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Abstract

Fuzzy quantifiers (like *many*, *few*, . . .) are an important research topic not only due to their abundance in natural language, but also because an adequate account of these quantifiers would provide a class of powerful yet human-understandable operators for information aggregation. In the report, we present an axiomatic approach to fuzzy quantification which builds on the novel concept of a Determiner Fuzzification Scheme (or DFS for short). DFS theory overcomes several limitations pertinent to the approaches to fuzzy quantification described in the literature. Unlike these approaches, DFS is

- a compatible extension of the Theory of Generalized Quantifiers [12, 13];
- a genuine theory of fuzzy multi-place quantification;
- not limited to absolute and proportional quantifiers;
- not limited to automorphism-invariant quantifiers;
- not limited to finite universes of discourse;
- based on a rigid axiomatic foundation ensuring important formal properties;
- fully compatible to the formation of negation, antonyms, and duals.

Due to the axiomatic rigor of its foundation, its nice theoretical properties, its broad coverage of quantificational phenomena, and the existence of computational models, DFS theory not only contributes to a better understanding of fuzzy quantification in general, but also allows for a *principled* account of fuzzy quantifiers in practical applications.[1]

[1]Parts of the work reported here were funded by the ministry of science and research of the state of Northrhine-Westphalia within the collaborative research initiative "Virtual Knowledge Factory".

1 Introduction

1.1 Fuzzy quantifiers: statement of the problem

It is almost impossible to form a statement of natural language which does not involve the (explicit or implicit) use of quantification. Moreover, the meaning of natural language depends *heavily* on its quantifying expressions, as e.g. witnessed by the different meaning of *Most Americans voted Clinton* vs. *No American voted Clinton*.

Many of these expressions are *vague* or *fuzzy* in nature.

a. **often**, **rarely**, **recently**, **mostly**, **almost always** , ...
(vague temporal specification)

b. **almost everywhere**, **hardly anywhere**, **partly** , ...
(vague local specification)

c. **(absolutely) many**, **few**, **a few**; **about ten**, ...
(approximate specification of the cardinality of a set)

d. **(relatively) many**, **few**; **almost all**, **about 40 percent**, ...
(approximate specification of a proportion of cardinalities)

e. **far more than**, **some more than**, ...
(approximate comparison of cardinalities)

These expressions are best modelled as resulting in *gradual* evaluations.

This gives rise to the problem of an *adequate description* of the underlying fuzzy quantification operators: what is needed is a representation of these operators which is both

A-1 *reduced* or *simple enough* to lend itself to human understanding;

A-2 *rich enough* to represent the whole class of such operators which are of relevance to natural language.

Zadeh [2, p.756] discerns two "views" of fuzzy quantifiers, viz.

Q-1 a fuzzy quantifier is a second-order fuzzy predicate;

Q-2 a fuzzy quantifier is a first-order fuzzy predicate whose denotation is a fuzzy subset of the unit interval (or of the non-negative reals).

We shall refer to fuzzy quantifiers in the sense of Q-1, i.e. viewed as (membership functions of) second order fuzzy relations, as *fuzzy determiners*.[2] These fuzzy determiners are the *target representations* of our approach, because they are directly applicable to argument tuples of fuzzy subsets, yielding $[0, 1]$-valued results. Fuzzy determiners, however, are often too complex to be easily understood and therefore violate our condition A-1 on adequate *descriptions* of fuzzy quantifiers.

The representation of fuzzy quantifiers in the sense of Q-2 lends itself better to human understanding (after all, these quantifiers can be displayed in graphical form). However, as we will show later,

[2]This terminology is analogous to that of the Theory of Generalized Quantifiers (TGQ), in which generalized quantifiers are usually called *determiners*.

this representation is not rich enough to cover the interesting class of quantifiers, and hence fails to satisfy A-2.

The *first problem* to be solved in this report is therefore to make available a representational medium which allows for a compact description of the whole class of considered quantificational operators. Such a medium will be provided by our novel concept of *fuzzy pre-determiners*. These serve as the *source representations* of fuzzy quantification operators. For the purpose of evaluating expressions involving fuzzy quantification, then, these fuzzy pre-determiners must be generalized to corresponding *target representations*, i.e. to appropriate fuzzy determiners.

The *second problem* to be solved is therefore how the transfer from fuzzy pre-determiners to corresponding fuzzy determiners can be accomplished in a *systematic* and *provably reasonable* way.

1.2 Outline of DFS solution

We tackle the problem of systematic interpretation by applying a fuzzification mechanism. The benefit of using such a mechanism is that one needs not decide on a case-by-case basis whether a proposed fuzzy analogon of a given fuzzy pre-determiner is reasonable: the fuzzification mechanism *itself* (as opposed to the instances of fuzzification) can be subjected to a critical inspection.

- We present axioms which govern the behaviour of reasonable fuzzification mechanisms which we call *determiner fuzzification schemes* (or DFSes for short).
- We then provide evidence that these axioms are useful and complete by establishing various properties of DFSes through formal proofs.
- Turning to the issue of DFS models, we firstly show that two commonly-used tools of fuzzy set theory, namely α-cuts and the resolution principle, can *not* be used to define DFSes. This failure is caused by their lack of symmetry with respect to complementation.
- We therefore utilize the concept of a three-valued cut (and the corresponding representation of fuzzy sets as a family of three-valued sets) which overcomes the asymmetry problem. This representation is still suited for efficient implementation because every three-valued cut can be represented by a pair of α-cuts.
- Building on the concept of three-valued cuts, we present three instances of fuzzification mechanisms and prove that these are DFS models.
- We then discuss interrelations among the models and show that they are consistent generalisations of the fuzzification mechanism proposed by Gaines [1] as a *foundation of fuzzy reasoning*.

Finally, we contrast DFS theory with other approaches to fuzzy quantification which have been described in the literature. DFS shares only few assumptions with these approaches because it is intended to be

- a compatible extension of TGQ,
- a genuine theory of fuzzy multi-place quantification,

- not limited to absolute and proportional quantifiers,
- concerned both with quantitative and non-quantitative determiners,
- not limited to finite universes of discourse,
- based on a rigid axiomatic foundation.

We shall detail on these aspects in the conclusion.

1.3 Applications of fuzzy quantifiers

Since in colloquial language, people are using natural language quantifiers with the same ease as the connectives "and", "or", and "not", an adequate account of these quantifiers would provide a rich yet *human understandable* class of operators for information aggregation and data fusion.
In *fuzzy information retrieval* (Miyamoto [3], Larsen & Yager [4], Glöckner & Knoll [5]), these operators could be used —in a similar fashion as the operators "and" and "or"— to provide for more natural (e.g., compromising) ways of aggregating over sets of search terms, and of focussing thesaurus search:

> *Show me all documents which contain as many terms associated with "fuzzy information retrieval" as possible, but only few terms associated with "probabilistic retrieval".*

But an adequate account of fuzzy quantifiers will also prove useful in other fields of application in which the use of fuzzy quantifiers has been reported in the literature, viz.

- in *fuzzy natural language processing* (Zadeh [6]); for handling the fuzziness and vagueness of quantifying natural language expressions;
- in *approximate reasoning* (Zadeh [7], Yager [8], Ralescu [9]); in order to interpret rules of the form

  If Q of the premises p_i are satisfied, then c.

  or

  If Q of the *relevant* premises p_i are satisfied, then c.

  where Q is a quantification operator, the premises p_i are satisfied to a degree $x_i \in \mathbf{I} = [0, 1]$, the relevance of a premise is $w_i \in \mathbf{I}$, and the objective is to compute the degree of truth (or "possibility") associated with the conclusion c;
- in *multi-criteria decision making* (Yager [10]);
- in *fuzzy controllers* (Thiele [11]);
- in *fuzzy data fusion* and *information aggregation* (Bloch [24]).

2 A theory of fuzzy generalized quantification

2.1 Determiners

Our solution to the problem of systematic interpretation of fuzzy quantifying expressions is based on the Theory of Generalized Quantifiers (TGQ) [12, 13, 14, 15], certainly the most elaborate extensional theory of natural language quantification. In TGQ, the semantic counterparts of quantifying expressions are generally called "determiners".

Definition 1 (Determiner) *An n-ary determiner on a nonempty set E of "entities" or "individuals" is a mapping $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$, where $\mathbf{2} = \{0, 1\}$ is the set of truth values and $\mathcal{P}(E) \cong \mathbf{2}^E$ is the set of subsets of E.*

A determiner D thus assigns to each n-tuple $(X_1, \ldots, X_n)$ of subsets of E a corresponding truth value $D(X_1, \ldots, X_n) \in \mathbf{2}$. E might be any nonempty set of objects; unless otherwise stated, we will always allow for base sets of *infinite cardinality* as well. In typical applications of fuzzy quantifiers such as fuzzy information retrieval (Glöckner & Knoll [5]) or multi-criteria decision making (Yager [10]), E might be a set of search expressions, or the set of premises of a fuzzy IF-THEN-rule, respectively.

Well-known examples of determiners are

$$\begin{aligned}
\forall_E(X) = 1 &\;\Leftrightarrow\; X = E \\
\exists_E(X) = 1 &\;\Leftrightarrow\; X \neq \varnothing \\
\textbf{no}_E(X_1, X_2) = 1 &\;\Leftrightarrow\; X_1 \cap X_2 = \varnothing \\
\textbf{all}_E(X_1, X_2) = 1 &\;\Leftrightarrow\; X_1 \subseteq X_2 \\
\textbf{some}_E(X_1, X_2) = 1 &\;\Leftrightarrow\; X_1 \cap X_2 \neq \varnothing \\
\textbf{atleast n}_E(X_1, X_2) = 1 &\;\Leftrightarrow\; \mathrm{card}\,(X_1 \cap X_2) \geq n
\end{aligned}$$

where $E \neq \varnothing$ is a base set, $X, X_1, X_2 \in \mathcal{P}(E)$, and card denotes cardinality. We will usually drop the subscript E.

On *finite* base sets $E \neq \varnothing$, we may define

$$\begin{aligned}
\textbf{atleast}\ r\ \textbf{percent}(X_1, X_2) = 1 &\;\Leftrightarrow\; \mathrm{card}\,(X_1 \cap X_2) \geq \frac{r\,\mathrm{card}\,X_1}{100} \\
\textbf{atmost}\ r\ \textbf{percent}(X_1, X_2) = 1 &\;\Leftrightarrow\; \mathrm{card}\,(X_1 \cap X_2) \leq \frac{r\,\mathrm{card}\,X_1}{100} \\
\textbf{most}(X_1, X_2) &= \begin{cases} 1 & : \; X_1 = \varnothing \text{ or } \mathrm{card}\,(X_1 \cap X_2) > \frac{\mathrm{card}\,X_1}{2} \\ 0 & : \; \text{else} \end{cases}
\end{aligned}$$

(where $r \in [0, 100]$).

The above definition of **most** covers its technical sense, as e.g. in *Most Americans voted Clinton*, where one vote can result in an all-or-nothing decision. In other contexts, a gradual modelling would be more appropriate.

To provide an example how expressions involving these two-valued determiners can be evaluated, suppose

$$E = \{\mathrm{John}, \mathrm{Lucas}, \mathrm{Mary}\}$$

is a set of persons, **men** $= \{\text{John, Lucas}\} \in \mathcal{P}(E)$ is the set of men in E, and **married** $= \{\text{Mary, Lucas}\}$ is the set of those persons in E who are married. Then by our above definitions,

$$\textbf{some}(\textbf{men}, \textbf{married}) = \textbf{some}(\{\text{John, Lucas}\}, \{\text{Mary, Lucas}\}) = 1\,,$$

but

$$\textbf{all}(\textbf{men}, \textbf{married}) = \textbf{all}(\{\text{John, Lucas}\}, \{\text{Mary, Lucas}\}) = 0\,.$$

Let us note that—at this stage—we are not yet able to evaluate statements like *All men are tall*, because the corresponding concept **tall** is fuzzy.

2.2 Fuzzy determiners

Now let us turn to the fuzzy case. Suppose E is some set. A *fuzzy subset* X of E assigns to each $x \in E$ a gradual membership value $\mu_X(x) \in \mathbf{I}$, where $\mathbf{I} = [0, 1]$. The set of fuzzy subsets of E, denoted by $\widetilde{\mathcal{P}}(E)$, is therefore in one-to-one correspondence with the set of membership functions $\mu : E \longrightarrow \mathbf{I}$, i.e. $\widetilde{\mathcal{P}}(E) \cong \mathbf{I}^E$.
We define fuzzy determiners as follows.

Definition 2 (Fuzzy determiner)
By an n-ary fuzzy determiner on a base set $E \neq \varnothing$ we denote a mapping $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

A fuzzy determiner thus maps any n-tuple $(X_1, \ldots, X_n)$ of fuzzy subsets of E to a corresponding fuzzy membership value $D(X_1, \ldots, X_n) \in \mathbf{I}$.
This definition is the obvious generalisation of two-valued determiners (in the sense of TGQ) to the fuzzy case. Nevertheless, these fuzzy determiners pose a problem. We all have strong intuitions about the proper definition of determiners on “crisp” subsets of E; for example, the definition of **all** $: \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ stated above is certainly uncontroversial. In the fuzzy case, however, such intuitions are lacking: which choice of **all** $: \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ should be considered the “correct” one and why?
The reader might now argue, say, that $\widetilde{\textbf{all}} : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, defined by

$$\widetilde{\textbf{all}}(X_1, X_2) = \inf\{\min(1 - \mu_{X_1}(v), \mu_{X_2}(v)) : v \in E\}$$

(for all $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $E \neq \varnothing$ is a given base set) be the proper fuzzy analogon[3] of **all** $: \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$. However, we find it tedious to justify *on a case-by-case basis* whether a proposed fuzzy analogon of a given (two-valued) determiner D is “reasonable”. In particular, this manual procedure would make it very difficult to ensure the desired systematicity and consistency in the transfer from “standard” determiners to their fuzzy analogons.
We therefore propose the use of a fuzzification mechanism. The benefit of using such a mechanism is that one needs not decide on a case-by-case basis whether the fuzzified analogon of a given (two-valued) determiner is reasonable. By contrast, the fuzzification mechanism itself (as opposed to the instances of fuzzification) can be subjected to a critical inspection.

[3]This is also our first example of a fuzzy determiner.

2.3 Fuzzy pre-determiners

Our basic idea is as follows. Suppose that in order to define a fuzzy determiner $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we would only have to specify its behaviour on "crisp" arguments, i.e. on $\mathcal{P}(E)^n$, because we had a consistent mechanism $\mathcal{F}$ which generalises our partial definition to a fuzzy determiner $\mathcal{F}(D)$ which is defined on fuzzy arguments also. For example, we could start with **all** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2} \subseteq \mathbf{I}$ and automatically obtain its fuzzy analogon $\mathcal{F}($**all**$) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$. Then the above problem of the proper definition of **all** as a fuzzy determiner would have a unique answer *relative to* $\mathcal{F}$, namely $\mathcal{F}($**all**$)$, and conditions on *reasonable* or *intended* interpretations of fuzzy quantifiers could be stated as axioms governing the behaviour of $\mathcal{F}$. To make this idea work, we firstly introduce fuzzy pre-determiners (these are the required "partial descriptions" of fuzzy determiners), and then detail the generalisation mechanism.

Definition 3 (Fuzzy pre-determiner) *An n-ary fuzzy pre-determiner on a set of entities E is a mapping $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$.*

In order to define a fuzzy pre-determiner, only results for the application of D to "crisp" arguments (n-tuples of two-valued sets) must be specified. Therefore, an appropriate interpretation of natural language determiners is much easier to obtain than in the case of fuzzy determiners. In particular, every two-valued determiner (i.e. determiner in the sense of TGQ, such as **all**), is a fuzzy pre-determiner by definition. We need not guess any new definitions for these two-valued determiners.

In our examples on (genuine) fuzzy pre-determiners, we shall focus on three generic types from which instances such as **many**, **almost all**, **often**, **almost everywhere** etc. can be derived. Let us call these fuzzy pre-determiners

- **abs many**$_{\rho,\tau} : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$
  ("many X_1's are X_2 compared to an absolute expected value $\rho \in \mathbb{R}$ with sharpness parameter $\tau \in [0, \rho]$"),
- **rel many**$_{\rho,\tau} : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$
  ("the proportion of X_1's that are X_2 is large compared to an expected proportion $\rho \in [0, 1]$ with sharpness parameter $\tau \in [0, \rho]$"), and
- **as many as possible** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$
  (which might be interpreted as denoting "the proportion of X_1's that are X_2"; it is un-parametrized).

Formally, we define these fuzzy pre-determiners as follows ($E \neq \varnothing$ finite).

$$\textbf{abs many}_{\rho,\tau}(X_1, X_2) = S(\mathrm{card}\,(X_1 \cap X_2), \rho - \tau, \rho + \tau)$$

$$\textbf{rel many}_{\rho,\tau}(X_1, X_2) = \begin{cases} 1 & : \mathrm{card}\,X_1 = 0 \\ S(\frac{\mathrm{card}\,(X_1 \cap X_2)}{\mathrm{card}\,X_1}, \rho - \tau, \rho + \tau) & : \mathrm{card}\,X_1 \neq 0 \end{cases}$$

$$\textbf{as many as possible}(X_1, X_2) = \begin{cases} 1 & : \mathrm{card}\,X_1 = 0 \\ \frac{\mathrm{card}\,(X_1 \cap X_2)}{\mathrm{card}\,X_1} & : \mathrm{card}\,X_1 \neq 0 \end{cases}$$

for all $X_1,\ X_2 \in \mathcal{P}(E)$, where S is Zadeh's S-function [2, pp. 611+], defined by

$$S(x,\,\alpha,\,\gamma) = \begin{cases} 0 & : \quad x \le \alpha \\ 2\cdot\left(\dfrac{x-\alpha}{\gamma-\alpha}\right)^2 & : \quad \alpha < x \le \dfrac{\alpha+\gamma}{2} \\ 1-2\cdot\left(\dfrac{x-\gamma}{\gamma-\alpha}\right)^2 & : \quad \dfrac{\alpha+\gamma}{2} < x \le \gamma \\ 1 & : \quad x > \gamma \end{cases}$$

for all $x,\ \alpha,\ \gamma \in \mathbf{I}$.
abs many$_{\rho,\,\tau}$, **rel many**$_{\rho,\,\tau}$ and **as many as possible** cover various meanings of **many** and similar quantifying expressions such as **often**, **relatively often**, **almost all** etc.
For example,

$$\textbf{almost all} : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$$

could be modelled as

$$\textbf{almost all} = \textbf{rel many}_{0.8,\,0.1}$$

The plot[4] of this choice of **almost all** is displayed in Fig. 1.

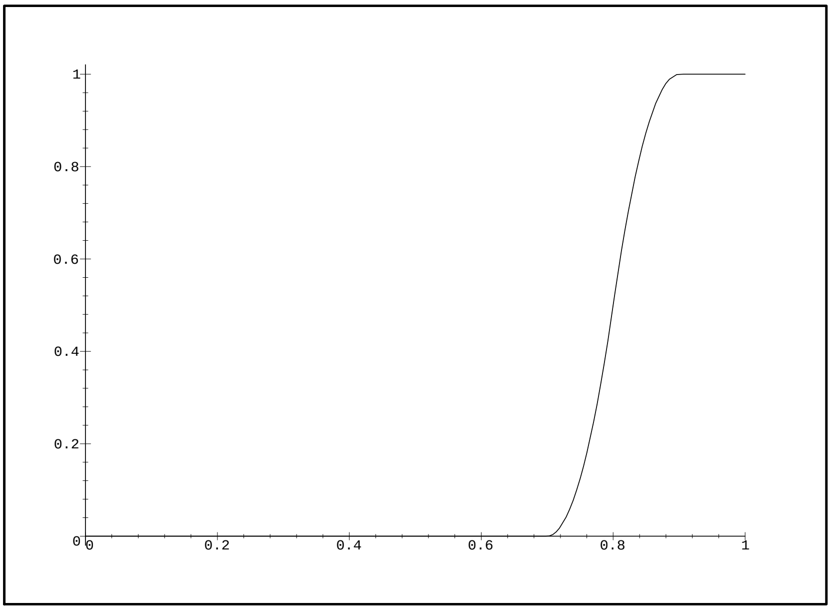

Figure 1: Plot of **almost all**$(X_1,\ X_2)$ as a function of $\frac{\mathrm{card}\,(X_1 \cap X_2)}{\mathrm{card}\,X_1}$

There is obviously some degree of freedom in the choice of these fuzzy pre-determiners (although their global shape should be uncontroversial) and some readers might wish to define them differently. We take this as providing evidence that to fuzzy pre-determiners (as opposed to fuzzy determiners), intuitions *do* apply!

[4]extrapolated to the continuous

Our theory, however, is in no respect dependent on any particular choice of these definitions. By contrast, it is supposed to ensure that *all* fuzzy pre-determiners (including, for example, the alternative definitions of the readers) are systematically generalised to corresponding fuzzy determiners. This is accomplished by means of a determiner fuzzification scheme.

By a determiner fuzzification scheme we mean a mechanism for the generalisation of fuzzy pre-determiners to corresponding fuzzy determiners which satisfies various axioms ensuring its "reasonable" and "systematic" behaviour.

Let us firstly introduce the *unrestricted* class of determiner fuzzification mechanisms, i.e. the initial class of considered fuzzification schemes which we will then narrow down to a "well-behaved" subclass by imposing axioms.

Definition 1 (Determiner fuzzification mechanisms)
A determiner fuzzification mechanism $\mathcal{F}$, $(D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$, assigns to each fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a corresponding fuzzy determiner $\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

Notes

- In particular, $\mathcal{F}(D)$ and D are defined on the same base set E and share the same arity n.
- We avoid to call $\mathcal{F}$ a "function" or "mapping" because the collection of fuzzy pre-determiners might form a proper class.

We will now address the problem of stating axioms which are making explicit what we think should be expected from a "reasonable" fuzzification scheme. This rigid axiomatic foundation is of particular importance in the context of fuzzy logic where—even for the propositional connectives—a lot of alternative interpretations have been proposed. The DFS-axioms, then, will provide a criterion to discern principled approaches to fuzzy quantification from ad-hoc ones.

The DFS axioms are mainly of the "preservation" or "compatibility" (homomorphism) type, i.e. we require that

- $\mathcal{F}$ maps particular (simple) fuzzy pre-determiners to their obvious fuzzy analogons;
- $\mathcal{F}$ preserves relevant properties of a determiner;
- $\mathcal{F}$ commutes with certain operations for building new determiners from given ones.[5]

We will now introduce these particular subclasses of "simple" fuzzy (pre-) determiners, the relevant properties to be preserved, and the constructions on fuzzy (pre-) determiners on which we will base the DFS axioms.

2.4 Special subclasses of fuzzy determiners and pre-determiners

Definition 2 (Constant fuzzy pre-determiner)
A fuzzy pre-determiner $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ is called a constant fuzzy pre-determiner.

[5] in the sense that it does not matter whether we first apply the operation and then apply $\mathcal{F}$, or whether we first apply $\mathcal{F}$ and then apply the (fuzzy analogon of the) considered operation.

Notes

- If A is some set and $n \in \mathbb{N}$, then A^n is commonly defined as the set of mappings from $n = \{0, \ldots, n-1\}$ to A, i.e. $A^n = A^{\{0, \ldots, n-1\}}$. Consequently, $A^0 = A^\varnothing = \{\varnothing\}$.

  A constant fuzzy pre-determiner $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$, i.e. $D : \{\varnothing\} \longrightarrow \mathbf{I}$, is therefore uniquely determined by its value $D(\varnothing) \in \mathbf{I}$, i.e. there is a one-to-one correspondence between constant fuzzy pre-determiners $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ and elements $D(\varnothing) = x \in \mathbf{I}$.

- To give an example, $c_{\frac{1}{2}} : \{\varnothing\} \longrightarrow \mathbf{I}$, defined by $c_{\frac{1}{2}}(\varnothing) = \frac{1}{2}$, is a constant fuzzy pre-determiner, and **true**, **false** : $\{\varnothing\} \longrightarrow \mathbf{I}$, defined by **true**$(\varnothing) = 1$ and **false**$(\varnothing) = 0$, are of course also constant fuzzy pre-determiners.

Definition 3 (Constant fuzzy determiner)
A fuzzy determiner $D : \widetilde{\mathcal{P}}(E)^0 \longrightarrow \mathbf{I}$ *is called a constant fuzzy determiner.*

Notes

- Again, $\widetilde{\mathcal{P}}(E)^0 = \widetilde{\mathcal{P}}(E)^\varnothing = \{\varnothing\}$, i.e. there is a one-to-one correspondence between constant fuzzy determiners $D : \{\varnothing\} \longrightarrow \mathbf{I}$ and elements $D(\varnothing) \in \mathbf{I}$.

- In particular, for any base set $E \neq \varnothing$, the set $\mathbf{I}^{(\mathcal{P}(E)^0)} = \mathbf{I}^{\{\varnothing\}}$ of constant fuzzy pre-determiners over E coincides with the set of constant fuzzy determiners $\mathbf{I}^{(\widetilde{\mathcal{P}}(E)^0)} = \mathbf{I}^{\{\varnothing\}}$.

Suppose E is some set and $A \in \mathcal{P}(E)$ a (crisp) subset of E. By $\chi_A : E \longrightarrow \mathbf{2}$ we denote the *characteristic function* of A, i.e. the function defined by

$$\chi_A(x) = \begin{cases} 1 & : \quad x \in A \\ 0 & : \quad x \notin A \end{cases}$$

for all $x \in E$.

We will now introduce a special type of fuzzy pre-determiners (and a corresponding type of fuzzy determiners) which allows us to treat membership assessments ("Is it true that $x \in X$?") as a special case of quantification.

Definition 4 (Fuzzy projection pre-determiner)
Let $E \neq \emptyset$, $n \in \mathbb{N}$. *Then for all* $x \in E$, $i = 1, \ldots, n$, *the fuzzy pre-determiner* $\pi_x^{n,i} : \mathcal{P}(E)^n \longrightarrow \mathbf{2} \subset \mathbf{I}$ *is defined by*

$$\begin{aligned} \pi_x^{n,i}(X_1, \ldots, X_n) &= \begin{cases} 1 & : \quad x \in X_i \\ 0 & : \quad x \notin X_i \end{cases} \\ &= \chi_{X_i}(x)\,, \end{aligned}$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.
If n *is clear from context, we will write* π_x^i *instead of* $\pi_x^{n,i}$. *If* $n = 1$, *we will simply write* π_x.

Notes

- Our use of the term "projection determiner" is motivated by the observation that $\mathcal{P}(E)^n \cong \mathbf{2}^{E\times n}$, i.e. we can view $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$ as an $(E \times n)$-tuple of elements of $\mathbf{2} = \{0, 1\}$, and $\pi_x^{n,i}$ as the corresponding projection onto the component (x, i).

- To present an example, suppose $E = \{\text{John}, \text{Lucas}, \text{Mary}\}$ is a set of persons and **married** $\in \mathcal{P}(E)$ the subset of married persons in E. Then **john** $= \pi_{\text{John}} : \mathcal{P}(E) \longrightarrow \mathbf{2}$ is a fuzzy projection pre-determiner, and

$$\textbf{john}(\textbf{married}) = \pi_{\text{John}}(\textbf{married}) = \begin{cases} 1 & : \text{John} \in \textbf{married} \\ 0 & : \text{John} \notin \textbf{married} \end{cases}$$

  i.e. π_{John} can be used to evaluate statements of the type *John is* X, where X is a crisp predicate.

Definition 5 (Fuzzy projection determiner)
Let $E \neq \emptyset$, $n \in \mathbb{N}$. For all $x \in E$, $i = 1, \ldots, n$, the fuzzy determiner $\widetilde{\pi}_x^{n,i} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{\pi}_x^{n,i}(X_1, \ldots, X_n) = \mu_{X_i}(x)$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.
Again if n is clear from context, we will write $\widetilde{\pi}_x^i$ instead of $\widetilde{\pi}_x^{n,i}$. If $n = 1$, we will simply write $\widetilde{\pi}_x$.

Notes

- Again, we use the term "fuzzy projection determiner" because $\widetilde{\mathcal{P}}(E)^n \cong \mathbf{I}^{E\times n}$, i.e. we can view $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ as an $(E \times n)$-tuple of elements of $\mathbf{I}$, and $\pi_x^{n,i}$ as the corresponding projection onto the component (x, i).

- The fuzzy projection determiners render it possible to view gradual membership assessments (i.e. "What is the value of $\mu_{X_i}(x)$?") as a special kind of fuzzy quantification.

- For example, if again $E = \{\text{John}, \text{Lucas}, \text{Mary}\}$, and if **bald** $\in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of persons in E who are bald, then

$$\widetilde{\pi}_{\text{John}}(\textbf{bald}) = \mu_{\textbf{bald}}(\text{John}),$$

  i.e. $\widetilde{\pi}_{\text{John}}$ allows us to treat statements of the form *John is* X, where X is a fuzzy predicate, as a special form of quantification.

- Due to their very similar roles, it is straightforward to suggest that $\widetilde{\pi}_x$ be the proper fuzzy analogon of π_x ($x \in E$).

2.5 Induced truth functions

By a canonical construction which we describe now, every determiner fuzzification mechanism $\mathcal{F}$ induces a unique fuzzy operator for each of the propositional connectives.[6]

Suppose $\{*\}$ is a singleton set. Then, $\pi_* : \mathcal{P}(\{*\}) \longrightarrow \mathbf{2}$ is a bijection with obvious inverse $\pi_*{}^{-1} : \mathbf{2} \longrightarrow \mathcal{P}(\{*\})$ defined by

$$\pi_*{}^{-1}(x) = \begin{cases} \{*\} & : \quad x = 1 \\ \varnothing & : \quad x = 0 \end{cases}$$

for all $x \in \mathbf{2} = \{0, 1\}$.

Likewise, $\widetilde{\pi}_* : \widetilde{\mathcal{P}}(\{*\}) \longrightarrow \mathbf{I}$ is a bijection, where the inverse $\widetilde{\pi}_*^{-1} : \mathbf{I} \longrightarrow \widetilde{\mathcal{P}}(\{*\})$ is defined

$$\widetilde{\pi}_*^{-1}(x) = \text{the unique } X \in \widetilde{\mathcal{P}}(\{*\}) \text{ with } \mu_X(*) = x,$$

for all $x \in \mathbf{I}$.

Definition 6 (Induced truth functions)

Suppose $\mathcal{F}$ is a determiner fuzzification mechanism and $f : \mathbf{2}^n \longrightarrow \mathbf{2}$ is a propositional function (e.g., $f = \wedge$). We can view f as a fuzzy pre-determiner $f^* : \mathcal{P}(\{*\})^n \longrightarrow \mathbf{2} \subseteq \mathbf{I}$ by defining

$$f^*(X_1, \ldots, X_n) = f(\pi_*(X_1), \ldots, \pi_*(X_n)) .$$

By applying $\mathcal{F}$, f^* is generalized to a fuzzy determiner $\mathcal{F}(f^*) : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \mathbf{I}$, from which we obtain a fuzzy truth function $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$ defined by

$$\widetilde{\mathcal{F}}(f)(x_1, \ldots x_n) = \mathcal{F}(f^*)(\widetilde{\pi}_*^{-1}(x_1), \ldots, \widetilde{\pi}_*^{-1}(x_n))$$

for all $x_1, \ldots, x_n \in \mathbf{I}$.

Often when $\mathcal{F}$ is clear from context, we we will abbreviate $\widetilde{f} = \widetilde{\mathcal{F}}(f)$.

Notes

- The above construction also applies to mappings $f : \mathbf{2}^n \longrightarrow \mathbf{I}$, yielding $\widetilde{\mathcal{F}}(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$.
- We shall *not* impose restrictions on the well-behavedness of the induced truth functions at this stage. These will result from the DFS axioms.

By pointwise application of the induced negation, conjunction, and disjunction, every determiner fuzzification mechanism $\mathcal{F}$ also induces a unique choice of fuzzy complement, fuzzy intersection, and fuzzy union.

Definition 7 (Induced operations on fuzzy sets)

Suppose $\mathcal{F}$ is a determiner fuzzification mechanism, and E is some set. $\mathcal{F}$ induces fuzzy set operators

$$\begin{aligned} \widetilde{\neg} &= \widetilde{\mathcal{F}}(\neg) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E) \\ \widetilde{\cap} &= \widetilde{\mathcal{F}}(\cap) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E) \\ \widetilde{\cup} &= \widetilde{\mathcal{F}}(\cup) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E) \end{aligned}$$

[6] This will also be our first technical example which demonstrates the practical usefulness of the projection determiners.

which are pointwise defined by

$$\mu_{\widetilde{\neg} X}(v) = \widetilde{\neg}\, \mu_X(v)$$
$$\mu_{X_1 \widetilde{\cap} X_2}(v) = \mu_{X_1}(v) \mathbin{\widetilde{\wedge}} \mu_{X_2}(v)$$
$$\mu_{X_1 \widetilde{\cup} X_2}(v) = \mu_{X_1}(v) \mathbin{\widetilde{\vee}} \mu_{X_2}(v)$$

for all $X, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ *and* $v \in E$*, where* $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg) : \mathbf{I} \longrightarrow \mathbf{I}$, $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge) : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *and* $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee) : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$.

Notes

- Our ambiguous use of $\widetilde{\neg}$ both as designating fuzzy negation and fuzzy complementation should not create confusion because the fuzzy negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ applies to gradual truth values $x \in \mathbf{I}$, while the fuzzy complement $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ applies to fuzzy subsets $X \in \widetilde{\mathcal{P}}(E)$ of a given set E. (Likewise for our notation of two-valued negation $\neg : \mathbf{2} \longrightarrow \mathbf{2}$ and crisp complementation $\neg : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$).
- Again, we can not expect any *a priori* restrictions on the induced fuzzy set operators—these will result from the DFS axioms.

In the following, we shall assume that an arbitrary, but fixed, determiner fuzzification mechanism $\mathcal{F}$ *is given. By* $\widetilde{\neg}$, $\widetilde{\wedge}$ *and* $\widetilde{\vee}$ *we shall always denote the fuzzy negation, fuzzy conjunction, and fuzzy disjunction induced by this particular* $\mathcal{F}$*; and by* $\widetilde{\neg}$ *(the fuzzy complement!),* $\widetilde{\cap}$, $\widetilde{\cup}$ *we shall always mean the fuzzy complement, fuzzy intersection, and fuzzy union, respectively, which are induced by this given* $\mathcal{F}$.

2.6 Fuzzy (pre-)determiners: constructions and properties

Definition 8 (External negation)

a. *Let* $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy pre-determiner and* $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$. *By* $\widetilde{\neg}\, D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *we denote the fuzzy pre-determiner defined by*

$$(\widetilde{\neg}\, D)(X_1, \ldots, X_n) = \widetilde{\neg}\,(D(X_1, \ldots, X_n))$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$ *(i.e.* $\widetilde{\neg}\, D = \widetilde{\neg} \circ D$*).*

b. *Likewise if* $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is a fuzzy determiner, the fuzzy determiner* $\widetilde{\neg}\, D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$(\widetilde{\neg}\, D)(X_1, \ldots, X_n) = \widetilde{\neg}\,(D(X_1, \ldots, X_n))$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

For example, **no** $= \widetilde{\neg}$ **some** is the external negation of **some** and **atleast n** is the external negation of **less than n**.[7,8]

[7] We have assumed here that $\widetilde{\neg}\, 0 = 1$ and $\widetilde{\neg}\, 1 = 0$, which will be ensured by the axioms to follow.

[8] In our examples, we will generally use simple two-valued determiners because these are the most familiar. All constructions of course apply to genuine fuzzy (pre-) determiners also. By contrast, Zadeh [29] uses **most**, which he models as being a genuine fuzzy quantifier, in order to explain the Σ-Count approach. This obscures the fact that when applied to two-valued quantifiers, the Σ-Count approach always yields "crisp" (i.e. two-valued) results, which is clearly inappropriate.

Definition 9 (Argument permutations)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $\beta : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ a permutation. By $D\beta : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the fuzzy pre-determiner defined by

$$D\beta(X_1, \ldots, X_n) = D(X_{\beta(1)}, \ldots, X_{\beta(n)}),$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.
In the case of fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we define $D\beta : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ analogously.

It is well-known that every permutation can be expressed as a composition of transpositions, which are defined as follows.

Definition 10 (Transpositions)
For all $n \in \mathbb{N}$ $(n > 0)$ and $i, j \in \{1, \ldots, n\}$, $\tau_{i,j} : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is defined by

$$\tau_{i,j}(k) = \begin{cases} i & : & k = j \\ j & : & k = i \\ k & : & \textit{else} \end{cases}$$

for all $k \in \{1, \ldots, n\}$.
By $\tau_i : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ we abbreviate $\tau_{i,n}$.

Notes

- Let us remark that it is even sufficient to consider the restricted class of transpositions τ_i ($i \in \{1, \ldots, n\}$) because every transposition $\tau_{i,j}$ can be expressed as

$$\tau_{i,j} = \tau_i \circ \tau_j \circ \tau_i .$$

- In the case of these simple transpositions, $D\tau_i(X_1, \ldots, X_n)$ becomes

$$D\tau_i(X_1, \ldots, X_n) = D(X_1, \ldots, X_{i-1}, X_n, X_{i+1}, \ldots, X_{n-1}, X_i).$$

We wish to illustrate the importance of argument permutations/transpositions by two examples.

1. There is a meaning of **only** $=$ $\textbf{only}_E : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ where **only** $=$ $\textbf{all}\tau_1$, i.e.

$$\textbf{only}(X_1, X_2) = \textbf{all}(X_2, X_1)$$

for all $X_1, X_2 \in \mathcal{P}(E)$. For example, if E is a set of persons, **men** $\in \mathcal{P}(E)$ the set of those $v \in E$ which are men, and **smokers** $\in \mathcal{P}(E)$ is the set[9] of persons who are smokers, then

$$\textbf{only}(\textbf{men}, \textbf{smokers}) = \textbf{all}\tau_1(\textbf{men}, \textbf{smokers}) = \textbf{all}(\textbf{smokers}, \textbf{men}),$$

i.e. the meaning of *Only men are smokers* coincides with that of *All smokers are men.*

[9] we shall assume that this set be crisp for the sake of argument

2. Argument transpositions render it possible to express symmetry properties of determiners. For example, the determiner **some** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ is symmetrical in its arguments, which can be stated as

$$\textbf{some} = \textbf{some}\tau_1\,.$$

Hence in the above domain of persons/smokers,

$$\textbf{some}(\textbf{men}, \textbf{smokers}) = \textbf{some}\tau_1(\textbf{men}, \textbf{smokers}) = \textbf{some}(\textbf{smokers}, \textbf{men})\,,$$

which expresses our intuition that the meanings of *Some men are smokers* and *Some smokers are men* coincide.

Definition 11 (Internal complementation)

a. *Let* $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy pre-determiner,* $n > 0$. *The fuzzy pre-determiner* $D\neg : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$D\neg(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, \neg X_n)$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

b. *Likewise if* $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is a fuzzy determiner,* $n > 0$, *we define* $D\widetilde{\neg} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *by*

$$D\widetilde{\neg}(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, \widetilde{\neg}\, X_n)$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Notes

- We recall our use of $\neg : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ as designating complementation (likewise for $\widetilde{\neg} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$).
- The fuzzy (pre-)determiner which results from the complementation of the i-th argument of a fuzzy (pre-) determiner D can be obtained as $D\tau_i\neg\tau_i$.
- Internal complementation corresponds to the formation of the *antonym* of a fuzzy (pre-) determiner. For example, **no** $=$ **all**$\neg$ is the antonym of **all**, and **not all** $=$ **some**$\neg$ is the antonym of **some**.
- By the *dual* of a fuzzy pre-determiner D we mean $\widetilde{\neg}D\neg$ (analogously for fuzzy determiners). For example, $\forall$ is the dual of $\exists$, and the dual of **no** is $\widetilde{\neg}$**no**$\neg$ $=$ **not all**.

The interdependencies of external negation, internal complementation (formation of antonym) and dualisation are summarized in the *Aristotelian square*.[10]

The Aristotelian square of the determiner **all** is displayed in Fig. 2.

[10]Sometimes also referred to as the *Square of Opposition*, see e.g. Gamut [16, p.238].

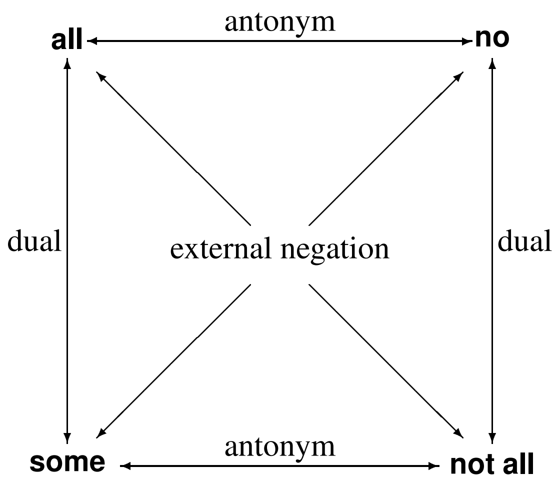

Figure 2: The Aristotelian square of **all**

The Aristotelian square expresses in graphical form that the operators $(\_)$ (identity), $\widetilde{\neg}(\_)$ (external negation), $(\_)\neg$ (antonym) and $\widetilde{\neg}(\_)\neg$ (dualisation) form a Klein group structure:

$\circ$	$(\_)$	$\widetilde{\neg}(\_)$	$(\_)\neg$	$\widetilde{\neg}(\_)\neg$
$(\_)$	$(\_)$	$\widetilde{\neg}(\_)$	$(\_)\neg$	$\widetilde{\neg}(\_)\neg$
$\widetilde{\neg}(\_)$	$\widetilde{\neg}(\_)$	$(\_)$	$\widetilde{\neg}(\_)\neg$	$(\_)\neg$
$(\_)\neg$	$(\_)\neg$	$\widetilde{\neg}(\_)\neg$	$(\_)$	$\widetilde{\neg}(\_)$
$\widetilde{\neg}(\_)\neg$	$\widetilde{\neg}(\_)\neg$	$(\_)\neg$	$\widetilde{\neg}(\_)$	$(\_)$

Or in the case of fuzzy determiners,

$\circ$	$(\_)$	$\widetilde{\neg}(\_)$	$(\_)\widetilde{\neg}$	$\widetilde{\neg}(\_)\widetilde{\neg}$
$(\_)$	$(\_)$	$\widetilde{\neg}(\_)$	$(\_)\widetilde{\neg}$	$\widetilde{\neg}(\_)\widetilde{\neg}$
$\widetilde{\neg}(\_)$	$\widetilde{\neg}(\_)$	$(\_)$	$\widetilde{\neg}(\_)\widetilde{\neg}$	$(\_)\widetilde{\neg}$
$(\_)\widetilde{\neg}$	$(\_)\widetilde{\neg}$	$\widetilde{\neg}(\_)\widetilde{\neg}$	$(\_)$	$\widetilde{\neg}(\_)$
$\widetilde{\neg}(\_)\widetilde{\neg}$	$\widetilde{\neg}(\_)\widetilde{\neg}$	$(\_)\widetilde{\neg}$	$\widetilde{\neg}(\_)$	$(\_)$

The square can be properly set up for arbitrary fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ provided that $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ is involutive (i.e. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$) and satisfies $\widetilde{\neg}\,|_{\mathbf{2}} = \neg$, i.e. $\widetilde{\neg}\, 1 = 0$ and $\widetilde{\neg}\, 0 = 1$. These conditions will of course be entailed by the axioms we will state.

Definition 12 (Internal meets)

a. *Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $n > 0$. The fuzzy pre-determiner $D\cap : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined by*

$$D\cap(X_1, \ldots, X_{n+1}) = D(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})$$

for all $(X_1, \ldots, X_{n+1}) \in \mathcal{P}(E)^{n+1}$.

b. *In the case of a fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, the fuzzy determiner $D\widetilde{\cap} : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is defined analogously, but "$\cap$" is replaced by "$\widetilde{\cap}$", and $(X_1, \ldots, X_{n+1}) \in \widetilde{\mathcal{P}}(E)^{n+1}$.*

Notes

- The fuzzy pre-determiner $D\cup : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ which results from the formation of union of the arguments X_n, X_{n+1} can be defined in a similar fashion. Likewise for fuzzy determiners, where $\cup$ is replaced by the fuzzy union $\widetilde{\cup}$ and the argument sets range over $\widetilde{\mathcal{P}}(E)$.
- To give an example, the two-place determiner **some** is defined by **some** $= \exists\cap$, and the two-place determiner **all** is defined by **all** $= \forall\cup\tau_1\neg\tau_1$.

Definition 13 (Insertion of arguments)

a. Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, $n > 0$. For all $A \in \mathcal{P}(E)$, by $D\triangleleft A : \mathcal{P}(E)^{n-1} \longrightarrow \mathbf{I}$ we denote the fuzzy pre-determiner defined by

$$D\triangleleft A(X_1, \ldots, X_{n-1}) = D(X_1, \ldots, X_{n-1}, A)$$

for all $(X_1, \ldots, X_{n-1}) \in \mathcal{P}(E)^{n-1}$.

b. If $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy determiner, $n > 0$ and $A \in \widetilde{\mathcal{P}}(E)$, the fuzzy determiner $D\triangleleft A : \widetilde{\mathcal{P}}(E)^{n-1} \longrightarrow \mathbf{I}$ is defined by

$$D\triangleleft A(X_1, \ldots, X_{n-1}) = D(X_1, \ldots, X_{n-1}, A)$$

for all $(X_1, \ldots, X_{n-1}) \in \widetilde{\mathcal{P}}(E)^{n-1}$.

Notes

- If $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $(X_1, \ldots, X_n) \in \mathcal{P}(E)$, then

$$D' = D\triangleleft X_n\triangleleft \ldots \triangleleft X_1 : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$$

  is the constant fuzzy pre-determiner defined by $D'(\varnothing) = D(X_1, \ldots, X_n)$. Likewise for fuzzy determiners.
- The fuzzy pre-determiner D' which results from the restriction of an argument i of a fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to a subset $A \subseteq E$ can be obtained as $D' = D\tau_i\cap\triangleleft A\tau_i$. Analogously for fuzzy determiners. In TGQ, this construction is usually called “adjectival restriction”.

The concept of argument insertion is important because it allows for an incremental interpretation of complex expressions in the sense of Frege’s *compositionality principle* (see e.g. Gamut [16, p.140]) which states that the meaning of a complex expression is a function of the meanings of its subexpressions. To provide an example (which also illustrates the above-mentioned mechanism of restricting an argument), let us consider the sentence

Most male persons are married.

Suppose that E is a base set, **person** $\in \mathcal{P}(E)$ is the extension of "person" in E, i.e. the subset of those $v \in E$ that are persons, **male** $\in \mathcal{P}(E)$ is the extension of "male" in E, and **married** $\in \mathcal{P}(E)$ is the extension of "married" in E.
Informally, we can think of this sentence as being interpreted in the following steps.[11]

$$\begin{aligned}
D &= \text{Most } X_1\text{'s are } X_2\text{'s} \\
D'' &= \text{Most male } Y_1\text{'s are } Y_2\text{'s} \\
D'' &= \text{Most male persons are } Z\text{'s} \\
D''' &= \text{Most male persons are married}
\end{aligned}$$

Ever since R. Montague published his influential work on "the proper treatment of quantification in ordinary English" [17], formal linguists would typically model the incremental interpretation process in some variant of the typed λ-calculus. This might look roughly[12] as follows:

$$\begin{aligned}
D &= \lambda X_1 X_2.\textbf{most}(X_1,\, X_2) = \textbf{most} \\
D' &= \lambda Y_1 Y_2.\textbf{most}(Y_1 \cap \textbf{male},\, Y_2) \\
D'' &= \lambda Z.D'(\textbf{person},\, Z) \\
D''' &= D''(\textbf{married})\,.
\end{aligned}$$

By means of our operators, we can recast this

$$\begin{aligned}
D &= \textbf{most} \\
D' &= \textbf{most}\tau_1 \cap \triangleleft \textbf{male}\tau_1 \\
D'' &= D'\tau_1 \triangleleft \textbf{person} \\
D''' &= D'' \triangleleft \textbf{married}\,.
\end{aligned}$$

We will now turn to monotonicity properties of fuzzy pre-determiners and fuzzy determiners. Let us first recall the definition of the fuzzy inclusion relation.

Definition 14 (Fuzzy inclusion relation)
Suppose E is some set and $X_1,\, X_2 \in \widetilde{\mathcal{P}}(E)$ are fuzzy subsets of E. We say that X_1 is contained in X_2 (in symbols, $X_1 \subseteq X_2$) iff

$$\mu_{X_1}(v) \le \mu_{X_2}(v)$$

for all $v \in E$.

Based on this concept, we are now able to define what it means for a fuzzy (pre-) determiner to be monotonically increasing (or decreasing) in one of its arguments.

Definition 15 (Monotonicity in i-th argument)
A fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is said to be monotonically increasing in its i-th argument ($i \in \{1,\, \ldots,\, n\}$, $n > 0$) iff for all $X_1,\, \ldots,\, X_n,\, X_i' \in \mathcal{P}(E)$ such that $X_i \subseteq X_i'$,

$$D(X_1,\, \ldots,\, X_n) \le D(X_1,\, \ldots,\, X_{i-1},\, X_i',\, X_{i+1},\, \ldots,\, X_n)\,.$$

[11] Our decomposition only serves illustrative purposes. A formal linguist might prefer a different decomposition of the above sentence.

[12] We are using a variant of the typed λ-calculus which offers product types. In monadic type theory, the expressions might look somewhat different.

D is said to be monotonically decreasing in its i-th argument if under the same conditions, it always holds that

$$D(X_1, \ldots, X_n) \geq D(X_1, \ldots, X_{i-1}, X_i', X_{i+1}, \ldots, X_n).$$

On fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, we use an analog definition, where $X_1, \ldots, X_n, X_i' \in \widetilde{\mathcal{P}}(E)$, and "$\subseteq$" is the fuzzy inclusion relation.

For example, **all** is monotonically decreasing in the first and monotonically increasing in the second argument, **most** : $\mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ is monotonically increasing in the second argument, etc.

2.7 The induced extension principle

Suppose E, E' are sets and $f : E \longrightarrow E'$ is a mapping from E to E'. f uniquely determines a powerset function $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ which maps subsets of E to subsets of E'. It is defined by

$$\widehat{f}(X) = \{f(x) : x \in X\},$$

for all $X \in \mathcal{P}(E)$.
Notes

- Normally, the same symbol (i.e., f) is used in order to denote $f : E \longrightarrow E'$ and the corresponding $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$. However, in order to unambiguously discern the original function and the induced powerset function, we will use the $\widehat{f}$-notation throughout.
- There is a closely related concept, namely that of the induced inverse $f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$ of a given $f : E \longrightarrow E'$, which is defined by

  $$f^{-1}(V) = \{x \in E : f(x) \in V\},$$

  for all $V \in \mathcal{P}(E')$. Often if V is a singleton, i.e. $V = \{v\}$ for some $v \in E'$, we will simply write $f^{-1}(v)$.

The underlying mechanism which transports f to $\widehat{f}$ can be generalized to the case of fuzzy sets; such a mechanism is then called an *extension principle*. Formally, we define (a very general class of) extension principles as follows.

Definition 16 (Extension principle)
An extension principle $\mathcal{E}$,

$$(f : E \longrightarrow E') \mapsto (\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')),$$

assigns to each mapping $f : E \longrightarrow E'$ (where $E, E' \neq \varnothing$), a corresponding mapping $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$.

Notes

- We do not impose any a priori restrictions on the well-behavedness of extension principles. These will later result from our axioms on determiner fuzzification schemes.

- We have excluded the case that $E = \varnothing$ or $E' = \varnothing$ because it is irrelevant in our context. However, every extension principle (in the sense of the above definition) can be extended to a "full" extension principle which is also defined in the case $E = \varnothing$ or $E' = \varnothing$ as follows. If $f : \varnothing \longrightarrow E'$, define

  $$\mathcal{E}(f) = c_\varnothing \,, \tag{*}$$

  where $c_\varnothing : \widetilde{\mathcal{P}}(\varnothing) \longrightarrow \widetilde{\mathcal{P}}(E')$ is the constant which to $\varnothing \in \widetilde{\mathcal{P}}(\varnothing) = \{\varnothing\}$ assigns the set $c_\varnothing(\varnothing) = \varnothing \in \widetilde{\mathcal{P}}(E')$.

  By the definition of functions, the case $E' = \varnothing$ can only occur when $E = \varnothing$ also; it is hence covered by (*).

The most prominent example of an extension principle is the extension principle as suggested by Zadeh [18], to which we will refer as the "standard extension principle". Recast in our notation, the standard extension principle

$$\hat{\hat{(\_)}} : (f : E \longrightarrow E') \mapsto (\hat{\hat{f}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E'))$$

is defined as follows.

Definition 17 (Standard extension principle)
Let $f : E \longrightarrow E'$ a mapping. $\hat{\hat{f}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ assigns to each $X \in \widetilde{\mathcal{P}}(E)$ the fuzzy subset $\hat{\hat{f}}(X) \in \widetilde{\mathcal{P}}(E')$ defined by the membership function

$$\mu_{\hat{\hat{f}}(X)}(v) = \sup\{\mu_X(x) : x \in f^{-1}(v)\} \,.$$

for all $v \in E'$.

Notes

- The extension principle can be generalized to n-ary functions

  $$f : \mathop{\times}_{i=1}^{n} E_i \longrightarrow E'$$

  which it takes to

  $$\hat{\hat{f}} : \mathop{\times}_{i=1}^{n} \widetilde{\mathcal{P}}(E_i) \longrightarrow \widetilde{\mathcal{P}}(E') \,,$$

  defined by

  $$\mu_{\hat{\hat{f}}(X_1, \ldots, X_n)}(v) = \sup\{\min_{i=1}^{n} \mu_{X_i}(x_i) : (x_1, \ldots, x_n) \in f^{-1}(v)\} \,,$$

  for all $X_1 \in \widetilde{\mathcal{P}}(E_1), \ldots, X_n \in \widetilde{\mathcal{P}}(E_n)$ and $v \in E'$.

  This generalized version of the extension principle will be of minor importance in this sequel.[13]

[13]However, the definition of $\widetilde{D}_v(X_1, \ldots, X_n)$ on p. 70 can be viewed as resulting from an infinitary variant of the generalized extension principle.

- By a *normalized* fuzzy subset of a set E we mean a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ such that $\mu_X(v) = 1$ for some choice of $v \in E$. We denote the collection of normalized subsets of a given set E by $\tilde{\tilde{\mathcal{P}}}(E)$.

  It is perhaps worth noting that for all $f : E \longrightarrow E'$ and $X \in \widetilde{\mathcal{P}}(E)$,

  $$X \in \tilde{\tilde{\mathcal{P}}}(E) \Rightarrow \hat{\hat{f}}(X) \in \tilde{\tilde{\mathcal{P}}}(E')\,,$$

  i.e. if X is normalized, then $\hat{\hat{f}}(X)$ is normalized also.

  It is therefore possible to restrict $\hat{\hat{(\_)}}$ to a "normalized extension principle"

  $$\hat{\hat{(\_)}}_{\mathrm{norm}} : (f : E \longrightarrow E') \mapsto (\hat{\hat{f}}_{\mathrm{norm}} : \tilde{\tilde{\mathcal{P}}}(E) \longrightarrow \tilde{\tilde{\mathcal{P}}}(E'))\,.$$

  We will need this normalized variant later (see p. 77).

Apart from the crucial role of the (standard) extension principle for the foundation of fuzzy set theory in general, the powerset functions $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$ and the corresponding fuzzy powerset functions $\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ obtained from a given extension principle $\mathcal{E}$ are important in our context because they can be applied to the argument sets of fuzzy pre-determiners (crisp case, $\widehat{f}$) and fuzzy determiners (fuzzy case, $\mathcal{E}(f)$).
For example, if $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, and $f_1, \ldots, f_n : E \longrightarrow E'$ ($E \neq \varnothing$) are mappings, we can define a fuzzy pre-determiner $D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$D'(X_1, \ldots, X_n) = D(\widehat{f_1}(X_1), \ldots, \widehat{f_n}(X_n))\,,$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$. This can be expressed more compactly if we recall the concept of *product mapping*. If $g_1, \ldots, g_m : A \longrightarrow B$ are any mappings, then $\overset{n}{\underset{i=1}{\times}} g_i : A^n \longrightarrow B^n$ is defined by

$$(\overset{n}{\underset{i=1}{\times}} g_i)(x_1, \ldots, x_n) = (g_1(x_1), \ldots, g_n(x_n))\,.$$

By using the product, D' as defined above becomes

$$D' = D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i}\,,$$

where "$\circ$" denotes functional composition.
Similarly in the case of fuzzy determiners $D : \widetilde{\mathcal{P}}(E')^n \longrightarrow \mathbf{I}$, we can define $D' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$D'(X_1, \ldots, X_n) = D(\mathcal{E}(f_1)(X_1), \ldots, \mathcal{E}(f_n)(X_n))\,,$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, or equivalently,

$$D' = D \circ \overset{n}{\underset{i=1}{\times}} \mathcal{E}(f_i)\,.$$

By a canonical construction which we describe now, every determiner fuzzification mechanism $\mathcal{F}$ induces a unique extension principle.

Definition 18 (Induced extension principle)
Suppose $\mathcal{F}$, $(D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$, *is a determiner fuzzification mechanism.* $\mathcal{F}$ *induces an extension principle* $\widehat{\mathcal{F}}$ *which to each* $f : E \longrightarrow E'$ *(where* $E, E' \neq \varnothing$*) assigns the mapping* $\widehat{\mathcal{F}}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ *defined by*

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(v) = \mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$*,* $v \in E'$*.*

We wish to add that the induced extension principle $\widehat{\mathcal{F}}$ and the corresponding mechanism of constructing fuzzy determiners $D' = D \circ \overset{n}{\underset{i=1}{\times}} \widehat{\mathcal{F}}(f_i)$ on a base set E from a given fuzzy determiner D on *another* base set E' is important especially because it connects the behaviour of $\mathcal{F}$ on different base sets.

2.8 Determiner fuzzification schemes (DFSes)

By using the concepts introduced in the previous section, we are now able to define what we mean by a reasonable determiner fuzzification mechanism.

Definition 19 (Determiner fuzzification schemes)
Suppose $\mathcal{F}$ *assigns to each fuzzy pre-determiner* $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy determiner* $\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$*.* $\mathcal{F}$ *is called a* determiner fuzzification scheme (DFS) *iff the following axioms are satisfied for all fuzzy pre-determiners* $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*.*

Preservation of constants	$\mathcal{F}(D) = D$ *if* $n = 0$	(DFS 1)
Compatibility with π_*	$\mathcal{F}(\pi_*) = \widetilde{\pi}_*$	(DFS 2)
External negation	$\mathcal{F}(\widetilde{\neg}\, D) = \widetilde{\neg}\, \mathcal{F}(D)$	(DFS 3)
Argument transposition	$\mathcal{F}(D\tau_i) = \mathcal{F}(D)\tau_i$ *if* $i \in \{1, \ldots, n\}$	(DFS 4)
Internal complementation	$\mathcal{F}(D\neg) = \mathcal{F}(D)\widetilde{\neg}$ *if* $n > 0$	(DFS 5)
Internal meets	$\mathcal{F}(D\cap) = \mathcal{F}(D)\widetilde{\cap}$ *if* $n > 0$	(DFS 6)
Argument insertion	$\mathcal{F}(D\triangleleft A) = \mathcal{F}(D)\triangleleft A$ *if* $n > 0$*,* A *crisp*	(DFS 7)
Preservation of monotonicity	D *decr. in* n*-th arg* $\Rightarrow \mathcal{F}(D)$ *decr. in* n*-th arg,* $n > 0$	(DFS 8)
Functional application	$\mathcal{F}(D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i}) = \mathcal{F}(D) \circ \overset{n}{\underset{i=1}{\times}} \widehat{\mathcal{F}}(f_i)$ *where* $f_1, \ldots, f_n : E' \longrightarrow E$*,* $E' \neq \varnothing$*.*	(DFS 9)

2.9 Comments on the DFS axioms

Informally, the relevance of the above axioms is the following.

Preservation of constants (DFS 1)

As we have already noted, the set of constant fuzzy pre-determiners coincides with the set of constant fuzzy-determiners. We therefore find it natural to require that the fuzzification scheme maps every constant fuzzy pre-determiner to itself (viewed as a fuzzy determiner). For example, the constant fuzzy pre-determiner **true** : $\{\varnothing\} \longrightarrow \mathbf{I}$, defined by **true**$(\varnothing) = 1$, would be mapped to $\mathcal{F}($**true**$) =$ **true**.

Compatibility with π_* (DFS 2)

The importance of the axiom "compatibility with π_*" arises from the fact that π_* is closely related to the identity truth function $\mathrm{id}_{\mathbf{2}} : \mathbf{2} \longrightarrow \mathbf{2}$ because $\mathrm{id}_{\mathbf{2}}^* = \pi_*$. As we shall later formally prove, the axiom ensures that in every DFS $\mathcal{F}$, $\widetilde{\mathrm{id}_{\mathbf{2}}} = \mathrm{id}_{\mathbf{I}}$, i.e. the fuzzy identity truth function induced by $\mathcal{F}$ is the identity $\mathrm{id}_{\mathbf{I}} : \mathbf{I} \longrightarrow \mathbf{I}$. In addition, our canonical procedure of obtaining induced truth functions is based on the idea that $\widetilde{\pi}_*$ be the fuzzy analogon of π_*. The induced truth functions are therefore guaranteed to be reasonable only if π_* is generalized to the the intended $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$.

External negation (DFS 3)

This axiom states that every DFS $\mathcal{F}$ commutes with the formation of (external) negations of arbitrary fuzzy pre-determiners. For example, from $\textbf{no} = \widetilde{\neg}\,\textbf{some}$ we obtain $\mathcal{F}(\textbf{no}) = \widetilde{\neg}\,\mathcal{F}(\textbf{some})$. Hence if E is a set of persons, $\textbf{bald} \in \widetilde{\mathcal{P}}(E)$ is the fuzzy extension of "bald", i.e. the fuzzy set of persons in E who are bald, and $\textbf{blonde} \in \widetilde{\mathcal{P}}(E)$ is the fuzzy extension of "blonde" in E, we obtain

$$\mathcal{F}(\textbf{no})(\textbf{bald}, \textbf{blonde}) = \widetilde{\neg}\,\mathcal{F}(\textbf{some})(\textbf{bald}, \textbf{blonde})\,,$$

which conforms to our intuition that the meanings of *No bald persons are blonde* and *It is not the case that some bald persons are blonde* coincide.

Argument transposition (DFS 4)

Because every permutation can be expressed as a sequence of transpositions, this axiom ensures that $\mathcal{F}$ commutes with arbitrary permutations of the arguments of a determiner. In particular, it guarantees that symmetry properties of a determiner D carry over to its fuzzified analogon $\mathcal{F}(D)$. For example, $\textbf{some} : \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ is a symmetrical fuzzy pre-determiner, i.e. $\mathcal{F}(\textbf{some})$ is also symmetrical by (DFS 4). Hence if E is again a set of persons, $\textbf{bald} \in \widetilde{\mathcal{P}}(E)$ the fuzzy set of bald persons in E, and $\textbf{tall} \in \widetilde{\mathcal{P}}(E)$ the fuzzy set of persons in E which are tall, then

$$\mathcal{F}(\textbf{some})(\textbf{bald}, \textbf{tall}) = \mathcal{F}(\textbf{some})(\textbf{bald}, \textbf{tall})$$

which reflects our intuitions that the (gradual) truth values of *Some bald persons are tall* and *Some tall persons are bald* should coincide.

Internal complementation (DFS 5)

The axiom states that every DFS is compatible to the formation of *antonyms*. By means of argument transpositions, it of course generalises to internal complementation in arbitrary argument positions. In particular, from $\textbf{no} = \textbf{all}\neg$ and $\textbf{not all} = \textbf{some}\neg$ we obtain $\mathcal{F}(\textbf{no}) = \mathcal{F}(\textbf{all})\widetilde{\neg}$ and $\mathcal{F}(\textbf{not all}) = \mathcal{F}(\textbf{some})\widetilde{\neg}$. Hence in our example domain of the bald and the blonde,

$$\mathcal{F}(\textbf{no})(\textbf{bald}, \textbf{blonde}) = \mathcal{F}(\textbf{all})(\textbf{bald}, \widetilde{\neg}\,\textbf{blonde})\,,$$

which states that *No bald persons are blonde* and *All bald persons are not blonde* are equivalent.

Combined with the axiom of "external negation" (DFS 3), we obtain that every DFS is also compatible to dualisation. Hence from $\textbf{some} = \widetilde{\neg}\,\textbf{all}\neg$ we obtain that $\mathcal{F}(\textbf{some}) = \widetilde{\neg}\,\mathcal{F}(\textbf{all})\widetilde{\neg}$. For example, *Some bald persons are tall* and *Not all bald persons are not tall* are equivalent in every DFS.

We can summarize (DFS 3) and (DFS 5) as ensuring that every DFS preserves Aristotelian squares.[14] I.e. if $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an arbitrary fuzzy pre-determiner, then the square

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[d, leftrightarrow, "\text{dual}"'] \arrow[dr, leftrightarrow, "\text{external negation}" description] & D\neg \arrow[d, leftrightarrow, "\text{dual}"] \arrow[dl, leftrightarrow] \\
\widetilde{\neg}\, D\neg \arrow[r, leftrightarrow, "\text{antonym}"] & \widetilde{\neg}\, D
\end{tikzcd}$$

is transported by $\mathcal{F}$ to the corresponding square on fuzzy determiners,

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}(D) \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[d, leftrightarrow, "\text{dual}"'] \arrow[dr, leftrightarrow, "\text{external negation}" description] & \mathcal{F}(D\neg) \arrow[d, leftrightarrow, "\text{dual}"] \arrow[dl, leftrightarrow] \\
\mathcal{F}(\widetilde{\neg}\, D\neg) \arrow[r, leftrightarrow, "\text{antonym}"] & \mathcal{F}(\widetilde{\neg}\, D)
\end{tikzcd}$$

where the indicated relations of antonymy, negation and duality are still valid, i.e. which coincides with

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{F}(D) \arrow[r, leftrightarrow, "\text{antonym}"] \arrow[d, leftrightarrow, "\text{dual}"'] \arrow[dr, leftrightarrow, "\text{external negation}" description] & \mathcal{F}(D)\widetilde{\neg} \arrow[d, leftrightarrow, "\text{dual}"] \arrow[dl, leftrightarrow] \\
\widetilde{\neg}\, \mathcal{F}(D)\widetilde{\neg} \arrow[r, leftrightarrow, "\text{antonym}"] & \widetilde{\neg}\, \mathcal{F}(D)
\end{tikzcd}$$

[14]Because all relations displayed in the Aristotelian square are bidirectional, it is also required that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ be involutive. But, as we will later prove, $\widetilde{\neg}$ is even guaranteed to be a strong negation operator (see p. 27).

Every DFS also commutes with the *Piaget group of transformations* (see e.g. [19, p. 158+]), which corresponds to the formation of

(identity)	$\mathrm{I}(D) = D$
(negation)	$\mathrm{N}(D) = \widetilde{\neg}\, D$
(reciprocity)	$\mathrm{R}(D) = D\tau_1\neg\tau_1 \ldots \tau_n\neg\tau_n$
(correlativity)	$\mathrm{C}(D) = \widetilde{\neg}\,\mathrm{R}(D)\,,$

where $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner (analogously for fuzzy determiners).[15] This is obvious from the definability of $\mathrm{I},\ \mathrm{N},\ \mathrm{R},\ \mathrm{C}$ in terms of external negation, internal complementation and argument transpositions, to which every DFS is compatible by (DFS 3), (DFS 5) and (DFS 4).

Internal meets (DFS 6)

This axiom states that every DFS is compatible to intersections in the last argument of a fuzzy pre-determiner. By (DFS 4), this generalises to intersections in arbitrary argument positions.
Recalling that the determiner **some** $: \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ can be expressed as **some** $= \exists\cap$, we obtain $\mathcal{F}(\textbf{some}) = \mathcal{F}(\exists)\widetilde{\cap}$. This ensures e.g. that $\mathcal{F}(\textbf{some})(\textbf{bald},\ \textbf{tall})$ can be interpreted as $\mathcal{F}(\exists)(\textbf{bald}\,\widetilde{\cap}\ \textbf{tall})$.

Argument insertion (DFS 7)

We have motivated the operation of argument insertion (fuzzy or crisp) by its ability a.) to model "adjectival restriction" and b.) to allow for a compositional interpretation of complex natural language expressions.
The axiom (DFS 7) states that every DFS commutes with the insertion of *crisp* arguments. It is restricted to crisp arguments because, given a fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and a *fuzzy* subset $A \in \widetilde{\mathcal{P}}(E)$, $D \triangleleft A$ would be undefined. (D is a fuzzy *pre*-determiner and hence defined on crisp arguments only!)
Nevertheless, the axiom ensures that

$$\mathcal{F}(\textbf{all}\tau_1\cap\triangleleft\textbf{married}\tau_1)(\textbf{men},\ \textbf{lucky}) = \mathcal{F}(\textbf{all})(\textbf{married}\,\widetilde{\cap}\ \textbf{men},\ \textbf{lucky})$$

provided that **married** $\in \mathcal{P}(E)$ is crisp, i.e. we obtain the same result when applying $\mathcal{F}$ to the composite determiner

$$D = \text{all married } Y_1\text{'s are } Y_2$$

and then evaluating $\mathcal{F}(D)(\textbf{men},\ \textbf{lucky})$, or when first applying $\mathcal{F}$ to **all** and then inserting the extensions of "married men" and "lucky".
The axiom is of particular importance because it generally ensures that *boundary conditions* (with respect to the crisp case) are valid. For example, the axiom ensures that

$$x_1\ \widetilde{\wedge}\ 1 = x_1$$

for all $x_1 \in \mathbf{I}$, which is one of the defining conditions of t-norms.

[15] The operator-based definition of $\mathrm{R}(D)$ might look complicated. Provided a choice of arguments $(X_1,\ \ldots,\ X_n) \in \mathcal{P}(E)^n$, this expression becomes

$$\mathrm{R}(D)(X_1,\ \ldots,\ X_n) = D(\neg X_1,\ \ldots,\ \neg X_n)\,.$$

Preservation of (decreasing) monotonicity (DFS 8)

The axiom states that $\mathcal{F}$ preserves decreasing monotonicity of a determiner in its last argument.
By applying argument transpositions (DFS 4), (DFS 8) generalises to the preservation of decreasing monotonicity in arbitrary argument positions. As we shall later see, it follows that $\mathcal{F}$ also preserves increasing monotonicity, and even monotonicity properties of determiners which hold only locally.
We have stated the axiom in terms of *decreasing* monotonicity for technical purposes; we need this to prove that $\widetilde{\mathcal{F}}(\neg)$ is a strong negation operator.
To provide an example where the axiom applies directly, let us consider the two-valued determiner **no**, which is decreasing in its last argument. By (DFS 8), then, $\mathcal{F}(\textbf{no})$ is decreasing in its last argument also.
Therefore if E is a set of persons, and **men**, **bald**, **tall** $\in \widetilde{\mathcal{P}}(E)$ are the fuzzy subsets of persons in E which are "men", "bald", and "tall", respectively, then **tall** $\widetilde{\cap}$ **bald** $\subseteq$ **tall** entails that

$$\mathcal{F}(\textbf{no})(\textbf{men}, \textbf{bald}) \leq \mathcal{F}(\textbf{no})(\textbf{men}, \textbf{bald} \,\widetilde{\cap}\, \textbf{tall}) \,,$$

which conforms to our intuition that *No men are bald* entails that *No men are bald and tall*.

