $\mu_X(1) = x$, and $\pi_1 : \mathcal{P}(\{1\}) \longrightarrow \mathbf{2}$. Because $x \in \mathbf{I}$ was chosen arbitrarily, this proves that

$$\mathcal{F}(\pi_1) = \widetilde{\pi}_1\,. \tag{40}$$

It is apparent from (15) that $\pi_1 \cap : \mathcal{P}(\{1\})^2 \longrightarrow \mathbf{2}$ is conservative. Hence by our assumption that $\mathcal{F}$ strongly preserves conservativity,

$$\begin{aligned}
\mathcal{F}(\pi_1 \cap) &= \mathcal{F}(\pi_1)\widetilde{\cap} && \text{by condition d.} \\
&= \widetilde{\pi}_1 \widetilde{\cap} && \text{by (40)}
\end{aligned}$$

is strongly conservative. Now suppose $a \in \mathbf{I}$ is given. Let us define $X_1, X_2 \in \widetilde{\mathcal{P}}(\{1\})$ by $\mu_{X_1}(1) = a$, $X_2 = \{1\}$. Then

$$\begin{aligned}
a &= a \mathbin{\widetilde{\wedge}} 1 && \text{by condition c., } \widetilde{\wedge} \text{ is } t\text{-norm} \\
&= \widetilde{\pi}_1(X_1 \mathbin{\widetilde{\cap}} X_2) && \text{by Def. 6, definition of } X_1, X_2 \\
&= \widetilde{\pi}_1\widetilde{\cap}(X_1, X_2) && \text{by Def. 15} \\
&= \widetilde{\pi}_1\widetilde{\cap}(X_1, X_1 \mathbin{\widetilde{\cap}} X_2) && \text{by assumed conservativity of } \widetilde{\pi}_1\widetilde{\cap} \\
&= \widetilde{\pi}_1(X_1 \mathbin{\widetilde{\cap}} (X_1 \mathbin{\widetilde{\cap}} X_2)) && \text{by Def. 15} \\
&= \mu_{X_1}(1) \mathbin{\widetilde{\wedge}} (\mu_{X_1}(1) \mathbin{\widetilde{\wedge}} \mu_{X_2}(1)) && \text{by Def. 6} \\
&= a \mathbin{\widetilde{\wedge}} (a \mathbin{\widetilde{\wedge}} 1) && \text{by definition of } X_1, X_2 \\
&= a \mathbin{\widetilde{\wedge}} a\,. && \text{because } \widetilde{\wedge}\ t\text{-norm by condition c.}
\end{aligned}$$

Because $a \in \mathbf{I}$ was chosen arbitrarily, this means that $\widetilde{\wedge}$ is an idempotent t-norm, i.e.

$$\widetilde{\wedge} = \min\,, \tag{41}$$

see e.g. [15, Th-3.9, p. 63].
Let us now consider another quantifier, viz $Q' = \widetilde{\pi}_1 \cap \square : \mathcal{P}(\{1\})^2 \longrightarrow \mathbf{2}$, i.e.

$$Q'(Y_1, Y_2) = \neg\pi_1(Y_1 \cap \neg Y_2) \tag{42}$$

for all $Y_1, Y_2 \in \mathcal{P}(E)$.[14] Q' is conservative because

$$\begin{aligned}
Q'(Y_1, Y_1 \cap Y_2) &= \neg\pi_1(Y_1 \cap \neg Y_1 \cap Y_2) && \text{by (42)} \\
&= \neg\pi_1(Y_1 \cap (\neg Y_1 \cup \neg Y_2)) && \text{by De Morgan's Law} \\
&= \neg\pi_1((Y_1 \cap \neg Y_1) \cup (Y_1 \cap \neg Y_2)) && \text{by distributivity of } \cup, \cap \\
&= \neg\pi_1(\varnothing \cup (Y_1 \cap \neg Y_2)) && \text{by law of contradiction} \\
&= \neg\pi_1(Y_1 \cap \neg Y_2) && \text{because } \varnothing \text{ is identity of } \cup \\
&= Q'(Y_1, Y_2)\,. && \text{by (42)}
\end{aligned}$$