Functional application (DFS 9)

This is the only axiom which refers to the induced extension principle $\widehat{\mathcal{F}}$. Nevertheless, it is sufficient to ensure that $\widehat{\mathcal{F}}$ is well-behaved in various ways. As we shall later prove, $\widehat{\mathcal{F}}$ is functorial, i.e. $\widehat{\mathcal{F}}(f \circ g) = \widehat{\mathcal{F}}(f) \circ \widehat{\mathcal{F}}(g)$ and $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$; $\widehat{\mathcal{F}}$ coincides with the standard extension principle on injective functions; $\widehat{\mathcal{F}}$ is closely related to existential quantification, etc.
Furthermore, the axiom of "functional application" is of particular importance because it is the only axiom which relates the behaviour of $\mathcal{F}$ on different domains E, E'. In particular, if $\beta : E \longrightarrow E'$ is a bijection, $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, and $D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$D'(X_1, \ldots, X_n) = D(\widehat{\beta}(X_1), \ldots, \widehat{\beta}(X_n))$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$, then

$$\mathcal{F}(D')(X_1, \ldots, X_n) = \mathcal{F}(D)(\widehat{\mathcal{F}}(\beta)(X_1), \ldots, \widehat{\mathcal{F}}(\beta)(X_n))$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, and

$$\mathcal{F}(D)(X_1, \ldots, X_n) = \mathcal{F}(D')(\widehat{\mathcal{F}}(\beta^{-1})(X_1), \ldots, \widehat{\mathcal{F}}(\beta^{-1})(X_n))$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E')^n$, which shows that $\mathcal{F}$ may not depend on any particular properties of elements of a base set E.

We have provided evidence that the DFS axioms are straightforward, mainly by means of illustrative natural language examples. But of course, an axiom set cannot be appropriately judged when looking at the axioms in isolation; we should rather investigate the properties of the axiom set *as a whole* by exploring the space of its consequences. The next chapter, "DFS properties", provides first results of such an investigation. It uncovers various interesting properties of DFSes (or specific subclasses of DFSes), which are all established through formal proofs. The proofs are listed in the appendix.

3 DFS Properties

3.1 Argument structure and propositional connectives

Let us firstly establish that $\mathcal{F}(D)$ coincides with the original fuzzy pre-determiner D when all arguments are crisp sets, i.e. that $\mathcal{F}(D)$ consistently extends D.

Theorem 1 (Correct generalisation)
Suppose $\mathcal{F}$ is a DFS and $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary fuzzy pre-determiner. Then

$$\mathcal{F}(D)|_{\mathcal{P}(E)^n} = D\,,$$

i.e. for all crisp *subsets* $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n \subset \widetilde{\mathcal{P}}(E)^n$,

$$\mathcal{F}(D)(X_1, \ldots, X_n) = D(X_1, \ldots, X_n)\,.$$

(Proof: A.2, p.87+)

For example, if E is a set of persons, and **women**, **married** $\in \mathcal{P}(E) \subset \widetilde{\mathcal{P}}(E)$ are the crisp sets of "women" and "married persons" (respectively) in E, then

$$\mathcal{F}(\textbf{some})(\textbf{women}, \textbf{married}) = \textbf{some}(\textbf{women}, \textbf{married})\,,$$

i.e. the "fuzzy some" obtained by applying $\mathcal{F}$ coincides with the (original) "crisp some" whenever the latter is defined, which is of course highly desirable.

Let us now turn to the fuzzy truth functions induced by a DFS. Our first theorem is concerned with the identity truth function $\mathrm{id}_{\mathbf{2}} : \mathbf{2} \longrightarrow \mathbf{2}$ defined by $\mathrm{id}_{\mathbf{2}}(0) = 0$, $\mathrm{id}_{\mathbf{2}}(1) = 1$.

Theorem 2 (Identity truth function)
In every DFS $\mathcal{F}$, $\widetilde{\mathrm{id}_2} = \mathrm{id}_{\mathbf{I}}$, where we abbreviate $\widetilde{\mathrm{id}_2} = \widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}})$.
(Proof: A.3, p.88+)

The identity truth function is hence generalised to the identity $\mathrm{id}_{\mathbf{I}} : \mathbf{I} \longrightarrow \mathbf{I}$. This is quite satisfactory.

As concerns negation, the standard choice in fuzzy logic is certainly $\neg : \mathbf{I} \longrightarrow \mathbf{I}$, defined by $\neg x = 1 - x$ for all $x \in \mathbf{I}$. The essential properties of this and other reasonable negation operators are captured by the following definition.

Definition 20 (Strong negation operators)
$\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ *is called a* strong negation operator *iff it satisfies*

a. $\widetilde{\neg}\, 0 = 1$ *(boundary condition)*

b. $\widetilde{\neg}\, x_1 \geq \widetilde{\neg}\, x_2$ *for all* $x_1, x_2 \in \mathbf{I}$ *such that* $x_1 < x_2$ *(i.e. $\widetilde{\neg}$ is monotonically decreasing)*

c. $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ *(i.e. $\widetilde{\neg}$ is involutive).*

Note. Further examples of negation operators are e.g. Sugeno's λ-complements [20].

As witnessed by the following theorem, the fuzzy negation induced by a DFS $\mathcal{F}$ is "reasonable" in the sense of belonging to the class of strong negation operators.

Theorem 3 (Negation)
In every DFS $\mathcal{F}$, $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ is a strong negation operator.
(Proof: A.4, p.88+)

In particular, $\widetilde{\neg}$ is strictly monotonically decreasing, and from $\widetilde{\neg} \circ \widetilde{\neg} = \mathrm{id}_{\mathbf{I}}$ we obtain that $\widetilde{\neg}\,\widetilde{\neg}\, D = D$ and $D\widetilde{\neg}\widetilde{\neg} = D$. We wish to recall that these properties are essential to the preservation of Aristotelian squares, and to the compliance of DFSes with the Piaget group of transformations (see p. 16, p. 24+).
It is now easy to establish the following theorem on the preservation of monotonicity properties.

Theorem 4 (Monotonicity in i-th argument)
Let $\mathcal{F}$ a DFS and $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$. Then D is monotonically increasing (decreasing) in its i-th argument ($i \leq n$) iff $\mathcal{F}(D)$ is monotonically increasing (decreasing) in its i-th argument.
(Proof: A.5, p.88+)

For example, **some** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ is monotonically increasing in both arguments. By the theorem, then, $\mathcal{F}(\textbf{some}) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ is increasing in both arguments also. In particular,

$$\mathcal{F}(\textbf{some})(\textbf{young\_men}, \textbf{very\_tall}) \leq \mathcal{F}(\textbf{some})(\textbf{men}, \textbf{tall}) ,$$

i.e. *Some young men are very tall* entails *Some men are tall*, if we assume that **young_men** $\subseteq$ **men** and **very_tall** $\subseteq$ **tall**.
Let us also note that redundant arguments have no effect on the results of $\mathcal{F}$.

Theorem 5 (Redundant arguments)
Suppose $\mathcal{F}$ is a DFS and $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $D' : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ are fuzzy pre-determiners such that

$$D'(X_1, \ldots, X_n, X_{n+1}) = D(X_1, \ldots, X_n)$$

for all $X_1, \ldots, X_{n+1} \in \mathcal{P}(E)$. Then also

$$\mathcal{F}(D')(X_1, \ldots, X_n, X_{n+1}) = \mathcal{F}(D)(X_1, \ldots, X_n)$$

for all $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$.
(Proof: A.6, p.89+)

Notes

- By iterating and transposing argument positions, this property of course generalises to any number of "redundant" or "vacuous" argument positions.
- The theorem is important because the situation described arises naturally through the use of *cylindrical extensions*. An example involving determiners is given on p. 36.

- We can now describe the induced truth functions of some (less usual) two-place connectives, which we denote by $\mathrm{pr}_1 : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$, $\mathrm{npr}_1 : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$, $\mathrm{pr}_2 : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$, $\mathrm{npr}_2 : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$, $0_{\mathbf{2}\times\mathbf{2}} : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$ and $1_{\mathbf{2}\times\mathbf{2}} : \mathbf{2}\times\mathbf{2}\longrightarrow\mathbf{2}$. These connectives are defined by

$$\begin{aligned}
&\mathrm{pr}_1(x_1, x_2) = x_1\\
&\mathrm{npr}_1(x_1, x_2) = \neg x_1\\
&\mathrm{pr}_2(x_1, x_2) = x_2\\
&\mathrm{npr}_2(x_1, x_2) = \neg x_2\\
&0_{\mathbf{2}\times\mathbf{2}}(x_1, x_2) = 0\\
&1_{\mathbf{2}\times\mathbf{2}}(x_1, x_2) = 1\,,
\end{aligned}$$

  for all $x_1, x_2 \in \mathbf{2}$. By applying the above theorem, we obtain that these connectives are transported to

$$\begin{aligned}
&\mathcal{F}(\mathrm{pr}_1)(x_1, x_2) = x_1\\
&\mathcal{F}(\mathrm{npr}_1)(x_1, x_2) = \widetilde{\neg}\, x_1\\
&\mathcal{F}(\mathrm{pr}_2)(x_1, x_2) = x_2\\
&\mathcal{F}(\mathrm{npr}_2)(x_1, x_2) = \widetilde{\neg}\, x_2\\
&\mathcal{F}(0_{\mathbf{2}\times\mathbf{2}})(x_1, x_2) = 0\\
&\mathcal{F}(1_{\mathbf{2}\times\mathbf{2}})(x_1, x_2) = 1\,,
\end{aligned}$$

  for all $x_1, x_2 \in \mathbf{I}$.

The above connectives are of course uninteresting for practical purposes because they depend on one argument (or none) only. Let us now consider the more interesting class of "genuine" two-place connectives, which depend on both of their arguments.
With conjunction, there are several common choices in fuzzy logic (although the standard is certainly $\wedge = \min$). All of these belong to the class of t-norms (cf. Schweizer & Sklar [21]), which seems to capture what one would expect of a reasonable conjunction operator.

Definition 21 (t-norms)
$\widetilde{\wedge} : \mathbf{I}\times\mathbf{I}\longrightarrow\mathbf{I}$ *is called a t-norm iff it satisfies the following conditions.*

a. $x \,\widetilde{\wedge}\, 0 = 0$, *for all* $x \in \mathbf{I}$

b. $x \,\widetilde{\wedge}\, 1 = x$, *for all* $x \in \mathbf{I}$

c. $x_1 \,\widetilde{\wedge}\, x_2 = x_2 \,\widetilde{\wedge}\, x_1$ *for all* $x_1, x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \le x_1'$, *then* $x_1 \,\widetilde{\wedge}\, x_2 \le x_1' \,\widetilde{\wedge}\, x_2$, *for all* $x_1, x_1', x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\wedge}$ *is monotonically increasing)*

e. $(x_1 \,\widetilde{\wedge}\, x_2) \,\widetilde{\wedge}\, x_3 = x_1 \,\widetilde{\wedge}\, (x_2 \,\widetilde{\wedge}\, x_3)$, *for all* $x_1, x_2, x_3 \in \mathbf{I}$ *(associativity).*

As noted above, the most prominent t-norm is the standard fuzzy conjunction $\wedge = \min$ (it is the largest t-norm). Other well-known examples of t-norms are the *algebraic product* $\widetilde{\wedge}_a$, and the *bounded product* $\widetilde{\wedge}_b$, which are defined by

$$\begin{aligned}
&x_1 \,\widetilde{\wedge}_a\, x_2 = x_1 \cdot x_2\\
&x_1 \,\widetilde{\wedge}_b\, x_2 = \max(0, x_1 + x_2 - 1)
\end{aligned}$$

for all $x_1, x_2 \in \mathbf{I}$. A less well-known example of a t-norm is $\widetilde{\wedge}_m$, defined by

$$x_1 \,\widetilde{\wedge}_m\, x_2 = \begin{cases} 2x_1x_2 & : \max(x_1, x_2) < \frac{1}{2} \\ 1 + 2x_1x_2 - x_1 - x_2 & : \min(x_1, x_2) \geq \frac{1}{2} \\ \min(x_1, x_2) & : \text{else} \end{cases}$$

for all $x_1, x_2 \in \mathbf{I}$. Anticipating the notation to be introduced in chapter 4., $\widetilde{\wedge}_m$ is the t-norm induced by the determiner fuzzification mechanism

$$\mathfrak{m}(D)(X_1, \ldots, X_n) = \int \cdots \int \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}_1^{\gamma_1}, \ldots, Y_n \in \mathcal{Y}_n^{\gamma_n}\}\, d\gamma_1 \ldots d\gamma_n \,,$$

which fails to be a DFS because it does not satisfy the axiom of "internal meets" (DFS 6).
The plots of these t-norms are displayed in Figs. 3–6.

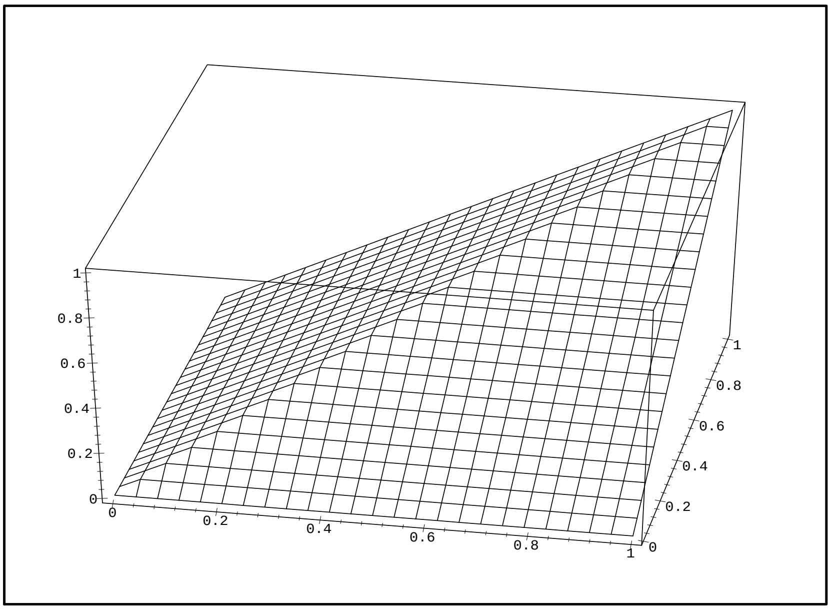

Figure 3: Plot of the t-norm $\wedge = \min$

Let us now state that the fuzzy conjunction induced by a DFS is "reasonable", in the sense of belonging to the class of t-norms.

Theorem 6 (Conjunction)
In every DFS $\mathcal{F}$, $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ is a t-norm.
(Proof: A.7, p.89+)

The dual concept of t-norm is that of an s-norm, which expresses the essential properties of fuzzy disjunction operators.

Definition 22 (s-norms)
$\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ *is called an s-norm iff it satisfies the following conditions.*

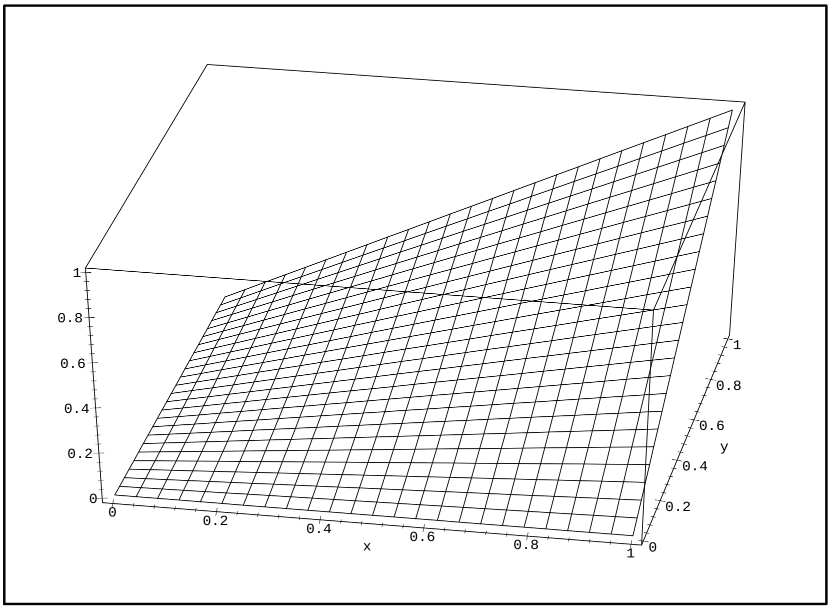

Figure 4: Plot of the algebraic product $\widetilde{\wedge}_a$

a. $x \mathbin{\widetilde{\vee}} 1 = 1$, *for all* $x \in \mathbf{I}$

b. $x \mathbin{\widetilde{\vee}} 0 = x$, *for all* $x \in \mathbf{I}$

c. $x_1 \mathbin{\widetilde{\vee}} x_2 = x_2 \mathbin{\widetilde{\vee}} x_1$ *for all* $x_1,\, x_2 \in \mathbf{I}$ *(commutativity)*

d. If $x_1 \leq x_1'$, *then* $x_1 \mathbin{\widetilde{\vee}} x_2 \leq x_1' \mathbin{\widetilde{\vee}} x_2$, *for all* $x_1,\, x_1',\, x_2 \in \mathbf{I}$ *(i.e.* $\widetilde{\vee}$ *is monotonically increasing)*

e. $(x_1 \mathbin{\widetilde{\vee}} x_2) \mathbin{\widetilde{\vee}} x_3 = x_1 \mathbin{\widetilde{\vee}} (x_2 \mathbin{\widetilde{\vee}} x_3)$, *for all* $x_1,\, x_2,\, x_3 \in \mathbf{I}$ *(associativity).*

Examples of s-norms are the standard choice $\vee = \max$ (it is the smallest s-norm), the algebraic sum $\widetilde{\vee}_a$ and the bounded sum $\widetilde{\vee}_b$ defined by

$$x_1 \mathbin{\widetilde{\vee}_a} x_2 = x_1 + x_2 - x_1 \cdot x_2$$
$$x_1 \mathbin{\widetilde{\vee}_b} x_2 = \min(1,\, x_1 + x_2)$$

(for all $x_1,\, x_2 \in \mathbf{I}$), and others.

Let us now state that the induced disjunction and induced implication of a DFS is reasonable.

Theorem 7 (Disjunction and Implication)
Let $\mathcal{F}$ *a DFS. Then*

a. $x_1 \mathbin{\widetilde{\vee}} x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, x_2)$, *i.e.* $\widetilde{\vee}$ *is the dual* s*-norm of* $\widetilde{\wedge}$ *under* $\widetilde{\neg}$,

b. $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2$

(Proof: A.8, p.90+)

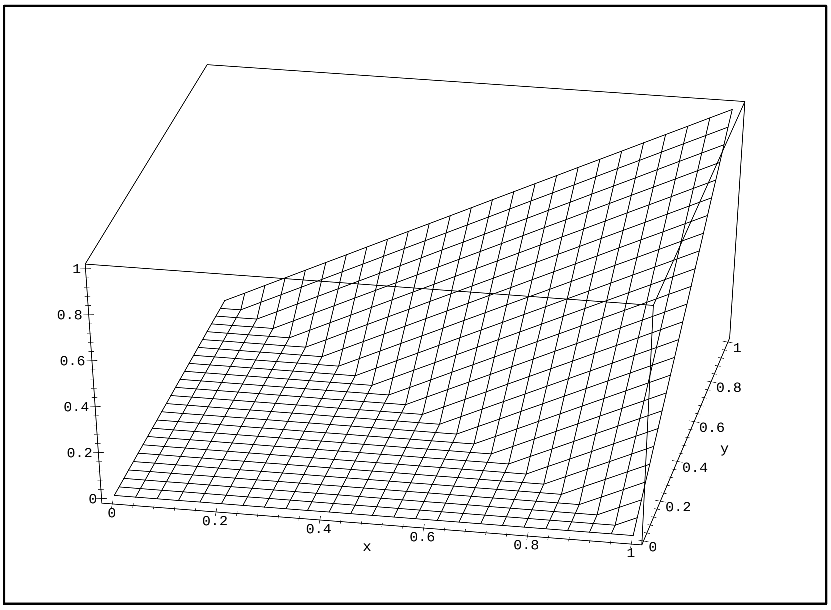

Figure 5: Plot of the bounded product $\widetilde{\wedge}_b$

The fuzzy disjunction induced by $\mathcal{F}$ is therefore definable in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$, and the fuzzy implication induced by $\mathcal{F}$ is definable in terms of $\widetilde{\vee}$ and $\widetilde{\neg}$ (and hence also in terms of $\widetilde{\wedge}$ and $\widetilde{\neg}$). A similar point can be made about *most* of the other two-place logical conectives.

$$\begin{aligned}
\widetilde{\mathcal{F}}(\not\rightarrow)(x_1,\,x_2) &= \widetilde{\neg}(x_1 \mathbin{\widetilde{\rightarrow}} x_2)\\
\widetilde{\mathcal{F}}(\leftarrow)(x_1,\,x_2) &= x_2 \mathbin{\widetilde{\rightarrow}} x_1\\
\widetilde{\mathcal{F}}(\not\leftarrow)(x_1,\,x_2) &= \widetilde{\neg}(x_2 \mathbin{\widetilde{\rightarrow}} x_1)\\
\widetilde{\mathcal{F}}(\mathrm{nand})(x_1,\,x_2) &= \widetilde{\neg}(x_1 \mathbin{\widetilde{\wedge}} x_2)\\
\widetilde{\mathcal{F}}(\mathrm{nor})(x_1,\,x_2) &= \widetilde{\neg}(x_1 \mathbin{\widetilde{\vee}} x_2)\,,
\end{aligned}$$

for all $x_1,\,x_2 \in \mathbf{I}$. These results are easily obtained from the definition of the above connectives in terms of $\wedge$ (or $\vee$ or $\rightarrow$), which only involves constructions with which $\mathcal{F}$ commutes (viz, negation and argument transposition). We shall discuss the connectives $\leftrightarrow$ and xor later; these require more effort.

We shall now discuss further homomorphism properties of DFSes with respect to operations on the argument sets.

On page 17, we have introduced the "internal join" $D\cup$ of a fuzzy pre-determiner (and $D\widetilde{\cup}$ for fuzzy determiners), which results from forming the union (induced fuzzy union, respectively) of argument sets. As an immediate consequence of (DFS 6), (DFS 5) and (DFS 4), we obtain

Theorem 8 (Internal joins)

Suppose $\mathcal{F}$ is a DFS and $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner ($n > 0$). Then

$$\mathcal{F}(D\cup) = \mathcal{F}(D)\widetilde{\cup}\,.$$

(Proof: A.9, p.91+)

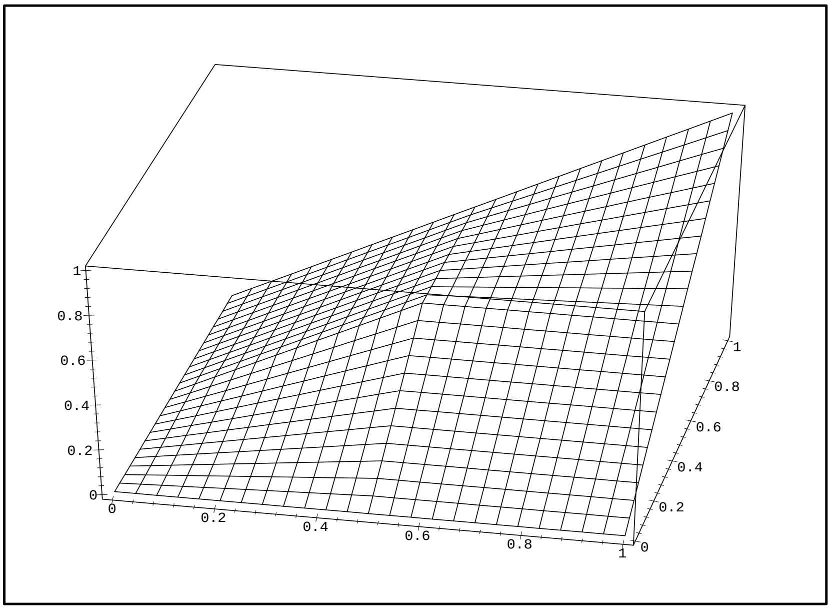

Figure 6: Plot of the t-norm $\widetilde{\wedge}_m$

To present an example, **all** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ is defined by

$$\textbf{all}(X_1,\, X_2) = \forall(\neg X_1 \cup X_2)\,,$$

i.e. **all** $= \forall \cup \tau_1 \neg \tau_1$. Then by the above theorem,

$$\mathcal{F}(\textbf{all}) = \mathcal{F}(\forall) \cup \tau_1 \neg \tau_1\,,$$

i.e.

$$\mathcal{F}(\textbf{all})(X_1,\, X_2) = \mathcal{F}(\forall)(\widetilde{\neg}\, X_1 \,\widetilde{\cup}\, X_2)$$

for all $X_1,\, X_2 \in \widetilde{\mathcal{P}}(E)$. Hence in our example domain E of persons,

$$\mathcal{F}(\textbf{all})(\textbf{women},\, \textbf{tall}) = \mathcal{F}(\forall)(\widetilde{\neg}\, \textbf{women} \,\widetilde{\cup}\, \textbf{tall})\,,$$

i.e. the gradual truth values of *All women are tall* and *The set of all persons which are either not women or tall is the set of all persons* are the same.

The axiom of internal complementation (DFS 5) states that every DFS $\mathcal{F}$ is compatible to complementation in the last argument of a determiner (and by (DFS 4), in arbitrary argument positions). This operation corresponds to the argument-wise negation of membership values: in the case of a fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$D\neg(X_1,\ \ldots,\ X_n) = D(X_1,\ \ldots,\ X_{n-1},\ Z)$$

where

$$\chi_Z(v) = \neg \chi_{X_n}(v)$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$ and $v \in E$. Analogously in the case of fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$,

$$D\widetilde{\neg}(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, Z)$$

where

$$\mu_Z(v) = \widetilde{\neg}\, \mu_{X_n}(v)\,,$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and $v \in E$.
We now wish to establish that $\mathcal{F}$ respects even more fine-grained application of the negation operator. In order to express this property, we need to introduce the (induced) fuzzy symmetrical difference.

Definition 23 (Symmetrical difference)
Let E a set. By $X_1 \triangle X_2$ we denote the symmetrical difference of subsets $X_1, X_2 \in \mathcal{P}(E)$, i.e.

$$X_1 \triangle X_2 = (X_1 \cap \neg X_2) \cup (X_2 \cap \neg X_1)\,.$$

Noting that $\triangle$ is the set operation induced by the "exclusive or" connective $\mathrm{xor} : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$, we define the fuzzy symmetrical difference $\widetilde{\triangle} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ as the operation on fuzzy sets which is induced by $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor})$, i.e. which to $X_1, X_2 \in \widetilde{\mathcal{P}}(E)$ assigns the fuzzy set $X_1 \,\widetilde{\triangle}\, X_2$ with membership function

$$\mu_{X_1 \widetilde{\triangle} X_2}(x) = \mu_{X_1}(x)\, \widetilde{\mathrm{xor}}\, \mu_{X_2}(x)$$

for all $x \in E$.

Definition 24
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner ($n > 0$) and $A \in \mathcal{P}(E)$ a crisp subset of E. By $D\triangle A : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the fuzzy pre-determiner defined by

$$D\triangle A(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, X_n \triangle A)$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.
Analogously, if $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy determiner and $A \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E, we define

$$D\widetilde{\triangle}A(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, X_n \,\widetilde{\triangle}\, A)$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

By using the symmetrical difference, we can negate the membership values of only *part* of the elements of an argument set (namely of those which are contained in A). I.e. if $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, then

$$D\triangle A(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, Z)$$

where

$$\chi_Z(v) = \begin{cases} \neg\chi_{X_n}(v) & : \quad v \in A \\ \chi_{X_n}(v) & : \quad v \notin A \end{cases}$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$, $A \in \mathcal{P}(E)$, $v \in E$. Likewise if $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy determiner and $A \in \mathcal{P}(E)$ is crisp, then

$$D \widetilde{\triangle} A(X_1, \ldots, X_n) = D(X_1, \ldots, X_{n-1}, Z)$$

where

$$\mu_Z(v) = \begin{cases} \widetilde{\neg}\, \mu_{X_n}(v) & : \quad v \in A \\ \mu_{X_n}(v) & : \quad v \notin A \end{cases}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, $v \in E$.
Let us now state that every DFS is compatible even to this more fine-structured type of negation.

Theorem 9
Let $\mathcal{F}$ a DFS. Then for every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ ($n > 0$) and every crisp subset A of E,

$$\mathcal{F}(D \triangle A) = \mathcal{F}(D)\, \widetilde{\triangle}\, A\,.$$

(Proof: A.10, p.91+)

Some further note on the propositional connectives, which may facilitate the proof that a given $\mathcal{F}$ is a DFS:

Theorem 10
Suppose a determiner fuzzification mechanism $\mathcal{F}$,

$$(D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$$

fulfills $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$, and $\neg' : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping such that for all fuzzy pre-determiners D,

$$\mathcal{F}(\neg' D) = \neg' \mathcal{F}(D)\,.$$

Then $\neg' = \widetilde{\mathcal{F}}(\neg)$, i.e. $\neg'$ is is the negation operator induced by $\mathcal{F}$.
(Proof: A.11, p.93+)

A similar point can be made about conjunction:

Theorem 11
Suppose a determiner fuzzification mechanism $\mathcal{F}$,

$$(D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$$

fulfills $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$, and $\wedge' : \mathbf{I} \longrightarrow \mathbf{I}$ is a mapping such that for all fuzzy pre-determiners D,

$$\mathcal{F}(D\cap) = \mathcal{F}(D)\cap'\,,$$

where $\cap'$ is the fuzzy intersection corresponding to $\wedge'$.
Then $\wedge' = \widetilde{\mathcal{F}}(\wedge)$, i.e. $\wedge'$ is the conjunction operator induced by $\mathcal{F}$.
(Proof: A.12, p.93+)

3.2 Further monotonicity properties

Let us now state that every DFS preserves monotonicity properties of fuzzy pre-determiners even if these hold only locally.

Definition 25 (Local monotonicity)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $U, V \in \mathcal{P}(E)^n$. We say that D is locally increasing in the range (U, V) *iff for all $(X_1, \ldots, X_n), (X'_1, \ldots, X'_n) \in \mathcal{P}(E)^n$ such that $U_i \subseteq X_i \subseteq X'_i \subseteq V_i$ ($i = 1, \ldots, n$),*

$$D(X_1, \ldots, X_n) \leq D(X'_1, \ldots, X'_n) .$$

We will say that D is locally decreasing in the range (U, V) if under the same conditions,

$$D(X_1, \ldots, X_n) \leq D(X'_1, \ldots, X'_n) .$$

On fuzzy determiners, local monotonicity is defined analogously, but $(X_1, \ldots, X_n), (X'_1, \ldots, X'_n)$ are taken from $\widetilde{\mathcal{P}}(E)^n$, and '$\subseteq$' is the fuzzy inclusion relation.

Theorem 12 (Preservation of local monotonicity)
Suppose $\mathcal{F}$ is a DFS, $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $U, V \in \mathcal{P}(E)^n$. Then D is locally increasing (decreasing) in the range (U, V) iff $\mathcal{F}(D)$ is locally increasing (decreasing) in the range (U, V).
(Proof: A.13, p.94+)

DFSes can also be shown to be *monotonic* in the sense of preserving inequations between determiners. Let us firstly define a partial order $\leq$ on fuzzy determiners and pre-determiners.

Definition 26
Suppose $D, D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are fuzzy pre-determiners of the same arity. Let us write $D \leq D'$ iff for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$,

$$D(X_1, \ldots, X_n) \leq D'(X_1, \ldots, X_n)$$

On fuzzy determiners, we define $\leq$ analogously, where $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Theorem 13 (Monotonicity)
Let $\mathcal{F}$ a DFS and $D, D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ fuzzy pre-determiners. Then

$$D \leq D' \Leftrightarrow \mathcal{F}(D) \leq \mathcal{F}(D') .$$

(Proof: A.14, p.94+)

We can now provide an example illustrating the utility of cylindrical extensions.
By Def. 26, $D \leq D'$ is defined only if D and D' have the same arity n. But it may be useful to express inequations also in the case of determiners involving a different number of arguments. For example, **all**$\cap : \mathcal{P}(E)^3 \longrightarrow \mathbf{2}$ can be said to be smaller than **all** $: \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ in the sense that

$$\textbf{all}\cap(X_1, X_2, X_3) = \textbf{all}(X_1, X_2 \cap X_3) \leq \textbf{all}(X_1, X_2) ,$$

for all $X_1, X_2, X_3 \in \mathcal{P}(E)$.

Let us now establish that this kind of inequation is preserved by every DFS $\mathcal{F}$. We define the cylindrical extension $D : \mathcal{P}(E)^3 \longrightarrow \mathbf{2}$ of $\textbf{all} : \mathcal{P}(E)^2 \longrightarrow \mathbf{2}$ by

$$D(X_1, X_2, X_3) = \textbf{all}(X_1, X_2)\,,$$

for all $X_1, X_2, X_3 \in \mathcal{P}(E)$. This derived fuzzy pre-determiner D has the same arity as $\textbf{all}\cap$, i.e. Th-13 is now applicable! We can then use the theorem on "redundant arguments" Th-5 to obtain

$$\begin{aligned}
\mathcal{F}(\textbf{all})(X_1, X_2) &= \mathcal{F}(D)(X_1, X_2, X_3) && \text{by Th-5}\\
&\leq \mathcal{F}(\textbf{all}\cap)(X_1, X_2, X_3) && \text{by Th-13}\\
&= \mathcal{F}(\textbf{all})(X_1, X_2 \mathbin{\widetilde{\cap}} X_3)
\end{aligned}$$

for all $X_1, X_2, X_3 \in \widetilde{\mathcal{P}}(E)$. In particular, if $\textbf{men}, \textbf{bald}, \textbf{tall} \in \widetilde{\mathcal{P}}(E)$ are the (fuzzy) extensions of "men", "bald" and "tall", respectively, in our example universe E, then

$$\mathcal{F}(\textbf{all})(\textbf{men}, \textbf{bald} \mathbin{\widetilde{\cap}} \textbf{tall}) \leq \mathcal{F}(\textbf{all})(\textbf{men}, \textbf{bald})\,,$$

i.e. *All men are bald and tall* expresses a stronger condition than *All men are bald.*[16]

Definition 27
Let $D_1, D_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ fuzzy pre-determiners and $U, V \in \mathcal{P}(E)^n$. We say that D_1 is (not necessary strictly) smaller than D_2 in the range (U, V) (in symbols: $D_1 \leq_{(U,V)} D_2$), iff for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$ such that $U_1 \subseteq X_1 \subseteq V_1, \ldots, U_n \subseteq X_n \subseteq V_n$,

$$D_1(X_1, \ldots, X_n) \leq D_2(X_1, \ldots, X_n)\,.$$

On fuzzy pre-determiners, we define $D_1 \leq_{(U,V)} D_2$ analogously (but $(X_1, \ldots, X_n)$ is taken from $\widetilde{\mathcal{P}}(E)^n$, and '$\subseteq$' denotes the fuzzy inclusion relation).

Again, every DFS preserves inequations between fuzzy pre-determiners even if these hold only locally.

Theorem 14
Let $\mathcal{F}$ a DFS, $D_1, D_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $U, V \in \mathcal{P}(E)^n$. Then

$$D_1 \leq_{(U,V)} D_2 \Leftrightarrow \mathcal{F}(D_1) \leq_{(U,V)} \mathcal{F}(D_2)\,.$$

(Proof: A.15, p.95+)

3.3 Properties of the induced extension principle

Theorem 15 (Projection determiners)
Let $\mathcal{F}$ a DFS, $E \neq \varnothing$ a base set and $n \in \mathbb{N}$. Then for all $x \in E$, $i \in \{1, \ldots, n\}$,

$$\mathcal{F}(\pi_x^{n,i}) = \widetilde{\pi}_x^{n,i}\,.$$

(Proof: A.16, p.95+)

[16] Of course, we already knew this from Th-4. Nevertheless, the example should be illustrative.

Hence in the example domain $E = \{\text{John, Lucas, Mary}\}$, the fuzzy projection pre-determiner **john** $= \pi_{\text{John}} : \mathcal{P}(E) \longrightarrow \mathbf{2}$ would be mapped to $\widetilde{\pi}_{\text{John}} : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$, as desired.

Theorem 16 (Compatibility with composition)
Let $\mathcal{F}$ a DFS and $\widehat{\mathcal{F}}$ the extension principle induced by $\mathcal{F}$. Then for all $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$ (where $E \neq \varnothing$, $E' \neq \varnothing$, $E'' \neq \varnothing$),

a. $\widehat{\mathcal{F}}(g \circ f) = \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)$

b. $\widehat{\mathcal{F}}(\mathrm{id}_E) = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}$

(Proof: A.17, p.98+)

Note. A reader familiar with category theory will recognize this as the statement that $\widehat{\mathcal{F}}$ is a covariant functor from the category of non-empty sets to the category of fuzzy power sets, provided that on objects E, we define $\widehat{\mathcal{F}}(E) = \widetilde{\mathcal{P}}(E)$.[17] It is a faithful functor because $f \neq g$ implies $\widehat{\mathcal{F}}(f)|_{\mathcal{P}(E)} = \widehat{f} \neq \widehat{g} = \widehat{\mathcal{F}}(g)|_{\mathcal{P}(E)}$, i.e. $\widehat{\mathcal{F}}(f) \neq \widehat{\mathcal{F}}(g)$. $\widehat{\mathcal{F}}$ is also injective on objects; $E \neq E'$ implies $\widetilde{\mathcal{P}}(E) \neq \widetilde{\mathcal{P}}(E')$.

Let us note that the induced extension principles of all DFSes coincide on injective mappings.

Theorem 17
Suppose $\mathcal{F}$ is a DFS and $f : E \longrightarrow E'$ is an injection. Then for all $X \in \widetilde{\mathcal{P}}(E)$, $v \in E'$,

$$\begin{aligned}\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \begin{cases} \mu_X(f^{-1}(v)) & : \quad v \in \operatorname{Im} f \\ 0 & : \quad v \notin \operatorname{Im} f \end{cases} \\ &= \begin{cases} \pi_{f^{-1}(v)}(X) & : \quad v \in \operatorname{Im} f \\ 0 & : \quad v \notin \operatorname{Im} f \end{cases}\end{aligned}$$

(Proof: A.18, p.98+)

We are now able to state that *quantitative* determiners remain quantitative when fuzzified by applying a DFS.

Definition 28 (Quantitative fuzzy (pre-) determiner)
A fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is called quantitative *iff for all automorphisms (i.e. bijections of E into itself) $\beta : E \longrightarrow E$ and all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$,*

$$D(X_1, \ldots, X_n) = D(\widehat{\beta}(X_1), \ldots, \widehat{\beta}(X_n)) .$$

Likewise, a fuzzy determiner $D : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is said to be quantitative iff for all automorphisms $\beta : E \longrightarrow E$ and all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$D(X_1, \ldots, X_n) = D(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_n)) ,$$

where $\hat{\hat{\beta}} : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E)$ is obtained by applying the standard extension principle.

[17] By the category of fuzzy power sets we mean the category in which the objects are fuzzy power sets $\widetilde{\mathcal{P}}(E)$, the morphisms are mappings $f : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ which to each fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ assign a fuzzy subset $f(X) \in \widetilde{\mathcal{P}}(E')$, and $\circ$ is the usual composition of functions.

Note. By Th-17, the induced extension principles of arbitrary DFSes coincide on injective mappings, in particular on bijections. Therefore, the explicit mention of the standard extension principle in the above definition does *not* limit the scope of this definition to any particular choice of extension principle.

Theorem 18
Suppose $\mathcal{F}$ is a DFS. Then for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$D \text{ quantitative } \Leftrightarrow \mathcal{F}(D) \text{ quantitative.}$$

(Proof: A.19, p.99+)

For example, the quantitative determiners **all**, **some** and **atleast n** are mapped to quantitative fuzzy determiners $\mathcal{F}(\textbf{all})$, $\mathcal{F}(\textbf{some})$ and $\mathcal{F}(\textbf{atleast n})$, respectively. On the other hand, the (non-quantitative) projection determiner **john** $= \pi_{\text{John}}$ is mapped to the fuzzy projection determiner $\mathcal{F}(\textbf{john}) = \widetilde{\pi}_{\text{John}}$, which is also non-quantitative.

We will now establish that every DFS is compatible with exactly one extension principle.

Theorem 19
Let $\mathcal{F}$ a DFS and $\mathcal{E}$ an extension principle such that for every fuzzy pre-determiner $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ and all $f_1 : E \longrightarrow E', \ \ldots, \ f_n : E \longrightarrow E'$,

$$\mathcal{F}(D \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(D) \circ \mathop{\times}_{i=1}^{n} \mathcal{E}(f_i) \,.$$

Then $\widehat{\mathcal{F}} = \mathcal{E}$. (Proof: A.20, p.99+)

The next theorem facilitates proofs that a given "DFS candidate" $\mathcal{F}$ is compatible to its own extension principle (i.e. that $\mathcal{F}$ satisfies (DFS 9))

Theorem 20
If a determiner fuzzification mechanism $\mathcal{F}$ fulfills (DFS 1-8) and there is an extension principle $\mathcal{E}$ such that

1. *for every fuzzy pre-determiner $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ and all $f_1 : E \longrightarrow E', \ \ldots, \ f_n : E \longrightarrow E'$,*

$$\mathcal{F}(D \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(D) \circ \mathop{\times}_{i=1}^{n} \mathcal{E}(f_i) \,,$$

2. *for all sets V and constants $c_v : \{*\} \longrightarrow V$, $* \mapsto c_v(*) = v$ (where $v \in V$),*

$$\mu_{\mathcal{E}(c_v)(\_)}(v) = \widetilde{\pi}_*$$

then $\widehat{\mathcal{F}} = \mathcal{E}$. In particular, $\mathcal{F}$ is a DFS.
(Proof: A.21, p.99+)

The extension principle $\widehat{\mathcal{F}}$ of a DFS $\mathcal{F}$ is uniquely determined by the fuzzy existential quantifiers $\mathcal{F}(\exists) = \mathcal{F}(\exists_E) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I}$ induced by $\mathcal{F}$.

Theorem 21
Suppose $\mathcal{F}$ is a DFS and $f : E \longrightarrow E'$. Then

$$\mu_{\widehat{\mathcal{F}}(f)(\_)}(v) = \mathcal{F}(\exists)\widetilde{\cap}\triangleleft f^{-1}(v)$$

for all $v \in E'$.
(Proof: A.22, p.100+)

The converse can also be shown: the fuzzy existential quantifiers obtained from a DFS $\mathcal{F}$ are uniquely determined by its extension principle $\widehat{\mathcal{F}}$.

Theorem 22
Let $\mathcal{F}$ a DFS, E any non-empty set and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Then

$$\mathcal{F}(\exists) = \widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!)\,,$$

where $! = !_E$ is the unique mapping $! : E \longrightarrow \{*\}$, i.e. defined by

$$!(x) = *$$

for all $x \in E$.
(Proof: A.23, p.100+)

A notion closely related to extension principles is that of *fuzzy inverse images*. We have already introduced the notion of (crisp) inverse images $f^{-1} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E)$ of a given $f : E \longrightarrow E'$, which are defined (as usual) by

$$f^{-1}(V) = \{x \in E : f(x) \in V\}\,,$$

for all $V \in \mathcal{P}(E')$. Unlike in the case of extension principles, there is only one reasonable definition of fuzzy inverse images: each $f : E \longrightarrow E'$ is mapped to that $\hat{\hat{f}}^{-1} : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ which to each $Y \in \widetilde{\mathcal{P}}(E')$ assigns the fuzzy subset $\hat{\hat{f}}^{-1}(Y) \in \widetilde{\mathcal{P}}(E)$ defined by

$$\mu_{\hat{\hat{f}}^{-1}(Y)}(x) = \mu_Y(f(x))$$

for all $x \in E$.
Now, every determiner fuzzification mechanism $\mathcal{F}$ induces fuzzy inverse images by means of the following construction.

Definition 29 (Induced fuzzy inverse images)
Suppose $\mathcal{F}$ is a determiner fuzzification mechanism and $f : E \longrightarrow E'$ is some mapping. $\mathcal{F}$ induces a fuzzy inverse image mapping $\widehat{\mathcal{F}}^{-1}(f) : \widetilde{\mathcal{P}}(E') \longrightarrow \widetilde{\mathcal{P}}(E)$ which to each $Y \in \widetilde{\mathcal{P}}(E')$ assigns the fuzzy subset $\widehat{\mathcal{F}}^{-1}(f)$ defined by

$$\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(x) = \mathcal{F}(\chi_{f^{-1}(\_)}(x))(Y)\,.$$

It is now easily shown that if $\mathcal{F}$ is a DFS, then its induced fuzzy inverse images coincide with the above "reasonable" definition.

Theorem 23 (Induced fuzzy inverse images)
Suppose $\mathcal{F}$ is a DFS, $f : E \longrightarrow E'$ is a mapping and $Y \in \widetilde{\mathcal{P}}(E')$. Then

$$\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(x) = \mu_Y(f(x)),$$

for all $x \in E$.
(Proof: A.24, p.101+)

These remarks on fuzzy inverse images conclude our discussion of the induced extension principle. Next, we investigate which fuzzy determiners are obtained when applying a DFS $\mathcal{F}$ to the standard quantifiers.

3.4 Properties with respect to the standard quantifiers

We will firstly consider the existential quantifiers.

Theorem 24
Let $\mathcal{F}$ a DFS, E a non-empty set and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

a. $\mathcal{F}(\exists)(\varnothing) = 0, \quad \mathcal{F}(\exists)(E) = 1$

b. $\mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} \{v\}) = \mu_X(v)$
for all $v \in E$

c. $\mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} A) = \mathop{\widetilde{\bigvee}}_{i=1}^{m} \mu_X(a_i)$
for all finite $A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E)$ ($a_i \neq a_j$ if $i \neq j$)

d. $\mathcal{F}(\exists)(X) \geq \sup \left\{ \mathop{\widetilde{\bigvee}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$

In particular, if E is finite, i.e. $E = \{v_1, \ldots, v_m\}$ where the v_i are pairwise distinct, then

$$\mathcal{F}(\exists)(X) = \mathop{\widetilde{\bigvee}}_{i=1}^{m} \mu_X(v_i).$$

(Proof: A.25, p.101+)

Note. By the last part of the theorem, the important relationship between $\exists$ and $\vee$, which holds in the finite case, is preserved when applying a DFS.
Th-24 has an important consequence.

Theorem 25
The induced disjunction $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ of a DFS $\mathcal{F}$ is uniquely determined by its induced extension principle, viz

$$x_1 \mathbin{\widetilde{\vee}} x_2 = (\widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!))(X)$$

for all $x_1, x_2 \in \mathbf{I}$, where $X \in \widetilde{\mathcal{P}}(\{1, 2\})$ is defined by $\mu_X(1) = x_1$ and $\mu_X(2) = x_2$, and $!$ is the unique mapping $! = !_{\{1,2\}} : \{1, 2\} \longrightarrow \{*\}$.
(Proof: A.26, p.104+)

Notes

- Consequently, if $\mathcal{F}$, $\mathcal{F}'$ are DFSes which are compatible to the same extension principle, they induce the same disjunction, i.e. $\widetilde{\mathcal{F}}(\vee) = \widetilde{\mathcal{F}'}(\vee)$.
- In particular, if $\widehat{\mathcal{F}}$ is the standard extension principle (see Def. 17), then $\widetilde{\vee} = \max$.

  Because of this fact, and because we did not want fuzzification schemes in which $\widetilde{\vee} \neq \max$ to be a priori excluded from consideration, we have stated axiom (DFS 9) in terms of the extension principle $\widehat{\mathcal{F}}$ induced by $\mathcal{F}$ (rather than requiring the compatibility of $\mathcal{F}$ to the standard extension principle).

This finishes our discussion of $\exists$, and we turn to the dual case of the quantifier $\forall$.

Theorem 26
Let F a DFS, E a non-empty set and $\forall = \forall_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

a. $\mathcal{F}(\forall)(\varnothing) = 0, \quad \mathcal{F}(\forall)(E) = 1$

b. $\mathcal{F}(\forall)(X \mathbin{\widetilde{\cup}} \widetilde{\neg} \{v\}) = \mu_X(v)$
for all $v \in E$

c. $\mathcal{F}(\forall)(X \mathbin{\widetilde{\cup}} \widetilde{\neg} A) = \mathop{\widetilde{\bigwedge}}\limits_{i=1}^{m} \mu_X(a_i)$
for all finite $A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E)$ ($a_i \neq a_j$ if $i \neq j$)

d. $\mathcal{F}(\forall)(X) \leq \inf \left\{ \mathop{\widetilde{\bigwedge}}\limits_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\}$

In particular, if E is finite, i.e. $E = \{v_1, \ldots, v_m\}$ where the v_i are pairwise distinct, then

$$\mathcal{F}(\forall)(X) = \mathop{\widetilde{\bigwedge}}\limits_{i=1}^{m} \mu_X(v_i)\,.$$

(Proof: A.27, p.105+)

Note. Again, the fuzzy universal quantifiers are reasonable in the sense that the important relationship between $\forall$ and $\wedge$, which holds in the finite case, is preserved when applying a DFS.

A final remark. We know by Th-24 that $\mathcal{F}(\exists)$ is uniquely determined by the extension principle $\widehat{\mathcal{F}}$ induced by $\mathcal{F}$. Recalling that

$$\begin{aligned} \mathcal{F}(\forall) &= \mathcal{F}(\widetilde{\neg}\,\exists\neg) \\ &= \widetilde{\neg}\,\mathcal{F}(\exists)\widetilde{\neg}\,, && \text{by (DFS 3), (DFS 5)} \end{aligned}$$

we see that $\mathcal{F}(\forall)$ is uniquely determined by the induced extension principle $\widehat{\mathcal{F}}$ of $\mathcal{F}$ and its induced negation $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$.
A similar point can be made about conjunction and implication because in every DFS,

$$x_1 \mathbin{\widetilde{\wedge}} x_2 = \widetilde{\neg}(\widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} \widetilde{\neg}\, x_2)$$

and

$$x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2$$

for all $x_1, x_2 \in \mathbf{I}$, i.e. $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ and $\widetilde{\rightarrow} = \widetilde{\mathcal{F}}(\rightarrow)$ are definable in terms of $\widetilde{\neg}$ and $\widetilde{\vee}$, and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is uniquely determined by the induced extension principle $\widehat{\mathcal{F}}$ by Th-25.
In particular, if two DFSes $\mathcal{F}$ and $\mathcal{F}'$ induce the same extension principle and the same negation operator, then $\mathcal{F}$ and $\mathcal{F}'$ coincide on the standard quantifiers $\exists$ and $\forall$ and on all propositional connectives—except possibly $\leftrightarrow$ and xor, about which we have no general results yet (for a result on an important subclass of DFSes, see p. 48).

3.5 Special DFS subclasses

We will now turn to subclasses of DFS models which satisfy some additional requirements.

3.5.1 DFSes which induce the standard complement

Definition 30
Let $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ a strong negation operator. A DFS $\mathcal{F}$ is called a $\widetilde{\neg}$-DFS if its induced negation coincides with $\widetilde{\neg}$, i.e.

$$\widetilde{\mathcal{F}}(\neg) = \widetilde{\neg}\,.$$

In particular, we will call $\mathcal{F}$ a $\neg$-DFS if it induces the standard negation $\neg : \mathbf{I} \longrightarrow \mathbf{I}$, $x \mapsto \neg x = 1 - x$.

In the following we will establish that without loss of generality, one can restrict attention to $\neg$-DFSes only. But first, we need to introduce a mechanism which allows us to transform a given DFS $\mathcal{F}$ into another.

Definition 31
Let $\mathcal{F}$ a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ a bijection. For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, we define

$$\mathcal{F}^{\sigma}(D)(X_1, \ldots, X_n) = \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n)\,,$$

where σD abbreviates $\sigma \circ D$, and $\sigma X_i \in \widetilde{\mathcal{P}}(E)$ is the fuzzy subset of E defined by the membership function

$$\mu_{\sigma X_i}(v) = \sigma(\mu_{X_i}(v))\,,$$

for all $v \in E$.

Theorem 27
If $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotonically increasing bijection, then $\mathcal{F}^{\sigma}$ is a DFS. The induced negation of $\mathcal{F}^{\sigma}$ is

$$\widetilde{\mathcal{F}^{\sigma}}(\neg)(x) = \sigma^{-1}\widetilde{\mathcal{F}}(\neg)(\sigma(x))$$

i.e. $\widetilde{\neg}^{\sigma} x = (\sigma^{-1} \widetilde{\neg} \sigma)(x)$ *for all* $x \in \mathbf{I}$*, and the induced conjunction is*

$$\widetilde{\mathcal{F}}^{\sigma}(\wedge)(x_1,\, x_2) = \sigma^{-1} \widetilde{\mathcal{F}}(\wedge)(\sigma(x_1),\, \sigma(x_2))\,,$$

for all $x_1,\, x_2 \in \mathbf{I}$*, i.e.* $x_1 \mathbin{\widetilde{\wedge}^{\sigma}} x_2 = \sigma^{-1}(\sigma x_1 \mathbin{\widetilde{\wedge}} \sigma x_2)$.
(Proof: A.28, p.105+)

It is well-known that for every strong negation $\widetilde{\neg} : \mathbf{I} \longrightarrow \mathbf{I}$ there is a monotonically increasing bijection $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ such that

$$\widetilde{\neg}\, x = \sigma^{-1}(1 - \sigma(x))\,.$$

(i.e. $\widetilde{\neg} = \sigma^{-1} \circ \neg \circ \sigma$). σ is called the *generator* of $\widetilde{\neg}$.
From the above theorem, it is then easy to obtain the following.

Theorem 28 ($\neg$-DFSes suffice)
Let $\mathcal{F}$ *a* $\widetilde{\neg}$*-DFS and* $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ *the generator of* $\widetilde{\neg}$*. Then* $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$ *is a* $\neg$*-DFS and* $\mathcal{F} = \mathcal{F}'^{\sigma}$*.*
(Proof: A.29, p.108+)

This means that we can freely move from an arbitrary $\widetilde{\neg}$-DFS to a corresponding $\neg$-DFS and vice versa. In particular, we can restrict our attention to $\neg$-DFSes because we would not gain anything (really) new when considering other DFSes also.

Definition 32 ($\mathcal{E}$-DFS)
A $\neg$*-DFS* $\mathcal{F}$ *which induces the extension principle* $\mathcal{E} = \widehat{\mathcal{F}}$ *will be called an* $\mathcal{E}$*-DFS.*

Theorem 29 (Combining $\mathcal{E}$-DFSes to form new $\mathcal{E}$-DFSes)
Suppose $\mathcal{J}$ *is a non-empty index set and* $(\mathcal{F}_j)_{j \in \mathcal{J}}$ *is a* $\mathcal{J}$*-indexed collection of* $\mathcal{E}$*-DFSes. Further suppose that* $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ *satisfies the following conditions:*

a. *If* $f \in \mathbf{I}^{\mathcal{J}}$ *is constant, i.e. if there is a* $c \in \mathbf{I}$ *such that* $f(j) = c$ *for all* $j \in \mathcal{J}$*, then* $\Psi(f) = c$*.*

b. $\Psi(1 - f) = 1 - \Psi(f)$*, where* $1 - f \in \mathbf{I}^{\mathcal{J}}$ *is point-wise defined by* $(1 - f)(j) = 1 - f(j)$*, for all* $j \in \mathcal{J}$*.*

c. Ψ *is monotonically increasing, i.e. if* $f(j) \le g(j)$ *for all* $j \in \mathcal{J}$*, then* $\Psi(f) \le \Psi(g)$*.*

If we define $\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}]$ *by*

$$\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}](D)(X_1,\, \ldots,\, X_n) = \Psi((\mathcal{F}_j(D)(X_1,\, \ldots,\, X_n))_{j \in \mathcal{J}})$$

for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and* $(X_1,\, \ldots,\, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, then* $\Psi[(\mathcal{F}_j)_{j \in \mathcal{J}}]$ *is an* $\mathcal{E}$*-DFS.*
(Proof: A.30, p.109+)

In particular, convex combinations (e.g., arithmetic mean) and stable symmetric sums [22] of $\mathcal{E}$-DFSes are again $\mathcal{E}$-DFSes.
The $\neg$-DFSes (and in particular, their subclasses of $\mathcal{E}$-DFSes), can be partially ordered by "specificity" or "cautiousness" (in the sense of closeness to $\frac{1}{2}$).

Definition 33 (Less specific)
Suppose $\mathcal{F}$, $\mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is (not necessarily strictly) less specific *than $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq \mathcal{F}'$, iff for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,*

$$|\mathcal{F}(D)(X_1, \ldots, X_n) - \tfrac{1}{2}| \leq |\mathcal{F}'(D)(X_1, \ldots, X_n) - \tfrac{1}{2}| .$$

Note. Obviously, $\preceq$ is a partial order on $\neg$-DFSes.

Definition 34 (Consistently less specific)
Suppose $\mathcal{F}$, $\mathcal{F}'$ are $\neg$-DFSes. We say that $\mathcal{F}$ is consistently less specific *that $\mathcal{F}'$, in symbols: $\mathcal{F} \preceq_c \mathcal{F}'$, iff for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,*

a. If $\mathcal{F}'(D)(X_1, \ldots, X_n) \geq \frac{1}{2}$, then $\mathcal{F}'(D)(X_1, \ldots, X_n) \geq \mathcal{F}(D)(X_1, \ldots, X_n) \geq \frac{1}{2}$

b. If $\mathcal{F}'(D)(X_1, \ldots, X_n) \leq \frac{1}{2}$, then $\mathcal{F}'(D)(X_1, \ldots, X_n) \leq \mathcal{F}(D)(X_1, \ldots, X_n) \leq \frac{1}{2}$.

Notes

- Again, $\preceq_c$ is a partial order on $\neg$-DFSes. Furthermore, $\mathcal{F} \preceq_c F' \Rightarrow \mathcal{F} \preceq \mathcal{F}'$ (this is apparent).
- $\preceq_c$ is closely related to the "ambiguity relation" $A \subseteq \mathbf{I} \times \mathbf{I}$ introduced by Mukaidono [23].

We now wish to establish the existence of consistently least specific $\mathcal{E}$-DFSes. In order to be able to state the theorem, we firstly need to introduce the *fuzzy median* $\mathrm{m}_{\frac{1}{2}}$.

Definition 35 (Fuzzy median)
The fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is defined by

$$\mathrm{m}_{\frac{1}{2}}(u_1, u_2) = \begin{cases} \min(u_1, u_2) & : \quad \min(u_1, u_2) > \frac{1}{2} \\ \max(u_1, u_2) & : \quad \max(u_1, u_2) < \frac{1}{2} \\ \frac{1}{2} & : \quad \textit{else} \end{cases}$$

The plot of $\mathrm{m}_{\frac{1}{2}}$ is displayed in Fig. 7.
$\mathrm{m}_{\frac{1}{2}}$ is an associative mean operator [24] and the only stable (i.e. idempotent) associative symmetric sum [22]. Due to its unique properties, the fuzzy median will play a central role throughout this report.
The fuzzy median can be extended to an operator $\mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ which accepts arbitrary subsets of $\mathbf{I}$ as its arguments.
Firstly, because it is associative, idempotent and commutative, $\mathrm{m}_{\frac{1}{2}}$ can be generalized to arbitrary finite sets of arguments (just apply $\mathrm{m}_{\frac{1}{2}}$ in any order). Noting that for all finite $X = \{x_1, \ldots, x_n\} \subseteq \mathbf{I}$, $n \geq 2$, it holds that

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}} (\min X, \max X) ,$$

the proper definition of $\mathrm{m}_{\frac{1}{2}} X$ in the case $n = 0$, $n = 1$ is

$$\mathrm{m}_{\frac{1}{2}} \varnothing = \mathrm{m}_{\frac{1}{2}}(\min \varnothing, \max \varnothing) = \mathrm{m}_{\frac{1}{2}}(1, 0) = \tfrac{1}{2} ,$$
$$\mathrm{m}_{\frac{1}{2}}\{u\} = \mathrm{m}_{\frac{1}{2}}(\min\{u\}, \max\{u\}) = \mathrm{m}_{\frac{1}{2}}(u, u) = u$$

We extend $\mathrm{m}_{\frac{1}{2}}$ to arbitrary subsets $X \subseteq \mathbf{I}$ as follows.

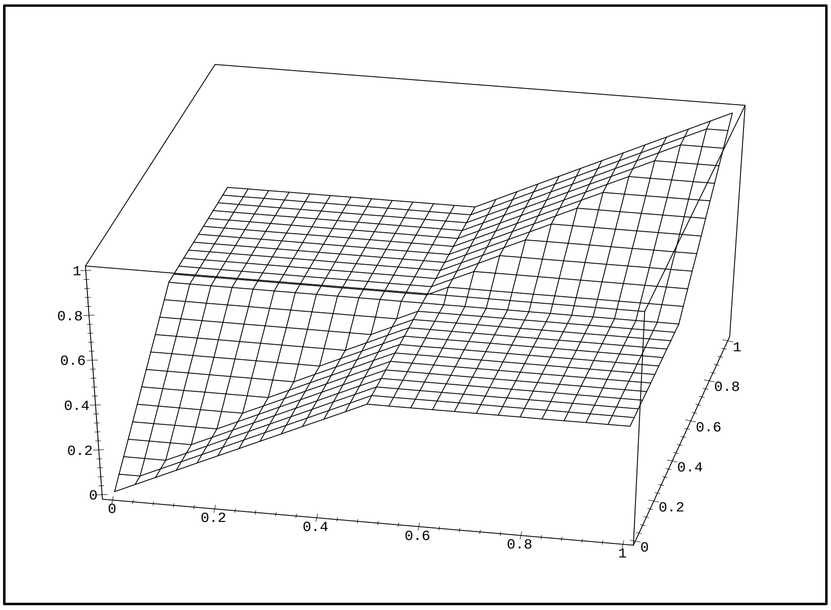

Figure 7: The fuzzy median $\mathrm{m}_{\frac{1}{2}}$

Definition 36 (Extending the fuzzy median to arbitrary subsets of $\mathbf{I}$)
The (extended) fuzzy median $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X,\ \sup X)\,,$$

for all $X \in \mathcal{P}(\mathbf{I})$.

Note. This definition is obviously compatible with the definition of $\mathrm{m}_{\frac{1}{2}}$ on finite subsets of $\mathbf{I}$. An important property of (the extended) $\mathrm{m}_{\frac{1}{2}}$ is stated in the following theorem.

Theorem 30
Suppose $X,\ Y \subseteq \mathbf{I}$ *and* $X \subseteq Y$.

a. If $\mathrm{m}_{\frac{1}{2}} X < \frac{1}{2}$, *then* $\mathrm{m}_{\frac{1}{2}} X \le \mathrm{m}_{\frac{1}{2}} Y \le \frac{1}{2}$.

b. If $\mathrm{m}_{\frac{1}{2}} X > \frac{1}{2}$, *then* $\frac{1}{2} \le \mathrm{m}_{\frac{1}{2}} Y \le \mathrm{m}_{\frac{1}{2}} X$.

c. If $\mathrm{m}_{\frac{1}{2}} X = \frac{1}{2}$ *and* $X \neq \varnothing$, *then* $\mathrm{m}_{\frac{1}{2}} Y = \frac{1}{2}$.

(Proof: A.31, p.110+)

By using the fuzzy median, we can now state the theorem concerning least specific DFSes that we are after.

Theorem 31 (Existence of consistently least specific $\mathcal{E}$-DFSes)
Let $\mathcal{E}$ *an extension principle induced by some* $\neg$*-DFS* $\mathcal{F}$. *There exists a consistently least specific* $\mathcal{E}$*-DFS, i.e. an* $\mathcal{E}$*-DFS* $\mathcal{L}$ *such that for every* $\mathcal{E}$*-DFS* $\mathcal{F}'$,

$$\mathcal{L} \preceq_c \mathcal{F}'\,.$$

$\mathcal{L}$ is defined by

$$\mathcal{L}(D)(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(D)(X_1, \ldots, X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\}$$

for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, where $\mathcal{E}$-$\mathbf{DFS}$ denotes the collection of all $\mathcal{E}$-DFSes.

(Proof: A.32, p.112+)

These first results finish our discussion of specificity. Further interesting issues, e.g. concerning the existence of most specific $\mathcal{E}$-DFSes, and the question whether $\preceq$ and $\preceq_c$ are equivalent once $\mathcal{E}$ is fixed, will be addressed in a later paper.

3.5.2 DFSes which induce the standard disjunction

We can also classify DFSes by means of their induced negation and disjunction.

Definition 37 ($(\widetilde{\neg}, \widetilde{\vee})$-DFS)
A DFS $\mathcal{F}$ such that $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ is called a $(\widetilde{\neg}, \widetilde{\vee})$-DFS.

In the following, we shall be concerned with $(\widetilde{\neg}, \max)$-DFSes, i.e. DFSes which induce the standard disjunction $\widetilde{\mathcal{F}}(\vee) = \vee = \max$. We will allow for arbitrary induced negation operators because this will not make the theorems and proofs more complicated.
Firstly let us remark that in every $(\widetilde{\neg}, \max)$-DFS $\mathcal{F}$,

$$\widetilde{\mathcal{F}}(\wedge) = \wedge = \min\,,$$

which is apparent from $\widetilde{\mathcal{F}}(\wedge)(x_1, x_2) = \widetilde{\neg}\max(\widetilde{\neg}\,x_1, \widetilde{\neg}\,x_2)$ and the fact that $\widetilde{\neg}$ is a strong negation (in particular, $\widetilde{\neg}$ is strictly monotonically decreasing).
In the case of $(\widetilde{\neg}, \max)$-DFSes, we can establish a theorem on conjunctions and disjunctions of fuzzy (pre-) determiners. The conjunction and disjunction of fuzzy (pre-) determiners are defined as follows.

Definition 38 (Conjunction $D \mathbin{\widetilde{\wedge}} D'$ and disjunction $D \mathbin{\widetilde{\vee}} D'$)
Suppose $\widetilde{\wedge} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ and $\widetilde{\vee} : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ are given.
For all fuzzy pre-determiners $D, D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the conjunction $D \mathbin{\widetilde{\wedge}} D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and the disjunction $D \mathbin{\widetilde{\vee}} D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are defined by

$$(D \mathbin{\widetilde{\wedge}} D')(X_1, \ldots, X_n) = D(X_1, \ldots, X_n) \mathbin{\widetilde{\wedge}} D'(X_1, \ldots, X_n)$$
$$(D \mathbin{\widetilde{\vee}} D')(X_1, \ldots, X_n) = D(X_1, \ldots, X_n) \mathbin{\widetilde{\vee}} D'(X_1, \ldots, X_n)$$

for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.
If $D, D' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ are fuzzy determiners, we define $D \mathbin{\widetilde{\wedge}} D' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ and $D \mathbin{\widetilde{\vee}} D' : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ analogously.

Note. Conjunctions and disjunctions of fuzzy (pre-) determiners of different arities can be formed by means of cylindrical extensions (add redundant arguments).
The theorem we can establish is the following.

Theorem 32 (Conjunction and disjunction)
Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS. Then for all $D,\ D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

a. $\mathcal{F}(D \wedge D') \leq \mathcal{F}(D) \wedge \mathcal{F}(D')$

b. $\mathcal{F}(D \vee D') \geq \mathcal{F}(D) \vee \mathcal{F}(D')$.

(Proof: A.33, p.113+)

We have so far not made any claims about the interpretation of $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow) : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ and $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor}) : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ in a given DFS $\mathcal{F}$. It is now easily shown how these connectives are interpreted in $(\widetilde{\neg}, \max)$-DFSes.

Theorem 33 (Bi-implication and exclusive-or in $(\widetilde{\neg}, \max)$-DFSes)
In every $(\widetilde{\neg}, \max)$-DFS $\mathcal{F}$,

a. $x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 = (x_1 \wedge x_2) \vee (\widetilde{\neg}\, x_1 \wedge \widetilde{\neg}\, x_2)$

b. $x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 = (x_1 \wedge \widetilde{\neg}\, x_2) \vee (\widetilde{\neg}\, x_1 \wedge x_2)$,

for all $x_1,\ x_2 \in \mathbf{I}$.
(Proof: A.34, p.113+)

The plot of the induced bi-implication $\widetilde{\leftrightarrow}$ of a $(\neg, \max)$-DFS is displayed in Fig. 8.

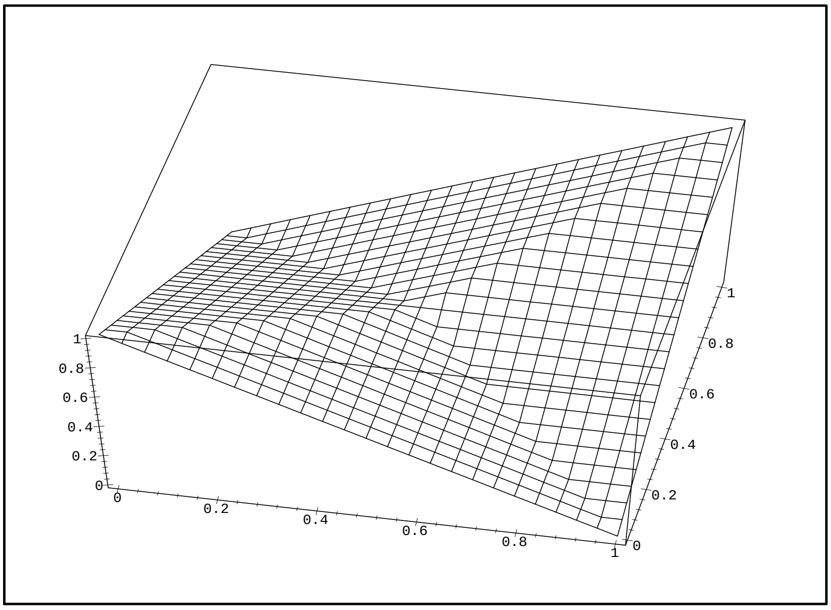

Figure 8: The induced bi-implication of a $(\neg, \max)$-DFS

3.5.3 DFSes which induce the standard extension principle

The most specific DFS subclass we will consider—and the most well-behaved—is that of *standard DFSes*.

Definition 39 (Standard DFS) *A DFS $\mathcal{F}$ is called a standard DFS iff it is a $\neg$-DFS and induces the standard extension principle.*

Building on our earlier theorems, we may now summarize the fuzzy truth functions induced by a standard DFS as follows.

Theorem 34 (Truth functions in standard DFSes)
In every standard DFS $\mathcal{F}$,

$$\begin{aligned} \widetilde{\neg}\, x_1 &= 1 - x_1 \\ x_1 \mathbin{\widetilde{\vee}} x_2 &= \max(x_1,\, x_2) \\ x_1 \mathbin{\widetilde{\wedge}} x_2 &= \min(x_1,\, x_2) \\ x_1 \mathbin{\widetilde{\rightarrow}} x_2 &= \max(1 - x_1,\, x_2) \\ x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 &= \max(\min(x_1,\, x_2),\ \min(1 - x_1,\, 1 - x_2)) \\ x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 &= \max(\min(x_1,\, 1 - x_2),\ \min(1 - x_1,\, x_2)) \end{aligned}$$

for all $x_1,\, x_2 \in \mathbf{I}$, where we have abbreviated $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$, $\widetilde{\vee} = \widetilde{\mathcal{F}}(\vee)$ etc.
(Proof: A.35, p.115+)

Notes

- The continuous-valued propositional logic induced by a standard DFS therefore coincides with the K-standard sequence logic first described by Dienes 1949 [25].[18]
- Some readers might prefer a different choice of the implication operator, namely $x_1 \mathbin{\widetilde{\rightarrow}} x_2 = \min(1,\, 1 - x_1 + x_2)$. However, it is clear that every determiner fuzzification mechanism with the highly desirable property of preserving Aristotelian squares will also preserve the interdefinability of the propositional connectives, and therefore be different from Łukasiewicz logic.

As concerns the standard quantifiers, we obtain

Theorem 35 (The standard quantifiers in standard DFSes)
In every standard DFS $\mathcal{F}$ and for all $E \neq \varnothing$,

$$\begin{aligned} \mathcal{F}(\exists)(X) &= \sup\{\mu_X(v) : v \in E\} \\ \mathcal{F}(\forall)(X) &= \inf\{\mu_X(v) : v \in E\} \\ \mathcal{F}(\textbf{all})(X_1,\, X_2) &= \inf\{\max(1 - \mu_{X_1}(v),\, \mu_{X_2}(v)) : v \in E\} \\ \mathcal{F}(\textbf{some})(X_1,\, X_2) &= \sup\{\min(\mu_{X_1}(v),\, \mu_{X_2}(v)) : v \in E\} \\ \mathcal{F}(\textbf{no})(X_1,\, X_2) &= \inf\{\max(1 - \mu_{X_1}(v),\, 1 - \mu_{X_2}(v)) : v \in E\}\,, \end{aligned}$$

for all $X,\, X_1,\, X_2 \in \widetilde{\mathcal{P}}(E)$.
(Proof: A.36, p.115+)

[18] On $\{0,\, \frac{1}{2},\, 1\}$-valued arguments, this logic coincides with with Kleene's three-valued logic.

This is quite satisfactory.
Standard DFSes represent a "boundary" case of DFSes, as witnessed by the following theorem.

Theorem 36
Suppose $\mathcal{F}$ is a DFS and $\mathcal{S}$ is a standard DFS. Then for all $E \neq \varnothing$, $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$,

$$\mathcal{F}(\exists) \geq \mathcal{S}(\exists)\,.$$

(Proof: A.37, p.116+)

This means that standard DFSes induce the smallest fuzzy existential quantifiers. It is immediate from Th-21, then, that standard DFSes also induce the smallest extension principle.
If $\mathcal{F}$ is a $\neg$-DFS and $\mathcal{S}$ a standard DFS, then clearly also

$$\mathcal{S}(\forall) \geq \mathcal{F}(\forall)\,,$$

i.e. standard DFSes are those $\neg$-DFSes which induce the largest fuzzy universal quantifiers.

3.6 A note on axioms *not* stated

We wish to add some notes on axioms *not* stated—because they would have ruled out the intended classes of DFS models.

Conjunctions and disjunctions of fuzzy pre-determiners

On page 47, we have introduced disjunctions and conjunctions of fuzzy pre-determiners and fuzzy determiners.
Why have we not required that $\mathcal{F}$ be a homomorphism with respect to the formation of conjunctions and disjunctions of fuzzy pre-determiners?
Suppose $\mathcal{F}$ is a DFS which satisfies

$$\mathcal{F}(D \mathbin{\widetilde{\wedge}} D') = \mathcal{F}(D) \mathbin{\widetilde{\wedge}} \mathcal{F}(D') \tag{1}$$

for all fuzzy pre-determiners $D,\ D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, where $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$.
Let $D = \mathrm{id}_{\mathbf{2}}{}^*$ and $D' = \neg^*$. Then

$$\mathrm{id}_{\mathbf{2}}{}^* \mathbin{\widetilde{\wedge}} \neg^* = 0\,,$$

and hence

$$\begin{aligned} 0 &= \mathcal{F}(0) && \text{by (DFS 1)} \\ &= \mathcal{F}(\mathrm{id}_{\mathbf{2}}{}^* \mathbin{\widetilde{\wedge}} \neg^*) && \text{by Th-5} \\ &= \mathcal{F}(\mathrm{id}_{\mathbf{2}}{}^*) \mathbin{\widetilde{\wedge}} \mathcal{F}(\neg^*) && \text{by (1)} \end{aligned}$$

i.e. $x \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, x = 0$ for all $x \in \mathbf{I}$.
A DFS $\mathcal{F}$ which is homomorphic with respect to conjunctions of determiners therefore induces a t-norm $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ which respects the law of excluded middle. This is clearly unacceptable since

it would exclude many interesting t-norms; in particular, the standard choice $\widetilde{\mathcal{F}}(\wedge) = \min$. (A similar point can be made about disjunctions.)
We have therefore *not* required that a DFS be homomorphic with respect to the conjunction and disjunction of determiners.
By the above reasoning we now know that in the case of $(\widetilde{\neg}, \max)$- and standard DFSes, which are the most well-behaved otherwise, there are choices of fuzzy pre-determiners $D, D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ where the inequations

$$\mathcal{F}(D \wedge D') \leq \mathcal{F}(D) \wedge \mathcal{F}(D')$$
$$\mathcal{F}(D \vee D') \geq \mathcal{F}(D) \vee \mathcal{F}(D')$$

are *proper*, i.e. the order of applying $\mathcal{F}$ might play a role if a fuzzy pre-determiner is defined as a conjunction or disjunction of other fuzzy pre-determiners.
Let us give an example. The two-valued determiner **exactly n** : $\mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{2}$ can be defined as a conjunction, viz.

$$\textbf{exactly n} = \textbf{atleast n} \wedge \textbf{atmost n} .$$

By our above reasoning, then, it would be possible that in a $(\widetilde{\neg}, \max)$- or standard DFS $\mathcal{F}$,

$$\mathcal{F}(\textbf{exactly n}) < \mathcal{F}(\textbf{atleast n}) \wedge \mathcal{F}(\textbf{atmost n}) .$$

It might therefore make a difference whether we first conjoin and then apply $\mathcal{F}$, or first apply $\mathcal{F}$ to **atleast n** and **atmost n** and then conjoin $\mathcal{F}$(**atleast n**) and $\mathcal{F}$(**atmost n**).
This phenomenon is of course well-known from other fuzzification mechanisms studied in the literature (see Menhardt [26, 4-20]), and by our above reasoning, we acknowledge this as an inevitable consequence of the fact that when one turns from two-valued logic to the continuous-valued case, it is impossible to retain all axioms of Boolean algebra (see e.g. Klir & Folger [27]).
We wish to emphasize that DFSes represent some kind of *optimum* in this respect because they are homomorphic with respect to negation, antonyms, dualisation, and the formation of intersections and unions in the arguments of fuzzy pre-determiners; in the case of these important constructions, fuzzification order will be irrelevant.
The order-dependence of DFSes with respect to conjunction and disjunction of course poses the problem of whether to first conjoin/disjoin and then apply $\mathcal{F}$, or vice versa. Basically, we think that it is desirable to fuzzify as late as possible. For example, we take $\mathcal{F}$(**exactly n**) to be the proper fuzzy analogon of **exactly n**, rather than $\mathcal{F}$(**atleast n**) $\widetilde{\wedge}$ $\mathcal{F}$(**atmost n**).[19]
But it would also be instructive to set up empirical tests. For example, one might investigate how gradual judgements of human subjects on questions

Are exactly n of the A also B?

where A, B are fuzzy concepts, are related to their gradual judgements on questions

Are at least n of the A also B?

and

Are at most n of the A also B?

Such an experiment could also include other determiners, for example **between** r_1 **and** r_2 **percent**, as compared to **atleast** r_1 **percent** and **atmost** r_2 **percent** (where $r_1, r_2 \in [0, 100]$, $r_1 < r_2$).

[19] after all, this is what $\mathcal{F}$ takes **exactly n** to!

Multiple occurences of variables

The reader will certainly have noticed our special treatment of the propositional connectives $\leftrightarrow$ and xor. The difficulties in proving properties of DFSes with respect to these connectives are caused by the fact that the definition of $\leftrightarrow : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ and $\mathrm{xor} : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ involves multiple occurrences of propositional variables, viz.

$$\begin{aligned} x_1 \leftrightarrow x_2 &= (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2) \\ &= (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \\ x_1 \mathrm{\,xor\,} x_2 &= (x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_2) \\ &= (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2)\,, \end{aligned}$$

for all $x_1, x_2 \in \mathbf{2} = \{0, 1\}$.

A brief glance at the DFS axioms shows that there is *no* axiom which describes this case of duplicated variables.

Why have we not required that a DFS be homomorphic with respect to duplication of variables, i.e. with occurrences of a variable in more than one argument position of a determiner?

Let us first generalize the concept of argument permutation such as to cover the case of multiple occurrences of a variable.

Definition 40

Suppose $D : \mathcal{P}(E)^m \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner ($m > 0$) and $\xi : \{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$ is a mapping ($n > 0$). By $D\xi : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ we denote the fuzzy pre-determiner defined by

$$D\xi(X_1, \ldots, X_n) = D(X_{\xi(1)}, \ldots, X_{\xi(m)}) \tag{2}$$

We use an analogue definition on fuzzy determiners $D : \widetilde{\mathcal{P}}(E)^m \longrightarrow \mathbf{I}$.

Notes

- compared to "argument transpositions" (DFS 4), we do not require that ξ be a permutation.
- Every DFS $\mathcal{F}$ is homomorphic with respect to the formation of $D\xi$ provided that ξ be injective. We obtain this by iterated application of argument transpositions (DFS 4) and elimination of redundant argument positions (by Th-5).

The interesting case is therefore that of non-injective ξ, which inserts the same variable in two (or more) argument positions of the original determiner D. As an example, let us consider the two-valued truth functions

$$\begin{aligned} f(x_1, x_2) &= x_1 \wedge \neg x_2 \\ g(x) &= x_1 \wedge \neg x_1 = 0\,, \end{aligned}$$

for all $x_1, x_2 \in \mathbf{2}$.

Clearly $g = f\xi$, where $\xi : \{1, 2\} \longrightarrow \{1\}$ is defined by $\xi(1) = 1, \xi(2) = 1$.

Now suppose $\mathcal{F}$ is a DFS. Then

$$\widetilde{\mathcal{F}}(f)(x_1, x_2) = x_1 \mathbin{\widetilde{\wedge}} \mathbin{\widetilde{\neg}} x_2 \qquad \text{by (DFS 5)}$$

and

$$\begin{aligned}\widetilde{\mathcal{F}}(g)(x_1) &= \widetilde{\mathcal{F}}(0)(x_1) \\ &= 0 && \text{by (DFS 1), Th-5}\end{aligned}$$

(for all $x_1, x_2 \in \mathbf{I}$). If we further assume that $\mathcal{F}$ commutes with (2), then obviously

$$\begin{aligned}x_1 \mathbin{\widetilde{\wedge}} \widetilde{\neg} x_1 &= \widetilde{\mathcal{F}}(f)(x_{\xi(1)}, x_{\xi(2)}) \\ &= \widetilde{\mathcal{F}}(f\xi)(x_1) && \text{(2) by assumption} \\ &= \widetilde{\mathcal{F}}(g)(x_1) \\ &= 0\end{aligned}$$

for all $x_1 \in \mathbf{I}$, i.e. the t-norm induced by $\mathcal{F}$ respects the law of excluded middle. Again, we find this too restrictive (in particular, because the standard choice $\wedge = \min$ of fuzzy conjunction would be excluded) and therefore have *not* required that $\mathcal{F}$ be homomorphic with respect to the duplication of variables.

3.7 Summary and open problems

We have presented first results of an investigation of formal properties of DFSes. In particular, we have shown that every DFS induces a "reasonable" choice of the fuzzy connectives $\widetilde{\neg}$, $\widetilde{\wedge}$, $\widetilde{\vee}$, $\widetilde{\rightarrow}, \ldots$; every DFS preserves monotonicity properties of fuzzy pre-determiners and inequations between fuzzy pre-determiners even if these hold only locally; the induced extension principle and the induced fuzzy inverse images are well-behaved; the projection determiners $\pi_v^{n,i}$ are mapped to their obvious fuzzy counterparts; and the standard quantifiers $\exists$ and $\forall$ are reasonably extended to the fuzzy case in the sense that the important relationship between $\exists$ and $\vee$ ($\forall$ and $\wedge$) is preserved.

We have also begun an investigation of specific subclasses of DFSes and provided evidence that it is sufficient to consider $\neg$-DFSes (which induce the standard negation $\neg x = 1 - x$). We have introduced the subclasses of $\mathcal{E}$-DFSes ($\mathcal{E}$ an extension principle) and shown that these are closed under an important class of operations which comprises stable symmetric sums, convex combinations, and other aggregation operators on $\mathcal{E}$-DFSes. We have also established the existence of least specific $\mathcal{E}$-DFSes. Furthermore, we have introduced the class of standard DFSes, which induce the well-known K-standard sequence logic.

Closing, we have justified our decision *not* to include certain conditions in the set of DFS axioms, namely a., compatibility with the formation of conjunctions/disjunctions of fuzzy pre-determiners, and b., compatibility with the identification of variables. These conditions entail the law of tertium non datur and hence rule out the standard choice of fuzzy connectives.

In the next chapter, we will answer positively the question of *consistency* of the DFS axioms, by presenting several standard DFSes. We are currently working on the issue of *independence* of the DFS axioms (i.e. irreducibility to a smaller axiom set).

We have taken pains to avoid the use of the standard extension principle in the DFS axioms (in favor of the *induced* extension principle of a DFS) because the requirement of being compatible to the standard extension principle would have excluded all DFSes which induce $\widetilde{\mathcal{F}}(\vee) \neq \max$ and $\widetilde{\mathcal{F}}(\wedge) \neq \min$. However, we have no results yet concerning the existence of DFSes which do *not* induce the standard extension principle (and, consequently, induce t-norms and s-norms different from the standard choices $\min$ and $\max$).

Other topics which might deserve further investigation are

- a precise characterisation of DFSes in terms of specificity; in particular: existence of most specific $\mathcal{E}$-DFSes; and
- definablity of $\widetilde{\leftrightarrow}$ and $\widetilde{\mathrm{xor}}$ in terms of $\widetilde{\neg}$ and $\widetilde{\wedge}$ in the general case.

A final note. In our comments on the DFS axioms, we have remarked that adjectival restriction with *fuzzy* adjectives cannot be modelled directly. I.e. if $A \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E, then only $\mathcal{F}(D) \triangleleft A$ is defined, but not $D \triangleleft A$. However, one could ask if $\mathcal{F}(D) \triangleleft A$ is obtained from a fuzzy pre-determiner D', i.e. if there exists D' such that $\mathcal{F}(D) \triangleleft A = \mathcal{F}(D')$. The obvious choice[20] of D' is

$$D' = (\mathcal{F}(D) \triangleleft A)|_{\mathcal{P}(E)^n} .$$

Therefore, it might be interesting to investigate if there exist DFSes $\mathcal{F}$ with the property that

$$\mathcal{F}(D) \triangleleft A = \mathcal{F}((\mathcal{F}(D) \triangleleft A)|_{\mathcal{P}(E)^n}) ,$$

for all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $n \in \mathbb{N}^+$ and all $A \in \widetilde{\mathcal{P}}(E)$.

It would also be instructive to impose a continuity condition. Informally speaking, it is often desirable that

- a small change in the membership degrees $\mu_{X_i}(v)$ of the argument sets does not change $\mathcal{F}(D)(X_1, \ldots, X_n)$ drastically; and
- a small change in the fuzzy pre-determiner D does not change $\mathcal{F}(D)(X_1, \ldots, X_n)$ drastically.

We will investigate these conditions which result in "continuous DFSes" in a later report.

[20] in fact, the only possible choice due to the restriction theorem Th-1

4 DFS Models

By stating the DFS axioms, we have made explicit our intuitions about "reasonable" mechanisms of fuzzy quantification. In order to show that these axioms are consistent (but also to make the theory useful for practical purposes), we will now present actual models. We firstly investigate the utility of α-cuts[21] in defining DFSes. These results will be negative because of the lack of symmetry of α-cuts with respect to negation. We then present DFS models based on the concept of a three-valued cut, which overcomes the asymmetry problem, and establish the relation of these models to the fuzzification scheme proposed by Gaines [1].

4.1 DFSes cannot be defined by alpha-cutting

Assume $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of some set E and $\alpha \in \mathbf{I}$.
By $(X)_{\geq\alpha} \in \mathcal{P}(E)$ we denote the α-cut

$$(X)_{\geq\alpha} = \{x \in E : \mu_X(x) \geq \alpha\}\,,$$

by $(X)_{>\alpha} \in \mathcal{P}(E)$ the strict α-cut

$$(X)_{>\alpha} = \{x \in E : \mu_X(x) > \alpha\}\,.$$

The effect of α-cutting relative to $\mu_X(v)$ is displayed in Fig. 9.

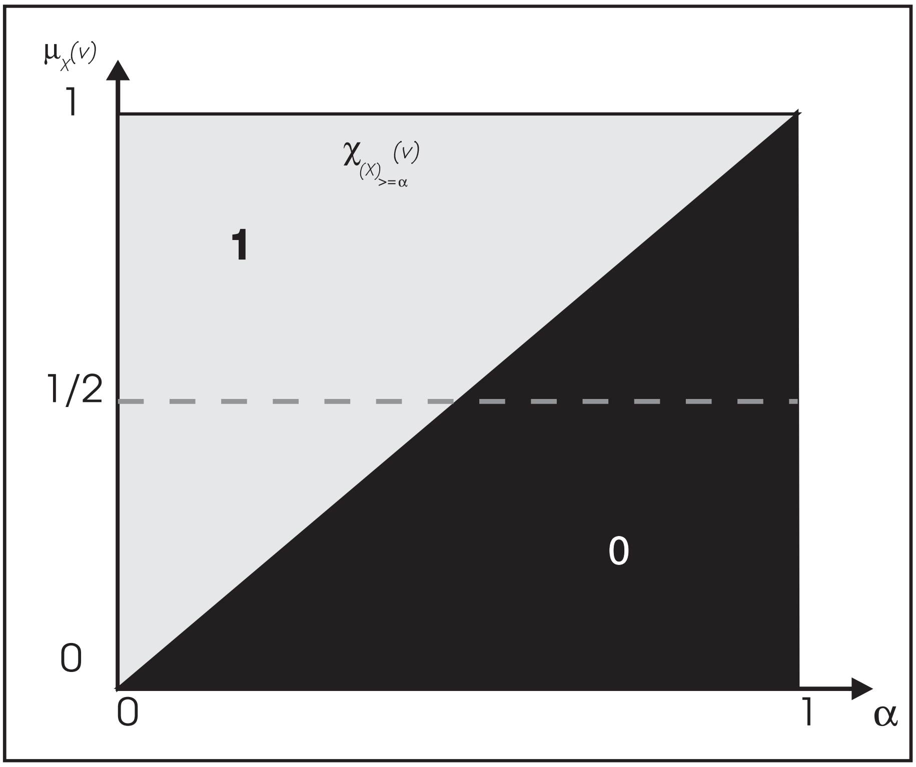

Figure 9: α-cut as a function of α and $\mu_X(v)$

Let us recall some important properties of α-cuts and strict α-cuts.

[21]which reduce fuzzy sets to crisp sets relative to a threshold α

Theorem 37 (Properties of α-cuts)
For all $X, Y \in \widetilde{\mathcal{P}}(E)$, $\alpha \in \mathbf{I}$,
Union: $(X \cup Y)_{\geq\alpha} = (X)_{\geq\alpha} \cup (Y)_{\geq\alpha}$
Intersection: $(X \cap Y)_{\geq\alpha} = (X)_{\geq\alpha} \cap (Y)_{\geq\alpha}$
Monotonicity: if $X \subseteq Y$, then $(X)_{\geq\alpha} \subseteq (Y)_{\geq\alpha}$
Decr. Monotonicity in α: if $\alpha \leq \alpha'$, then $(X)_{\geq\alpha} \supseteq (X)_{\geq\alpha'}$
If A is crisp, then $(A)_{\geq\alpha} = A$ for all $\alpha > 0$.
Extension: $\widehat{f}((X)_{\geq\alpha}) \subseteq \left(\hat{\hat{f}}(X)\right)_{\geq\alpha}$
(Proof: B.1, p.117+)

Theorem 38 (Properties of strict α-cuts)
For all $X, Y \in \widetilde{\mathcal{P}}(E)$, $\alpha \in \mathbf{I}$,
Union: $(X \cup Y)_{>\alpha} = (X)_{>\alpha} \cup (Y)_{>\alpha}$
Intersection: $(X \cap Y)_{>\alpha} = (X)_{>\alpha} \cap (Y)_{>\alpha}$
Monotonicity: if $X \subseteq Y$, then $(X)_{>\alpha} \subseteq (Y)_{>\alpha}$
Decr. Monotonicity in α: if $\alpha \leq \alpha'$, then $(X)_{>\alpha} \supseteq (X)_{>\alpha'}$
If A is crisp, then $(A)_{>\alpha} = A$ for all $\alpha < 1$.
Extension: $\widehat{f}((X)_{>\alpha}) = \left(\hat{\hat{f}}(X)\right)_{>\alpha}$.
(Proof: B.2, p.117+)

α-cuts and strict α-cuts are linked by negation, as witnessed by the following theorem.

Theorem 39 (α-cuts and complementation)
For all sets E, $X \in \widetilde{\mathcal{P}}(E)$, $\alpha \in \mathbf{I}$,

$$(\neg X)_{\geq\alpha} = \neg(X)_{>\neg\alpha}$$

Or, equivalently,

$$(\neg X)_{>\alpha} = \neg(X)_{\geq\neg\alpha}\,.$$

(Proof: B.3, p.117+)

As a first attempt to find a DFS, let us consider

$$\mathcal{A}(D)(X_1, \ldots, X_n) \quad = \quad \int_0^1 D((X_1)_{\geq\alpha}, \ldots, (X_n)_{\geq\alpha})\, d\alpha\,,$$

where $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.
$\mathcal{A}$ fails to be a DFS because the above integral may not exist (we have not imposed any restrictions on the "well-behavedness" of D).[22]
But even in the case that $\mathcal{A}$ is defined, it fails to satisfy the axiom of internal complementation (DFS 5) because $(\neg X)_{\geq\alpha} = \neg(X)_{>1-\alpha}$, which is different from $\neg(X_{\geq\alpha})$ in most cases.
The failure of the above attempt to define a DFS by integration over α-cuts is just an instance of the more general fact that it is *impossible* to define DFSes based on α-cuts.

[22]To see this, choose a non-measurable function $\mathrm{f} : \mathbf{I} \longrightarrow \mathbf{I}$ and let $E = \mathbf{I}$. Define $D : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ by $D(X) = \mathrm{f}(\inf X)$ for all $X \in \mathcal{P}(\mathbf{I})$. $\mathcal{A}$ is undefined on the fuzzy subset $X^* \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by $\mu_{X^*}(x) = x$ for all $x \in \mathbf{I}$, because $D((X^*)_{\geq\alpha}) = \mathrm{f}(\alpha)$.

Theorem 40 (Nonexistence of alpha-cut based DFSes)
Define $\mathcal{C}$, $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I} \quad \mapsto \quad \mathcal{C}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$ *by*

$$\mu_{\mathcal{C}(D)(X_1,\dots,X_n)}(\alpha) = D((X_1)_{\geq\alpha}, \dots, (X_n)_{\geq\alpha})$$

for all $\alpha \in \mathbf{I}$.
There exists no $\mathcal{S} : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ *such that* $\mathcal{F} = \mathcal{S} \circ \mathcal{C}$ *satisfies*

- $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$
- $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ *is a strong negation operator*
- $\mathcal{F}$ *satisfies the law of internal complementation* (DFS 5)

In particular, there exists no $\mathcal{S} : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ *such that* $\mathcal{F} = \mathcal{S} \circ \mathcal{C}$ *is a DFS.*
(Proof: B.4, p.117+)

Note. We could have replaced the first condition by the equivalent

$$\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}}) = \mathrm{id}_{\mathbf{I}}\,,$$

a condition which it is certainly natural to require.

Definition 41 (Resolution principle)
Let $f : \mathcal{P}(E) \longrightarrow V$ *a mapping. By* $\mathcal{R}(f)$ *we denote the mapping* $\mathcal{R}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(V)$ *which to each* $X \in \widetilde{\mathcal{P}}(E)$ *associates the fuzzy set* $\mathcal{R}(f)(X)$ *defined by*

$$\mu_{\mathcal{R}(f)(X)}(v) = \sup\{\alpha : f((X)_{\geq\alpha}) = v\}$$

for all $v \in V$. *We then say that* $\mathcal{R}(f)$ *is obtained from* f *by applying the the resolution principle.*

Note. We have stated the resolution principle for one-place functions only. Because $\mathcal{P}(E)^n \cong \mathcal{P}(E \times n)$ and $\widetilde{\mathcal{P}}(E)^n \cong \widetilde{\mathcal{P}}(E \times n)$, this generalises to n-place functions $f : \mathcal{P}(E)^n \longrightarrow V$ in the obvious way (i.e. by component-wise α-cutting).
By applying the resolution principle, then, an n-ary fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is mapped to that $\mathcal{R}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$ which to each $(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ assigns the fuzzy subset $\mathcal{R}(D)(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by

$$\mu_{\mathcal{R}(D)(X_1,\dots,X_n)}(v) = \sup\{\alpha : D((X_1)_{\geq\alpha}, \dots, (X_n)_{\geq\alpha}) = v\}\,,$$

for all $v \in \mathbf{I}$. As a corollary of the above theorem on α-cut based attempts to define DFSes, we now obtain

Theorem 41 (Nonexistence of DFSes based on the resolution principle)
There is no DFS based on the resolution principle, i.e. there is no $\mathcal{S} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ *such that*

$$\mathcal{F} = \mathcal{S} \circ \mathcal{R}$$

is a DFS.
(Proof: B.5, p.118+)

By the α-cut representation of $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ we mean the fuzzy set $\mathrm{RES}(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(\mathcal{P}(E))^n$ defined by

$$\mu_{\mathrm{RES}(X_1, \ldots, X_n)}((Y_1, \ldots, Y_n)) = \sup\{\alpha \in \mathbf{I} : (X_1)_{\geq\alpha} = Y_1, \ldots, (X_n)_{\geq\alpha} = Y_n\},$$

for all all $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$.

Of course, DFSes kann be defined based on this α-cut representation of $(X_1, \ldots, X_n)$ (if DFSes exist at all—and they do). But this is a trivial statement, since $(X_1, \ldots, X_n)$ can be recovered from its α-cut representation.

4.2 DFS models based on three-valued cuts and fuzzy median

We will now present models of the DFS axioms which are based on the novel concept of a three-valued cut, and on the use of the fuzzy median as an aggregation operator over sets of gradual evaluations.

4.2.1 From two-valued logic to Kleene's logic, and beyond

As witnessed by Th-34, the $\{0, \frac{1}{2}, 1\}$-valued portion of the logic induced by a standard DFS coincides with Kleene's three-valued logic (see e.g. [27, p. 29]), and this observation will serve as our starting point for defining DFS models.

Kleene's logic can be obtained from two-valued logic by the following mechanism of generalizing a propositional function $\mathrm{f} : \mathbf{2}^n \longrightarrow \mathbf{2}$ to the three-valued $\breve{\mathrm{f}} : \{0, \frac{1}{2}, 1\}^n \longrightarrow \{0, \frac{1}{2}, 1\}$.

Suppose $x_1, \ldots, x_n \in \{0, \frac{1}{2}, 1\}$ are given. Associate to each x_i a set $\mathbf{y}_i \in \mathcal{P}(\mathbf{2})$ as follows:

$$\mathbf{y}_i = \begin{cases} \{0\} & : x_i = 0 \\ \{0, 1\} & : x_i = \frac{1}{2} \\ \{1\} & : x_i = 1 \end{cases}$$

Then define $\breve{\mathrm{f}} : \{0, \frac{1}{2}, 1\}^n \longrightarrow \{0, \frac{1}{2}, 1\}$ by

$$\breve{\mathrm{f}}(x_1, \ldots, x_n) = \begin{cases} 1 & : \mathrm{f}(y_1, \ldots, y_n) = 1 \text{ for all } y_i \in \mathbf{y}_i,\ i = 1, \ldots, n \\ 0 & : \mathrm{f}(y_1, \ldots, y_n) = 0 \text{ for all } y_i \in \mathbf{y}_i,\ i = 1, \ldots, n \\ \frac{1}{2} & : \text{else} \end{cases}$$

for all $x_1, \ldots, x_n \in \{0, \frac{1}{2}, 1\}$.

In this definition, the "indeterminate" value $\frac{1}{2}$ is treated by considering both alternatives $0, 1$. The truth-values 0 and 1 do not induce any indeterminacy. We recall the definition of the (extended) fuzzy median (see Def. 36). By using $\mathrm{m}_{\frac{1}{2}} : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$, we can now state the above definition more compactly,

$$\breve{\mathrm{f}}(x_1, \ldots, x_n) = \mathrm{m}_{\frac{1}{2}}\{\mathrm{f}(y_1, \ldots, y_n) : y_i \in \mathbf{y}_i,\ i = 1, \ldots, n\}.$$

This median-based definition has the advantage that it can also be applied to functions $\mathrm{f} : \mathbf{2}^n \longrightarrow \mathbf{I}$, which it maps to $\breve{\mathrm{f}} : \{0, \frac{1}{2}, 1\}^n \longrightarrow \mathbf{I}$, and thus takes us beyond Kleene's logic.

Application of the generalisation mechanism to determiners. Using the same mechanism, we can generalize two-valued determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ to three-valued determiners $\breve{D} : \breve{\mathcal{P}}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$, where $\breve{\mathcal{P}}(E) \cong \{0, \frac{1}{2}, 1\}^E$ is the set of all three-valued subsets of E (each three-valued subset X of E is uniquely determined by its membership function $\nu_X : E \longrightarrow \{0, \frac{1}{2}, 1\}$). Assume $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ and $X_1, \ldots, X_n \in \breve{\mathcal{P}}(E)$ are given. To each of the three-valued sets $X_i \in \breve{\mathcal{P}}(E)$ we associate a set of two-valued sets $\mathcal{Y}_i \subseteq \mathcal{P}(E)$,

$$\mathcal{Y}_i = \{Y \subseteq E : X_i^{\min} \subseteq Y \subseteq X_i^{\max}\}$$

where

$$\begin{aligned} X_i^{\min} &= \{x \in E : \nu_{X_i}(x) = 1\}, \\ X_i^{\max} &= \{x \in E : \nu_{X_i}(x) = \tfrac{1}{2} \text{ or } \nu_{X_i}(x) = 1\} \end{aligned}$$

Notes

- If $x \in E$ such that $\mu_{X_i}(x) = 1$, then all $Y \in \mathcal{Y}_i$ contain x; if $x \in E$ such that $\mu_{X_i}(x) = 0$, then no $Y \in \mathcal{Y}_i$ contains x, and for those $x \in E$ with $\mu_{X_i} = \frac{1}{2}$, $\mathcal{Y}_i$ contains all combinations of the alternatives $x \in Y$, $x \notin Y$.
- The pairs $(X_i^{\min}, X_i^{\max})$, $X_i^{\min} \subseteq X_i^{\max}$ form a representation of the three-valued sets X_i because every X_i can be recovered from $(X_i^{\min}, X_i^{\max})$ by

$$\nu_{X_i}(v) = \begin{cases} 1 & : \ v \in X_i^{\min} \\ 0 & : \ v \notin X_i^{\max} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

  for all $v \in E$.

Using $\mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$, we can now define

$$\breve{D}(X_1, \ldots, X_n) = \begin{cases} 1 & : \ D(Y_1, \ldots, Y_n) = 1 \text{ for all } (Y_1, \ldots, Y_n) \in \mathcal{Y} \\ 0 & : \ D(Y_1, \ldots, Y_n) = 0 \text{ for all } (Y_1, \ldots, Y_n) \in \mathcal{Y} \\ \frac{1}{2} & : \ \text{else} \end{cases}$$

Again, we can restate this

$$\breve{D}(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}\}.$$

The $\mathrm{m}_{\frac{1}{2}}$-based mechanism is defined on $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ also, yielding $\breve{D} : \breve{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$.

4.2.2 Three-valued cuts at a given level of cautiousness

Now let us introduce three-valued cuts on fuzzy sets in order to build on these definitions. Assume $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of some set E and $\gamma \in (0, 1]$. Denote the membership function of X by $\mu_X : E \longrightarrow \mathbf{I}$. We can cut X to a three-valued set $\breve{X} \in \breve{\mathcal{P}}(E)$ with membership function $\nu_{\breve{X}} = \mu_X^\gamma : E \longrightarrow \{0, \frac{1}{2}, 1\}$ by defining

$$\mu_X^\gamma(x) = \begin{cases} 0 & : \ \mu_X(x) \le \frac{1}{2} - \frac{1}{2}\gamma \\ \frac{1}{2} & : \ \frac{1}{2} - \frac{1}{2}\gamma < \mu_X(x) < \frac{1}{2} + \frac{1}{2}\gamma \\ 1 & : \ \mu_X(x) \ge \frac{1}{2} + \frac{1}{2}\gamma \end{cases}$$

for all $x \in E$. For $\gamma = 0$, let

$$\mu_X^0(x) \quad = \quad \begin{cases} 0 & : \quad \mu_X(x) < \frac{1}{2} \\ \frac{1}{2} & : \quad \mu_X(x) = \frac{1}{2} \\ 1 & : \quad \mu_X(x) > \frac{1}{2} \end{cases}$$

for all $x \in E$.

γ can be thought of as a parameter of "cautiousness". If $\gamma = 0$, the set of indeterminates[23] contains only those $x \in E$ with $\mu_X(x) = \frac{1}{2}$; all other elements of E are mapped to the closest truth value in $\{0, 1\}$. As γ increases, the set of indeterminates is increasing. For $\gamma = 1$, then, the level of maximal cautiousness is reached where all elements of E except those with $\mu_X(x) \in \{0, 1\}$ are interpreted as indeterminates (see Fig. 10).

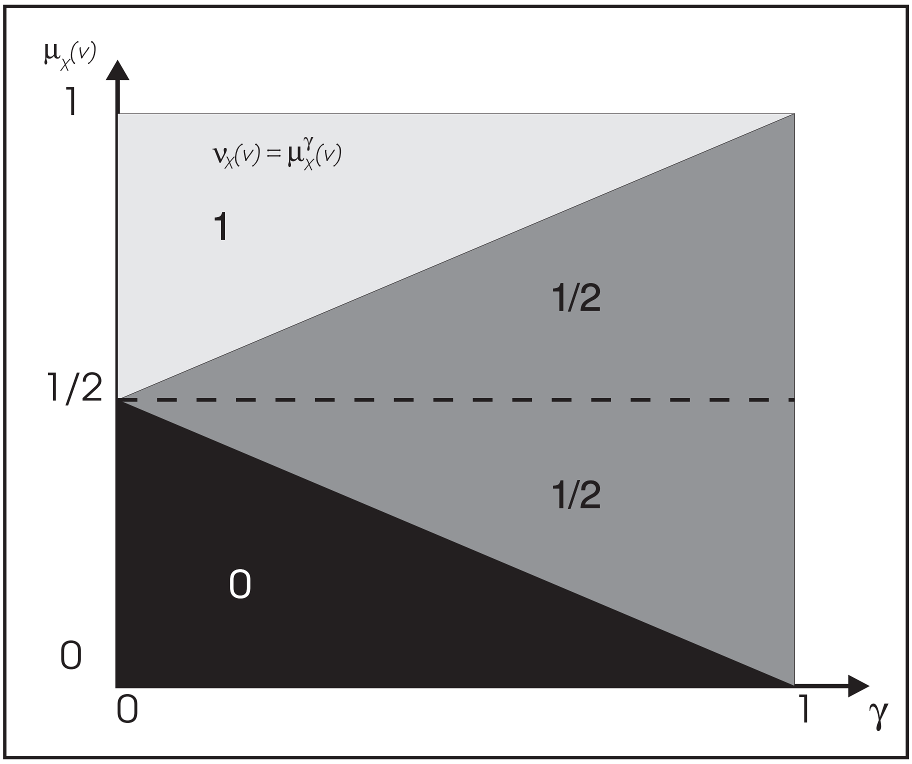

Figure 10: Three-valued cut as a function of γ and $\mu_X(v)$

4.2.3 Evaluation of determiners at a given level of cautiousness

In order to interpret a fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ at a given level of cautiousness γ, we apply the above mechanism to D, yielding $\breve{D} : \breve{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$, and then insert the three-valued cuts of the argument sets at γ. This approach is summarized in our following definition.

Definition 42

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $\gamma \in \mathbf{I}$. Define $D_\gamma : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ by

$$D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\},$$

[23] i.e. of those $x \in E$ such that $\nu_{\breve{X}}(x) = \mu_X^\gamma(x) = \frac{1}{2}$

where $\mathcal{Y}^\gamma = \mathcal{Y}_i^\gamma \times \cdots \times \mathcal{Y}_n^\gamma$, *and* $\mathcal{Y}_i^\gamma$ *is obtained from the three-valued cut of* X_i *under* γ *(*$i = 1, \ldots, n$*), i.e.*

$$\begin{aligned} \mathcal{Y}_i^\gamma &= \{Y \subseteq E : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ (X_i)_\gamma^{\min} &= \begin{cases} (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} & : \gamma \in (0,\,1] \\ (X_i)_{>\frac{1}{2}} & : \gamma = 0 \end{cases} \\ (X_i)_\gamma^{\max} &= \begin{cases} (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma \in (0,\,1] \\ (X_i)_{\geq\frac{1}{2}} & : \gamma = 0 \end{cases} \end{aligned}$$

Notes

- Building on our above remark, the three-valued set which results from the three-valued cut of X_i at $\gamma \in \mathbf{I}$ can be represented by the pair $((X_i)_\gamma^{\min},\, (X_i)_\gamma^{\max})$, i.e. by a pair of α-cuts.

  The γ-indexed family $((X_i)_\gamma^{\min},\, (X_i)_\gamma^{\max})_{\gamma\in\mathbf{I}}$ faithfully represents X_i because X_i can recovered from this representation by means of

  $$\mu_{X_i}(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}\sup\{\gamma \in \mathbf{I} : x \in (X_i)_\gamma^{\min}\} & : x \in (X_i)_0^{\min} \\ \frac{1}{2} - \frac{1}{2}\sup\{\gamma \in \mathbf{I} : x \notin (X_i)_\gamma^{\max}\} & : x \notin (X_i)_0^{\min} \end{cases}$$

  for all $x \in E$.

- If the base set E is finite, then the number of *distinct* three-valued cuts of $(X_1,\, \ldots,\, X_n) \in \widetilde{\mathcal{P}}(E)^n$ is finite also. In this case, $D(\gamma) = D_\gamma(X_1,\, \ldots,\, X_n)$ is a step function (i.e. piecewise constant in γ, except for a finite number of discontinuities). It is this property which makes the concept of three-valued cuts (and our definition of $(\_)_\gamma$) still suited for efficient implementation: results need to be computed only for the (usually small) number of distinct three-valued cuts.

- Two typical plots of $D(\gamma) = D_\gamma(X_1,\, \ldots,\, X_n)$ (in the case of finite E) are displayed in Fig. 11 and Fig. 12.

- The typical shape of $D(\gamma) = D_\gamma(X_1,\, \ldots,\, X_n)$ in the case of constant fuzzy pre-determiners $D : \{\varnothing\} \longrightarrow \mathbf{I}$, and of three-valued fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \{0,\, \frac{1}{2},\, 1\}$, is displayed in Fig. 13 and Fig. 14, respectively.

Let us now make explicit some monotonicity properties of $(\_)_\gamma$.

Theorem 42 (Monotonicity properties of $(\_)_\gamma$***)***
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy pre-determiner and* $(X_1,\, \ldots,\, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. *If* $D_0(X_1,\, \ldots,\, X_n) > \frac{1}{2}$, *then* $D_\gamma(X_1,\, \ldots,\, X_n)$ *is monotonically decreasing in* γ *and* $D_\gamma(X_1,\, \ldots,\, X_n) \geq \frac{1}{2}$ *for all* $\gamma \in \mathbf{I}$.

b. *If* $D_0(X_1,\, \ldots,\, X_n) = \frac{1}{2}$, *then* $D_\gamma(X_1,\, \ldots,\, X_n) = \frac{1}{2}$ *for all* $\gamma \in \mathbf{I}$.

c. *If* $D_0(X_1,\, \ldots,\, X_n) < \frac{1}{2}$, *then* $D_\gamma(X_1,\, \ldots,\, X_n)$ *is monotonically increasing in* γ *and* $D_\gamma(X_1,\, \ldots,\, X_n) \leq \frac{1}{2}$ *for all* $\gamma \in \mathbf{I}$.

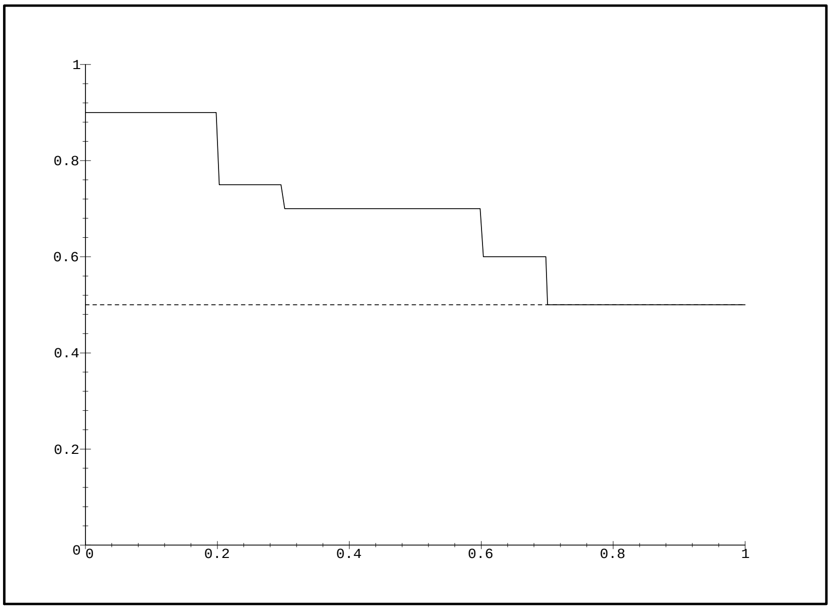

Figure 11: Typical plot of $D(\gamma) = D_\gamma(X_1, \ldots, X_n)$ if $D(0) > \frac{1}{2}$

These monotonicity properties reflect our intuition that as soon as one becomes more cautious (i.e. the "level of cautiousness" γ increases), the results obtained will become less specific (in the sense of being closer to $\frac{1}{2}$).
We wish to add that the assignment $(\_)_\gamma : (D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (D_\gamma : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$ satisfies *almost all* of the DFS-axioms. Its specific properties are described (and established by proofs) in B.8, p.123+.
However, $(\_)_\gamma$ is not a DFS yet; the fuzzy median suppresses to much structure. Notably, $(\pi_*)_\gamma \neq \widetilde{\pi}_*$ for all $\gamma \in \mathbf{I}$, i.e. (DFS 2) does not hold.

4.2.4 Combining the results of each level of cautiousness

We must hence take into account the results obtained at each level of cautiousness. There are several ways of accomplishing this combination, which all result in a DFS.

Definition 43
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the fuzzy determiner $\mathcal{M}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{M}(D)(X_1, \ldots, X_n) = \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma\,.$$

We should perhaps note that the integral exists, regardless of D.

Theorem 43 (Well-definedness of $\mathcal{M}$)
$\mathcal{M}$ is well-defined, i.e. the integral

$$\int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma$$

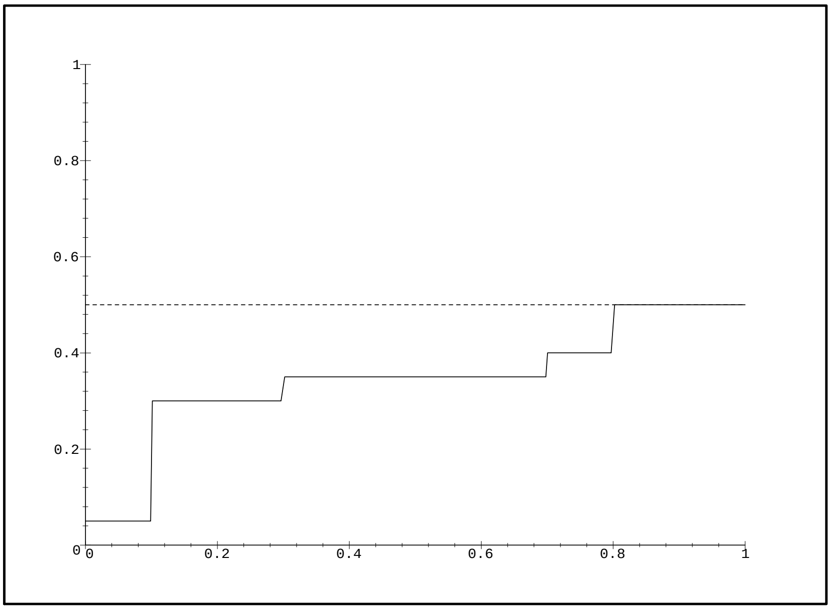

Figure 12: Typical plot of $D(\gamma) = D_\gamma(X_1, \ldots, X_n)$ if $D(0) < \frac{1}{2}$

exists for every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and every choice of* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.
(Proof: B.9, p.138+)

Theorem 44 $\mathcal{M}$ *is a standard DFS.*
(Proof: B.10, p.138+)

Definition 44 ($\mathcal{M}^*$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*, the fuzzy determiner* $\mathcal{M}^*(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{M}^*(D)(X_1, \ldots, X_n) = (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, where use abbreviations*

$$D_0^* = \lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n)$$
$$\gamma^* = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}\,.$$

Fig. 15 and Fig. 16 illustrate how $\mathcal{M}^*$ is related to $D(\gamma) = D_\gamma(X_1, \ldots, X_n)$ (smallest rectangle "outside" the graph of $D(\_)$, viewed from $y = \frac{1}{2}$, ignoring $D(0)$).

Theorem 45 $\mathcal{M}^*$ *is a standard DFS.*
(Proof: B.11, p.144+)

The reader might find the definition of $\mathcal{M}^*$ somewhat complicated.
Why have we used $\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n)$ instead of $D_0(X_1, \ldots, X_n)$?

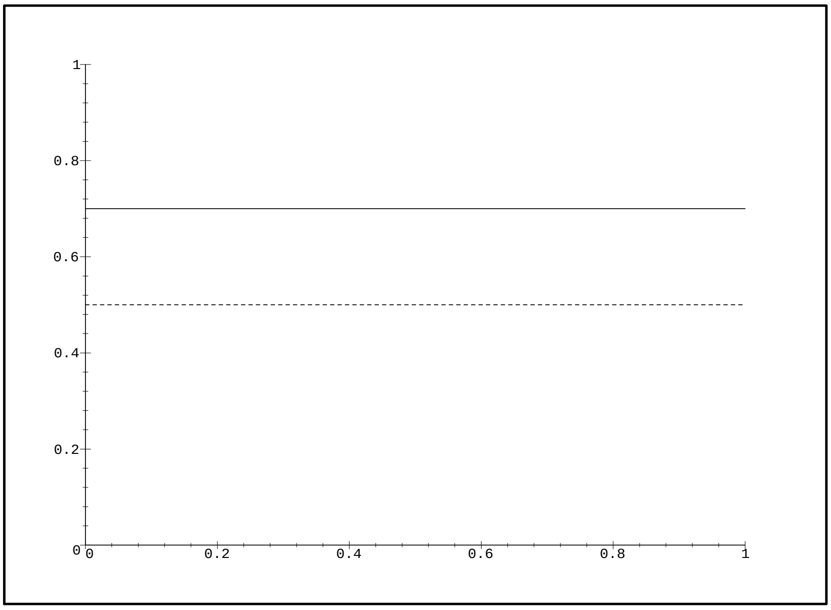

Figure 13: Typical plot of $D(\gamma) = D_\gamma(\varnothing)$ for a constant fuzzy pre-determiner

Theorem 46 ($\mathcal{M}^\bullet$)
Suppose $\mathcal{M}^\bullet(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ assigns to each fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ the fuzzy determiner $\mathcal{M}^\bullet(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ defined by

$$\mathcal{M}^\bullet(D)(X_1, \ldots, X_n) = (D_0(X_1, \ldots, X_n) - \tfrac{1}{2})\gamma^* + \tfrac{1}{2}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.
$\mathcal{M}^\bullet$ is not a DFS.
(Proof: B.12, p.152+)

Definition 45 ($\mathcal{M}_*$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the fuzzy determiner $\mathcal{M}_*(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \frac{1}{2}) : \gamma \in \mathbf{I}\} & : \quad D_0 \geq \frac{1}{2} \\ \frac{1}{2} - \sup\{\gamma(\frac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} & : \quad D_0 < \frac{1}{2} \end{cases}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, where we have abbreviated

$$D_0 = D_0(X_1, \ldots, X_n)\,.$$

Fig. 17 and Fig. 18 illustrate how $\mathcal{M}_*(D)(X_1, \ldots, X_n)$ is determined from $D(\gamma) = D_\gamma(X_1, \ldots, X_n)$ (largest rectangle “inside” the graph of $D(\_)$, relative to $y = \frac{1}{2}$).

Theorem 47 *$\mathcal{M}_*$ is a standard DFS.*
(Proof: B.13, p.154+)

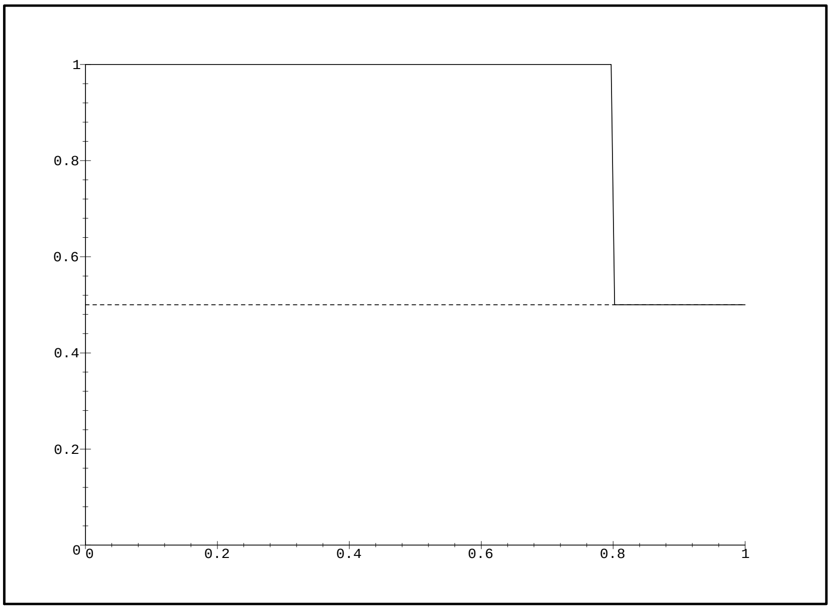

Figure 14: Typical plot of $D(\gamma) = D_\gamma(X_1, \dots, X_n)$ for a three-valued fuzzy pre-determiner

We will now establish some relationships among these DFS models.

Theorem 48
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy pre-determiner and* $(X_1, \dots, X_n) \in \breve{\mathcal{P}}(E)^n \subset \widetilde{\mathcal{P}}(E)^n$ *an* n*-tuple of* $\{0, \frac{1}{2}, 1\}$*-valued subsets of* E*. Then*

$$\begin{aligned} D_0(X_1, \dots, X_n) &= \mathcal{M}(D)(X_1, \dots, X_n) \\ &= \mathcal{M}^*(D)(X_1, \dots, X_n) \\ &= \mathcal{M}_*(D)(X_1, \dots, X_n) \end{aligned}$$

(Proof: B.14, p.161+)

There is another case in which these schemes coincide, namely on two- or three-valued determiners.

Theorem 49
Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ *is a three-valued fuzzy pre-determiner. Then for all* $(X_1, \dots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$\begin{aligned} \mathcal{M}(D)(X_1, \dots, X_n) &= \mathcal{M}^*(D)(X_1, \dots, X_n) \\ &= \mathcal{M}_*(D)(X_1, \dots, X_n) \\ &= \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad D_0(X_1, \dots, X_n) \geq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad D_0(X_1, \dots, X_n) < \frac{1}{2} \end{cases} \end{aligned}$$

(Proof: B.15, p.161+)

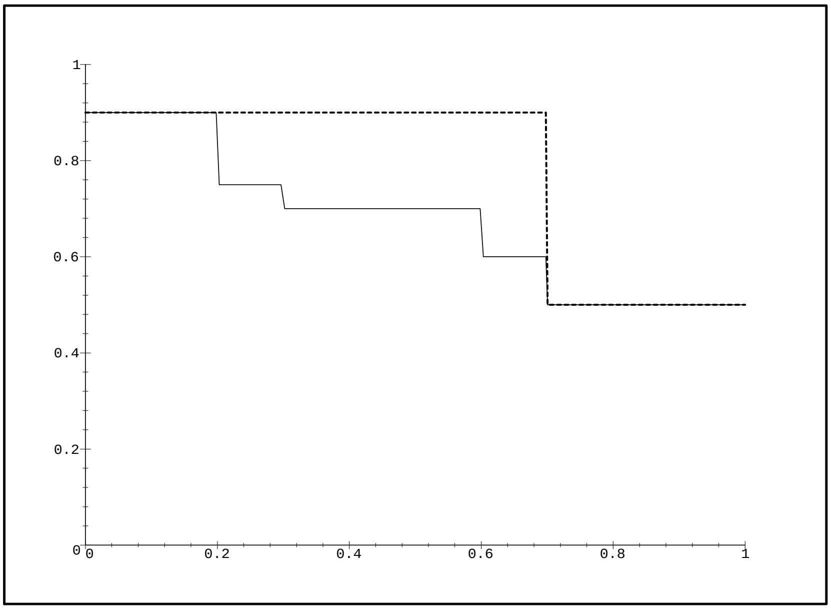

Figure 15: What $\mathcal{M}^*$ measures (dotted) if $D(0) > \frac{1}{2}$

The median-based DFS-models are ordered by (consistent) specificity, as stated by the following theorem.

Theorem 50

$$\mathcal{M}_* \preceq_c \mathcal{M} \preceq_c \mathcal{M}^*$$

(Proof: B.16, p.161+)

4.3 DFS models based on the extension principle

In this section we will show how—with some additional assumptions—DFSes can be constructed by applying the extension principle. We start by reconsidering Gaines 1976 [1], in which a fuzzification scheme is presented which can be used to generalize truth functions of 2-valued logic to corresponding truth functions of an $\mathbf{I}$-valued logic. This scheme can be shown to be closely related to the (standard) extension principle; and Gaines suggests that it might provide a foundation of fuzzy reasoning. After stating the scheme as a mechanism for the fuzzification of two-valued determiners, we present several extensions of this basic approach to the general case of fuzzy pre-determiners. We then establish by proofs that the fuzzification schemes obtained are just a re-formulation of $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ from a different perspective.

4.3.1 The fuzzification mechanism proposed by Gaines

In order to present the Gainesian scheme, we have to introduce some notation.

Let us firstly define the "compatibility" $\theta(x, y)$ of a gradual truth value $x \in \mathbf{I}$ to a "crisp" truth value $y \in \mathbf{2} = \{0, 1\}$.

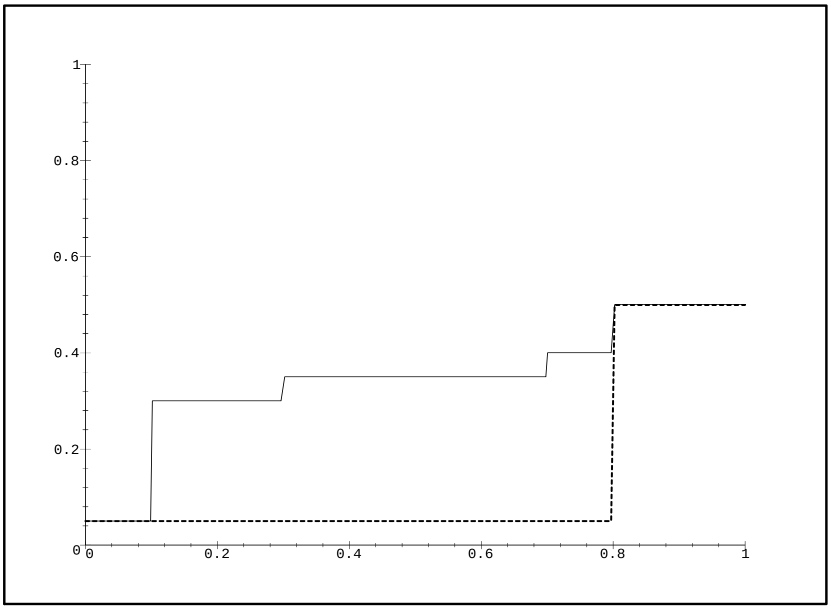

Figure 16: What $\mathcal{M}^*$ measures (dotted) if $D(0) < \frac{1}{2}$

Definition 46 (θ)
$\theta : \mathbf{I} \times \mathbf{2} \longrightarrow \mathbf{I}$ is defined by

$$\theta(x,\, y) \;=\; \begin{cases} 2x & : \; x \leq \frac{1}{2} \textit{ and } y = 1 \\ 2 - 2x & : \; x \geq \frac{1}{2} \textit{ and } y = 0 \\ 1 & : \; \textit{else} \end{cases}$$

for all $x \in \mathbf{I}$, $y \in \mathbf{2} = \{0,\, 1\}$.

Notes

- Hence a gradual truth value $x \leq \frac{1}{2}$ is considered fully compatible to “false” (i.e. $y = 0$), but only gradually compatible to “true” ($y = 1$), and a gradual truth value $x' \geq \frac{1}{2}$ is considered fully compatible to “true” ($y = 1$), but only gradually compatible to “false” ($y = 0$).
- The plots of $\theta(\_,\, 1) : \mathbf{I} \longrightarrow \mathbf{I}$ and $\theta(\_,\, 0) : \mathbf{I} \longrightarrow \mathbf{I}$ are displayed in Fig. 19 and Fig. 20.

θ can be applied to compare membership grades $\mu_X(v)$ ($X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E) with “crisp” membership values $\chi_Y(v)$ (i.e. “Is $v \in Y$?”, $Y \in \mathcal{P}(E)$ crisp), where $v \in E$ is some element of the universe E. This gives rise to the following definition of the compatibility $\Theta(X,\, Y)$ of a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ to a crisp subset $Y \in \mathcal{P}(E)$.

Definition 47 (Θ)
Let E a nonempty set. The mapping $\Theta = \Theta_E : \widetilde{\mathcal{P}}(E) \times \mathcal{P}(E) \longrightarrow \mathbf{I}$ is defined by

$$\Theta(X,\, Y) \;=\; \inf_{v \in E} \theta(\mu_X(v),\, \chi_Y(v))\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$, $Y \in \mathcal{P}(E)$.

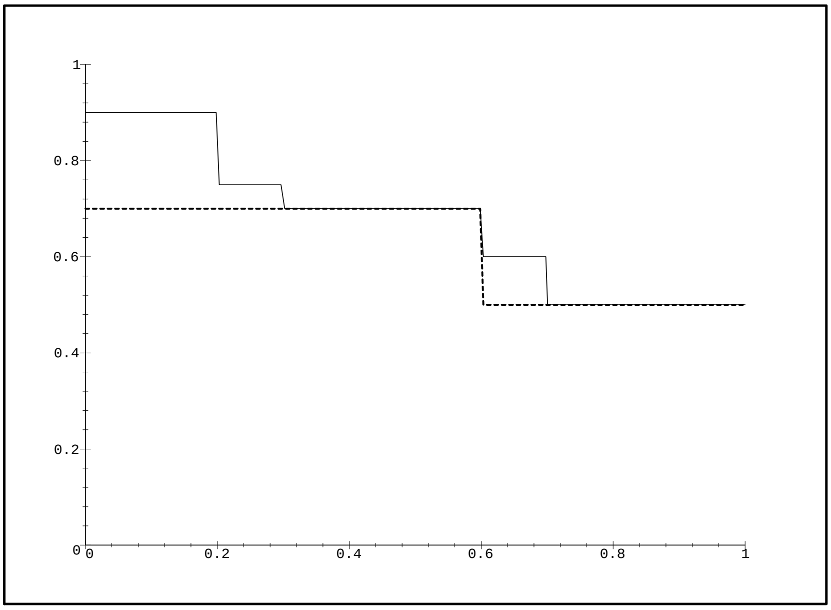

Figure 17: What $\mathcal{M}_*$ measures (dotted) if $D(0) > \frac{1}{2}$

Notes

- The compatibility $\Theta(X, Y)$ of a fuzzy set $X \in \widetilde{\mathcal{P}}(E)$ to a crisp set $Y \in \mathcal{P}(E)$ is therefore the minimal degree of element-wise compatibility of the membership function of X and the characteristic function of Y.
- To present an example, suppose $E = \{a, b\}$, and define $X \in \widetilde{\mathcal{P}}(E)$ by $\mu_X(a) = \frac{1}{3}$, $\mu_X(b) = \frac{3}{4}$. Then

$$\begin{aligned}
\Theta(X, \varnothing) &= \min\{\theta(\mu_X(a), \chi_\varnothing(a)), \theta(\mu_X(b), \chi_\varnothing(b))\} \\
&= \min\{\theta(\tfrac{1}{3}, 0), \theta(\tfrac{3}{4}, 0)\} \\
&= \min\{1, \tfrac{1}{2}\} \\
&= \tfrac{1}{2} \\
\Theta(X, \{a\}) &= \min\{\theta(\tfrac{1}{3}, 1), \theta(\tfrac{3}{4}, 0)\} \\
&= \min\{1, \tfrac{1}{2}\} \\
&= \tfrac{1}{2} \\
\Theta(X, \{b\}) &= \min\{\theta(\tfrac{1}{3}, 0), \theta(\tfrac{3}{4}, 1)\} \\
&= \min\{1, 1\} \\
&= 1 \\
\Theta(X, \{a, b\}) &= \min\{\theta(\tfrac{1}{3}, 1), \theta(\tfrac{3}{4}, 1)\} \\
&= \min\{\tfrac{2}{3}, 1\} \\
&= \tfrac{2}{3}\,.
\end{aligned}$$

We are now able to judge the compatibility of $x \in \mathbf{I}$ to $y \in \{0, 1\}$ by applying θ, and we are able to evaluate the compatibility or similarity of a fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$ of E to crisp subsets

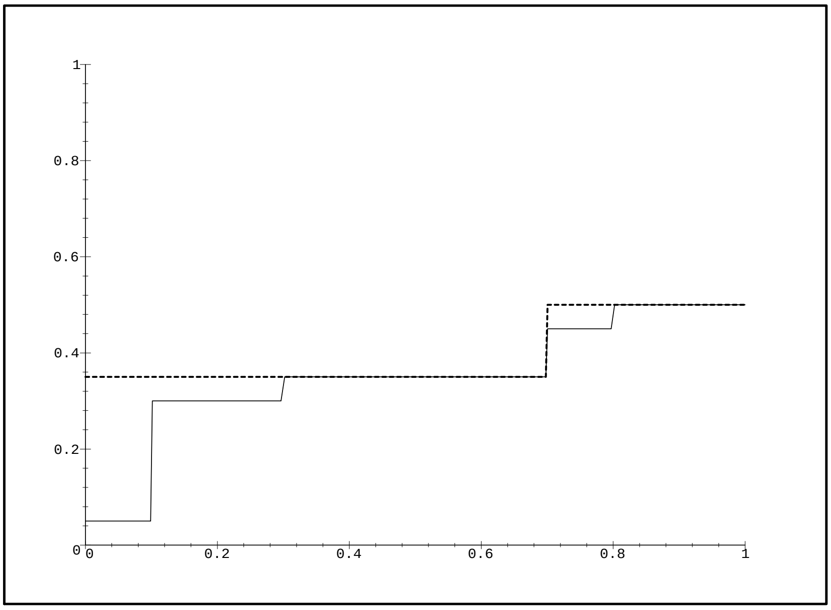

Figure 18: What $\mathcal{M}_*$ measures (dotted) if $D(0) < \frac{1}{2}$

$Y \in \mathcal{P}(E)$ of E by applying Θ. What we still need is a concept converse to that of θ, which converts a pair (x, y) of compatibility assessments (x denoting compatibility to "true", and y denoting compatibility to "false") to a gradual truth value $\mathrm{K}(x, y) \in \mathbf{I}$. We shall assume that (x, y) be normalised in the sense that either $x = 1$ or $y = 1$.

Definition 48 (K)
The mapping $\mathrm{K} : (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$\begin{aligned} \mathrm{K}(x, y) &= \frac{1 + x - y}{2} \\ &= \begin{cases} 1 - \frac{1}{2}y & : \quad x = 1 \\ \frac{1}{2}x & : \quad y = 1 \end{cases} \end{aligned}$$

Notes

- K is a bijection. The inverse, $\mathrm{K}^{-1} : \mathbf{I} \longrightarrow (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I})$, is defined by

$$\begin{aligned} \mathrm{K}^{-1}(z) &= \begin{cases} (2z, 1) & : \quad z \leq \frac{1}{2} \\ (1, 2 - 2z) & : \quad z > \frac{1}{2} \end{cases} \\ &= (\theta(z, 1), \theta(z, 0)) \end{aligned}$$

  for all $z \in \mathbf{I}$, which reveals the close relation between K and θ.[24]

We can now apply these concepts to fuzzy pre-determiners. Let us firstly define $\widetilde{D}_v(X_1, \ldots, X_n)$, the compatibility of $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ to the gradual truth value $v \in \mathbf{I}$, given a choice $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ of fuzzy argument sets.

[24] This relationship is even more apparent in Gaines' original exposition. However, we found our notation better suited for stating the proofs.

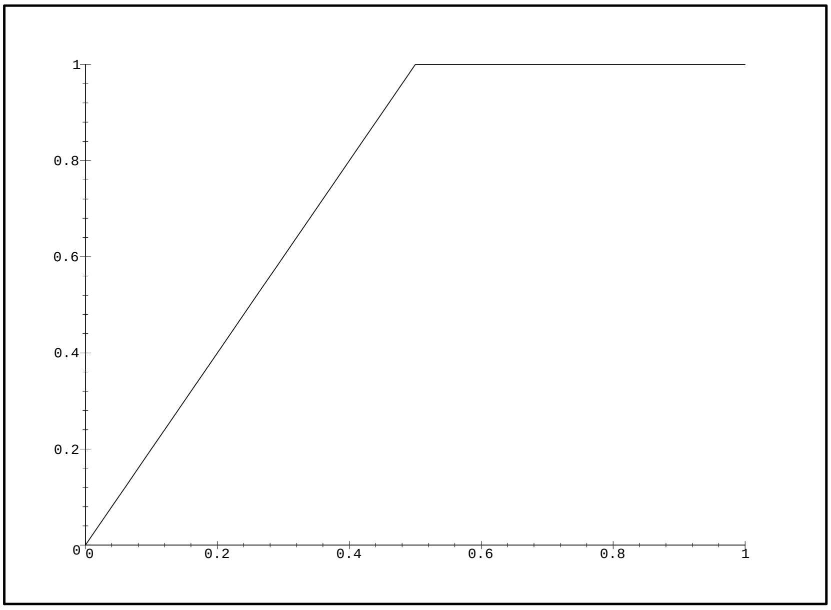

Figure 19: Plot of $\theta(\_, 1)$

Definition 49 ($\widetilde{D}_v$)
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $v \in \mathbf{I}$.
The fuzzy determiner $\widetilde{D}_v : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$\widetilde{D}_v(X_1, \ldots, X_n) = \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : Y = (Y_1, \ldots, Y_n) \in D^{-1}(v)\},$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Notes

- Apparently, the definition of $\widetilde{D}_v(X_1, \ldots, X_n)$ is closely related to the standard extension principle (see Def. 17).
- $\widetilde{D}_v(X_1, \ldots, X_n)$ can be viewed as a measure of the maximal similarity or compatibility of $(X_1, \ldots, X_n)$ to those $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$ which are mapped to $D(Y_1, \ldots, Y_n) = v$.

Building on our definition of $\widetilde{D}_v$ and K, we can now re-state the fuzzification mechanism proposed by Gaines.

Definition 50 (G on two-valued fuzzy pre-determiners)
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued (fuzzy pre-)determiner. The fuzzy determiner $G(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$G(D)(X_1, \ldots, X_n) = \mathrm{K}(\varphi_{\mathrm{T}}, \varphi_{\mathrm{F}}),$$

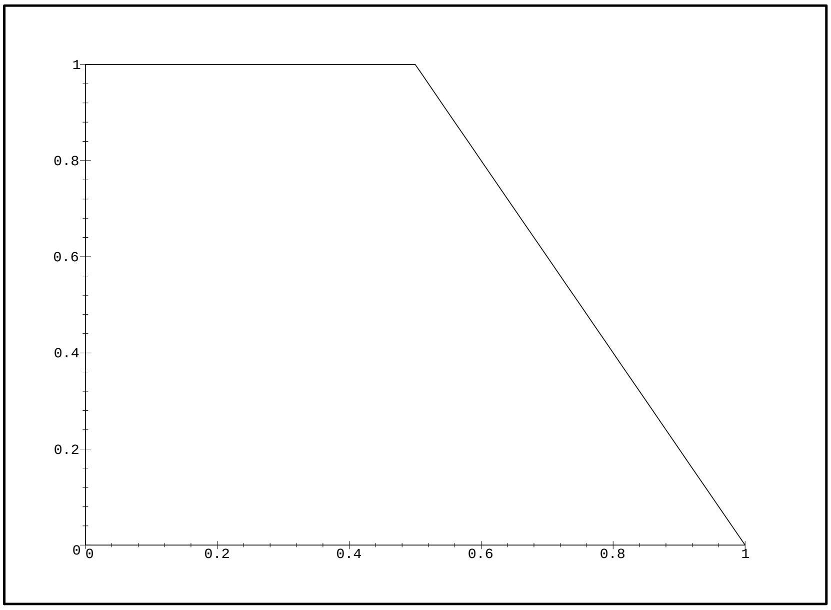

Figure 20: Plot of $\theta(\_, 0)$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, where by* φ_{T} *and* φ_{F} *we abbreviate*

$$\varphi_{\mathrm{T}} = \widetilde{D}_1(X_1, \ldots, X_n) = \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(1)\},$$
$$\varphi_{\mathrm{F}} = \widetilde{D}_0(X_1, \ldots, X_n) = \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(0)\}.$$

Notes

- G applies to arbitrary two-valued determiners of arbitrary arities (i.e. to all determiners in the sense of TGQ), which it transports to corresponding fuzzy determiners. However, it does not cover the case of genuine fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, which also assume values different from $\{0, 1\}$.
- To be precise, Gaines has stated his scheme for propositional functions $f : \mathbf{2}^n \longrightarrow \mathbf{2}$ only, which are transported to $G(f) : \mathbf{I}^n \longrightarrow \mathbf{I}$. He provides results of applying the scheme to the existential and universal quantifiers without *explicitely* stating the required re-formulation to use it as a fuzzification mechanism on two-valued determiners. However, these modifications are apparent and coincide with our definition of G.

We should perhaps add that G is well-defined, i.e. φ_{T} and φ_{F} are contained in the domain of K.