Therefore $\mathcal{F}(Q')$ is assumed to be strongly conservative. Now let $a, b \in \mathbf{I}$ and define $X_1, X_2 \in \widetilde{\mathcal{P}}(\{1\})$ by $\mu_{X_1}(1) = a$, $\mu_{X_2}(1) = b$. Then

$$\begin{aligned}
\mathcal{F}(Q')(X_1, X_2) &= \widetilde{\neg}\,\widetilde{\pi}_{(}X_1 \,\widetilde{\cap}\, \widetilde{\neg}\, X_2) && \text{by (40) and conditions d.,e.} \\
&= \widetilde{\neg}(a \,\widetilde{\wedge}\, \widetilde{\neg}\, b) && \text{by Def. 6, definition of } X_1, X_2 \\
&= \widetilde{\neg}\min(a, \widetilde{\neg}\, b) && \text{because } \widetilde{\wedge} = \min \text{ by (41)} \\
&= \max(\widetilde{\neg}\, a, b)\,, && \text{because } \widetilde{\neg} \text{ is strong negation by b.}
\end{aligned}$$

i.e.

$$\mathcal{F}(Q')(X_1, X_2) = \max(\widetilde{\neg}\, a, b)\,. \tag{43}$$

Similarly

$$\begin{aligned}
\mathcal{F}(Q')(X_1, X_1 \,\widetilde{\cap}\, X_2) &= \widetilde{\neg}\,\widetilde{\pi}_{(}X_1 \,\widetilde{\cap}\, \widetilde{\neg}(X_1 \,\widetilde{\cap}\, X_2)) && \text{by (40) and conditions d.,e.} \\
&= \widetilde{\neg}(a \,\widetilde{\wedge}\, \widetilde{\neg}(a \,\widetilde{\wedge}\, b)) && \text{by Def. 6, definition of } X_1, X_2 \\
&= \widetilde{\neg}\min(a, \widetilde{\neg}\min(a, b)) && \text{because } \widetilde{\wedge} = \min \text{ by (41)} \\
&= \max(\widetilde{\neg}\, a, \min(a, b))\,, && \text{because } \widetilde{\neg} \text{ is strong negation by b.}
\end{aligned}$$

i.e.

$$\mathcal{F}(Q')(X_1, X_1 \,\widetilde{\cap}\, X_2) = \max(\widetilde{\neg}\, a, \min(a, b))\,. \tag{44}$$

Because $a, b \in \mathbf{I}$ were chosen arbitrarily, we conclude from (43), (44) and the assumed strong conservativity of $\mathcal{F}(Q')$, i.e. $\mathcal{F}(Q')(X_1, X_2) = \mathcal{F}(Q')(X_1, X_1 \,\widetilde{\cap}\, X_2)$, that in fact

$$\max(\widetilde{\neg}\, a, b) = \max(\widetilde{\neg}\, a, \min(a, b))\,, \tag{45}$$

for all $a, b \in \mathbf{I}$. Because $\widetilde{\neg}$ is a strong negation by condition b. of the theorem, we know that there exists an equilibrium point $h \in (0, 1)$ such that $\widetilde{\neg}\, h = h$, see [15, Th-3.4, p. 58]. Now let $h < a < b$. Then $\widetilde{\neg}\, h = h > \widetilde{\neg}\, a$, i.e. $\widetilde{\neg}\, a < h < a < b$ and hence

$$\begin{aligned}
&\max(\widetilde{\neg}\, a, b) = b \\
&\max(\widetilde{\neg}\, a, \min(a, b)) = a \neq b\,,
\end{aligned}$$

which contradicts (45). We conclude that the assumption that $\mathcal{F}$ strongly preserves conservativity is false.

[14] because $\widetilde{\neg}$ is a strong negation by condition b. of the theorem, it coincides with the standard negation $\neg$ on $\{0, 1\}$-valued arguments.

Bisher erschienene Reports an der Technischen Fakultät
Stand: 15. December 1999

2000–01 Advances in DFS Theory
(Ingo Glöckner)

2000–02 A Broad Class of DFS Models
(Ingo Glöckner)

2000–03 An Axiomatic Theory of Fuzzy Quantifiers in Natural Languages
(Ingo Glöckner)