Theorem 51 (*G *is well-defined)
For all two-valued (fuzzy pre-)determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ *and all* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*,*

$$(\varphi_{\mathrm{T}}, \varphi_{\mathrm{F}}) \in (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I}).$$

(Proof: B.17, p.163+)

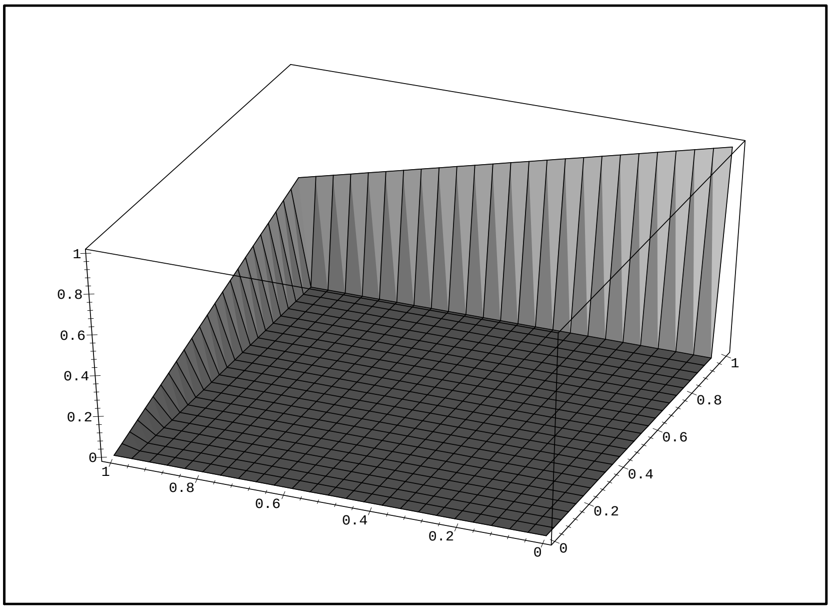

Figure 21: Plot of K

4.3.2 Generalising the Gainesian mechanism to obtain DFS models

We will now generalise this basic mechanism to arbitrary fuzzy pre-determiners. Firstly we will allow for its application to $\{0, \frac{1}{2}, 1\}$-valued fuzzy pre-determiners.

Definition 51 (Extension of G to three-valued fuzzy determiners)
Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is a three-valued fuzzy pre-determiner. The fuzzy determiner $G(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is defined by

$$G(D)(X_1, \ldots, X_n) = \mathrm{K}(\varphi_{\mathrm{T}}, \varphi_{\mathrm{F}}) ,$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, where we abbreviate

$$\begin{aligned}\varphi_{\mathrm{T}} &= \max(\widetilde{D}_1(X_1, \ldots, X_n), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n))\\ \varphi_{\mathrm{F}} &= \max(\widetilde{D}_0(X_1, \ldots, X_n), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) .\end{aligned}$$

Notes

- If D assumes the value $\frac{1}{2}$ at $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$, i.e. $D(Y_1, \ldots, Y_n) = \frac{1}{2}$, this is treated as if *both* $D(Y_1, \ldots, Y_n) = 0$ and $D(Y_1, \ldots, Y_n) = 1$.
- Apparently if D is two-valued, then $\widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n) = 0$ for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and hence

$$\begin{aligned}\max(\widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n), \widetilde{D}_1(X_1, \ldots, X_n)) &= \widetilde{D}_1(X_1, \ldots, X_n)\\ \max(\widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n), \widetilde{D}_0(X_1, \ldots, X_n)) &= \widetilde{D}_0(X_1, \ldots, X_n)\end{aligned}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. The extended scheme therefore coincides with the original scheme whenever the latter is defined.

Let us remark that the extended scheme is still well-defined.

Theorem 52 (Well-definedness of extended scheme)
For all three-valued fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ *and all* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$(\varphi_{\mathrm{T}}, \varphi_{\mathrm{F}}) \in (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I}) .$$

(Proof: B.18, p.165+)

In order to make the extended scheme applicable to arbitrary fuzzy pre-determiners, we shall again resort to the concept of three-valued cut.

Definition 52 (Three-valued cut of a fuzzy pre-determiner)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *a fuzzy pre-determiner and* $\ell \in \mathbf{I}$. *The three-valued fuzzy pre-determiner* $D_{\langle \ell \rangle} : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ *("three-valued cut of* D *at* ℓ*") is defined as follows.*

- *If* $\ell = 0$, *then*

$$D_{\langle \ell \rangle}(X_1, \ldots, X_n) = \begin{cases} 1 & : \quad D(X_1, \ldots, X_n) > \frac{1}{2} \\ \frac{1}{2} & : \quad D(X_1, \ldots, X_n) = \frac{1}{2} \\ 0 & : \quad D(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

  for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

- *If* $\ell > 0$, *then*

$$D_{\langle \ell \rangle}(X_1, \ldots, X_n) = \begin{cases} 1 & : \quad D(X_1, \ldots, X_n) \geq \frac{1}{2} + \frac{1}{2}\ell \\ \frac{1}{2} & : \quad \frac{1}{2} - \frac{1}{2}\ell < D(X_1, \ldots, X_n) < \frac{1}{2} + \frac{1}{2}\ell \\ 0 & : \quad D(X_1, \ldots, X_n) \leq \frac{1}{2} - \frac{1}{2}\ell \end{cases}$$

  for all $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$.

We can then apply the extended Gaines scheme.

Definition 53 ($\mathcal{G}_\ell(\_)$)
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *is a fuzzy pre-determiner and* $\ell \in \mathbf{I}$. *The fuzzy determiner* $\mathcal{G}_\ell(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = G(D_{\langle \ell \rangle})(X_1, \ldots, X_n) ,$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Notes

- Although both $\mathcal{G}_\ell(\_)$ and $(\_)_\gamma$ are defined in terms of three-valued cuts, $\mathcal{G}_\ell(\_)$ is conceptually different from $(\_)_\gamma$. This is because $D_\gamma(X_1, \ldots, X_n)$ is obtained by cutting the argument sets to three-valued sets, while $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is obtained by cutting the determiner itself to a three-valued determiner.

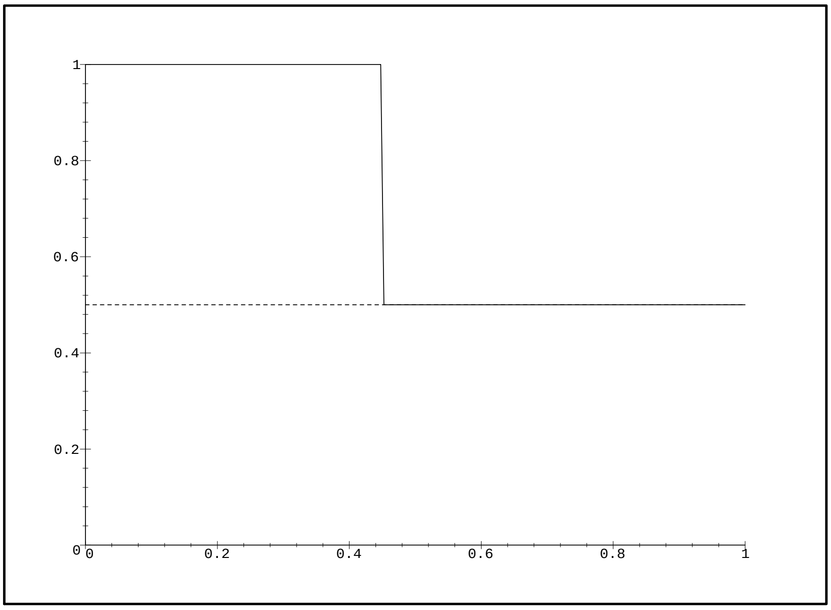

Figure 22: Typical plot of $G(\ell) = \mathcal{G}_\ell(D)(\varnothing)$ for a constant fuzzy pre-determiner

- The conceptual difference of $\mathcal{G}_\ell(\_)$ and $(\_)_\gamma$ is obvious from Fig. 22 and Fig. 23: compared to $(\_)_\gamma$, the behaviour with respect to constant fuzzy pre-determiners $D : \{\varnothing\} \longrightarrow \mathbf{I}$ and three-valued fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is exactly *reversed*.

Before we state our DFS-models, we will show that $\mathcal{G}_\ell(D)$ can be stated somewhat simpler. Our above presentation was mainly to show that $\mathcal{G}_\ell(D)$ is based on three-valued cuts. This might have been obscured by the re-formulation to follow.

Definition 54 ($\varphi_{\mathrm{T}}(\ell)$ and $\varphi_{\mathrm{F}}(\ell)$)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. We introduce abbreviations

$$\varphi_{\mathrm{T}}(\ell) = \begin{cases} \sup\{\min_{i=1}^n \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}([\frac{1}{2}, 1])\} & : \ell = 0 \\ \sup\{\min_{i=1}^n \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1])\} & : \ell > 0 \end{cases}$$
$$\varphi_{\mathrm{F}}(\ell) = \begin{cases} \sup\{\min_{i=1}^n \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}([0, \frac{1}{2}])\} & : \ell = 0 \\ \sup\{\min_{i=1}^n \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}([0, \frac{1}{2} + \frac{1}{2}\ell))\} & : \ell > 0 \end{cases}$$

Using these abbreviations, we can now present $\mathcal{G}_\ell(\_)$ in a simpler way.

Theorem 53 (A simpler presentation of $\mathcal{G}_\ell(\_)$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and all $\ell \in \mathbf{I}$,

$$\varphi_{\mathrm{T}} = \varphi_{\mathrm{T}}(\ell)$$

and

$$\varphi_{\mathrm{F}} = \varphi_{\mathrm{F}}(\ell)\,,$$

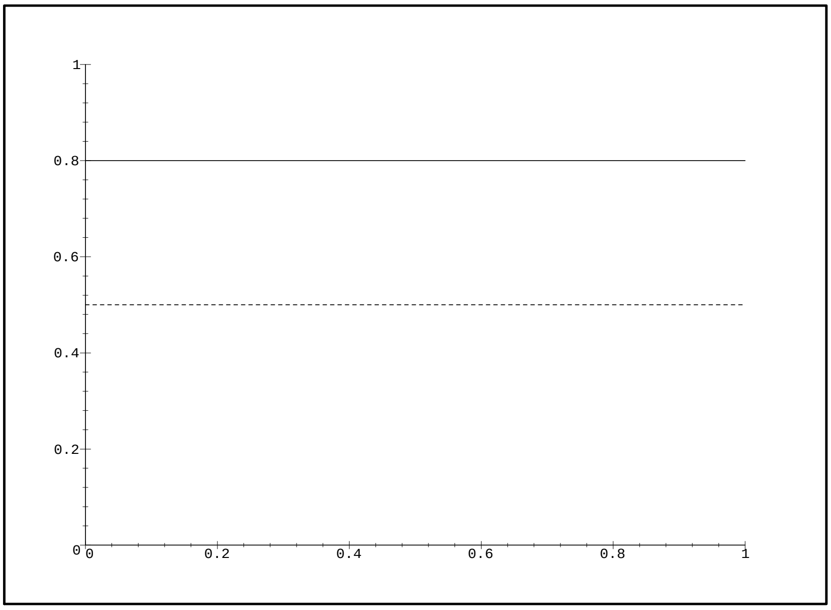

Figure 23: Typical plot of $G(\ell) = \mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ for a three-valued fuzzy pre-determiner

where φ_T and φ_F are obtained from $D_{\langle \ell \rangle}$ by applying the extended Gaines scheme (see Def. 51). In particular,

$$\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \mathrm{K}(\varphi_\mathrm{T}(\ell), \varphi_\mathrm{F}(\ell)) .$$

(Proof: B.19, p.166+)

It is now easy to prove that $\mathcal{G}_\ell(\_)$ has monotonicity properties similar to those of $(\_)_\gamma$.

Theorem 54 (Monotonicity properties of $\mathcal{G}_\ell(\_)$)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

a. if $\mathcal{G}_0(D)(X_1, \ldots, X_n) > \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is monotonically decreasing in ℓ and $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $\ell \in \mathbf{I}$.

b. if $\mathcal{G}_0(D)(X_1, \ldots, X_n) = \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\ell \in \mathbf{I}$.

c. if $\mathcal{G}_0(D)(X_1, \ldots, X_n) < \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is monotonically increasing in ℓ and $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) \leq \frac{1}{2}$ for all $\ell \in \mathbf{I}$.

(Proof: B.20, p.167+)

Like $(\_)_\gamma$, $\mathcal{G}_\ell(\_)$ satisfies "almost all" DFS-axioms. $\mathcal{G}_\ell(\_)$, however, fails on (DFS 1). We therefore need to aggregate over the results obtained for different values of the "cautiousness parameter" ℓ.

Definition 55 ($\mathcal{G}$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, the fuzzy determiner $\mathcal{G}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is

defined by

$$\mathcal{G}(X_1, \ldots, X_n) = \int_0^1 \mathcal{G}_\ell(D)(X_1, \ldots, X_n)\, d\ell\,, \tag{3}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Note. If $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ is given, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is either monotonically increasing or decreasing in ℓ by Th-54 and hence Riemann integrable.

Definition 56 ($\mathcal{G}^*$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*, the fuzzy determiner* $G_0^*(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{G}^*(D)(X_1, \ldots, X_n) = (G_0^* - \tfrac{1}{2})\ell^* + \tfrac{1}{2}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, where we use abbreviations*

$$\begin{aligned} G_0^* &= \lim_{\ell \to 0^+} \mathcal{G}_\ell(D)(X_1, \ldots, X_n) \\ \ell^* &= \inf\{\ell : \mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \tfrac{1}{2}\}\,. \end{aligned}$$

Definition 57 ($\mathcal{G}_*$)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$*, the fuzzy determiner* $\mathcal{G}_*(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ *is defined by*

$$\mathcal{G}_*(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \sup\{\ell(\mathcal{G}_\ell(D)(X_1, \ldots, X_n) - \frac{1}{2}) : \ell \in \mathbf{I}\} & : \quad G_0 \geq \frac{1}{2} \\ \frac{1}{2} - \sup\{\ell(\frac{1}{2} - \mathcal{G}_\ell(D)(X_1, \ldots, X_n)) : \ell \in \mathbf{I}\} & : \quad G_0 < \frac{1}{2} \end{cases}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$*, where we have abbreviated*

$$G_0 = \mathcal{G}_0(D)(X_1, \ldots, X_n)\,.$$

We will not give a direct proof that $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ are DFS models. Instead, we will show that these models correspond to $\mathcal{M}$, $\mathcal{M}^*$, and $\mathcal{M}_*$, respectively, which we have already shown to be DFSes. Before turning to these equivalence proofs, we wish to note that $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ coincide with the original scheme G on two-valued determiners (and with its extension to three-valued fuzzy pre-determiners) whenever these are defined. This will result from the theorems presented in the next section.

Let us also remark that $\mathcal{G}$, $\mathcal{G}^*$ and $\mathcal{G}_*$ all factor through the standard extension principle (i.e. the definition of these schemes can be seen to be based on the standard extension principle). This is immediate from their definition in terms of $\mathcal{G}_\ell(\_)$ and the fact that $\mathcal{G}_\ell(\_)$ factors through the extension principle:

Theorem 55 ($\mathcal{G}_\ell(\_)$ factors through the extension principle)
For every fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *and every* $\ell \in \mathbf{I}$*,*

$$\mathcal{G}_\ell(D) = Q_\ell \circ \hat{\hat{D}}_{\mathrm{norm}} \circ \widetilde{\Theta}$$

where $\widetilde{\Theta} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \tilde{\tilde{\mathcal{P}}}(\mathcal{P}(E)^n)$ *is the mapping which to each* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ *assigns the normalized fuzzy subset* $\widetilde{\Theta}(X_1, \ldots, X_n)$ *of* $\mathcal{P}(E)^n$ *defined by*

$$\mu_{\widetilde{\Theta}(X_1,\ldots,X_n)}(Y_1, \ldots, Y_n) = \min_{i=1}^{n} \Theta(X_i, Y_i)$$

for all $(Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$, $\hat{\hat{D}}_{\mathrm{norm}} : \tilde{\tilde{\mathcal{P}}}(\mathcal{P}(E)^n) \longrightarrow \tilde{\tilde{\mathcal{P}}}(\mathbf{I})$ *is obtained from* $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ *by applying the standard extension principle (restricted to normalized fuzzy sets), and* $Q_\ell : \tilde{\tilde{\mathcal{P}}}(\mathbf{I}) \longrightarrow \mathbf{I}$ *is defined by*

$$Q_\ell(Z) = \mathrm{K}(\sup\{\mu_Z(v) : v \in T_\ell\}, \ \sup\{\mu_Z(v) : v \in F_\ell\}),$$

for all $Z \in \tilde{\tilde{\mathcal{P}}}(\mathbf{I})$, *where*

$$T_\ell = \begin{cases} [\frac{1}{2}, 1] & : \ \ell = 0 \\ (\frac{1}{2} - \frac{1}{2}\ell, 1] & : \ \ell > 0 \end{cases}$$
$$F_\ell = \begin{cases} [0, \frac{1}{2}] & : \ \ell = 0 \\ [0, \frac{1}{2} + \frac{1}{2}\ell) & : \ \ell > 0 \end{cases}$$

(Proof: B.21, p.168+)

4.3.3 The resulting models coincide with the median-based DFSes

As we have already noted, our extensions of the Gainesian scheme can be shown to correspond to the median-based models.

Theorem 56 $\mathcal{M} = \mathcal{G}$.
(Proof: B.23, p.177+)

Theorem 57 $\mathcal{M}^* = \mathcal{G}^*$.
(Proof: B.24, p.183+)

Theorem 58 $\mathcal{M}_* = \mathcal{G}_*$.
(Proof: B.25, p.187+)

We should add that $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ coincide with the original scheme of Gaines (and its extension to three-valued fuzzy pre-determiners) whenever these are defined. $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ are therefore consistent extensions of the Gainesian fuzzification scheme.

Theorem 59 (Extensions of** G**)
Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ *a three-valued fuzzy pre-determiner. Then*

$$\mathcal{M}_*(D) = \mathcal{M}(D) = \mathcal{M}^*(D) = G(D),$$

where G *is the "extended" Gaines scheme which applies to three-valued fuzzy pre-determiners.*
(Proof: B.26, p.189+)

Note. In particular, if D is two-valued, then $\mathcal{M}_*(D)$, $\mathcal{M}(D)$ and $\mathcal{M}^*(D)$ coincide with the original scheme as proposed by Gaines.

In addition, because $\mathcal{G}_*$, $\mathcal{G}$ and $\mathcal{G}^*$ factor through the extension principle by Th-55, we now know that $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ factor through the extension principle also.[25] From this we see that, although it is not possible to define DFSes based on the commonly-used tools of α-cutting and resolution principle, it is still possible to define DFSes based on the extension principle.

This is in accordance with our intuition that the extension principle is more foundational to fuzzy set theory than are α-cuts and the resolution principle.

We wish to remark, however, that DFSes are even *more* foundational than the extension principle because every DFS not only induces its "own" extension principle, but also provides a unique interpretation of all fuzzy connectives and (of course) of arbitrary fuzzy pre-determiners.

$\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$, for example, induce the standard connectives $\min$, $\max$ and $1 - x$ and the standard extension principle. In this sense, it could be argued that $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ capture the essence of fuzzy logic in a single formula.

4.4 Evaluation of fuzzy pre-determiners in the DFS models

Our theory has been designed to establish a principled account of quantified expressions in NL statements, and we have ensured this by stating it in the form of axioms. As we have also provided actual models of the theory, we are now able to evaluate arbitrary fuzzy quantifying expressions on fuzzy subsets of arbitrary base sets. We only need to provide a fuzzy pre-determiner D which reflects the semantics of the considered operation on "crisp" sets, and then implement the corresponding fuzzy determiner $\mathcal{M}'(D)$ (where $\mathcal{M}'$ is one of $\mathcal{M}_*$, $\mathcal{M}$, $\mathcal{M}^*$).

Let us firstly consider the standard quantifiers. By Th-35 and the fact that $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ are standard DFSes, we obtain that

$$\begin{aligned}
\mathcal{M}'(\exists)(X) &= \sup\{\mu_X(v) : v \in E\} \\
\mathcal{M}'(\forall)(X) &= \inf\{\mu_X(v) : v \in E\} \\
\mathcal{M}'(\textbf{all})(X_1, X_2) &= \inf\{\max(1 - \mu_{X_1}(v), \mu_{X_2}(v)) : v \in E\} \\
\mathcal{M}'(\textbf{some})(X_1, X_2) &= \sup\{\min(\mu_{X_1}(v), \mu_{X_2}(v)) : v \in E\} \\
\mathcal{M}'(\textbf{no})(X_1, X_2) &= \inf\{\max(1 - \mu_{X_1}(v), 1 - \mu_{X_2}(v)) : v \in E\} ,
\end{aligned}$$

for all base sets $E \neq \varnothing$ and $X, X_1, X_2 \in \widetilde{\mathcal{P}}(E)$, where $\mathcal{M}' \in \{\mathcal{M}_*, \mathcal{M}, \mathcal{M}^*\}$.

For the absolute quantifiers **atleast n**, **atmost n** $= \neg$**atleast n+1**, **more than n** $=$ **atleast n+1** and **less than n** $=$ **atmost n-1**, we obtain the following results.

Theorem 60

Suppose $E \neq \varnothing$ is a finite set, $\mathcal{M}' \in \{\mathcal{M}_*, \mathcal{M}, \mathcal{M}^*\}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\mathcal{M}'(\textbf{atleast n})(X_1, X_2) &= \mu_{(n)} \\
\mathcal{M}'(\textbf{atmost n})(X_1, X_2) &= \neg\mu_{(n+1)} \\
\mathcal{M}'(\textbf{more than n})(X_1, X_2) &= \mu_{(n+1)} \\
\mathcal{M}'(\textbf{less than n})(X_1, X_2) &= \neg\mu_{(n)}
\end{aligned}$$

[25] In the sense that for all $E \neq \emptyset$ and $n \in \mathbb{N}$ there are mappings $A : \mathcal{P}(E)^n \longrightarrow \tilde{\tilde{\mathcal{P}}}(\mathcal{P}(E)^n)$, $B : \tilde{\tilde{\mathcal{P}}}(\mathbf{I}) \longrightarrow \mathbf{I}$ such that for all $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $\mathcal{M}_*(D) = B \circ \hat{\hat{D}} \circ A$ (likewise for $\mathcal{M}$ and $\mathcal{M}^*$).

for all $X_1,\ X_2 \in \widetilde{\mathcal{P}}(E)$*, where* $\mu_{(i)}$ *is the* i*-th largest element in the ordered sequence of membership values of* $X_1 \cap X_2$*, including duplicates. (In the case that* $i > \operatorname{card} E$*, define* $\mu_{(i)} = 0$*).*

(Proof: B.27, p.191+)

In the general case, $\mathcal{M}_*(D)$, $\mathcal{M}(D)$ and $\mathcal{M}^*(D)$ cannot be described by such closed-form expressions. However, the resulting fuzzy determiners can often be efficiently implemented on the basis of histogram computations if D is quantitative. In an experimental multimedia retrieval system (Glöckner & Knoll [5]), which utilises fuzzy determiners for various aggregation and information fusion purposes, we have implemented 28 fuzzy determiners in this fashion. Implementation aspects are described in a separate report [28].

Which DFS should be used in a given application?

Availability of more than one DFS of course poses the problem of which DFS to choose in a given application. We think that $\mathcal{M}$ and $\mathcal{M}^*$ be preferable to $\mathcal{M}_*$ in most cases because one is usually interested in obtaining *specific* results. We tend to view $\mathcal{M}_*$ as a boundary case and conjecture that it might coincide with the least specific standard DFS.

$\mathcal{M}^*$ yields the most specific results. However (as witnessed by Fig. 15 and Fig. 16), $\mathcal{M}^*$ ignores the fine-structure of $D_\gamma(X_1,\ \ldots,\ X_n)$, while $\mathcal{M}$ considers *all* results of $D_\gamma(X_1,\ \ldots,\ X_n)$ by means of integration. Because of its more compromising and aggregative behaviour, we suppose that $\mathcal{M}$ might often be preferable to $\mathcal{M}^*$ in practical applications.

These remarks on $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ are of course only first, and informal, hints which might facilitate the choice. We intend to investigate further properties of $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ in the future, in order to better understand the characteristics of their aggregation behaviour.

5 Conclusion

This axiomatic account of fuzzy quantification offers clear advantages compared to the approaches described in the literature. Ever since Zadeh published his basic papers on fuzzy quantification [29, 2, 30], fuzzy quantifiers have generally been treated as fuzzy subsets $Q \in \widetilde{\mathcal{P}}(\mathbb{R}^+)$ of the non-negative reals (so-called "absolute quantifiers" like **atleast n**, **about n**) or of the unit interval ($Q \in \widetilde{\mathcal{P}}(\mathbf{I})$, "proportional quantifiers" like **most** or **more than** r **percent**). In order to make these fuzzy numbers applicable to fuzzy sets for the purpose of quantification, a mechanism (which we denote by $\mathcal{Z}$) is needed which transports Q to a fuzzy determiner

$$\mathcal{Z}(Q) : \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I} \quad \text{(unrestricted use, relative to } E\text{), or}$$
$$\mathcal{Z}(Q) : \widetilde{\mathcal{P}}(E) \times \widetilde{\mathcal{P}}(E) \longrightarrow \mathbf{I} \quad \text{(restricted use, relative to first argument).}$$

Zadeh has also formulated the idea that in order to evaluate a statement "$Q\,X's$ are A" (in our notation: to compute $\mathcal{Z}(Q)(A)$), one should instead evaluate the statement "$\mathrm{card}\,X$ is Q", where card is a scalar or fuzzy measure of the cardinality of the fuzzy set $A \in \widetilde{\mathcal{P}}(E)$ associated with the linguistic variable X.[26] The approaches described in the literature mainly differ in the measure of fuzzy cardinality used and in the way that the required comparison of fuzzy cardinalities is accomplished. Zadehs original Σ-count approach was shown to be implausible (see e.g. Ralescu [9, 31] and Yager [32]). To present an example, if $E = \{\mathrm{John}, \mathrm{Lucas}\}$ is a set of persons and

$$\mu_{\textbf{bald}}(\mathrm{John}) = 0.5, \quad \mu_{\textbf{bald}}(\mathrm{Lucas}) = 0.5\,,$$

then the Σ-Count approach yields that *Exactly one person is bald*, which is clearly inappropriate. Another point to be made is that the Σ-Count approach, when applied to two-valued determiners such as **all** or **atleast n**, always generates "crisp" (i.e. $\{0, 1\}$-valued) results, which is also inappropriate.

Others have built on Zadehs idea of representing fuzzy quantifiers by fuzzy numbers, and of utilizing a (scalar or fuzzy) measure of the cardinality of fuzzy sets. Yager [8] describes an FG-Count-based approach, which is however limited to monotone quantifiers. Ralescu [9, 31] presents a possibilistic approach which is based on the computation of FE-Counts.[27] Dubois & Prade [33] improve on the FE-Count approach (which discerns less structure than possible) and on Zadeh's original definition of fuzzy cardinality [29] (which may generate non-convex fuzzy cardinalities) by presenting an alternative measure of fuzzy cardinality with nicer formal properties. Yager [10, 34, 35] models fuzzy quantifiers by Ordered Weighted Averaging (OWA) operators.

Compared to these approaches, DFS theory offers the following conceptual advantages.

No restriction to absolute and proportional determiners

Although absolute and proportional determiners form natural subclasses, it should be noted that not every determiner (in the sense of TGQ) corresponds to these classes, not even every two-place determiner. Quite the reverse, the comprehensive classification of natural language determiners by Keenan & Stavi [15, pp. 253-256] distinguishes 16 main classes of determiners (among others, determiners of exception like **all except one**, definites like **the ten**, bounding determiners like **only**, etc.). It should be noted that TGQ can model all these very different classes of determiners

[26] On proportional quantifiers, the cardinality measure is replaced by a measure of fuzzy proportion, usually referred to as "relative cardinality".

[27] FG-Count and FE-Count are measures of fuzzy cardinality, cf. Zadeh [2].

with the same basic mechanism, just as our DFS approach does: by a given DFS, *every* determiner in the sense of TGQ can be generalized to a fuzzy determiner which also accepts fuzzy subsets of the universe E as its arguments.

By contrast, the fuzzy approaches to quantification cited above rely on the distinction of proportional and absolute determiners, which they model by *different* mechanisms. There is no simple or straightforward way to generalise these approaches to arbitrary determiners; just as there is no uniform account of cardinal (absolute) and proportional determiners in these approaches.

No restriction to finite universes

In addition, these approaches are based on the assumption that the base set E be finite.[28] DFS, however, requires a proper treatment of quantification over infinite base sets as well (and the models $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ can handle this case—without any assumptions on measurability).

Although the universes of quantification are likely to be finite in most applications of fuzzy quantification (e.g. in information retrieval [5] and multi-criteria decision making [10]), a proper treatment of quantification over infinite base sets is still of great practical importance. This is because in natural language, there are not only count nouns (which correspond to discrete objects), but also *mass nouns* like *water* or *wine*, which correspond to continuous and possibly atomless masses. In order to provide for an adequate account of quantification over such masses (*some water*, *much wine*, etc.), it is inevitable to consider the case of non-finite sets (of which masses are instances) as well.

No restriction to quantitative determiners

Unlike the approaches to fuzzy quantification described in the literature, DFS also provides an adequate account of non-quantitative determiners, e.g. of projection determiners like **john** $= \pi_{\mathrm{John}}$, non-quantitative composite determiners, as expressed by *All X's except the men are Y* or *All male X's are Y*, and others. By Th-18, a DFS $\mathcal{F}$ maps quantitative fuzzy pre-determiners to quantitative fuzzy determiners. But the theorem also states that $\mathcal{F}$ maps non-quantitative fuzzy pre-determiners to corresponding *non-quantitative* fuzzy determiners.

By contrast, the existing approaches to fuzzy quantification *always* result in quantitative determiners (in the sense of automorphism invariance), and are therefore not suited to model non-quantitative classes of determiners. We acknowledge this as a consequence of their definition in terms of a (scalar or fuzzy) measure of cardinality of fuzzy sets, because any such measure *must* be invariant under automorphisms in order to adequately capture the notion of "cardinality". Let us remark that Yager's OWA approach, which does not make direct use of a cardinality measure, is also subject to this criticism because all OWA operators are automorphism-invariant.

The relevance of this failure to model non-quantitative determiners may be enhanced by the fact that, as van Benthem [13, p.473+] shows, *most* important quantifiers (including e.g., **most**) become non-quantitative (in the sense of automorphism invariance) as soon as one allows for non-finite universes of quantification!

A theory of fuzzy quantification which also intends to cover the case of infinite universes of quantification (and in particular, quantification over masses) therefore *must* be able to represent non-quantitative determiners as well, or will fail to satisfy an important adequacy condition.

[28] The Σ-Count approach can be generalised to measurable fuzzy subsets of sets of infinite cardinality by replacing summation with integration. But this approach is inadequate for independent reasons.

Adequate account of restricted quantification

No need to introduce ad-hoc devices such as degrees of Andness/Orness (cf. Yager [10], Arnould & Ralescu [36]); DFS treats restricted quantification with the same mechanisms (and ease) as unrestricted quantification. This is possible because DFS is a

Genuine theory of multi-place quantification

DFS theory is concerned with fuzzy determiners of arbitrary arities (and the models $\mathcal{M}_*$, $\mathcal{M}$ and $\mathcal{M}^*$ can handle the case that $n > 2$ without any difficulty), which are only mentioned—but never actually accounted for—in the literature on fuzzy quantifiers.

We wish to add that the Theory of Generalized Quantifiers wisely discerns determiners of different arities n because there are genuine multiplace determiners such as **more** in *More men than women are smokers* ($n = 3$). These cannot be interpreted as absolute or proportional quantifiers (and hence cannot be accounted for by the approaches described in the literature), simply because they are known to be irreducible to two-place determiners (Hamm [37, pp. 23+]).

We have compared DFS to the existing approaches to fuzzy quantification, which are based on Zadeh's representation of fuzzy quantifiers by fuzzy numbers, and an evaluation of "$\mathrm{card}\, X$ is Q" by means of some cardinality measure on fuzzy sets. Because of the severe limitations of these approaches both with regard to multi-place and non-quantitative fuzzy pre-determiners, and with regard to base sets of infinite cardinality—limitations which DFS can easily overcome because it does not share their common assumptions —we feel justified to conclude that *only DFS* is

Compatible to linguistic theory

This is because the (widely-accepted) Theory of Generalized Quantifiers can be consistently embedded into DFS, but *not* into the other approaches to fuzzy quantification (which is obvious from their limitations, which are unknown to TGQ). The coverage of DFS with respect to quantificational phenomena[29] compared to that of TGQ and of existing approaches to fuzzy quantification, is displayed in Fig. 24.

Axiomatic foundation

DFS theory is based on a rigid axiomatic foundation which ensures the reasonable behaviour of the resulting models. This axiomatic approach becomes possible by the use of a fuzzification mechanism which abstracts from all particular instances of quantification. Its axiomatic foundation ensures that the generalisations produced by a given DFS are intuitively plausible. The uniform approach (one DFS is sufficient) also guarantees the desired systematicity.

In particular, the DFS axioms ensure the

Preservation of Aristotelian squares

Every DFS is fully compatible to the formation of external negation, antonyms, and duals; a property which is offered by *none* of the approaches mentioned above. Furthermore, every DFS is

[29]Excluding intensional and branching types of quantification, see e.g. Barwise & Cooper [12]

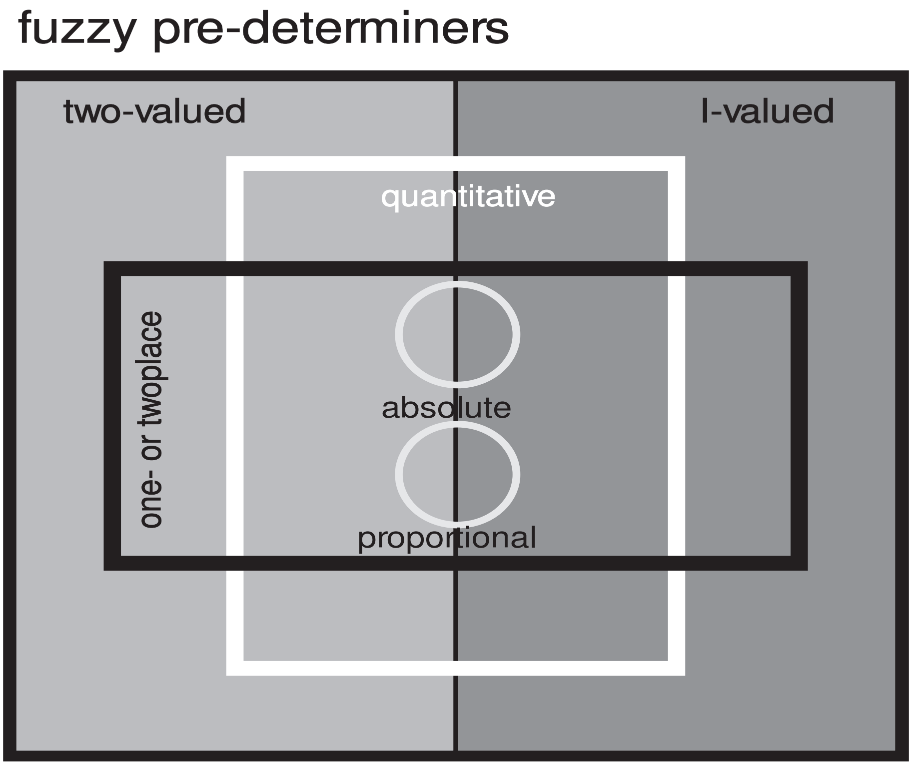

Figure 24: Coverage of quantificational phenomena: DFS (outer rectange) vs. TGQ (left half) and existing approaches (circles)

compatible to the Piaget group of transformations, the relevance of which stems from empirical findings in developmental psychology.[30]

Consistent view of fuzzy set theory

Another advantage is that by choosing a DFS, a unique definition of all fuzzy connectives is induced. For example, each DFS is compatible with exactly one choice of conjunction, disjunction, implication, weighted conjunction[31] and weighted disjunction—there is no degree of freedom left in the definition of these connectives. In addition, every DFS induces a unique choice of extension principle and of fuzzy inverse images.

Concluding, DFS theory provides a consistent view of fuzzy set theory and simultaneously *extends* the Theory of Generalized Quantifiers in a uniform and systematic way to the fuzzy case. By contrast, the approaches described in the literature are essentially *limited* compared to TGQ and usually lack an axiomatic foundation.

Due to the axiomatic rigor of its foundation, its nice formal properties, its broad coverage of quantificational phenomena, and the existence of computational models, DFS theory not only con-

[30]For a note on the importance of the Piaget group of transformations to fuzzy logic, see Dubois & Prade [19, p. 158+].

[31]We take a weighted conjunction $\tilde{\bigwedge}_{i=1}^{n} w_i{:}x_i$ as expressing *All relevant conditions are true*, i.e. we interpret it as $\mathcal{F}(\textbf{all})(W, X)$, where $W \in \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ is the fuzzy subset of relevant conditions, $\mu_W(i) = w_i$, and $X \in \widetilde{\mathcal{P}}(\{1, \ldots, n\})$ is the fuzzy set of conditions which are (gradually) true, i.e. $\mu_X(i) = x_i$ ($i \in \{1, \ldots, n\}$). Analogously for weighted disjunctions, using the two-place fuzzy determiner $\mathcal{F}(\textbf{some})$.

tributes to a better understanding of fuzzy quantification in general, but also allows for a *principled* account of fuzzy quantifiers in practical applications.

Appendix: Proofs

We have adopted the convention of calling any proposition which occurs in the main text a *theorem*, and of calling any proposition which only occurs in the proofs a *lemma*. Theorems are referred to as Th-n, where n is the number of the theorem, and lemmata are referred to as L-n, where n is the number of the lemma. Equations which are embedded in proofs are referred to as (n), where n is the number of the equation.
The main text has been organized such that the dependency order of the proofs is respected, i.e. the proofs may be read linearly.

A Proof of the theorems in chapter "DFS Properties"

A.1 Preliminaries

In the proofs, we will often need the following.

Lemma 1 ($\widetilde{\pi}_*^{-1}$ on pointwise-defined operations)
Suppose $f : \mathbf{I}^n \longrightarrow \mathbf{I}$ and $F : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \widetilde{\mathcal{P}}(\{*\})$ is pointwise defined by

$$\mu_{F(X_1,\dots,X_n)}(*) = f(\mu_{X_1}(*),\ \dots \mu_{X_n}(*))$$

for all $(X_1,\ \dots,\ X_n) \in \widetilde{\mathcal{P}}(\{*\})^n$.
Then for all $x_1,\ \dots,\ x_n \in \mathbf{I}$,

$$\widetilde{\pi}_*^{-1}(f(x_1,\ \dots,\ x_n)) = F(\widetilde{\pi}_*^{-1}(x_1),\ \dots,\ \widetilde{\pi}_*^{-1}(x_n))\,.$$

Proof Suppose $f : \mathbf{I}^n \longrightarrow \mathbf{I}$ and $F : \widetilde{\mathcal{P}}(\{*\})^n \longrightarrow \widetilde{\mathcal{P}}(\{*\})$ is defined by

$$\mu_{F(X_1,\dots,X_n)}(*) = f(\mu_{X_1}(*),\ \dots \mu_{X_n}(*)) \tag{4}$$

for all $(X_1,\ \dots,\ X_n) \in \widetilde{\mathcal{P}}(\{*\})^n$.
Recalling that $\widetilde{\pi}_*$ is a bijection, it is sufficient to show that

$$f(x_1,\ \dots,\ x_n) = \widetilde{\pi}_*(F(\widetilde{\pi}_*^{-1}(x_1),\ \dots,\ \widetilde{\pi}_*^{-1}(x_n)))\,.$$

Now,

$$\begin{aligned}
\widetilde{\pi}_*(F(\widetilde{\pi}_*^{-1}(x_1),\ \dots,\ \widetilde{\pi}_*^{-1}(x_n))) &= \mu_{F(\widetilde{\pi}_*^{-1}(x_1),\dots,\widetilde{\pi}_*^{-1}(x_n))}(*) && \text{by definition}\\
&= f(\mu_{\widetilde{\pi}_*^{-1}(x_1)}(*),\ \dots,\ \mu_{\widetilde{\pi}_*^{-1}(x_n)}(*)) && \text{by (4)}\\
&= f(\widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(x_1)),\ \dots,\ \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(x_n))) && \text{by definition of } \widetilde{\pi}_*\\
&= f(x_1,\ \dots,\ x_n)\,.
\end{aligned}$$

A.2 Proof of Theorem 1

As we have already noted, $D(X_1,\ \dots,\ X_n) = (D\triangleleft X_n \dots \triangleleft X_1)(\varnothing)$. Therefore,

$$\begin{aligned}
\mathcal{F}(D)(X_1,\ \dots,\ X_n) &= (\mathcal{F}(D)\triangleleft X_n \dots \triangleleft X_1)(\varnothing)\\
&= \mathcal{F}(D\triangleleft X_n \dots \triangleleft X_1)(\varnothing) && \text{by (DFS 7)}\\
&= (D\triangleleft X_n \dots \triangleleft X_1)(\varnothing) && \text{by (DFS 1)}\\
&= D(X_1,\ \dots,\ X_n)\,.
\end{aligned}$$

A.3 Proof of Theorem 2

$$\begin{aligned}
\widetilde{\mathcal{F}}(\mathrm{id}_{\mathbf{2}})(x) &= \mathcal{F}(\mathrm{id}^*_{\mathbf{2}})(\widetilde{\pi}_*^{-1}(x)) \\
&= \mathcal{F}(\mathrm{id}_{\mathbf{2}} \circ \pi_*)(\widetilde{\pi}_*^{-1}(x)) \\
&= \mathcal{F}(\pi_*)(\widetilde{\pi}_*^{-1}(x)) \\
&= \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(x)) && \text{by (DFS 2)} \\
&= x \\
&= \mathrm{id}_{\mathbf{I}}(x)\,.
\end{aligned}$$

A.4 Proof of Theorem 3

a. $\widetilde{\neg}\, 0 = 1$ by Th. 1.

b. $\widetilde{\neg}$ is monotocically decreasing by (DFS 8) because $\neg$ is monotonically decreasing.

c. $\widetilde{\neg}\,\widetilde{\neg}\, x = x$.
To see this, consider

$$\begin{aligned}
\widetilde{\neg}\,\widetilde{\neg}\, x &= \widetilde{\neg}\,\widetilde{\neg}\, \mathrm{id}_{\mathbf{I}}(x) \\
&= \mathcal{F}(\widetilde{\neg}\,\widetilde{\neg}\, \mathrm{id}_{\mathbf{2}})(x) && \text{by (DFS 3)} \\
&= \mathcal{F}(\neg\neg\mathrm{id}_{\mathbf{2}})(x) && \text{by \textbf{b.} and \textbf{c.}, } \widetilde{\neg}\, 0 = \neg 0,\ \widetilde{\neg}\, 1 = \neg 1 \\
&= \mathcal{F}(\mathrm{id}_{\mathbf{2}})(x) \\
&= \mathrm{id}_{\mathbf{I}} x && \text{by Th-2} \\
&= x
\end{aligned}$$

A.5 Proof of Theorem 4

Lemma 2 (Increasing monotonicity in n-th argument)
If $\mathcal{F}$ is a DFS and the fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $n > 0$ is increasing in its n-th argument, then $\mathcal{F}(D)$ is also increasing in its n-th argument.

Proof If $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is increasing in the n-th argument, then $\widetilde{\neg}\, D$ is decreasing in the n-th argument because $\widetilde{\neg}$ is decreasing. By (DFS 8), $\mathcal{F}(\widetilde{\neg}\, D)$ is decreasing in its n-th argument. Then because $\widetilde{\neg}$ is decreasing, $\widetilde{\neg}\, \mathcal{F}(\widetilde{\neg}\, D)$ is increasing in the n-th argument. But $\widetilde{\neg}\, \mathcal{F}(\widetilde{\neg}\, D) = \mathcal{F}(\widetilde{\neg}\,\widetilde{\neg}\, D) = \mathcal{F}(D)$ by (DFS 3), i.e. $\mathcal{F}(D)$ is monotonically increasing.

Proof of Theorem 4

"$\Rightarrow$" Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $i \in \{1, \ldots, n\}$. Then apparently

$$\begin{aligned}
&D(X_1, \ldots, X_{i_1}, X_i, X_{i+i}, \ldots, X_n) \\
&\leq D(X_1, \ldots, X_{i_1}, X', X_{i+i}, \ldots, X_n)
\end{aligned}$$

iff

$$\begin{aligned}
&(D\tau_i)(X_1, \ldots, X_{i_1}, X_n, X_{i+i}, \ldots, X_{n-1}, X_i) \\
&\leq (D\tau_i)(X_1, \ldots, X_{i_1}, X_n, X_{i+i}, \ldots, X_{n-1}, X')\,,
\end{aligned}$$

for all $X_1, \ldots, X_n, X' \in \mathcal{P}(E)$, i.e. D is monotonically increasing (decreasing) in the i-th argument iff $D\tau_i$ is monotonically increasing (decreasing) in the n-th argument. For similar reasons, a fuzzy determiner $\widetilde{D} : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I}$ is increasing (decreasing) in its i-th argument iff $\widetilde{D}\tau_i$ is increasing (decreasing) in the n-th argument. Therefore,

$$\begin{array}{llll} D \ \text{decr. in } i\text{-th arg.} & \Leftrightarrow D\tau_i & \text{decr. in } n\text{-th arg.} & \\ & \Rightarrow \mathcal{F}(D\tau_i) & \text{decr. in } n\text{-th arg.} & \text{by (DFS 8)} \\ & \Leftrightarrow \mathcal{F}(D)\tau_i & \text{decr. in } n\text{-th arg.} & \text{by (DFS 4)} \\ & \Leftrightarrow \mathcal{F}(D) & \text{decr. in } i\text{-th arg.} & \end{array}$$

Likewise for inceasing monotonicity, applying L-2.

"$\Leftarrow$" If $\mathcal{F}(D)$ is monotonically increasing (decreasing) in its i-th argument, then so is $\mathcal{F}(D)|_{\mathcal{P}(E)^n}$. But by Th. 1, $\mathcal{F}(D)|_{\mathcal{P}(E)^n} = D$.

A.6 Proof of Theorem 5

Since $D'(X_1, \ldots, X_n, X_{n+1}) = D'(X_1, \ldots, X_n, X'_{n+1})$ for all $X_1, \ldots, X_{n+1}, X'_{n+1} \in \widetilde{\mathcal{P}}(E)$, D' is both monotonically decreasing and increasing in the n-th argument. From Th. 4 we obtain that $\mathcal{F}(D') : \widetilde{\mathcal{P}}(E)^{n+1} \longrightarrow \mathbf{I}$ is also both monotonically increasing and decreasing in its n-th argument. Hence we know that $\mathcal{F}(D')$ is constant in the n-th argument, i.e.

$$\mathcal{F}(D')(X_1, \ldots, X_n, X_{n+1}) = \mathcal{F}(D')(X_1, \ldots, X_n, X'_{n+1}) \tag{5}$$

for all $X_1, \ldots, X_{n+1}, X'_{n+1} \in \widetilde{\mathcal{P}}(E)$.
Now choose an arbitrary crisp subset $A \in \mathcal{P}(E)$. Then

$$\begin{aligned} (D' \triangleleft A)(X_1, \ldots, X_n) &= D'(X_1, \ldots, X_n, A) && (6) \\ &= D(X_1, \ldots, X_n) && (7) \end{aligned}$$

i.e. $D' \triangleleft A = D$. Therefore

$$\begin{aligned} \mathcal{F}(D)(X_1, \ldots, X_n) &= \mathcal{F}(D' \triangleleft A)((X_1, \ldots, X_n)) && \text{by (6)} \\ &= \mathcal{F}(D')(X_1, \ldots, X_n, A) && \text{by (DFS 7)} \\ &= \mathcal{F}(D')(X_1, \ldots, X_n, X_{n+1}) && \text{by (5)} \end{aligned}$$

which finishes the proof.

A.7 Proof of Theorem 6

a. $x_1 \mathbin{\widetilde{\wedge}} 0 = 0$

$$\begin{aligned} x_1 \mathbin{\widetilde{\wedge}} 0 &= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1), \widetilde{\pi}_*^{-1}(0)) && \\ &= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1), \varnothing) && \\ &= \mathcal{F}(\wedge^* \triangleleft \varnothing)(\widetilde{\pi}_*^{-1}(x_1)) && \text{by (DFS 7)} \\ &= \mathcal{F}(0)(\widetilde{\pi}_*^{-1}(x_1)) && \text{because } 0 \wedge 0 = 0, 1 \wedge 0 = 0 \\ &= 0 && \text{by Th-5 and (DFS 1)} \end{aligned}$$

b. $x_1 \mathbin{\widetilde{\wedge}} 1 = x_1$

Noting that $x \wedge 1 = x = \mathrm{id}_{\mathbf{2}}(x)$, we obtain

$$\begin{aligned}
x_1 \mathbin{\widetilde{\wedge}} 1 &= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(1)) \\
&= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1),\, \{*\}) \\
&= \mathcal{F}(\wedge^*)\triangleleft\{*\}(\widetilde{\pi}_*^{-1}(x_1)) \\
&= \mathcal{F}(\wedge^*\triangleleft\{*\})(\widetilde{\pi}_*^{-1}(x_1)) && \text{by (DFS 7)} \\
&= \mathcal{F}(\mathrm{id}_{\mathbf{2}}^*)(\widetilde{\pi}_*^{-1}(x_1)) \\
&= \widetilde{\mathrm{id}_2}(x_1) \\
&= x_1 && \text{by Th-2}
\end{aligned}$$

c. $x_1 \mathbin{\widetilde{\wedge}} x_2 = x_2 \mathbin{\widetilde{\wedge}} x_1$ by (DFS 4) and commutativity of $\wedge$.

d. Because $\wedge$ is monotonically increasing in both arguments, $\widetilde{\wedge}$ is also increasing in both arguments by Th-4

e. $\widetilde{\wedge}$ is associative.

To see this, let $x_1,\, x_2,\, x_3 \in \mathbf{I}$. We compute

$$\begin{aligned}
x_1 \mathbin{\widetilde{\wedge}} (x_2 \mathbin{\widetilde{\wedge}} x_3) &= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2 \mathbin{\widetilde{\wedge}} x_3)) \\
&= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2) \mathbin{\widetilde{\cap}} \widetilde{\pi}_*^{-1}(x_3)) && \text{by L-1} \\
&= (\mathcal{F}(\wedge^*)\widetilde{\cap})(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2),\, \widetilde{\pi}_*^{-1}(x_3)) \\
&= \mathcal{F}(\wedge^*\cap)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2),\, \widetilde{\pi}_*^{-1}(x_3)) && \text{by (DFS 6)} \\
&= \mathcal{F}(\wedge^*\,\tau_1\cap\tau_2\tau_1)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2),\, \widetilde{\pi}_*^{-1}(x_3)) && \text{by assoc of } \cap \\
&= \mathcal{F}(\wedge^*)\tau_1\widetilde{\cap}\tau_2\tau_1(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2),\, \widetilde{\pi}_*^{-1}(x_3)) && \text{by (DFS 4), (DFS 6)} \\
&= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1) \mathbin{\widetilde{\cap}} \widetilde{\pi}_*^{-1}(x_2),\, \widetilde{\pi}_*^{-1}(x_3)) \\
&= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1 \mathbin{\widetilde{\wedge}} x_2),\, \widetilde{\pi}_*^{-1}(x_3)) && \text{by L-1} \\
&= (x_1 \mathbin{\widetilde{\wedge}} x_2) \mathbin{\widetilde{\wedge}} x_3\,.
\end{aligned}$$

A.8 Proof of Theorem 7

a. $\vee^* = \neg(\wedge^*\,\tau_1\neg\tau_1\neg)$ by DeMorgan's law and hence

$$\begin{aligned}
x_1 \mathbin{\widetilde{\vee}} x_2 &= \mathcal{F}(\vee^*)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2) \\
&= \mathcal{F}(\neg(\wedge^*\,\tau_1\neg\tau_1\neg))(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2) \\
&= \widetilde{\neg}(\mathcal{F}(\wedge^*)\tau_1\widetilde{\neg}\tau_1\widetilde{\neg})(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2) && \text{by (DFS 3), (DFS 5), (DFS 4)} \\
&= \widetilde{\neg}\,\mathcal{F}(\wedge^*)(\widetilde{\neg}\,\widetilde{\pi}_*{}^{-1}(x_1),\, \widetilde{\neg}\,\widetilde{\pi}_*{}^{-1}(x_2)) \\
&= \widetilde{\neg}\,\mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(\widetilde{\neg}\,x_1),\, \widetilde{\pi}_*^{-1}(\widetilde{\neg}\,x_2)) && \text{by L-1} \\
&= \widetilde{\neg}(\widetilde{\neg}\,x_1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\,x_2)\,.
\end{aligned}$$

Because $\widetilde{\vee}$ is the dual of a t-norm under a strong negation, it follows that $\widetilde{\vee}$ is an s-norm.

b. $\rightarrow^* = \vee^* \tau_1 \neg \tau_1$ and hence

$$\begin{aligned}
x_1 \mathbin{\widetilde{\rightarrow}} x_2 &= \mathcal{F}(\rightarrow^*)(\widetilde{\pi}_*^{-1}(x_1), \widetilde{\pi}_*^{-1}(x_2)) \\
&= \mathcal{F}(\vee^* \tau_1 \neg \tau_1)(\widetilde{\pi}_*^{-1}(x_1), \widetilde{\pi}_*^{-1}(x_2)) \\
&= \mathcal{F}(\vee^*) \tau_1 \widetilde{\neg} \tau_1 (\widetilde{\pi}_*^{-1}(x_1), \widetilde{\pi}_*^{-1}(x_2)) && \text{by (DFS 7), (DFS 5)} \\
&= \mathcal{F}(\vee^*)(\widetilde{\neg}\, \widetilde{\pi}_*^{-1}(x_1), \widetilde{\pi}_*^{-1}(x_2)) \\
&= \mathcal{F}(\vee^*)(\widetilde{\pi}_*^{-1}(\widetilde{\neg}\, x_1), \widetilde{\pi}_*^{-1}(x_2)) && \text{by L-1} \\
&= \widetilde{\neg}\, x_1 \mathbin{\widetilde{\vee}} x_2 \,.
\end{aligned}$$

A.9 Proof of Theorem 8

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner ($n > 0$) and $\mathcal{F}$ a DFS. Then

$$\begin{aligned}
\mathcal{F}(D\cup) &= \mathcal{F}(D \neg \cap \neg \tau_n \neg \tau_n) \\
&= \mathcal{F}(D) \widetilde{\neg} \widetilde{\cap} \widetilde{\neg} \tau_n \widetilde{\neg} \tau_n && \text{by (DFS 4), (DFS 5) and (DFS 6)} \\
&= \mathcal{F}(D) \widetilde{\cup} \,.
\end{aligned}$$

A.10 Proof of Theorem 9

Lemma 3
Let $\mathcal{F}$ a DFS, $\widetilde{\triangle} : \widetilde{\mathcal{P}}(E)^2 \longrightarrow \widetilde{\mathcal{P}}(E)$ the fuzzy symmetrical difference induced by $\widetilde{\mathrm{xor}} = \widetilde{\mathcal{F}}(\mathrm{xor})$ and $A \in \mathcal{P}(E)$ a crisp subset of E. Then for every fuzzy subset $X \in \widetilde{\mathcal{P}}(E)$,

$$X \mathbin{\widetilde{\triangle}} A = (X \mathbin{\widetilde{\cap}} \widetilde{\neg}\, A) \mathbin{\widetilde{\cup}} (A \mathbin{\widetilde{\cap}} \widetilde{\neg}\, X) \,.$$

Proof Let $x \in E$. Then

$$\begin{aligned}
\mu_{X \widetilde{\triangle} A}(x) &= \mu_X(x) \mathbin{\widetilde{\mathrm{xor}}} \mu_A(x) \\
&= \mu_X(x) \mathbin{\widetilde{\mathrm{xor}}} \chi_A(x) && \text{because } A \text{ is crisp}
\end{aligned}$$

Now if $x \notin A$, i.e. $\chi_A(x) = 0$, we compute

$$\begin{aligned}
\mu_X(x) \mathbin{\widetilde{\mathrm{xor}}} 0 &= \widetilde{\mathrm{id}_{\mathbf{2}}}(\mu_X(x)) && \text{by (DFS 7)} \\
&= \mu_X(x) && \text{by Th-2} \\
&= \mu_X(x) \mathbin{\widetilde{\wedge}} 1 && \widetilde{\wedge}\ t\text{-norm} \\
&= \mu_X(x) \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, 0 && \widetilde{\neg} \text{ strong negation} \\
&= (\mu_X(x) \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, 0) \mathbin{\widetilde{\vee}} 0 && \widetilde{\vee}\ s\text{-norm} \\
&= (\mu_X(x) \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, 0) \mathbin{\widetilde{\vee}} (\widetilde{\neg}\, \mu_X(x) \mathbin{\widetilde{\wedge}} 0) && \widetilde{\wedge}\ t\text{-norm} \\
&= \mu_{(X \widetilde{\cap} \widetilde{\neg} A) \widetilde{\cup} (A \widetilde{\cap} \widetilde{\neg} X)}(x) && \text{by L-1}
\end{aligned}$$

Likewise if $\chi_A(x) = 1$,

$$\begin{aligned}
\mu_X(x) \mathbin{\widetilde{\text{xor}}} 1 &= \widetilde{\neg}\, \mu_X(x) && \text{by (DFS 7)}\\
&= 1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, \mu_X(x) && \widetilde{\wedge}\ t\text{-norm}\\
&= 0 \mathbin{\widetilde{\vee}} (1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, \mu_X(x)) && \widetilde{\vee}\ s\text{-norm}\\
&= (\mu_X(x) \mathbin{\widetilde{\wedge}} 0) \mathbin{\widetilde{\vee}} (1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, \mu_X(x)) && \widetilde{\wedge}\ t\text{-norm}\\
&= (\mu_X(x) \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, 1) \mathbin{\widetilde{\vee}} (1 \mathbin{\widetilde{\wedge}} \widetilde{\neg}\, \mu_X(x)) && \widetilde{\neg}\ \text{strong negation}\\
&= \mu_{(X \widetilde{\cap} \widetilde{\neg} A) \widetilde{\cup} (A \widetilde{\cap} \widetilde{\neg} X)}(x) && \text{by L-1}
\end{aligned}$$

Hence $\mu_{X \widetilde{\triangle} A}(x) = \mu_X(x) \mathbin{\widetilde{\text{xor}}} A = \mu_{(X \widetilde{\cap} \widetilde{\neg} A) \widetilde{\cup} (A \widetilde{\cap} \widetilde{\neg} X)}(x)$ for all $x \in E$, i.e. $X \mathbin{\widetilde{\triangle}} A = (X \mathbin{\widetilde{\cap}} \widetilde{\neg}\, A) \mathbin{\widetilde{\cup}} (A \mathbin{\widetilde{\cap}} \widetilde{\neg}\, X)$, as desired.

Proof of Theorem 9

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner ($n > 0$) and $A \in \mathcal{P}(E)$ a crisp subset of E. Define $D' : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ by

$$D'(X_1, \ldots, X_{n+1}) = D(X_1, \ldots, X_{n-1}, (X_n \cap \neg A) \cup (A \cap \neg X_{n+1}))\,, \tag{8}$$

for all $X_1, \ldots, X_{n+1} \in \mathcal{P}(E)$. Then

$$D'(X_1, \ldots, X_{n+1}) = (D \triangle A)(X_1, \ldots, X_{n-1}, (X_n \cap \neg A) \cup (X_{n+1} \cap A)) \tag{9}$$

for all $X_1, \ldots, X_{n+1} \in \mathcal{P}(E)$ because

$$\begin{aligned}
&(D \triangle A)(X_1, \ldots, X_{n-1}, (X_n \cap \neg A) \cup (X_{n+1} \cap A))\\
&= D(X_1, \ldots, X_{n-1}, [\neg((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap A]\\
&\quad \cup [((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap \neg A]) && \text{by L-3}\\
&= D(X_1, \ldots, X_{n-1}, (X_n \cap \neg A) \cup (A \cap \neg X_{n+1})) && (*)
\end{aligned}$$

where (*) can be shown by the following computation.

$$\begin{aligned}
&[\neg((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap A] \cup [((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap \neg A]\\
&= [(\neg(X_n \cap \neg A) \cap \neg(A \cap X_{n+1})) \cap A]\\
&\quad \cup [((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap \neg A] && \text{by DeMorgan's law}\\
&= [((\neg X_n \cup A) \cap (\neg A \cup \neg X_{n+1})) \cap A]\\
&\quad \cup [((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap \neg A] && \text{by DeMorgan's law}\\
&= [((\neg X_n \cup A) \cap A) \cap ((\neg A \cup \neg X_{n+1}) \cap A)]\\
&\quad \cup [((X_n \cap \neg A) \cup (A \cap X_{n+1})) \cap \neg A]\\
&= [((\neg X_n \cap A) \cup (A \cap A)) \cap ((\neg A \cap A) \cup (\neg X_{n+1} \cap A))\\
&\quad \cup [(X_n \cap \neg A \cap \neg A) \cup (A \cap X_{n+1} \cap \neg A)] && \text{by distributivity}\\
&= (A \cap \neg X_{n+1}) \cup (X_n \cap \neg A)\,,
\end{aligned}$$

where the last equation holds because idempotence, absorption and law of excluded middle are valid in this case (all involved sets are crisp).

Noting that D' is obtained from D by constructions with which $\mathcal{F}$ commutes, we can prove

$$\begin{aligned}
\mathcal{F}(D\triangle A)(X_1,\ \dots,\ X_n) &= \mathcal{F}(D\triangle A)(X_1,\ \dots,\ X_{n-1},\ (X_n \mathbin{\widetilde{\cap}} \mathbin{\widetilde{\neg}} A) \mathbin{\widetilde{\cup}} (X_n \mathbin{\widetilde{\cap}} A)) && \text{because } A \text{ crisp}\\
&= \mathcal{F}(D')(X_1,\ \dots,\ X_{n-1},\ X_n,\ X_n) && \text{by (9)}\\
&= \mathcal{F}(D)(X_1,\ \dots,\ X_{n-1},\ (X_n \mathbin{\widetilde{\cap}} \mathbin{\widetilde{\neg}} A) \mathbin{\widetilde{\cup}} (A \mathbin{\widetilde{\cap}} \mathbin{\widetilde{\neg}} X_n)) && \text{by (8)}\\
&= \mathcal{F}(D)(X_1,\ \dots,\ X_{n-1},\ X_n \mathbin{\widetilde{\triangle}} A) && \text{by L-3}\\
&= \mathcal{F}(D)\widetilde{\triangle}A(X_1,\ \dots,\ X_n)\,.
\end{aligned}$$

A.11 Proof of Theorem 10

Suppose $\mathcal{F}$ satisfies

$$\mathcal{F}(\pi_*) = \widetilde{\pi}_* \tag{10}$$

and

$$\mathcal{F}(\neg' D) = \neg' \mathcal{F}(D) \tag{11}$$

for every fuzzy pre-determiner D. Then

$$\begin{aligned}
\mathcal{F}(\neg^*) &= \mathcal{F}(\neg' \circ \pi_*) && \text{(def. of } \neg^*)\\
&= \mathcal{F}(\neg' \pi_*) && \text{(def. of external negation)}\\
&= \neg' \mathcal{F}(\pi_*) && \text{by (11)}\\
&= \neg' \widetilde{\pi}_* && \text{by (10)}
\end{aligned}$$

and hence for all $x \in \mathbf{I}$,

$$\begin{aligned}
\widetilde{\neg}\, x &= \mathcal{F}(\neg^*)(\widetilde{\pi}_*^{-1}(x))\\
&= \neg' \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(x))\\
&= \neg' x\,.
\end{aligned}$$

A.12 Proof of Theorem 11

Suppose $\mathcal{F}$ satisfies

$$\mathcal{F}(\pi_*) = \widetilde{\pi}_* \tag{12}$$

and let $\wedge' : \mathbf{I}^2 \longrightarrow \mathbf{I}$ such that for every fuzzy pre-determiner D with arity $n > 0$,

$$\mathcal{F}(D\cap) = \mathcal{F}(D)\cap'\,, \tag{13}$$

where $\cap'$ is the fuzzy intersection induced by $\wedge'$. Then for all $a,\ b \in \mathbf{I}$,

$$\begin{aligned}
a \mathbin{\widetilde{\wedge}} b &= \mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(a),\ \widetilde{\pi}_*^{-1}(b))\\
&= \mathcal{F}(\pi_*\cap)(\widetilde{\pi}_*^{-1}(a),\ \widetilde{\pi}_*^{-1}(b))\\
&= \mathcal{F}(\pi_*)\cap'(\widetilde{\pi}_*^{-1}(a),\ \widetilde{\pi}_*^{-1})(b)) && \text{by (13)}\\
&= \mathcal{F}(\pi_*)(\widetilde{\pi}_*^{-1}(a) \cap' \widetilde{\pi}_*^{-1}(b))\\
&= \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(a) \cap' \widetilde{\pi}_*^{-1}(b)) && \text{by (12)}\\
&= \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(a \wedge' b)) && \text{by L-1}\\
&= a \wedge' b\,.
\end{aligned}$$

A.13 Proof of Theorem 12

"$\Rightarrow$" Firstly, if D is increasing in the range $(U,\ V)$, then the fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ defined by

$$D'(X_1,\ \ldots,\ X_n) = D((X_1 \cup U_1) \cap V_1,\ \ldots,\ (X_n \cup U_n) \cap V_n)$$

for all $(X_1,\ \ldots,\ X_n) \in \mathcal{P}(E)^n$, is increasing in all arguments. By Th-4, $\mathcal{F}(D')$ is increasing in all arguments. Furthermore,

$$\mathcal{F}(D')(X_1,\ \ldots,\ X_n) = \mathcal{F}(D)((X_1 \mathbin{\widetilde{\cup}} U_1) \mathbin{\widetilde{\cap}} V_1,\ \ldots,\ (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) \tag{14}$$

by (DFS 6), (DFS 7), (DFS 4) and Th-8.

Now let $(X_1,\ \ldots,\ X_n), (X'_1,\ \ldots,\ X'_n) \in \widetilde{\mathcal{P}}(E)^n$ such that $U_i \subseteq X_i \subseteq X'_i \subseteq V_i$ for all $i = 1,\ \ldots,\ n$. Then

$$(X_i \mathbin{\widetilde{\cup}} U_i) \mathbin{\widetilde{\cap}} V_i = X_i$$
$$(X'_i \mathbin{\widetilde{\cup}} U_i) \mathbin{\widetilde{\cap}} V_i = X'_i$$

for $i = 1,\ \ldots,\ n$ (because $U_i,\ V_i$ are crisp!) and hence

$$\begin{aligned}
\mathcal{F}(D)(X_1,\ \ldots,\ X_n) &= \mathcal{F}(D)((X_1 \mathbin{\widetilde{\cup}} U_1) \mathbin{\widetilde{\cap}} V_1,\ \ldots,\ (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) \\
&= \mathcal{F}(D')(X_1,\ \ldots,\ X_n) && \text{by (14)} \\
&\le \mathcal{F}(D')(X'_1,\ \ldots,\ X'_n) \\
&= \mathcal{F}(D)((X'_1 \mathbin{\widetilde{\cup}} U_1) \mathbin{\widetilde{\cap}} V_1,\ \ldots,\ (X'_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) && \text{by (14)} \\
&= \mathcal{F}(D)(X'_1,\ \ldots,\ X'_n)\,.
\end{aligned}$$

(Likewise for decreasing monotonicity.)

"$\Leftarrow$" The converse direction is an immediate consequence of Th-1.

A.14 Proof of Theorem 13

"$\Rightarrow$" Let $D,\ D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ fuzzy pre-determiners and $D \le D'$. Define $D'' : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ by

$$D''(X_1,\ \ldots,\ X_{n+1}) = \begin{cases} D(X_1,\ \ldots,\ X_n) & : \quad X_{n+1} = \varnothing \\ D'(X_1,\ \ldots,\ X_n) & : \quad X_{n+1} \neq \varnothing \end{cases}$$

Obviously, $D'' \triangleleft \varnothing = D$, $D'' \triangleleft E = D'$. Furthermore, D'' is monotonically increasing in its last argument (because $D \le D'$) and hence by Th-4, $\mathcal{F}(D'')$ is also increasing in its last argument. Therefore,

$$\begin{aligned}
\mathcal{F}(D)(X_1,\ \ldots,\ X_n) &= \mathcal{F}(D'' \triangleleft \varnothing)(X_1,\ \ldots,\ X_n) \\
&= \mathcal{F}(D'') \triangleleft \varnothing(X_1,\ \ldots,\ X_n) && \text{by (DFS 7)} \\
&= \mathcal{F}(D'')(X_1,\ \ldots,\ X_n,\ \varnothing) \\
&\le \mathcal{F}(D'')(X_1,\ \ldots,\ X_n,\ E) && \text{because } \mathcal{F}(D'') \text{ incr. in last arg} \\
&= \mathcal{F}(D'') \triangleleft E(X_1,\ \ldots,\ X_n) \\
&= \mathcal{F}(D'' \triangleleft E)(X_1,\ \ldots,\ X_n) && \text{by (DFS 7)} \\
&= \mathcal{F}(D')(X_1,\ \ldots,\ X_n)\,.
\end{aligned}$$

"$\Leftarrow$" The converse implication is a direct consequence of Th-1.

A.15 Proof of Theorem 14

"$\Rightarrow$" $D_1, D_2 : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ and $U, V \in \mathcal{P}(E)^n$ such that $D_1 \leq_{(U,V)} D_2$.
Define $D_1^*, D_2^* : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ by

$$\begin{aligned} D_1^*(X_1, \ldots, X_n) &= D_1((X_1 \cup U_1) \cap V_1, \ldots, (X_n \cup U_n) \cap V_n) \\ D_2^*(X_1, \ldots, X_n) &= D_2((X_1 \cup U_1) \cap V_1, \ldots, (X_n \cup U_n) \cap V_n) \end{aligned}$$

Then $D_1^* \leq D_2^*$ unconditionally. In addition by (DFS 6), (DFS 7), (DFS 4) and Th-8,

$$\mathcal{F}(D_1^*)(X_1, \ldots, X_n) = \mathcal{F}(D_1)((X_1 \mathbin{\widetilde{\cup}} U_1 \mathbin{\widetilde{\cap}} V_1), \ldots, (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) \tag{15}$$
$$\mathcal{F}(D_2^*)(X_1, \ldots, X_n) = \mathcal{F}(D_2)((X_1 \mathbin{\widetilde{\cup}} U_1 \mathbin{\widetilde{\cap}} V_1), \ldots, (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) \tag{16}$$

for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

Now let $(X_1, \ldots, X_n), (X_1', \ldots, X_n') \in \widetilde{\mathcal{P}}(E)^n$ such that $U_i \subseteq X_i \subseteq X_i' \subseteq V_i$ for all $i = 1, \ldots, n$. Then

$$\begin{aligned} (X_i \mathbin{\widetilde{\cup}} U_i) \mathbin{\widetilde{\cap}} V_i &= X_i \\ (X_i' \mathbin{\widetilde{\cup}} U_i) \mathbin{\widetilde{\cap}} V_i &= X_i' \end{aligned}$$

for $i = 1, \ldots, n$ (because U_i, V_i are crisp!) and hence

$$\begin{aligned} \mathcal{F}(D_1)(X_1, \ldots, X_n) &= \mathcal{F}(D_1)((X_1 \mathbin{\widetilde{\cup}} U_1) \mathbin{\widetilde{\cap}} V_1, \ldots, (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) \\ &= \mathcal{F}(D_1^*)(X_1, \ldots, X_n) && \text{by (15)} \\ &\leq \mathcal{F}(D_2^*)(X_1, \ldots, X_n) && \text{by Th-13} \\ &= \mathcal{F}(D_2)((X_1 \mathbin{\widetilde{\cup}} U_1) \mathbin{\widetilde{\cap}} V_1, \ldots, (X_n \mathbin{\widetilde{\cup}} U_n) \mathbin{\widetilde{\cap}} V_n) && \text{by (16)} \\ &= \mathcal{F}(D_2)(X_1, \ldots, X_n) . \end{aligned}$$

"$\Leftarrow$" The converse implication again follows from Th-1.

A.16 Proof of Theorem 15

We will first consider the case that $n = 1$. The lemma is stated a bit more general than needed here but in this formulation, it will later prove useful in another context.

Lemma 4 *Suppose the determiner fuzzification mechanism $\mathcal{F}$,*

$$(D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}) \mapsto (\mathcal{F}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \mathbf{I})$$

satisfies (DFS 2)*,* (DFS 6) *and* (DFS 7)*, and suppose that $\mathcal{F}(\wedge)$ is a t-norm.*
Further suppose that $\mathcal{E}$, $(f : E \longrightarrow E') \mapsto (\mathcal{E}(f) : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E'))$
is an extension principle which satisfies

$$\mu_{\mathcal{E}(c_v)(X)}(v) = \mu_X(v) = \widetilde{\pi}_*(X) \tag{17}$$

for all constants $c_v : \{*\} \longrightarrow V$, $c_v(*) = v$ *(*$v \in V$*) and all* $X \in \widetilde{\mathcal{P}}(\{*\})$*, and that* $\mathcal{E}$ *is compatible with* $\mathcal{F}$ *in the sense that for all fuzzy pre-determiners* $D : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ *and* $f_1 : E \longrightarrow E'$, $\dots$, $f_n : E \longrightarrow E'$,

$$\mathcal{F}(D \circ \mathop{\times}_{i=1}^{n} \widehat{f_i}) = \mathcal{F}(D) \circ \mathop{\times}_{i=1}^{n} \mathcal{E}(f_i) \tag{18}$$

Then for all $V \neq \varnothing$, $v \in V$,

$$\mathcal{F}(\pi_v) = \widetilde{\pi}_v \,.$$

Proof Noting that for all $X \in \mathcal{P}(E)$,

$$\begin{aligned} \pi_v(X) &= \pi_v(X \cap \{v\}) \\ &= \pi_v \cap\!\triangleleft \{v\}\,, \end{aligned}$$

(DFS 6) and (DFS 7) yield

$$\mathcal{F}(\pi_v)(X) = \mathcal{F}(\pi_v)(X \mathbin{\widetilde{\cap}} \{v\})\,. \tag{19}$$

Let $g : \mathbf{I} \longrightarrow \widetilde{\mathcal{P}}(E)$ the function which to every $a \in \mathbf{I}$ assigns the fuzzy subset $g(a)$ defined by the membership function

$$\mu_{g(a)}(x) = \begin{cases} a & : \quad x = v \\ 0 & : \quad \text{else} \end{cases} \tag{20}$$

Then by the assumption that $\widetilde{\wedge} = \widetilde{\mathcal{F}}(\wedge)$ be a t-norm,

$$X \mathbin{\widetilde{\cap}} \{v\} = g(\mu_X(v))$$

i.e.

$$X \mathbin{\widetilde{\cap}} \{v\} = (g \circ \widetilde{\pi}_v)(X)\,. \tag{21}$$

Summarizing, $\mathcal{F}(\pi_v)$ is a function of $\widetilde{\pi}_v$ because for all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned} \mathcal{F}(\pi_v)(X) &= \mathcal{F}(\pi_v)(X \mathbin{\widetilde{\cap}} \{v\}) && \text{by (19)} \\ &= \mathcal{F}(\pi_v)(g(\widetilde{\pi}_v(X))) && \text{by (21)} \\ &= (\mathcal{F}(\pi_v) \circ g)(\widetilde{\pi}_v(X)) \\ &= (f \circ \widetilde{\pi}_v)(X), \end{aligned}$$

where $f : \mathbf{I} \longrightarrow \mathbf{I}$ is defined by $f = \mathcal{F}(\pi_v) \circ g$.
We will now show that $f = \mathrm{id}_{\mathbf{I}}$.
Let $c_v : \{*\} \longrightarrow E$ the constant $* \mapsto c_v(*) = v$. Obviously, $\pi_* = \pi_v \circ \widehat{c_v}$ and hence

$$\begin{aligned} \widetilde{\pi}_* &= \mathcal{F}(\pi_*) && \text{by (DFS 2)} \\ &= \mathcal{F}(\pi_v \circ \widehat{c_v}) \\ &= \mathcal{F}(\pi_v) \circ \mathcal{E}(c_v) && \text{by assumption (18)} \\ &= f \circ \widetilde{\pi}_v \circ \mathcal{E}(c_v) && (*) \end{aligned}$$

Now let $a \in \mathbf{I}$ and $X = \widetilde{\pi}_*^{-1}(a)$, i.e. $\widetilde{\pi}_*(X) = \mu_X(*) = a$. Then

$$\begin{aligned} a &= \widetilde{\pi}_v(X) \\ &= (f \circ \widetilde{\pi}_v \circ \mathcal{E}(c_v))(X)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} (\widetilde{\pi}_v \circ \mathcal{E}(c_v))(X) &= \mu_{\mathcal{E}(c_v)(X)}(v) \\ &= \widetilde{\pi}_*(X) && \text{by assumption (17)} \\ &= a && \text{(+)} \end{aligned}$$

Combining these results, we obtain

$$\begin{aligned} \mathrm{id}_{\mathbf{I}}(a) &= \widetilde{\pi}_*(X) \\ &= (f \circ \widetilde{\pi}_v \circ \mathcal{E}(c_v))(X) && \text{by (*)} \\ &= f((\widetilde{\pi}_v \circ \mathcal{E}(c_v))(X)) \\ &= f(a) && \text{by (+)}\,, \end{aligned}$$

i.e. $f = \mathrm{id}_{\mathbf{I}}$.

Lemma 5 *Let $\mathcal{F}$ a DFS, $E \neq \varnothing$ a base set and $v \in E$. Then $\mathcal{F}(\pi_v) = \widetilde{\pi}_v$.*

Proof The conditions of L-4 are fulfilled: by Th-6, $\widetilde{\wedge}$ is a t-norm, and by (DFS 9), $\mathcal{F}$ combines with $\mathcal{E} = \widehat{\mathcal{F}}$. Furthermore, $\widehat{\mathcal{F}}$ has the required property that

$$\mu_{\widehat{\mathcal{F}}(c_v)(X)}(v) = \widetilde{\pi}_*(X)\,,$$

because

$$\chi_{\widehat{c_v}(X)}(v) = \pi_*(X) \tag{22}$$

for all $X \subseteq \{*\}$ and hence

$$\begin{aligned} \mu_{\widehat{\mathcal{F}}(c_v)(X)}(v) &= \mathcal{F}(\chi_{\widehat{c_v}(\_)}(v))(X) \\ &= \mathcal{F}(\pi_*)(X) && \text{by (22)} \\ &= \widetilde{\pi}_*(X) && \text{by (DFS 2)} \end{aligned}$$

Proof of Theorem 15

Suppose $\mathcal{F}$ is a DFS, $E \neq \varnothing$ a base set and $n \in \mathbb{N}$. Furthermore let $v \in E$ and $i \in \{1, \ldots, n\}$. Noting that $\pi_v^{n,i}$ and $\widetilde{\pi}_v^{n,i}$ depend only on their i-th argument, we obtain for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$\begin{aligned} \mathcal{F}(\pi_v^{n,i})(X_1, \ldots, X_n) &= \mathcal{F}(\pi_v)(X_i) && \text{by Th-5} \\ &= \widetilde{\pi}_v(X_i) && \text{by L-5} \\ &= \widetilde{\pi}_v^{n,i}(X_1, \ldots, X_n)\,. \end{aligned}$$

A.17 Proof of Theorem 16

a. Suppose E, E', E'' are non-empty sets and $f : E \longrightarrow E'$, $g : E' \longrightarrow E''$. Firstly, let us recall that $\widehat{f} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E')$, $\widehat{g} : \mathcal{P}(E') \longrightarrow \mathcal{P}(E'')$ satisfy

$$\widehat{g \circ f} = \widehat{g} \circ \widehat{f}\,. \tag{23}$$

Secondly, let us note that for every $h : E \longrightarrow E''$, $v \in E''$,

$$\chi_{\widehat{h}}(v) = \pi_v \circ \widehat{h} \tag{24}$$

because

$$\begin{aligned} \chi_{\widehat{h}(X)}(v) &= \pi_v(\widehat{h}(X)) && \text{by definition of } \pi_v \\ &= (\pi_v \circ h)(X), \end{aligned}$$

for all $X \in \mathcal{P}(E)$.
Therefore,

$$\begin{aligned} \mu_{\widehat{\mathcal{F}}(g \circ f)}(v) &= \mathcal{F}(\chi_{\widehat{g \circ f}}(v)) \\ &= \mathcal{F}(\pi_v \circ \widehat{g \circ f}) && \text{by (24)} \\ &= \mathcal{F}(\pi_v \circ (\widehat{g} \circ \widehat{f})) && \text{by (23)} \\ &= \mathcal{F}((\pi_v \circ \widehat{g}) \circ \widehat{f}) && \text{by associativity of } \circ \\ &= \mathcal{F}(\pi_v \circ \widehat{g}) \circ \widehat{\mathcal{F}}(f) && \text{by (DFS 9)} \\ &= \mathcal{F}(\pi_v) \circ \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f) && \text{by (DFS 9)} \\ &= \widetilde{\pi}_v \circ \widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f) && \text{by Th-15} \\ &= \mu_{\widehat{\mathcal{F}}(g) \circ \widehat{\mathcal{F}}(f)}(v)\,. && \text{by definition of } \widetilde{\pi}_v \end{aligned}$$

b. Suppose E is a non-empty set. Then for all $X \in \widetilde{\mathcal{P}}(E)$, $v \in E$,

$$\begin{aligned} \mu_{\widehat{\mathcal{F}}(\mathrm{id}_E)(X)}(v) &= \mathcal{F}(\chi_{\widehat{\mathrm{id}}_E}(()v))(X) \\ &= \mathcal{F}(\pi_v)(X) \\ &= \widetilde{\pi}_v(X) && \text{by Th-15} \\ &= \mu_X(v)\,, \end{aligned}$$

i.e. $\widehat{\mathrm{id}_E}(X) = X = \mathrm{id}_{\widetilde{\mathcal{P}}(E)}(X)$.

A.18 Proof of Theorem 17

Let $f : E \longrightarrow E'$ an injection, $X \in \widetilde{\mathcal{P}}(E)$ and $v \in E'$.
By definition of $\widehat{\mathcal{F}}$,

$$\mu_{\widehat{\mathcal{F}}(f)(X)}(v) = \mathcal{F}(\chi_{\widehat{f}(\_)}(v))(X)\,.$$

Now if $Y \in \mathcal{P}(E)$ is a crisp subset of E, we obviously have

$$\begin{aligned} \chi_{\widehat{f}(Y)}(v) &= \begin{cases} \chi_Y(f^{-1}(v)) & : \quad v \in \operatorname{Im} f \\ 0 & : \quad v \notin \operatorname{Im} f \end{cases} \\ &= \begin{cases} \pi_{f^{-1}(v)}(Y) & : \quad v \in \operatorname{Im} f \\ 0 & : \quad v \notin \operatorname{Im} f \end{cases} \end{aligned}$$

a. $v \in \operatorname{Im} f$. Then

$$\begin{aligned}\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(X) \\ &= \mathcal{F}(\pi_{f^{-1}(v)})(X) \\ &= \widetilde{\pi}_{\pi_{f^{-1}(v)}}(X) && \text{by Th-15} \\ &= \mu_X(f^{-1}(v))\,.\end{aligned}$$

b. $v \notin \operatorname{Im} f$. Then

$$\begin{aligned}\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(X) \\ &= \mathcal{F}(0)(X) \\ &= 0 && \text{by Th-5 and (DFS 1)}\end{aligned}$$

A.19 Proof of Theorem 18

"⇒" Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ quantitative, $\beta : E \longrightarrow E$ a permutation of E and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then,

$$\begin{aligned}\mathcal{F}(D)(\hat{\hat{\beta}}(X_1), \ldots, \hat{\hat{\beta}}(X_n)) &= \mathcal{F}(D)(\widehat{\mathcal{F}}(\beta)(X_1), \ldots, \widehat{\mathcal{F}}(\beta)(X_n)) && \text{by Th-17} \\ &= (\mathcal{F}(D \circ \mathop{\times}_{i=1}^{n} \widehat{\beta}))(X_1, \ldots, X_n) && \text{by (DFS 9)} \\ &= \mathcal{F}(D)(X_1, \ldots, X_n)\,. && D \text{ quantitative}\end{aligned}$$

"⇐" The converse implication is immediate from Th-1.

A.20 Proof of Theorem 19

Suppose $\mathcal{F}$ is a DFS and $\mathcal{E}$ an extension principle which is compatible to $\mathcal{F}$. Let $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $v \in E'$. Then

$$\begin{aligned}\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(X) \\ &= \mathcal{F}(\pi_v \circ \widehat{f})(X) \\ &= (\mathcal{F}(\pi_v) \circ \mathcal{E}(f))(X) && \text{by assumption} \\ &= (\widetilde{\pi}_v \circ \mathcal{E}(f))(X) && \text{by Th-15} \\ &= \mu_{\mathcal{E}(f)(X)}(v)\end{aligned}$$

i.e. $\mathcal{E}$ and $\widehat{\mathcal{F}}$ coincide.

A.21 Proof of Theorem 20

Suppose $\mathcal{F}$ satisfies (DFS 1-8) and $\mathcal{E}$ is an extension principle which is compatible to $\mathcal{F}$ and respects

$$\mu_{\mathcal{E}(c_v)(\text{-})}(v) = \widetilde{\pi}_*$$

for all constants $c_v : \{*\} \longrightarrow E'$.

In particular, the conditions of L-4 are fulfilled (the lemma requires that $\widetilde{\mathcal{F}}(\wedge)$ be a t-norm, but the proof of this depends on (DFS 1-8) only).
Now let $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $v \in E'$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(X) \\
&= \mathcal{F}(\pi_v \circ \widehat{f})(X) \\
&= (\mathcal{F}(\pi_v) \circ \mathcal{E}(f))(X) && \text{by assumption} \\
&= (\widetilde{\pi}_v \circ \mathcal{E}(f))(X) && \text{by L-4} \\
&= \mu_{\mathcal{E}(f)(X)}(v)
\end{aligned}$$

i.e. $\mathcal{E}$ and $\widehat{\mathcal{F}}$ coincide.
In particular, $\mathcal{F}$ satisfies (DFS 9) (which is obtained from the compatibility condition on $\mathcal{E}$ if $\widehat{\mathcal{F}} = \mathcal{E}$).

A.22 Proof of Theorem 21

Suppose $\mathcal{F}$ is a DFS and $f : E \longrightarrow E'$. Firstly let us note that for all crisp sets $Y \in \mathcal{P}(E)$ and $v \in E'$,

$$\begin{aligned}
\chi_{\widehat{f}(Y)}(v) &= \begin{cases} 1 & : \text{ there exists } y \in Y \text{ such that } v = f(y) \\ 0 & : \text{ there exists no } y \in Y \text{ such that } v = f(y) \end{cases} \\
&= \begin{cases} 1 & : f^{-1}(v) \cap Y \neq \varnothing \\ 0 & : \text{else} \end{cases} \\
&= \exists(Y \cap f^{-1}(v)) \\
&= \exists \cap \triangleleft f^{-1}(v)(Y)\,,
\end{aligned}$$

i.e. $\chi_{\widehat{f}}(v) = \exists \cap \triangleleft f^{-1}(v)$.
Now if $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E and $v \in E'$, we obtain

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(f)(X)}(v) &= \mathcal{F}(\chi_{\widehat{f}}(v))(X) \\
&= \mathcal{F}(\exists \cap \triangleleft f^{-1}(v))(X) \\
&= \mathcal{F}(\exists) \widetilde{\cap} \triangleleft f^{-1}(v)(X) && \text{by (DFS 6) and (DFS 7)} \\
&= \mathcal{F}(\exists)(X \,\widetilde{\cap}\, f^{-1}(v))
\end{aligned}$$

i.e. $\mu_{\widehat{\mathcal{F}}(f)}(v) = \mathcal{F}(\exists) \widetilde{\cap} \triangleleft f^{-1}(v)$, which we wanted to prove.

A.23 Proof of Theorem 22

Let $\mathcal{F}$ a DFS, E any non-empty set and $\exists = \exists_E : \mathcal{P}(E) \longrightarrow \mathbf{2}$, i.e. $\exists = \neg\chi_\varnothing$. Furthermore, let $! = !_E$ the unique mapping $! : E \longrightarrow \{*\}$, i.e. defined by

$$!(x) = *$$

for all $x \in E$.

Then for every crisp subset $Y \in \mathcal{P}(E)$ of E,

$$\begin{aligned}
\exists(Y) &= \begin{cases} 1 & : \quad Y \neq \varnothing \\ 0 & : \quad Y = \varnothing \end{cases} \\
&= \begin{cases} 1 & : \quad \widehat{!}(Y) \neq \varnothing \\ 0 & : \quad \widehat{!}(Y) = \varnothing \end{cases} \\
&= \begin{cases} 1 & : \quad \widehat{!}(Y) = \{*\} \\ 0 & : \quad \widehat{!}(Y) = \varnothing \end{cases} && \text{because } !(Y) \subseteq \{*\} \\
&= \pi_*(\widehat{!}(Y)) \,,
\end{aligned}$$

i.e. $\exists = \pi_* \circ !$.
From this we can easiliy prove our claim:

$$\begin{aligned}
\mathcal{F}(\exists) &= \mathcal{F}(\pi_* \circ \widehat{!}) \\
&= \mathcal{F}(\pi_*) \circ \widehat{\mathcal{F}}(!) && \text{by (DFS 9)} \\
&= \widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!) \,. && \text{by (DFS 2)}
\end{aligned}$$

A.24 Proof of Theorem 23

Suppose $\mathcal{F}$ is a DFS, $f : E \longrightarrow E'$ is a function and $Y \in \widetilde{\mathcal{P}}(E')$ is a fuzzy subset of E'. Noting that for every crisp set $Z \in \mathcal{P}(E)$,

$$\begin{aligned}
\chi_{f^{-1}(Z)}(x) &= \begin{cases} 1 & : \quad f(x) \in Z \\ 0 & : \quad f(x) \notin Z \end{cases} \\
&= \pi_{f(x)}(Z)
\end{aligned}$$

for all $x \in E$, Th-15 immediately yields

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}^{-1}(f)(Y)}(x) &= \mathcal{F}(\chi_{f^{-1}(_)}(x))(Y) && \text{by definition} \\
&= \mathcal{F}(\pi_{f(x)})(Y) && \text{see above} \\
&= \widetilde{\pi}_{f(x)}(Y) && \text{by Th-15} \\
&= \mu_Y(f(x)) && \text{by def. of } \widetilde{\pi}_{f(x)} \\
&= \mu_{\hat{f}^{-1}(Y)}(x) \,,
\end{aligned}$$

i.e. $\widehat{\mathcal{F}}^{-1}(f)$ coincides with our definition of "standard" fuzzy inverse images, as desired.

A.25 Proof of Theorem 24

Lemma 6 *Suppose $\mathcal{F}$ is a DFS, E a non-empty set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $a \in E$. Let $c_a : \{*\} \longrightarrow E$ the constant $* \mapsto c_a(*) = a$.*
Then

$$\mu_{\widehat{\mathcal{F}}(c_a)(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_a(X)))}(v) = \begin{cases} \mu_X(a) & : \quad v = a \\ 0 & : \quad v \neq a \end{cases}$$

for all $v \in E$.

Proof Noting that $c_a : \{*\} \longrightarrow E$ is an injection and $\operatorname{Im} c_a = \{a\}$, Th-17 yields that

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}}(c_a)(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_a(X)))}(v) &= \begin{cases} \widetilde{\pi}_*(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_a(X))) & : \quad v = a \\ 0 & : \quad \text{else} \end{cases} \\
&= \begin{cases} \widetilde{\pi}_a(X) & : \quad v = a \\ 0 & : \quad \text{else} \end{cases} \\
&= \begin{cases} \mu_X(a) & : \quad v = a \\ 0 & : \quad \text{else} \end{cases}
\end{aligned}$$

which we intended to show.

Lemma 7

Suppose $\mathcal{F}$ is a DFS, E a non-empty set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E. Furthermore let $A \in \mathcal{P}(E)$ a finite subset of E, i.e. $A = \{a_1, \ldots, a_m\}$ where the a_i's are pairwise distinct. Then

$$X \mathbin{\widetilde{\cap}} A = \mathop{\widetilde{\cup}}_{i=1}^{m} \widehat{\mathcal{F}}(c_{a_i})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X)))\,.$$

Proof Let E a nonempty set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $A = \{a_1, \ldots, a_n\} \in \mathcal{P}(E)$ a finite set (where the a_i's are pairwise distinct). Then for all $v \in E$,

$$\begin{aligned}
\mu_{X \widetilde{\cap} A}(v) &= \mu_X(v) \mathbin{\widetilde{\wedge}} \mu_A(v) \\
&= \mu_X(v) \mathbin{\widetilde{\wedge}} \chi_A(v) && \text{because } A \text{ is crisp} \\
&= \begin{cases} \mu_X(v) & : \quad v \in A \\ 0 & : \quad v \notin A \end{cases} && \widetilde{\wedge} \text{ is } t\text{-norm by Th-6} \\
&= \mathop{\widetilde{\vee}}_{i=1}^{m} \begin{cases} \mu_X(a_i) & : \quad v = a_i \\ 0 & : \quad v \neq A \end{cases} && \widetilde{\vee} \text{ is } s\text{-norm by Th-7} \\
&= \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_{\widehat{\mathcal{F}}(c_{a_i})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X)))}(v)\,, && \text{by L-6}
\end{aligned}$$

i.e.

$$X \mathbin{\widetilde{\cap}} A = \mathop{\widetilde{\cup}}_{i=1}^{n} \widehat{\mathcal{F}}(c_{a_i})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X)))\,,$$

as desired.

Lemma 8

Let E a non-empty set and $a_1, \ldots, a_m \in E$ pairwise distinct elements of E. By $c_{a_i} : \{*\} \longrightarrow E$ denote the constant function $* \mapsto c_{a_i}(*) = a_i$. Then

$$\exists \underbrace{\cup \cdots \cup}_{m-1} \circ \mathop{\times}_{i=1}^{m} \widehat{c_{a_i}} = \pi_* \underbrace{\cup \cdots \cup}_{m-1}$$

Proof Let $(Y_1, \ldots, Y_m) \in \mathcal{P}(\{*\})^m$. Then

$$\begin{aligned}
(\exists \underbrace{\cup \cdots \cup}_{m-1} \circ \mathop{\times}_{i=1}^{m} \widehat{c_{a_i}})(Y_1, \ldots, Y_n) &= \exists(\bigcup_{i=1}^{m} \widehat{c_{a_i}}(Y_i)) \\
&= \begin{cases} 1 & : \bigcup_{i=1}^{m} \widehat{c_{a_i}}(Y_i) \neq \varnothing \\ 0 & : \bigcup_{i=1}^{m} \widehat{c_{a_i}}(Y_i) = \varnothing \end{cases} \\
&= \begin{cases} 1 & : \text{for one of the } i\text{'s, } \widehat{c_{a_i}}(Y_i) \neq \varnothing \\ 0 & : \text{else} \end{cases} \\
&= \begin{cases} 1 & : \text{for one of the } i\text{'s, } Y_i \neq \varnothing \\ 0 & : \text{else} \end{cases} \\
&= \begin{cases} 1 & : \bigcup_{i=1}^{m} Y_i \neq \varnothing \\ 0 & : \text{else} \end{cases} \\
&= \pi_*(\bigcup_{i=1}^{m} Y_i) \\
&= (\pi_* \underbrace{\cup \cdots \cup}_{m-1})(Y_1, \ldots, Y_m)\,.
\end{aligned}$$

Proof of Theorem 24

a. $\mathcal{F}(\exists)(\varnothing) = 0$ and $\mathcal{F}(\exists)(E) = 1$ are immediately obtained from $\exists(\varnothing) = 0$, $\exists(E) = 1$ by Th-1.

b. For all $v \in E$ and $Y \in \mathcal{P}(E)$ crisp,

$$\begin{aligned}
\exists(Y \cap \{v\}) &= \begin{cases} 1 & : Y \cap \{v\} \neq \varnothing \\ 0 & : Y \cap \{v\} = \varnothing \end{cases} \\
&= \begin{cases} 1 & : v \in Y \\ 0 & : v \notin Y \end{cases} \\
&= \pi_v(Y)\,,
\end{aligned}$$

i.e. $\exists \cap \triangleleft \{v\} = \pi_v$.
Now if $X \in \widetilde{\mathcal{P}}(E)$ is a fuzzy subset of E and $v \in E$, we obtain

$$\begin{aligned}
\mathcal{F}(\exists)(X \cap \{v\}) &= \mathcal{F}(\exists)\widetilde{\cap}\triangleleft\{v\}(X) \\
&= \mathcal{F}(\exists\cap\triangleleft\{v\})(X) && \text{by (DFS 7), (DFS 6)} \\
&= \mathcal{F}(\pi_v)(X) && \text{see above} \\
&= \widetilde{\pi}_v(X) && \text{by Th-15} \\
&= \mu_X(v)\,,
\end{aligned}$$

as desired.

c. Let E a non-empty set, $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of E and $A \in \mathcal{P}(E)$ a crisp finite subset of E, i.e. $A = \{a_1, \ldots, a_m\}$ where the a_i are pairwise distinct. Then

$$\begin{aligned}
\mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} A) &= \mathcal{F}(\exists)(\mathop{\widetilde{\cup}}_{i=1}^{m} \widehat{\mathcal{F}}(c_{a_i})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X)))) && \text{by L-7} \\
&= [\mathcal{F}(\exists)\underbrace{\widetilde{\cup}\cdots\widetilde{\cup}}_{m-1} \circ \mathop{\times}_{i=1}^{m} \widehat{\mathcal{F}}(c_{a_i})](\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_1}(X)), \ldots, \widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_m}(X))) \\
&= \mathcal{F}(\exists\underbrace{\cup\cdots\cup}_{m-1} \circ \mathop{\times}_{i=1}^{m} \widehat{c_{a_i}})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_1}(X)), \ldots, \widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_m}(X))) && \text{by Th-8, (DFS 9)} \\
&= \mathcal{F}(\pi_*\underbrace{\cup\cdots\cup}_{m-1})(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_1}(X)), \ldots, \widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_m}(X))) && \text{by L-8} \\
&= \mathcal{F}(\pi_*)\underbrace{\widetilde{\cup}\cdots\widetilde{\cup}}_{m-1}(\widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X)), \ldots, \widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X))) && \text{by Th-8} \\
&= \widetilde{\pi}_*(\mathop{\widetilde{\cup}}_{i=1}^{m} \widetilde{\pi}_*^{-1}(\widetilde{\pi}_{a_i}(X))) && \text{by Th-15} \\
&= \widetilde{\pi}_*\widetilde{\pi}_*^{-1}(\mathop{\widetilde{\vee}}_{i=1}^{m} \widetilde{\pi}_{a_i}(X)) && \text{by L-1} \\
&= \mathop{\widetilde{\vee}}_{i=1}^{m} \widetilde{\pi}_{a_i}(X) \\
&= \mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i)\,.
\end{aligned}$$

d. Let us abbreviate

$$\begin{aligned}
S &= \left\{ \mathop{\widetilde{\cup}}_{i=1}^{m} \mu_X(a_i) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\} \\
&= \left\{ \mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} A) : A = \{a_1, \ldots, a_m\} \in \mathcal{P}(E) \text{ finite, } a_i \neq a_j \text{ if } i \neq j \right\} && \text{by part } \mathbf{c.}
\end{aligned}$$

Now for *all* $A \in \mathcal{P}(E)$, it holds that

$$\mathcal{F}(\exists)(X) \geq \mathcal{F}(\exists)(X \mathbin{\widetilde{\cap}} A)$$

simply because $\mathcal{F}(\exists)$ is increasing in its argument (this follows from $\exists$ being increasing by Th-4) and $X \supseteq X \mathbin{\widetilde{\cap}} A$. In particular, $\mathcal{F}(\exists)(X) \geq s$ for all $s \in S$, i.e. $\mathcal{F}(\exists)$ is an upper bound on S, and therefore $\mathcal{F}(\exists)(X) \geq \sup S$, which finishes the proof.

A.26 Proof of Theorem 25

Suppose $\mathcal{F}$ is a DFS, $\widetilde{\vee} : \mathbf{I} \longrightarrow \mathbf{I}$ the induced fuzzy disjunction of $\mathcal{F}$ and $x_1, x_2 \in \mathbf{I}$. Let $X \in \widetilde{\mathcal{P}}(\{1, 2\})$ the fuzzy subset of $\{1, 2\}$ defined by the membership function

$$\begin{aligned}
\mu_X(1) &= x_1 \\
\mu_X(2) &= x_2
\end{aligned}$$

Then

$$\begin{aligned}
x_1 \mathbin{\widetilde{\vee}} x_2 &= \mu_X(1) \mathbin{\widetilde{\vee}} \mu_X(2) \\
&= \mathcal{F}(\exists)(X) && \text{by Th-24} \\
&= \widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!)(X)\,. && \text{by Th-22}
\end{aligned}$$

A.27 Proof of Theorem 26

All claims follow from the corresponding claims for $\exists$, noting that $\forall = \neg\exists\neg$ and $\mathcal{F}(\forall) = \widetilde{\neg}\,\mathcal{F}(\forall)\,\widetilde{\neg}$ by (DFS 3) and (DFS 5).

A.28 Proof of Theorem 27

Lemma 9
Suppose $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotonically increasing bijection. Then

a. $\sigma(0) = 0$

b. $\sigma(1) = 1.$

In particular, if $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2} \subset \mathbf{I}$ is a two-valued fuzzy pre-determiner, then

$$\sigma D = D\,.$$

Proof Trivial.

a. Suppose $\sigma(0) = s_0 > 0$. Then for all $x > 0$, $\sigma(x) \geq \sigma(0) = s_0$ by the monotonicity of σ. It follows that $\operatorname{Im}\sigma \subseteq [s_0,\, 1]$, which contradicts our assumption that σ be a bijection.

b. Analogous.

Lemma 10
Suppose $\mathcal{F}$ is a DFS and $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ a monotonically increasing bijection. Then

a. For all $x \in \mathbf{I}$, $\widetilde{\mathcal{F}^\sigma}(\neg)(x) = \sigma^{-1}(\widetilde{\mathcal{F}}(\neg)(\sigma(x)))$

b. For all $x_1,\, x_2 \in \mathbf{I}$, $\widetilde{\mathcal{F}^\sigma}(\wedge)(x_1,\, x_2) = \sigma^{-1}(\widetilde{\mathcal{F}}(\wedge)(\sigma(x_1),\, \sigma(x_2)))\,.$

Proof

a. Let $x \in \mathbf{I}$. Then

$$\begin{aligned}
\widetilde{\mathcal{F}^\sigma}(\neg)(x) &= \mathcal{F}^\sigma(\neg^*)(\widetilde{\pi}_*^{-1}(x)) \\
&= \sigma^{-1}\mathcal{F}(\sigma\,\neg^*)(\sigma\widetilde{\pi}_*^{-1}(x)) \\
&= \sigma^{-1}\mathcal{F}(\neg^*)(\sigma\widetilde{\pi}_*^{-1}(x)) && \text{by L-9} \\
&= \sigma^{-1}\mathcal{F}(\neg^*)(\widetilde{\pi}_*^{-1}(\sigma(x))) && \text{by L-1} \\
&= \sigma^{-1}(\widetilde{\mathcal{F}}(\neg)(\sigma(x)))\,.
\end{aligned}$$

b. Let $x_1,\, x_2 \in \mathbf{I}$. Then

$$\begin{aligned}
\widetilde{\mathcal{F}^\sigma}(\wedge)(x_1,\, x_2) &= \mathcal{F}^\sigma(\wedge^*)(\widetilde{\pi}_*^{-1}(x_1),\, \widetilde{\pi}_*^{-1}(x_2)) \\
&= \sigma^{-1}\mathcal{F}(\sigma \wedge^*)(\sigma\widetilde{\pi}_*^{-1}(x_1),\, \sigma\widetilde{\pi}_*^{-1}(x_2)) \\
&= \sigma^{-1}\mathcal{F}(\wedge^*)(\sigma\widetilde{\pi}_*^{-1}(x_1),\, \sigma\widetilde{\pi}_*^{-1}(x_2)) && \text{by L-9} \\
&= \sigma^{-1}\mathcal{F}(\wedge^*)(\widetilde{\pi}_*^{-1}(\sigma(x_1)),\, \widetilde{\pi}_*^{-1}(\sigma(x_2))) && \text{by L-1} \\
&= \sigma^{-1}\widetilde{\mathcal{F}}(\wedge)(\sigma(x_1),\, \sigma(x_2))\,.
\end{aligned}$$

Lemma 11
Suppose $\mathcal{F}$ is a DFS, $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ is a monotonically increasing bijection and $f : E \longrightarrow E'$ where $E,\, E'$ are non-empty sets. Then

$$\widehat{\mathcal{F}^\sigma}(f)(X) = \sigma^{-1}\widehat{\mathcal{F}}(f)(\sigma X)\,,$$

for all $X \in \widetilde{\mathcal{P}}(E)$.

Proof Let $f : E \longrightarrow E'$, $X \in \widetilde{\mathcal{P}}(E)$ and $v \in E'$. Then

$$\begin{aligned}
\mu_{\widehat{\mathcal{F}^\sigma}(f)(X)}(v) &= \mathcal{F}^\sigma(\chi_{\widehat{f}(\text{-})}(v))(X) \\
&= \sigma^{-1}\mathcal{F}(\sigma\chi_{\widehat{f}(\text{-})}(v))(\sigma X) \\
&= \sigma^{-1}\mathcal{F}(\chi_{\widehat{f}(\text{-})}(v))(\sigma X) && \text{by L-9} \\
&= \sigma^{-1}\mu_{\widehat{\mathcal{F}}(f)(\sigma X)}(v) \\
&= \mu_{\sigma^{-1}\widehat{\mathcal{F}}(f)(\sigma X)}(v)\,,
\end{aligned}$$

i.e. $\widehat{\mathcal{F}^\sigma}(f)(X) = \sigma^{-1}\widehat{\mathcal{F}}(f)(\sigma X)$.

Proof of Theorem 27

Preservation of constants (DFS 1) Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then

$$\begin{aligned}
\mathcal{F}^\sigma(D)(\varnothing) &= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma\varnothing) \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\varnothing) && \text{by pointwise def.} \\
&= \sigma^{-1}\sigma D(\varnothing) && \text{by (DFS 1): } \sigma D \in \mathcal{P}(E)^0 \\
&= D(\varnothing)\,.
\end{aligned}$$

Compatibility to π_* (DFS 2) Let $X \in \widetilde{\mathcal{P}}(\{*\})$. Then

$$\begin{aligned}
\mathcal{F}^\sigma(\pi_*)(X) &= \sigma^{-1}\mathcal{F}(\sigma\pi_*)(\sigma X) \\
&= \sigma^{-1}\mathcal{F}(\pi_*)(\sigma X) && \text{by L-9} \\
&= \sigma^{-1}\widetilde{\pi}_*(\sigma X) && \text{by (DFS 2)} \\
&= \sigma^{-1}\sigma\widetilde{\pi}_*(X) && \text{by L-1} \\
&= \widetilde{\pi}_*(X)\,.
\end{aligned}$$

External negation (DFS 3) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Abbreviating $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and $\widetilde{\neg}^\sigma = \widetilde{\mathcal{F}^\sigma}(\neg)$, we obtain

$$\begin{aligned}
\mathcal{F}^\sigma(\widetilde{\neg}^\sigma D)(X_1, \ldots, X_n) &= \sigma^{-1}\mathcal{F}(\sigma \widetilde{\neg}^\sigma D)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}\mathcal{F}(\sigma\sigma^{-1} \widetilde{\neg}\, \sigma D)(\sigma X_1, \ldots, \sigma X_n) && \text{by L-10} \\
&= \sigma^{-1}\mathcal{F}(\widetilde{\neg}\, \sigma D)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1} \widetilde{\neg}\, \mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n) && \text{by (DFS 3)} \\
&= \sigma^{-1} \widetilde{\neg}\, \sigma\sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n) && \text{by L-10} \\
&= \widetilde{\neg}^\sigma\, \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n) \\
&= \widetilde{\neg}^\sigma\, \mathcal{F}^\sigma(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

Argument transposition (DFS 4) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $i \in \{1, \ldots, n\}$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}
\mathcal{F}^\sigma(D\tau_i)(X_1, \ldots, X_n) &= \sigma^{-1}\mathcal{F}(\sigma D\tau_i)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}\sigma\mathcal{D}(()\sigma X_1, \ldots, \sigma X_{i-1}, \sigma X_n, \sigma X_{i+1}, \ldots, \sigma X_n) && \text{by (DFS 4)} \\
&= \mathcal{F}^\sigma(D)\tau_i(\sigma X_1, \ldots, \sigma X_n)\,.
\end{aligned}$$

Internal complementation (DFS 5) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. By $\widetilde{\neg}$ and $\widetilde{\neg}^\sigma$ we denote the fuzzy complements induced by the negation operators $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$ and and $\widetilde{\neg}^\sigma = \widetilde{\mathcal{F}^\sigma}(\neg)$, respectively. Then

$$\begin{aligned}
\mathcal{F}^\sigma(D\neg)(X_1, \ldots, X_n) &= \sigma^{-1}\mathcal{F}(\sigma D\neg)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_{n-1}, \widetilde{\neg}\, \sigma X_{n-1}) && \text{by (DFS 5)} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_{n-1}, \sigma\, \widetilde{\neg}^\sigma\, \sigma^{-1}\sigma X_{n-1}) && \text{by L-10} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_{n-1}, \sigma(\widetilde{\neg}^\sigma X_n)) \\
&= \mathcal{F}^\sigma(D)(X_1, \ldots, X_{n-1}, \widetilde{\neg}^\sigma X_n)\,.
\end{aligned}$$

Internal meets (DFS 6) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_{n+1}) \in \widetilde{\mathcal{P}}(E)^{n+1}$. By $\widetilde{\cap}$ we denote the fuzzy intersection induced by $\widetilde{\mathcal{F}}(\wedge)$ and by $\widetilde{\cap}^\sigma$ the fuzzy intersection induced by $\widetilde{\mathcal{F}^\sigma}(\wedge)$. Then

$$\begin{aligned}
\mathcal{F}^\sigma(D\cap)(X_1, \ldots, X_{n+1}) &= \sigma^{-1}\mathcal{F}(\sigma D\cap)(\sigma X_1, \ldots, \sigma X_{n+1}) \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_{n-1}, (\sigma X_n)\, \widetilde{\cap}\, (\sigma X_{n+1})) && \text{by (DFS 6)} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_{n-1}, \sigma(X_n\, \widetilde{\cap}^\sigma\, X_{n+1})) && \text{by L-10} \\
&= \mathcal{F}^\sigma(D)(X_1, \ldots, X_n\, \widetilde{\cap}^\sigma\, X_{n+1})\,.
\end{aligned}$$

Argument insertion (DFS 7) Let $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and $A \in \mathcal{P}(E)$. Then

$$\begin{aligned}
\mathcal{F}^\sigma(D\triangleleft A)(X_1, \ldots, X_n) &= \sigma^{-1}\mathcal{F}(\sigma D\triangleleft A)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n, A) && \text{by (DFS 7)} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n, \sigma A) && \text{by L-9, } A \text{ crisp} \\
&= \mathcal{F}^\sigma(D)(X_1, \ldots, X_n, A)\,.
\end{aligned}$$

Preservation of decreasing monotonicity (DFS 8) Suppose $\mathcal{F}$ is a DFS and $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is monotonically decreasing in its n-th argument. Then σD is decreasing in the n-th argument also because σ is monotonically increasing. Now let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $X_n \subseteq X'_n$. Again by the monotonicity of σ, it follows that $\sigma X_n \subseteq \sigma X'_n$. But $\mathcal{F}$ is a DFS and by (DFS 8),

$$\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n) \geq \mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X'_n)\,. \tag{25}$$

Now, as σ is increasing, $\sigma^{-1} : \mathbf{I} \longrightarrow \mathbf{I}$ is increasing and hence preserves inequalities. We therefore obtain from (25) that

$$\sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X_n) \geq \sigma^{-1}\mathcal{F}(\sigma D)(\sigma X_1, \ldots, \sigma X'_n)\,,$$

which we desired to show.

Functional application (DFS 9) Let $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $f_1, \ldots f_n : E \longrightarrow E'$, where $E, E' \neq \varnothing$. Furthermore, let $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}
&\mathcal{F}^\sigma(D \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})(X_1, \ldots, X_n) \\
&= \sigma^{-1}\mathcal{F}(\sigma D \circ \mathop{\times}_{i=1}^{n} \widehat{f_i})(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\widehat{\mathcal{F}}(f_1)(\sigma X_1), \ldots, \widehat{\mathcal{F}}(f_n)(\sigma X_n)) && \text{by (DFS 9)} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma\widehat{\mathcal{F}^\sigma}(f_1)(\sigma^{-1}\sigma X_1), \ldots, \sigma\widehat{\mathcal{F}^\sigma}(f_n)(\sigma^{-1}\sigma X_n)) && \text{by L-11} \\
&= \sigma^{-1}\mathcal{F}(\sigma D)(\sigma\widehat{\mathcal{F}^\sigma}(f_1)(X_1), \ldots, \sigma\widehat{\mathcal{F}^\sigma}(f_n)(X_n)) \\
&= \mathcal{F}^\sigma(D)(\widehat{\mathcal{F}^\sigma}(f_1)(X_1), \ldots, \widehat{\mathcal{F}^\sigma}(f_n)(X_n))\,.
\end{aligned}$$

A.29 Proof of Theorem 28

Let $\mathcal{F}$ a $\widetilde{\neg}$-DFS, $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ the generator of $\widetilde{\neg}$, i.e.

$$\widetilde{\neg}x = \sigma^{-1}(1 - \sigma(x))\,, \tag{26}$$

and $\mathcal{F}' = \mathcal{F}^{\sigma^{-1}}$. Let us note that from σ being a monotonically increasing bijection it follows that σ^{-1} is a monotonically increasing bijection also.

a. From L-10, we immediately obtain that

$$\begin{aligned}
\widetilde{\mathcal{F}^{\sigma^{-1}}}(\neg)(x) &= (\sigma^{-1})^{-1}(\widetilde{\neg}(\sigma^{-1}(x))) \\
&= (\sigma^{-1})^{-1}(\sigma^{-1}(1 - \sigma(\sigma^{-1}(x)))) && \text{by (26)} \\
&= \sigma(\sigma^{-1}(1 - \sigma(\sigma^{-1}(x)))) \\
&= 1 - x\,,
\end{aligned}$$

i.e. $\mathcal{F}^{\sigma^{-1}}$ induces the standard negation.

b. Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}
\mathcal{F}'^\sigma(D)(X_1, \ldots, X_n) &= \sigma^{-1}\mathcal{F}'(\sigma D)(\sigma X_1, \ldots, \sigma X_n) \\
&= \sigma^{-1}(\sigma^{-1})^{-1}\mathcal{F}(\sigma^{-1}\sigma D)(\sigma^{-1}\sigma X_1, \ldots, \sigma^{-1}\sigma X_n) && \text{as } \mathcal{F}' = \mathcal{F}^{\sigma^{-1}} \\
&= \mathcal{F}(D)(X_1, \ldots, X_n)\,,
\end{aligned}$$

which finishes the proof.

A.30 Proof of Theorem 29

Let $\mathcal{J}$ a non-empty index set and $(\mathcal{F}_j)_{j\in\mathcal{J}}$ a $\mathcal{J}$-indexed family of $\mathcal{E}$-DFSes. Furthermore, let us suppose that $\Psi : \mathbf{I}^{\mathcal{J}} \longrightarrow \mathbf{I}$ is an operator which satisfies the conditions stated in the theorem. We shall refer to these as cond-a, cond-b and cond-c, respectively.

Preservation of constants (DFS 1) Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(\varnothing) &= \Psi((\mathcal{F}_j(D)(\varnothing))_{j\in\mathcal{J}}) \\ &= \Psi((D(\varnothing))_{j\in\mathcal{J}}) && \text{by (DFS 1)} \\ &= D(\varnothing)\,. && \text{by cond-a}\end{aligned}$$

Compatibility with π_* (DFS 2) Likewise if $X \in \widetilde{\mathcal{P}}(\{*\})$,

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](\pi_*)(X) &= \Psi((\mathcal{F}_j(\pi_*)(X))_{j\in\mathcal{J}}) \\ &= \Psi((\widetilde{\pi}_*(X))_{j\in\mathcal{J}}) && \text{by (DFS 2)} \\ &= \widetilde{\pi}_*(X)\,. && \text{by cond-a}\end{aligned}$$

External negation (DFS 3) By assumption, all of the $\mathcal{F}_j$ are $\neg$-DFSes. We will show that $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ is compatible to the standard negation.
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](\neg D)(X_1, \ldots, X_n) &= \Psi((\mathcal{F}_j(\neg D)(X_1, \ldots, X_n))_{j\in\mathcal{J}}) \\ &= \Psi((1 - \mathcal{F}_j(D)(X_1, \ldots, X_n))_{j\in\mathcal{J}}) && \text{by (DFS 3)} \\ &= 1 - \Psi((\mathcal{F}_j(D)(X_1, \ldots, X_n))_{j\in\mathcal{J}})\,. && \text{by cond-b}\end{aligned}$$

By Th-10, the induced negation of $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ is the standard negation.

Argument transposition (DFS 4) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $i \in \{1, \ldots, n\}$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D\tau_i)(X_1, \ldots, X_n) &= \Psi((\mathcal{F}_j(D\tau_i)(X_1, \ldots, X_n))_{j\in\mathcal{J}}) \\ &= \Psi((\mathcal{F}_j(D)\tau_i(X_1, \ldots, X_n))_{j\in\mathcal{J}}) && \text{by (DFS 4)} \\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)\tau_i(X_1, \ldots, X_n)\,.\end{aligned}$$

Internal complementation (DFS 5) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D\neg)(X_1, \ldots, X_n) &= \Psi((\mathcal{F}_j(D\neg)(X_1, \ldots, X_n))_{j\in\mathcal{J}}) \\ &= \Psi((\mathcal{F}_j(D)(X_1, \ldots, \neg X_n))_{j\in\mathcal{J}}) && \text{by (DFS 5)} \\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(X_1, \ldots, \neg X_n)\,.\end{aligned}$$

Internal meets (DFS 6) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_{n+1}) \in \widetilde{\mathcal{P}}(E)^{n+1}$. We know by Th-25 that the induced fuzzy union operators of all $\mathcal{F}_j$ coincide, which carries over to the induced fuzzy intersections because these are definable in terms of the induced fuzzy union and $\neg$. Denoting this fuzzy intersection by $\widetilde{\cap}$, we obtain:

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D\cap)(X_1, \ldots, X_{n+1}) &= \Psi((\mathcal{F}_j(D\cap)(X_1, \ldots, X_{n+1}))_{j\in\mathcal{J}})\\ &= \Psi((\mathcal{F}_j(D)(X_1, \ldots, X_n \mathbin{\widetilde{\cap}} X_{n+1}))_{j\in\mathcal{J}}) \quad \text{by (DFS 6)}\\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(X_1, \ldots, X_n \mathbin{\widetilde{\cap}} X_{n+1})\,.\end{aligned}$$

By Th-11, it follows that the fuzzy intersection induced by $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ coincides with $\widetilde{\cap}$.

Argument insertion (DFS 7) Let $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and $A \in \mathcal{P}(E)$. Then

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D\triangleleft A)(X_1, \ldots, X_n) &= \Psi((\mathcal{F}_j(D\triangleleft A)(X_1, \ldots, X_n))_{j\in\mathcal{J}})\\ &= \Psi((\mathcal{F}_j(D)(X_1, \ldots, X_n, A)_{j\in\mathcal{J}}) \qquad \text{by (DFS 7)}\\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(X_1, \ldots, X_n, A)\,.\end{aligned}$$

Preservation of decreasing monotonicity (DFS 8) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner which is monotonically decreasing in the n-th argument, and $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ such that $X_n \subseteq X'_n$. Then, because all of the $\mathcal{F}_j$ are DFSes and hence satisfy (DFS 8),

$$\mathcal{F}_j(D)(X_1, \ldots, X_n) \geq \mathcal{F}_j(D)(X_1, \ldots, X'_n)\,. \tag{27}$$

From (27) we obtain, by applying cond-c,

$$\begin{aligned}\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(X_1, \ldots, X_n) &= \Psi((\mathcal{F}_j(D)(X_1, \ldots, X_n))_{j\in\mathcal{J}})\\ &\geq \Psi((\mathcal{F}_j(D)(X_1, \ldots, X'_n))_{j\in\mathcal{J}}) \qquad \text{by (27) and cond-c}\\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(X_1, \ldots, X'_n)\,.\end{aligned}$$

Functional application (DFS 9) Suppose $f_1, \ldots, f_n : E \longrightarrow E'$ where E, E' are non-empty sets, $D : \mathcal{P}(E')^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)$. Recalling that $\widehat{\mathcal{F}}_i = \mathcal{E}$ for all $j \in \mathcal{J}$ by assumption, we compute

$$\begin{aligned}&\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n)\\ &= \Psi((\mathcal{F}_j(D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n))_{j\in\mathcal{J}})\\ &= \Psi((\mathcal{F}_j(D)(\mathcal{E}(f_1)(X_1), \ldots, \mathcal{E}(f_n)(X_n))_{j\in\mathcal{J}}) \qquad \text{by (DFS 9)}\\ &= \Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}](D)(\mathcal{E}(f_1)(X_1), \ldots \mathcal{E}(f_n)(X_n))\,.\end{aligned}$$

By Th-20, then, the extension principle induced by $\Psi[(\mathcal{F}_j)_{j\in\mathcal{J}}]$ coincides with $\mathcal{E}$.

A.31 Proof of Theorem 30

Proof

a. $\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X, \sup X) < \frac{1}{2}$. Then $\inf X \le \sup X < \frac{1}{2}$. In particular, $X \neq \varnothing$ and $\mathrm{m}_{\frac{1}{2}} X = \sup X$.
Now as $X \subseteq Y$ and $X \neq \varnothing$,

$$\inf Y \le \inf X < \tfrac{1}{2}$$

and

$$\sup X \le \sup Y\,.$$

1. $\sup Y < \frac{1}{2}$. Then

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} Y &= \mathrm{m}_{\frac{1}{2}}(\inf Y,\ \sup Y)\\ &= \sup Y && \text{as } \inf Y < \tfrac{1}{2} \text{ and } \sup Y < \tfrac{1}{2}\\ &\ge \sup X\\ &= \mathrm{m}_{\frac{1}{2}} X\,.\end{aligned}$$

2. $\sup Y \ge \frac{1}{2}$. Then

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} Y &= \mathrm{m}_{\frac{1}{2}}(\inf Y,\ \sup Y)\\ &= \tfrac{1}{2} && \text{as } \inf Y < \tfrac{1}{2} \text{ and } \sup Y \ge \tfrac{1}{2}\\ &> \mathrm{m}_{\frac{1}{2}} X\,.\end{aligned}$$

b. $\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X, \sup X) > \frac{1}{2}$. Then $\frac{1}{2} < \inf X \le \sup X$. In particular, $X \neq \varnothing$ and $\mathrm{m}_{\frac{1}{2}} X = \inf X$.
Now as $X \subseteq Y$ and $X \neq \varnothing$,

$$\tfrac{1}{2} < \sup X \le \sup Y$$

and

$$\inf X \le \inf Y\,.$$

1. $\inf Y > \frac{1}{2}$. Then

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} Y &= \mathrm{m}_{\frac{1}{2}}(\inf Y,\ \sup Y)\\ &= \inf Y && \text{as } \inf Y > \tfrac{1}{2} \text{ and } \sup Y > \tfrac{1}{2}\\ &\le \inf X\\ &= \mathrm{m}_{\frac{1}{2}} X\,.\end{aligned}$$

2. $\inf Y \le \frac{1}{2}$. Then

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} Y &= \mathrm{m}_{\frac{1}{2}}(\inf Y,\ \sup Y)\\ &= \tfrac{1}{2} && \text{as } \inf Y \le \tfrac{1}{2} \text{ and } \sup Y > \tfrac{1}{2}\\ &< \mathrm{m}_{\frac{1}{2}} X\,.\end{aligned}$$

c. $\mathrm{m}_{\frac{1}{2}} X = \frac{1}{2}$ and $X \neq \varnothing$.
Because $X \neq \varnothing$, we know that $\inf X \leq \sup X$. From $\mathrm{m}_{\frac{1}{2}} X = \mathrm{m}_{\frac{1}{2}}(\inf X,\ \sup X) = \frac{1}{2}$ it then follows that $\inf X \leq \frac{1}{2}$ and $\sup X \geq \frac{1}{2}$. But $X \subseteq Y$, i.e. $\inf Y \leq \inf X \leq \frac{1}{2}$ and $\sup Y \geq \sup X \geq \frac{1}{2}$. Therefore, $\mathrm{m}_{\frac{1}{2}} Y = \mathrm{m}_{\frac{1}{2}}(\inf Y,\ \sup Y) = \frac{1}{2}$.

A.32 Proof of Theorem 31

Suppose $\mathcal{L}$ is defined by

$$\mathcal{L}(D)(X_1,\ \dots,\ X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(D)(X_1,\ \dots,\ X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\}$$

where $\mathcal{E} = \widehat{\mathcal{F}}$ for some DFS $\mathcal{F}$.

a. We firstly prove that $\mathcal{L}$ is a DFS.
Let $\mathcal{J} = \mathcal{E}\text{-}\mathbf{DFS}$ and define a $\mathcal{J}$-index family of $\mathcal{E}$-DFSes as

$$(\mathcal{F})_{\mathcal{F}\in\mathcal{J}}\,.$$

Noting that $\Psi = \mathrm{m}_{\frac{1}{2}}$ satisfies the requirements of Th-29, we obtain that

$$\mathcal{L} = \Psi[(\mathcal{F})_{\mathcal{F}\in\mathcal{E}\text{-}\mathbf{DFS}}]$$

is an $\mathcal{E}$-DFS.

b. We are left to show that $\mathcal{L}$ is the consistently least specific $\mathcal{E}$-DFS. To see this, suppose $\mathcal{F}' \in \mathcal{E}\text{-}\mathbf{DFS}$, $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1,\ \dots,\ X_n) \in \widetilde{\mathcal{P}}(E)^n$. Apparently,

$$\{\mathcal{F}'(D)(X_1,\ \dots,\ X_n)\} \subseteq \{\mathcal{F}(D)(X_1,\ \dots,\ X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\}\,.$$

Following Th-30, we discern three cases.

1. $\mathcal{F}'(D)(X_1,\ \dots,\ X_n) > \frac{1}{2}$.
   Then by Th-30,
   $$\begin{aligned}\tfrac{1}{2} &\leq \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(D)(X_1,\ \dots,\ X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\} \leq \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}'(D)(X_1,\ \dots,\ X_n)\}\\ &= \mathcal{F}'(D)(X_1,\ \dots,\ X_n)\,,\end{aligned}$$
   i.e.
   $$\tfrac{1}{2} \leq \mathcal{L}(D)(X_1,\ \dots,\ X_n) \leq \mathcal{F}'(D)(X_1,\ \dots,\ X_n)\,.$$
2. $\mathcal{F}'(D)(X_1,\ \dots,\ X_n) = \frac{1}{2}$. Then by Th-30 (because $\{\mathcal{F}'(D)(X_1,\ \dots,\ X_n)\} \neq \varnothing$),
   $$\mathcal{L}(D)(X_1,\ \dots,\ X_n) = \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(D)(X_1,\ \dots,\ X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\} = \tfrac{1}{2}\,.$$
3. $\mathcal{F}'(D)(X_1,\ \dots,\ X_n) < \frac{1}{2}$. Then by the same theorem,
   $$\begin{aligned}\tfrac{1}{2} &\geq \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}(D)(X_1,\ \dots,\ X_n) : \mathcal{F} \in \mathcal{E}\text{-}\mathbf{DFS}\} \geq \mathrm{m}_{\frac{1}{2}}\{\mathcal{F}'(D)(X_1,\ \dots,\ X_n)\}\\ &= \mathcal{F}'(D)(X_1,\ \dots,\ X_n)\,,\end{aligned}$$
   i.e.
   $$\tfrac{1}{2} \geq \mathcal{L}(D)(X_1,\ \dots,\ X_n) \geq \mathcal{F}'(D)(X_1,\ \dots,\ X_n)\,.$$

Combining 1.–3., it follows that $\mathcal{L} \preceq_c \mathcal{F}'$, which we wanted to prove.

A.33 Proof of Theorem 32

Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS and $D,\ D' : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ are fuzzy pre-determiners.

a. Conjunction.
From $D \wedge D' \leq D$ and $D \wedge D' \leq D'$ we obtain

$$\begin{aligned} \mathcal{F}(D \wedge D') &\leq \mathcal{F}(D) \\ \mathcal{F}(D \wedge D') &\leq \mathcal{F}(D') \,. \end{aligned}$$

by Th-13. Because $\wedge = \min$ is monotonically increasing (and hence preserves inequations), it follows that

$$\mathcal{F}(D \wedge D') \wedge \mathcal{F}(D \wedge D') \leq \mathcal{F}(D) \wedge \mathcal{F}(D') \,,$$

which is equivalent to the desired

$$\mathcal{F}(D \wedge D') \leq \mathcal{F}(D) \wedge \mathcal{F}(D')$$

because $\wedge = \min$ is idempotent.

b. Disjunction.
The proof is completely analogous to that of the dual case **a.**

A.34 Proof of Theorem 33

a. Bi-implication operator "$\widetilde{\leftrightarrow}$".
Suppose $\mathcal{F}$ is a $(\widetilde{\neg}, \max)$-DFS and $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow)$.
Firstly, let us observe that the two-valued equivalence operator $\leftrightarrow : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ satisfies

$$\begin{aligned} x_1 \leftrightarrow x_2 &= (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2) \\ x_1 \leftrightarrow x_2 &= (\neg x_1 \vee x_2) \wedge (x_1 \vee \neg x_2) \end{aligned}$$

for all $x_1,\ x_2 \in \mathbf{2}$. By applying Th-32, we obtain

$$(x_1 \wedge x_2) \vee (\widetilde{\neg}\, x_1 \wedge \widetilde{\neg}\, x_2) \leq x_1 \mathbin{\widetilde{\leftrightarrow}} x_2 \leq (\widetilde{\neg}\, x_1 \vee x_2) \wedge (x_1 \vee \widetilde{\neg}\, x_2)$$

for all $x_1,\ x_2 \in \mathbf{I}$ (where $\wedge = \min = \widetilde{\mathcal{F}}(\wedge)$ is the standard conjunction and $\vee = \max = \widetilde{\mathcal{F}}(\vee)$ is the standard disjunction).
It is therefore sufficient to prove that

$$(x_1 \wedge x_2) \vee (\widetilde{\neg}\, x_1 \wedge \widetilde{\neg}\, x_2) = (\widetilde{\neg}\, x_1 \vee x_2) \wedge (x_1 \vee \widetilde{\neg}\, x_2) \,, \tag{28}$$

for all $x_1,\ x_2 \in \mathbf{I}$.

1. $\mathcal{F}$ is a $\neg$-DFS (i.e. induces the standard negation).

   In this case, the left- and righthand sides of (28) are polynomials in $\min$, $\max$ and $\neg$.

It is well-known from Preparata & Yeh [38] and Mukaidono [23] that if $f : \mathbf{I}^n \longrightarrow \mathbf{I}$ is a *fuzzy expression*, i.e. a polynomial in $\min$, $\max$ and $\neg$, then f is uniquely determined by its behaviour on $\{0, \frac{1}{2}, 1\}$-valued arguments, i.e. by $f|_{\{0, \frac{1}{2}, 1\}^n}$.

By Th-1, we only need to consider arguments $x_1, x_2 \in \{0, \frac{1}{2}, 1\}$ where at least one of the components is $\frac{1}{2}$. Furthermore, by the commutativity of both sides of (28) and the apparent symmetry with respect to simultaneous negation in x_1 and x_2, we only need to consider the cases $(x_1, x_2) \in \{(0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$. But apparently,

$$\begin{aligned}
(0 \wedge \tfrac{1}{2}) \vee (\neg 0 \wedge \neg \tfrac{1}{2}) &= (0 \wedge \tfrac{1}{2}) \vee (1 \wedge \tfrac{1}{2}) \\
&= \tfrac{1}{2} \\
&= (1 \vee \tfrac{1}{2}) \wedge (0 \vee \tfrac{1}{2}) \\
&= (\neg 0 \vee \tfrac{1}{2}) \wedge (0 \vee \neg \tfrac{1}{2})
\end{aligned}$$

and

$$\begin{aligned}
(\tfrac{1}{2} \wedge \tfrac{1}{2}) \vee (\neg \tfrac{1}{2} \wedge \neg \tfrac{1}{2}) &= (\tfrac{1}{2} \wedge \tfrac{1}{2}) \vee (\tfrac{1}{2} \wedge \tfrac{1}{2}) \\
&= \tfrac{1}{2} \\
&= (\tfrac{1}{2} \vee \tfrac{1}{2}) \wedge (\tfrac{1}{2} \vee \tfrac{1}{2}) \\
&= (\neg \tfrac{1}{2} \vee \tfrac{1}{2}) \wedge (\tfrac{1}{2} \vee \neg \tfrac{1}{2}) ,
\end{aligned}$$

which finishes the proof of case 1.

2. $\mathcal{F}$ does not induce the standard negation. In this case, let $\sigma : \mathbf{I} \longrightarrow \mathbf{I}$ the generator of $\widetilde{\neg} = \widetilde{\mathcal{F}}(\neg)$. It is easily verified that (because σ^{-1} is strictly monotonically increasing), $\mathcal{F}^{\sigma^{-1}}$ is a $(\neg, \max)$-DFS. In particular, for all $x_1, x_2 \in \mathbf{I}$,

$$x_1 \widetilde{\leftrightarrow}^{\sigma} x_2 = (x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_2)$$

by case 1, where $\neg$, $\wedge = \min$ and $\vee = \max$ are the induced truth functions of $\mathcal{F}^{\sigma^{-1}}$. Abbreviating $\widetilde{\leftrightarrow} = \widetilde{\mathcal{F}}(\leftrightarrow)$, we obtain for the original $\mathcal{F}$,

$$\begin{aligned}
x_1 \widetilde{\leftrightarrow} x_2 &= \sigma^{-1}(\sigma x_1 \widetilde{\leftrightarrow}^{\sigma} \sigma x_2) && \text{by Th-28} \\
&= \sigma^{-1}((\sigma x_1 \wedge \sigma x_2) \vee (\neg \sigma x_1 \wedge \neg \sigma x_2)) && \text{by case 1.} \\
&= \sigma^{-1}((\sigma x_1 \wedge \sigma x_2) \vee (\sigma \widetilde{\neg} \sigma^{-1} \sigma x_1 \wedge \sigma \widetilde{\neg} \sigma^{-1} \sigma x_2)) && \text{by L-10} \\
&= \sigma^{-1}((\sigma x_1 \wedge \sigma x_2) \vee (\sigma \widetilde{\neg} x_1 \wedge \sigma \widetilde{\neg} x_2)) \\
&= \sigma^{-1}(\sigma(x_1 \wedge x_2) \vee \sigma(\widetilde{\neg} x_1 \wedge \widetilde{\neg} x_2)) && \sigma \text{ incr., } \wedge = \min \\
&= \sigma^{-1}\sigma((x_1 \wedge x_2) \vee (\widetilde{\neg} x_1 \wedge \widetilde{\neg} x_2)) && \sigma \text{ incr., } \vee = \max \\
&= (x_1 \wedge x_2) \vee (\widetilde{\neg} x_1 \vee \widetilde{\neg} x_2) ,
\end{aligned}$$

for all $x_1, x_2 \in \mathbf{I}$, as desired.

b. Exclusive or.

The claim is apparent from case **a.**, noting that $\mathrm{xor}^* = \leftrightarrow^* \neg$ and hence $\widetilde{\mathcal{F}}(\mathrm{xor})(x_1, x_2) = x_1 \mathbin{\widetilde{\mathrm{xor}}} x_2 = x_1 \widetilde{\leftrightarrow} \widetilde{\neg} x_2$.

A.35 Proof of Theorem 34

Suppose $\mathcal{F}$ is a standard DFS and $x_1, x_2 \in \mathbf{I}$.

1. $\widetilde{\neg}\, x_1 = 1 - x_1$ by definition of a standard DFS as a special kind of $\neg$-DFS.
2. as concerns disjunction, define $X \in \widetilde{\mathcal{P}}(\{1, 2\})$ by $\mu_X(1) = x_1$, $\mu_X(2) = x_2$ and let $! = !_{\{1,2\}} : \{1, 2\} \longrightarrow \{*\}$. Then by Th-25,

$$\begin{aligned}
x_1 \,\widetilde{\vee}\, x_2 &= \widetilde{\pi}_* \circ \widehat{\mathcal{F}}(!)(X) && \text{by Th-25} \\
&= \widetilde{\pi}_* \circ \hat{\hat{!}}(X) && \text{because } \mathcal{F} \text{ standard DFS} \\
&= \sup\{\mu_X(v) : v \in !^{-1}(*)\} \\
&= \sup\{\mu_X(v) : v \in \{1, 2\}\} \\
&= \sup\{\mu_X(1), \mu_X(2)\} \\
&= \max(\mu_X(1), \mu_X(2)) \\
&= \max(x_1, x_2)\,. && \text{by definition of } X
\end{aligned}$$

3. The claims on $\widetilde{\wedge}$ and $\widetilde{\rightarrow}$ are immediate from the definability of the (crisp) truth functions $\wedge$ and $\rightarrow$ in terms of $\vee$ and $\neg$, which is preserved because $\mathcal{F}$ satisfies (DFS 3) and (DFS 5).
4. From $\widetilde{\mathcal{F}}(\vee) = \max$ we know that every standard DFS is a $(\widetilde{\neg}, \max)$-DFS; the claims on $\widetilde{\leftrightarrow}$ and $\widetilde{\mathrm{xor}}$ are then immediate from Th-33.

A.36 Proof of Theorem 35

Suppose $\mathcal{F}$ is a standard DFS and $E \neq \varnothing$.

- $\mathcal{F}(\exists)(X) = \sup\{\mu_X(v) : v \in E\}$ for all $X \in \widetilde{\mathcal{P}}(E)$ is apparent from Th-22.
- $\mathcal{F}(\forall)(X) = \inf\{\mu_X(v) : v \in E\}$ for all $X \in \widetilde{\mathcal{P}}(E)$ is obtained from

$$\begin{aligned}
\mathcal{F}(\forall) &= \mathcal{F}(\neg\exists\neg) \\
&= \neg\mathcal{F}(\exists)\neg\,. && \text{by (DFS 3), (DFS 5)}
\end{aligned}$$

- The results on $\mathcal{F}(\textbf{all})$, $\mathcal{F}(\textbf{some})$ and $\mathcal{F}(\textbf{no})$ are apparent from their definitions in terms of $\exists$ and $\forall$,

$$\begin{aligned}
\textbf{all} &= \forall \cup \tau_1 \neg \tau_1 \\
\textbf{some} &= \exists \cap \\
\textbf{no} &= \textbf{all}\neg
\end{aligned}$$

which only involve constructions with which $\mathcal{F}$ commutes.

A.37 Proof of Theorem 36

Suppose $\mathcal{F}$ is a DFS, $\mathcal{S}$ is a standard DFS, and $E \neq \varnothing$, $\exists = \exists_E$. Then for all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}
\mathcal{F}(\exists)(X) &\geq \sup\{\mathop{\widetilde{\vee}}_{i=1}^{m} \mu_X(a_i) : \{a_1, \ldots, a_m\} \subseteq E \text{ finite, } a_i \neq a_j \text{ if } i \neq j\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Th-24} \\
&\geq \sup\{\max\{\mu_X(a_1), \ldots, \mu_X(a_m)\} : \{a_1, \ldots, a_m\} \subseteq E \text{ finite, } a_i \neq a_j \text{ if } i \neq j\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \max \text{ is smallest } s\text{-norm} \\
&= \sup\{\mu_X(v) : v \in E\} \\
&= \mathcal{S}(\exists)(X) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by Th-35.}
\end{aligned}$$

B Proof of the theorems in chapter "DFS models"

B.1 Proof of Theorem 37

Consult any textbook on fuzzy set theory, e.g. Klir & Folger 1988 [27].

B.2 Proof of Theorem 38

We again refer to Klir & Folger 1988 [27] or other textbooks.

B.3 Proof of Theorem 39

Again consult Klir & Folger 1988 [27] or other textbooks.

B.4 Proof of Theorem 40

For all $x \in I$, $X = \widetilde{\pi}_*^{-1}(x) \in \widetilde{\mathcal{P}}(\{*\})$,

$$\mu_{\mathcal{C}(\pi_*)(X)}(\alpha) = \begin{cases} 1 & : \quad \alpha \leq x \\ 0 & : \quad \alpha > x \end{cases}$$

i.e.

$$\mathcal{C}(\pi_*)X = [0,\, x]\,.$$

But by assumption, $\mathcal{F}(\pi_*) = \widetilde{\pi}_*$, i.e.

$$\mathcal{S}([0,\, x]) = x \tag{29}$$

for all $x \in I$.
For all $x_1,\, x_2 \in I$, $X_1 = \widetilde{\pi}_*^{-1}(x_1),\ X_2 = \widetilde{\pi}_*^{-1}(x_2)$,

$$\begin{aligned} \mu_{\widetilde{\mathcal{C}}(\wedge)(x_1,\,x_2)}(\alpha) &= \mu_{\mathcal{C}(\wedge^*)(X_1,\,X_2)}(\alpha) \\ &= \begin{cases} 1 & : \quad \alpha \leq \min(x_1,\, x_2) \\ 0 & : \quad \alpha > \min(x_1,\, x_2) \end{cases} \end{aligned}$$

i.e.

$$\widetilde{\mathcal{C}}(\wedge)(x_1,\, x_2) = [0,\ \min(x_1,\, x_2)]\,. \tag{30}$$

By applying (29), we obtain

$$\widetilde{\mathcal{F}}(\wedge)(x_1,\, x_2) = \mathcal{S}([0,\ \min(x_1,\, x_2)]) = \min(x_1,\, x_2), \tag{31}$$

i.e. $\widetilde{\mathcal{F}}(\wedge) = \min$. In particular,

$$\widetilde{\mathcal{F}}(\wedge)(x_1,\ \widetilde{\neg}\, x_2) = \min(x_1,\ \widetilde{\neg}\, x_2) \tag{32}$$

for all $x_1,\, x_2 \in I$. However,

$$\mu_{\widetilde{\mathcal{C}}(\wedge\neg)(x_1,x_2)}(\alpha) = \begin{cases} 1 & : \quad x_1 \geq \alpha \text{ and } x_2 < \alpha \\ 0 & : \quad x_1 < \alpha \text{ or } x_2 \geq \alpha \end{cases} \tag{33}$$

for all $x_1, x_2 \in I$. Because $\widetilde{\neg}$ is a strong negation operator, it is a monotonically decreasing bijection. Consequently, there exist $0 < x_1 < x_2 < 1$ such that

$$0 < \widetilde{\neg}\, x_2 < x_1 < 1 \tag{34}$$

We compute

$$\begin{aligned}
\widetilde{\mathcal{F}}(\wedge\neg)(x_1, x_2) &= \mathcal{S}(\widetilde{\mathcal{C}}(\wedge\neg)(x_1, x_2)) && (*)\\
&= \mathcal{S}([0, 0]) && \text{by (33)}\\
&= 0 && \text{by (29)}
\end{aligned}$$

However,

$$\begin{aligned}
\widetilde{\mathcal{F}}(\wedge)(x_1, \widetilde{\neg}\, x_2) &= \min(x_1, \widetilde{\neg}\, x_2) \\
&= \widetilde{\neg}\, x_2 && \text{by (32)}\\
&\neq 0 && \text{by (34)}\\
&= \widetilde{\mathcal{F}}(\wedge\neg)(x_1, x_2)\,. && \text{by (*)}
\end{aligned}$$

Hence $\mathcal{F}$ does not satisfy the axiom of internal complementation.

B.5 Proof of Theorem 41

For all fuzzy pre-determiners $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$,

$$\mathcal{R}(D) : \widetilde{\mathcal{P}}(E)^n \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$$

assigns to each $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ the fuzzy subset $\mathcal{R}(D)(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(\mathbf{I})$ defined by

$$\begin{aligned}
\mu_{\mathcal{R}(D)(X_1,\ldots,X_n)}(v) &= \sup\{\alpha \in \mathbf{I} : D((X_1)_{\geq\alpha}, \ldots, (X_n)_{\geq\alpha}) = v\}\\
&= \sup\{\alpha \in \mathbf{I} : \mu_{\mathcal{C}(D)(X_1,\ldots,X_n)}(\alpha) = v\}\,,
\end{aligned}$$

for all $v \in \mathbf{I}$. Therefore,

$$\mathcal{R} = \mathcal{R}' \circ \mathcal{C}\,,$$

where $\mathcal{R}' : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \widetilde{\mathcal{P}}(\mathbf{I})$ is defined by

$$\mu_{\mathcal{R}'(X)}(v) = \sup\{\alpha \in \mathbf{I} : \mu_X(\alpha) = v\}$$

for all $X \in \widetilde{\mathcal{P}}(\mathbf{I})$, $v \in \mathbf{I}$.
Now suppose there is an $\mathcal{S} : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ such that $\mathcal{F} = \mathcal{S} \circ \mathcal{R}$ is a DFS. Then

$$\begin{aligned}
\mathcal{F} &= \mathcal{S} \circ \mathcal{R}\\
&= \mathcal{S} \circ (\mathcal{R}' \circ \mathcal{C})\\
&= (\mathcal{S} \circ \mathcal{R}') \circ \mathcal{C} && \text{by associativity of } \circ\\
&= \mathcal{S}' \circ \mathcal{C}\,,
\end{aligned}$$

where $\mathcal{S}' = \mathcal{S} \circ \mathcal{R}' : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$.
Hence if $\mathcal{F}$ was a DFS, there would exists an $\mathcal{S}' : \widetilde{\mathcal{P}}(\mathbf{I}) \longrightarrow \mathbf{I}$ such that $\mathcal{S}' \circ \mathcal{C}$ is a DFS, which contradicts Th-40.

B.6 Preliminaries (fuzzy median-based models)

Lemma 12 ($(X)^{\min}_\gamma$, $(X)^{\max}_\gamma$ on three-valued sets)
Let E a nonempty set. If $X \in \breve{\mathcal{P}}\,(E) \subset \widetilde{\mathcal{P}}(E)$ is a $\{0,\, \frac{1}{2},\, 1\}$-valued subset of E, then

$$\begin{aligned}(X)^{\min}_\gamma &= \{v \in E : \mu_X(v) = 1\}\\ (X)^{\max}_\gamma &= \{v \in E : \mu_X(v) = \tfrac{1}{2} \text{ or } \mu_X(v) = 1\}\end{aligned}$$

for all $\gamma \in \mathbf{I}$ (where we have denoted the membership function of X by $\mu_X : E \longrightarrow \{0,\, \frac{1}{2},\, 1\}$).
In particular, if $X \in \mathcal{P}(E) \subset \breve{\mathcal{P}}\,(E)$ is a crisp subset of E, then $(X)^{\min}_\gamma = (X)^{\max}_\gamma = X$ for all $\gamma \in \mathbf{I}$.

Proof Let $X \in \breve{\mathcal{P}}\,(E) \subset \widetilde{\mathcal{P}}(E)$ a $\{0,\, \frac{1}{2},\, 1\}$-valued subset of E.

a. Firstly,

$$\begin{aligned}(X)^{\min}_0 &= (X)_{>\frac{1}{2}}\\ &= \{v \in E : \mu_X(v) > \tfrac{1}{2}\}\\ &= \{v \in E : \mu_X(v) = 1\} && \text{as } \mu_X(v) \in \{0,\, \tfrac{1}{2},\, 1\}\\ (X)^{\max}_0 &= (X)_{\geq\frac{1}{2}}\\ &= \{v \in E : \mu_X(v) \geq \tfrac{1}{2}\}\\ &= \{v \in E : \mu_X(v) = \tfrac{1}{2} \text{ or } \mu_X(v) = 1\}\,. && \text{as } \mu_X(v) \in \{0,\, \tfrac{1}{2},\, 1\}\end{aligned}$$

b. If $\gamma > 0$,

$$\begin{aligned}(X)^{\min}_\gamma &= (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma}\\ &= \{v \in E : \mu_X(v) \geq \tfrac{1}{2} + \tfrac{1}{2}\gamma\}\\ &= \{v \in E : \mu_X(v) = 1\} && \text{as } \mu_X(v) \in \{0,\, \tfrac{1}{2},\, 1\}\\ (X)^{\max}_\gamma &= (X)_{\geq\frac{1}{2}-\frac{1}{2}\gamma}\\ &= \{v \in E : \mu_X(v) > \tfrac{1}{2} - \tfrac{1}{2}\gamma\}\\ &= \{v \in E : \mu_X(v) = \tfrac{1}{2} \text{ or } \mu_X(v) = 1\}\,. && \text{as } \mu_X(v) \in \{0,\, \tfrac{1}{2},\, 1\}\end{aligned}$$

The claim on crisp subsets is immediate from the fact that when $X \in \mathcal{P}(E)$ is crisp, the case $\mu_X(v) = \frac{1}{2}$ never occurs and hence can be omitted from the above expressions.

Lemma 13 ($(X)^{\min}_\gamma$ and $(X)^{\max}_\gamma$ on singletons) *Let $X \in \widetilde{\mathcal{P}}(\{*\})$ and $x = \widetilde{\pi}_*(X)$.*

a. If $x < \frac{1}{2}$, then

$$\begin{aligned}(X)^{\min}_\gamma &= \varnothing\\ (X)^{\max}_\gamma &= \begin{cases} \varnothing & : \quad \gamma \leq 1 - 2x\\ \{*\} & : \quad \gamma > 1 - 2x\end{cases}\end{aligned}$$

for all $\gamma \in \mathbf{I}$.

b. *If* $x = \frac{1}{2}$, *then* $(X)^{\min}_\gamma = \varnothing$ *and* $(X)^{\max}_\gamma = \{*\}$ *for all* $\gamma \in \mathbf{I}$.

c. *If* $x > \frac{1}{2}$, *then*

$$(X)^{\min}_\gamma = \begin{cases} \{*\} & : \gamma \leq 2x - 1 \\ \varnothing & : \gamma > 2x - 1 \end{cases}$$
$$(X)^{\max}_\gamma = \{*\}$$

for all $\gamma \in \mathbf{I}$.

Proof

a. $x < \frac{1}{2}$
Then $(X)^{\min}_0 = (X)_{>\frac{1}{2}} = \varnothing$ because $\mu_X(*) = \widetilde{\pi}_*(X) = x < \frac{1}{2}$. But $(X)^{\min}_\gamma$ is decreasing in γ and therefore $(X)^{\min}_\gamma = \varnothing$ for all $\gamma \in \mathbf{I}$.
$(X)^{\max}_0 = (X)_{\geq\frac{1}{2}} = \varnothing$ as $\mu_X(*) < \frac{1}{2}$. Similarly for $\gamma \in (0,\, 1 - 2x]$, $(X)^{\max}_0 = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} = \varnothing$ because $\frac{1}{2} - \frac{1}{2}\gamma \geq \frac{1}{2} - \frac{1}{2}(1 - 2x) = x$.
However if $\gamma \in (1-2x,\, 1]$, $(X)^{\max}_\gamma = (X)_{>\frac{1}{2}-\frac{1}{2}\gamma} = \{*\}$ because then $\frac{1}{2} - \frac{1}{2}\gamma < \frac{1}{2} - \frac{1}{2}(1-2x) = x$.

b. $x = \frac{1}{2}$
Then $(X)^{\min}_0 = (X)_{>\frac{1}{2}} = \varnothing$. But $(X)^{\min}_\gamma$ is decreasing in γ, i.e. $(X)^{\min}_\gamma = \varnothing$ for all $\gamma \in \mathbf{I}$.
$(X)^{\max}_0 = (X)_{\geq\frac{1}{2}} = \{*\}$ because $\mu_X(*) = \frac{1}{2}$. But $(X)^{\max}_\gamma \subset \{*\}$ is increasing in γ, i.e. $(X)^{\max}_\gamma = \{*\}$ for all $\gamma \in I$.

c. $x > \frac{1}{2}$
Then $(X)^{\max}_0 = (X)_{\geq\frac{1}{2}} = \{*\}$ and again because $(X)^{\max}_\gamma$ is increasing in γ, $(X)^{\max}_\gamma = \{*\}$ for all $\gamma \in I$.
$(X)^{\min}_0 = (X)_{>\frac{1}{2}} = \{*\}$ because $\mu_X(*) = x > \frac{1}{2}$. Also $(X)^{\min}_\gamma = (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = \{*\}$ for $\gamma \in (0,\, 2x - 1]$ because $\frac{1}{2} + \frac{1}{2}\gamma \leq \frac{1}{2} + \frac{1}{2}(2x - 1) = x$. But if $\gamma \in (2x - 1,\, 1]$, $\frac{1}{2} + \frac{1}{2}\gamma > \frac{1}{2} + \frac{1}{2}(2x - 1) = x = \mu_X(*)$ and hence $(X)^{\min}_\gamma = \varnothing$.

Lemma 14 ($\mathcal{Y}^\gamma_i$ on singletons)
Let $(X_1,\, \ldots,\, X_n) \in \widetilde{\mathcal{P}}(\{*\})^n$, $i \in \{1,\, \ldots,\, n\}$, $x_i = \widetilde{\pi}_*(X_i)$.

a. *If* $x_i = 0$, *then*

$$\mathcal{Y}^\gamma_i = \begin{cases} \{\varnothing\} & : \gamma \leq 1 - 2x_i \\ \{\varnothing, \{*\}\} & : \gamma > 1 - 2x_i \end{cases}$$

for all $\gamma \in I$.

b. *If* $x_i = \frac{1}{2}$, *then* $\mathcal{Y}^\gamma_i = \{\varnothing,\, \{*\}\}$ *for all* $\gamma \in \mathbf{I}$.

c. *If* $x_i > \frac{1}{2}$, *then*

$$\mathcal{Y}^\gamma_i = \begin{cases} \{\{*\}\} & : \gamma \leq 2x_i - 1 \\ \{\varnothing, \{*\}\} & : \gamma > 2x_i - 1 \end{cases}$$

for all $\gamma \in \mathbf{I}$.

Proof

a. $x_i < \frac{1}{2}$.
If $\gamma \leq 1 - 2x_i$, then

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : \varnothing \subseteq Y \subseteq \varnothing\} && \text{by L-13} \\ &= \{\,\varnothing\,\}\,.\end{aligned}$$

If $\gamma > 1 - 2x$, then

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : \varnothing \subseteq Y \subseteq \{*\}\} && \text{by L-13} \\ &= \{\,\varnothing,\ \{*\}\,\}\,.\end{aligned}$$

b. $x_i = \frac{1}{2}$.

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : \varnothing \subseteq Y \subseteq \{*\}\} && \text{by L-13} \\ &= \{\,\varnothing,\ \{*\}\,\}\,.\end{aligned}$$

c. $x_i > \frac{1}{2}$.
If $\gamma \leq 2x_i - 1$, then

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : \{*\} \subseteq Y \subseteq \{*\}\} && \text{by L-13} \\ &= \{\,\{*\}\,\}\,.\end{aligned}$$

If $\gamma > 2x_i - 1$, then

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : \varnothing \subseteq Y \subseteq \{*\}\} && \text{by L-13} \\ &= \{\,\varnothing,\ \{*\}\,\}\,.\end{aligned}$$

Lemma 15
For every base set $E \neq \varnothing$, all $n \in \mathbb{N}$ and all $(X_1,\ \ldots,\ X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$\mathcal{Y}_i^\gamma \neq \varnothing$$

for all $i \in \{1,\ \ldots,\ n\}$.

Proof Suppose $E \neq \varnothing$, $n \in \mathbb{N}$, $X \in \widetilde{\mathcal{P}}(E)^n$ and $i \in \{1, \ldots, n\}$. By definition,

$$\mathcal{Y}_i^\gamma = \{Y \in \mathcal{P}(E) : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\}.$$

It is apparent from the definition of $(X_i)_\gamma^{\min}$ and $(X_i)_\gamma^{\max}$ that

$$(X_i)_\gamma^{\min} \subseteq (X_i)_\gamma^{\max}$$

for all $\gamma \in \mathbf{I}$ and hence

$$(X_i)_\gamma^{\min} \in \mathcal{Y}_i^\gamma,$$

as desired.

B.7 Proof of Theorem 42

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Now for all $\gamma, \gamma' \in \mathbf{I} \setminus \{0\}$,

$$(X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma'} \subseteq (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}} \subseteq (X_i)_{\geq \frac{1}{2}} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma'}$$

i.e.

$$(X_i)_{\gamma'}^{\min} \subseteq (X_i)_\gamma^{\min} \subseteq (X_i)_0^{\min} \subseteq (X_i)_0^{\max} \subseteq (X_i)_\gamma^{\max} \subseteq (X_i)_{\gamma'}^{\max}$$

and hence

$$\mathcal{Y}_i^0 \subseteq \mathcal{Y}_i^\gamma \subseteq \mathcal{Y}_i^{\gamma'}, \qquad (*)$$

for $i = 1, \ldots, n$. Let us abbreviate

$$Z_\gamma = \{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\}.$$

Then by (*),

$$\varnothing \neq Z_0 \subseteq Z_\gamma \subseteq Z_{\gamma'}, \qquad (35)$$

for all $0 < \gamma \leq \gamma'$.

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$.
Then $D_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_0 > \frac{1}{2}$, $Z_0 \subseteq Z_\gamma$ and by Th-30, $\frac{1}{2} \leq D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_\gamma \leq D_0(X_1, \ldots, X_n)$.
Likewise if $\gamma' > \gamma$, then $Z_\gamma \subseteq Z_{\gamma'}$ and by Th-30, $\frac{1}{2} \leq D_{\gamma'}(X_1, \ldots, X_n) \leq D_\gamma(X_1, \ldots, X_n)$, i.e. $D_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing in γ.

b. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$. Then $D_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_0 = \frac{1}{2}$. $Z_0 \subseteq Z_\gamma$ and $Z_0 \neq \varnothing$ (by L-15) yields $D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_\gamma = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ (again by Th-30).

c. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$.
Then $D_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_0 < \frac{1}{2}$, $Z_0 \subseteq Z_\gamma$ and by Th-30, $\frac{1}{2} \geq D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_\gamma \geq D_0(X_1, \ldots, X_n)$.
Likewise if $\gamma' > \gamma$, then $Z_\gamma \subseteq Z_{\gamma'}$ and by Th-30, $\frac{1}{2} \geq D_{\gamma'}(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X_n)$, i.e. $D_\gamma(X_1, \ldots, X_n)$ is monotonically increasing in γ.

B.8 Properties of $(\_)_\gamma$

Lemma 16 *Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Let us abbreviate*

$$D_\gamma^{\min} = D_\gamma^{\min}(X_1, \ldots, X_n) = \inf\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\}$$
$$D_\gamma^{\max} = D_\gamma^{\max}(X_1, \ldots, X_n) = \sup\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\}$$

for all $\gamma \in \mathbf{I}$. Then $D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}(D_\gamma^{\min}(X_1, \ldots, X_n), D_\gamma^{\max}(X_1, \ldots, X_n))$. Furthermore,

a. if $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$, then $D_\gamma(X_1, \ldots, X_n) = D_\gamma^{\max}(X_1, \ldots, X_n)$

b. if $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$, then $D_\gamma(X_1, \ldots, X_n) = D_\gamma^{\min}(X_1, \ldots, X_n)$

c. if $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$, then $D_\gamma^{\min}(X_1, \ldots, X_n) \leq \frac{1}{2}$ and $D_\gamma^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$.

Proof Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then for all $\gamma \in \mathbf{I}$,

$$\begin{aligned} D_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots Y_n) \in \mathcal{Y}^\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots Y_n) \in \mathcal{Y}^\gamma\}, \\ &\qquad \sup\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots Y_n) \in \mathcal{Y}^\gamma\}) \qquad \text{by Def. 36} \\ &= \mathrm{m}_{\frac{1}{2}}(D_\gamma^{\min}(X_1, \ldots, X_n), D_\gamma^{\max}(X_1, \ldots, X_n)) \,. \end{aligned}$$

a. $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$.
In this case

$$\begin{aligned} D_\gamma(X_1, \ldots, X_n) < \tfrac{1}{2} &\Leftrightarrow \mathrm{m}_{\frac{1}{2}}(D_\gamma^{\min}(X_1, \ldots, X_n), D_\gamma^{\max}(X_1, \ldots, X_n)) < \tfrac{1}{2} \\ &\Leftrightarrow D_\gamma^{\max}(X_1, \ldots, X_n) < \tfrac{1}{2} \end{aligned}$$

where the last equivalence holds by Def. 35 because $D_\gamma^{\min}(X_1, \ldots, X_n) \leq D_\gamma^{\max}(X_1, \ldots, X_n)$.

b. $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$.
Then

$$\begin{aligned} D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} &\Leftrightarrow \mathrm{m}_{\frac{1}{2}}(D_\gamma^{\min}(X_1, \ldots, X_n), D_\gamma^{\max}(X_1, \ldots, X_n)) > \tfrac{1}{2} \\ &\Leftrightarrow D_\gamma^{\min}(X_1, \ldots, X_n) > \tfrac{1}{2} \end{aligned}$$

where the last equivalence is again valid by Def. 35 because

$$D_\gamma^{\min}(X_1, \ldots, X_n) \leq D_\gamma^{\max}(X_1, \ldots, X_n) \,.$$

c. $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$.
Then

$$D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2} \Leftrightarrow \mathrm{m}_{\frac{1}{2}}(D_\gamma^{\min}(X_1, \ldots, X_n), D_\gamma^{\max}(X_1, \ldots, X_n)) = \tfrac{1}{2}$$
$$\Leftrightarrow D_\gamma^{\min}(X_1, \ldots, X_n) \leq \tfrac{1}{2} \text{ and } D_\gamma^{\max}(X_1, \ldots, X_n) \geq \tfrac{1}{2}\,.$$

The last step is again obtained from $D_\gamma^{\min}(X_1, \ldots, X_n) \leq D_\gamma^{\max}(X_1, \ldots, X_n)$.

Lemma 17 *Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $X_1, \ldots, X_n \in \breve{\mathcal{P}}(E) \subseteq \widetilde{\mathcal{P}}(E)$ are $\{0, \frac{1}{2}, 1\}$-valued subsets of E. Then*

$$D_\gamma(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$, i.e. $D_\gamma(X_1, \ldots, X_n)$ is constant.

Proof If $(X_1, \ldots, X_n)$ is an n-tuple of $\{0, \frac{1}{2}, 1\}$-valued subsets of E, then for all $i \in \{1, \ldots, n\}$,

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\}\\ &= \{Y : (X_i)_0^{\min} \subseteq Y \subseteq (X_i)_0^{\max}\} && \text{by L-12}\\ &= \mathcal{Y}_i^0\end{aligned}$$

for all $\gamma \in \mathbf{I}$ and hence

$$\begin{aligned}D_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\}\\ &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^0\}\\ &= D_0(X_1, \ldots, X_n)\,.\end{aligned}$$

Preservation of constants (DFS 1)

Lemma 18
Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then $D_\gamma = D$ for all $\gamma \in \mathbf{I}$.

Proof Because $\mathcal{P}(E)^0 = \{\varnothing\} = \widetilde{\mathcal{P}}(E)^0$, we only have to show that $D_\gamma(\varnothing) = D(\varnothing)$. Now $\mathcal{Y}^\gamma$ is defined

$$\mathcal{Y}^\gamma = \{\mathbf{Y} = (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n : \text{for all } 1 \leq i \leq n,\ Y_i \in \mathcal{Y}_i^\gamma\}$$

which in the case $n = 0$ becomes

$$\begin{aligned}\mathcal{Y}^\gamma &= \{\mathbf{Y} \in \{\varnothing\} : \text{for all } i \in \varnothing,\ Y_i \in \mathcal{Y}_i^\gamma\}\\ &= \{\mathbf{Y} \in \{\varnothing\} : \text{true}\}\\ &= \{\varnothing\}\,,\end{aligned}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}D_\gamma(\varnothing) &= \mathrm{m}_{\frac{1}{2}}\{D(Y) : Y \in \{\varnothing\}\}\\ &= \mathrm{m}_{\frac{1}{2}}\{D(\varnothing)\}\\ &= D(\varnothing)\end{aligned}$$

The projection determiner under $(\_)_\gamma$

Lemma 19
Let $X \in \widetilde{\mathcal{P}}(\{*\})$ *and* $x = \widetilde{\pi}_*(X)$*. Then for all* $\gamma \in \mathbf{I}$*,*

a. *If* $x < \frac{1}{2}$*, then*

$$\pi_{*\gamma}(X) = \begin{cases} 0 & : \quad \gamma \leq 1 - 2x \\ \frac{1}{2} & : \quad \gamma > 1 - 2x \end{cases}$$

b. *If* $x = \frac{1}{2}$*, then* $\pi_{*\gamma}(X) = \frac{1}{2}$

c. *If* $x > \frac{1}{2}$*, then*

$$\pi_{*\gamma}(X) = \begin{cases} 1 & : \quad \gamma \leq 2x - 1 \\ \frac{1}{2} & : \quad \gamma > 2x - 1 \end{cases}$$

Proof

a. $x < \frac{1}{2}$
If $\gamma \leq 1 - 2x$, then $\mathcal{Y}^\gamma = \{\varnothing\}$ by L-14 and hence

$$\begin{aligned} \pi_{*\gamma}(X) &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(Y) : Y \in \{\varnothing\}\} \\ &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(\varnothing)\} \\ &= \mathrm{m}_{\frac{1}{2}}\{0\} \\ &= 0\,. \end{aligned}$$

If $\gamma > 1 - 2x$, then $\mathcal{Y}^\gamma = \{\varnothing, \{*\}\}$ by L-14. Therefore,

$$\begin{aligned} \pi_{*\gamma}(X) &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(\varnothing), \pi_*(\{*\})\} \\ &= \mathrm{m}_{\frac{1}{2}}\{0, 1\} \\ &= \tfrac{1}{2}\,. \end{aligned}$$

b. $x = \frac{1}{2}$
By lemma L-14, $\mathcal{Y}^\gamma = \{\,\varnothing, \{*\}\,\}$ for all $\gamma \in \mathbf{I}$, i.e.

$$\begin{aligned} \pi_{*\gamma}(X) &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(\varnothing), \pi_*(\{*\})\} \\ &= \mathrm{m}_{\frac{1}{2}}\{0, 1\} \\ &= \tfrac{1}{2}\,. \end{aligned}$$

c. $x > \frac{1}{2}$
If $\gamma \leq 2x - 1$ then again by L-14, $\mathcal{Y}^\gamma = \{\{*\}\}$. Therefore

$$\begin{aligned} \pi_{*\gamma}(X) &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(\{*\})\} \\ &= \mathrm{m}_{\frac{1}{2}}\{1\} \\ &= 1\,. \end{aligned}$$

By the same lemma if $\gamma > 2x - 1$, $\mathcal{Y}^\gamma = \{\varnothing, \{*\}\}$ and hence again

$$\begin{aligned}\pi_{*\gamma}(X) &= \mathrm{m}_{\frac{1}{2}}\{\pi_*(\varnothing), \pi_*(\{*\})\}\\ &= \tfrac{1}{2}\,.\end{aligned}$$

(DFS 3) ***(External negation)***

Lemma 20
If $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$, then

$$(\neg D)_\gamma(X_1, \ldots, X_n) = \neg(D_\gamma(X_1, \ldots, X_n))$$

for all $\gamma \in \mathbf{I}$, where $\neg : \mathbf{I} \longrightarrow \mathbf{I}$ is the standard negation $\neg x = 1 - x$.

Proof Noting that $\mathrm{m}_{\frac{1}{2}} \neg Z = \neg\, \mathrm{m}_{\frac{1}{2}} Z$ because

$$\begin{aligned}\mathrm{m}_{\frac{1}{2}} \neg Z &= \mathrm{m}_{\frac{1}{2}}\{1 - z : z \in Z\}\\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{1 - z : z \in Z\},\ \sup\{1 - z : z \in Z\}) && \text{by Def. 36}\\ &= \mathrm{m}_{\frac{1}{2}}(1 - \sup\{z : z \in Z\}, 1 - \inf\{z : z \in Z\})\\ &= \mathrm{m}_{\frac{1}{2}}(1 - \sup Z, 1 - \inf Z)\\ &= 1 - \mathrm{m}_{\frac{1}{2}}(\sup Z,\ \inf Z) && (\mathrm{m}_{\frac{1}{2}} \text{ is symmetric sum})\\ &= \neg\, \mathrm{m}_{\frac{1}{2}} Z\,,\end{aligned}$$

we obtain

$$\begin{aligned}(\neg D)_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{(\neg D)(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_\gamma^n\}\\ &= \mathrm{m}_{\frac{1}{2}}\{1 - D(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_n^\gamma\}\\ &= 1 - \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_n^\gamma\}\\ &= 1 - D_\gamma(X_1, \ldots, X_n)\end{aligned}$$

for all $\gamma \in \mathbf{I}$.

(DFS 4) ***(Argument transposition)***

Lemma 21
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, $1 \leq k \leq n$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$D_\gamma(X_1, \ldots, X_{k-1}, X_n, X_{k+1}, \ldots, X_{n-1}, X_k) = (D\tau_k)_\gamma(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$.

Proof Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $k \in \{1, \ldots, n\}$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$D_\gamma(X_1, \ldots, X_{k-1}, X_n, X_{k+1}, \ldots, X_{n-1}, X_k) = \mathrm{m}_{\frac{1}{2}}\{D(Z_1, \ldots, Z_n) : Z_i \in \mathcal{Z}_i^\gamma\};$$

$$\mathcal{Z}_i^\gamma = \begin{cases} \{Z \subseteq E : (X_i)_\gamma^{\min} \subseteq Z \subseteq (Z)_\gamma^{\max}\} & : \quad i \neq k,\ i \neq n \\ \{Z \subseteq E : (X_n)_\gamma^{\min} \subseteq Z \subseteq (X_n)_\gamma^{\max}\} & : \quad i = k \\ \{Z \subseteq E : (X_k)_\gamma^{\min} \subseteq Z \subseteq (X_k)_\gamma^{\max}\} & : \quad i = n \end{cases}$$

The right-hand side of the equation is

$$\begin{aligned} &(D\tau_k)_\gamma(X_1, \ldots, X_n) \\ &= \mathrm{m}_{\frac{1}{2}}\{(D\tau_k)(Y_1, \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma,\ i = 1, \ldots, n\} \\ &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_{k-1}, Y_n, Y_{k+1}, \ldots, Y_{n-1}, Y_k) : Y_i \in \mathcal{Y}_i^\gamma,\ i = 1, \ldots, n\} \\ &= \mathrm{m}_{\frac{1}{2}}\{D(Z_1, \ldots, Z_n) : Z_i \in \mathcal{Z}'^\gamma_i,\ i = 1, \ldots, n\} \end{aligned}$$

where

$$\mathcal{Y}_i^\gamma = \{Y \subseteq E : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\}, \quad i = 1, \ldots, n$$

and the $\mathcal{Z}'^\gamma_i$ are obtained by the renaming

$$\mathcal{Z}'^\gamma_i = \begin{cases} \mathcal{Y}_i^\gamma & : \quad i \neq k \text{ and } i \neq n \\ \mathcal{Y}_n^\gamma & : \quad i = k \\ \mathcal{Y}_k^\gamma & : \quad i = n \end{cases}$$

But obviously, $\mathcal{Z}_i^\gamma = \mathcal{Z}'^\gamma_i$ for all $i = 1, \ldots, n$, which finishes the proof.

(DFS 5) ***(Internal complementation)***

Lemma 22
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$(D\neg)_\gamma(X_1, \ldots, X_n) = D_\gamma(X_1, \ldots, \neg X_n)$$

for all $\gamma \in (0, 1]$.

Proof Suppose $\gamma \in (0, 1]$. Then

$$\begin{aligned} (D\neg)_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{(D\neg)(Y_1, \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_{n-1}, \neg Y_n) : Y_i \in \mathcal{Y}_i^\gamma\}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Y : (X_1)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \subseteq (X_i)_{> \frac{1}{2}-\frac{1}{2}\gamma}\}, \end{aligned}$$

as we have assumed $\gamma \in (0, 1]$.

On the other hand,

$$D_\gamma(X_1, \ldots, X_{n-1}, \neg X_n) \;=\; \mathrm{m}_{\frac{1}{2}}\{D(Z_1, \ldots, Z_n) : Z_i \in \mathcal{Z}_i^\gamma\}$$

where $\mathcal{Z}_i^\gamma = \mathcal{Y}_i^\gamma$ for all $i < n$ and

$$\begin{aligned}\mathcal{Z}_n^\gamma &= \{Z : (\neg X_n)_\gamma^{\min} \subseteq Z \subseteq (\neg X_n)_\gamma^{\max}\} \\ &= \{Z : (\neg X_n)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \subseteq Z \subseteq (\neg X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}\}\,,\end{aligned}$$

again as we have assumed that $\gamma > 0$.
We only have to show that $\neg\mathcal{Y}_n^\gamma = \mathcal{Z}_n^\gamma$, where $\neg\mathcal{Y}_n^\gamma = \{\neg Y : Y \in \mathcal{Y}_n^\gamma\}$.
Recalling that

$$A \subseteq Y \subseteq B \;\Leftrightarrow\; \neg B \subseteq \neg Y \subseteq \neg A\,,$$

for all (crisp) subsets A, Y, B of E, we obtain:

$$\begin{aligned} Z \in \neg\mathcal{Y}_n^\gamma &\Leftrightarrow \text{ exists } Y \in \mathcal{Y}_n^\gamma \text{ s.th. } Z = \neg Y \\ &\Leftrightarrow \text{ exists } Y,\ (X_n)_\gamma^{\min} \subseteq Y \subseteq (X_n)_\gamma^{\max} \text{ s.th. } Z = \neg Y \\ &\Leftrightarrow \neg(X_n)_\gamma^{\max} \subseteq Z \subseteq \neg(X_n)_\gamma^{\min} \\ &\overset{(*)}{\Leftrightarrow} (\neg X_n)_\gamma^{\min} \subseteq Z \subseteq (\neg X_n)_\gamma^{\max} \\ &\Leftrightarrow Z \in \mathcal{Z}_n^\gamma\,. \end{aligned}$$

To see $(*)$, we need that by Th-39,

$$\begin{aligned}(\neg X)_{\geq\alpha} &= \neg(X)_{>1-\alpha} \\ (\neg X)_{>\alpha} &= \neg(X)_{\geq 1-\alpha}\end{aligned}$$

for all $X \in \widetilde{\mathcal{P}}(E)$ and $\alpha \in \mathbf{I}$. Now as we have assumed $\gamma \in (0, 1]$,

$$\begin{aligned}\neg(X_n)_\gamma^{\min} &= \neg(X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \\ &= (\neg X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma} \\ &= (\neg X_n)_\gamma^{\max}\end{aligned}$$

and

$$\begin{aligned}\neg(X_n)_\gamma^{\max} &= \neg(X_n)_{\geq\frac{1}{2}-\frac{1}{2}\gamma} \\ &= (\neg X_n)_{>\frac{1}{2}+\frac{1}{2}\gamma} \\ &= (\neg X_n)_\gamma^{\min}\,,\end{aligned}$$

which completes the proof.

(DFS 6) ***(Internal meets)***

Lemma 23
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-ary fuzzy pre-determiner and $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then for all $\gamma \in (0, 1]$,

$$(D\cap)_\gamma(X_1, \ldots, X_{n+1}) = D_\gamma(X_1, \ldots, X_{n-1}, X_n \cap X_{n+1})\,.$$

Note. If $\gamma = 0$, the condition also holds, but we shall drop this case because it is not needed lateron.

Proof The left-hand side of the equation is defined

$$\begin{aligned}(D\cap)_\gamma(X_1,\ \ldots,\ X_{n+1}) &= \mathrm{m}_{\frac{1}{2}}\{D\cap(Z_1,\ \ldots,\ Z_{n+1}) : Z_1 \in \mathcal{Z}_1^\gamma,\ \ldots,\ Z_{n+1} \in \mathcal{Z}_{n+1}^\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}\{D(Z_1,\ \ldots,\ Z_{n-1},\ Z_n \cap Z_{n+1}) : Z_1 \in \mathcal{Z}_1^\gamma,\ \ldots,\ Z_{n+1} \in \mathcal{Z}_{n+1}^\gamma\}\end{aligned}$$

where for $i = 1,\ \ldots,\ n+1$,

$$\begin{aligned}\mathcal{Z}_i^\gamma &= \{Z : (X_i)_\gamma^{\min} \subseteq Z \subseteq (X_i)_\gamma^{\max}\} \\ &= \{Z : (X_i)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Z \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma}\},\end{aligned}$$

as we have assumed $\gamma \in (0,\ 1]$.
The right-hand side becomes

$$D_\gamma(X_1,\ \ldots,\ X_{n-1},\ X_n \cap X_{n+1}) = \mathrm{m}_{\frac{1}{2}}\{D(Y_1,\ \ldots,\ Y_n) : Y_1 \in \mathcal{Y}_1^\gamma,\ \ldots,\ Y_n \in \mathcal{Y}_n^\gamma\},$$

where

$$\begin{aligned}\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\ &= \mathcal{Z}_i^\gamma\end{aligned}$$

for all $i = 1,\ \ldots,\ n-1$, and

$$\begin{aligned}\mathcal{Y}_n^\gamma &= \{Y : (X_n \cap X_{n+1})_\gamma^{\min} \subseteq Y \subseteq (X_n \cap X_{n+1})_\gamma^{\max}\} \\ &= \{Y : (X_n \cap X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \subseteq (X_n \cap X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}\},\end{aligned}$$

again by the assumption that $\gamma \in (0,\ 1]$.
Abbreviating

$$\mathcal{Y'}_n^\gamma = \{Z_n \cap Z_{n+1} : Z_n \in \mathcal{Z}_n^\gamma,\ Z_{n+1} \in \mathcal{Z}_{n+1}^\gamma\},$$

we are left to show that $\mathcal{Y'}_n^\gamma = \mathcal{Y}_n^\gamma$.

a. $\mathcal{Y'}_n^\gamma \subseteq \mathcal{Y}_n^\gamma$.
   To see this, let $Z_n \in \mathcal{Z}_n^\gamma$, $Z_{n+1} \in \mathcal{Z}_{n+1}^\gamma$.
   We have to prove that $(X_n \cap X_{n+1})_\gamma^{\min} \subseteq Z_n \cap Z_{n+1}$ and $Z_n \cap Z_{n+1} \subseteq (X_n \cap X_{n+1})_\gamma^{\max}$. Due to our assumption that $\gamma \in (0,\ 1]$, these conditions become $(X_n \cap X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Z_n \cap Z_{n+1}$ and $Z_n \cap Z_{n+1} \subseteq (X_n \cap X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}$.
   As concerns the first condition, let us recall that by Th-37,

$$(A \cap B)_{\geq\alpha} = (A)_{\geq\alpha} \cap (B)_{\geq\alpha}.$$

   for all $A,\ B \in \widetilde{\mathcal{P}}(E)$ and $\alpha \in \mathbf{I}$. Therefore,

$$\begin{aligned}(X_n \cap X_{n+1})_\gamma^{\min} &= (X_n \cap X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \\ &= (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \cap (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \\ &= (X_n)_\gamma^{\min} \cap (X_{n+1})_\gamma^{\min} \\ &\subseteq Z_n \cap Z_{n+1},\end{aligned}$$

as $(X_n)^{\min}_\gamma \subseteq Z_n$ and $(X_{n+1})^{\min}_\gamma \subseteq Z_{n+1}$ by the definition of $\mathcal{Z}^\gamma_n$ and $\mathcal{Z}^\gamma_{n+1}$.

Also by the monotonicity of $\cap$, from $Z_n \subseteq (X_n)^{\max}_\gamma$ and $Z_{n+1} \subseteq (X_{n+1})^{\max}_\gamma$ it follows that

$$Z_n \cap Z_{n+1} \;\subseteq\; (X_n)^{\max}_\gamma \cap X_{n+1}\gamma\,.$$

Noting that for all $A,\, B \in \widetilde{\mathcal{P}}(E)$,

$$(A \cap B)_{>\alpha} \;=\; (A)_{>\alpha} \cap (B)_{>\alpha}\,,$$

we obtain (by the assumption that $\gamma \in (0,\, 1]$):

$$\begin{aligned}(X_n \cap X_{n+1})^{\max}_\gamma &= (X_n \cap X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}\\ &= (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma} \cap (X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}\\ &= (X_n)^{\max}_\gamma \cap (X_{n+1})^{\max}_\gamma\\ &\supseteq Z_n \cap Z_{n+1}\,.\end{aligned}$$

b. $\mathcal{Y}^\gamma_n \subseteq \mathcal{Y}'^\gamma_n$.

Let $Y \in \mathcal{Y}^\gamma_n$, i.e. $(X_n \cap X_{n+1})^{\min}_\gamma \subseteq Y \subseteq (X_n \cap X_{n+1})^{\max}_\gamma$. As $\gamma \in (0,\, 1]$, this condition becomes $(X_n \cap X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \subseteq (X_n \cap X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}$.

Let $Z_n = Y \cup (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma}$, $Z_{n+1} = Y \cup (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma}$. Firstly, let us note that $Y = Z_n \cap Z_{n+1}$:

$$\begin{aligned}&Z_n \cap Z_{n+1}\\ &= (Y \cup (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma}) \cap (Y \cup (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma})\\ &= (Y \cap Y) \cup (Y \cap (X)_{\geq\frac{1}{2}+\frac{1}{2}\gamma}) \cup ((X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \cap Y) \cup ((X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \cap (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma})\\ &= Y\,.\end{aligned}$$

We have used the distributivity of $\cap$ and $\cup$ and the obvious inclusions $Y \cap (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y$, $Y \cap (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y$, and $(X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \cap (X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = (X_n \cap X_{n+1})_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y$.

We still have to show that $Z_n \in \mathcal{Z}^\gamma_n$, $Z_{n+1} \in \mathcal{Z}^\gamma_{n+1}$. We will only consider Z_n as the other case can be treated analogously.

We recall the definition of $\mathcal{Z}^\gamma_n$,

$$\begin{aligned}\mathcal{Z}^\gamma_n &= \{Z : (X_n)^{\min}_\gamma \subseteq Z \subseteq (X_n)^{\max}_\gamma\}\\ &= \{Z : (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Z \subseteq (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}\} \text{ as } \gamma > 0\,.\end{aligned}$$

Now obviously $(X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \cup (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} = Z_n$. To see why the second inclusion $Z_n \subseteq (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}$ holds, note that

$$\begin{aligned}Y &\subseteq (X_n \cap X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}\\ &= (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma} \cap (X_{n+1})_{>\frac{1}{2}-\frac{1}{2}\gamma}\\ &\subseteq (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}\end{aligned}$$

and $(X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}$ ($\gamma > 0$), hence $Y \cup (X_n)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_n)_{>\frac{1}{2}-\frac{1}{2}\gamma}$ also.

(DFS 7) ***(Argument insertion)***

Lemma 24 *Suppose $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ is an $(n+1)$-ary fuzzy pre-determiner, $A \in \mathcal{P}(E)$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,*

$$(D \triangleleft A)_\gamma(X_1, \ldots, X_n) = D_\gamma(X_1, \ldots, X_n, A)$$

for all $\gamma \in \mathbf{I}$.

Proof Let $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
(D \triangleleft A)_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{D \triangleleft A(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_n^\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n, A) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_n^\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n, Y_{n+1}) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_n \in \mathcal{Y}_n^\gamma, Y_{n+1} \in \{A\}\} \\
&= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_{n+1}) : Y_i \in \mathcal{Y}_i^\gamma\} \\
&= D_\gamma(X_1, \ldots, X_n, A)\,,
\end{aligned}$$

noting that

$$\begin{aligned}
\mathcal{Y}_{n+1}^\gamma &= \{Y : (A)_\gamma^{\min} \subseteq Y \subseteq (A)_\gamma^{\max}\} \\
&= \{Y : A \subseteq Y \subseteq A\} && \text{by L-12} \\
&= \{A\}\,.
\end{aligned}$$

(DFS 8) ***(Preservation of monotonicity)***

Lemma 25
Assume $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is monotonically decreasing in its n-th argument. Let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ where $X_n \subseteq X'_n$. Then for all $\gamma \in \mathbf{I}$,

$$D_\gamma(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X_{n-1}, X'_n)\,.$$

Proof As stated by L-16,

$$\begin{aligned}
D_\gamma(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}(D_{\min}, D_{\max})\,,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\
D_{\min} &= \inf\{D(Y_1, \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma\} \\
D_{\max} &= \sup\{D(Y_1, \ldots, Y_n) : Y_i \in \mathcal{Y}_i^\gamma\}\,.
\end{aligned}$$

Analogously,

$$\begin{aligned}
D_\gamma(X_1, \ldots, X_{n-1}, X'_n) &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_{n-1}, Y'_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma, Y'_n \in \mathcal{Y'}_n^\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}(D'_{\min}, D'_{\max})
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \quad i = 1, \ldots, n-1 \\
\mathcal{Y'}_n^\gamma &= \{Y : (X'_n)_\gamma^{\min} \subseteq Y \subseteq (X'_n)_\gamma^{\max}\} \\
D'_{\min} &= \inf\{D(Y_1, \ldots, Y_{n-1}, Y'_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma, Y'_n \in \mathcal{Y'}_n^\gamma\} \\
D'_{\max} &= \sup\{D(Y_1, \ldots, Y_{n-1}, Y'_n) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma, Y'_n \in \mathcal{Y'}_n^\gamma\}\,.
\end{aligned}$$

By the monotonicity of fuzzy median, it is sufficient to show that $D_{\min} \geq D'_{\min}$ and $D_{\max} \geq D'_{\max}$. As D is monotonically decreasing in the n-th argument, we obtain

$$\begin{aligned}
D_{\min} &= \inf\{D(Y_1, \ldots, Y_{n-1}, (X_n)_\gamma^{\max}) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma\} \\
D_{\max} &= \sup\{D(Y_1, \ldots, Y_{n-1}, (X_n)_\gamma^{\min}) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma\} \\
D'_{\min} &= \inf\{D(Y_1, \ldots, Y_{n-1}, (X'_n)_\gamma^{\max}) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma\} \\
D'_{\max} &= \sup\{D(Y_1, \ldots, Y_{n-1}, (X'_n)_\gamma^{\min}) : Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma\}\,.
\end{aligned}$$

By the monotonicity of α-cut and strict α-cut, $X_n \subseteq X'_n$ implies that $(X_n)_{\geq\alpha} \subseteq (X'_n)_{\geq\alpha}$ and $(X_n)_{>\alpha} \subseteq (X'_n)_{>\alpha}$ for all $\alpha \in \mathbf{I}$. Hence by the definition of $(\_)_\gamma^{\min}$ and $(\_)_\gamma^{\max}$,

$$\begin{aligned}
(X_n)_\gamma^{\min} &\subseteq X'_n\gamma \\
(X_n)_\gamma^{\max} &\subseteq (X'_n)_\gamma^{\max}
\end{aligned}$$

for all $\gamma \in \mathbf{I}$. Because D is monotonically decreasing in the n-th argument,

$$\begin{aligned}
D(Y_1, \ldots, Y_{n-1}, (X_n)_\gamma^{\max}) &\geq D(Y_1, \ldots Y_{n-1}, (X'_n)_\gamma^{\max}) \\
D(Y_1, \ldots, Y_{n-1}, (X_n)_\gamma^{\min}) &\geq D(Y_1, \ldots, Y_{n-1}, (X'_n)_\gamma^{\min})\,,
\end{aligned}$$

for all $Y_1 \in \mathcal{Y}_1^\gamma, \ldots, Y_{n-1} \in \mathcal{Y}_{n-1}^\gamma$ and hence

$$\begin{aligned}
D_{\min} &\geq D'_{\min} \\
D_{\max} &\geq D'_{\max}\,.
\end{aligned}$$

(DFS 9) **(Functional application)**
For technical reasons, we need to introduce an alternative definition of three-valued cuts which is slightly different from the original one. For these modified cuts it is easy to prove that (DFS 9) holds, which is not generally the case for the original definition. Let us remark that it is inevitable to introduce both alternatives because the modified cuts fail to be symmetric under negation (i.e. had we based our definitions on the modified cuts, we would have had to introduce our other definition in order to prove (DFS 5)).

Definition 58 (Alternative definition of three-valued cuts)

$$\begin{aligned}
D_\gamma^\blacktriangledown(X_1, \ldots, X_n) &= \mathrm{m}_{\frac{1}{2}}\{D(Y_1, \ldots, Y_n) : Y_1 \in \mathcal{Y}^{\blacktriangledown\gamma}_1, \ldots, \mathcal{Y}^{\blacktriangledown\gamma}_n\} \\
\mathcal{Y}^{\blacktriangledown\gamma}_i &= \{Y : (X_i)^{\blacktriangledown\min}_\gamma \subseteq Y \subseteq (X_i)^{\blacktriangledown\max}_\gamma\} \\
(X_i)^{\blacktriangledown\min}_\gamma &= (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} \\
(X_i)^{\blacktriangledown\max}_\gamma &= \begin{cases} (X_i)_{\geq\frac{1}{2}} & : \gamma = 0 \\ (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma > 0 \end{cases}
\end{aligned}$$

Lemma 26
Suppose $f : E \longrightarrow E'$, where $E,\ E' \neq \varnothing$, and $X^{\min} \subseteq X^{\max} \in \mathcal{P}(E)$. Then

$$\{\widehat{f}(Y) : X^{\min} \subseteq Y \subseteq X^{\max}\} = \{Z : \widehat{f}(X^{\min}) \subseteq Z \subseteq \widehat{f}(X^{\max})\}\,.$$

Proof

“$\supseteq$” Clearly if $X^{\min} \subseteq Y \subseteq X^{\max}$, then $\widehat{f}(X^{\min}) \subseteq \widehat{f}(Y) \subseteq \widehat{f}(X^{\max})$ by the monotonicity of $\widehat{f}$ and hence

$$f(Y) \in \{Z : \widehat{f}(X^{\min}) \subseteq Z \subseteq \widehat{f}X^{\max}\}$$

for all $Y \in \{Y : X^{\min} \subseteq Y \subseteq X^{\max}\}$.

“$\subseteq$” Let $Z \in \mathcal{P}(E')$ such that $\widehat{f}(X^{\min}) \subseteq Z \subseteq \widehat{f}(X^{\max})$.
$f^{-1}(Z) \cap X^{\max} \subseteq X^{\max}$ is immediate. Furthermore, $\widehat{f}(X^{\min}) \subseteq Z$ entails

$$X^{\min} \subseteq f^{-1}\widehat{f}(X^{\min}) \subseteq f^{-1}(Z)$$

and

$$X^{\min} = X^{\min} \cap X^{\max} \subseteq f^{-1}(Z) \cap X^{\max}\,.$$

Therefore

$$f^{-1}(Z) \cap X^{\max} \in \{Y : X^{\min} \subseteq Y \subseteq X^{\max}\}\,.$$

Finally,

$$\begin{aligned}
\widehat{f}(f^{-1}(Z) \cap X^{\max}) &= \widehat{f}(f^{-1}(Z)) \cap \widehat{f}X^{\max} \\
&= Z \cap \widehat{f}(X^{\max}) && \text{because } Z \subseteq \operatorname{Im} f \\
&= Z\,, && \text{because } Z \subseteq \widehat{f}(X^{\max})
\end{aligned}$$

which finishes the proof.

Lemma 27
Let $D : \mathcal{P}(E)'^{n} \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $f_1,\ \ldots,\ f_n : E \longrightarrow E'$ and $(X_1,\ \ldots,\ X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then for all $\gamma \in \mathbf{I} \setminus \{0\}$,

$$(D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)^{\blacktriangledown}_{\gamma}(X_1,\ \ldots,\ X_n) = D^{\blacktriangledown}_{\gamma}(\hat{\hat{f}}_1(X_1),\ \ldots,\ \hat{\hat{f}}_n(X_n))\,,$$

i.e. $(D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)^{\blacktriangledown}_{\gamma} = D^{\blacktriangledown}_{\gamma} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i$.

Proof Let $D : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $f_1, \ldots, f_n : E \longrightarrow E'$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Let us abbreviate

$$\begin{aligned}
\mathcal{Z}_i^\gamma &= \{Z : (\hat{\hat{f}}_i(X_i))^{\blacktriangledown \min}_\gamma \subseteq Z \subseteq (\hat{\hat{f}}_i(X_i))^{\blacktriangledown \max}_\gamma\} && i = 1, \ldots, n \\
&= \{Z : \left(\hat{\hat{f}}_i(X_i)\right)_{> \frac{1}{2} + \frac{1}{2}\gamma} \subseteq Z \subseteq \left(\hat{\hat{f}}_i(X_i)\right)_{> \frac{1}{2} - \frac{1}{2}\gamma}\} && \text{as } \gamma > 0 \text{ by assumption} \\
\mathcal{Z} &= \mathcal{Z}_1^\gamma \times \cdots \times \mathcal{Z}_n^\gamma \\
\mathcal{Y}_i^\gamma &= \{Y : (X_i)^{\blacktriangledown \min}_\gamma \subseteq Y \subseteq (X_i)^{\blacktriangledown \max}_\gamma\} && i = 1, \ldots, n \\
&= \{Y : (X_i)_{> \frac{1}{2} + \frac{1}{2}\gamma} \subseteq Y \subseteq (X_i)_{> \frac{1}{2} - \frac{1}{2}\gamma}\} && \text{as } \gamma > 0 \text{ by assumption} \\
\mathcal{Y}^\gamma &= \mathcal{Y}_1^\gamma \times \cdots \times \mathcal{Y}_n^\gamma\,.
\end{aligned}$$

where $\hat{\hat{f}}_i : \widetilde{\mathcal{P}}(E) \longrightarrow \widetilde{\mathcal{P}}(E')$ is obtained from $f_i : E \longrightarrow E'$ by applying the standard extension principle (see Def. 17).
By Th-38,

$$\left(\hat{\hat{f}}_i(X_i)\right)_{> \frac{1}{2} + \frac{1}{2}\gamma} = \widehat{f}_i((X_i)_{> \frac{1}{2} + \frac{1}{2}\gamma}$$

and

$$\left(\hat{\hat{f}}_i(X_i)\right)_{> \frac{1}{2} - \frac{1}{2}\gamma} = \widehat{f}_i((X_i)_{> \frac{1}{2} - \frac{1}{2}\gamma}$$

for $i = 1, \ldots, n$, i.e.

$$\begin{aligned}
\mathcal{Z}_i^\gamma &= \{Z : \widehat{f}_i((X_i)_{> \frac{1}{2} + \frac{1}{2}\gamma}) \subseteq Z \subseteq \widehat{f}_i((X_i)_{> \frac{1}{2} - \frac{1}{2}\gamma})\} \\
&= \{\widehat{f}_i(Y) : (X_i)_{> \frac{1}{2} + \frac{1}{2}\gamma} \subseteq Y \subseteq X_i \tfrac{1}{2} - \tfrac{1}{2}\gamma\} && \text{by L-26} \\
&= \{\widehat{f}_i(Y) : Y \in \mathcal{Y}_i^\gamma\}\,. && (*)
\end{aligned}$$

Therefore

$$\begin{aligned}
D^{\blacktriangledown}_\gamma(\hat{\hat{f}}_1(X_1), \ldots, \hat{\hat{f}}_n(X_n)) &= \mathrm{m}_{\frac{1}{2}}\{D(Z_1, \ldots, Z_n) : (Z_1, \ldots, Z_n) \in \mathcal{Z}^\gamma\} \\
&= \mathrm{m}_{\frac{1}{2}}\{D(\widehat{f}_1(Y_1), \ldots, \widehat{f}_n(Y_n)) : (Y_1, \ldots Y_n) \in \mathcal{Y}\} && \text{by (*)} \\
&= (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n)\,.
\end{aligned}$$

Lemma 28
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.
Noting that $D^{\blacktriangledown}_0(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$,

a. if $D_0(X_1, \ldots, X_n) > \frac{1}{2}$, then $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing and

$$D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) \geq \tfrac{1}{2}$$

for all $\gamma \in \mathbf{I}$.

b. if $D_0(X_1, \ldots, X_n) = \frac{1}{2}$*, then* $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ *for all* $\gamma \in \mathbf{I}$.

c. if $D_0(X_1, \ldots, X_n) < \frac{1}{2}$*, then* $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$ *is monotonically increasing and*

$$D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) \le \tfrac{1}{2}$$

for all $\gamma \in \mathbf{I}$.

Proof Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. $D^{\blacktriangledown}_0(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$ is immediate from the definition of $D^{\blacktriangledown}_0$. Now for all $\gamma, \gamma' \in \mathbf{I} \setminus \{0\}$,

$$(X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma'} \subseteq (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}} \subseteq (X_i)_{\ge\frac{1}{2}} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma'}$$

i.e.

$$(X_i)^{\blacktriangledown\min}_{\gamma'} \subseteq (X_i)^{\blacktriangledown\min}_{\gamma} \subseteq (X_i)^{\blacktriangledown\min}_{0} \subseteq (X_i)^{\blacktriangledown\max}_{0} \subseteq (X_i)^{\blacktriangledown\max}_{\gamma} \subseteq (X_i)^{\blacktriangledown\max}_{\gamma'}$$

and hence

$$\mathcal{Y}^{\blacktriangledown}{}^{0}_{i} \subseteq \mathcal{Y}^{\blacktriangledown}{}^{\gamma}_{i} \subseteq \mathcal{Y}^{\blacktriangledown}{}^{\gamma'}_{i}\,, \tag{*}$$

for $i = 1, \ldots, n$. Let us abbreviate

$$Z^{\blacktriangledown}_\gamma = \{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^{\blacktriangledown\gamma}\}\,.$$

Then by (*),

$$\varnothing \ne Z^{\blacktriangledown}_0 \subseteq Z^{\blacktriangledown}_\gamma \subseteq Z^{\blacktriangledown}_{\gamma'}\,, \tag{36}$$

for all $0 < \gamma \le \gamma'$.

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$.
Then $D_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_0 > \frac{1}{2}$, $Z^{\blacktriangledown}_0 \subseteq Z^{\blacktriangledown}_\gamma$ and by Th-30, $\frac{1}{2} \le D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_\gamma \le D_0(X_1, \ldots, X_n)$.
Likewise if $\gamma' > \gamma$, then $Z^{\blacktriangledown}_\gamma \subseteq Z^{\blacktriangledown}_{\gamma'}$ and by Th-30, $\frac{1}{2} \le D^{\blacktriangledown}_{\gamma'}(X_1, \ldots, X_n) \le D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$, i.e. $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing in γ.

b. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$. Then $D^{\blacktriangledown}_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_0 = \frac{1}{2}$. $Z^{\blacktriangledown}_0 \subseteq Z^{\blacktriangledown}_\gamma$ and $Z^{\blacktriangledown}_0 \ne \varnothing$ (by L-15 and $\mathcal{Y}^0 = \mathcal{Y}^{\blacktriangledown 0}$) yields $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_\gamma = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ (again by Th-30).

c. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$.
Then $D_0(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_0 < \frac{1}{2}$, $Z^{\blacktriangledown}_0 \subseteq Z^{\blacktriangledown}_\gamma$ and by Th-30, $\frac{1}{2} \ge D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_\gamma \ge D_0(X_1, \ldots, X_n)$.
Likewise if $\gamma' > \gamma$, then $Z^{\blacktriangledown}_\gamma \subseteq Z^{\blacktriangledown}_{\gamma'}$ and by Th-30, $\frac{1}{2} \ge D^{\blacktriangledown}_{\gamma'}(X_1, \ldots, X_n) \ge D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$, i.e. $D^{\blacktriangledown}_\gamma(X_1, \ldots, X_n)$ is monotonically increasing in γ.

Lemma 29
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then for all $\gamma \in \mathbf{I} \setminus \{0\}$,

a. if $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$, then $\frac{1}{2} \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n) \leq D_\gamma(X_1, \ldots, X_n)$

b. if $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$, then $D_\gamma(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n) \leq \frac{1}{2}$

c. if $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$, then $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$.

Proof Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$, $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ and $\gamma \in \mathbf{I} \setminus \{0\}$. We use abbreviations

$$\begin{aligned}
\mathcal{Y}_i^\gamma &= \{Y : (X_i)_\gamma^{\min} \subseteq Y \subseteq (X_i)_\gamma^{\max}\} \\
&= \{Y : (X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \subseteq (X_i)_{> \frac{1}{2}-\frac{1}{2}\gamma}\} \qquad (\gamma > 0 \text{ by assumption}) \\
\mathcal{Y}^\gamma &= \mathcal{Y}_1^\gamma \times \cdots \times \mathcal{Y}_n^\gamma \\
Z_\gamma &= \{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\} \\
\mathcal{Y}^{\blacktriangledown}{}_i^\gamma &= \{Y : (X_i)^{\blacktriangledown}{}_\gamma^{\min} \subseteq Y \subseteq (X_i)^{\blacktriangledown}{}_\gamma^{\max}\} \\
&= \{Y : (X_i)_{> \frac{1}{2}+\frac{1}{2}\gamma} \subseteq Y \subseteq (X_i)_{> \frac{1}{2}-\frac{1}{2}\gamma}\} \qquad (\gamma > 0 \text{ by assumption}) \\
\mathcal{Y}^{\blacktriangledown\gamma} &= \mathcal{Y}^{\blacktriangledown}{}_1^\gamma \times \cdots \times \mathcal{Y}^{\blacktriangledown}{}_n^\gamma \\
Z^\blacktriangledown_\gamma &= \{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^{\blacktriangledown\gamma}\}\,.
\end{aligned}$$

Clearly, $\mathcal{Y}_i^\gamma \subseteq \mathcal{Y}^{\blacktriangledown}{}_i^\gamma$ and hence $Z_\gamma \subseteq Z^\blacktriangledown_\gamma$ for all $\gamma > 0$.

a. $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$. Then $D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_\gamma > \frac{1}{2}$ and $Z_\gamma \subseteq Z^\blacktriangledown_\gamma$. Th-30 yields $\frac{1}{2} \leq \mathrm{m}_{\frac{1}{2}} Z^\blacktriangledown_\gamma = D^\blacktriangledown_\gamma(X_1, \ldots, X_n) \leq D_\gamma(X_1, \ldots, X_n)$.

b. $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$. Then $D_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z_\gamma < \frac{1}{2}$ and $Z_\gamma \subseteq Z^\blacktriangledown_\gamma$. Th-30 yields $\frac{1}{2} \geq \mathrm{m}_{\frac{1}{2}} Z^\blacktriangledown_\gamma = D^\blacktriangledown_\gamma(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X_n)$.

c. $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$. Then $\mathrm{m}_{\frac{1}{2}} Z_\gamma = \frac{1}{2}$, where $Z_\gamma \neq \varnothing$ by L-15. From Th-30 we obtain $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}} Z^\blacktriangledown_\gamma = \frac{1}{2}$.

Lemma 30
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then for all $\gamma \in \mathbf{I}$,

a. if $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$, then $D_{\gamma'}(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ for all $\gamma' > \gamma$

b. if $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$, then $D_{\gamma'}(X_1, \ldots, X_n) \geq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ for all $\gamma' > \gamma$

c. if $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$, then $D_{\gamma'}(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma' > \gamma$.

Proof We use the same abbreviations as in the previous proof. Let us note that if $\gamma' > \gamma$, then

$$(X_i)_{\geq \frac{1}{2}+\frac{1}{2}\gamma'} \subseteq (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma'}$$

i.e.

$$(X_i)^{\min}_{\gamma'} \subseteq (X_i)^{\blacktriangledown\min}_{\gamma} \subseteq (X_i)^{\blacktriangledown\max}_{\gamma} \subseteq (X_i)^{\max}_{\gamma'}$$

and hence $\mathcal{Y}^{\blacktriangledown\gamma}_i \subseteq \mathcal{Y}^{\gamma'}_i$ for all $i = 1, \ldots, n$. In particular, $Z^{\blacktriangledown}_{\gamma} \subseteq Z_{\gamma'}$ for all $\gamma' > \gamma$.

a. $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) > \frac{1}{2}$, i.e. $\mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_{\gamma} > \frac{1}{2}$. By Th-30, $Z^{\blacktriangledown}_{\gamma} \subseteq Z_{\gamma'}$ yields

$$\tfrac{1}{2} \leq D_{\gamma'}(X_1, \ldots, X_n) \leq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n)\,.$$

b. $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) < \frac{1}{2}$, i.e. $\mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_{\gamma} < \frac{1}{2}$. By Th-30, $Z^{\blacktriangledown}_{\gamma} \subseteq Z_{\gamma'}$ yields

$$\tfrac{1}{2} \geq D_{\gamma'}(X_1, \ldots, X_n) \geq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n)\,.$$

c. $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) = \frac{1}{2}$, i.e. $\mathrm{m}_{\frac{1}{2}}\, Z^{\blacktriangledown}_{\gamma} = \frac{1}{2}$, where $Z^{\blacktriangledown}_{\gamma} \neq \varnothing$. Then it follows from Th-30 that

$$D_{\gamma'}(X_1, \ldots, X_n) = \mathrm{m}_{\frac{1}{2}}\, Z_{\gamma'} = \tfrac{1}{2}$$

also.

Lemma 31
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. If $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$, then $D_{\gamma}(X_1, \ldots, X_n)$ and $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n)$ are monotonically decreasing in γ and for all $\gamma \in \mathbf{I}$,

$$\begin{aligned} D_{\gamma}(X_1, \ldots, X_n) &\geq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \\ D_{\gamma'}(X_1, \ldots, X_n) &\leq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \qquad \textit{for all } \gamma' > \gamma. \end{aligned}$$

b. If $D_0(X_1, \ldots, X_n) \leq \frac{1}{2}$, then $D_{\gamma}(X_1, \ldots, X_n)$ and $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n)$ are monotonically increasing in γ and for all $\gamma \in \mathbf{I}$,

$$\begin{aligned} D_{\gamma}(X_1, \ldots, X_n) &\leq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \\ D_{\gamma'}(X_1, \ldots, X_n) &\geq D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n) \qquad \textit{for all } \gamma' > \gamma. \end{aligned}$$

Proof

a. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. We already know by Th-42 and L-28 that $D_{\gamma}(X_1, \ldots, X_n)$ and $D^{\blacktriangledown}_{\gamma}(X_1, \ldots, X_n)$ are decreasing in this case. Furthermore, $D_0(X_1, \ldots, X_n) = D^{\blacktriangledown}_0(X_1, \ldots, X_n)$, i.e. the case $\gamma = 0$ is trivially fulfilled because then $D_0(X_1, \ldots, X_n) \geq D^{\blacktriangledown}_0(X_1, \ldots, X_n)$ and

$$D_{\gamma'}(X_1, \ldots, X_n) \leq D_0(X_1, \ldots, X_n) = D^{\blacktriangledown}_0(X_1, \ldots, X_n)$$

for all $\gamma' > 0$.
If $\gamma > 0$, there are two cases to consider.

1. If $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$, then $D_\gamma(X_1, \ldots, X_n) \geq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ by L-29.a and
$$D_{\gamma'}(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$$
for all $\gamma' > \gamma$ by L-30.a or L-30.c .

2. If $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$, then $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ by L-29.c and $D_{\gamma'}(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma' > \gamma$ by Th-42. In particular, $D_\gamma(X_1, \ldots, X_n) \geq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ and $D_{\gamma'}(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ for all $\gamma' > \gamma$.

b. $D_0(X_1, \ldots, X_n) \leq \frac{1}{2}$. We already know by Th-42 and L-28 that $D_\gamma(X_1, \ldots, X_n)$ and $D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ are increasing in this case. Furthermore, $D_0(X_1, \ldots, X_n) = D^\blacktriangledown_0(X_1, \ldots, X_n)$, i.e. if $\gamma = 0$, then $D_0(X_1, \ldots, X_n) \leq D^\blacktriangledown_0(X_1, \ldots, X_n)$ and
$$D_{\gamma'}(X_1, \ldots, X_n) \geq D_0(X_1, \ldots, X_n) = D^\blacktriangledown_0(X_1, \ldots, X_n)$$
for all $\gamma' > 0$.
If $\gamma > 0$, there are two cases to consider.

1. If $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$, then $D_\gamma(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ by L-29.b and
$$D_{\gamma'}(X_1, \ldots, X_n) \geq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$$
for all $\gamma' > \gamma$ by L-30.b or L-30.c .

2. If $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$, then $D^\blacktriangledown_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ by L-29.c and $D_{\gamma'}(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma' > \gamma$ by Th-42. In particular, $D_\gamma(X_1, \ldots, X_n) \leq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ and $D_{\gamma'}(X_1, \ldots, X_n) \geq D^\blacktriangledown_\gamma(X_1, \ldots, X_n)$ for all $\gamma' > \gamma$.

B.9 Proof of Theorem 43

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then the integral in
$$\mathcal{M}(D) = \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma$$
ranges over the function $D(\_) : \mathbf{I} \longrightarrow \mathbf{I}$ defined by
$$D(\gamma) = D_\gamma(X_1, \ldots, X_n)$$
By Th-42, we know that $D(\_)$ is monotonically decreasing if $D(0) > \frac{1}{2}$, constant if $D(0) = \frac{1}{2}$, and monotonically increasing if $D(0) < \frac{1}{2}$. Hence, as every monotonic function on a closed interval, $D(\_)$ is Riemann integrable.

B.10 Proof of Theorem 44

Firstly, we will state two lemmas which allow for a simplification of the proofs for $\mathcal{M}^*$ and $\mathcal{M}_*$. This is mainly for reference purposes since the proofs of the lemmas are rather trivial.

Lemma 32 ($\mathcal{M}$ on constant D_γ)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. If $D_\gamma(X_1, \ldots, X_n)$ does not depend on γ, i.e.
$$D_\gamma(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$$
for all $\gamma \in \mathbf{I}$, then
$$\mathcal{M}(D)(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n).$$

Proof Let $D : \mathcal{P}(E)^0 \longrightarrow I$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ such that $D_\gamma(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$ for all $\gamma \in \mathbf{I}$. Then

$$\begin{aligned}
\mathcal{M}(D)(X_1, \ldots, X_n) &= \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma \\
&= \int_0^1 D_0(X_1, \ldots, X_n)\, d\gamma && \text{by assumption} \\
&= D_0(X_1, \ldots, X_n) \int_0^1 1\, d\gamma \\
&= D_0(X_1, \ldots, X_n)\,.
\end{aligned}$$

Lemma 33 ($\mathcal{M}$ on three-valued determiners)
Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ a three-valued fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\mathcal{M}(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad D(0) > \frac{1}{2} \\ \frac{1}{2} & : \quad D(0) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad D(0) < \frac{1}{2} \end{cases}$$

where we have used abbreviations $D(0) = D_0(X_1, \ldots, X_n)$ and

$$\gamma^* = \inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}\,.$$

Proof

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $\gamma^* > 0$.
Then $D_{\gamma'}(X_1, \ldots, X_n) > \frac{1}{2}$ for all $\gamma' \in [0, \gamma^*)$. But $D_{\gamma'} \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 1$. Therefore

$$\begin{aligned}
\mathcal{M}(D)(X_1, \ldots, X_n) &= \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma \\
&= \int_0^{\gamma^*} D_\gamma(X_1, \ldots, X_n)\, d\gamma + \int_{\gamma^*}^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma \\
&= \int_0^{\gamma^*} 1\, d\gamma + \int_{\gamma^*}^1 \tfrac{1}{2}\, d\gamma \\
&= \gamma^* + \tfrac{1}{2}(1 - \gamma^*) \\
&= \tfrac{1}{2} + \tfrac{1}{2}\gamma^*\,.
\end{aligned}$$

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$ and $\gamma^* > 0$.
Then $D_{\gamma'}(X_1, \ldots, X_n) < \frac{1}{2}$ for all $\gamma' \in [0, \gamma^*)$. But $D_{\gamma'} \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 0$.

Therefore

$$\begin{aligned}
\mathcal{M}(D)(X_1,\ \ldots,\ X_n) &= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^{\gamma^*} D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma + \int_{\gamma^*}^1 D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^{\gamma^*} 0\, d\gamma + \int_{\gamma^*}^1 \tfrac{1}{2}\, d\gamma \\
&= \tfrac{1}{2}(1-\gamma^*) \\
&= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* .
\end{aligned}$$

c. $\gamma^* = 0$ (In particular, this is the case if $D_0(X_1,\ \ldots,\ X_n) = 0$). Then $D_\gamma(X_1,\ \ldots,\ X_n) = \frac{1}{2}$ for all $\gamma > 0$ and hence

$$\begin{aligned}
\mathcal{M}(D)(X_1,\ \ldots,\ X_n) &= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^1 \tfrac{1}{2}\, d\gamma \\
&= \tfrac{1}{2} \\
&= \tfrac{1}{2} + \gamma^* \\
&= \tfrac{1}{2} - \gamma^* .
\end{aligned}$$

We need one more lemma which may simplify the proof that (DFS 9) is satisfied by $\mathcal{M}$.

Lemma 34

a. *Let $f,\ g : \mathbf{I} \longrightarrow \mathbf{I}$ monotonically increasing functions such that for all $x \in \mathbf{I}$,*

$$\begin{aligned}
&f(x) \le g(x) \\
&f(y) \ge g(x) \quad \text{for all } y > x .
\end{aligned}$$

Then

$$\int_0^1 f(x)\, dx = \int_0^1 g(x)\, dx .$$

b. *Likewise if $f,\ g : \mathbf{I} \longrightarrow \mathbf{I}$ monotonically decreasing and for all $x \in \mathbf{I}$,*

$$\begin{aligned}
&f(x) \ge g(x) \\
&f(y) \le g(x) \quad \text{for all } y > x .
\end{aligned}$$

Then

$$\int_0^1 f(x)\, dx = \int_0^1 g(x)\, dx .$$

Proof

a. Firstly let us note that both integrals exist as $f,\, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically increasing. Let $\varepsilon > 0$. Choose a partition $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ of $\mathbf{I}$ such that the lower Riemann sum of the partition fulfills

$$\begin{aligned} L_g = \sum_{i=0}^{n-1}(t_{i+1} - t_i)g(t_i) &\geq \int_0^1 g(x)\,dx - \varepsilon \\ L_f = \sum_{i=0}^{n-1}(t_{i+1} - t_i)f(t_i) &\geq \int_0^1 f(x)\,dx\,. \end{aligned}$$

Choose additional $s_i \in \mathbf{I}$, $i = 0,\, \ldots,\, n-1$ such that $t_i < s_i < t_{i+1}$ and $s_i - t_i < \frac{\varepsilon}{n}$, for all $i = 1,\, \ldots,\, n$.
Now consider the refinement

$$t_0 < s_0 < t_1 < s_1 \cdots < t_{n-1} < s_{n-1} < t_n$$

The lower sum of the refinement is

$$\begin{aligned} L'_f &= \underbrace{\sum_{i=0}^{n-1}(s_i - t_i)f(t_i)}_{\geq 0} + \sum_{i=0}^{n-1} \underbrace{(t_{i+1} - s_i)}_{\geq (t_{i+1}-t_i)-\frac{\varepsilon}{n}} \underbrace{f(s_i)}_{\geq g(t_i)} \\ &\geq \left(\sum_{i=0}^{n-1}(t_{i+1} - t_i)g(t_i)\right) - \varepsilon \end{aligned}$$

Hence $L'_f \geq L_g - \varepsilon \leq \int_0^1 g(x)\,dx - 2\varepsilon$. But being a lower sum, $L'_f \leq \int_0^1 f(x)\,dx$, i.e.

$$\int_0^1 f(x)\,dx \;\geq\; \int_0^1 g(x)\,dx - 2\varepsilon\,.$$

from which we obtain $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$ for $\varepsilon \to 0$.

b. If $f,\, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically decreasing and for all $x \in \mathbf{I}$,

$$\begin{aligned} f(x) &\geq g(x) \\ f(y) &\leq g(x) \quad \text{for all } y > x\,, \end{aligned}$$

then also $\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx$ because

$$\begin{aligned} \int_0^1 f(x)\,dx &= \int_0^1 1 - (1 - f(x))\,dx \\ &= 1 - \int_0^1 1 - f(x)\,dx \\ &= 1 - \int_0^1 1 - g(x)\,dx \quad \text{(by the above lemma)} \\ &= \int_0^1 g(x)\,dx\,. \end{aligned}$$

Proof of Theorem 44

Preservation of constants (DFS 1) Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then by L-18, $D_\gamma(\varnothing) = D(\varnothing) = D_0(\varnothing)$. L-32 yields $\mathcal{M}(D)(\varnothing) = D_0(\varnothing) = D(\varnothing)$, i.e. $\mathcal{M}(D) = D$, as desired.

Compatibility with π_* (DFS 2) Let $X \in \widetilde{\mathcal{P}}(\{*\})$ and $x = \widetilde{\pi}_* X$. We wish to prove that

$$\mathcal{M}(\pi_*)(X) = \widetilde{\pi}_*(X) = x\,. \tag{37}$$

Following L-19, we discern three cases.

a. $x > \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}(\pi_*)(X) &= \int_0^1 \left\{ \begin{array}{ccc} 1 & : & \gamma \leq 2x-1 \\ \frac{1}{2} & : & \gamma > 2x-1 \end{array} \right. d\gamma && \text{by L-19} \\
&= \int_0^{2x-1} 1\, d\gamma + \int_{2x-1}^1 \tfrac{1}{2}\, d\gamma \\
&= 2x - 1 + (1-(2x-1))\tfrac{1}{2} \\
&= 2x - 1 + 1 - x \\
&= x\,.
\end{aligned}$$

b. $x = \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}(\pi_*)(X) &= \int_0^1 \pi_{*\gamma}\, d\gamma \\
&= \int_0^1 \tfrac{1}{2}\, d\gamma && \text{by L-19} \\
&= \tfrac{1}{2}\,.
\end{aligned}$$

c. $x < \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}(\pi_*)(X) &= \int_0^1 \left\{ \begin{array}{ccc} 0 & : & \gamma \leq 1-2x \\ \frac{1}{2} & : & \gamma > 1-2x \end{array} \right. d\gamma && \text{by L-19} \\
&= \int_0^{1-2x} 0\, d\gamma + \int_{1-2x}^1 \tfrac{1}{2}\, d\gamma \\
&= (1-(1-2x))\tfrac{1}{2} \\
&= x\,.
\end{aligned}$$

External negation (DFS 3) Firstly, let us show that $\mathcal{M}$ is compatible to the standard negation $\neg x = 1 - x$.

$$\begin{aligned}
\mathcal{M}(\neg D)(X_1,\ \ldots,\ X_n) &= \int_0^1 (\neg D)_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^1 1 - D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma && \text{by L-20} \\
&= 1 - \int_0^1 D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \neg\mathcal{M}(D)(X_1,\ \ldots,\ X_n)\,.
\end{aligned}$$

From Th-11, it follows that $\widetilde{\mathcal{M}}(\neg)$ is the standard negation. In particular, $\mathcal{M}$ satisfies (DFS 3).

Argument transposition (DFS 4) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, $1 \leq k \leq n$ and $X_1,\ \ldots,\ X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$\begin{aligned}
\mathcal{M}(D\tau_k)(X_1,\ \ldots,\ X_n) &= \int_0^1 (D\tau_k)_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_{k-1},\ X_n,\ X_{k+1},\ \ldots,\ X_{n-1},\ X_k)\, d\gamma && \text{by L-21} \\
&= (\mathcal{M}(D)\tau_k)(X_1,\ \ldots,\ X_n)\,.
\end{aligned}$$

Internal complementation (DFS 5) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $X_1,\ \ldots,\ X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$\begin{aligned}
\mathcal{M}(D\neg)(X_1,\ \ldots,\ X_n) &= \int_0^1 (D\neg)_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\
&= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_{n-1},\ \neg X_n)\, d\gamma && \text{by L-22} \\
&= (\mathcal{M}(D)\neg)(X_1,\ \ldots,\ X_n)\,.
\end{aligned}$$

Internal meets (DFS 6) Firstly, we will show that $\mathcal{M}$ is compatible to the standard fuzzy intersection $\cap$ induced by the t-norm $\min : \mathbf{I} \longrightarrow \mathbf{I}$.
Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is an n-ary fuzzy pre-determiner and $X_1,\ \ldots,\ X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}
\mathcal{M}(D\cap)(X_1,\ \ldots,\ X_{n+1}) &= \int_0^1 (D\cap)_\gamma(X_1,\ \ldots,\ X_{n+1})\, d\gamma \\
&= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_{n-1},\ X_n \cap X_{n+1})\, d\gamma && \text{by L-23} \\
&= (\mathcal{M}(D)\cap)(X_1,\ \ldots,\ X_{n+1}\,.
\end{aligned}$$

From Th-11, it follows that $\widetilde{\mathcal{M}}(\wedge) = \min$. In particular, $\mathcal{M}$ satisfies (DFS 6).

Argument insertion (DFS 7) Let $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ an $(n+1)$-ary fuzzy pre-determiner, $A \in \mathcal{P}(E)$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$\begin{aligned}\mathcal{M}(D \triangleleft A)(X_1, \ldots, X_n) &= \int_0^1 (D \triangleleft A)_\gamma(X_1, \ldots, X_n)\, d\gamma \\ &= \int_0^1 D_\gamma(X_1, \ldots, X_n, A)\, d\gamma && \text{by L-24} \\ &= \mathcal{M}(D) \triangleleft A(X_1, \ldots, X_n)\,.\end{aligned}$$

Preservation of decreasing monotonicity (DFS 8) Assume $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is monotonically decreasing in its n-th argument. Let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ where $X_n \subseteq X'_n$. By L-25,

$$D_\gamma(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X_{n-1}, X'{}_n)$$

for all $\gamma \in \mathbf{I}$. Therefore

$$\begin{aligned}\mathcal{M}(D)(X_1, \ldots, X_n) &= \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma \\ &\geq \int_0^1 D_\gamma(X_1, \ldots, X_{n-1}, X'_n)\, d\gamma \\ &= \mathcal{M}(D)(X_1, \ldots, X_{n-1}, X'_n)\end{aligned}$$

by the monotonicity of the Riemann integral.

Functional application (DFS 9) Let $D : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $f_1, \ldots, f_n : E \longrightarrow E'$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}\mathcal{M}(D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n) &= \int_0^1 (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)_\gamma (X_1, \ldots, X_n) \\ &= \int_0^1 (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)_\gamma^{\blacktriangledown} (X_1, \ldots, X_n) && \text{by L-31, L-34} \\ &= \int_0^1 (D_\gamma^{\blacktriangledown} \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n) && \text{by L-27} \\ &= \int_0^1 (D_\gamma \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n) && \text{by L-31, L-34} \\ &= (\mathcal{M}(D) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n)\,.\end{aligned}$$

We have shown that $\mathcal{M}$ is compatible to the standard extension principle. From Th-20 we obtain that $\mathcal{M}$ is compatible to its own extension principle, which coincides with the standard extension principle.

B.11 Proof of Theorem 45

Lemma 35 ($\mathcal{M}^*$ on constant D_γ)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and suppose that given a choice of $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$D_0(X_1, \ldots, X_n) = D_\gamma(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$.
Then

$$\mathcal{M}^*(D)(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n).$$

Proof

a. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$. Then

$$\begin{aligned} \gamma^* &= \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \\ &= \inf\{\gamma : D_0(X_1, \ldots, X_n) = \tfrac{1}{2}\} && \text{by assumption} \\ &= \inf \mathbf{I} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{M}^*(D)(X_1, \ldots, X_n) &= (D_0^* - \tfrac{1}{2}) \cdot 0 + \tfrac{1}{2} \\ &= \tfrac{1}{2}\,. \end{aligned}$$

b. $D_0(X_1, \ldots, X_n) \neq \frac{1}{2}$. Then

$$\begin{aligned} D_0^* &= \lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) \\ &= \lim_{\gamma \to 0^+} D_0(X_1, \ldots, X_n) && \text{by assumption} \\ &= D_0(X_1, \ldots, X_n) \end{aligned}$$

and

$$\begin{aligned} \gamma^* &= \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \\ &= \inf\{\gamma : D_0(X_1, \ldots, X_n) = \tfrac{1}{2}\} && \text{by assumption} \\ &= \inf \varnothing && \text{as } D_0(X_1, \ldots, X_n) \neq \tfrac{1}{2} \\ &= 1 \end{aligned}$$

i.e.

$$\begin{aligned} \mathcal{M}^*(D)(X_1, \ldots, X_n) &= (D_0(X_1, \ldots, X_n) - \tfrac{1}{2}) \cdot 1 + \tfrac{1}{2} \\ &= D_0(X_1, \ldots, X_n)\,. \end{aligned}$$

Lemma 36 ($\mathcal{M}^*$ on three-valued determiners)
Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ *a three-valued fuzzy pre-determiner. Then for all* $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$\mathcal{M}^*(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) > \frac{1}{2} \\ \frac{1}{2} & : \quad D_(X_1, \ldots, X_n) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

Proof

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $\gamma^* \neq 0$.
Then for all $\gamma' < \gamma^*$, $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2}$.
But $D_{\gamma'}(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 1$. In particular,

$$\begin{aligned} D_0^* &= \lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) \\ &= 1 \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{M}^*(D)(X_1, \ldots, X_n) &= (1 - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* . \end{aligned}$$

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$ and $\gamma^* \neq 0$.
Then for all $\gamma' < \gamma^*$, $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$.
But $D_{\gamma'}(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 0$. In particular,

$$\begin{aligned} D_0^* &= \lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) \\ &= 0 \end{aligned}$$

and hence

$$\begin{aligned} \mathcal{M}^*(D)(X_1, \ldots, X_n) &= (0 - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* . \end{aligned}$$

c. $\gamma^* = 0$ (in particular, this is the case if $D_0(X_1, \ldots, X_n) = \frac{1}{2}$). Then

$$\begin{aligned} \mathcal{M}^*(D)(X_1, \ldots, X_n) &= (D_0^* - \tfrac{1}{2}) \cdot 0 + \tfrac{1}{2} \\ &= \tfrac{1}{2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* \\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* . \end{aligned}$$

Lemma 37 ($\mathcal{M}^*$ and functional application)

a. Suppose $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically increasing and for all $x \in \mathbf{I}$,

$$\begin{aligned} &f(x) \leq g(x) \\ &f(y) \geq g(x) \qquad \text{for all } y > x. \end{aligned}$$

Then

$$\begin{aligned} &\inf\{x : f(x) \geq a\} = \inf\{x : g(x) \geq a\} \qquad \text{for all } a \in \mathbf{I} \\ &\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} g(x) . \end{aligned}$$

b. If $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically decreasing and for all $x \in \mathbf{I}$,

$$\begin{aligned} f(x) &\geq g(x) \\ f(y) &\leq g(x) && \textit{for all } y > x\,. \end{aligned}$$

Then

$$\begin{aligned} &\inf\{x : f(x) \leq a\} = \inf\{x : g(x) \leq a\} && \textit{for all } a \in \mathbf{I} \\ &\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} g(x)\,. \end{aligned}$$

Proof

a. Suppose $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically increasing and satisfy the conditions stated in **a.**

1. We abbreviate

$$\begin{aligned} F &= \inf\{x : f(x) \geq a\} \\ G &= \inf\{x : g(x) \geq a\} \end{aligned}$$

Clearly $\{x : f(x) \geq a\} \subseteq \{x : g(x) \geq a\}$ (because $f \geq g$) and hence $F \geq G$.

To see that $F \leq G$, let $\varepsilon > 0$, $x \in [G, G + \varepsilon)$ and $y \in (x, G + \varepsilon)$. Then $g(x) \geq a$ and $f(y) \geq g(x) \geq a$ as $y > x$.

Hence $F = \inf\{x : f(x) \geq a\} \leq y < G + \varepsilon$. $\varepsilon \to 0$ yields $F \leq G$.

2. In order to prove the second claim, let us abbreviate

$$\begin{aligned} F' &= \lim_{x \to 0^+} f(x) \\ &= \inf\{f(x) : x > 0\} && \text{because } f \text{ is increasing} \\ G' &= \lim_{x \to 0^+} g(x) \\ &= \inf\{g(x) : x > 0\} && \text{because } g \text{ is increasing} \end{aligned}$$

From $f \leq g$ we immediately obtain $F' \leq G'$.

To see the converse, let $\varepsilon > 0$. Then there exists $y_0 \in \mathbf{I}$ such that $f(y_0) \leq F + \varepsilon$. Then

$$\begin{aligned} G &= \inf\{g(x) : x > 0\} \\ &\leq g(\tfrac{y_0}{2}) \\ &\leq f(y_0) && \text{because } \tfrac{y_0}{2} < y_0 \\ &\leq F + \varepsilon\,. \end{aligned}$$

$\varepsilon \to 0$ yields $G \leq F$.

b. The case that $g, f : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically decreasing can be reduced to **a.** by noting that

$$\begin{aligned} &\inf\{x : f(x) \leq a\} = \inf\{x : 1 - f(x) \geq 1 - a\} \\ &\inf\{x : g(x) \leq a\} = \inf\{x : 1 - g(x) \geq 1 - a\} \\ &\lim_{x \to 0^+} f(x) = 1 - \lim_{x \to 0^+} (1 - f(x)) \\ &\lim_{x \to 0^+} g(x) = 1 - \lim_{x \to 0^+} (1 - g(x))\,. \end{aligned}$$

Proof of Theorem 45

Preservation of constants (DFS 1) Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then $D_\gamma(\varnothing) = D(\varnothing)$ for all $\gamma \in \mathbf{I}$ by L-18. L-35 yields $\mathcal{M}^*(D)(\varnothing) = D_0(\varnothing) = D(\varnothing)$.

Compatibility with π_* (DFS 2) π_* is a two-valued fuzzy pre-determiner. Hence for all $X \in \widetilde{\mathcal{P}}(E)$,

$$\begin{aligned}\mathcal{M}^*(\pi_*)(X) &= \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad \pi_{*0}(X) > \frac{1}{2} \\ \frac{1}{2} & : \quad \pi_{*0}(X) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad \pi_{*0}(X) < \frac{1}{2} \end{cases} \\ &= \mathcal{M}(\pi_*)(X) && \text{by L-33} \\ &= \widetilde{\pi}_*(X)\,. && \text{by (37)}\end{aligned}$$

External negation (DFS 3) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Abbreviating

$$\begin{aligned}D_0^* &= \lim_{\gamma\to 0^+} D_\gamma(X_1, \ldots, X_n) \\ \gamma^* &= \inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \\ D_0^{*\prime} &= \lim_{\gamma\to 0^+} (\neg D)_\gamma(X_1, \ldots, X_n) \\ \gamma^{*\prime} &= \inf\{\gamma \in \mathbf{I} : \neg D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}\end{aligned}$$

i.e.

$$\begin{aligned}\mathcal{M}^*(D)(X_1, \ldots, X_n) &= (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ \mathcal{M}^*(\neg D)(X_1, \ldots, X_n) &= (D_0^{*\prime} - \tfrac{1}{2})\gamma^{*\prime} + \tfrac{1}{2}\end{aligned}$$

we obtain firstly

$$\begin{aligned}D_0^{*\prime} &= \lim_{\gamma\to 0^+} (\neg D)_\gamma(X_1, \ldots, X_n) \\ &= \lim_{\gamma\to 0^+} 1 - D_\gamma(X_1, \ldots, X_n) && \text{by L-20} \\ &= 1 - \lim_{\gamma\to 0^+} D_\gamma(X_1, \ldots, X_n) \\ &= 1 - D_0^*\,.\end{aligned}$$

and secondly,

$$\begin{aligned}\gamma^{*\prime} &= \inf\{\gamma \in \mathbf{I} : (\neg D)_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \\ &= \inf\{\gamma \in \mathbf{I} : 1 - D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} && \text{by L-20} \\ &= \inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \\ &= \gamma^*\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{M}^*(\neg D)(X_1, \ldots, X_n) &= (D_0^{*\prime} - \tfrac{1}{2})\gamma^{*\prime} + \tfrac{1}{2} \\ &= ((1 - D_0^*) - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= -(D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= 1 - ((D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2}) \\ &= 1 - \mathcal{M}^*(D)(X_1, \ldots, X_n)\,.\end{aligned}$$

Argument transposition (DFS 4) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, $1 \leq k \leq n$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$.

$$\begin{aligned}\mathcal{M}^*(D\tau_i)(X_1, \ldots, X_n) &= ((\lim_{\gamma\to 0^+} (D\tau_i)_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\ &\quad \cdot(\inf\{\gamma : (D\tau_i)_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &= ((\lim_{\gamma\to 0^+} (D_\gamma)\tau_i(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\ &\quad \cdot(\inf\{\gamma : (D_\gamma)\tau_i(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-21} \\ &= \mathcal{M}^*(D)\tau_i(X_1, \ldots, X_n)\,.\end{aligned}$$

By Th-10 it follows that $\mathcal{M}^*$ satisfies (DFS 3) and $\widetilde{\mathcal{M}^*}(\neg) = \neg$.

Internal complementation (DFS 5) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$\begin{aligned}\mathcal{M}^*(D\neg)(X_1, \ldots, X_n) &= ((\lim_{\gamma\to 0^+} (D\neg)_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\ &\quad \cdot(\inf\{\gamma : (D\neg)_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &= ((\lim_{\gamma\to 0^+} (D_\gamma)\neg(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\ &\quad \cdot(\inf\{\gamma : (D_\gamma)\neg(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &= \mathcal{M}^*(D)\neg(X_1, \ldots, X_n)\,.\end{aligned}$$

Internal meets (DFS 6) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-ary fuzzy pre-determiner and $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Then

$$\begin{aligned}\mathcal{M}^*(D\cap)(X_1, \ldots, X_{n+1}) &= ((\lim_{\gamma\to 0^+} (D\cap)_\gamma(X_1, \ldots, X_{n+1})) - \tfrac{1}{2}) \\ &\quad \cdot \inf\{\gamma : (D\cap)_\gamma(X_1, \ldots, X_{n+1}) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &= ((\lim_{\gamma\to 0^+} (D_\gamma)\cap(X_1, \ldots, X_{n+1})) - \tfrac{1}{2}) \\ &\quad \cdot \inf\{\gamma : (D_\gamma)\cap(X_1, \ldots, X_{n+1}) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-23} \\ &= \mathcal{M}^*(D)\cap(X_1, \ldots, X_{n+1})\,.\end{aligned}$$

We have shown that $\mathcal{M}^*$ is compatible to the fuzzy intersection induced by $\wedge = \min$. By Th-11, it follows that $\widetilde{\mathcal{M}^*}(\wedge) = \min$. In particular, $\mathcal{M}^*$ satisfies (DFS 6).

Argument insertion (DFS 7) Let $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ an $(n+1)$-ary fuzzy pre-determiner, $A \in \mathcal{P}(E)$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then,

$$\begin{aligned}
\mathcal{M}^*(D \triangleleft A) &= ((\lim_{\gamma \to 0^+} (D \triangleleft A)_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot (\inf\{\gamma : (D \triangleleft A)_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\
&= ((\lim_{\gamma \to 0^+} ((D_\gamma) \triangleleft A)(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot (\inf\{\gamma : ((D_\gamma) \triangleleft A)(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-24} \\
&= \mathcal{M}^*(D) \triangleleft A(X_1, \ldots, X_n)\,.
\end{aligned}$$

Preservation of decreasing monotonicity (DFS 8) Assume $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is monotonically decreasing in its n-th argument. Let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ where $X_n \subseteq X'_n$. By L-25, $D_\gamma(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X'_n)$ for all $\gamma \in \mathbf{I}$. We therefore distinguish three cases.

1. $D_0(X_1, \ldots, X_n) \geq D_0(X_1, \ldots, X'_n) \geq \frac{1}{2}$. Then

$$\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \subseteq \{\gamma : D_\gamma(X_1, \ldots, X'_n) = \tfrac{1}{2}\}$$

and hence

$$\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \geq \inf\{\gamma : D_\gamma(X_1, \ldots, X'_n) = \tfrac{1}{2}\}\,.$$

Therefore

$$\begin{aligned}
\mathcal{M}^*(D)(X_1, \ldots, X_n) &= ((\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot (\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\
&\geq ((\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X'_n)) - \tfrac{1}{2}) \\
&\quad \cdot (\inf\{\gamma : D_\gamma(X_1, \ldots, X'_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\
&= \mathcal{M}^*(D)(X_1, \ldots, X'_n)\,.
\end{aligned}$$

2. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2} > D_0(X_1, \ldots, X'_n)$. Then

$$\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) \geq \tfrac{1}{2}$$

and hence

$$\begin{aligned}
\mathcal{M}^*(D)(X_1, \ldots, X_n) &= (\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\
&\geq (\tfrac{1}{2} - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\
&= \tfrac{1}{2}\,.
\end{aligned}$$

On the other hand,

$$\lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X'_n) \leq \tfrac{1}{2}$$

and hence

$$\begin{aligned}\mathcal{M}^*(D)(X_1, \ldots, X_n') &= ((\lim_{\gamma\to 0^+} D_\gamma(X_1, \ldots, X_n') - \tfrac{1}{2})\gamma^{*\prime} + \tfrac{1}{2} \\ &\leq (\tfrac{1}{2} - \tfrac{1}{2})\gamma^{*\prime} + \tfrac{1}{2} \\ &= \tfrac{1}{2}\end{aligned}$$

i.e. $\mathcal{M}^*(D)(X_1, \ldots, X_n) \geq \frac{1}{2} \geq \mathcal{M}^*(D)(X_1, \ldots, X_n')$.

3. $\frac{1}{2} > D_0(X_1, \ldots, X_n) \geq D_0(X_1, \ldots, X_n')$.
   From

$$D_\gamma(X_1, \ldots, X_n) \geq D_\gamma(X_1, \ldots, X_n')$$

for all $\gamma \in \mathbf{I}$ we obtain

$$\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \supseteq \{\gamma : D_\gamma(X_1, \ldots, X_n') = \tfrac{1}{2}\}$$

and hence

$$\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\} \leq \inf\{\gamma : D_\gamma(X_1, \ldots, X_n') = \tfrac{1}{2}\}\,.$$

Now

$$\begin{aligned}\mathcal{M}^*(D)(X_1, \ldots, X_n) &= ((\lim_{\gamma\to 0^+} D_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\ &\quad\cdot(\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &\geq ((\lim_{\gamma\to 0^+} D_\gamma(X_1, \ldots, X_n')) - \tfrac{1}{2}) \\ &\quad\cdot(\inf\{\gamma : D_\gamma(X_1, \ldots, X_n') = \tfrac{1}{2}\}) + \tfrac{1}{2} \\ &= \mathcal{M}^*(D)(X_1, \ldots, X_n')\,.\end{aligned}$$

Functional application (DFS 9) Let $D : \mathcal{P}(E)'^{n} \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $f_1, \ldots, f_n : E \longrightarrow E'$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}
&\mathcal{M}^*(D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n) \\
&= ((\lim_{\gamma\to 0^+} (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)_\gamma (X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot(\inf\{\gamma : (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)_\gamma (X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} \\
&= ((\lim_{\gamma\to 0^+} (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)^{\blacktriangledown}_\gamma (X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot(\inf\{\gamma : (D \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)^{\blacktriangledown}_\gamma (X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-31, L-37} \\
&= ((\lim_{\gamma\to 0^+} (D^{\blacktriangledown}_\gamma \circ \mathop{\times}_{i=1}^{n} \widehat{f}_i)(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot(\inf\{\gamma : (D^{\blacktriangledown}_\gamma \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-27} \\
&= ((\lim_{\gamma\to 0^+} (D_\gamma \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n)) - \tfrac{1}{2}) \\
&\quad \cdot(\inf\{\gamma : (D_\gamma \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n) = \tfrac{1}{2}\}) + \tfrac{1}{2} && \text{by L-31, L-37} \\
&= (\mathcal{M}^*(D) \circ \mathop{\times}_{i=1}^{n} \hat{\hat{f}}_i)(X_1, \ldots, X_n)\,.
\end{aligned}$$

Hence $\mathcal{M}^*$ is compatible to the standard extension principle. From Th-20 we obtain that the extension principle induced by $\mathcal{M}^*$ coincides with the standard extension principle.

B.12 Proof of Theorem 46

Lemma 38 ($\mathcal{M}^\bullet$ on three-valued determiners)
Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ a three-valued fuzzy pre-determiner. Then for all $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$,

$$\mathcal{M}^\bullet(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) > \frac{1}{2} \\ \frac{1}{2} & : \quad D_(X_1, \ldots, X_n) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

Proof

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $\gamma^* \neq 0$. Then $D_0(X_1, \ldots, X_n) = 1$ because $D_0(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, and hence

$$\begin{aligned}\mathcal{M}^\bullet(D)(X_1, \ldots, X_n) &= (1 - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^*\,.\end{aligned}$$

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$ and $\gamma^* \neq 0$. Then $D_0(X_1, \ldots, X_n) = 0$ because $D_0(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, from which we obtain

$$\begin{aligned}\mathcal{M}^\bullet(D)(X_1, \ldots, X_n) &= (0 - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^*\,.\end{aligned}$$

c. $\gamma^* = 0$ (in particular, this is the case if $D_0(X_1, \ldots, X_n) = \frac{1}{2}$). Then

$$\begin{aligned}\mathcal{M}^\bullet(D)(X_1, \ldots, X_n) &= (D_0(X_1, \ldots, X_n) - \tfrac{1}{2}) \cdot 0 + \tfrac{1}{2} \\ &= \tfrac{1}{2} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* \\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* .\end{aligned}$$

Proof of Theorem 46

Suppose that $\mathcal{M}^\bullet$ is a DFS. Then by Th-21, L-36 and L-38, the extension principle of $\mathcal{M}^\bullet$ coincides with the extension principle induced by $\mathcal{M}^*$, i.e. with the standard extension principle. We will prove that functional application (DFS 9) fails to hold.
Let $D : \mathcal{P}(\mathbf{I}) \longrightarrow \mathbf{I}$ the monadic fuzzy pre-determiner defined by

$$D(X) = \begin{cases} \frac{3}{4} & : \frac{1}{2} \in X \\ 1 & : \text{else} \end{cases}$$

for all $X \subseteq \mathbf{I}$.
Define $f : \mathbf{I} \longrightarrow \mathbf{I}$ by

$$f(x) = \begin{cases} x & : x > \frac{1}{2} \\ \frac{1}{2} & : \text{else} \end{cases}$$

and let $D'(X) = D(\widehat{f}(X))$ for all $X \in \mathcal{P}(\mathbf{I})$.
Furthermore, suppose $Z \in \widetilde{\mathcal{P}}(\mathbf{I})$ is defined by the membership function

$$\mu_Z(x) = \begin{cases} x & : x \in [0, \frac{1}{2}) \\ 0 & : \text{else} \end{cases}$$

Then,

$$\begin{aligned}(Z)^{\min}_\gamma &= \varnothing \\ (Z)^{\max}_\gamma &= \begin{cases} \varnothing & : \gamma = 0 \\ (\frac{1}{2} - \frac{1}{2}\gamma, \frac{1}{2}) & : \gamma > 0 \end{cases}\end{aligned}$$

and hence

$$\begin{aligned}\widehat{f}((Z)^{\min}_\gamma) &= \varnothing \\ \widehat{f}((Z)^{\max}_\gamma) &= \begin{cases} \varnothing & : \gamma = 0 \\ \{\frac{1}{2}\} & : \gamma > 0 \end{cases}\end{aligned}$$

from which we obtain

$$D'_\gamma(Z) = \begin{cases} 1 & : \gamma = 0 \\ \frac{3}{4} & : \text{else} \end{cases}$$

i.e.

$$\begin{aligned} D'_0(Z) &= 1 \\ \gamma^* &= \inf\{\gamma : D'_\gamma(Z) = \tfrac{1}{2}\} = \inf \varnothing = 1 \\ \mathcal{M}^\bullet(D')(Z) &= (1 - \tfrac{1}{2})1 + \tfrac{1}{2} = 1\,. \end{aligned}$$

On the other hand,

$$\mu_{\hat{f}(Z)}(x) = \begin{cases} \frac{1}{2} & : \quad x = \frac{1}{2} \\ 0 & : \quad x \neq \frac{1}{2} \end{cases}$$

And hence

$$\begin{aligned} (\hat{\hat{f}}(Z))^{\min}_\gamma &= \varnothing \\ (\hat{\hat{f}}(Z))^{\max}_\gamma &= \{\tfrac{1}{2}\} \end{aligned}$$

for all $\gamma \in \mathbf{I}$. Therefore

$$D_\gamma(\hat{\hat{f}}(Z)) = \tfrac{3}{4}$$

for all $\gamma \in \mathbf{I}$, i.e.

$$\begin{aligned} D_0(\hat{\hat{f}}(Z)) &= \tfrac{3}{4} \\ \gamma^* &= \inf\{\gamma : D_\gamma(\hat{\hat{f}}(Z)) = \tfrac{1}{2}\} = \inf \varnothing = 1 \\ \mathcal{M}^\bullet(D)(\hat{\hat{f}}(Z)) &= (\tfrac{3}{4} - \tfrac{1}{2})1 + \tfrac{1}{2} = \tfrac{3}{4}\,. \end{aligned}$$

Hence $\mathcal{M}^\bullet(D')(Z) \neq \mathcal{M}^\bullet(D)(\hat{\hat{f}}(Z))$, i.e. the axiom of functional application (DFS 9) does not hold, which contradicts our assumption that $\mathcal{M}^\bullet$ be a DFS.

B.13 Proof of Theorem 47

Lemma 39 ($\mathcal{M}_*$ on constant D_γ)
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. If

$$D_\gamma(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$$

for all $\gamma \in \mathbf{I}$, then

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)\,.$$

Proof

a. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. Then

$$\begin{aligned} \mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} + \sup\{\gamma(D_0(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by assumption} \\ &= \tfrac{1}{2} + (D_0(X_1, \ldots, X_n) - \tfrac{1}{2}) \\ &= D_0(X_1, \ldots, X_n)\,. \end{aligned}$$

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$. Then

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_0(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \qquad \text{by assumption} \\ &= \tfrac{1}{2} - (\tfrac{1}{2} - D_0(X_1, \ldots, X_n)) \\ &= D_0(X_1, \ldots, X_n)\,.\end{aligned}$$

Lemma 40 ($\mathcal{M}_*$ on three-valued determiners)
Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ a three-valued fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) > \frac{1}{2} \\ \frac{1}{2} & : \quad D_(X_1, \ldots, X_n) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* & : \quad D_0(X_1, \ldots, X_n) < \frac{1}{2} \end{cases}$$

Proof

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $\gamma^* > 0$.
Then for all $\gamma' < \gamma^*$, $D_{\gamma'}(X_1, \ldots, X_n) > \frac{1}{2}$.
But $D_{\gamma'}(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 1$. Therefore

$$\begin{aligned}\sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} &= \max(\sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in [0, \gamma^*)\}, \\ &\qquad\quad \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in (\gamma^*, 1]\}) \\ &= \max(\sup\{\gamma(1 - \tfrac{1}{2}) : \gamma \in [0, \gamma^*)\}, \\ &\qquad\quad \sup\{\gamma(\tfrac{1}{2} - \tfrac{1}{2}) : \gamma \in (\gamma^*, 1]\}) \\ &= \max(\tfrac{1}{2}\gamma^*, 0) \\ &= \tfrac{1}{2}\gamma^*\,,\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^*\,.\end{aligned}$$

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$ and $\gamma^* > 0$.
Then for all $\gamma' < \gamma^*$, $D_{\gamma'}(X_1, \ldots, X_n) < \frac{1}{2}$.
But $D_{\gamma'}(X_1, \ldots, X_n) \in \{0, \frac{1}{2}, 1\}$, i.e. $D_{\gamma'}(X_1, \ldots, X_n) = 0$. Therefore

$$\begin{aligned}\sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} &= \max(\sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in [0, \gamma^*)\}, \\ &\qquad\quad \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in (\gamma^*, 1]\}) \\ &= \max(\sup\{\gamma(\tfrac{1}{2} - 0) : \gamma \in [0, \gamma^*)\}, \\ &\qquad\quad \sup\{\gamma(\tfrac{1}{2} - \tfrac{1}{2}) : \gamma \in (\gamma^*, 1]\}) \\ &= \max(\tfrac{1}{2}\gamma^*, 0) \\ &= \tfrac{1}{2}\gamma^*\,,\end{aligned}$$

and hence

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\}\\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^*\,.\end{aligned}$$

c. $\gamma^* = 0$ (In particular, this is the case if $D_0(X_1, \ldots, X_n) = \frac{1}{2}$).
Then $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma > 0$ and hence

$$\begin{aligned}\sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} &= \max(0 \cdot (D_0(X_1, \ldots, X_n) - \tfrac{1}{2}),\ \sup\{\gamma(\tfrac{1}{2} - \tfrac{1}{2}) : \gamma > 0\})\\ &= 0\\ \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} &= \max(0 \cdot (\tfrac{1}{2} - D_0(X_1, \ldots, X_n)),\ \sup\{\gamma(\tfrac{1}{2} - \tfrac{1}{2}) : \gamma > 0\})\\ &= 0\,.\end{aligned}$$

In any case,

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2}\\ &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^*\\ &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^*\,.\end{aligned}$$

Lemma 41 ($\mathcal{M}_*$ and functional application)
Suppose $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ are monotonically decreasing and for all $x \in \mathbf{I}$,

$$f(x) \geq g(x) \tag{38}$$

$$f(y) \leq g(x) \qquad \textit{for all } y > x\,. \tag{39}$$

Then

$$\sup\{xf(x) : x \in \mathbf{I}\} = \sup\{xg(x) : x \in \mathbf{I}\}\,.$$

Proof Suppose $f, g : \mathbf{I} \longrightarrow \mathbf{I}$ satisfy the requirements of the lemma. Let us abbreviate

$$\begin{aligned}F &= \sup\{xf(x) : x \in \mathbf{I}\}\\ G &= \sup\{xg(x) : x \in \mathbf{I}\}\,.\end{aligned}$$

$F \geq G$ is immediate from $f \geq g$ by (38).
To see that $F \leq G$, let $\varepsilon > 0$. Then there exists $y \in \mathbf{I}$ such that

$$yf(y) > F - \tfrac{\varepsilon}{2}\,. \tag{40}$$

We compute

$$\begin{aligned}\sup\{xg(x) : x \in \mathbf{I}\} &\geq (y - \tfrac{\varepsilon}{2})g(y - \tfrac{\varepsilon}{2})\\ &\geq (y - \tfrac{\varepsilon}{2})f(y) && \text{by assumption (39)}\\ &= yf(y) - \tfrac{\varepsilon}{2}\underbrace{f(y)}_{\leq 1}\\ &\geq yf(y) - \tfrac{\varepsilon}{2}\\ &> (F - \tfrac{\varepsilon}{2}) - \tfrac{\varepsilon}{2} && \text{by (40)}\\ &= F - \varepsilon\,.\end{aligned}$$

$\varepsilon \to 0$ yields $G \geq F$, as desired.

Proof of Theorem 47

Preservation of constants (DFS 1) Let $D : \mathcal{P}(E)^0 \longrightarrow \mathbf{I}$ a constant fuzzy pre-determiner. Then $D_\gamma(\varnothing) = D(\varnothing)$ for all $\gamma \in \mathbf{I}$ by L-18 and hence

$$\begin{aligned} \mathcal{M}_*(D)(\varnothing) &= D_0(\varnothing) && \text{by L-39} \\ &= D(\varnothing)\,. && \text{by L-18} \end{aligned}$$

Compatibility with π_* (DFS 2) π_* is two-valued and hence for all $X \in \widetilde{\mathcal{P}}(\{*\})$,

$$\begin{aligned} \mathcal{M}_*(\pi_*)(X) &= \begin{cases} \frac{1}{2} + \frac{1}{2}\gamma^* : \pi_{*0}(X) > \frac{1}{2} & : \\ \frac{1}{2} & : \quad \pi_{*0}(X) = \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2}\gamma^* : \pi_{*0}(X) < \frac{1}{2} & : \end{cases} && \text{by L-40} \\ &= \mathcal{M}(\pi_*)(X) && \text{by L-33} \\ &= \widetilde{\pi}_*(X) && \text{by (37)} \end{aligned}$$

External negation (DFS 3) We will prove that $\mathcal{M}_*$ is compatible with the standard negation $\neg : \mathbf{I} \longrightarrow \mathbf{I}$ defined by $\neg x = 1 - x$ for all $x \in \mathbf{I}$. By Th-10 we then know that $\mathcal{M}_*$ satisfies (DFS 3) and $\widetilde{\mathcal{M}_*}(\neg) = \neg$.

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. If $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$, then

$$(\neg D)_0(X_1, \ldots, X_n) = 1 - D_0(X_1, \ldots, X_n) \leq \tfrac{1}{2} \qquad \text{by L-20} \qquad (41)$$

In this case,

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}$$

i.e.

$$\begin{aligned} & 1 - \mathcal{M}_*(D)(X_1, \ldots, X_n) \\ & = 1 - (\tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}) \\ & = \tfrac{1}{2} - \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ & = \tfrac{1}{2} - \sup\{\gamma(1 - (1 - (D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}))) : \gamma \in \mathbf{I}\} \\ & = \tfrac{1}{2} - \sup\{\gamma(1 - ((1 - D_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2})) : \gamma \in \mathbf{I}\} \\ & = \tfrac{1}{2} - \sup\{\gamma(1 - ((\neg D)_\gamma(X_1, \ldots, X_n)) - \tfrac{1}{2})) : \gamma \in \mathbf{I}\} && \text{by L-20} \\ & = \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (\neg D)_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\ & = \mathcal{M}_*(\neg D)(X_1, \ldots, X_n)\,. && \text{by (41)} \end{aligned}$$

b. Likewise if $D_0(X_1, \ldots, X_n) < \frac{1}{2}$, then

$$(\neg D)_0(X_1, \ldots, X_n) = 1 - D_0(X_1, \ldots, X_n) > \tfrac{1}{2} \qquad \text{by L-20} \qquad (42)$$

and we obtain

$$\begin{aligned}
&1 - \mathcal{M}_*(D)(X_1, \ldots, X_n) \\
&= 1 - (\tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\}) \\
&= \tfrac{1}{2} + \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} + \sup\{\gamma(\tfrac{1}{2} - (1 - (1 - D_\gamma(X_1, \ldots, X_n)))) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} + \sup\{\gamma(\tfrac{1}{2} - (1 - (\neg D)_\gamma(X_1, \ldots, X_n))) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} + \sup\{\gamma(\tfrac{1}{2} + (\neg D)_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\
&= \mathcal{M}_*(\neg D)(X_1, \ldots, X_n)\,. && \text{by (42)}
\end{aligned}$$

Argument transposition (DFS 4) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner, $1 \leq k \leq n$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Then

$$(D\tau_i)_0(X_1, \ldots, X_n) = (D_\gamma)\tau_i(X_1, \ldots, X_n)\,. \qquad \text{by L-21} \tag{43}$$

1. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}_*(D\tau_i)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma((D\tau_i)_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma(D_\gamma)\tau_i(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-21} \\
&= \mathcal{M}_*(D)\tau_i(X_1, \ldots, X_n)\,. && \text{by (43)}
\end{aligned}$$

2. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}_*(D\tau_i)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\gamma(\tfrac{1}{2} - (D\tau_i)_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\gamma(\tfrac{1}{2} - (D_\gamma)\tau_i(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-21} \\
&= \mathcal{M}_*(D)\tau_i(X_1, \ldots, X_n)\,. && \text{by (43)}
\end{aligned}$$

Internal complementation (DFS 5) Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Again,

$$(D\neg)_0(X_1, \ldots, X_n) = (D_0)\neg(X_1, \ldots, X_n)\,. \qquad \text{by L-22} \tag{44}$$

1. $(D\neg)_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}_*(\neg D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma((\neg D)_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} + \sup\{\gamma(\neg(D_\gamma)(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-22} \\
&= \neg\mathcal{M}_*(D)(X_1, \ldots, X_n)\,. && \text{by (44)}
\end{aligned}$$

2. $(D\neg)_0(X_1, \ldots, X_n) < \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{M}_*(\neg D)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (\neg D)_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\
&= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - \neg(D_\gamma)(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-22} \\
&= \neg\mathcal{M}_*(D)(X_1, \ldots, X_n)\,. && \text{by (44)}
\end{aligned}$$

Internal meets (DFS 6) Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ an n-ary fuzzy pre-determiner and $X_1, \ldots, X_{n+1} \in \widetilde{\mathcal{P}}(E)$. Noting that

$$(D\cap)_0(X_1, \ldots, X_{n+1}) = (D_0)\cap(X_1, \ldots, X_{n+1}) \qquad \text{by L-23} \qquad (45)$$

we obtain

1. If $(D\cap)_0(X_1, \ldots, X_{n+1}) \geq \frac{1}{2}$,

$$\begin{aligned} \mathcal{M}_*(D\cap) &= \tfrac{1}{2} + \sup\{\gamma((D\cap)_\gamma(X_1, \ldots, X_{n+1}) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} + \sup\{\gamma((D_\gamma)\cap(X_1, \ldots, X_{n+1}) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-23} \\ &= \mathcal{M}_*(D)\cap(X_1, \ldots, X_{n+1}) . && \text{by (45)} \end{aligned}$$

2. $(D\cap)_0(X_1, \ldots, X_{n+1}) < \frac{1}{2}$. Then

$$\begin{aligned} \mathcal{M}_*(D\cap) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D\cap)_\gamma(X_1, \ldots, X_{n+1})) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D_\gamma)\cap(X_1, \ldots, X_{n+1})) : \gamma \in \mathbf{I}\} && \text{by L-23} \\ &= \mathcal{M}_*(D)\cap(X_1, \ldots, X_{n+1}) . && \text{by (45)} \end{aligned}$$

We have shown that $\mathcal{M}_*$ is compatible to the standard fuzzy intersection $\cap$ which corresponds to the t-norm $\wedge = \min$. By Th-11, we obtain that $\widetilde{\mathcal{M}_*}(\wedge) = \min$.

Argument insertion (DFS 7) Let $D : \mathcal{P}(E)^{n+1} \longrightarrow \mathbf{I}$ an $(n+1)$-ary fuzzy pre-determiner, $A \in \mathcal{P}(E)$ and $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. Again we should note that

$$(D\triangleleft A)_0(X_1, \ldots, X_n) = (D_\gamma)\triangleleft A(X_1, \ldots, X_n) . \qquad \text{by L-24} \qquad (46)$$

1. $(D\triangleleft A)_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. Then

$$\begin{aligned} \mathcal{M}_* D\triangleleft A(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma((D\triangleleft A)_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} + \sup\{\gamma((D_\gamma)\triangleleft A(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-24} \\ &= \mathcal{M}_*(D)\triangleleft A(X_1, \ldots, X_n) . && \text{by (46)} \end{aligned}$$

2. $(D\triangleleft A)_0(X_1, \ldots, X_n) < \frac{1}{2}$. In this case

$$\begin{aligned} \mathcal{M}_* D\triangleleft A(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D\triangleleft A)_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} \\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D_\gamma)\triangleleft A(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-24} \\ &= \mathcal{M}_*(D)\triangleleft A(X_1, \ldots, X_n) . && \text{by (46)} \end{aligned}$$

Preservation of decreasing monotonicity (DFS 8) Assume $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is monotonically decreasing in its n-th argument. Let $X_1, \ldots, X_n, X'_n \in \widetilde{\mathcal{P}}(E)$ where $X_n \subseteq X'_n$. By L-25, we know that $D_0(X_1, \ldots, X_n) \geq D_0(X_1, \ldots, X'_n)$.

1. $D_0(X_1, \ldots, X_n) \geq (X_1, \ldots, X'_n) \geq \frac{1}{2}$. Then

$$\begin{aligned} \mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \\ &\geq \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X'_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-25} \\ &= \mathcal{M}_*(D)(X_1, \ldots, X'_n) . \end{aligned}$$

2. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2} > D_0(X_1, \ldots, X'_n)$. In this case,

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}\\ &\geq \tfrac{1}{2}\\ &\geq \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X'_n)) : \gamma \in \mathbf{I}\}\\ &= \mathcal{M}_*(D)(X_1, \ldots, X'_n)\,.\end{aligned}$$

3. $\frac{1}{2} > D_0(X_1, \ldots, X_n) \geq D_0(X_1, \ldots, X'_n)$. Then

$$\begin{aligned}\mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\}\\ &\geq \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - D_\gamma(X_1, \ldots, X'_n) : \gamma \in \mathbf{I}\} && \text{by L-25}\\ &= \mathcal{M}_*(D)(X_1, \ldots, X'_n)\,.\end{aligned}$$

Functional application (DFS 9) Let $D : \mathcal{P}(E)'^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner, $f_1, \ldots, f_n : E \longrightarrow E'$ and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

1. $(D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_0 \geq \frac{1}{2}$. Then

$$\begin{aligned}&\mathcal{M}_*(D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})(X_1, \ldots, X_n)\\ &= \tfrac{1}{2} + \sup\{\gamma((D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_\gamma (X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}\\ &= \tfrac{1}{2} + \sup\{\gamma((D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_\gamma^\blacktriangledown (X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-31, L-41}\\ &= \tfrac{1}{2} + \sup\{\gamma((D_\gamma^\blacktriangledown \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-27}\\ &= \tfrac{1}{2} + \sup\{\gamma((D_\gamma \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} && \text{by L-31, L-41}\\ &= (\mathcal{M}_*(D) \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n)\,.\end{aligned}$$

2. $(D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_0 < \frac{1}{2}$. In this case,

$$\begin{aligned}&\mathcal{M}_*(D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})(X_1, \ldots, X_n)\\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_\gamma (X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\}\\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D \circ \overset{n}{\underset{i=1}{\times}} \widehat{f_i})_\gamma^\blacktriangledown (X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-31, L-41}\\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D_\gamma^\blacktriangledown \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-27}\\ &= \tfrac{1}{2} - \sup\{\gamma(\tfrac{1}{2} - (D_\gamma \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n)) : \gamma \in \mathbf{I}\} && \text{by L-31, L-41}\\ &= (\mathcal{M}_*(D) \circ \overset{n}{\underset{i=1}{\times}} \hat{\hat{f}}_i)(X_1, \ldots, X_n)\,.\end{aligned}$$

Hence $\mathcal{M}_*$ is compatible to the standard extension principle. From Th-20 we obtain that the induced extension principle of $\mathcal{M}_*$ coincides with the standard extension principle.

B.14 Proof of Theorem 48

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \breve{\mathcal{P}}(E)^n \subset \widetilde{\mathcal{P}}(E)^n$ an n-tuple of $\{0, \frac{1}{2}, 1\}$-valued subsets of E.
Then by L-17, $D_\gamma(X_1, \ldots, X_n) = D_0(X_1, \ldots, X_n)$ for all $\gamma \in \mathbf{I}$. From L-32, L-35, and L-39 it then follows that

$$\begin{aligned} D_0(X_1, \ldots, X_n) &= \mathcal{M}(D)(X_1, \ldots, X_n) \\ &= \mathcal{M}^*(D)(X_1, \ldots, X_n) \\ &= \mathcal{M}_*(D)(X_1, \ldots, X_n) \,. \end{aligned}$$

B.15 Proof of Theorem 49

This is immediately obtained from L-33, L-36, and L-40.

B.16 Proof of Theorem 50

Lemma 42
Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.
If $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$, then

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = \sup\{\int_0^1 f_x(\gamma)\, d\gamma : x \in \mathbf{I}\} \,,$$

where for all $x \in \mathbf{I}$, $f_x : \mathbf{I} \longrightarrow \mathbf{I}$ is the step function defined by

$$f_x(\gamma) = \begin{cases} D_x(X_1, \ldots, X_n) & : \quad \gamma < x \\ \frac{1}{2} & : \quad \gamma \geq x \end{cases}$$

for all $\gamma \in \mathbf{I}$.

Proof Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$ such that $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$.
For all $x \in \mathbf{I}$,

$$\begin{aligned} \int_0^1 f_x(\gamma)\, d\gamma &= \int_0^x D_x(X_1, \ldots, X_n)\, d\gamma + \int_x^1 \tfrac{1}{2}\, d\gamma \\ &= x D_x(X_1, \ldots, X_n) + (1-x)\tfrac{1}{2} \\ &= \tfrac{1}{2} + (x(D_x(X_1, \ldots, X_n) - \tfrac{1}{2}))\,, \end{aligned}$$

from which we immediately obtain that

$$\begin{aligned} \mathcal{M}_*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\} \quad \text{as } D_0(X_1, \ldots, X_n) \geq 0 \\ &= \sup\{\int_0^1 f_x(\gamma)\, d\gamma : x \in \mathbf{I}\} \,. \end{aligned}$$

Proof of Theorem 50

a. Firstly let us show that $\mathcal{M} \preceq \mathcal{M}^*$.

$$\begin{aligned}\mathcal{M}^*(D)(X_1,\ \ldots,\ X_n) &= (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\ &= \gamma^* D_0^* + (1-\gamma^*)\tfrac{1}{2} \\ &= \sup\{\int_0^1 g_x(\gamma)\, d\gamma : x \in \mathbf{I}\}\,,\end{aligned}$$

where we have abbreviated

$$g_x(\gamma) = \begin{cases} D_0^* & : \quad \gamma \leq x \\ \frac{1}{2} & : \quad \gamma > x \end{cases}$$

for all $x \in \mathbf{I}$, $\gamma \in \mathbf{I}$.

1. If $D_0(X_1,\ \ldots,\ X_n) \geq \frac{1}{2}$, $D_\gamma(X_1,\ \ldots,\ X_n)$ is monotonically decreasing in γ by Th-42. In this case

$$\begin{aligned}D_0^* &= \lim_{\gamma\to 0^+} D_\gamma(X_1,\ \ldots,\ X_n) \\ &= \sup\{D_\gamma(X_1,\ \ldots,\ X_n) : \gamma > 0\}\,,\end{aligned}$$

and hence

$$D_0^* \geq D_\gamma(X_1,\ \ldots,\ X_n)$$

for all $\gamma > 0$. In particular,

$$g_{x_0}(\gamma) \geq D_\gamma(X_1,\ \ldots,\ X_n)$$

for all $\gamma \in \mathbf{I}$ provided that $x_0 \geq \gamma^*$. Then,

$$\begin{aligned}\mathcal{M}(D)(X_1,\ \ldots,\ X_n) &= \int_0^1 D_\gamma(X_1,\ \ldots,\ X_n)\, d\gamma \\ &\leq \int_0^1 g_{x_0}(\gamma)\, d\gamma && \text{by monotonicity of } \textstyle\int \\ &\leq \sup\{\int_0^1 g_x(\gamma)\, d\gamma : x \in \mathbf{I}\} \\ &= \mathcal{M}^*(D)(X_1,\ \ldots,\ X_n)\,.\end{aligned}$$

2. $D_0(X_1,\ \ldots,\ X_n) \leq \frac{1}{2}$. Then

$$\begin{aligned}\mathcal{M}(D)(X_1,\ \ldots,\ X_n) &= \mathcal{M}(\neg\neg D)(X_1,\ \ldots,\ X_n) && \text{by (DFS 3) andTh-44} \\ &= 1 - \mathcal{M}(\neg D)(X_1,\ \ldots,\ X_n) \\ &\geq 1 - \mathcal{M}^*(\neg D)(X_1,\ \ldots,\ X_n) && \text{case } \textbf{a.1} \text{ above} \\ &= \mathcal{M}^*(\neg\neg D)(X_1,\ \ldots,\ X_n) && \text{by (DFS 3) and Th-45} \\ &\mathcal{M}^*(D)(X_1,\ \ldots,\ X_n)\,.\end{aligned}$$

b. We are left to show that $\mathcal{M}_* \preceq \mathcal{M}$.

1. $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$. In this case, $D_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing in γ and $D_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}$ by Th-42. Hence for all $x \in \mathbf{I}$, $\gamma \in \mathbf{I}$

$$f_x(\gamma) \leq D_\gamma(X_1, \ldots, X_n) .$$

(using the abbreviations of the above lemma). By the monotonicity of the Riemann integral,

$$\int_0^1 f_x(\gamma)\, d\gamma \leq \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma , \tag{47}$$

for all $x \in \mathbf{I}$, and hence

$$\begin{aligned} \mathcal{M}_*(D)(X_1, \ldots, X_n) &= \sup\{\int_0^1 f_x(\gamma)\, d\gamma : x \in \mathbf{I}\} && \text{by L-42} \\ &\leq \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma && \text{by (47)} \\ &= \mathcal{M}(D)(X_1, \ldots, X_n) . \end{aligned}$$

2. $D_0(X_1, \ldots, X_n) \leq \frac{1}{2}$. Analogous to **a.2**; reduce to case **b.1** by means of negation.

B.17 Proof of Theorem 51

Lemma 43
Suppose $X \in \widetilde{\mathcal{P}}(E)$, where $E \neq \varnothing$ is a non-empty set, and

$$Y = (X)_{\geq \frac{1}{2}} .$$

Then

$$\Theta(X, Y) = 1$$

Proof Let $X \in \widetilde{\mathcal{P}}(E)$ a fuzzy subset of $E \neq \varnothing$

$$Y = (X)_{\geq \frac{1}{2}} ,$$

i.e.

$$v \in Y \Leftrightarrow \mu_X(v) \geq \tfrac{1}{2} \tag{48}$$

for all $v \in E$.
Firstly, we will show that $\theta(\mu_X(v), \chi_Y(v)) = 1$ for all $v \in E$. To see this, choose any $v \in E$.

a. $\mu_X(v) \geq \frac{1}{2}$.
Then $v \in Y$ by (48) and hence

$$\begin{aligned} \theta(\mu_X(v), \chi_Y(v)) &= \theta(\mu_X(v), 1) && \text{because } v \in Y \\ &= 1 . && \text{by Def. 46: } \mu_X(v) \geq \tfrac{1}{2} \end{aligned}$$

b. $\mu_X(v) < \frac{1}{2}$.
Then $v \notin Y$ by (48) and hence

$$\begin{aligned}\theta(\mu_X(v),\, \chi_Y(v)) &= \theta(\mu_X(v),\, 0) && \text{because } v \notin Y \\ &= 1\,. && \text{by Def. 46: } \mu_X(v) < \tfrac{1}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\Theta(X,\, Y) &= \inf\{\theta(\mu_X(v),\, \chi_Y(v)) : v \in E\} \\ &= \inf\{1 : v \in E\} && \text{by a. and b. above} \\ &= 1\,,\end{aligned}$$

as desired.

Proof of Theorem 51

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{2}$ is a two-valued determiner and $(X_1,\, \dots,\, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Define $(Y'_1,\, \dots,\, Y'_n) \in \mathcal{P}(E)^n$ by

$$Y'_i = (X_i)_{\geq\frac{1}{2}}$$

for all $i \in \{1,\, \dots,\, n\}$. By L-43,

$$\Theta(X_i,\, Y'_i) = 1 \tag{49}$$

for all $i \in \{1,\, \dots,\, n\}$ and hence

$$\min_{i=1}^{n} \Theta(X_i,\, Y'_i) = 1\,.$$

Because D is two-valued, there are two cases to consider.

a. $D(Y'_1,\, \dots,\, Y'_n) = 0$. Then $(Y'_1,\, \dots,\, Y'_n) \in D^{-1}(0)$ and hence

$$\begin{aligned}\varphi_{\mathrm{F}} &= \widetilde{D}_0(X_1,\, \dots,\, X_n) \\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \dots,\, Y_n) \in D^{-1}(0)\} \\ &\leq \min_{i=1}^{n} \Theta(X_i,\, Y'_i) && (Y'_1,\, \dots,\, Y'_n) \in D^{-1}(0) \\ &= 1\,. && \text{by (49)}\end{aligned}$$

b. $D(Y'_1,\, \dots,\, Y'_n) = 1$. Then $(Y'_1,\, \dots,\, Y'_n) \in D^{-1}(1)$ and hence

$$\begin{aligned}\varphi_{\mathrm{T}} &= \widetilde{D}_1(X_1,\, \dots,\, X_n) \\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \dots,\, Y_n) \in D^{-1}(1)\} \\ &\leq \min_{i=1}^{n} \Theta(X_i,\, Y'_i) && (Y'_1,\, \dots,\, Y'_n) \in D^{-1}(1) \\ &= 1\,. && \text{by (49)}\end{aligned}$$

In any case, $(\varphi_{\mathrm{T}},\, \varphi_{\mathrm{F}}) \in (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I})$.

B.18 Proof of Theorem 52

Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is a threes-valued fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Again define $(Y_1', \ldots, Y_n') \in \mathcal{P}(E)^n$ by

$$Y_i' = (X_i)_{\geq \frac{1}{2}}$$

for all $i \in \{1, \ldots, n\}$. By L-43,

$$\Theta(X_i, Y_i') = 1 \tag{50}$$

for all $i \in \{1, \ldots, n\}$ and hence

$$\min_{i=1}^{n} \Theta(X_i, Y_i') = 1 .$$

There are three cases to consider.

a. $D(Y_1', \ldots, Y_n') = 0$. Then $(Y_1', \ldots, Y_n') \in D^{-1}(0)$ and hence

$$\begin{aligned}
\varphi_{\mathrm{F}} &= \max(\widetilde{D}_0(X_1, \ldots, X_n), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&= \max(\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(0)\}, \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\leq \max(\min_{i=1}^{n} \Theta(X_i, Y_i'), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\qquad \text{as } (Y_1', \ldots, Y_n') \in D^{-1}(0) \\
&= \max(1, \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\qquad \text{by (50)} \\
&= 1 .
\end{aligned}$$

b. $D(Y_1', \ldots, Y_n') = 1$. Then $(Y_1', \ldots, Y_n') \in D^{-1}(1)$ and hence

$$\begin{aligned}
\varphi_{\mathrm{T}} &= \max(\widetilde{D}_1(X_1, \ldots, X_n), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&= \max(\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(1)\}, \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\leq \max(\min_{i=1}^{n} \Theta(X_i, Y_i'), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\qquad \text{as } (Y_1', \ldots, Y_n') \in D^{-1}(1) \\
&= \max(1, \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&\qquad \text{by (50)} \\
&= 1 .
\end{aligned}$$

c. $D(Y'_1, \ldots, Y'_n) = \frac{1}{2}$. Then

$$\begin{aligned}
\varphi_{\mathrm{F}} &= \max(\widetilde{D}_0(X_1, \ldots, X_n), \widetilde{D}_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&= \max(\widetilde{D}_0(X_1, \ldots, X_n), \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(\tfrac{1}{2})\}) \\
&\leq \max(\widetilde{D}_0(X_1, \ldots, X_n), \min_{i=1}^{n} \Theta(X_i, Y'_i)) \\
&\qquad \text{as } (Y'_1, \ldots, Y'_n) \in D^{-1}(\frac{1}{2}) \\
&= \max(\widetilde{D}_0(X_1, \ldots, X_n), 1) \\
&\qquad \text{by (50)} \\
&= 1\,.
\end{aligned}$$

In any case, $(\varphi_{\mathrm{T}}, \varphi_{\mathrm{F}}) \in (\mathbf{I} \times \{1\}) \cup (\{1\} \times \mathbf{I})$, as desired.

B.19 Proof of Theorem 53

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned}
\varphi_{\mathrm{T}} &= \max((\widetilde{D_{\langle\ell\rangle}})_1(X_1, \ldots, X_n), (\widetilde{D_{\langle\ell\rangle}})_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&= \max(\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(1)\}, \\
&\qquad \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(\tfrac{1}{2})\}) \\
&= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(\{\tfrac{1}{2}, 1\})\}\,. \qquad (*)
\end{aligned}$$

Likewise

$$\begin{aligned}
\varphi_{\mathrm{F}} &= \max((\widetilde{D_{\langle\ell\rangle}})_0(X_1, \ldots, X_n), (\widetilde{D_{\langle\ell\rangle}})_{\frac{1}{2}}(X_1, \ldots, X_n)) \\
&= \max(\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(0)\}, \\
&\qquad \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(\tfrac{1}{2})\}) \\
&= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle\ell\rangle}}^{-1}(\{0, \tfrac{1}{2}\})\}\,. \qquad (**)
\end{aligned}$$

Now let $\ell \in \mathbf{I}$.

a. $\ell = 0$.

Then

$$\begin{aligned}
\varphi_{\mathrm{T}} &= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in {D_{\langle 0\rangle}}^{-1}(\{\tfrac{1}{2}, 1\})\} && \text{by } (*) \\
&= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(\{\tfrac{1}{2}\}) \cup D^{-1}((\tfrac{1}{2}, 1])\} && \text{by Def. 52} \\
&= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}([\tfrac{1}{2}, 1])\} \\
&= \varphi_{\mathrm{T}}(0)
\end{aligned}$$

and

$$\begin{aligned}\varphi_{\mathrm{F}} &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in {D_{\langle 0\rangle}}^{-1}(\{0,\, \tfrac{1}{2}\})\} && \text{by (**)}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2})) \cup D^{-1}(\{\tfrac{1}{2}\})\} && \text{by Def. 52}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2}])\}\\ &= \varphi_{\mathrm{F}}(0)\end{aligned}$$

b. $\ell > 0$. Then

$$\begin{aligned}\varphi_{\mathrm{T}} &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in {D_{\langle \ell\rangle}}^{-1}(\{\tfrac{1}{2},\, 1\})\}\\ & \qquad \text{by (*)}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell,\, \tfrac{1}{2} + \tfrac{1}{2}\ell) \cup D^{-1}([\tfrac{1}{2} + \tfrac{1}{2}\ell,\, 1])\}\\ & \qquad \text{by Def. 52}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell,\, 1])\}\\ &= \varphi_{\mathrm{T}}(\ell)\end{aligned}$$

and

$$\begin{aligned}\varphi_{\mathrm{F}} &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in {D_{\langle \ell\rangle}}^{-1}(\{0,\, \tfrac{1}{2}\})\}\\ & \qquad \text{by (**)}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2} - \tfrac{1}{2}\ell]) \cup D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell,\, \tfrac{1}{2} + \tfrac{1}{2}\ell))\}\\ & \qquad \text{by Def. 52}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i,\, Y_i) : (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2} + \tfrac{1}{2}\ell))\}\\ &= \varphi_{\mathrm{F}}(\ell)\end{aligned}$$

B.20 Proof of Theorem 54

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1,\, \ldots,\, X_n) \in \widetilde{\mathcal{P}}(E)^n$. By Th-53,

$$\mathcal{G}_\ell(D)(X_1,\, \ldots,\, X_n) = \mathrm{K}(\varphi_{\mathrm{T}}(\ell),\, \varphi_{\mathrm{F}}(\ell))$$

for all $\ell \in \mathbf{I}$, where

$$\varphi_{\mathrm{T}}(\ell) = \begin{cases} \sup\{\min_{i=1}^n \Theta(X_i,\, Y_i) : Y = (Y_1,\, \ldots,\, Y_n) \in D^{-1}([\tfrac{1}{2},\, 1])\} & : \ \ell = 0\\ \sup\{\min_{i=1}^n \Theta(X_i,\, Y_i) : Y = (Y_1,\, \ldots,\, Y_n) \in D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell,\, 1])\} & : \ \ell > 0\end{cases}$$

$$\varphi_{\mathrm{F}}(\ell) = \begin{cases} \sup\{\min_{i=1}^n \Theta(X_i,\, Y_i) : Y = (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2}])\} & : \ \ell = 0\\ \sup\{\min_{i=1}^n \Theta(X_i,\, Y_i) : Y = (Y_1,\, \ldots,\, Y_n) \in D^{-1}([0,\, \tfrac{1}{2} + \tfrac{1}{2}\ell))\} & : \ \ell > 0\end{cases}$$

It is apparent from these formulae that both $\varphi_{\mathrm{T}}(\ell)$ and $\varphi_{\mathrm{F}}(\ell)$ are monotonically increasing in ℓ because the intervals to which D^{-1} is applied are increasing when ℓ increases.

By Th-52 and Th-53, we further know that $\varphi_{\mathrm{T}}(0) = 1$ or $\varphi_{\mathrm{F}}(0) = 1$.

a. $\varphi_{\mathrm{T}}(0) = 1$ and $\varphi_{\mathrm{F}}(0) < 1$.
Then $\varphi_{\mathrm{T}}(\ell) = 1$ for all $\ell \in \mathbf{I}$ because $\varphi_{\mathrm{T}}(\ell)$ is monotonically increasing in ℓ and $1 \geq \varphi_{\mathrm{T}}(\ell) \geq \varphi_{\mathrm{T}}(0) = 1$. In particular,

$$\begin{aligned}\mathcal{G}_\ell(D)(X_1, \ldots, X_n) &= \mathrm{K}(\varphi_{\mathrm{T}}(\ell), \varphi_{\mathrm{F}}(\ell)) && \text{by Th-53}\\ &= \mathrm{K}(1, \varphi_{\mathrm{F}}(\ell))\\ &= 1 - \tfrac{1}{2}\varphi_{\mathrm{F}}(\ell)\,. && (*)\end{aligned}$$

Therefore

- $\mathcal{G}_0(D)(X_1, \ldots, X_n) = 1 - \frac{1}{2}\varphi_{\mathrm{F}}(0) > \frac{1}{2}$, as $\varphi_{\mathrm{F}}(0) < 1$ by assumption;
- $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is monotonically decreasing in ℓ because $\varphi_{\mathrm{F}}(\ell)$ is monotonically increasing in ℓ;
- $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) \geq \frac{1}{2}$ because $\varphi_{\mathrm{F}}(\ell) \leq 1$ and hence $1 - \frac{1}{2}\varphi_{\mathrm{F}}(\ell) \geq \frac{1}{2}$, for all $\ell \in \mathbf{I}$.

This proves part **a.** of the theorem.

b. $\varphi_{\mathrm{T}}(0) = 1$ and $\varphi_{\mathrm{F}}(0) = 1$.
Then $\varphi_{\mathrm{T}}(\ell) = \varphi_{\mathrm{F}}(\ell) = 1$ for all $\ell \in \mathbf{I}$ because $\varphi_{\mathrm{T}}(\ell)$ and $\varphi_{\mathrm{F}}(\ell)$ are monotonically increasing in ℓ and $1 \geq \varphi_{\mathrm{T}}(\ell) \geq \varphi_{\mathrm{T}}(0) = 1$, $1 \geq \varphi_{\mathrm{F}}(\ell) \geq \varphi_{\mathrm{F}}(0) = 1$. It follows that

$$\begin{aligned}\mathcal{G}_\ell(D)(X_1, \ldots, X_n) &= \mathrm{K}(\varphi_{\mathrm{T}}(\ell), \varphi_{\mathrm{F}}(\ell)) && \text{by Th-53}\\ &= \mathrm{K}(1, 1)\\ &= \tfrac{1}{2}\,, && \text{by Def. 48}\end{aligned}$$

for all $\ell \in \mathbf{I}$.

c. $\varphi_{\mathrm{T}}(0) < 1$ and $\varphi_{\mathrm{F}}(\ell) = 1$.
(Analogous to **a.**)

B.21 Proof of Theorem 55

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Let us abbreviate

$$Z = \hat{\hat{D}}(\widetilde{\Theta}(X_1, \ldots, X_n))\,.$$

Then for all $v \in \mathbf{I}$,

$$\begin{aligned}\mu_Z(v) &= \sup\{\mu_{\widetilde{\Theta}(X_1,\ldots,X_n)}(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in D^{-1}(v)\}\\ &= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(v)\}\\ &= \widetilde{D}_v(X_1, \ldots, X_n) && (*)\end{aligned}$$

and hence

$$\begin{aligned}
&(Q_\ell \circ \hat{\hat{D}} \circ \widetilde{\Theta})(X_1, \ldots, X_n)\\
&= Q_\ell(Z)\\
&= \mathrm{K}(\sup\{\mu_Z(v) : v \in T_\ell\}, \sup\{\mu_Z(v) : v \in F_\ell\})\\
&= \mathrm{K}(\sup\{\widetilde{D}_v(X_1, \ldots, X_n) : v \in T_\ell\}, \sup\{\widetilde{D}_v(X_1, \ldots, X_n) : v \in F_\ell\}) && \text{by (*)}\\
&= \mathrm{K}(\sup\{\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(v)\} : v \in T_\ell\},\\
&\qquad \sup\{\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(v)\} : v \in F_\ell\})\\
&= \mathrm{K}(\sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(T_\ell)\},\\
&\qquad \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : (Y_1, \ldots, Y_n) \in D^{-1}(F_\ell)\})\\
&= \mathrm{K}(\varphi_{\mathrm{T}}(\ell), \varphi_{\mathrm{F}}(\ell)) && \text{by Def. 54}\\
&= \mathcal{G}_\ell(D)(X_1, \ldots, X_n) && \text{by Th-53}
\end{aligned}$$

B.22 Preliminaries (equivalence proofs)

For technical reasons, we need to introduce another variant of three-valued cuts.

Definition 59
Let $\gamma \in \mathbf{I}$, $X_1, \ldots, X_n \in \widetilde{\mathcal{P}}(E)$. For all $i = 1, \ldots, n$,

$$\begin{aligned}
\widehat{(X_i)}_\gamma^{\min} &= \begin{cases} (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} & : \gamma < 1 \\ (X_i)_{\geq 1} & : \gamma = 1 \end{cases}\\
\widehat{(X_i)}_\gamma^{\max} &= \begin{cases} (X_i)_{\geq\frac{1}{2}-\frac{1}{2}\gamma} & : \gamma < 1 \\ (X_i)_{>0} & : \gamma = 1 \end{cases}\\
\widehat{\mathcal{Y}}_i^\gamma &= \{Y : \widehat{(X_i)}_\gamma^{\min} \subseteq Y \subseteq \widehat{(X_i)}_\gamma^{\max}\}\\
\widehat{\mathcal{Y}} &= \widehat{\mathcal{Y}}_1 \times \cdots \times \widehat{\mathcal{Y}}_n\,.
\end{aligned}$$

Lemma 44

a. For all $\gamma \in \mathbf{I}$, $\mathcal{Y}^\gamma \subseteq \widehat{\mathcal{Y}}^\gamma$.

b. For all $\gamma, \gamma' \in \mathbf{I}$, if $\gamma' > \gamma$ then $\widehat{\mathcal{Y}}^\gamma \subseteq \mathcal{Y}^{\gamma'}$.

Proof Suppose E is a nonempty set and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. Let $\gamma \in \mathbf{I}$.

1. $\gamma = 0$. Clearly $\mathcal{Y}_i^0 = \widehat{\mathcal{Y}}_i^0$ for all $i \in \{1, \ldots, n\}$ and hence $\mathcal{Y}^0 = \widehat{\mathcal{Y}}^0$.

2. If $\gamma \in (0,\, 1)$, then

$$(X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{\geq\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma} \subseteq (X_i)_{\geq\frac{1}{2}-\frac{1}{2}\gamma}$$

i.e.

$$\widehat{(X_i)}_\gamma^{\min} \subseteq (X_i)_\gamma^{\min} \subseteq (X_i)_\gamma^{\max} \subseteq \widehat{(X_i)}_\gamma^{\max}$$

and therefore $\mathcal{Y}_i^\gamma \subseteq \widehat{\mathcal{Y}}_i^\gamma$ for all $i \in \{1,\, \ldots,\, n\}$, in particular $\mathcal{Y}^\gamma \subseteq \widehat{\mathcal{Y}}^\gamma$.

3. $\gamma = 1$. Noting that $(X_i)_1^{\min} = (X_i)^{\blacktriangledown\min}_1$ and $(X_i)_1^{\max} = (X_i)^{\blacktriangledown\max}_1$, we immediately obtain $\mathcal{Y}^\gamma = \widehat{\mathcal{Y}}^\gamma$.

b. Let $\gamma,\, \gamma' \in \mathbf{I}$, $\gamma' > \gamma$.

1. $\gamma \in [0,\, 1)$. Then

$$(X_i)_{\geq\frac{1}{2}+\frac{1}{2}\gamma'} \subseteq (X_i)_{>\frac{1}{2}+\frac{1}{2}\gamma} \subseteq (X_i)_{\geq\frac{1}{2}-\frac{1}{2}\gamma} \subseteq (X_i)_{>\frac{1}{2}-\frac{1}{2}\gamma'}$$

i.e.

$$(X_i)_{\gamma'}^{\min} \subseteq (X_i)^{\blacktriangledown\min}_\gamma \subseteq (X_i)^{\blacktriangledown\max}_\gamma \subseteq (X_i)_{\gamma'}^{\max}$$

for $i \in \{1,\, \ldots,\, n\}$, and therefore $\widehat{\mathcal{Y}}^\gamma \subseteq \mathcal{Y}^{\gamma'}$.

2. $\gamma = 1$. This case is trivial.

Lemma 45
Let $\gamma < 1$, $Y_i \in \mathcal{P}(E)$. Then $Y_i \in \widehat{\mathcal{Y}}_i^\gamma$ iff for all $x \in E$, $\theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) \geq 1 - \gamma$.

Proof

a. $Y_i \in \widehat{\mathcal{Y}}_i^\gamma \Rightarrow \theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) \geq 1 - \gamma$. Let $x \in E$.

1. $x \in \widehat{(X_i)}_\gamma^{\min}$, i.e. $x \in Y_i$ and $\mu_{X_i}(x) \geq \frac{1}{2}$ by Def. 59. Therefore,

$$\begin{aligned}\theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) &= \theta(\mu_{X_i}(x),\, 1)\\ &= 1\end{aligned}$$

by the definition of θ.

2. $x \in \widehat{(X_i)}_\gamma^{\max}$, $x \notin \widehat{(X_i)}_\gamma^{\min}$.

If $\mu_{X_i}(x) \leq \frac{1}{2}$, then

$$\theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) \geq \theta(\mu_{X_i}(x),\, 1) = 2\mu_{X_i} \geq 2(\tfrac{1}{2} - \tfrac{1}{2}\gamma) = 1 - \gamma$$

If $\mu_{X_i}(x) > \frac{1}{2}$, then

$$\theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) \geq \theta(\mu_{X_i}(x),\, 0) = 2 - 2\mu_{X_i}(x) \geq 2 - 2(\tfrac{1}{2} + \tfrac{1}{2}\gamma) = 1 - \gamma$$

3. $x \notin \widehat{(X_i)}_\gamma^{\max}$. Then $x \notin Y_i$ and $\mu_{X_i}(x) < \frac{1}{2} - \frac{1}{2}\gamma \leq \frac{1}{2}$. Therefore

$$\theta(\mu_{X_i}(x),\, \chi_{Y_i}(x)) = \theta(\mu_{X_i}(x),\, 0) = 1 \geq 1 - \gamma\,.$$

b. $Y_i \notin \widehat{\mathcal{Y}}_i^\gamma \Rightarrow \theta(\mu_{X_i}(x), \chi_{Y_i}(x)) < 1 - \gamma$ for some $x \in E$. Suppose $Y_i \notin \widehat{\mathcal{Y}}_i^\gamma$.

1. $\widehat{(X_i)}_\gamma^{\min} \nsubseteq Y_i$, i.e. there exists $x \in E$ such that $\mu_{X_i}(x) > \frac{1}{2} + \frac{1}{2}\gamma$ and $x \notin Y_i$. Hence
$$\theta(\mu_{X_i}(x), \chi_{Y_i}(x)) = \theta(\mu X_i(x), 0) = 2 - 2\mu_{X_i}(x) < 2 - 2(\tfrac{1}{2} + \tfrac{1}{2}\gamma) = 1 - \gamma\,.$$
2. $Y_i \nsubseteq \widehat{(X_i)}_\gamma^{\max}$. Then, there exists $x \in Y_i$ such that $\mu_{X_i}(x) < \frac{1}{2} - \frac{1}{2}\gamma$.
$$\theta(\mu_{X_i}(x), \chi_{Y_i}(x)) = \theta(\mu_{X_i}(x), 1) = 2\mu_{X_i}(x) < 2(\tfrac{1}{2} - \tfrac{1}{2}\gamma) = 1 - \gamma\,.$$

Lemma 46
Let $\gamma \in [0, 1)$.

a. For all $Y_i \in \mathcal{P}(E)$, $Y_i \in \widehat{\mathcal{Y}}_i^\gamma$ iff $\Theta(X_i, Y_i) \geq 1 - \gamma$.

b. For all $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$, $Y \in \widehat{\mathcal{Y}}^\gamma$ iff $\min_{i=1}^n \Theta(X_i, Y_i) \geq 1 - \gamma$.

Proof

a. Let $\gamma \in [0, 1)$, $Y_i \in \mathcal{P}(E)$.

1. $Y_i \in \widehat{\mathcal{Y}}_i^\gamma \Rightarrow \Theta(X_i, Y_i) \geq 1 - \gamma$.

   Suppose $Y_i \in \widehat{\mathcal{Y}}_i^\gamma$, $x \in E$. We know from L-45 that for all $x \in E$, $\theta(\mu_{X_i}(x), \chi_{Y_i}(x)) \geq 1-\gamma$. Therefore,
$$\Theta(X_i, Y_i) = \inf_{x \in E} \theta(\mu_{X_i}(x), \chi_{Y_i}(x)) \geq 1 - \gamma\,.$$
2. $Y_i \notin \widehat{\mathcal{Y}}_i^\gamma \Rightarrow \Theta(X_i, Y_i) < 1 - \gamma$.

   Suppose $Y_i \notin \widehat{\mathcal{Y}}_i^\gamma$. By L-45 there exists $x' \in E$ such that $\theta(\mu_{X_i}(x), \chi_{Y_i}(x')) < 1 - \gamma$. It follows that
$$\Theta(X_i, Y_i) = \inf_{x \in E} \theta(\mu_{X_i}(x), \chi_{Y_i}(x)) \leq x' < 1 - \gamma\,.$$

b. Let $\gamma \in [0, 1)$, $Y = (Y_1, \ldots, Y_n) \in \mathcal{P}(E)^n$.

1. $Y \in \widehat{\mathcal{Y}}^\gamma$. Then $Y_i \in \widehat{\mathcal{Y}}_i^\gamma$ for $i = 1, \ldots, n$, i.e. $\Theta(X_i, Y_i) \geq 1 - \gamma$ by part **a.** of this lemma. Let i_0 such that $\min_{i=1}^n \Theta(X_i, Y_i) = \Theta(X_{i_0}, Y_{i_0})$. We obtain
$$\min_{i=1}^n \Theta(X_i, Y_i) = \Theta(X_{i_0}, Y_{i_0}) \geq 1 - \gamma\,.$$
2. $Y \notin \widehat{\mathcal{Y}}^\gamma$. Then there is an $i_0 \in \{1, \ldots, n\}$ such that $Y_{i_0} \notin \widehat{\mathcal{Y}}_{i_0}^\gamma$. Hence,
$$\min_{i=1}^n \Theta(X_i, Y_i) \leq \Theta(X_{i_0}, Y_{i_0}) < 1 - \gamma,$$
   where the last inequation holds by part **a.** of this lemma.

Lemma 47
For all $v \in \mathbf{I}$, $\widetilde{D}_v(X_1, \ldots, X_n) = \sup\{1 - \gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\}$.

Proof

a. $\widetilde{D}_v(X_1, \ldots, X_n) \leq \sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\}$
The case $D^{-1}(v) = \emptyset$ is trivial because it implies that $\widetilde{D}_v(X_1, \ldots, X_n) = \sup \varnothing = 0$.
Now let us consider the case $D^{-1}(v) \neq \emptyset$. We have to show that for all $Y \in D^{-1}(v)$,

$$\min_{i=1}^{n} \Theta(X_i, Y_i) \leq \sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(x) \neq \varnothing\}\,. \tag{51}$$

Let $Y = (Y_1, \ldots, Y_n) \in D^{-1}(v)$.

1. $\min_{i=1}^n \Theta(X_i, Y_i) = 0$. Then (51) trivially holds.
2. $\min_{i=1}^n \Theta(X_i, Y_i) > 0$. Let $\gamma' = 1 - \widetilde{D}_v(X_1, \ldots, X_n) < 1$. Then $\Theta(X, Y) = 1 - \gamma'$ and by L-46, $Y \in \widehat{\mathcal{Y}}^{\gamma'}$. Therefore, $\widehat{\mathcal{Y}}^{\gamma'} \cap D^{-1}(x) \neq \varnothing\}$, i.e.

$$1 - \gamma' = \Theta(X, Y) \in \{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(x) \neq \varnothing\}\,.$$

   It follows that

$$\sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(x) \neq \varnothing\} \geq \Theta(X, Y)\,.$$

b. $\widetilde{D}_v(X_1, \ldots, X_n) \geq \sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\}$.
If $\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\} = \varnothing$, the condition holds trivially since $\widetilde{D}_v(X_1, \ldots, X_n) \geq 0 = \sup \varnothing$.
So let us consider the case that $\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\} \neq \varnothing$. Suppose $\gamma \in \mathbf{I}$ such that $\widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing$.

1. $\gamma = 1$. Trivially, $\widetilde{D}_v(X_1, \ldots, X_n) \geq 0 = 1 - \gamma$.
2. $\gamma < 1$. Let $Y' \in \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing$. Then by L-46, $\min_{i=1}^n \Theta(X_i, Y'_i) \geq 1 - \gamma$, yielding

$$\begin{aligned}\widetilde{D}_v(X_1, \ldots, X_n) &= \sup\{\min_{i=1}^{n} \Theta(X_i, Y_i) : Y \in D^{-1}(v)\}\\ &\geq \min_{i=1}^{n} \Theta(X_i, Y'_i)\\ &\geq 1-\gamma\,.\end{aligned}$$

Lemma 48
For all $v \in \mathbf{I}$, $\widetilde{D}_v(X_1, \ldots, X_n) = \sup\{1-\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\}$.

Proof By L-47,

$$\widetilde{D}_v(X_1, \ldots, X_n) = \sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\}.$$

Therefore, we only need to show that

$$\sup\{1-\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} = \sup\{1-\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\}$$

which is obviously equivalent to the condition

$$\inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} = \inf\{\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\} \tag{52}$$

In the following, the left-hand side of (52) is abbreviated by L, its right-hand side by R.

1. $L \geq R$.
   By L-44, $\mathcal{Y}^\gamma \subseteq \widehat{\mathcal{Y}}^\gamma$ and hence
   $$\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} \subseteq \{\gamma : \widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing\} .$$
   It follows that $L \geq R$.
2. $L \leq R$. Firstly, if $R = 1$, then from $L \geq R$, $L \in \mathbf{I}$ we obtain $L = R$.
   Now let us consider the case $R < 1$.
   Let $\varepsilon > 0$. Then there exists $\gamma \in [R,\ R + \varepsilon)$ such that $\widehat{\mathcal{Y}}^\gamma \cap D^{-1}(v) \neq \varnothing$.
   Let $\gamma' \in (\gamma, R + \varepsilon)$. Then by L-44, $\widehat{\mathcal{Y}}^\gamma \subseteq \mathcal{Y}^{\gamma'}$, in particular, $\mathcal{Y}^{\gamma'} \cap D^{-1}(v) \neq \varnothing$. Therefore
   $$\inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} \leq \gamma' < R + \varepsilon .$$
   As $\varepsilon \to 0$, we obtain $L \leq R$, as desired.

Definition 60
For all $\ell \in \mathbf{I}$, we define
$$g_{\mathrm{T}}(\ell) = \begin{cases} \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell,\ 1]) \neq \varnothing\} & : \quad \ell > 0 \\ \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}([\frac{1}{2},\ 1]) \neq \varnothing\} & : \quad \ell = 0 \end{cases}$$
$$g_{\mathrm{F}}(\ell) = \begin{cases} \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}([0,\ \frac{1}{2} + \frac{1}{2}\ell)) \neq \varnothing\} & : \quad \ell > 0 \\ \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}([0,\ \frac{1}{2}]) \neq \varnothing\} & : \quad \ell = 0 \end{cases}$$

Lemma 49
For all $\ell \in \mathbf{I}$,

a. $\varphi_{\mathrm{T}}(\ell) = 1 - g_{\mathrm{T}}(\ell)$

b. $\varphi_{\mathrm{F}}(\ell) = 1 - g_{\mathrm{F}}(\ell)$

Proof Let $A \subseteq \mathbf{I}$. Then
$$\begin{aligned} \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}(A) \neq \varnothing\} &= \inf\{\gamma : \exists v \in A \text{ s.th. } \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} && (*) \\ &= \inf\{\inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} : v \in A\} \\ &= 1 - \sup\{\sup\{1 - \gamma : \mathcal{Y}^\gamma \cap D^{-1}(v) \neq \varnothing\} : v \in A\} \\ &= 1 - \sup\{\widetilde{D}_v(X_1,\ \ldots,\ X_n) : v \in A\} && \text{by L-48} \end{aligned}$$

a. If $\ell = 0$,
$$\begin{aligned} \varphi_{\mathrm{T}}(0) &= \sup\{\widetilde{D}_v(X_1,\ \ldots,\ X_n) : v \in [\tfrac{1}{2},\ 1]\} \\ &= 1 - (1 - \sup\{\widetilde{D}_v(X_1,\ \ldots,\ X_n) : v \in [\tfrac{1}{2},\ 1]\} \\ &= 1 - \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}([\tfrac{1}{2},\ 1]) \neq \varnothing\} && \text{by } (*) \\ &= 1 - g_{\mathrm{T}}(0) \end{aligned}$$
Similarly if $\ell > 0$,
$$\begin{aligned} \varphi_{\mathrm{T}}(\ell) &= \sup\{\widetilde{D}_v(X_1,\ \ldots,\ X_n) : v \in (\tfrac{1}{2} - \tfrac{1}{2}\ell,\ 1]\} \\ &= 1 - (1 - \sup\{\widetilde{D}_v(X_1,\ \ldots,\ X_n) : v \in (\tfrac{1}{2} - \tfrac{1}{2}\ell,\ 1]\} \\ &= 1 - \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell,\ 1]) \neq \varnothing\} && \text{by } (*) \\ &= 1 - g_{\mathrm{T}}(\ell) \end{aligned}$$

b. Analogously. Use $A = [0, \frac{1}{2}]$ for $\ell = 0$, $A = [0, \frac{1}{2} + \frac{1}{2}\ell)$ for $\ell > 0$ and apply (*).

Definition 61

$$h_{\mathrm{T}}(\ell) = \begin{cases} \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell\} & : \quad \ell > 0 \\ \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}\} & : \quad \ell = 0 \end{cases}$$
$$h_{\mathrm{F}}(\ell) = \begin{cases} \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < \frac{1}{2} + \frac{1}{2}\ell\} & : \quad \ell > 0 \\ \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) \leq \frac{1}{2}\} & : \quad \ell = 0 \end{cases}$$

First a (trivial) note:

Lemma 50

a. If $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$, then $h_{\mathrm{T}}(\ell) = 0$ for all $\ell \in \mathbf{I}$.

b. If $D_0(X_1, \ldots, X_n) \leq \frac{1}{2}$, then $h_{\mathrm{F}}(\ell) = 0$ for all $\ell \in \mathbf{I}$.

Proof

a. Because $D_0(X_1, \ldots, X_n) \geq \frac{1}{2}$, $\gamma = 0$ satisfies $D_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}$, and hence

$$h_{\mathrm{T}}(0) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) \geq \tfrac{1}{2}\} \leq \inf\{0\} = 0\,,$$

i.e. $h_{\mathrm{T}}(0) = 0$ because $h_{\mathrm{T}}(\ell) \in \mathbf{I}$.
For all $\ell > 0$, $D_0(X_1, \ldots, X_n) = \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\ell$, yielding

$$h_{\mathrm{T}}(\ell) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} \leq \inf\{0\} = 0,$$

i.e. $h_{\mathrm{T}}(\ell) = 0$ because $h_{\mathrm{T}}(\ell) \in \mathbf{I}$.

b. Analogously. Just reverse every inequation in a., replace h_{T} by h_{F} and $\frac{1}{2} - \frac{1}{2}\ell$ by $\frac{1}{2} + \frac{1}{2}\ell$.

Lemma 51
For all $\ell \in (0, 1]$,

a. $g_{\mathrm{T}}(\ell) = h_{\mathrm{T}}(\ell)$

b. $g_{\mathrm{F}}(\ell) = h_{\mathrm{F}}(\ell)$

Proof Let $\ell > 0$.

a. $g_{\mathrm{T}}(\ell) = h_{\mathrm{T}}(\ell)$.

1. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$.
   Then $D_0^{\min}(X_1, \ldots, X_n) > \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\ell$ by L-16, i.e.
   $$\inf\{D_\gamma(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^0\} \geq \tfrac{1}{2} \tag{53}$$
   Hence there exists $(Y_1, \ldots, Y_n) \in \mathcal{Y}^0$ such that $D(Y_1, \ldots, Y_n) > \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\ell$, i.e. $\gamma = 0$ satisfies $\mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing$. Therefore
   $$g_{\mathrm{T}}(\ell) = \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell, 1]) \neq \varnothing\} = 0 = h_{\mathrm{T}}(\ell)$$
   by L-50.

2. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$.
   Then by L-16, $D_0^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$ and $D_0^{\min}(X_1, \ldots, X_n) \leq \frac{1}{2}$.
   But $D_0^{\max}(X_1, \ldots, X_n) = \sup\{D_\gamma(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^0\}$, i.e. for all $\varepsilon > 0$, there exists $(Y_1, \ldots, Y_n) \in \mathcal{Y}^0$ such that $D(Y_1, \ldots, Y_n) > \frac{1}{2} - \varepsilon$. In particular, $\gamma = 0$ satisfies $\mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing$ (simply set $\varepsilon = \frac{1}{2}\ell$), from which we obtain
   $$g_{\mathrm{T}}(\ell) = \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\tfrac{1}{2} - \tfrac{1}{2}\ell, 1]) \neq \varnothing\} = 0 = h_{\mathrm{T}}(\ell)$$
   by L-50.

3. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$.
   Firstly, let us prove that $g_{\mathrm{T}}(\ell) \leq h_{\mathrm{T}}(\ell)$. If $\{\gamma : D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell\} = \varnothing$, then
   $$g_{\mathrm{T}}(\ell) \leq 1 = \inf \varnothing = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} = h_{\mathrm{T}}(\ell)\,.$$
   Now suppose $\{\gamma : D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell\} \neq \varnothing$. Because
   $$h_{\mathrm{T}}(\ell) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\}\,,$$
   it is sufficient to show that for all γ satisfying $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell$, it also holds that $g_{\mathrm{T}}(\ell) \leq \gamma$.
   Let $\gamma \in \mathbf{I}$ such that $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell$.

   - $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$.
     Then $D_\gamma^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$ by L-16, i.e.
     $$\sup\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\} \geq \tfrac{1}{2} > \tfrac{1}{2} - \tfrac{1}{2}\ell\,.$$
     In particular, there exists $(Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma$ such that $D(Y_1, \ldots, Y_n) > \frac{1}{2} - \frac{1}{2}\ell$. Thus $\mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing$ and hence $g_{\mathrm{T}}(\ell) \leq \gamma$.
   - $D_\gamma(X_1, \ldots, X_n) < \frac{1}{2}$.
     Then $D_\gamma(X_1, \ldots, X_n) = D_\gamma^{\max}(X_1, \ldots, X_n)$, again by L-16.
     But $D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell$ by assumption. Therefore
     $$\begin{aligned}\sup\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma\} &= D_\gamma^{\max}(X_1, \ldots, X_n)\\ &= D_\gamma(X_1, \ldots, X_n)\\ &> \tfrac{1}{2} - \tfrac{1}{2}\ell\,.\end{aligned}$$
     Hence there exists $(Y_1, \ldots, Y_n) \in \mathcal{Y}^\gamma$ such that $D(Y_1, \ldots, Y_n) > \frac{1}{2} - \frac{1}{2}\ell$, i.e. $g_{\mathrm{T}}(\ell) \leq \gamma$.

We still have to show that $g_{\mathrm{T}}(\ell) \geq h_{\mathrm{T}}(\ell)$. This holds trivially if $g_{\mathrm{T}}(\ell) = 1$.

Now suppose $g_{\mathrm{T}}(\ell) < 1$. In particular, $\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing\} \neq \varnothing$.

To prove that $g_{\mathrm{T}}(\ell) \geq h_{\mathrm{T}}(\ell)$, it is sufficient to show that for all $\varepsilon > 0$, $g_{\mathrm{T}}(\ell) + \varepsilon > h_{\mathrm{T}}(\ell)$.

So let $\varepsilon > 0$. Let us abbreviate $g^* = g_{\mathrm{T}}(\ell) + \varepsilon$. Because $g_{\mathrm{T}}(\ell) = \inf\{\gamma : \mathcal{Y}^\gamma \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing\}$ and $g^* > g_{\mathrm{T}}(\ell)$, there exists $\gamma' < g^*$ such that $\mathcal{Y}^{\gamma'} \cap D^{-1}((\frac{1}{2} - \frac{1}{2}\ell, 1]) \neq \varnothing\}$. Hence there exists $(Y_1', \ldots, Y_n') \in \mathcal{Y}^{\gamma'}$ such that $D(Y_1', \ldots, Y_n') > \frac{1}{2} - \frac{1}{2}\ell$, yielding

$$\begin{aligned} D_{\gamma'}^{\max}(X_1, \ldots, X_n) &= \sup\{D(Y_1, \ldots, Y_n) : (Y_1, \ldots, Y_n) \in \mathcal{Y}^{\gamma'}\} \\ &\geq D(Y_1', \ldots, Y_n') \\ &> \tfrac{1}{2} - \tfrac{1}{2}\ell\,. \end{aligned} \tag{+}$$

- $D_{\gamma'}^{\max}(X_1, \ldots, X_n) < \frac{1}{2}$.
  Then $D_{\gamma'}(X_1, \ldots, X_n) = D_{\gamma'}^{\max}(X_1, \ldots, X_n)$ by L-16.
  Therefore $D_{\gamma'}(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell$ by (+), and
  $$h_{\mathrm{T}}(\ell) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} \leq \gamma' < g^* = g_{\mathrm{T}}(\ell) + \varepsilon\,.$$
- $D_{\gamma'}^{\max}(X_1, \ldots, X_n) \geq \frac{1}{2}$.
  Then $D_{\gamma'}(X_1, \ldots, X_n) = \frac{1}{2} > \frac{1}{2} - \frac{1}{2}\ell$ by Th-42 and again
  $$h_{\mathrm{T}}(\ell) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} \leq \gamma' < g^* = g_{\mathrm{T}}(\ell) + \varepsilon\,.$$

b. The proof is completely analogous to that of part **a.**

Lemma 52
Let $\ell > 0$. Then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \mathrm{K}(1 - h_{\mathrm{T}}(\ell), 1 - h_{\mathrm{F}}(\ell))$

Proof By Th-53, $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \mathrm{K}(\varphi_{\mathrm{T}}(\ell), \varphi_{\mathrm{F}}(\ell))$. The lemma is then obtained as an immediate consequence of

$$\begin{aligned} \varphi_{\mathrm{T}}(\ell) &= 1 - g_{\mathrm{T}}(\ell) && \text{by L-49} \\ &= 1 - h_{\mathrm{T}}(\ell) && \text{by L-51} \end{aligned}$$

and

$$\begin{aligned} \varphi_{\mathrm{F}}(\ell) &= 1 - g_{\mathrm{F}}(\ell) && \text{by L-49} \\ &= 1 - h_{\mathrm{F}}(\ell) && \text{by L-51} \end{aligned}$$

Lemma 53
Let $\ell > 0$.

a. If $D_0(X_1, \ldots, X_n) < \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2} - \frac{1}{2}h_{\mathrm{T}}(\ell)$.

b. If $D_0(X_1, \ldots, X_n) = \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}$.

c. If $D_0(X_1, \ldots, X_n) > \frac{1}{2}$, then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2} + \frac{1}{2}h_{\mathrm{F}}(\ell)$.

Proof

a. $D_0(X_1,\ \dots,\ X_n) < \frac{1}{2}$.

$$\begin{aligned} \mathcal{G}_\ell(D)(X_1,\ \dots,\ X_n) &= \mathrm{K}(1 - h_{\mathrm{T}}(\ell),\ 1) && \text{by L-50, L-52} \\ &= \tfrac{1}{2}(1 - h_{\mathrm{T}}(\ell)) && \text{by Def. 48} \\ &= \tfrac{1}{2} - \tfrac{1}{2}h_{\mathrm{T}}(\ell)\,. \end{aligned}$$

b. $D_0(X_1,\ \dots,\ X_n) = \frac{1}{2}$.

$$\begin{aligned} \mathcal{G}_\ell(D)(X_1,\ \dots,\ X_n) &= \mathrm{K}(1,\ 1) && \text{by L-50, L-52} \\ &= \tfrac{1}{2}\,. && \text{by Def. 48} \end{aligned}$$

c. $D_0(X_1,\ \dots,\ X_n) > \frac{1}{2}$.

$$\begin{aligned} \mathcal{G}_\ell(D)(X_1,\ \dots,\ X_n) &= \mathrm{K}(1,\ 1 - h_{\mathrm{F}}(\ell)) && \text{by L-50, L-52} \\ &= 1 - \tfrac{1}{2}(1 - h_{\mathrm{F}}(\ell)) && \text{by Def. 48} \\ &= \tfrac{1}{2} + \tfrac{1}{2}h_{\mathrm{F}}(\ell)\,. \end{aligned}$$

B.23 Proof of Theorem 56

In order to show that $\mathcal{M} = \mathcal{G}$, we will firstly prove some technical lemmas.

Lemma 54
Let $f : \mathbf{I} \longrightarrow \mathbf{I}$. *The function* $f^* : \mathbf{I} \longrightarrow \mathbf{I}$*, defined by*

$$f^*(v) \quad = \quad \inf\{x \in \mathbf{I} : f(x) < v\}$$

for all $v \in \mathbf{I}$*, is monotonically decreasing.*

Proof As $v \le v'$, $f(x) < v$ implies $f(x) < v'$. Therefore,

$$\{x : f(x) < v\} \quad \subseteq \quad \{x : f(x) < v'\}\,,$$

and hence

$$\inf\{x : f(x) < v\} \quad \ge \quad \inf\{x : f(x) < v'\}\,.$$

Definition 62 (Step function)
Suppose $\varphi : \mathbf{I} \longrightarrow \mathbf{I}$. φ *is called a* step function *if there exist* $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ *in* $\mathbf{I}$ *such that for all* $k = 0,\ \dots,\ n-1$, $\varphi|_{(x_k,x_{k+1})}$ *is constant, i.e. for all* $x,\ y \in (x_k,\ x_{k+1})$,

$$\varphi(x) = \varphi(y)\,.$$

The step functions are of course Riemann integrable (indeed, they serve to define Riemann integrability). If $\varphi : \mathbf{I} \longrightarrow \mathbf{I}$ is a step function with $0 = x_0 < \cdots < x_n = 1$ as above, then

$$\int_0^1 \varphi(x)\, dx \quad = \quad \sum_{k=0}^{n-1} (x_{k+1} - x_k)\varphi(x'_k), \tag{54}$$

where $x'_k \in (x_k,\, x_{k+1})$, e.g. $x'_k = (x_k + x_{k+1})/2$.

Lemma 55
Let $\varphi : \mathbf{I} \longrightarrow \mathbf{I}$ a monotonically decreasing step function. Then $\varphi^* : \mathbf{I} \longrightarrow \mathbf{I}$, defined by

$$\varphi^*(v) = \inf\{x : \varphi(x) < v\}$$

for all $v \in \mathbf{I}$, is also a step function, and

$$\int_0^1 \varphi(x)\, dx \quad = \quad \int_0^1 \varphi^*(v)\, dv\,.$$

Proof By definition 62, there exist $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ such that for all $k = 0,\ \ldots,\, n-1$,

$$\varphi(x) = \varphi(y) \tag{55}$$

for all $x,\, y \in (x_k,\, x_{k+1})$. Without loss of generality let us assume that for all $k = 0,\ \ldots,\, n-2$,

$$\varphi(x) \neq \varphi(y) \tag{56}$$

for all $x \in (x_k,\, x_{k+1})$, $y \in (x_{k+1},\, x_{k+2})$.
Now suppose

$$0 = v_0 \leq v_1 < \cdots < v_n \leq v_{n+1} = 1$$

are defined by

$$v_j \quad = \quad \begin{cases} 0 & = \quad j = 0 \\ \varphi(\frac{x_{n-j} + x_{n-j+1}}{2}) & = \quad j \in \{1,\ \ldots,\, n\} \\ 1 & = \quad j = n+1 \end{cases} \tag{57}$$

Firstly, we will show that for all $k = 0,\ \ldots,\, n$ and all $v \in (v_k,\, v_{k+1})$,

$$\varphi^*(v) = x_{n-k}\,, \tag{58}$$

i.e. φ^* is a step function.

1. $k = 0$

   If $v_0 = v_1$, then (58) is trivially fulfilled, since quantification ranges over the empty set $(v_0,\, v_1) = (0,\, 0) = \varnothing$.

   Suppose $v_0 < v_1$ and $v \in (v_1,\, v_2) \neq \varnothing$. Then for all $x < x_n = 1$, $\varphi(x) \geq \varphi(\frac{x_{n-1} + x_n}{2}) = v_1$, i.e. $\{x : \varphi(x) < v\} \subseteq \{1\}$.

a. $\varphi(1) < v$. Then

$$\inf\{x : \varphi(x) < v\} = \inf\{1\} = 1 = x_n\,.$$

b. $\varphi \geq v$. Then

$$\inf\{x : \varphi(x) < v\} = \inf \varnothing = 1 = x_n\,.$$

2. $k \in \{1, \ldots, n-1\}$

Let $v \in (v_k, v_{k+1} = (\varphi(\frac{x_{n-k}+x_{n-k+1}}{2}), \varphi(\frac{x_{n-(k+1)}+x_{n-k}}{2}))$, i.e. for all $x < x_{n-k}$,

$$\varphi(x) \geq v$$

since φ is decreasing (and constant in (v_k, v_{k+1})). Therefore, $\{x : f(x) < v\} \subseteq [x_{n-k}, 1]$. Now, for all $x > x_{n-k}$, $\varphi(x) \leq \varphi(\frac{x_{n-k}+x_{n-k+1}}{2}) < v$, i.e. $(x_{n-k}, 1] \subseteq \{x : \varphi(x) < v\}$.

a. $\varphi(x_{n-k}) < v$. Then $\{x : \varphi(x) < v\} = [x_{n-k}, 1]$, i.e.

$$\varphi^*(v) = \inf\{x : \varphi(x) < v\} = \inf[x_{n-k}, 1] = x_{n-k}\,.$$

b. $\varphi(x_{n-k}) \geq v$. Then $\{x : \varphi(x) < v\} = (x_{n-k}, 1]$ and hence

$$\varphi^*(v) = \inf\{x : \varphi(x) < v\} = \inf(x_{n-k}, 1] = x_{n-k}\,.$$

3. $k = n$.

Then $v_n = \varphi(\frac{x_{n-n}+x_{n-n+1}}{2}) = \varphi(\frac{x_0+x_1}{2})$ and $v_{n+1} = 1$.

If $v_n = v_{n+1} = 1$, v ranges over the empty set $(v_n, v_{n+1}) = (1, 1) = \varnothing$ and condition (58) is trivially fulfilled.

If $v_n < v_{n+1}$, suppose $v \in (v_n, v_{n+1})$. Then as φ is decreasing and constant in (v_n, v_{n+1}),

$$\varphi(x) \leq \frac{x_0 + x_1}{2} < v$$

for all $x > 0$, i.e. $(0, 1] \subseteq \{x : \varphi(x) < v\}$.

a. $\varphi(0) < v$. Then $\{x : \varphi(x) < v\} = [0, 1]$ and hence

$$\varphi^*(v) = \inf\{x : \varphi(x) < v\} = \inf[0, 1] = 0 = x_0\,.$$

b. $\varphi(0) \geq v$. Then $\{x : \varphi(x) < v\} = (0, 1]$, i.e.

$$\varphi^*(v) = \inf\{x : \varphi(x) < v\} = \inf(0, 1] = 0 = x_0\,.$$

This finishes the proof that φ^* is a step function.

Applying (58), we obtain

$$\begin{aligned}
\int_0^1 \varphi^*(v)\,dv &= \sum_{k=0}^{n}(v_{k+1}-v_k)x_{n-k} \\
&= \sum_{k=0}^{n}\sum_{j=1}^{n-k}(x_j - x_{j-1}) \\
&= \sum_{l=0}^{n}\sum_{j=1}^{l}(v_{n-l+1}-v_{n-l})(x_j - x_{j-1}) \\
&= \sum_{j=1}^{l}(x_j - x_{j-1})\left(\sum_{l=j}^{n}(v_{n-l+1}-v_{n-l})\right) \\
&= \sum_{j=1}^{n}(x_j - x_{j-1})\left(\sum_{i=0}^{n-j}(v_{i+1}-v_i)\right) \\
&= \sum_{j=1}^{n}(x_j - x_{j-1})v_{n-j+1} \\
&= \sum_{j=1}^{n}(x_j - x_{j-1})\varphi(\frac{x_{j-1}+x_j}{2}) \\
&= \int_0^1 \varphi(x)\,dx\,.
\end{aligned}$$

Lemma 56

Let $f,\ g : \mathbf{I} \longrightarrow \mathbf{I}$ and $f \leq g$. Then $f^* \leq g^*$, where $f^*,\ g^* : \mathbf{I} \longrightarrow \mathbf{I}$ are defined by

$$\begin{aligned}
f^*(v) &= \inf\{x : f(x) < v\} \\
g^*(v) &= \inf\{x : g(x) < v\}
\end{aligned}$$

for all $v \in \mathbf{I}$.

Proof Suppose $g(x) < v$. Then also $f(x) \leq g(x) < v$ (since $f \leq g$) and hence

$$\{x : g(x) < v\} \subseteq \{x : f(x) < v\}\,,$$

from which we obtain

$$f^*(v) = \inf\{x : f(x) < v\} \leq \inf\{x : g(x) < v\} = g^*(v)\,.$$

Lemma 57

Suppose $f : \mathbf{I} \longrightarrow \mathbf{I}$ is monotonically decreasing. Then

$$\int_0^1 f(x)\,dx \quad = \quad \int_0^1 f^*(v)\,dv\,,$$

where $f^* : \mathbf{I} \longrightarrow \mathbf{I}$ is defined by $f^*(v) = \inf\{x : f(x) < v\}$.

Proof Let $\varepsilon > 0$. Because f (being monotonic) is Riemann integrable, there exist step functions $\varphi,\ \psi : \mathbf{I} \longrightarrow \mathbf{I}$ such that

$$\varphi \leq f \leq \psi \tag{59}$$

and

$$\int_0^1 \psi(x)\,dx - \int_0^1 \varphi\,dx < \varepsilon\,. \tag{60}$$

Without loss of generality, we may assume that ψ and φ are monotonically decreasing.
By L-56,

$$\varphi^* \leq f^* \leq \psi^*\,.$$

Due to the monotonicity of the Riemann integral and by applying L-55, we obtain

$$\int_0^1 \varphi(x)\,dx = \int_0^1 \varphi^*(v)\,dv \leq \int_0^1 f^*(v)\,dv \leq \int_0^1 \psi^*(v)\,dv = \int_0^1 \psi(x)\,dx$$

(f^* is Riemann integrable as it is monotonically decreasing by L-54). Now, (60) yields

$$\left|\int_0^1 f^*(v)\,dv - \int_0^1 f(x)\,dx\right| < \varepsilon\,.$$

As $\varepsilon \to 0$, we obtain $\int_0^1 f^*(v)\,dv = \int_0^1 f(x)\,dx$.

Proof of Theorem 56

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $X_1,\ \ldots,\ X_n \in \widetilde{\mathcal{P}}(E)$.

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$.

$$\begin{aligned}
\mathcal{G}(D)(X_1, \ldots, X_n) &= \int_0^1 (\tfrac{1}{2} + \tfrac{1}{2} h_{\mathrm{F}}(\ell))\, d\ell && \text{by L-53} \\
&= \tfrac{1}{2} + \int_0^1 \tfrac{1}{2} h_{\mathrm{F}}(\ell)\, d\ell \\
&= \tfrac{1}{2} + \int_{\frac{1}{2}}^1 h_{\mathrm{F}}(2v-1)\, dv \\
&= \int_0^{\frac{1}{2}} 1\, dv \\
&\quad + \int_{\frac{1}{2}}^1 \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < \tfrac{1}{2} + \tfrac{1}{2}(2v-1)\}\, dv \\
&= \int_0^{\frac{1}{2}} \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v\}\, dv \\
&\quad + \int_{\frac{1}{2}}^1 \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v\}\, dv \\
&= \int_0^1 \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v\}\, dv \\
&= \int_0^1 D_\gamma(X_1, \ldots, X_n)\, d\gamma && \text{by L-57 and } D_\gamma \text{ decr.} \\
&= \mathcal{M}(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

$\int_0^{1/2} 1\, dv$ may be replaced by $\int_0^{1/2} \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v\}\, dv$ because $D_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ and hence for all $v \leq \frac{1}{2}$, $\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v\} = \inf \varnothing = 1$.

b. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$. Then by Th-42 and L-32, $\mathcal{M}(D)(X_1, \ldots, X_n) = \frac{1}{2}$.

$$\begin{aligned}
\mathcal{G}(D)(X_1, \ldots, X_n) &= \int_0^1 \tfrac{1}{2}\, d\ell && \text{by L-53} \\
&= \tfrac{1}{2}\,.
\end{aligned}$$

c. $D_0(X_1,\ \ldots,\ X_n) < \frac{1}{2}$.

$$\begin{aligned}
\mathcal{G}(D)(X_1,\ \ldots,\ X_n) &= \int_0^1 \mathrm{K}(1 - h_{\mathrm{T}}(\ell),\, 1 - h_{\mathrm{F}}(\ell))\, d\ell && \text{by Th-53}\\
&= \int_0^1 (\tfrac{1}{2} - \tfrac{1}{2} h_{\mathrm{T}}(\ell)\, d\ell && \text{by L-53}\\
&= \tfrac{1}{2} - \int_0^{\frac{1}{2}} h_{\mathrm{T}}(1 - 2w)\, dw\\
&= 1 - (\int_{\frac{1}{2}}^1 1\, dw + \int_0^1 \inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\}\, dw)\\
&= 1 - \big(\int_{\frac{1}{2}}^1 \inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\}\, dw\\
&+ \int_0^{\frac{1}{2}} \inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\}\, dw\big)\\
&= 1 - \int_0^1 \inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\}\, dw\\
&= 1 - \int_0^1 \inf\{\gamma : 1 - (1 - D_\gamma)(X_1,\ \ldots,\ X_n) > w\}\, dw\\
&= 1 - \int_0^1 \inf\{\gamma : 1 - (\neg D)_\gamma)(X_1,\ \ldots,\ X_n) > w\}\, dw\\
&= 1 - \int_0^1 \inf\{\gamma : (\neg D)_\gamma)(X_1,\ \ldots,\ X_n) < 1 - w\}\, dw\\
&= 1 - \int_0^1 \inf\{\gamma : (\neg D)_\gamma)(X_1,\ \ldots,\ X_n) < v\}\, dv\\
&= 1 - \mathcal{M}(\neg D)(X_1,\ \ldots,\ X_n)\\
&= 1 - (1 - \mathcal{M}(D)(X_1,\ \ldots,\ X_n)) && \text{(DFS 3), Th-44}\\
&= \mathcal{M}(D)(X_1,\ \ldots,\ X_n)\,.
\end{aligned}$$

Note that $\int_{1/2}^1 1\, dw$ may be replaced by

$$\int_{1/2}^1 \inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\}\, dw$$

because $D_\gamma(X_1,\ \ldots,\ X_n) \leq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ and hence for $w \geq \frac{1}{2}$,

$$\inf\{\gamma : D_\gamma(X_1,\ \ldots,\ X_n) > w\} = \inf \varnothing = 1\,.$$

B.24 Proof of Theorem 57

Lemma 58

a. If $D_0(X_1,\ \ldots,\ X_n) < \frac{1}{2}$, *then* $\ell^* = 1 - 2D_0^*$.

b. If $D_0(X_1, \ldots, X_n) = \frac{1}{2}$*, then* $D_0^* = \frac{1}{2}$ *and* $\ell^* = 0$*.*

c. If $D_0(X_1, \ldots, X_n) > \frac{1}{2}$*, then* $\ell^* = 2D_0^* - 1$*.*

Proof By definition, $\ell^* = \inf\{\ell \in \mathbf{I} : \mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}\}$. Suppose $\ell^* = 0$. Then $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\ell \geq \ell^* = 0$. It follows that $\inf\{\ell > 0 : \mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}\} = 0 = \ell^*$, i.e. we can assume $\ell > 0$ (in the infimum expression) throughout the proof.

a. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$
By L-53, we know that for all $\ell > 0$,

$$\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \tfrac{1}{2} - \tfrac{1}{2} h_{\mathrm{T}}(\ell)\,,$$

i.e.

$$\ell^* = \inf\{\ell > 0 : h_{\mathrm{T}}(\ell) = 0\}\,. \tag{61}$$

Now $h_{\mathrm{T}}(\ell) = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \frac{1}{2} - \frac{1}{2}\ell\}$, i.e.

$$\begin{aligned}\ell^* &= \inf\{\ell > 0 : \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} = 0\}\\ &= \inf\{1 - 2v > 0 : \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > v\} = 0\}\\ &= 1 - 2\sup\{v \in [0, \tfrac{1}{2}) : \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > v\} = 0\}\,.\end{aligned}$$

(The last equation holds because $D_\gamma(X_1, \ldots, X_n)$ is monotonically increasing by Th-42). Let us note that

$$\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > v\} = \inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\}$$

for all $v \in [0, \frac{1}{2})$, again because $D_\gamma(X_1, \ldots, X_n)$ is monotonically increasing in γ. (If $0 = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > v\}$, then $D_\gamma(X_1, \ldots, X_n) > v$ for all $\gamma > 0$, i.e. $\inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\} = 0$ also.) We are left to show that

$$D_0^* = \sup\{v \in [0, \tfrac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\} = 0\}\,. \tag{62}$$

1. "$\leq$" Let $\varepsilon > 0$, $v' = D_0^* - \varepsilon$. Noting that

$$D_0^* = \lim_{\gamma \to 0^+} D_\gamma(X_1, \ldots, X_n) = \inf\{D_\gamma(X_1, \ldots, X_n) : \gamma \in (0, 1]\}$$

(because $D_\gamma(X_1, \ldots, X_n)$ is monotonically increasing by Th-42), we obtain that for all $\gamma \in (0, 1]$,

$$D_\gamma(X_1, \ldots, X_n) \geq D_0^* > v'\,,$$

i.e.

$$v' \in \{v : \inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\} = 0\}$$

and hence

$$\sup\{v \in [0, \tfrac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\} = 0\} \geq v' = D_0^* - \varepsilon\,.$$

As $\varepsilon \to 0$, we obtain

$$\sup\{v \in [0, \tfrac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v\} = 0\} \geq D_0^*\,.$$

2. "$\geq$"

Firstly if $\sup\{v \in [0, \frac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma > v\} = 0\} = 0$ the inequation trivially holds. So let us assume that $\sup\{v \in [0, \frac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma > v\} = 0\} > 0$.

Let $\varepsilon > 0$, $v^* = \sup\{v \in [0, \frac{1}{2}) : \inf\{\gamma \in (0, 1] : D_\gamma > v\} = 0\}$. Then there exists $v' \in (v^* - \varepsilon, v^*)$ such that

$$\inf\{\gamma \in (0, 1] : D_\gamma(X_1, \ldots, X_n) > v'\} = 0 \,.$$

Because $D_\gamma(X_1, \ldots, X_n)$ is monotonically increasing, it follows that

$$D_\gamma(X_1, \ldots, X_n) > v'$$

for all $\gamma \in (0, 1]$, i.e. $D_0^* = \inf\{D_\gamma(X_1, \ldots, X_n) : \gamma \in (0, 1]\} \geq v' > v^* - \varepsilon$.

Now $\varepsilon \to 0$ yields

$$D_0^* \geq v^* = \sup\{v \in [0, \tfrac{1}{2}) : \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > v\} = 0\} \,,$$

as desired.

b. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$
Then $D_\gamma(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\gamma \in \mathbf{I}$ by Th-42, i.e. $D_0^* = \frac{1}{2}$. By L-53, $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\ell > 0$. Therefore,

$$\ell^* = \inf\{\ell > 0 : \mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \tfrac{1}{2}\} = \inf(0, 1] = 0 \,.$$

c. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$
The proof is entirely analogous to that of the dual case a.

Lemma 59

a. If $D_0(X_1, \ldots, X_n) < \frac{1}{2}$, then $G_0^* = \frac{1}{2} - \frac{1}{2}\gamma^*$.

b. If $D_0(X_1, \ldots, X_n) = \frac{1}{2}$, then $\gamma^* = 0$ and $G_0^* = \frac{1}{2}$.

c. If $D_0(X_1, \ldots, X_n) > \frac{1}{2}$, then $G_0^* = \frac{1}{2} + \frac{1}{2}\gamma^*$.

Proof

a. Then for $\ell > 0$, $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2} - \frac{1}{2} h_{\mathrm{T}}(\ell)$ by L-53. In particular, $\mathcal{G}_\ell(D)(X_1, \ldots, X_n)$ is monotonically increasing in ℓ and hence

$$G_0^* = \lim_{\ell \to 0^+} \mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \inf\{\mathcal{G}_\ell(D)(X_1, \ldots, X_n) : \ell \in (0, 1]\} \,,$$

from which we obtain

$$\begin{aligned}
G_0^* &= \inf\{\mathcal{G}_\ell(D)(X_1, \dots, X_n) : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{h_{\mathrm{T}}(\ell) : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\inf\{\gamma : D_\gamma > \tfrac{1}{2} - \tfrac{1}{2}\ell\} : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\inf\{\gamma : D_\gamma > v\} : v \in [0, \tfrac{1}{2})\} && (*) \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\sup\{\gamma : D_\gamma \le v\} : v \in (\tfrac{1}{2}, 1]\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\gamma : \exists v \in [0, \tfrac{1}{2}) \text{ s.th } D_\gamma(X_1, \dots, X_n) \le v\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\sup\{\gamma : D_\gamma(X_1, \dots, X_n) < \tfrac{1}{2}\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\inf\{\gamma : D_\gamma(X_1, \dots, X_n) \ge \tfrac{1}{2}\} && (**) \\
&= \tfrac{1}{2} - \tfrac{1}{2}\inf\{\gamma : D_\gamma(X_1, \dots, X_n) = \tfrac{1}{2}\} \\
&= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* .
\end{aligned}$$

Equations (*) and (**) hold because $D_\gamma(X_1, \dots, X_n)$ is monotonically increasing by Th-42. The second-last equation holds because $D_0(X_1, \dots, X_n) < \frac{1}{2}$ and hence $D_\gamma(X_1, \dots, X_n) \le \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

b. $D_0(X_1, \dots, X_n) = \frac{1}{2}$.
If $D_0(X_1, \dots, X_n) = \frac{1}{2}$, then $D_\gamma(X_1, \dots, X_n) = \frac{1}{2}$ for all γ (see Th-42) and hence $\gamma^* = \inf\{\gamma : D_\gamma(X_1, \dots, X_n) = \frac{1}{2}\} = 0$.
We already know by L-53 that $\mathcal{G}_\ell(D)(X_1, \dots, X_n) = \frac{1}{2}$ for all $\ell > 0$. Therefore,

$$\begin{aligned}
G_0^* &= \lim_{\ell \to 0^+} \mathcal{G}_\ell(D)(X_1, \dots, X_n) \\
&= \lim_{\ell \to 0^+} \tfrac{1}{2} \\
&= \tfrac{1}{2} .
\end{aligned}$$

c. $D_0(X_1, \dots, X_n) > \frac{1}{2}$.
The proof is analogous to that of the dual case a.
Suppose $D_0(X_1, \dots, X_n) > \frac{1}{2}$. Then for $\ell > 0$, $\mathcal{G}_\ell(D)(X_1, \dots, X_n) = \frac{1}{2} + \frac{1}{2}h_{\mathrm{F}}(\ell)$ by L-53. In particular, $\mathcal{G}_\ell(D)(X_1, \dots, X_n)$ is monotonically decreasing in ℓ and hence

$$G_0^* = \lim_{\ell \to 0^+} \mathcal{G}_\ell(D)(X_1, \dots, X_n) = \sup\{\mathcal{G}_\ell(D)(X_1, \dots, X_n) : \ell \in (0, 1]\} ,$$

from which we obtain

$$\begin{aligned}
G_0^* &= \sup\{\mathcal{G}_\ell(D)(X_1, \dots, X_n) : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{h_{\mathrm{F}}(\ell) : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\inf\{\gamma : D_\gamma < \tfrac{1}{2} + \tfrac{1}{2}\ell\} : \ell \in (0, 1]\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\inf\{\gamma : D_\gamma < v\} : v \in (\tfrac{1}{2}, 1]\} && (*) \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\sup\{\gamma : D_\gamma \ge v\} : v \in (\tfrac{1}{2}, 1]\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma : \exists v \in (\tfrac{1}{2}, 1] \text{ s.th } D_\gamma(X_1, \dots, X_n) \ge v\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\sup\{\gamma : D_\gamma(X_1, \dots, X_n) > \tfrac{1}{2}\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\inf\{\gamma : D_\gamma(X_1, \dots, X_n) \le \tfrac{1}{2}\} && (**) \\
&= \tfrac{1}{2} + \tfrac{1}{2}\inf\{\gamma : D_\gamma(X_1, \dots, X_n) = \tfrac{1}{2}\} \\
&= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* .
\end{aligned}$$

Equations (*) and (**) hold because $D_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing by Th-42. The second-last equation holds because $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and hence $D_\gamma(X_1, \ldots, X_n) \geq \frac{1}{2}$ for all $\gamma \in \mathbf{I}$.

Proof of Theorem 57

Let $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

1. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$.

$$\begin{aligned}
\mathcal{G}^*(D)((X_1, \ldots, X_n) &= (G_0^* - \tfrac{1}{2})\ell^* + \tfrac{1}{2} \\
&= (\tfrac{1}{2} - \tfrac{1}{2}\gamma^* - \tfrac{1}{2})(1 - 2D_0^*) + \tfrac{1}{2} && \text{(by L-59, L-58)} \\
&= -\tfrac{1}{2}\gamma^*(1 - 2D_0^*) + \tfrac{1}{2} \\
&= (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\
&= \mathcal{M}^*(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

2. $D_0(X_1, \ldots, X_n) = \frac{1}{2}$. Then

$$\begin{aligned}
\mathcal{G}^*(D)(X_1, \ldots, X_n) &= (G_0^* - \tfrac{1}{2})\ell^* + \tfrac{1}{2} \\
&= (\tfrac{1}{2} - \tfrac{1}{2})0 + \tfrac{1}{2} && \text{by L-59, L-58} \\
&= \tfrac{1}{2} \\
&= (\tfrac{1}{2} - \tfrac{1}{2})0 + \tfrac{1}{2} \\
&= (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} && \text{by L-59, L-58} \\
&= \mathcal{M}^*(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

3. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$.

$$\begin{aligned}
\mathcal{G}^*(D)(X_1, \ldots, X_n) &= (G_0^* - \tfrac{1}{2})\ell^* + \tfrac{1}{2} \\
&= (\tfrac{1}{2} + \tfrac{1}{2}\gamma^* - \tfrac{1}{2})(2D_0^* - 1) + \tfrac{1}{2} && \text{by L-59, L-58} \\
&= (D_0^* - \tfrac{1}{2})\gamma^* + \tfrac{1}{2} \\
&= \mathcal{M}^*(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

B.25 Proof of Theorem 58

Suppose $D : \mathcal{P}(E)^n \longrightarrow \mathbf{I}$ is a fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$.

a. $D_0(X_1, \ldots, X_n) > \frac{1}{2}$ and $\gamma^* > 0$.

Then $G_0^* = \frac{1}{2} + \frac{1}{2}\gamma^* > \frac{1}{2}$ by L-59 and hence $\mathcal{G}_0(D)(X_1, \ldots, X_n) > \frac{1}{2}$ by Th-54. Therefore,

$$\begin{aligned}
&\mathcal{G}_*(D)(X_1, \ldots, X_n)\\
&= \tfrac{1}{2} + \sup\{\ell(\mathcal{G}_\ell(D)(X_1, \ldots, X_n) - \tfrac{1}{2}) : \ell \in \mathbf{I}\}\\
&= \tfrac{1}{2} + \sup\{\ell(\mathcal{G}_\ell(D)(X_1, \ldots, X_n) - \tfrac{1}{2}) : \ell \in \mathbf{I} \setminus \{0\}\}\\
&= \tfrac{1}{2} + \sup\{\ell(\tfrac{1}{2} + \tfrac{1}{2}h_{\mathrm{F}}(\ell) - \tfrac{1}{2}) : \ell \in \mathbf{I} \setminus \{0\}\} && \text{by L-53}\\
&= \tfrac{1}{2} + \sup\{\tfrac{1}{2}\ell\, h_{\mathrm{F}}(\ell) : \ell \in \mathbf{I} \setminus \{0\}\}\\
&= \tfrac{1}{2} + \sup\{\tfrac{1}{2}\ell \inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) < \tfrac{1}{2} + \tfrac{1}{2}\ell\} : \ell \in \mathbf{I} \setminus \{0\}\}\\
&= \tfrac{1}{2} + \sup\{\tfrac{1}{2}(2v-1) \inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) < v\} : v \in (\tfrac{1}{2}, 1]\}\\
&= \tfrac{1}{2} + \sup\{\inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) < v\} \cdot (v - \tfrac{1}{2}) : v \in (\tfrac{1}{2}, 1]\}\,.
\end{aligned}$$

Furthermore

$$\mathcal{M}_*(D)(X_1, \ldots, X_n) = \tfrac{1}{2} + \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}\,.$$

Abbreviating

$$\begin{aligned}
F &= \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}\\
G &= \sup\{\inf\{\gamma \in \mathbf{I} : D_\gamma(X_1, \ldots, X_n) < v\} \cdot (v - \tfrac{1}{2}) : v \in (\tfrac{1}{2}, 1]\}
\end{aligned}$$

we are left to prove that $G = F$.

"$\geq$" To see that $G \geq F$, let $\gamma' \in \mathbf{I}$ and $v' = D_{\gamma'}(X_1, \ldots, X_n)$. Because $D_\gamma(X_1, \ldots, X_n)$ is decreasing in γ (by Th-42),

$$\gamma' \leq \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v'\}$$

and hence

$$\gamma'(D_{\gamma'}(X_1, \ldots, X_n) - \tfrac{1}{2}) \leq \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v'\}(v' - \tfrac{1}{2})\,,$$

which proves that $G \geq F$.

"$\leq$" To prove that $G \leq F$, we will show that $F \geq G - \varepsilon$ for all $\varepsilon > 0$.

Let $\varepsilon > 0$. Then there exists $v_0 \in (\frac{1}{2}, 1]$ such that

$$G - \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\}(v_0 - \tfrac{1}{2}) < \tfrac{\varepsilon}{2}\,, \tag{63}$$

i.e. $\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\}(v_0 - \frac{1}{2}) > G - \frac{\varepsilon}{2}$. Let $\gamma_0 = \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\} - \frac{\varepsilon}{2}$. Then

$$D_{\gamma_0}(X_1, \ldots, X_n) \geq v_0\,,$$

because $D_\gamma(X_1, \ldots, X_n)$ is monotonically decreasing by Th-42. We compute:

$$\begin{aligned}
F &= \sup\{\gamma(D_\gamma(X_1, \ldots, X_n) - \tfrac{1}{2}) : \gamma \in \mathbf{I}\}\\
&\geq \gamma_0 D_{\gamma_0}(X_1, \ldots, X_n)\\
&\geq (\inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\} - \tfrac{\varepsilon}{2}) \cdot v_0\\
&= \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\} \cdot v_0 - \tfrac{v_0\varepsilon}{2}\\
&\geq \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) < v_0\} \cdot v_0 - \tfrac{\varepsilon}{2}\\
&> (G - \tfrac{\varepsilon}{2}) - \tfrac{\varepsilon}{2} && \text{by (63)}\\
&= G - \varepsilon\,.
\end{aligned}$$

$\varepsilon \to 0$ yields $G \leq F$.

b. $D_0(X_1, \ldots, X_n) < \frac{1}{2}$ and $\gamma^* > 0$.
Then $G_0^* = \frac{1}{2} - \frac{1}{2}\gamma^* < \frac{1}{2}$ by L-59 and hence $\mathcal{G}_0(D)(X_1, \ldots, X_n) < \frac{1}{2}$ by Th-54. Therefore,

$$\begin{aligned}
\mathcal{G}^*(D)(X_1, \ldots, X_n) &= \tfrac{1}{2} - \sup\{\ell(\tfrac{1}{2} - \mathcal{G}_\ell(D)(X_1, \ldots, X_n)) : \ell \in \mathbf{I}\} \\
&= \tfrac{1}{2} - \sup\{\ell(\tfrac{1}{2} - \mathcal{G}_\ell(D)(X_1, \ldots, X_n)) : \ell > 0\} \\
&= \tfrac{1}{2} - \sup\{\ell(\tfrac{1}{2} - (\tfrac{1}{2} - \tfrac{1}{2}h_{\mathrm{T}}(\ell))) : \ell > 0\} \\
&\qquad \text{by L-53} \\
&= \tfrac{1}{2} - \sup\{\tfrac{1}{2}\ell h_{\mathrm{T}}(\ell) : \ell > 0\} \\
&= \tfrac{1}{2} - \sup\{\tfrac{1}{2}\ell \cdot \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > \tfrac{1}{2} - \tfrac{1}{2}\ell\} : \ell > 0\} \\
&= \tfrac{1}{2} - \sup\{\tfrac{1}{2}(1 - 2w) \inf\{\gamma : D_\gamma(X_1, \ldots, X_n) > w\} : v \in [0, \tfrac{1}{2})\} \\
&= \tfrac{1}{2} - \sup\{(\tfrac{1}{2} - w) \inf\{\gamma : 1 - D_\gamma(X_1, \ldots, X_n) < 1 - w\} : w \in [0, \tfrac{1}{2})\} \\
&= \tfrac{1}{2} - \sup\{(v - \tfrac{1}{2}) \inf\{\gamma : (\neg D)_\gamma(X_1, \ldots, X_n) < v\} : v \in (\tfrac{1}{2}, 1]\} \\
&= 1 - (\tfrac{1}{2} + \sup\{(v - \tfrac{1}{2}) \inf\{\gamma : (\neg D)_\gamma(X_1, \ldots, X_n) < v\} : v \in (\tfrac{1}{2}, 1]\}) \\
&= 1 - \mathcal{M}_*(\neg D)(X_1, \ldots, X_n) \\
&\qquad \text{by case } \textbf{a.} \\
&= 1 - (1 - \mathcal{M}_*(D)(X_1, \ldots, X_n)) \\
&\qquad \text{by } \widetilde{\mathcal{M}_*}(\neg) = \neg, \mathcal{M}_* \text{ DFS} \\
&= \mathcal{M}_*(D)(X_1, \ldots, X_n)\,.
\end{aligned}$$

c. $\gamma^* = 0$ (In particular, this is the case if $D_0(X_1, \ldots, X_n) = \frac{1}{2}$.)
Then $\mathcal{M}_*(D)(X_1, \ldots, X_n) = \frac{1}{2}$ because in this case, $D_\gamma(X_1, \ldots, X_n) = 0$ for all $\gamma > 0$. Furthermore, $\gamma^* = 0$ entails that $G_0^* = \frac{1}{2}$ by L-59, i.e. $\mathcal{G}_\ell(D)(X_1, \ldots, X_n) = \frac{1}{2}$ for all $\ell > 0$. This proves that $\mathcal{G}^*(D)(X_1, \ldots, X_n) = \frac{1}{2}$ also.

B.26 Proof of Theorem 59

Lemma 60
Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is a three-valued fuzzy pre-determiner. Then

$$D_{\langle \ell \rangle} = D\,,$$

for all $\ell \in \mathbf{I}$.

Proof Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is a three-valued fuzzy pre-determiner, $(X_1, \ldots, X_n) \in \mathcal{P}(E)^n$ and $\ell \in \mathbf{I}$.

a. $\ell = 0$. Then

$$\begin{aligned}
D_{\langle 0 \rangle}(X_1, \ldots, X_n) &= \begin{cases} 1 & : \quad D(X_1, \ldots, X_n) > \frac{1}{2} \\ \frac{1}{2} & : \quad D(X_1, \ldots, X_n) = \frac{1}{2} \\ 0 & : \quad D(X_1, \ldots, X_n) < \frac{1}{2} \end{cases} \\
&= \begin{cases} 1 & : \quad D(X_1, \ldots, X_n) = 1 \\ \frac{1}{2} & : \quad D(X_1, \ldots, X_n) = \frac{1}{2} \\ 0 & : \quad D(X_1, \ldots, X_n) = 0 \end{cases} \qquad \text{as } D \text{ three-valued} \\
&= D(X_1, \ldots, X_n)\,.
\end{aligned}$$

b. $\ell > 0$. In this case,

$$\begin{aligned} D_{\langle\ell\rangle}(X_1, \ldots, X_n) &= \begin{cases} 1 & : & D(X_1, \ldots, X_n) \geq \frac{1}{2} + \frac{1}{2}\ell \\ \frac{1}{2} & : & \frac{1}{2} - \frac{1}{2}\ell < D(X_1, \ldots, X_n) < \frac{1}{2} + \frac{1}{2}\ell \\ 0 & : & D(X_1, \ldots, X_n) < \frac{1}{2} - \frac{1}{2}\ell \end{cases} \\ &= \begin{cases} 1 & : & D(X_1, \ldots, X_n) = 1 \\ \frac{1}{2} & : & D(X_1, \ldots, X_n) = \frac{1}{2} \\ 0 & : & D(X_1, \ldots, X_n) = 0 \end{cases} && \text{as } D \text{ three-valued} \\ &= D(X_1, \ldots, X_n)\,. \end{aligned}$$

Lemma 61

For every three-valued fuzzy pre-determiner $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$,

$$\mathcal{G}(D) = G(D)\,.$$

Proof Let $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ a three-valued fuzzy pre-determiner and $(X_1, \ldots, X_n) \in \widetilde{\mathcal{P}}(E)^n$. Then

$$\begin{aligned} \mathcal{G}(D)(X_1, \ldots, X_n) &= \int_0^1 \mathcal{G}_\ell(D)(X_1, \ldots, X_n)\, d\ell \\ &= \int_0^1 G(D_{\langle\ell\rangle})(X_1, \ldots, X_n)\, d\ell \\ &= \int_0^1 G(D)(X_1, \ldots, X_n)\, d\ell && \text{by L-60} \\ &= G(D)(X_1, \ldots, X_n) \int_0^1 1\, d\ell \\ &= G(D)(X_1, \ldots, X_n)\,. \end{aligned}$$

Proof of Theorem 59

Suppose $D : \mathcal{P}(E)^n \longrightarrow \{0, \frac{1}{2}, 1\}$ is a three-valued fuzzy pre-determiner. Then by L-61,

$$\mathcal{G}(D) = G(D)\,,$$

by Th-56,

$$\mathcal{G}(D) = \mathcal{M}(D)\,,$$

and by Th-48,

$$\mathcal{M}(D) = \mathcal{M}_*(D) = \mathcal{M}^*(D)\,,$$

which finishes the proof.

B.27 Proof of Theorem 60

We will firstly consider the one-place determiner $(\geq \mathsf{n}) : \mathcal{P}(E) \longrightarrow \mathbf{2}$ defined by

$$(\geq \mathsf{n})(X) = \begin{cases} 1 & : \quad \mathrm{card}\,(X) \geq n \\ 0 & : \quad \text{else} \end{cases}$$

for all $X \in \mathcal{P}(E)$.
Suppose $X \in \widetilde{\mathcal{P}}(E)$. We firstly observe that

$$\begin{aligned} & (\geq \mathsf{n})_\gamma(X) \\ &= \mathrm{m}_{\frac{1}{2}}\{(\geq \mathsf{n})(Y) : Y \in \mathcal{Y}^\gamma\} \\ &= \mathrm{m}_{\frac{1}{2}}(\inf\{\{(\geq \mathsf{n})(Y) : Y \in \mathcal{Y}^\gamma\},\ \sup\{(\geq \mathsf{n})(Y) : Y \in \mathcal{Y}^\gamma\}) \\ &= \mathrm{m}_{\frac{1}{2}}((\geq \mathsf{n})((X)^{\min}_\gamma),\ (\geq \mathsf{n})((X)^{\max}_\gamma)) \end{aligned} \tag{*}$$

because $(\geq \mathsf{n})$ is monotonically increasing in its argument.
By assumption, E is finite, i.e. $|E| = m$ where $m > 0$.
Choose pairwise distinct $v_j \in E$ ($j = 1, \ldots, m$) such that $E = \{v_1, \ldots, v_m\}$ and

$$\mu_X(v_1) \geq \mu_X(v_2) \geq \cdots \geq \mu_X(v_{m-1}) \geq \mu_X(v_m)\,.$$

Further define

$$\mu_{(j)} = \mu_X(v_j)$$

for all $j = 1, \ldots, m$.

a. $n > m$.
   Then $(\geq \mathsf{n})(Y) = 0$ for all $Y \in \mathcal{P}(E)$ and hence $\mathcal{M}(\geq \mathsf{n})(X) = 0$ (by Th-5 and (DFS 1)).

b. $n \leq m$ and $\mu_{(n)} \geq \frac{1}{2}$.
   Then

$$\{v_1, \ldots, v_n\} \subseteq (X)^{\max}_\gamma$$

   and hence

$$(\geq \mathsf{n})((X)^{\max}_\gamma) = 1 \tag{64}$$

   for all $\gamma \in \mathbf{I}$.
   Let us define

$$\gamma' = 2\mu_{(n)} - 1\,.$$

   Then for all $\gamma \leq \gamma'$,

$$\{v_1, \ldots, v_n\} \subseteq (X)^{\min}_\gamma\,,$$

i.e. $(\geq \mathbf{n})((X)^{\min}_\gamma) = 1$ and hence

$$(\geq \mathbf{n})_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(1,\, 1) = 1\,, \tag{65}$$

and for all $\gamma > \gamma'$,

$$(X)^{\min}_\gamma \subseteq \{v_1,\, \dots,\, v_{n-1}\}\,,$$

i.e. $(\geq \mathbf{n})((X)^{\min}_\gamma) = 0$ and hence

$$(\geq \mathbf{n})_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(0,\, 1) = \tfrac{1}{2}\,. \tag{66}$$

Combining (65) and (66), we see that $\gamma' = \gamma^*$ and hence

$$\begin{aligned} \mathcal{M}(\geq \mathbf{n})(X) &= \tfrac{1}{2} + \tfrac{1}{2}\gamma^* && \text{by Th-49} \\ &= \tfrac{1}{2} + \tfrac{1}{2}(2\mu_{(n)} - 1) && \text{by def. of } \gamma' \\ &= \tfrac{1}{2} + \mu_{(n)} - \tfrac{1}{2} \\ &= \mu_{(n)}\,. \end{aligned}$$

c. $n \leq m$ and $\mu_{(n)} < \frac{1}{2}$.
Then

$$(X)^{\min}_\gamma \subseteq \{v_1,\, \dots,\, v_{n-1}\}$$

and hence

$$(\geq \mathbf{n})((X)^{\min}_\gamma) = 0 \tag{67}$$

for all $\gamma \in \mathbf{I}$.
Let us define

$$\gamma'' = 1 - 2\mu_{(n)}\,.$$

Then for all $\gamma \leq \gamma''$,

$$(X)^{\max}_\gamma \subseteq \{v_1,\, \dots,\, v_{n-1}\}\,,$$

i.e. $(\geq \mathbf{n})((X)^{\max}_\gamma) = 0$ and hence

$$(\geq \mathbf{n})_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(0,\, 0) = 0\,, \tag{68}$$

and for all $\gamma > \gamma''$,

$$\{v_1,\, \dots,\, v_n\} \subseteq (X)^{\max}_\gamma\,,$$

i.e. $(\geq \mathbf{n})((X)^{\max}_\gamma) = 1$ and hence

$$(\geq \mathbf{n})_\gamma(X) = \mathrm{m}_{\frac{1}{2}}(0,\, 1) = \tfrac{1}{2}\,. \tag{69}$$

Combining (68) and (69), we see that $\gamma'' = \gamma^*$ and hence

$$\begin{aligned} \mathcal{M}(\geq \mathbf{n})(X) &= \tfrac{1}{2} - \tfrac{1}{2}\gamma^* && \text{by Th-49} \\ &= \tfrac{1}{2} - \tfrac{1}{2}(1 - 2\mu_{(n)}) && \text{by def. of } \gamma'' \\ &= \tfrac{1}{2} - \tfrac{1}{2} + \mu_{(n)} \\ &= \mu_{(n)}\,. \end{aligned}$$

The claims of the theorem are now immediate from

$$\begin{aligned} \textbf{atleast n} &= (\geq n)\cap \\ \textbf{atmost n} &= \neg\textbf{atleast n+1} \\ \textbf{more than n} &= \textbf{atleast n+1} \\ \textbf{less than n} &= \textbf{atmost n-1} \end{aligned}$$

by applying (DFS 6) and (DFS 3).

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